Tree traversals

- Systematic visiting of all nodes in the tree.
- Each node visited once, does not specify order.
- Breadth-first traversal: visit each node, starting from the lowest level and moving down by level, visiting nodes from left to right.
- Depth-first traversals: go in subtree as deep as you can, backtrack. Differ on the order of visiting root, left, right subtrees.
- preorder: VLR.
- inorder: LVR.
- postorder: LRV.
- these definitions are recursive.

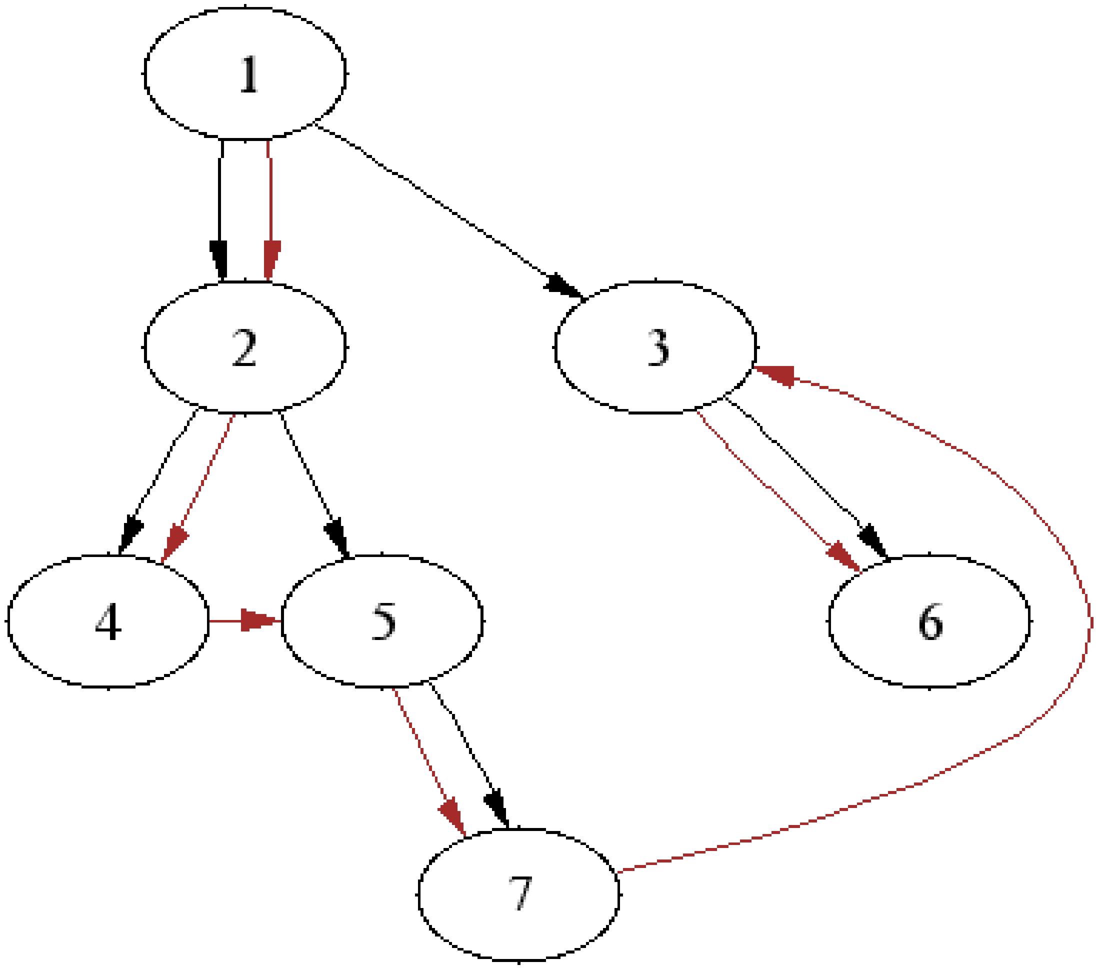
BFS traversal of a tree

```
void BST::breadthFirst(){
  Queue<BSTNode> q;
  BSTNode *p = root;
  if (p != 0){
   q.enqueue(p);
   while (!q.empty()){
       p = q.dequeue();
       visit(p);
       if (p-> left != 0)
         q.enqueue(p->left);
       if (p-> right != 0)
         q.enqueue(p->left);
   }
  }
}
```

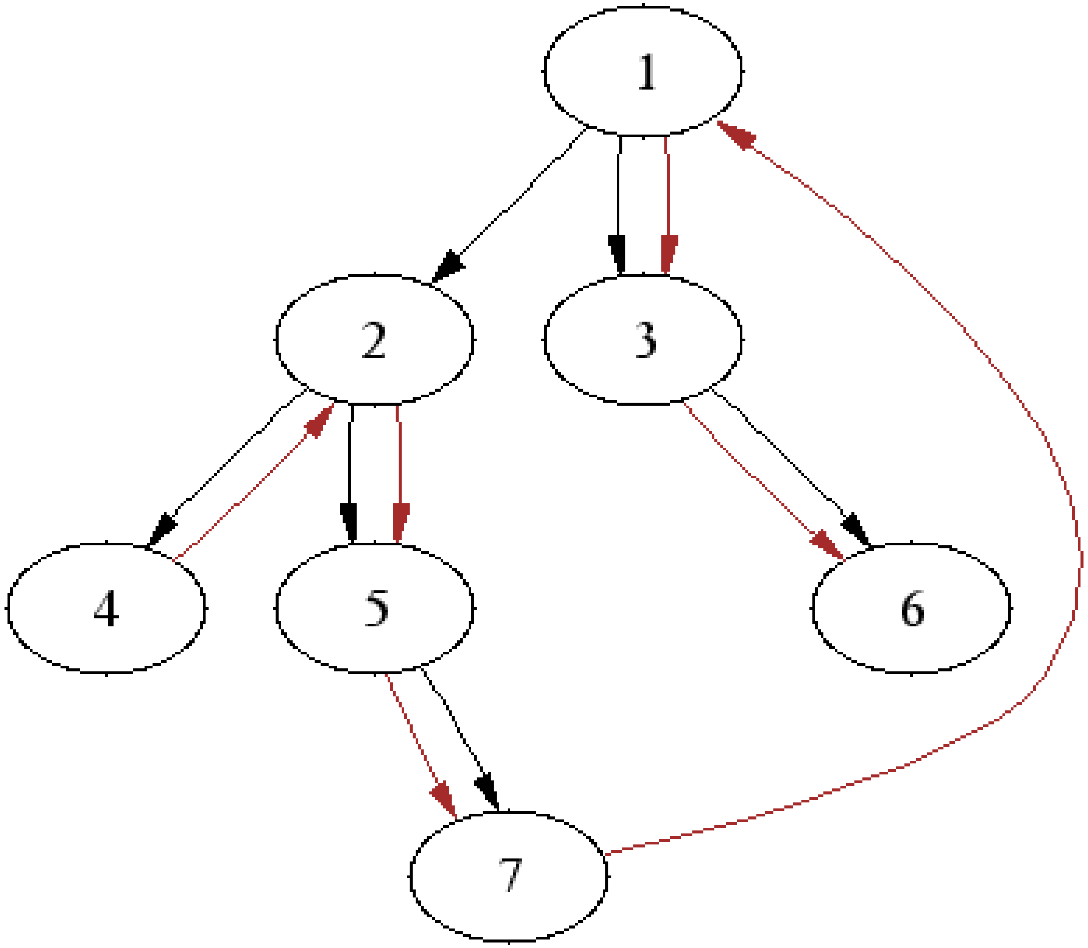
DFS traversals of a tree

```
void BST::inorder(BSTNode *p){
  if (p!=0){
   inorder(p->left);
   visit(p);
   inorder(p->right);
  }}
void BST::preorder(BSTNode *p){
  if (p!=0){
   visit(p);
   preorder(p->left);
   preorder(p->right);
  }}
void BST::postorder(BSTNode *p){
  if (p!=0){
   postorder(p->left);
   postorder(p->right);
```

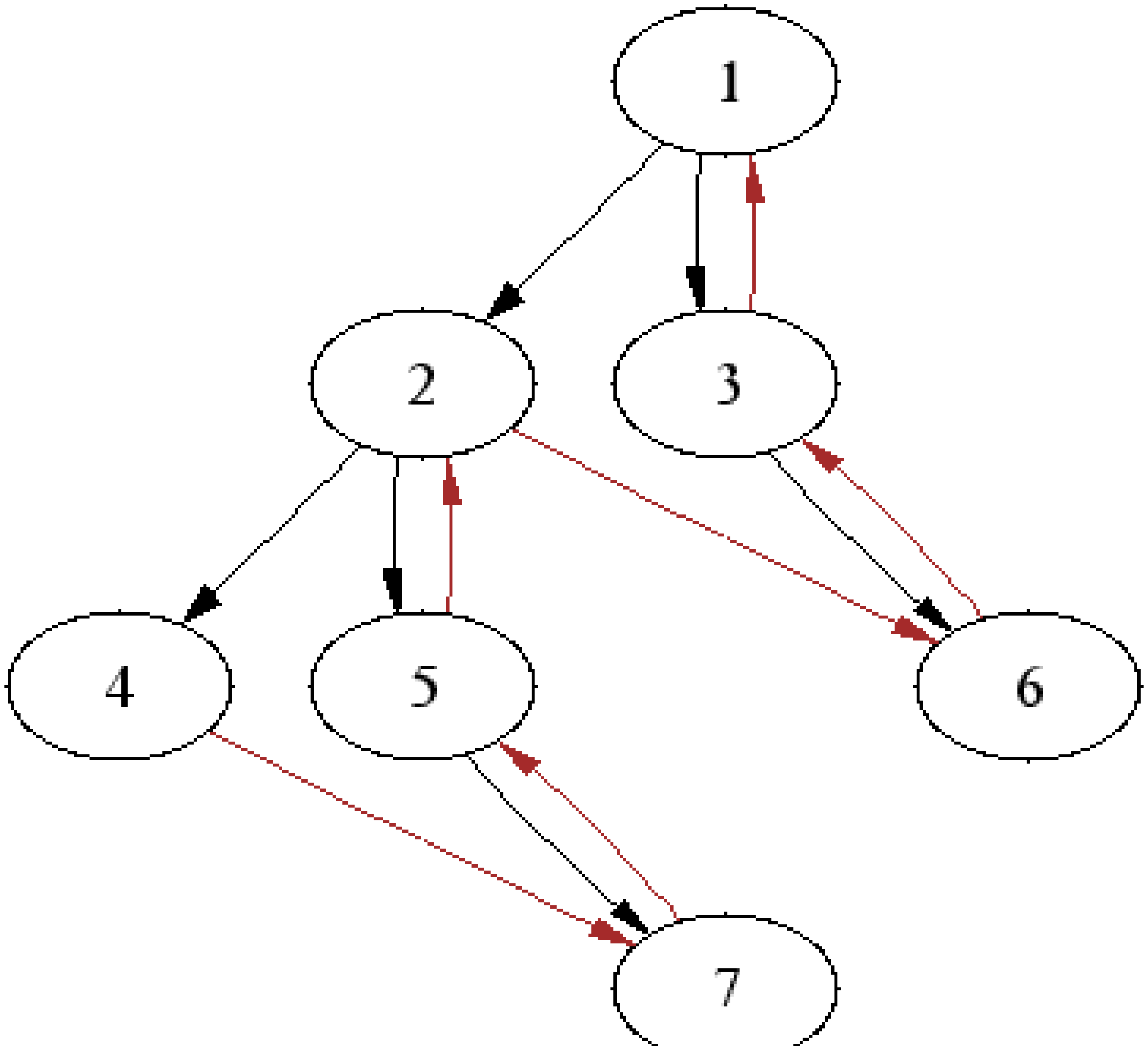
Preorder traversal



Inorder traversal



Postorder traversal



Intemezzo: Recursion

- One of the basic rules: objects/concepts in terms of simpler objects/concepts.
- However: many programming concepts "define themselves". recursive definitions.
- A recursive definition consists of two parts: anchor(ground) case, rules for construction of objects out of basic elements/objects already constructed.
- Example: natural numbers.
  - (i) $0 \in \mathbf{N}$.
  - (ii) $(x \in \mathbf{N}) \implies (x + 1 \in \mathbf{N})$.
  - (iii) these are all natural numbers.
- Example: natural numbers in base 10.
  - (i) $0, 1, 2, \ldots, 9 \in \mathbf{N}$.
  - (ii) $(x \in \mathbf{N}) \implies (x0, x1, \ldots, x9 \in \mathbf{N})$.
  - (iii) these are all natural numbers.
- Example: factorial.

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1, \end{cases}$$

Function calls and recursive implementation

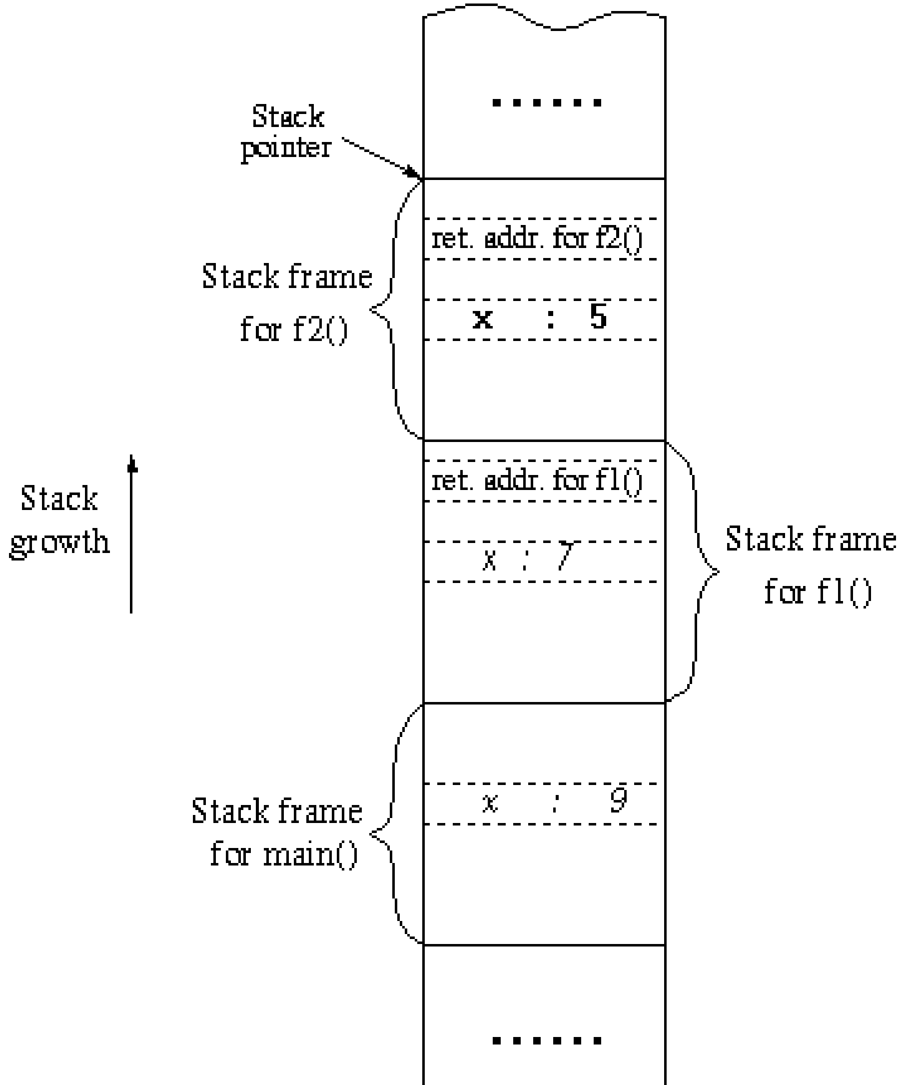
```
unsigned int factorial(unsigned int n){
if (n==0)
  return 1;
else
  return n*factorial(n-1);
};
```

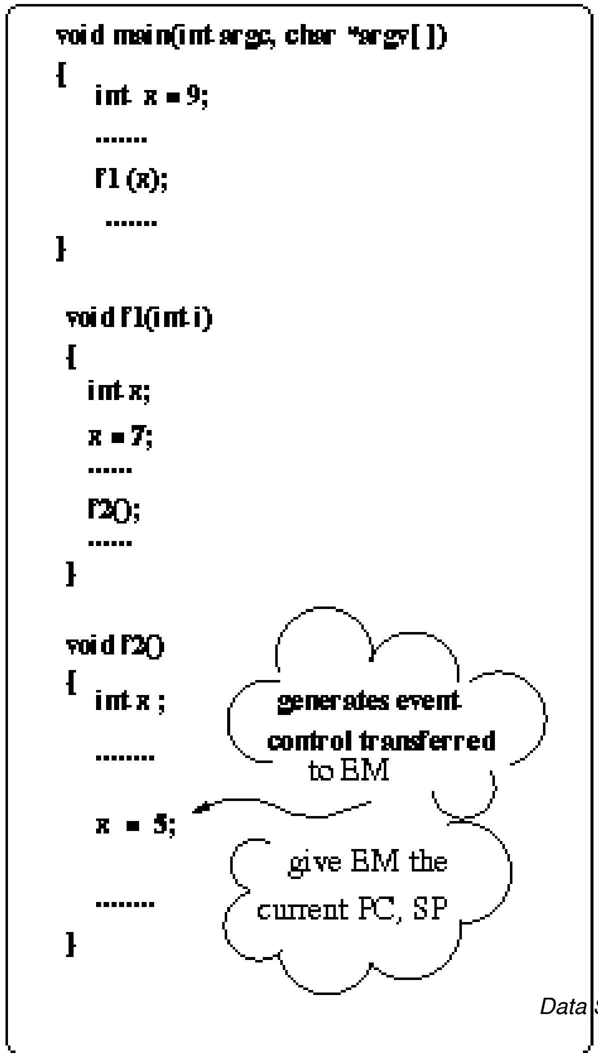
- What happens when you call function ?
- If function has formal parameters, they have to be initialized to the values passed as actual parameters.
- System has to know where to resume execution after function has finished.
- System has to store context of the call.
- Variable x might exist in both called context and calling context.

Stack frame

- Stack frame (activation record): data area containing this information.
- values for all parameters of the function, address of the first entry in an array (if passed).
- Local variables: values can be stored elsewhere, descriptor, pointer to locations where they are stored.
- Dynamic link, pointer to caller's activation record
- Return address to resume control by the caller, the address of the caller's instruction immediately following the call.
- Return value for a function not declared as void.

Example: stack frame





Anatomy of recursive call

```
double power(double x, unsigned int n){
if (n==0)
  return 1.0;
return x*power(x,n-1);
}
```

```
power(x,4)
  power(x,3)
   power(x,2)
       power(x,1)
         power(x,0)
         1
       x
```

$x \cdot x$

$x \cdot x \cdot x$

$x \cdot x \cdot x \cdot x$

Non-recursive implementation of power

```
double nonRecPower(double x,unsigned int n){
double result = 1;
for(result = x; n> 1; --n)
  result *= x;
return result;
```

- Recursion: more intuitive.
- Shorter than the iterative version.
- Costlier than the iterative version.

Excessive recursion

- Recursion is logically simple and yields readable code, but has high overhead (stack).
- Can sometimes overflow the stack.
- Many times nonoptimal.
- Example: Fibonacci numbers.

$$fib(n) = \begin{cases} 1 & \text{if } n = 0, 1 \\ fib(n-1) + fib(n-2) & \text{otherwise.} \end{cases}$$

- Recursive implementation: immediate.
- Fib(6) calls fib(5) and fib(4). Fib(5) also calls fib(4). Different stack frames, so different computations !
- Exponential number of calls to fib.

Good case: Tail recursion

```
void tail(int i){
  if (i>0){
   cout << i<< " ";
   tail(i-1);
  }
}
void iterativeEquivalentOfTail(int i){
for( ; i>0;i--)
  cout << i<< " ";
}
```

- Function: recursive call at the end.
- Basically a loop.
- Tail recursion: can be replaced with iteration.

Nontail recursion

```
void reverse(){
char ch;
cin.get(ch);
if (ch != '\n'){
  reverse();
  /* 204 */ cin.put(ch);
  }
}
```

- main calls reverse() with parameter "ABC".
- an activation record created for parameter ch and return address. Not for the result since the function return type is void.
- stack frame: ('a', (to main))->('b',(204),'a',(to main))->('c',(204),'b',(204),'a',(to main))-> ('\n',(204),'c',(204),'b',(204),'a',(to main)).

Nonrecursive implementation

```
void iterativeReverse(){
char stack[80];
register int top = 0;
cin.get(stack[top]);
while(stack[top]!= \n)
  cin.get(stack[++top]);
for (top -=2; top >=0;cout.put(stack[top--]));
}
```

Nonrecursive implementation: comments

- Name stack for array not accidental. Our stack takes over the run-time stack's duty.
- The transformation of nontail recursion into tail recursion explicitly involves handling a stack.

Indirect recursion

- Preceding slides: f calls itself. However, f can call itself indirectly, via chain of other functions. Chain can have arbitrary length e.g. $f()- > f_1()- > \ldots - > f_n()- > f()$. Also: f can call itself through different chains.
- E.g. receive()->decode()->store()->receive()->decode()->store()->. . ..

```
receive(buffer)
  while(buffer is not filled up)
   if information still incoming
       get a character and store it in the buffer
   else exit()
  decode(buffer);

decode(buffer)
  decode information in buffer;
  store(buffer);

store(buffer)
  transfer information from buffer to file;
  receive(buffer);
```

Nested recursion

- More complicated case: function not only defined in terms of itself, but used as a parameter.
- Example

$$h(n) = \begin{cases} 0 & \text{if } n = 0, \\ n & \text{if } n > 4, \\ h(2 + h(2n) & \text{if } n \leq 4. \end{cases}$$

- Famous example: Ackerman's function.

$$A(n, m) = \begin{cases} m + 1 & \text{if } n = 0, \\ A(n-1, 1) & \text{if } n > 0, m = 0, \\ A(n-1, A(n, m-1)) & \text{otherwise.} \end{cases}$$

- $A(3, m) = 2^{m+3} - 3$, $A(4, m) = 2^{2^{\cdot^{\cdot^{\cdot^{2^{16}}}}}} - 3$, $A(4, 1)$ exceeds the number of atoms in the universe.
- nice recursive expression, difficult iterative one.

Alternatives to (excessive) recursion

- Memoization: store previous results in a (hash) table. When function called recursively check first whether needed value is in the table.
- Of course, iterative solution. Need two previous values, so update two variables.

```
unsigned int iterativeFib(unsigned int n){
  if (n<2)
   return 1;
  else{
   register int i=2, tmp, current = 1, last =0;
   for(;i<=n;++i){
       tmp = current;
       current+=last;
       last=tmp;
   }
  }
  return current;
}
```

Recursion: concluding remarks

- Should be used with good judgement. No general rules when (not) to use it.
- Recursion usually less efficient than its iterative equivalent. But: if recursion 100 ms and iterative version 10ms, difference hardly perceivable.
- Recursion often simpler than its iterative equivalent and more consistent with logic of original algorithm.
- If nontail recursion, a stack has to be used.
- Two situations in which a nonrecursive implementation preferred.
- real-time systems. Systems where an immediate response time vital for proper functioning of the program.
- Programs that are executed hundreds of times. E.g.: compiler.
- Avoid duplicating calls.

Back to BST: Iterative preorder

```
void BST::iterativePreorder(){
  Stack<BSTNode *> travStack;
  BSTNode *p = root;
  if(p != 0){
   travStack.push(p);
   while(!travStack.empty()){
        p = travStack.pop();
        visit(p);
        if (p->right !=0)
        travStack.push(p->right);
        if (p->left !=0)
        travStack.push(p->left);
   }
  }}
```

Nonrecursive postorder tree traversal

- Recursive preorder and postorder only differ by order of operations.
- Can we easily transform iterative preorder into iterative postorder ? NO.
- iterativePreorder(): visiting before both children pushed to the stack.
- children pushed first, then node visited: still preorder traversal.
- what matters: visit() has to follow pop(), the latter precedes both calls of push().
- Preorder: want to visit left child first so push right child first. STACK: last in first out.

Nonrecursive postorder tree traversal

- Sequence generated by left-to-right postoder traversal is the same as the reversed sequence generated by right-to-left preorder traversal (VRL order).
- Can use two stacks: one to visit each node in the reverse order after right-to-left preorder traversal finished.
- However: can develop function for postorder traversal that pushes onto stack a node that has two descendants, once before traversing its left subtree, once before traversing right subtree.
- Auxiliary pointer q is used to distinguish between these two cases.
- Nodes with one descendant pushed only once, leaves don't need to be pushed at all.

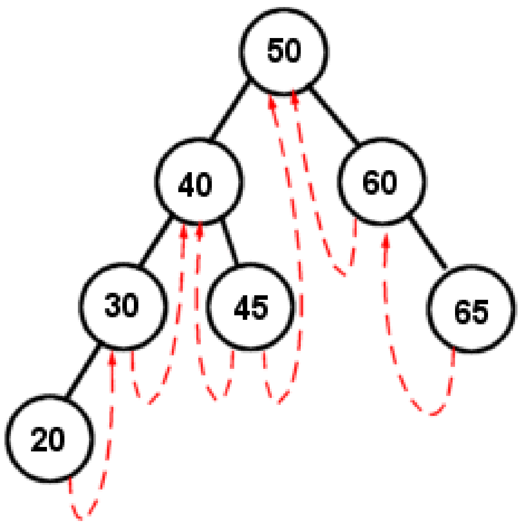
Iterative postorder

```
void BST::iterativePostorder(){
  Stack<BSTNode *> travStack;
  BSTNode *p = root; *q = root;
  while(p != 0){
     travStack.push(p);
     while(!travStack.empty()){
        for(   ;p->left != 0; p=p->left)// work in left subtree
           travStack.push(p);
        while(p!=0 && (p->right==0 || p->right == q)){
           visit(p); // right child:  none or last visited node
           q=p; // q is last visited node
           if(travStack.empty()) return;
           p = travStack.pop();
        }
        travStack.push(p);
        p = p->right; // work in right subtree

     }
  }
```

Nonrecursive inorder. Stackless DF traversal

- Nonrecursive inorder: Very difficult. Only justified when speed is really paramount.
- Can eliminate use of stack if we use threaded trees.
- Threaded trees: stack is "part of the tree". Pointers to predecessor and successor of a node according to an inorder traversal.
- Alternative: overload pointer meaning. Left pointer: pointer to child or predecessor.
- Need new data member to indicate current meaning of the pointers.
- One thread may be sufficient.



Threaded trees

- Threaded trees: stack is "part of the tree". Pointers to predecessor and successor of a node according to an inorder traversal.
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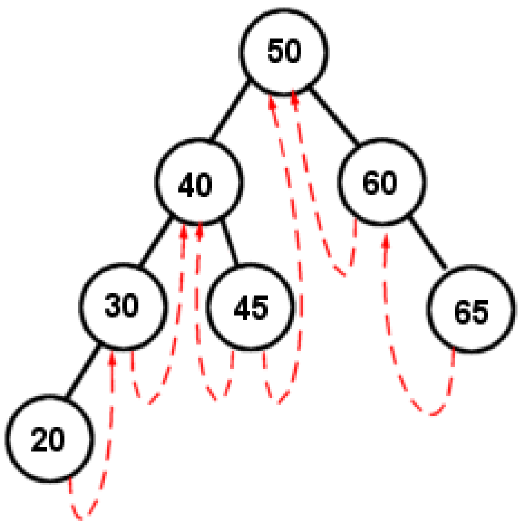


Class ThreadedNode

```
class ThreadedNode{
public:
  ThreadedNode(){
     left = right = 0;
  }
  ThreadedNode(int el,ThreadedNode *l=0,ThreadedNode *r=0){
  key = el; left = l; right = l; successor = 0;
  }
int key;
ThreadedNode *left,*right;
unsigned int successor :  1;
}
class ThreadedTree{
public:
  ThreadedTree(){
     root = 0;
  }
```

Class ThreadedTree

```
  void insert(int);
  void inorder();
  ......
protected:
  ThreadedNode *root;
};
void ThreadedTree::inorder(){
  ThreadedNode *prev,*p=root;
  if (p!=0){ // process only nonempty trees;
     while(p->left != 0) // start at leftmost node
        p = p->left;
     while(p!=0){
        visit(p); prev = p; // prev= last visited node
        p = p->right; // after visiting go to the right
        // or successor node
        if (p != 0 && prev->successor ==0) //if descendent
           while(p->left != 0) // go to the
           p = p->left; // leftmost node
        // otherwise will visit the successor next time;
        }
```

Threaded Trees: Preorder (idea)

- Can be used also for preorder and postorder traversals.
- Preorder: current node is visited first and then traversal continues with its left descendant, if any, or right descendant, if any.
- If current node is a leaf, threads are used to go through the chain of already visited inorder successors to restart traversal with the right descendant of the last successor.

Threaded Trees: Postorder (idea)

- Postorder: a dummy node created that has root as left descendant.
- A variable can be used to check type of current action.
- If action is left traversal and current node has a left descendant, then descendant is traversed. Otherwise action changed to right traversal.
- If action is right traversal and current node has a left descendant, action changed to left traversal. Otherwise action changed to visiting a node.
- If action is visiting node: current node is visited, afterwards its postorder successor has to be found.
- If current node's parent accessible through a thread (i.e. current node is parent's left child) then traversal is set to continue with the right descendant of parent.
- If current node has no right descendant, this is the end of the right-extended chain of nodes.
- *First:* the beginning of the chain is reached through the thread of the current node.
- *Second:* right references of nodes in the chain is reversed.
- *Finally:* chain is scanned backward, each node is visited, then right references are restored to previous settings.

Traversal through tree transformation

- Possible to traverse a tree without using any stack or threads by making temporary changes in trees during traversal.
- Changes: reassign some pointers.
- Tree might lose temporarily tree structure, needs to be restored before traversal finished.
- Algorithm, due to J. Morris, for inorder traversal.
- If tree has no left successors, inorder trivial.
- Temporarily transforms the tree so no left subtree. Has to keep information to restore it.
- Transformation: make current node the right child of the rightmost node in its left descendant.
- We retain the left pointer of the node moved down right subtree.

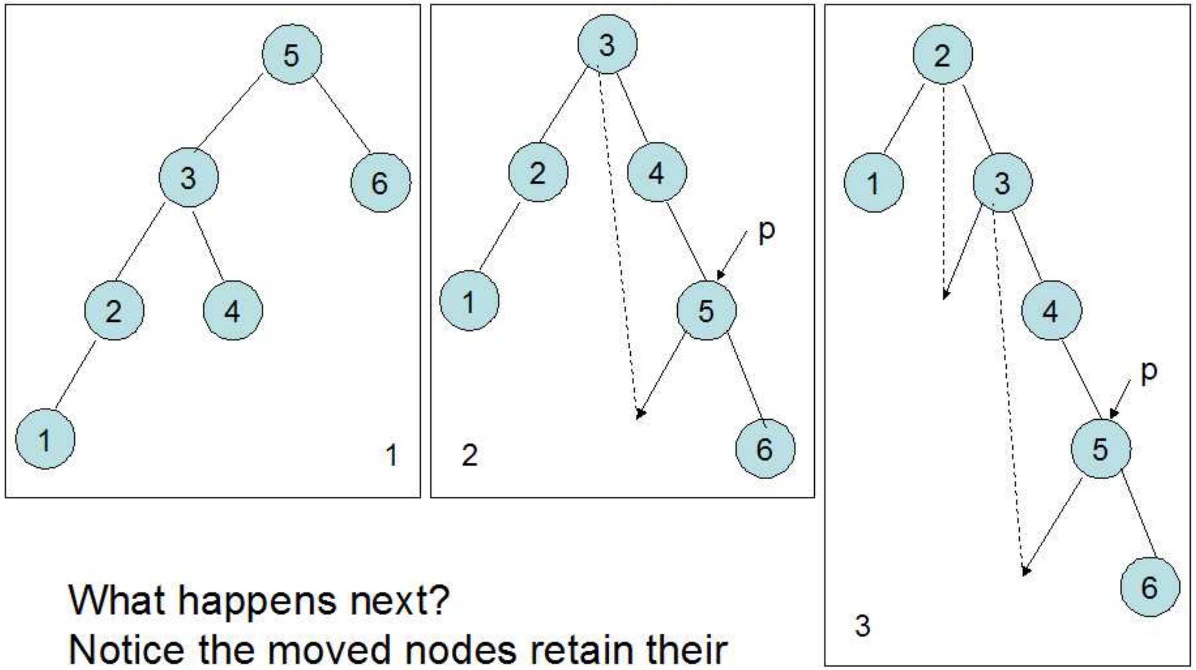
Morris's algorithm: Pseudocode

MorrisInorder()

```
while (not finished)
   if (node has no left descendant)
      visit it;
      go to the right;
   else
      make this node the right child of the rightmost node
      in its left descendant; // leaf !
      go to this left descendant;
```

Morris's algorithm: illustration

Morris's Algorithm



What happens next?
Notice the moved nodes retain their
left pointers so the original shape can be regained

Morris inorder: implementation

```
void BST::MorrisInorder(){
BSTNode *p = root, *tmp;
while(p!=0)
  if (p->left == 0){
     visit(p);
     p = p-> right;
  } else
     {
        tmp = p->left;
        while(tmp->right != 0 && // go to the rightmost node
        tmp->right != p) // of the left subtree or
           tmp = tmp-> right; // to the temporary parent
        if (tmp->right == 0){ // of p; if 'true'
           tmp->right = p; // rightmost node was
           p = p->left; // reached, make it a
        } // temporary parent of the current root
```

Morris inorder: implementation

```
        else { // current root, else a temporary
           visit(p); // parent has been found; visit node p
           tmp->right = 0; // and then cut right pointer of
           p = p->right; // current parent, whereby it
        } // ceases to be a parent;
     }
  }
```

Morris's algorithm: Efficiency

- Notice: time depends on the number of loops.
- Number of loops: depends on number of left pointers.
- Some trees more efficient than others.
- Experimentally: 5 to 10% savings on randomly generated tree, but great space improvement.
- Preorder (idea): move visit() from the inner else clause to the inner if clause. A node visited before transformation.
- Postorder (idea): first create dummy node whose left descendant tree being processed. Then perform inorder traversal. In the inner else clause, after finding temporary parent, nodes between p->left and p (excluded) processed in reversed order.

Insertion

- searching does not modify the tree.
- To insert a new node with key el, a tree node with a dead end has to be reached, new node attached to it.
- found using same procedure as searching: compare key of currently scanned node to *el*. If el less than the key try left child; otherwise try right child.
- If the child is empty, discontinue search and make the child point to a new node of key el.

Node insertion

```
void BST::insert(int el){
BSTNode *p = root, *prev=0;
while (p != 0){
  prev = p;
  if (el > p-> key)
     p = p->right;
  else
     p = p-> left;
  }
  if (root == 0)
     root = new BSTNode(el);
  else
     if (prev->key < el)
        prev->right = new BSTNode(el);
     else prev->left = new BSTNode(el);
}
```

Inserting in threaded tree

- stack traversal: does not change the tree, Morris: restores it after traversal.
- second method: preparatory actions (threads) needed before traversal.
- Threads can be created before traversal and removed each time it’s finished. If traversal infrequent a viable option.
- What if this is not the case ? Need algorithm to update threads when inserting.
- Update function: for inorder, only takes care of successors.
- Node with a right child: has its successor somewhere in the right subtree, does not need a thread.
- Why ? Threads are for "climbing up the tree", not for going down.
- A node with no right child has its successor somewhere. Inherits successor from parent.
- If a node becomes a left node, its parent is successor.

Inserting in threaded trees

```
void ThreadedTree::insert(int el){
ThreadedNode *p, *prev = 0, *newNode;
newNode = new ThreadedNode(el);
if (root == 0) {
  root = newNode;
  return;
}
p = root;
while (p!= 0){
  prev = p;
  if (p->key > el)
     p = p->left;
  else if (p->successor == 0) // go to the right node only if
     p = p-> right; // it is a descendant, not a successor;
  else break;
}
```

Inserting in threaded trees (II)

```
 if(prev ->key > el){ // if newNode is left child of
  prev->left = newNode; // its parent, the parent
  newNode->successor = 1; // also becomes its successor
  newNode->right = prev;
}
else if (prev-> successor == 1){ // if the parent of newNode
  newNode->successor = 1; // is not the rightmost node,
  prev->successor = 0; // make parent’s successor
  newNode->right = prev-> right; // newNode’s successor
  prev-> right = newNode;
}
else prev->right = newNode; // otherwise it has no successor
}
```

Deleting nodes

- Level of complexity of deletion depends on the position of the node in the tree. Three cases:
- The node is a leaf: it has no children. Set appropriate pointer of parent to null, dispose of node.
- The node has one child: Parent's pointer is reset to point to the node's child. This way nodes's children are lifted up one level. Then node is disposed of.
- Node has two children: No one step operation can be made, because parent's pointer cannot point to both children at the same time.
- More than one solution.
- Deletion by merging: Make one tree out of left and right subtree and then attach to parent.

Deletion by merging: Idea

- How can one merge the trees ? By tree property every key in left subtree smaller than every key in the right subtree.
- SOLUTION: Find in the left subtree the node with the largest key and make it a parent of the right subtree.
- Symmetrically: can find node with smallest key in the right subtree and make it a parent of left subtree.
- Desired node: rightmost node of left subtree.
- To locate it: move along this subtree, take right pointers until null encountered.
- This means the node has no right child, no danger of violating the BST property by merging trees.

Deletion by merging: implementation

```
void BST::deleteByMerging(BSTNode *& node){
BSTNode *tmp = node;
if (node != 0) {
  if (!node->right) // node has no right child:  its left
     node = node->left; // child (if any) is attached to its parent
  else if (node->left == 0) // node has no left child:  its right
     node = node->right; // child is attached to its parent;
  else{ // have to merge subtrees
     tmp = node->left; //1.  move left
     while(tmp-> right != 0) //2.  and then right as far as
        tmp = tmp->right; // possible
     tmp-> right = node-> right; //establish link between the
        // rightmost node of the left subtree and
        // the right subtree
     tmp = node; //4.
     node = node->left; //5.
     }
  delete tmp; // 6.
  }
}
```

Delete by merging (II)

```
void BST::findAndDeleteByMerging(int el){
 BSTNode *node = root, *prev = 0;
 while(node != 0){
    if (node->key == el)
       break;
    prev = node;
    if(node->key < el)
       node = node-> right;
    else node = node-> left;
 }
 if(node != 0 && node->key == el)
    if(node == root)
       deleteByMerging(root);
    else if(prev->left == node)
       deleteByMerging(prev->left);
    else deleteByMerging(prev->right);
 else if (root != 0)
    cout << "key "<< el<< " is not in the tree."<< endl;
 else cout << "the tree is empty"<< endl;
}
```