Real Elementary (De)Convolution

Oleg Soloviev

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> A mathematician is asked to design a table. He first designs a table with no legs. Then he designs a table with infinitely many legs. He spend the rest of his life generalizing the results for the table with N legs (where N is not necessarily a natural number).
>
> *A fieldistic joke*

In this presentation, I'm going to tell you about my N-leg table (or O-leg table, if you want). Although it began from a multiframe blind deconvolution problem, which can be formulated as:

> Given N images I_n of the same object O obtained through different unknown wavefront aberrations, obtain O,

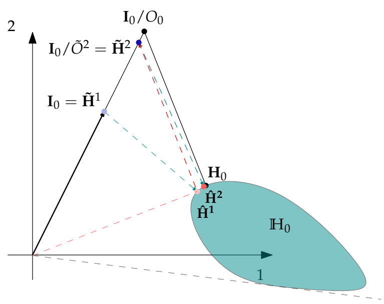
somehow it converted to the next two puzzles, on which I've spent quite a lot time during the last year.

Puzzle 1: Take some (convex) figure $\mathcal{H}$ in a plane and any point I in that plane (outside of $\mathcal{H}$).

Consider the following algorithm:

1. set $I_0 = I$
2. set $H_k = \mathcal{P}_{\mathcal{H}} I_k$
3. restore perpendicular to the radius vector OH ans set its intersection with line OI as I_k
4. go to step 2.

Prove that the algorithm converges to pair of points $\hat{H}, \hat{I}$ and $\angle IO\hat{H} = \arg\min_{H\in\mathcal{H}} \angle IOH$.



Puzzle 2: Let h be a positive function with a finite support. Formulate the conditions on h such that from $h = h_1 * h_2$ [1] it follows that either h_1 or h_2 coincides with h.

[1] where $*$ denotes convolution operation

Blind multiframe deconvolution and TIP algorithm

Blind multiframe deconvolution problem is formulated in the following way.

Let i_n be several images of the same object o obtained by convolution with different PSFs h_n. Let the *a priori* knowledge on the object and PSFs be represented by to sets $\mathcal{O}$, $\mathcal{H}$. Find the best estimate for o.

$$\begin{aligned} i_n = h_n * o, \quad n = 1, \ldots, N \\ \text{s.t. } o \geq 0, h_n \geq 0 \\ o \in \mathcal{O}, h \in \mathcal{H}. \end{aligned} \tag{1}$$

TIP algorithm uses iterative procedure described below. It was shown to work well for some particular sets $\mathcal{O}$ and $\mathcal{H}$.

Figure 1: Current version of the TIP algorithm

The TIP Algorithm

Input: $\{i_n, \hat{h}_n^0\}$
Output: $\{\hat{h}_n, \hat{o}\}$

Initialisation :
$\{I_n, \hat{H}_n^0\} = \mathcal{F}(\{i_n, \hat{h}_n^0\})$
Main Loop :
for $k = 0$ to $K-1$ **do**
    $\hat{O}^k = \frac{\sum_{n=1}^{N} \hat{H}_n^{*k-1} I_n}{\sum_{n=1}^{N} |\hat{H}_n^{k-1}|^2}$
    for $n = 1$ to N **do**
        $\tilde{H}_n^k = \frac{I_n}{\hat{O}^k}$
        $\hat{H}_n^k = \mathcal{P}_{\mathcal{H}}(\tilde{H}_n^k)$
    end for
end for
Final Reconstruction :
$\{\hat{h}_n\} = \mathcal{P}_{\mathcal{H}}(\mathcal{F}(\{\hat{H}_n^K\}))$
$\hat{o} = \mathcal{P}_{\mathcal{O}}(\mathcal{F}(\frac{\sum_{n=1}^{N} \hat{H}_n^{*K} I_n}{\sum_{n=1}^{N} |\hat{H}_n^K|^2}))$

Despite its simplicity, TIP appeared quite difficult for a solid mathematical confirmation. After some unsuccessful attempts to reduce TIP to to projections on the convex sets methods or, more generally, to prove that TIP corresponds to some contraction mapping, I decided to investigate the problem on very simple examples first.

The goal of the following section is to apply all convolution-related aspects on a simple example to get more understanding and insight that can be generalised to a higher dimensionality of the problem.

Simple example

In this example, let's limit ourselves to total number of pixels in each image $M = 3 \times 1$, that is our space is just three-points functions (see Fig. 2) with convolution understood cyclically. The problem of deconvolution is that I give you a vector of length 3, $\mathbf{i} = [i_1, i_2, i_3]$ with all values positive, and ask you to represent it as a convolution of two non-negative vectors $\mathbf{o}$ and $\mathbf{h}$ which might satisfy also some conditions. Depending on these conditions the problem might or might not have a solution. Non-negativeness is also a constraint.

Deconvolution

A simple example of constraint is "known PSF", that is $\mathbf{h} = \mathbf{h}_0$. Then the problem is just called deconvolution.

Often the image we have is know to be obtained as a convolution of $\mathbf{h}$ with some unknown function $\mathbf{o}$, but usually it's also corrupted with noise, that is

$$\mathbf{i} = \mathbf{h} * \mathbf{o} + \mathbf{n}, \tag{2}$$

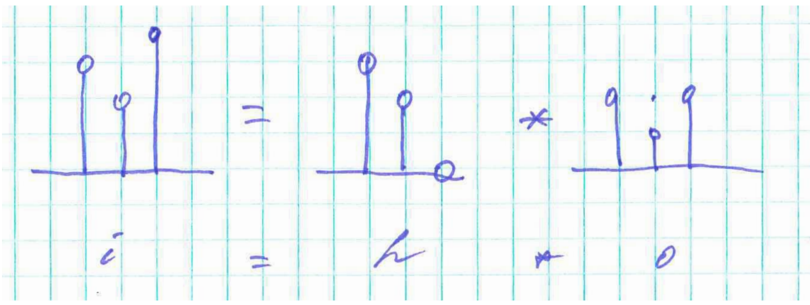


Figure 2: Functions from 3 points

and depending on the noise term it might happen that now there is no exact solution for $\mathbf{o}$ that satisfies the constraints $\mathbf{o} \in \mathcal{O}$. For three-dimensional vectors it should be easy to understand, let's have a closer look at what happens.

Range of a convolution operator $\mathbf{H}$

First of all, convolution is a bi-linear operation, so for a fixed $\mathbf{h}$ we have a linear operator $\mathbf{H}$ associated with it

$$\mathbf{Hx} = \mathbf{h} * \mathbf{x}. \tag{3}$$

If our $\mathbf{x}$ is limited to non-negative vectors, $\mathbf{x} \in \mathbb{R}^3_+$, then the result of convolution is also non-negative, and image lies in some rotationally symmetrical around vector $(1, 1, 1)$ cone formed by three vectors $[h_1, h_2, h_3]$, $[h_2, h_3, h_1]$, $[h_3, h_1, h_2]$, which are the images of the basis vectors (see Fig. 3), and which we denote as $\mathcal{K}_\mathbf{h}$:

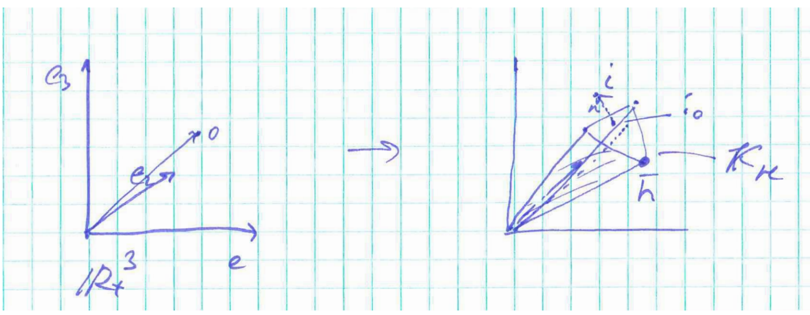


Figure 3: Cone $\mathcal{K}_\mathbf{h}$

$$\mathcal{K}_\mathbf{h} = \mathbf{H}\mathbb{R}^3_+. \tag{4}$$

Now we can easily see that a strong enough noise can take the image out of the cone $\mathcal{K}_\mathbf{h}$. In this case, we most probably accept the closest to $\mathbf{i}$ point in cone as approximation to our data (and if we know that our noise is white Gaussian, this is what we should do, I guess). In other words, we solve a least squares problem

$$\begin{aligned} \|\mathbf{i} - \mathbf{h}_0 * \mathbf{o}\| &\longrightarrow \min_{\mathbf{o}} \\ \text{s.t.} \quad \mathbf{o} &\in \mathbb{R}^3_+, \end{aligned} \tag{5}$$

which we might want to rewrite immediately in a more geral form

$$\begin{aligned} \|\mathbf{i} - \mathbf{h} * \mathbf{o}\| &\longrightarrow \min_{\mathbf{h},\mathbf{o}} \\ \text{s.t.} \quad \mathbf{o} \in \mathcal{O}, &\mathbf{h} \in \mathcal{H}. \end{aligned} \tag{6}$$

More on a range of operator **H** *and connection to semi-ring of ideals*

From the definition of the range cone $\mathcal{K}_\mathbf{h}$, its convolution with any non-negative vector $\mathbf{x}$ lies completely inside it:

$$\mathbf{x} * \mathcal{K}_\mathbf{h} \subseteq \mathcal{K}_\mathbf{h} \quad \forall \mathbf{x} \in \mathbb{R}^3_+, \tag{7}$$

and equality is achieved if and only if $\mathbf{x} = ce_i$, a delta function.

Indeed this follows from that if $\mathbf{y} \in \mathcal{K}_\mathbf{h}$, then $\mathbf{y} = \mathbf{h} * \mathbf{o}$ for some non-negative $\mathbf{o}$. Then $\mathbf{y} * \mathbf{x} = \mathbf{h} * (\mathbf{x} * \mathbf{o}) \in \mathcal{K}_\mathbf{h}$.

This absorption property of $\mathcal{K}_\mathbf{H}$ is similar to the defining property of an ideal in abstract algebra. And indeed, as convolution of finite numerical sequences is related to the multiplication of polynomials with coefficients formed by these sequences, there is a bijection between $\mathbb{R}^3$ and polynomial factor ring $\mathbb{R}[x]/(x^3 - 1)$. Range $\mathcal{K}_\mathbf{h}$ by this bijection is mapped into principal ideals in this ring. If we limit ourselves only to the non-negative coefficients, the ideals form a semi-ring. We'll discuss it later.

Simple(x) representation

Because of bi-linearity of convolution, we can have multiple solutions, $i = (c\mathbf{h}) * (\frac{1}{c}\mathbf{o})$, where c is any positive constant, $c > 0$. Let's exclude them by demanding a unit norm, for instance. We can use 2-norm and thus limit ourselves to a piece of sphere, or 1-norm and limit to a simplex. In both cases we just consider rays defined by a non-zero point. Let's stick with simplex as it's easier to draw. And we shall draw it as seen from direction $(1,1,1)$ — see Figure 4.

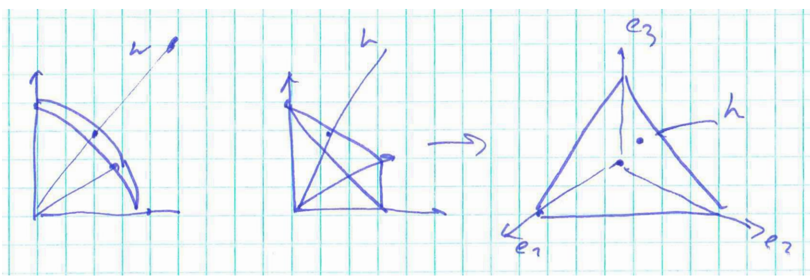


Figure 4: Representation of the space

Simplex representation is appropriate also because convolution of two non-negative functions with 1-norm equal to 1 has also unit norm:

$$\|\mathbf{h} * \mathbf{o}\|_1 = \|\mathbf{h}\|_1 \|\mathbf{o}\|_1 \quad \forall \mathbf{h}, \mathbf{o} \in \mathbb{R}^3_+. \tag{8}$$

It's easy to see from the definition of 1-norm as an integral or sum of all points. Our simplex is then defined as

$$S = \{\mathbf{x} : \|\mathbf{x}\|_1 = 1\}. \tag{9}$$

In signal processing, the 1-norm is often referred to as DC-component of the signal.[2]

In this representation, for 3-dimension example, operator $\mathbf{H}$ becomes a rotational homothety *with coefficient smaller than* 1 (see Figure 5). One can easily recognise here a parallel with the multiplication by complex numbers, and we discuss it later.

For the moment just notice that

- reduced to the simplex, the convolution with a fixed function $\mathbf{H}$ remains the linear operator $\mathbf{H}_s$ and
- the range of the operator $\mathbf{H}_s$ is also a cyclically symmetrical simplex.

Let's also remove another solution ambiguity related to the rotation by fixing the position of maximum element, for instance, of $\mathbf{o}$ and $\mathbf{h}$.

[2] thus our normalisation corresponds to a constraint to functions wit a fixed mean value

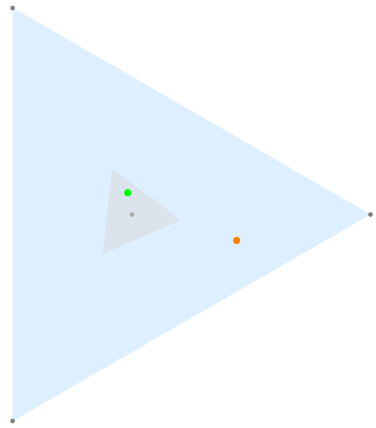


Figure 5: $\mathbf{H}$ as a rotational homothety

From Cartesian to simplex coordinates

In simplex, one can introduce barycentric coordinates. To get the simplex coordinates of any point h, $h(x) > 0$, one needs just to divide by the DC component

$$h \mapsto h/(h_1 + h_2 + h_3) \tag{10}$$

The simplex centre we'll call $\mathbf{h}_{\mathrm{DC}} = [1, 1, 1]$.

Simplex generated by $\mathbf{h}$

The range of the operator $\mathbb{H}$ is represented in S as a smaller simplex $S_{\mathbf{h}}$ with vertices $\mathbf{h}, T\mathbf{h}, T^2\mathbf{h}$, where T is translation operator. The maximal simplex S can be named S_e, the unit simplex.

One can also introduce barycentric coordinates defined by simplex $S_{\mathbf{h}}$. Then if a point $\mathbf{i}$ has coordinates $[o_1, o_2, o_3]$ in $S_{\mathbf{h}}$, (denoted as $\mathbf{i} = [o_1, o_2, o_3]_{\mathbf{h}}$,), then $\mathbf{i}$ is the convolution of point $\mathbf{o}$ with the same coordinates in the unit simplex with $\mathbf{h}$:

$$\mathbf{i} = \mathbf{h} * \mathbf{o}. \tag{11}$$

This follows form the linearity of operator $\mathbb{H}$.

In other words, convolution can be seen change of barycentric system of coordinates. *To deconvolve an image* $\mathbf{i}$ *with some PSF* $\mathbf{h}$ *is equivalent to finding the barycentric coordinates of* $\mathbf{i}$ *in simplex* $S_{\mathbf{h}}$.

For a point inside simplex $S_{\mathbf{h}}$ this is as easy as drawing Cevians from each of the simplex vertices through $\mathbf{i}$. This is a linear operation.

For a point outside the simplex, the situation is more complicated as we need first to project in R^3_+ point $\mathbf{i}$ on $\mathcal{K}_{\mathbf{H}}$ and to then to find the barycentric coordinates of the projection.

How to make this projection directly in the simplex space will be described further.

Contrast definition and its decrease by convolution

Any point $\mathbf{x}$ from simplex can b represented as $\mathbf{h}_{\mathrm{DC}} + (\mathbf{x} - \mathbf{h}_{\mathrm{DC}})$. Note that as $\mathbf{h}_{\mathrm{DC}}$ is a constant signal, all the information is contained in the signal $(\mathbf{x} - \mathbf{h}_{\mathrm{DC}})$.

Distance between $\mathbf{x}$ and $\mathbf{h}_{\mathrm{DC}}$ defines the contrast, that is what percentage of the energy contained *not* in the DC component.[3]

[3] For function with a fixed mean value this corresponds also to half of peak to valley amplitude

Any signal of type contains the same amount of information, but signals with smaller α (smaller contrast) are more sensitive to noise.

Global (or uniform) contrast maximisation is operation of finding maximal α such that $\mathbf{h}_{\mathrm{DC}} + \alpha(\mathbf{x} - \mathbf{h}_{\mathrm{DC}}) \geq 0$. It corresponds to drawing a line from the simplex centre through the point $\mathbf{x}$ to the simplex edge. It is non-linear operation. Every $\mathbf{x}$ obtained by contrast maximisation contains at least one zero value (because it belongs to a simplex edge).

Thresholding (setting to zero some components lower than certain value)can be considered as local contrast maximisation.

From representation of the previous figure we see that any convolution decreases the contrast. The extreme case is when $\mathbf{h} = \mathbf{h}_{\mathrm{DC}}$. In this case the contrast is decreased to zero and all information is lost.

Convolution with even N-dimensional PSF is non-uniform contrast decrease (without rotation).

Maximum contrast and uniqueness of deconvolution

Suppose that original object o didn't have maximal contrast. Then in the simplex representation there exists a uniform scaling operator $s, s > 1$ (which is equivalent with convolution with a scaling function $\{s, \frac{1-s}{N}, \ldots, \frac{1-s}{N}\} =$ such that $o' = s * o \in \mathrm{Re}^N_{\geq 0}$. Thus $s^{-1} = \{s^{-1}, \frac{1-s^{-1}}{N}, \ldots, \frac{1-s^{-1}}{N}\} \in S$ – function from the simplex and one has $o = o' * s^{-1}$ that is o can be represented as convolution of two nonnegative functions

$$i = o * h = o' * s * h = o' * h'. \tag{12}$$

From this we can conclude that for the uniqueness of the solution we can also require that both o and h have maximum contrast.

As maximum contrast implies that the function reaches zero in one of its points, this might be motivating to consider functions with a limited support.

And vice versa, the limited support constraint guaranties the maximal contrast. As by the convolution the contrast decreases (it seems that it can stay at maximal only for a delta function or some rare case of periodic functions) this can provide for some kind uniqueness of the solution.

This is not absolutely true, as consider convolution of two small rectangles – the results is a triangle, with finite support and thus with maximal contrast. if the mean value is kept at the same level, it means its PV amplitude is decreased, which is obvious, as for the non-negative functions with limited support, the PV amplitude is equal to its maximal value. Thus it might be a better approach to the uniquenss of the solution consider only solutions with the maximal amplitude. This is related to the maximisation of the number of zero terms in the function. This might speak in favour of thresholding and speckle PSFs.

Limited support and uniqueness

It's easy to come up with an example of non-unique deconvolution even in the presence of limited support constraint. Consider for instance, any function h_2 with support consisting of two neighbour pixels. Then $h^2 h * h$ has support of three neighbour pixels, h^3 of 4 pixels and so on. Obviously they will all satisfy the constrain on M pixels, and they will provide nonunique solution.

From polynomial (or Fourier) perspective, this corresponds to the fact that polynomials h^n have the same roots, and thus it's like collecting maximum power on irreducible polynomials below some degree.

In other words, if $i = p_1^{k_1} p_2^{k_2} \cdots p_l^{k_l}$, and there is a constraint on limited support for h, which ca be reformulated in terms of degree of h as $\deg h \leq m$, then one gets as many solutions as there are ways to solve

$$m_1 + m_2 + \ldots + m_l \leq m, \quad m_i \leq k_i. \tag{13}$$

In general case thus, the uniqueness of the solution is not guaranteed. In our particular case of multifram deconvolution, however, we have intersting supports in two directions – namely, having separated t and $\mathbf{x}$, we have

$$i(\mathbf{x}, t) = o(\mathbf{x}) h(\mathbf{x}, t), \tag{14}$$

where independence o on t is equivalent to the constraint support $deg_t o = 0$. Thus o has minimal possible constraint on degrees of t and thus o is given by a free from t common factor of $i(\mathbf{x}, t)$.

By introducing no limitation on the degree of o in $\mathbf{x}$, we can implicitly mean that $deg_{\mathbf{x}} o$ can be as large as needed to satisfy eq. (14), which is equivalent to minimising degree of h.

Please note there is still no guarantee of uniqueness, as ther ecan be more than one polynomial of the same minimal degree.

Another approach would be also minimise the degree of o in $\mathbf{x}$, which is equivalent to finding a polynomial of a maximal degree in $\mathbf{x}$ satisfying eq. (13).

The last approach might be motivated by the following subsection and by the fact that actually degrees of h are not limited (they are infinite series which we truncate and approximate by polynomials.

Dependence of the solution on the initial conditions

TIP illustration as tangential projections on a convex set $\mathcal{H}$ can have more than one stable point if th boundary of $\mathcal{H}$ contains a patch of a linear subset.

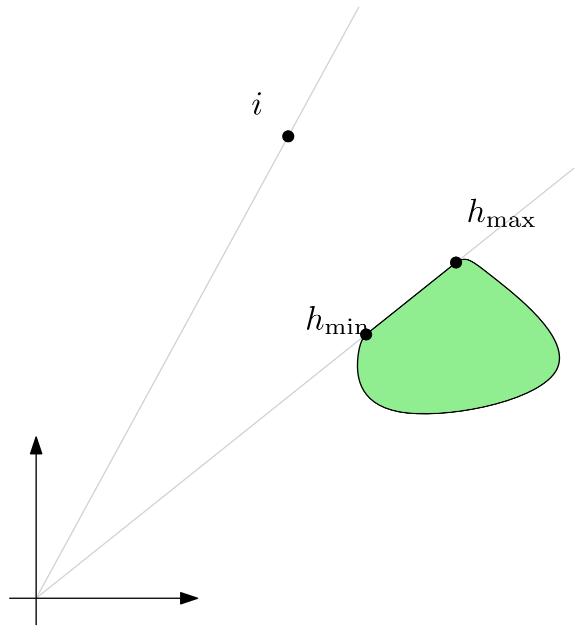


Figure 6: Illustration of non-unique solution in case of a linear subset patch presence in teh boundary of $\mathcal{H}$

Mayb this in not the greatest example, as this linear patch is actually equivalent (in this example) as multiplication by a constant, which we have already removed.

Nevertheless, if we follow here the algorithm of the projections, we can land on any point between $h_{\min}$ and $h_{\max}$. This might be (I'm not sure yet) equivalent to getting any of h_i from decomposition of i into irreducible multipliers.

But we will alway s get the same solution of we start from the origin (first projecting on $\mathcal{H}$ and so on; this might corresponds to starting from o with a maximum degree in $\mathbf{x}$ and going down.

Different projections on the simplex side

We can discriminate three different projection on a side of a simplex $S_{\mathbf{h}}$. Namely, if the side σ of the simplex is given by equation

$$\sigma = \{\mathbf{h} * \mathbf{x} | x_{k+1} = \ldots = x_N = 0\}, \tag{15}$$

then for any point $i \in S_{\mathbf{h}}$, we can define three projections i_1, i_2, i_3 on σ.

1. i_1 is given by l_2 projection on the side of the cone $\mathcal{K}_{\mathbf{h}}$ and then rescaling to unit l_1 norm;

2. i_2 is given by direct l_2 projection on σ

3. i_3 is given intersection of σ with the subspace defined by $h_{k+1}, \ldots, h_N$ and point i.

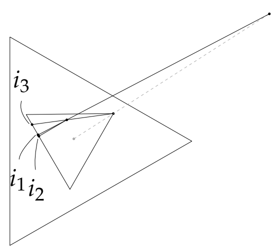


Figure 7: Projection on simplex $S_{\mathbf{h}}$

In case of the unit simplex ($\mathbf{h} = \delta$) the first and last projections coincide.

As all points i_1, i_2, i_3 lie inside the cone $\mathcal{K}_{\mathbf{h}}$, they are given by convolution of o_1, o_2, o_3 with $\mathbf{h}$. Note that o_1 is then a (nonlinear) deconvolution of i with $\mathbf{h}$ with the only constraint on non-negativeness of the object, and o_2 is the convolution with the constraint that $o \in \sigma$.

Motivation for these projections

All three projections give an estimate for an object o with its support in sigma. We can motivate each of them as follows.

The first projection i_1 on the side of the cone gives the renormalised MAP estimator for an image formed by convolution of $\mathbf{h}$ with o with a finite support and the additive Gaussian noise. Here $\|o\|_1$ is not necessary equal to 1, that is o might not be not in the simplex:

$$o = \frac{o^*}{\|o^*\|_1}, \text{ where } o^* = \arg\min_{o \in \mathcal{O}} \|i - o * \mathbf{h}\|_2 . \tag{16}$$

The second projection uses modified prior on o that $\|o\|_1 = 1$ and o has finite support and is the MAP estimator for an image obtained by convolution of $\mathbf{h}$ with o and the additive Gaussian noise, thus

$$o = \arg\min_{o \in \mathcal{O},\ \|o\|_1 = 1} \|i - o * \mathbf{h}\|_2 . \tag{17}$$

The last projection is the most difficult to explain. It uses the result of linear deconvolution of i with $\mathbf{h}$, then finds the closest $o \in \mathcal{O}$ and then renormalise it to unit in l_1 norm. It can be motivated as follows: any point in the simplex $S_{\mathbf{h}}$ can be represented as convex combination of a point from σ and some other point from $S_{\mathbf{h}}$:

$$i = (1 - \alpha) i_\sigma + \alpha i', \quad i' \notin \sigma, \alpha \in [0, 1]. \tag{18}$$

Then i_3 corresponds to the minimum α, in other words it maximises the weight of σ-part of the exact solution:

$$o = \frac{o^*}{\|o^*\|_1}, \text{ where } o^* = \arg\min_{o \in \mathcal{O},\ i - o * \mathbf{h} \in \mathcal{K}_{\mathbf{h}}} \|i - o * \mathbf{h}\|_1 . \tag{19}$$

This can be explained in terms of normal and tangential error components; but maybe the main motivation is that a) this projection is easy to calculate and b) for the points *inside* the simplex $S_{\mathbf{h}}$ this operation is linear (while it might be not for the other two projections).

It seems that here we must to discriminate between ppoint inside simplex $S\mathbf{h}$ and outside it. For a point outside, eq.(19) should be re-interpreted. The interesting thing (conjecture) is that the points outside the simplex are in the "feasible" region (but then how to deal with the degenerative case of tetrahedor, where almost all the points are outside of it?) as the non-feasilbe set convex and

Projection on unit simplex edges

This operation is similar to thresholding. To project **h** on side i of simplex, set $\mathbf{h}_i = 0$. This is quite obvious from Fig. 8.

If drawn in simplex itself, projection on the simplex side is performed by drawing Cevian through point **h** from vertex i.

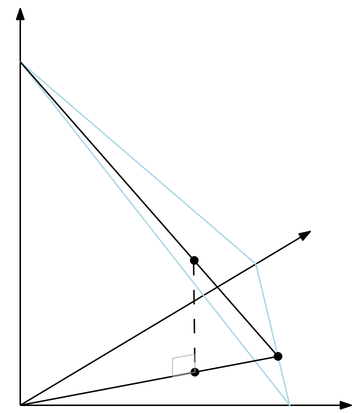


Figure 8: Projection on the unit simplex side

Projection on side of simplex $S_{\mathbf{h}}$

Projection of point **i** on side of simplex defined by point **h** is given by drawing a line trough **i** from a point inverse to **h**.

Linear deconvolution

If $\mathbf{h} \neq \mathbf{h}_{DC}$, one can make an inverse transform[4], that is rotation backwards and dilation with the same coefficient to restore the original **o**. Because now the homothety coefficient is greater than 1, noise amplification becomes evident. Degree of noise amplification is proportional to the contrast decrease.

The linear deconvolution might result in a solution with negative values.

[4] it will be also a linear operator, hence the name *linear deconvolution*

Deconvolution via re-blurring

Deconvolution via re-blurring is a technique of convolving with a reversed (or symmetrical) PSF $\bar{\mathbf{h}} = [h_3, h_2, h_1]$ and then discarding the DC component to restore the contrast (figure)

More detailed and dimension-independent: $\bar{\mathbf{h}} = [h_{N-i}]$. Then $\bar{\mathbf{h}} * \mathbf{h}$ is symmetrical PSF, and its matrix in the Fourier basis is diagonalised. In other words, the eigenvalues of $\mathbf{H}_{\bar{\mathbf{h}} * \mathbf{h}}$ are real an positive (and less then one).

In this case, for $N = 3$, $\bar{\mathbf{h}} * i$ is scaled (in the simplex domain) version of **o**, that is it's **o** with reduced contrast. **o** can be simply restored by contrast maximisation.

For $N \neq 3$, the contrast decrease might be non-uniform, and simple contrast maximisation will not work.

Constrained optimisation problem

Non-linear deconvolution is an attempt to solve the constrained optimisation problem when deconvolved by the previous method **i** lies outside the simplex. Usually just simply project back to the simplex.

The projection on simplex is not linear operation, because the side on which the projection is performed is not fixed.

Consequence: because (for an invertible psf) the projection lies on the boundary of the simplex, it gives a kind of "maximum contrast solution". Please note that maximum contrast solution cannot be represented as a non-trivial convolution of two positive functions (related to Puzzle 2).

Blind deconvolution

Now **h** is not known exactly, but we have some guess about it, for instance that its last component h_3 vanishes. This corresponds to all **h** lying on the edge of simplex. As we can see from Fig. 9, any point inside of simplex can be represented as such convolution, and for any *internal* point this representation is not unique.

Interesting that if we have noise and need to project back on the feasible images, we get a boundary point and unique solution.

We should limit the possibilities for **o** to get more meaningful results, so we introduce constraints on **o**.

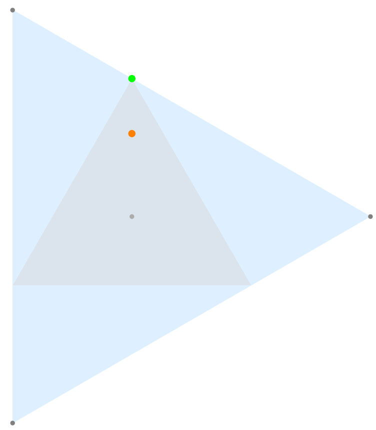


Figure 9: Inner point as i and a straightforward two-point h candidate

CONVEXITY OF CONSTRAINTS on object and PSF do not automatically means convexity of the image set — see two examples. We do prefer to have convexity on the image constraints because of the feasiblity problem — if the image region is not convex, we might not have unique projection back to it.

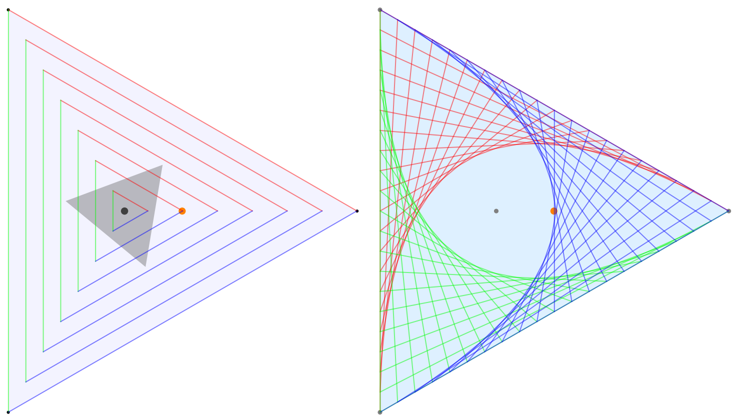


Figure 10: Two examples of constraints on the object set $\mathcal{O}$. In the left case it was $o_2 = o_3$, in the right $o_3 = 0$ (red lines). Green and blue lines correspond to the translated constraints.

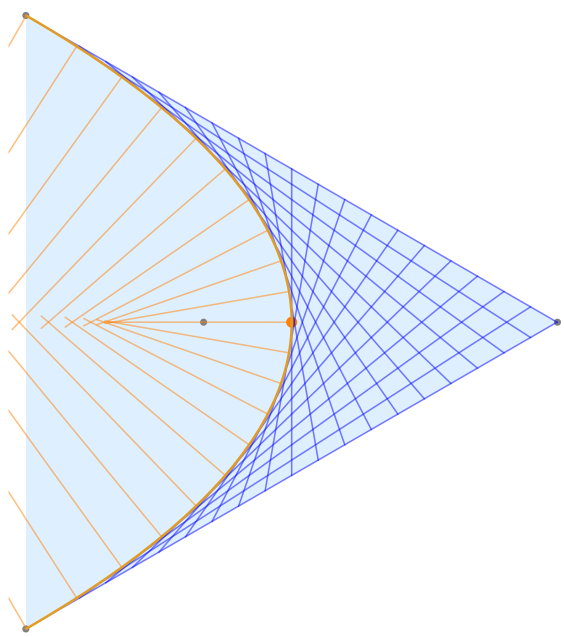


Figure 11: Convolutions of two-point PSF with two-point object. Although the surface is not convex, if the point is **i** is close enough to it, the projection is uniquely defined.

Sum of cones and union of cones

Often a direct sum of cones is considered instead of union of cones. This is actually a convex hull and is of course convex set. The problem is that the solutin is than given as a sum of two solutions, each consisting of different pair psf-object.

The direct sum of cones corresponds to the sum of ideals.

Approach through inversion

We can rewrite the problem as

$$\mathbf{h} = \mathbf{u} * \mathbf{i}, \tag{20}$$

where $\mathbf{u}$ — deconvolution operator for $\mathbf{o}$:

$$\mathbf{o} * \mathbf{u} = 1. \tag{21}$$

Sidenote: $\mathbf{u}$ is given by inversion and symmetry (again, it's just complex inverse) of $\mathbf{o}$.

Then $\mathbf{u}$ belong to some set U, and $\mathbf{u} * \mathbf{i}$ — is a scaled and rotated copy of it.

Now we can find a point in their intersection.

Effect of noise here will be that this scaled and rotated copy of U will be blurred.

The question is: Why it seems easier in this approach? The answer is not that we inverted the object, but because we have separated the ifluence of the psf and object. This leads to the next approach.

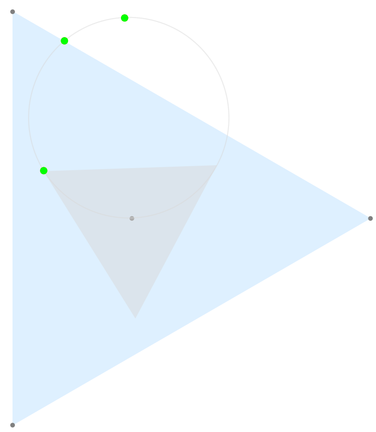


Figure 12: h can be found as intersection of the simlex edge and inverted, rotated, and scaled simplex edge

Direct product of psf and object spaces

Here, we concentrate on pairs (o, h) (we will write them just as points, without underlying their vector nature). i now becomes a set of feasible pairs. It's difficult now to draw even for 3 dimesnions, so we switch to 1D case.

We have direct product of constraints which form than a convex box (for convex constraints) and set I, which can be represented as a boundary of a convex set (epigraph of function $h = i/o$.

Two convex sets — we can use alternative projections! The algorithm than finds either crossection or closest to I pair:

$$\begin{aligned} &|ho - h_i o_i| \to \min \\ &\text{s.t. } (o, h) \in \mathcal{O} \times \mathcal{H}, (o_i, h_i) \in \mathcal{I}' \end{aligned} \tag{22}$$

which is actually equivalent to problem of deconvolution.

Nice! but how to project on $\mathcal{I}$?

TIP algorithm

TIP says that(o_i, h_i) is a useless knowledge and waste of time. Indeed, to find (h, o), we don't need to know them. In case of intersection (and maybe multiple solutions), any would do (we get actually what we asked), and in case of no intersection — we need just the closest to $\mathcal{I}$ point.

TIP behaves consequently and insists on separation of o and h. each projection is done along either o or h. see Figure. Finally we find the closest point.

General case

Now we move from 3D case to any dimensions. (It's a good ecxercise to try first on 4-points function, for which simplex representation is a 3D tetrahedron.)

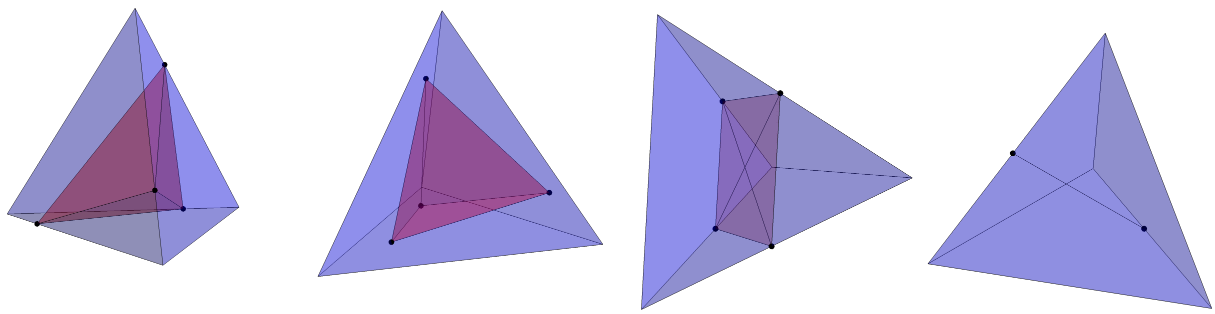


Figure 13: Simplex representation for the range of convolution for $M = 4$ and different h (left to right): $h = \{1/4, 3/4, 0, 0\}, \{.1, .2, .7\}, \{1/2, 1/2, 0, 0\}$, and $\{1/2, 0, 1/2, 0\}$.

First we note that any sinusoid is transformed by convolution to the sine wave of the same frequency, only amplitude and phase are changed (this can be proved by elementary methods too). So any pair of $(\cos(kx), sin(kx))$ with added DC component would behave in the same way as our 3D example. The scaling coefficient and rotation angle would depend, of course on the wavevector k. This will make all the direct projection as in example 1 even more interesting.

The simplex representation still remains valid.

If none of the frequencies scaling coefficients is equal to zero, the convolution can be inverted.

Example with 4-point simplex and uninvertable aberration.

Please note also that for univertable aberration, there is also a uniqueness problem — due to reduced dimensionality of the simplex $S_{\mathbf{h}}$, every point has more than one possible coordinate set.

Idempotent semiring of ideals

and can define partial order $\leq$ here as usual. This order might be a formalisation of the contrast introduced before.

Convergence of TIP

In the most general representation, one iteration of TIP consists of four steps:

1. $(o_k, h_k) \rightarrow (i/h_k, h_k)$
2. $(i/h_k, h_k) \rightarrow (\mathcal{P}_{\mathcal{O}}(i/h_k), h_k) \equiv (o_{k+1}, h_{k+1})$
3. $(o_{k+1}, h_{k+1}) \rightarrow (o_{k+1}, i/o_{k+1})$
4. $(o_{k+1}, i/o_{k+1}) \rightarrow (o_{k+1}, \mathcal{P}_{\mathcal{H}}(i/o_{k+1})) \equiv (o_{k+2}, h_{k+2})$

From step 1 we see that the value of o_k is simply discarded. Let's trace what happens with h_k then:

$$h_{k+2} = \mathcal{P}_{\mathcal{H}} \left(\frac{i}{\mathcal{P}_{\mathcal{O}}\left(\frac{i}{h_k}\right)} \right). \tag{23}$$

It's typical procedure to prove the convergence of iterative algorithm by proving that it's base on a contraction transform. However, it ca be shown that $h_{k+2} = T(h_k)$ is *not* a contraction. For this, we can provide two examples that differs only by the starting conditions. In its simplest form, it can be reduced to $i = x * y$ with starting points $h_0 = x$ or $h_0 = y$. In the first case, we get all $h_k \equiv x$, and in the second $h_k \equiv y \ \forall k$. Of course, imposing stricter constraints on $\mathcal{H}$ and $\mathcal{O}$ or considering the convergence locally can help, but at the moment we leave it aside.[5]

[5] These two solutions correspond to two intersection points of Fig. 12

Namely, we can show that for some constraints TIP is equivalent to the coordinated descent optimisation.

Coordinate descent method

The essence of the method: keep one coordinate fixed and optimise by the second one. Exchange the coordinates and repeat.

In our case: $\|i - h * o\| \rightarrow$ min:

0. set some h_o, then repeat
1. $o_k = \arg\min_{o \in \mathcal{O}} \|i - h_k * o\|$
2. $h_{k+1} = \arg\min_{o \in \mathcal{H}} \|i - o_k * h\|$

Clearly it's a convex optimisation problem for each step as $\mathcal{K}_h$ is a convex set, and for $i_k = h_k * o_k$, $\|i_k - i\| \geq \|i_{k+1} - i\|$, and thus there is some limiting value of the norm of approximation error, but we cannot speak about convergence here.

Normal and tangential errors

The ideal noiseless image is given by exact result of convolution $\mathbf{i} = \mathbf{o} * \mathbf{h}$. The measured image is affected by noise. The statistics of noise and its type (additive, multiplicative, etc) defines the way the problem is formulated and solved. For instance, for additive Gaussian noise, the measured image $\mathbf{i}'$ is written as

$$\mathbf{i}' = \mathbf{i} + n, \tag{24}$$

where n is the noise term. Using the knowledge on the noise distributution (Gaussian in this case), and on the original variable $\mathbf{i}$ one can obtain the statistics of the random variable $\mathbf{i}'$ — in this case it will remain also gaussian with mean at $\mathbf{i}$. MAP methods are applicable where some information (called prior) on $\mathbf{i}$ (often in form of priors on $\mathbf{o}$ and $\mathbf{h}$) is available — one can then calculated a parametrised probability distribution (that is conditional probability) of $\mathbf{i}'_{|\mathbf{o},\mathbf{h}}$ and to find such values of parameters $\mathbf{o}, \mathbf{h}$ so the maximum probability will coincide with the measurement result $\mathbf{i}'$. The method is widely used, because even if it fails for some particluar measurement $\mathbf{i}$, it will work very good in average.

As illustration, consider non-linear deconvolution with a known PSF. In the case of additive gaussian noise, MAP estimator is equivalent to the least sqaure projection of $\mathbf{i}'$ on cone $\mathcal{K}_{\mathbf{h}}$.

Here we propose another approach to this problem, which is less dependent on the particular statistics of the noise and uses only the prior knowledge on $\mathbf{i}$ (or on $\mathbf{o}$ and $\mathbf{h}$). Consider the same example of non-linear deconvolution with known PSF and let the measured value $\mathbf{i}'$ happen to lie inside cone $\mathcal{K}_{\mathbf{h}}$. Then we can linearly deconvolve $\mathbf{i}'$ to obtain $\mathbf{o}'$,

$$\mathbf{i}' = \mathbf{o}' * \mathbf{h}, \tag{25}$$

with the exact equality. Without any additional information, we cannot make now any preference for $\mathbf{o}$ against the "faulty" object $\mathbf{o}'$. This happened because the noise hasn't taken $\mathbf{i}'$ out of the feasibility region $\mathcal{K}_{\mathbf{h}}$. Thus we can say that the difference $\Delta\mathbf{i}$ between the measurement results and the original image can be seen as consisting of two components, n_t and n_n, a normal and tangential to the feasibility set components.

$$\mathbf{i}' = \mathbf{i} + n_n + n_t \tag{26}$$

Please note two things here: 1) although it looks similar to expression of the additive noise, we do not say anything about the nature of the noise, which can be also multiplicative 2) terms normal and tangential should not be understood literally, but in more general sense as component that lies in the feasible set and component that does not. Of course this division is not unique, so I think let's say that it is normal component of the noise and perpendicular to it. Well, here I need to think more. The problem is that the feasibility set is not (or is it) a polyhedron. If it does contain a line in some direction (because it's always a cone, for instance), than it may contain tangential component in this direction, and if it doesn't have a line in some direction y, than the tangential component of noise in this direction by definition is zero. Or we can define it as in the example here — draw a line at some direction. Then to the point of intersection with the feasiblity region it will be a tnagential component, and the rest – normal.

And if the feasibility set contains a whole line at some direction, than we want of course to chose a minimal normal component. The

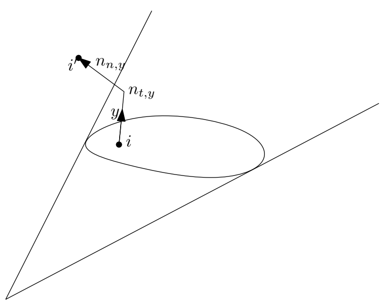


Figure 14: Normal and tangential error in the direction of y

least squares method tries to find the point which gives the minimal normal component (in $l-2$ norm). We propose to consider here corresponding decomposition of the result of linear deconvolution $\mathbf{o}'$ into normal and tangential component

$$\mathbf{o}' = vo_n + \mathbf{o}_t \tag{27}$$

Or (just got this idea) — we consider as in MAP all feasible objects (subset of all non-negative objects and in our case of limited support even the linear subspace)

(TBC)

TIME 1h 16m, 649 word +1 drawing/

Noise vs uncertainty

In contrast whit MAP methods which minimise the total distance to the feasible set, TIP minimises uncertainty. To explain this concept, consider the PSF. Every PSF obtained with a limited aperture will have unlimited support (by property of the Fourier Transform). Thus when looking for PSF with the limited support, we limit ourselves to a wrong set. The true image (noiseless) is obtained by convolution of an unlimited PSF with some object. Or, in case of multiframe blind deconvolution, or assumption on the nonchanging object can be wrong. In this case instead of feasibility set with a strictly defined boundary, one gets a blurred boundary. For limited support constraints, for instance, the gradient of the density of this boundary (for initial set, side of the simplex) is directed to the other vertices of the simplex. For the product feasibility set (the parabola inside the simplex) the direction of the gradient will be in the direction of the image of the corresponding vertices.

From this description it is clear now that TIP minimizes uncertainty.

We can be wrong in our *a priori* assumptions,k and the tangential projection will minimise our error.

The same principle can be applied also to normal deconvolution, when we can say that our psf is $\mathbf{h}$ plus some uncertain term. Then we should also use tangential projection in the direction of minimal uncertainty.

So uncertainty is when our constraints are expressed in terms of probability. MAP is when our measurements are random variable. It is an open questio how to combine both.

Tangential noise can be now explained as part of the noise that doesn't change the level of uncertainty. and the normal noise component is that part that goes in the direction of the uncertainty change.

relation to hierarchical MAP

This might be reduced (not sure still) to hierarchical MAP, with an additional model of hyperparameters on PSF. The hyperparameter here is the width of the support of the PSF. We know for sure, for instance, from the size of a diffraction-limited PSF that it's more than 3 pixels, but not sure is it 4, 5 and so on. For each of the values of the parameters we can build corresponding pdf and then sum them. But it will be equivalent to setting immediately pdf on all domain.

Another way to think about it that by some reason we are interesting only in some central part of a PSF. If we have no noise, this part will be given by TIP projection. The value of this projection will be independent on what number of pixel we are projecting on (thus if we project on 4-pixels psf and on 3-pixels psf, then the 3-pixel part in both psf will coincide after renormalisation). If we have noise and use l_2 or l_1 projection, than the result will be dependent on the number of pixles in the support. For instance, if we project in the least-squares sense on the side of ortant or if we project on the side of the simplex (like projecton i1 and i2 discussed before (actually in some different way, but OK. This need to be shown on a tetrahedron)). As we are not sure in our prior (or even we are pretty much sure we have cut some pixels form the psf), we need to weight-average all these projections. I'm not sure whether it will be the same as TIP projection, but here are some thoughts in support of it in case of l1 norm.

Our *a priori* knowledge about PSF can be formalised that it it's some function where at least several pixels are not equal to zero (like diffraction-limited psf or Pakhomov's speckle PSF within some support), say in some support $\mathcal{H}$. Then we can say that our PSF can be represented as a sum of a psf from $\mathcal{H}$ plus some non-negative psf which is equal to zero in $\mathcal{H}$. Now I don't know yet how to formulate our desire to make the weight inside $\mathcal{H}$ as large as possible. Maybe we can say that in this way we want to minimize the possible PSF, but not to a single-pixel-one or that we want to maximize power-in-the-bucket (but why?). The explanation through uncertainty says that we want it to match with the cropped value of the original PSF. If it happened that the original PSF was itself a a convolution of two other PSFs, TIP (in general, there might be special cases of exception) will find result corresponding to a more compact PSF, as it will have more weighted part in the bucket. Because at every step TIP finds PSF subdivided in the form of (18) and maximizing thus the norm of part in the bucket.

For the object in the multiframe presentation there might be some difficulties, as the generalised object is not represented in this way (outside the delta function, there might be any complex values. But I think that Gonsalves formula will work also in a general way).

Simplex in one Fourier frequency plane

Consider a projection of N-dimensional simplex on one frequency plane, that is a plane $F_k$formed by two vectors $coskx$ and $sinkx$.

Operation T of translation of the sequence x transforms this plane into itself[6]. Thus $T^N = I$, where I is identity transform, and as T keeps the norm, T should be a rotation on some angle. So if N is a prime number, then simplex S is projected as a regular N-gone on F_k.

[6] it's obvious for the continuous case and slightly less trivial for the discrete one

It's interesting to note how T is represented on each of f_k for different k, which is equivalent to numbering of the vertices of a N-gone. For $k = \pm 1$ it is clock-wise/anticlockwise numbering, for $k = 2$ – over one and so on. It can be related to the phase and to the expression of the shift operator T as a phase shift. In general, T corresponds to numbering over k/N of the whole cycle. And if k/N is unreducible fraction, then all of the vertices are numbered.

What follows from this Fourier plane interpretation that for N prime, convolution with any PSF is either 0 or invertible. This might make things easier (to define inverse to a convolution for instance, and with the tangential projections on a simplex side)[7] as for finite objects/psf one can always pad to some prime number, but computationally it just makes it less advantageous[8].

[7] or just not, as if one of the fequencies goes to zero, it's impossible to define exactly on what simplex side to project

[8] need to refresh here on how FFT is done

Multiframe TIP as Gröbner basis search

If we interpret convolution as polynomial multiplications, then each i_k defines some ideal I_k in the semiring. One can consider an ideal formed by all images $\mathcal{I} = \langle i_1, \ldots, i_M \rangle$. Then in the ring of polynomials there will be one minimal polynomial o that will contain all this polynomials and in the semiring there will be some finite sum (I guess), equivalent to Gröbner basis. It's also very similar to system identification (a lot of points form a simplex S_o that contains them all).

Here, TIP algorithm can be considered not as proposed before, where no essential difference between $\mathcal{O}$ and $\mathcal{H}$ was made, and where sequence of 2-dimensional images (a 3D array, actually) was vectorised and treated as one dimensional, but keeping its 3D structure by introducing independent variables for x, y and t, with t corresponding to the number of image. Difference between x and y is not so interesting at the moment, so we let it aside.

By introducing t we make a polynomial in the form $i(x,t) = i_1(x)t^0 + i_2(x)t^1 + \ldots$. By assuming each of the images were obtained as product of the same object $o(x)$ with different PSFs $h_i(x)$ we should receive

$$i(x,t) = o(x) \sum_{i=1}^{N} h_i(x) t^{i-1}, \tag{28}$$

that is there should be a common factor not dependent on t.[9] If the noise is present, the equation changes to

[9] This also nicely corresponds to our former illustration with o having a compact(1-pixel-wide) support

$$i(x,t) = o(x) \sum_{i=1}^{N} h_i(x) t^{i-1} + \sum_{i=1}^{N} n_i(x) t^{i-1}, \tag{29}$$

and can we say here that we are looking for the largest $o(x)$ satisfying this?

To return everything to the semi-ring of non-negative polynomials, we can group positive and negative coefficients of noise and look for something like

$$i(x,t) + \sum_{i=1}^{N} n_{i,-}(x)t^{i-1} = o(x)\sum_{i=1}^{N} h_i(x)t^{i-1} + \sum_{i=1}^{N} n_{i,+}(x)t^{i-1}, \quad (30)$$

but this is too advance for me at the moment.

Multiframe deconvolution as one frame deconvolution with support constrain

Let's show that multiframe deconvolution problem can be reduced to a one-frame blind deconvolution problem with additional dimensionality of the domain and additional support constraint.

Let image space be defined over some $\mathbf{x}$ (e.g. $\mathbf{x} = (x, y)$ for two-dimensional images), and let $i_k = i_k(\mathbf{x})$ be the results of convolution of the same object o with different PSFs h_k:

$$i_k = o * h_k. \quad (31)$$

Consider an image defined on a domain $(\mathbf{x}, k)$

$$i(\mathbf{x}, k) \stackrel{\text{def}}{=} i_k(\mathbf{x}). \quad (32)$$

Define in a similar way $h(\mathbf{x}, k), n(\mathbf{x}, k)$ and define o_δ as

$$o_\delta(\mathbf{x}, k) = \begin{cases} o(\mathbf{x}), & k = 1 \\ 0, & k \neq 1 \end{cases}. \quad (33)$$

Then obviously

$$i(\mathbf{x}, k) = h(\mathbf{x}, k) * o_\delta(\mathbf{x}, k), \quad (34)$$

and the original multiframe problem can be reformulated as a one-frame deconvolution

$$i(\mathbf{x}, k) = h(\mathbf{x}, k) * o(\mathbf{x}, k) + n(\mathbf{x}, k), \quad (35)$$

with constraints

$$h \in \mathcal{H}' \stackrel{\text{def}}{=} \mathcal{H} \times K, \quad o \in \mathcal{O}' \stackrel{\text{def}}{=} \mathcal{O} \times \{o(\mathbf{x}, k) = 0 \text{ for } k \neq 1\}. \quad (36)$$

If $\mathcal{H}$ and $\mathcal{O}$ are finite support constraints, so will be $\mathcal{H}'$ and $\mathcal{O}'$.

Or we can say it differently: let initial constraint on o be only its non-negativeness. Then $\mathcal{O}'$ is intersection of non-negativeness constraint and finite support constraint. As one-frame image formation model (35) is symmetrical with respect to o and h, it's natural to set similar type of constraint on both of them. This motivates our limitation to support constraint only.

Outlines as part of GWS-II

Outline for section Maximum contrast and uniqueness of deconvolution

12.09.2017

written in 45 minutes, 297 words, but it was combined with thinking of new ideas, so speed reduce of twice is not bad.

- A concept of maximal contrast (or better to say of function with not improvable contrast) is introduced.
- It is simply to show that if either h or o are functions with improvable contrast, the solution is not unique.
- This supports an idea of limited support constraints as every member of such class of functions is not improvable in contrast.
- If image is given as convolution of two contrast-maximal functions, there still can be non-uniqueness of the solution. This can be reduced by choosing from the function class one with the maximum contrast, that is with the maximum value. One may want to choose for the maximum contrast of the object or of the PSF, depending on the *a priory* knowledge.

Outline for section Limited support and uniqueness

written in 54 minutes, parallel with thinking new ideas again.

543 words, very good!

- degrees of a function with limited support as example of nin-unique solution under limited support constraint.
- Link to the roots of polynomials
- Number of different solutions as set of solution of equation obtained form sum of degrees of the irreducible multipliers of i
- Uniqueness of o as consequence of the constraint $\deg_t o = 0$
- it's not clear at the moment should the degree of h be minimized or maximized to provide the unique solution. From the equation on the degrees of irreducible multipliers, it's clear that in both cases it doesn't guarantee the uniqueness.
- in this case the solution is dependent on the initial point of iterations, which is however always fixed in TIP. This is illustrated by the geometrical TIP example; however non uniqueness of the example is caused by multipliyng by a scalar factor, which was exluded before

Outlook

- to consider h as infinite series which are truncated by the constraints. In this case we want of course to maximize the degree

- To consider dependence of the zeroes of the polynomials (or the zeroes of the Fourier spectra) on the coefficients of the series/poly. Although it's continuous, it's definitely more sensitive on the higher degrees. Thus we might consider the higher degrees as less reliable and try to minimize their presence
- these two things combined may mean that our task is to find the polynomial with zeroes as close to that of i from the measurement, and of a minimum possible degree.
- very simple proof of multiframe deconvolution as one frame via eq. (14)