Type Vigilance and the Truth About Transient Gradual Typing

OLEK GIERCZAK, Northeastern University, USA
LUCY MENON, Northeastern University, USA
CHRISTOS DIMOULAS, Northwestern University, USA
AMAL AHMED, Northeastern University, USA

Gradually typed languages (GTLs) permit statically and dynamically typed code to coexist in a single language and allow programmers to add type annotations to enforce precise types on values produced by code with imprecise type annotations. But different gradual languages dynamically check the same types via different semantics for type casts. For instance, the Natural semantics turns type casts into type-enforcing proxies, while the Transient semantics does simple tag checks when it evaluates type casts and elimination forms. This raises the question of whether the type annotations developers add to their code are indeed enforced, either statically or dynamically. Neither type soundness, nor complete monitoring, nor any other meta-theoretic property of gradually typed languages provides a satisfying answer.

In response, we present *vigilance*, a semantic property that ensures developers can rely on type annotations in a GTL, by requiring the semantics of the GTL to supplement static checks with explicit type casts that perform sufficient checks. Technically, vigilance asks if a semantics enforces the complete *run-time typing history* of a value, which consists of all the types that casts should enforce on the value at run time. We show that Natural is *vigilant* for a simple type system, but Transient is not, while Transient is *vigilant* for a tag type system, but Natural is not, hence, clarifying their comparative type-level reasoning power. Furthermore, using vigilance as a guide, we develop a new gradual type system for Transient, dubbed *truer*, that is stronger than tag typing and more faithfully reflects the type-level reasoning power guaranteed by Transient semantics.

Additional Key Words and Phrases: gradual typing, semantics, logical relations, vigilance, transient

1 CAN DEVELOPERS RELY ON ANNOTATIONS IN GRADUALLY TYPED PROGRAMS?

In gradually typed languages, even sound ones, developers cannot always take type annotations at face value. Since gradual type systems have to accommodate source programs without all the annotations that a typical type system would expect, gradually typed languages (GTLs) aim to detect dynamically whether a program's behavior and its existing annotations are at odds. Typically, the behavior of a gradually typed program is defined by elaboration into an *internal gradually typed language* (IGTL) equipped with an operational semantics. The IGTL contains *type casts* and *type assertions*, which perform runtime checks that bridge the gap between what the type system of the GTL can statically assure on its own and what the developer claims through the type annotations of the program.[1] Hence, developers should still be able to rely on the type annotations in a source program as they reason about their code; if there is a mismatch between an annotation and the program's behavior, the type casts and assertions of the translated program should reveal it by raising a dynamic type error. Unfortunately, there is a problem: due to concerns about performance overheads and implementation difficulty, the evaluation of a translated program may take place in an IGTL that does not perform all the checks needed by the type system of the GTL, e.g., [27].

Because the implementation of a GTL may not dynamically enforce types, the meaning of types differs from what is conventional in strongly typed languages. In fact, prior work [10] shows there is a spectrum of type soundness results for GTLs, varying by IGTL semantics: the Natural semantics

[1]The notion of IGTL is related to *cast calculi* [6, 21, 27], a term used for multiple systems with different conditions and theorems, sometimes orthogonal to the concerns in this paper. Therefore, to avoid association with technical details in the various incarnations of cast calculi, we use the term IGTL.

Authors' addresses: Olek Gierczak, Northeastern University, Boston, MA, USA, gierczak.o@northeastern.edu; Lucy Menon, Northeastern University, Boston, MA, USA, semiotics@ccs.neu.edu; Christos Dimoulas, Northwestern University, Evanston, IL, USA, chrdimo@northwestern.edu; Amal Ahmed, Northeastern University, Boston, MA, USA, amal@ccs.neu.edu.

satisfies a full type soundness theorem, while the Transient semantics satisfies a tag (or top-level type constructor) soundness theorem. These results, however, say nothing about whether either semantics enforces type annotations ascribed to values that, during the evaluation of a program, "flow" to parts of the program with less precise annotations than the ascribed ones. Hence, type soundness does not capture the flow-sensitive type-based reasoning that is possible in gradually typed languages where GTL types and IGTL semantics are in sync.

Complete monitoring for gradual types [9, 11] aims to bridge this gap by ensuring checks are performed by an IGTL semantics at certain key points during the evaluation of a program. However, complete monitoring does not ensure the performed checks match the types ascribed to a value; it only determines that *a* check happens on the value. Furthermore, at a technical level, complete monitoring relies on an instrumented operational semantics which uses a syntactic, brittle, and difficult-to-generalize system of annotations that adds and discharges so-called *ownership obligations* as it reduces a program. As a result, complete monitoring is lacking in both the strength and applicability of its result.

Contributions. In this paper, we revisit the gap between GTL type systems and IGTL semantics, and make the following contributions:

1. As a prerequisite to comparing how different type systems and dynamic semantics fit together, we introduce a uniform semantic framework for GTLs and IGTLs. Greenman and Felleisen [10] compare IGTL semantics by defining a single syntax and type system for an IGTL, and varying its semantics. We extend this idea to build a uniform framework that uses a single syntax for all our GTLs and a single syntax for all our IGTLs. We isolate the variation between GTLs to the definitions of their type systems, and the variation between IGTLs to the definitions of their type systems and the reduction rules of their semantics. Moreover, the latter differ only in certain parameterized reduction rules.

In detail, we present typing rules for three GTL and IGTL type systems—simple typing, tag typing, and truer typing—and IGTL reduction rules for two semantics—Natural and Transient. We combine these into pairs of GTLs and IGTLs, covering all combinations. Furthermore, we connect each GTL with its corresponding IGTL, through a type-preserving translation. This last step is critical; it allows us to lift properties of IGTLs to GTLs. Specifically, it enables us to state general properties about the "fit" between a GTL type system and an IGTL semantics, investigate which combinations satisfy these properties at the IGTL level, and then project the results back to the GTL.

2. We introduce *vigilance*, a semantic property to describe when the type system of a GTL and the semantics of an IGTL are a good fit for each other. Vigilance captures the flow-sensitive type-based reasoning implied by the annotations, which type soundness lacks, and ensures that annotations are meaningfully enforced, unlike complete monitoring. Vigilance is defined with a *step-indexed unary logical relation*, using an instrumented semantics where every value that arises during computation is "allocated" to a unique label, and each label is associated with type information derived from annotations that the value has flowed through during computation. Unlike complete monitoring, this instrumentation is simple, and adapting an operational semantics to the instrumentation is purely mechanical. Vigilance says that each value satisfies, or "behaves like", the types associated with its label.

3. We evaluate vigilance:

- We prove that the Natural semantics is vigilant for a standard type system while the Transient one is not. These are expected analogues of the type soundness and complete monitoring results already known from previous work.
- We show that a tag type system, previously used in syntactic soundness results for Transient [10, 27], is not strictly semantically weaker than a standard type system, because well typed

functions can be applied in more contexts. As a result, we prove Natural is not vigilant for such a tag type system, while Transient is.
- We show that there exists a vigilant type system for Transient that is semantically stronger than the tag type system, and call this type system *truer typing*. We show that this stronger type system can be used to justify an optimization similar to prior work [26], by writing the optimization as a simple type-preserving translation and proving it correct.

Outline. The remainder of the paper is organized as follows. §2 describes the background and key ideas of the paper in detail. §3 presents the formal linguistic setting of the paper and §4 builds on that to define vigilance and prove that Natural is vigilant for simple typing. §5 develops the truer type system, shows that Transient is vigilant for tag typing and truer typing, and proves that truer renders unnecessary some of the checks that Transient performs. §6 describes related work not already covered in §2, and §7 discusses future directions and concludes.

2 MAIN IDEAS

2.1 An Example of Why Programmers Can't Always Trust Annotations

Let us take a closer look at Natural and Transient to see why a programmer cannot always rely on type annotations in code. Consider Reticulated Python, a gradually typed variant of Python [25]. Notably, Reticulated Python has two back ends: Natural Reticulated relies on proxies that are difficult (though not impossible [15]) to implement in a performant manner, while Transient Reticulated offers a lightweight alternative. That is, the behavior of a Reticulated Python program can be described through its translation and evaluation into two IGTLs that both aim to enforce the types of Reticulated programs but one has Natural semantics and the other Transient semantics.

The example in Figure 1 demonstrates the workings of Natural Reticulated Python. Here, the developer applies `encode`, which expects as its second argument a function from strings to integers, to `rolling_hash`, which has no type annotations. Even though Reticulated Python's type checker does not have at its disposal the necessary annotations to derive that `rolling_hash` indeed has type `Function([String],Int)`, it accepts the program as well-typed. To compensate for the partial type checking, the translation of the call to `encode` at the bottom of the snippet injects a cast around `rolling_hash` from the dynamic type `Dyn`[2] to the expected type `Function([String],Int)`. The role of the cast is to check whether `rolling_hash` behaves as a function from strings to integers whenever it is called by the body of `encode`. Consequently, in Natural, the cast checks that `rolling_hash` is a function and then wraps it in a proxy that, in turn, checks that the results of calls to `rolling_hash` are integers.

```
def rolling_hash(el):
    ...
def encode(file_path, word_coder: Function([String],Int)):
    ...
encode("~\paper.tex",rolling_hash)
```

Fig. 1. Dynamically Typed Argument for a Typed Parameter

The same code evaluates differently in Transient Reticulated Python. As before, the translation injects a cast from type `Dyn` to type `Function([String],Int)` around `rolling_hash`. However, the Transient cast only checks that `rolling_hash` is a function and does not create a proxy. To counter for the absence of the proxy, the translation further re-writes the body of `encode` and injects type assertions to ensure that the results of any calls to `word_coder` are integers. In other words, Transient performs lightweight tag checks instead of wrapping values in proxies, but seems to establish the same type-level facts for a program as Natural with its costly proxies.

[2]Code without annotations implicitly has type `Dyn`.

However, the above conclusion is misleading. This becomes clear when developers attempt to write dynamically typed code that uses a function with type other than Dyn. This situation arises ubiquitously in gradually typed languages [12, 14, 30], where developers routinely opt to import libraries with typed interfaces even while writing dynamically typed code.

Unfortunately, many libraries with typed interfaces are in fact dynamically typed themselves, and so the interface is nothing more than a thin veneer of possibly incorrect type annotations. The casts and assertions injected by the translation of the developer's code to one of the IGTLs may not perform all the checks that are necessary to validate these annotations, and the developer may only discover that they are incorrect after a painful debugging process; it all depends on whether the semantics of the IGTL enforce the types the developer relied upon.

The variant of our running example in Figure 2 demonstrates this situation. In this example, the developer opts not to add any type annotations to encode. Instead the developer uses type_and_launder, a hand-rolled type adapter, to get some of the benefits of types while implementing the dynamically typed encode. In particular, the developer expects that assigning the result of type_and_launder to word_coder_laundered makes unnecessary the use of defensive code that inspects the results of calls to word_coder to determine whether they are integers. However, whether Reticulated Python ends up checking that calls to word_coder_laundered produce integers depends on the semantics of the IGTL the program translates to. In Natural, word_coder_laundered has the proxy that inspects the results of the function, ensuring the type Int from the annotation; in Transient, there is no proxy and, since word_coder is an identifier of type Dyn, the compiler injects no type assertions for the results of calls to word_coder. Hence, in Transient, a developer that trusts the annotations of the program and does not code defensively may discover that the results of word_coder_laundered are not integers when some other part of the code handles them.

```
def type_and_launder(f: Fun([String],Int)):
    return f
def encode(file_path, word_coder):
    word_coder_laundered = type_and_launder(word_coder)
    ...
encode("~\oopsla23\paper.tex",rolling_hash)
```

Fig. 2. Applying a Hand-Rolled Type Adapter

2.2 Type Soundness is Not Enough

At first glance, this difference between the proxy enforcement in Natural and tag-based checks in Transient seems like an issue that type soundness for Reticulated Python should clarify. Type soundness does distinguish between the two [10], but falls short of fully characterizing this difference in their behavior. As a result, the developer needs to know something about the semantics of the IGTL beyond type soundness to be able to trust that Reticulated Python is going to enforce the annotation.

Intuitively, in Transient, the use context of a function is expected to inspect the results of the function via type assertions at call sites to make sure the function behaves as its type describes. Since, for the final result of a program there is no use context, Transient can say little about programs that produce functions. In contrast, in Natural, the proxy around a function enforces the expected type independently of the use context.

Formally, the picture becomes a bit more nuanced. If a source program is well-typed at type τ, then, it translates to a Natural program with casts that has type τ. When the translated program runs to a value, the value also has type τ. In contrast, the differently-translated Transient program and its result value have types that match the tag of τ (i.e. its top type constructor), but that are not necessarily exactly equal to τ (tag soundness). Hence, type soundness seems to reveal that the choice of IGTL affects the predictive power of Reticulated Python's type system.

However, type soundness stops short of explaining whether the semantics of either IGTL performs all the checks the Reticulated Python type system, and hence developers, rely on. After all, type soundness only connects the type of the source program with the type of the translated IGTL program's result. Therefore, all annotations that do not contribute to that goal are immaterial. And this is exactly the situation with the running example. First, given the pervasive lack of annotations in the example, Reticulated Python's type-checker determines that the type of `word_coder_laundered` is `Dyn`, which is inhabited by all values, not just functions from strings to integers. So type soundness says nothing about `word_coder_laundered` directly. Second, the developer cannot rely on the compositional nature of the typing rules of the type system and type soundness to deduce transitively some more precise type-level information than `Dyn` for `word_coder_laundered`. While the typing rules of Reticulated Python and type soundness do stipulate that the result of `type_and_launder(word_coder)` needs to behave as a function from integers to strings, this is a fact that the type system cannot prove without the help of the semantics of the underlying IGTL, again due to the sparsity of type annotations.

As discussed above, in Natural, which employs a proxy around `type_and_launder(word _coder)`, the expected type is indeed enforced. But, in Transient, which does not create the proxy, the type, and the corresponding annotation, are "forgotten". In general, for Transient, the avoidance of proxies comes at the price of a weaker (tag) soundness guarantee, but in this case, this weaker guarantee is not the reason that the developer cannot use type-level reasoning alone to justify the elision of defensive checks on the results of `word_coder_laundered`. In fact, the Forgetful [7] and Amnesic [11] variants of Natural create the same proxy as Natural but then remove it upon the assignment to `word_coder_laundered` to reduce the running time and memory cost from proxies. Hence, their net effect is exactly the same as that of Transient in this example but they are as type sound as Natural. The root of the issue is that from the perspective of type soundness, it does not matter whether the annotation on the result of `type_and_launder(word_coder)` is ever enforced, meaning the developer must reason beyond type soundness to ensure their annotation is meaningful.

2.3 Complete Monitoring is Not Enough

A first attempt at going beyond type soundness is complete monitoring for gradual types [9, 11], which adapts the notion of complete monitoring from work on contract systems [5]. The starting point for complete monitoring is a collection of semantics for an IGTL, i.e., different semantics with the same syntax and the same simple type system. The goal is to determine which of the semantics enforces the types of IGTL programs completely. However, complete monitoring establishes a weaker property. Intuitively, an IGTL semantics is a complete monitor if it "has complete control over every typed-induced channel of communication between two components." [9].

Formally, complete monitoring relies on a brittle notion of ownership of values and code by components. In detail, in the complete-monitoring framework, components are encoded as label annotations on expressions; all expressions that "belong" to the same component have the label of the component. A system of axioms determines how values may accumulate labels, (and therefore component-owners), as they 'flow" from one component to another during evaluation. Another set of axioms describes how values lose labels due to checks from type casts. Given this formal setup, complete monitoring becomes preservation of a single-ownership invariant for all values during the evaluation of a program: a value can either have a single label, or multiple that are separated by type casts. Hence, the single-ownership property entails that the IGTL semantics is in control of all flows of every value from one component to another as it imposes checks that regulate such flows. In that sense, Natural is a complete monitor, but Transient is not.

From the perspective of a developer, complete monitoring says that checks are happening where they are supposed to happen. Back to the running example, since Natural is a complete monitor, every "channel" is checked. Hence as `word_coder` flows through `type_and_launder`, it accumulates labels and becomes `word_coder_laundered`. But these labels are separated by casts, and, furthermore, when `word_coder_laundered` is applied in the body of `encode`, the casts perform checks that discharge the accumulated ownership labels for the result of the function and allow it to obtain the same label as its calling component.

Note, though, that the description above is intentionally vague about what checks exactly have to happen at each point in the evaluation where complete monitoring prescribes a check. This is because the formal framework of complete monitoring does not specify that much: an IGTL semantics would still be a complete monitor if all checks were equivalent to a no-op. Hence, a developer cannot reason about what types a semantics enforces based solely on complete monitoring.

In summary, while complete monitoring offers a dimension that type soundness lacks, i.e., that casts should mediate the interactions of components, it is not strong enough to determine whether an IGTL semantics is a good match for a GTL type system. First, it considers a single (simple) *IGTL* type system. So it cannot say anything about GTLs and their different possible type systems. Second, even in this restricted setting, it entails a rather weak relation between the types of the IGTL and the checks that its semantics perform.

2.4 Vigilance: Type Annotations You Can Trust

Intuitively, a GTL should enable developers to reliably take into account type annotations when they reason about code, in the same way that they rely on type annotations when they reason about code in strongly typed languages, or the way they rely on manually placed type-level predicates in dynamically typed languages. Returning to the example, Reticulated Python, under either Natural or Transient, should empower the developer to deduce that `type_and_launder(word_coder)` behaves according to the return type of `type_and_launder`, and so does `word_coder_laundered` after the assignment.

To capture this intuition, this paper develops a new semantic property called *vigilance*. The goal of vigilance is to describe how the semantics of an IGTL impacts the enforcement of the type annotations in a GTL program. Vigilance as a property serves two purposes: on one hand, given a type system for a GTL, vigilance examines whether the semantics of an IGTL can mislead developers who assume that the language does its due diligence to enforce types as the GTL type system promises; on the other hand, given a semantics for an IGTL, vigilance guides the design of a type system for a GTL such that the IGTL semantics and the GTL type system work together to truly enforce the developer's type annotations.

Vigilance as a semantic property goes beyond (semantic) type soundness and complete monitoring. By requiring that the semantics of an IGTL enforce every type from annotations a value flows through during the evaluation of a program, **vigilance considers flow-sensitive and compositional type information from annotations that type soundness ignores**. And, instead of only specifying the location of checks as in complete monitoring, **vigilance ensures and allows for exactly enough dynamic enforcement to enforce the type-based properties of terms**. And by varying the type system and translation of a GTL, and thereby varying the typing histories for expressions, **vigilance is applicable to more systems than complete monitoring**.

2.5 Vigilance: Language Framework

In order to specify when type annotations in a GTL program are enforced, we need a language framework that lets us refer to values and the type obligations they collect at run time. The language framework that we use to develop vigilance is inspired by that of Greenman and Felleisen [10],

who show how a framework that irons away the differences between different GTLs enables an apples-to-apples comparison of their semantics. Our formal development starts with the definition of syntax for a GTL based on the gradually typed λ-calculus [20]. For each type system that we wish to consider (simple, tag, truer), we define a single type-preserving translation that maps well-typed (under the relevant type system) programs in this GTL to programs in a family of IGTLs that share a syntax and type system but differ in their reduction semantics. Since the translation is type preserving, we reduce the relationship between IGTL semantics and GTL types to the relationship between IGTL semantics and IGTL types.

Vigilance relies on naming values in order to associate with each value a list of types it must satisfy, so we instrument our IGTL operational semantics. We allocate every value v that arises during evaluation (including the results of casts and assertions) to a fresh label ℓ in a *value log* Σ, and modify elimination forms to act on labels ℓ and eliminate the value associated with the label $\Sigma(\ell)$. On every cast that evaluates to a label, the type from the cast is also stored. Unlike complete monitoring, the instrumented semantics in our framework are just a system for naming values and tracking types over a predefined semantics.

2.6 Vigilance: Technical Definition and Implications

After equipping our IGTL family with two different reduction semantics, for Natural and Transient respectively, we define vigilance using a (step-indexed) unary logical relation that models IGTL types τ as sets of values v that inhabit them. In contrast to a typical logical relation for type soundness [2], vigilance comes with an extra index, the *type history* Ψ, which can be thought of as a value-log typing. Specifically, the type history Ψ is a log that collects the types present on each cast or assertion in the program that a particular value has passed through.

Semantic type soundness says that a language is semantically type sound if and only if any well-typed expression $e : \tau$ "behaves like" τ. We say that the semantics of an IGTL are *vigilant* for a type system if this remains true when the latter constraint is strengthened to say that in any well-formed type history Ψ (capturing potential casts and assertions from a context), e behaves like τ. The latter means that, if evaluating e produces a label ℓ (as well as a potentially larger type history Ψ', capturing casts and assertions present in e), then $\Sigma(\ell)$ not only behaves like τ (in the conventional sense), but also like all of the types in $\Psi'(\ell)$.

2.7 Vigilance: By Example

Vigilance tells the developer that every value that flows through an annotation must satisfy that annotation, or more precisely, that every value must satisfy its typing history. To see this concretely, we can approximate the value log by partially annotating sub expressions of an example in Reticulated Python with labels[3]. We will write `{e}_l` to denote an expression `e` annotated with label `l`. Figure 3 presents an example program to illustrate the value log and typing histories. Both Natural and Transient evaluate this example identically (barring differences in elided code represented by `...`). The developer starts with three functions: `make_nat`, which casts an integer to a natural number, `use_nat`, which does something with a natural number, and `use_int`, which does something with an integer.

The developer writes a function `case_by_case`, which takes an integer `i` and a boolean `b`, and performs two case analyses on `b` with some unspecified code in between. In the first conditional, the developer casts `i` to a natural or leaves it as is if the boolean is true or false respectively, and

[3]The type preserving translations will only add expressions (usually casts) around particular subexpressions of the GTL term, and each subexpression in these simple examples will, barring errors, evaluate to a value, so this approximation is sound.

assigns the result to x. In the second conditional, the developer treats x as a natural or as an integer if the boolean is true or false respectively, and returns the result.

In the second conditional, the developer performs flow-sensitive reasoning to ensure that x must be a natural number, and therefore use_nat(x) must not fail on its argument. To illustrate and justify that flow-sensitive reasoning, consider the annotations l0, l1 and l2. In order to determine what value in the value log and type in the typing history gets assigned to l0, the developer performs case analysis on b. In the case that b = **False**, the conditional evaluates to i, so l0 maps to i in the value log, and Int in the typing history. Then in the second conditional, x is safely used as an integer. In the case that b = **True**, the conditional evaluates to the first branch. The allocating semantics first allocates i in l1 with typing history Int. It then evaluates the function application, which asserts that i must be a natural number. In the case that i is indeed a natural number, the cast succeeds, and l1 is allocated in l2 with typing history Nat as well as Int (from the typing history of l1). Finally, the conditional expression finishing evaluating, and l2 is allocated in l0, with typing history Nat as well as Int. Then in the second conditional, x is safely used as a natural number, since the typing history of l0 contains Nat.

```
def make_nat(i : Int) -> Nat:
    return i
def use_nat(n : Nat):
    ...
def use_int(i : Int):
    ...
def case_by_case(i : Int, b : Bool):
    x = {{make_nat({i}_l1)}_l2 if b else i}_l0
    ... # code that does not use x or b
    return use_nat(x) if b else use_int(x)
```

Fig. 3. Example to Illustrate Labels and Typing History

Vigilance tells the developer that at each point in this analysis, the value at a label satisfies every type in its typing history. However, since this example is evaluated identically between Natural and Transient, the developer hasn't learned anything new about their guarantees. Back to the example in Figure 2, vigilance communicates to the developer the difference between Natural and Transient Reticulated Python regarding the guarantees each offers about the type annotations in the code. In order for either Natural or Transient to be vigilant for the simple type system of Reticulated Python, the value bound to word_coder_laundered after the assignment should behave not only according to type Dyn, but also, due to the type history, according to type Fun([String],Int). Only Natural meets this standard, and only in Natural can the developer rely on the type annotation of type_and_launder to avoid programming encode defensively.

2.8 Vigilance: An Examination of Transient

Besides informing developers, vigilance is a guiding tool for language designers. To demonstrate this aspect of vigilance, the paper revisits Transient and asks for what type system is the Transient semantics vigilant. An initial answer, which confirms prior work on Transient and tag soundness, is that one such type system ascribes type tags (top-level type constructors) to expressions rather than types. Hence, this *tag type system* accepts programs that have severely imprecise types, which makes plain the difference between what Reticulated Python's simple type system promises and what its Transient semantics achieves.

However, the fact that Transient is vigilant for tag typing, also reveals that Transient can provide strong support when developers reason about Reticulated Python code. For example, consider the code in Figure 4. Here, a developer uses a dynamically-typed image library that provides two functions: crop, which crops images to a particular size, and segment which segments an image into a pair of a "foreground" and a "background" image.

Similarly to the previous examples, the developer creates a typed interface for this library that restricts it to PNGs. Unlike before, in this

```
# Typed interface
def png_crop(img : PNG) -> PNG:
    return crop(img)
def segment_png_small(img : PNG) -> (PNG, PNG):
    let (a, b) = segment(img)
    in png_crop(a), png_crop(b)

let foreground = segment_png_small(my_png)[0]
```

example, the typed interface does a bit more than just acting as a veneer of types; it uses `crop` to reduce the sizes of the pair of images that `segment` produces. If the example is evaluated in Transient, due to vigilance, a developer can rely on the return type annotation for `segment_png_small`: vigilance for tag typing guarantees that Transient checks that the results of the calls to `png_small` in the body of `segment_png_small` are `PNG`s, and hence, the result of `segment_png_small` has type `(PNG, PNG)`.

2.9 Vigilance: Towards Truer Transient Types

More interesting is the fact that vigilance can also point the way for designing a new gradual type system for which Transient is vigilant, and that maps expressions to more precise types than the tag type system. In detail, this *truer type system* makes limited use of union and intersection types in order to reflect the outcomes of Transient casts and assertions to the types of expressions, similarly to the way a developer can use these tag checks to reason about code as we discuss above.

A consequence of this more precise type system is that some of the checks that are necessary so that Transient is vigilant for the tag type system can be elided in a provably correct manner when pairing Transient with the truer type system. For example, the truer type system can stitch together type information from the type assertions on the results of the calls to `png_crop` in the body of `segment_png_small` to deduce statically that `segment_png_small` indeed has a type that precisely matches its type annotation (rather than that it simply returns a pair that should be checked further at run time). Moreover, this precise truer type makes unnecessary a type assertion on the outcome of the left-pair projection that gets bound to `foreground`; it is statically known that it is a `PNG`.

2.10 Technical Contributions

The following table summarizes the vigilance results of the paper. Each cell of the table corresponds to a pairing of a semantics and a type system. When the cell contains a ✓, that semantics is vigilant for that type system; otherwise it is not.

	Simple Typing	Tag Typing	Truer Typing
Natural	✓	×	×
Transient	×	✓	✓

In addition, to the results discussed above, the table includes two negative results about Natural and the tag and truer type systems. It turns out that the simple type system rejects some programs that the other two accept, and these programs expose that tag and truer typing are not type sound for Natural. Since vigilance strengthens type soundness, Natural is also not vigilant for the two type systems.

3 FROM A GTL TO A FAMILY OF IGTLS

The top portion of Figure 5 presents the syntax of λ^{GTL}, our GTL, which is inspired by the gradually-typed λ-calculus [20]. Most of its features are the same as the corresponding features of a simply-typed λ-calculus extended with constants, pairs and their relevant elimination forms. The one unconventional syntactic form is that for anonymous functions. In particular, anonymous functions come with type annotations that describe both the type of their arguments and the type of their result — matching the way that Reticulated Python developers can annotate function definitions. The type annotations τ range over *simple types* with the addition of $*$, the dynamic type, which, as

$$
\begin{array}{rcl}
t & ::= & x \mid n \mid i \mid \mathsf{True} \mid \mathsf{False} \mid \lambda(x:\tau) \to \tau'.\, t \mid \langle t, t \rangle \mid \mathsf{if}\ t\ \mathsf{then}\ t\ \mathsf{else}\ t \\
& \mid & \mathit{binop}\, t\, t \mid t\ t \mid \mathsf{fst}\, t \mid \mathsf{snd}\, t \\
\tau & ::= & \mathsf{Nat} \mid \mathsf{Int} \mid \mathsf{Bool} \mid \tau \times \tau \mid \tau \to \tau \mid * \\
\\
e & ::= & x \mid n \mid i \mid \mathsf{True} \mid \mathsf{False} \mid \lambda(x:\tau).\, e \mid \langle e, e \rangle \mid \mathsf{if}\ e\ \mathsf{then}\ e\ \mathsf{else}\ e \\
& \mid & \mathit{binop}\, e\, e \mid \mathsf{app}\{\tau\}\, e\ e \mid \mathsf{fst}\{\tau\}\, e \mid \mathsf{snd}\{\tau\}\, e \mid \mathsf{cast}\,\{\tau \Leftarrow \tau\}\, e
\end{array}
\qquad
\begin{array}{rcl}
\mathit{binop} & ::= & \mathsf{sum} \mid \mathsf{quotient} \\
n \in \mathbb{N} & , & i \in \mathbb{Z}
\end{array}
$$

Fig. 5. Syntax of λ^{GTL} (top) and λ^{IGTL} (bottom)

usual in the gradual typing setting, indicates imprecise or missing type information. For example, the expression $\lambda(x:* \to *) \to *.\, t$ represents an anonymous function that consumes functions and may return anything.

Since λ^{GTL} expressions t do not evaluate directly but, similar to Reticulated Python, are translated to an IGTL, before delving into the type checking and evaluation of λ^{GTL} expressions, we discuss briefly the syntax of λ^{IGTL}, our family of IGTLs. The bottom portion of Figure 5 shows the syntax of λ^{IGTL} expressions e. Its features correspond to those of λ^{GTL} with a few important differences. First, functions $\lambda(x:\tau).\, e$ come with type annotations for their arguments but not their results. Second, pair projections and function applications also have type annotations. Third, λ^{IGTL} has a new syntactic form compared to λ^{GTL}: *cast* expressions. Specifically, $\mathsf{cast}\,\{\tau_1 \Leftarrow \tau_2\}\ e$ represents a cast from type τ_2 to τ_1 for the result of expression e. In other words, while in λ^{GTL} all type annotations are on functions, in λ^{IGTL}, they are spread over applications, pair projections, function parameters, and casts. This is because the first three are the syntactic loci in a program that correspond to "boundaries" between pieces of code that can have types with different precision according to the type system of GTL. Hence, the translation injects type assertions and casts exactly at these spots.

Figure 6 presents type checking for λ^{GTL} expressions t and their translation to λ^{IGTL} expressions e with a single judgment $\Gamma \vdash_{\mathsf{sim}} t : \tau \rightsquigarrow e$ — the sim annotation indicates simple typing to distinguish it from the tag and tru type systems we present later. A λ^{GTL} function type checks at its type annotation $\tau \to \tau'$ if its body type checks at some type τ''. To bridge the potential gap between τ'' and τ', the translation of the function produces a λ^{IGTL} function whose body is wrapped in a cast from τ'' to τ', if needed. Specifically, metafunction $[\tau \swarrow \tau']e$ inserts a cast around e when τ is compatible with but not a subtype of τ' (subtyping is standard). Compatibility is the reflexive and symmetric relation that rules out non-convertible type casts, or type casts that will always error. Unlike standard definitions, compatibility includes $\mathsf{Nat} \sim \mathsf{Int}$[4],to allow programmers to freely convert between Naturals and Integers, and have the translation insert appropriate checks.

Conditionals type check and translate in a recursive manner. The type of the conditional is the consistent subtype join $\widetilde{\sqcup}$ of the types of its two branches. The consistent subtype join definition is standard [3], and gives the least upper bound of the types with respect to subtyping, as well as more precise types in place of $*$. To bridge the potential gap between the type of the branch and the consistent join, the translation may wrap each branch in a cast with the same metafunction as above. Translated applications obtain ascriptions for the return type of the applied function, along with (possible) casts around the argument expression that make sure the domain of the applied function jives with the type of the provided argument.

As an example of how the translation works, consider the λ^{GTL} expression

$$t = (\lambda(f:\mathsf{Nat} \to \mathsf{Nat}) \to *.\, f\ 42)\ \lambda(x:*) \to *.\, x$$

[4]Compatibility is a symmetric version of consistent subtyping [3], and if we used compatible subtyping here, we would not be able to convert from Int to Nat.

$\boxed{\Gamma \vdash_{\mathsf{sim}} t : \tau \rightsquigarrow e}$ (selected rules)

$$\frac{\Gamma,\ (x{:}\tau) \vdash_{\mathsf{sim}} t : \tau'' \rightsquigarrow e}{\Gamma \vdash_{\mathsf{sim}} \lambda(x{:}\tau) \to \tau'.\, t : \tau \to \tau' \rightsquigarrow \lambda(x{:}\tau).\, ([\tau' \swarrow \tau'']e)}$$

$$\frac{\Gamma \vdash_{\mathsf{sim}} t_1 : \tau \to \tau' \rightsquigarrow e_1 \quad \Gamma \vdash_{\mathsf{sim}} t_2 : \tau'' \rightsquigarrow e_2}{\Gamma \vdash_{\mathsf{sim}} t_1\ t_2 : \tau' \rightsquigarrow \mathsf{app}\{\tau'\}\ e_1\ ([\tau \swarrow \tau'']e_2)}$$

$$[\tau \swarrow \tau']e = \begin{cases} e & \text{if } \tau \leqslant: \tau' \\ \mathsf{cast}\ \{\tau \Leftarrow \tau'\}\ e & \text{if } \tau \not\leqslant: \tau' \text{ and } \tau \sim \tau' \end{cases}$$

$$\frac{\Gamma \vdash_{\mathsf{sim}} t_1 : * \rightsquigarrow e_1 \quad \Gamma \vdash_{\mathsf{sim}} t_2 : \tau' \rightsquigarrow e_2}{\Gamma \vdash_{\mathsf{sim}} t_1\ t_2 : * \rightsquigarrow \mathsf{app}\{*\}\ (\mathsf{cast}\ \{* \to * \Leftarrow *\}\ e_1)\ ([* \swarrow \tau']e_2)}$$

$$\frac{\Gamma \vdash_{\mathsf{sim}} t_b : \mathsf{Bool} \rightsquigarrow e_b \quad \Gamma \vdash_{\mathsf{sim}} t_1 : \tau_1 \rightsquigarrow e_1 \quad \Gamma \vdash_{\mathsf{sim}} t_2 : \tau_2 \rightsquigarrow e_2}{\Gamma \vdash_{\mathsf{sim}} \mathsf{if}\ t_b\ \mathsf{then}\ t_1\ \mathsf{else}\ t_2 : \tau_1 \mathbin{\widetilde{\sqcup}} \tau_2 \rightsquigarrow \mathsf{if}\ e_b\ \mathsf{then}\ ([\tau_1 \mathbin{\widetilde{\sqcup}} \tau_2 \swarrow \tau_1]e_1)\ \mathsf{else}\ ([\tau_1 \mathbin{\widetilde{\sqcup}} \tau_2 \swarrow \tau_2]e_2)}$$

$\boxed{\tau \sim \tau'}$ (selected rules)

$$\frac{}{\tau \sim *} \qquad \frac{}{\mathsf{Nat} \sim \mathsf{Int}} \qquad \frac{\tau_0 \sim \tau_2 \quad \tau_1 \sim \tau_3}{\tau_0 \times \tau_1 \sim \tau_2 \times \tau_3} \qquad \frac{\tau_0 \sim \tau_2 \quad \tau_1 \sim \tau_3}{\tau_0 \to \tau_1 \sim \tau_2 \to \tau_3}$$

Fig. 6. Simple Typing Translation From λ^{GTL} to λ^{IGTL}

and its λ^{IGTL} image

$$e = \mathsf{app}\{*\}\ (\lambda f : \mathsf{Nat} \to \mathsf{Nat}.\, \mathsf{cast}\ \{* \Leftarrow \mathsf{Nat}\}\ (\mathsf{app}\{\mathsf{Nat}\}\ f\ 42))$$
$$(\mathsf{cast}\ \{\mathsf{Nat} \to \mathsf{Nat} \Leftarrow * \to *\}\ \lambda x : *.\, x).$$

The example is reminiscent of the one from Figure 1; it involves a higher-order function whose parameter f has an annotation. Hence, the translation injects in e a cast to ensure that the argument of the higher-order function behaves as prescribed by the annotation. The translation also introduces a trivial cast around the body of the higher-order function to bridge the gap between Nat, the deduced type for the body of the function, and $*$, the expected type according to the type annotation in t. Conversely, the translation does not introduce a cast around 42, the argument to f in the body of the higher-order function, as its deduced and expected type (Nat) are the same. Finally, the translation annotates all applications with their expected types, affirming the return types from the type annotation in t.

Figure 7 gives some rules for the typing judgment for λ^{IGTL}: $\Gamma \vdash_{\mathsf{sim}} e : \tau$. In general, the type system of λ^{IGTL} is straightforward and closely follows the translation of λ^{GTL}.

The translation has a key property: it maps well-typed λ^{GTL} expressions to well-typed λ^{IGTL} expressions with the same type.

Theorem 3.1 (The Translation is Type-Preserving). *If* $\Gamma \vdash_{\mathsf{sim}} t : \tau \rightsquigarrow e$ *then* $\Gamma \vdash_{\mathsf{sim}} e : \tau$.

IGTL Type Guarantees Lift to the GTL. As discussed in §2, type preservation allows us to focus on λ^{IGTL} to determine the impact of Natural and Transient on the enforcement of the types of a well typed λ^{GTL} expression t. The semantics of a λ^{GTL} expression is defined by elaboration into λ^{IGTL}, and the types of λ^{GTL} expressions match the types of elaborated λ^{IGTL} expressions. Therefore, if a semantics enforces all the types of λ^{IGTL} expressions, then the semantics enforces all of the types of λ^{GTL} expressions.

Note: The complete formal development of λ^{GTL} and λ^{IGTL} along with all the definitions and proofs of theorems from the paper are in the supplemental material.

$\boxed{\Gamma \vdash_{\mathsf{sim}} e : \tau}$ (selected rules)

$$\frac{\Gamma_0 \vdash_{\mathsf{sim}} e_0 : \tau_0 \to \tau_1 \qquad \Gamma_0 \vdash_{\mathsf{sim}} e_1 : \tau_0}{\Gamma_0 \vdash_{\mathsf{sim}} \mathsf{app}\{\tau_1\}\, e_0\, e_1 : \tau_1} \qquad \frac{\Gamma_0 \vdash_{\mathsf{sim}} e_0 : \tau_0}{\Gamma_0 \vdash_{\mathsf{sim}} \mathsf{cast}\,\{\tau_1 \Leftarrow \tau_0\}\, e_0 : \tau_1} \qquad \frac{\Gamma_0 \vdash_{\mathsf{sim}} e_0 : \tau_0 \qquad \tau_0 \leqslant: \tau_1}{\Gamma_0 \vdash_{\mathsf{sim}} e_0 : \tau_1}$$

Fig. 7. Simple Typing for λ^{IGTL}

3.1 A Natural and a Transient Semantics for λ^{IGTL}

The definition of vigilance, which is the centerpiece of this paper, requires an apparatus for determining the types associated with the value of each (sub)expression in a program — intuitively, all the types in casts applied to that value — so that vigilance can decide if the semantics of the IGTL indeed enforces these types. Such an apparatus needs to be dynamic in order to be precise in a higher-order gradually typed setting, such as λ^{IGTL}. Consider, for instance the expression $e_1 = \mathsf{if}\ e_b\ \mathsf{then}\ \mathsf{cast}\,\{* \Leftarrow \tau\}\,\mathsf{cast}\,\{\tau \Leftarrow *\}\ e_0\ \mathsf{else}\ e_0$. If the result of e_0 is a value v_0, then depending on the result of e_b, v_0 is associated with different types: if e_b evaluates to True, then v_0 is associated with τ, and otherwise it is not.

To record these types, we devise a *log-based* reduction semantics for λ^{IGTL}. This reduction semantics creates fresh labels ℓ for each intermediate value during the evaluation of a program to distinguish between different values that are structurally the same, and then uses the labels to track the (two) types from any casts that a label encounters during the evaluation of a program. Formally, the semantics populates a *value log* Σ, which is a map from labels ℓ to values v and potential types $option(\tau \times \tau)$. The type information is optional because a value may never go through a cast.

The definition of the *log-based* reduction semantics requires an extension of the syntax of λ^{IGTL} with values, labels, unannotated applications, errors and, most importantly, expressions that correspond to the run-time representations of type casts and assertions. Essentially, these act as hooks that allow us to define either a Natural or a Transient semantics for λ^{IGTL} while leaving the rest of the formalism unchanged. The top left of Figure 8 depicts these extensions. The *monitor* expression $\mathsf{mon}\,\{\tau \Leftarrow \tau\}\,e$ regulates the evaluation of cast expressions; it is an intermediate term that separates the tag checks performed by a cast from the creation of a proxy. An assert expression, $\mathsf{assert}\,\tau\,e$, reifies type annotations on applications and function parameters as type assertions. Unannotated applications correspond to applications whose annotation has been reified as a type assertion. There are two kinds of errors in the evaluation language of λ^{IGTL}: $\mathsf{Err}^{\bullet}$ are expected errors and include failures due to failed type casts and assertions, and Err° are unexpected errors that indicate a failure of type soundness such as a call to a value that is not a function. Err ranges over these.

The two semantics of λ^{IGTL} are defined with the reduction relation $\longrightarrow_L^*$ that is the transitive, compatible closure (over evaluation contexts) of the notions of reductions $\hookrightarrow_L$, where L is either N (for Natural) or T (for Transient). The only difference between the two notions of reduction is in their *compatibility* metafunction $\propto_c^L$, where the parameter c represents the kind of check being performed by the semantics. The metafunction consumes a value and a type, and either immediately returns True or invokes $v \propto \lfloor\tau\rfloor$ that checks whether v matches the *tag* $\lfloor\tau\rfloor$ of the given type τ. Put differently, $v \propto_c^L \tau$ either performs a tag check or is a no-op — which of the two depends on its c and L parameters, that is, the λ^{IGTL} construct that triggers a possible tag check and the particular semantics of λ^{IGTL}. Specifically, in both semantics, a cast expression performs a tag check. However, assert expressions perform tag checks only in Transient since in Natural all dynamic type checking takes place via proxies. Conversely, monitor expressions perform tag checks only in Natural since Transient does not rely on proxies for dynamic type checking.

The bottom part of Figure 8 presents a few selected rules of $\hookrightarrow_L$. When an expression reduces a value, $\hookrightarrow_L$ replaces it with a fresh label ℓ and updates Σ accordingly. An annotated application

becomes an unannotated one but wrapped in an assert expression that reifies the annotation as a type assertion. Unannotated applications delegate to the compatibility metafunction a potential check of the argument against the type of the parameter — Transient performs such tag tests to "protect" the bodies of functions from arguments of the wrong type in the absence of proxies. When the compatibility metafunction returns True the evaluation proceeds with a beta-reduction; otherwise it raises a TypeErr, i.e., a dynamic type error. Since all values are stored in the value log and these rules need to inspect values, they employ metafunction pointsto$(\cdot,\cdot)$. Given a value log Σ and a label ℓ, the metafunction traverses Σ starting from ℓ through labels that point to other labels until it reaches a non-label value. The case where an application does not involve a function is one of the cases that the type system of λ^{IGTL} should prevent. Hence, the corresponding reduction rule raises Wrong to distinguish this unexpected error from dynamic type errors.

Assert and cast expressions also delegate any tag checks they perform to the compatibility metafunction. If answer of the latter is positive, an assert expression simplifies to its label-body, while a cast expression wraps its value-body into a monitor with the same type annotations.

Monitor expressions essentially implement proxies, if the semantics of λ^{IGTL} relies on them. Specifically, a monitor expression performs any checks a proxy would perform using the compatibility metafunction, and produces a fresh label to record in the value log that after the fresh label is associated with two additional types. Upon an application of a label, $\hookrightarrow_L$ retrieves the types associated with it, and creates a monitor expression to enforce them. Hence, if the compatibility metafunction does perform tag checks for monitor expressions, monitors implement the two steps of checking types with proxies: checking first-order properties of the monitored value, and creating further proxies upon the use of a higher-order value. If the compatibility metafunction does not perform tag checks then all these reduction rules are essentially void of computational significance; they are just a convenient way for keeping the semantics syntactically uniform across Natural and Transient.

The sequence of reductions in Figure 9 gives a taste of the log-based semantics of λ^{IGTL} through the evaluation of the example expression e from above. The reduction sequence is the same for both Natural and Transient except for the step marked with (†). Up to that point, both semantics store intermediate values in the value log Σ, check with a cast that ℓ_2 points to a function, and, after the check succeeds, create a new label ℓ_3 that the updated Σ associates with the types from the cast. For step (†), both semantics perform a beta-reduction. But via the compatibility metafunction, Transient also checks that the argument of ℓ_1 is indeed a function. The two semantics get out of sync again after the last step of the shown reduction sequence. Specifically, for the remainder of the evaluation, Natural performs checks due to the monitor expressions such as the ones around ℓ_3 and ℓ_4, while Transient performs the checks stipulated by assert expressions.

4 TYPE VIGILANCE

In this section, we define *vigilance*, a semantic property of an IGTL that says that every value must satisfy *both* the type ascribed to it by the type system *and* all the types from casts that were evaluated to produce this value. We refer to the latter list of types as the run-time typing history for the value. The first of these two conditions is essentially (semantic) type soundness which can be captured using a unary logical relation indexed by types and inhabited by values that satisfy the type. For the second condition, we must extend the logical relation to maintain a *type history* Ψ that keeps track of the run-time typing history h for each value v in the log Σ, and then require that each v satisfy all the types in its history h.

We start with the more standard semantic-type-soundness part of our step-indexed logical relation. Figure 10 presents the value and expression relations. Ignoring, for the moment, the highlighted terms in the figure, the value relation $\mathcal{V}^L[\![\tau]\!]$ specifies when a value stored at label ℓ in Σ satisfies the type τ for k steps — or, in more technical terms, when a Σ, ℓ pair belongs to τ. But

$$
\begin{array}{lcl}
v & ::= & \ell \mid n \mid i \mid \mathsf{True} \mid \mathsf{False} \mid \langle \ell, \ell \rangle \mid \lambda(x{:}\tau).\, e \\
e & ::= & \ldots \mid \ell \mid e\ e \mid \mathsf{mon}\ \{\tau \Leftarrow \tau\}\ e \mid \mathsf{assert}\ \tau\ e \mid \mathsf{Err} \\
\Sigma & : & \mathbb{L} \to \mathbb{V} \times \mathsf{option}(\mathbb{T} \times \mathbb{T})
\end{array}
$$

$\boxed{\mathsf{pointsto}(\Sigma, \ell)}$

$$
\mathsf{pointsto}(\Sigma, \ell) = \begin{cases} \mathit{fst}(\Sigma(\ell)) & \text{if } \mathit{fst}(\Sigma(\ell)) \neq \ell' \\ \mathsf{pointsto}(\Sigma, \ell') & \text{if } \mathit{fst}(\Sigma(\ell)) = \ell' \end{cases}
$$

$\boxed{\Sigma, e \hookrightarrow_L \Sigma, e}$ (selected rules)

$$
\begin{array}{l}
\Sigma, v \\
\hookrightarrow_L \Sigma[\ell \mapsto (v, \mathsf{none})], \ell \\
\quad \text{where } \ell \notin \mathit{dom}(\Sigma)
\end{array}
$$

$$
\begin{array}{l}
\Sigma, \mathsf{app}\{\tau_0\}\ \ell_0\ \ell_1 \\
\hookrightarrow_L \Sigma, \mathsf{assert}\ \tau_0\ (\ell_0\ \ell_1)
\end{array}
$$

$$
\begin{array}{l}
\Sigma, \ell_0\ \ell_1 \\
\hookrightarrow_L \Sigma, e_0[x_0 \leftarrow \ell_1] \\
\quad \text{if } \Sigma(\ell_0) = (\lambda(x_0{:}\tau_1).\, e_0, \_) \\
\quad \text{and } \mathsf{pointsto}(\Sigma, \ell_1) \propto^L_{\mathit{assert}} \tau_1
\end{array}
$$

$$
\begin{array}{l}
\Sigma, \ell_0\ \ell_1 \hookrightarrow_L \Sigma, \mathsf{TypeErr}(\tau_1,\ \ell_1) \\
\quad \text{if } \Sigma(\ell_0) = (\lambda(x_0{:}\tau_1).\, e_0, \_) \\
\quad \text{and } \neg\mathsf{pointsto}(\Sigma, \ell_1) \propto^L_{\mathit{assert}} \tau_1
\end{array}
$$

$$
\begin{array}{l}
\Sigma, \ell_0\ \ell_1 \\
\hookrightarrow_L \mathsf{Wrong} \\
\quad \text{if } \Sigma(\ell_0) = (v, \_) \\
\quad \text{and } v \notin \lambda(x{:}\tau).\, e \cup \ell \\
\quad \text{or } \Sigma(\ell_0) = (\ell_0', \mathsf{none})
\end{array}
$$

$$
\begin{array}{l}
\Sigma, \mathsf{assert}\ \tau_0\ \ell_0 \\
\hookrightarrow_L \Sigma, \ell_0 \\
\quad \text{if } \mathsf{pointsto}(\Sigma, \ell_0) \propto^L_{\mathit{assert}} \tau_0
\end{array}
$$

$$
\begin{array}{l}
\Sigma, \mathsf{assert}\ \tau_0\ \ell_0 \\
\hookrightarrow_L \Sigma, \mathsf{TypeErr}(\tau_0,\ \ell_0) \\
\quad \text{if } \neg\mathsf{pointsto}(\Sigma, \ell_0) \propto^L_{\mathit{assert}} \tau_0
\end{array}
$$

$$
\begin{array}{l}
\Sigma, \mathsf{cast}\ \{\tau_1 \Leftarrow \tau_2\}\ \ell_0 \\
\hookrightarrow_L \Sigma, \mathsf{mon}\ \{\tau_1 \Leftarrow \tau_2\}\ \ell_0 \\
\quad \text{if } \mathsf{pointsto}(\Sigma, \ell_0) \propto^L_{\mathit{cast}} \tau_1 \\
\quad \text{and } \mathsf{pointsto}(\Sigma, \ell_0) \propto^L_{\mathit{cast}} \tau_0
\end{array}
$$

$$
\begin{array}{l}
\Sigma, \mathsf{cast}\ \{\tau_1 \Leftarrow \tau_2\}\ \ell_0 \\
\hookrightarrow_L \Sigma, \mathsf{TypeErr}(\tau_1,\ \ell_0) \\
\quad \text{if } \neg\mathsf{pointsto}(\Sigma, \ell_0) \propto^L_{\mathit{cast}} \tau_1
\end{array}
$$

$$
\begin{array}{l}
\Sigma, \mathsf{mon}\ \{\tau_1 \Leftarrow \tau_2\}\ \ell_0 \\
\hookrightarrow_L \Sigma[\ell_1 \mapsto (\ell_0, \mathsf{some}(\tau_1, \tau_2))], \ell_1 \\
\quad \text{if } \ell_1 \notin \mathit{dom}(\Sigma) \\
\quad \text{and } \mathsf{pointsto}(\Sigma, \ell_0) = v \\
\quad \text{where } v = i \text{ or True or False} \\
\quad \text{and } v \propto^L_{\mathit{mon}} \tau_1 \wedge v \propto^L_{\mathit{mon}} \tau_2
\end{array}
$$

$$
\begin{array}{l}
\Sigma, \mathsf{mon}\ \{\tau_1 \Leftarrow \tau_2\}\ \ell_0 \\
\hookrightarrow_L \langle \mathsf{mon}\ \{\mathit{fst}(\tau_1) \Leftarrow \mathit{fst}(\tau_2)\}\ \ell_1, \\
\qquad \mathsf{mon}\ \{\mathit{snd}(\tau_1) \Leftarrow \mathit{snd}(\tau_2)\}\ \ell_2 \rangle \\
\quad \text{if } \Sigma(\ell_0) = (\langle \ell_1, \ell_2 \rangle, \_)
\end{array}
$$

$$
\begin{array}{l}
\Sigma, \mathsf{mon}\ \{\tau_1 \Leftarrow \tau_2\}\ \ell_0 \\
\hookrightarrow_L \Sigma[\ell_1 \mapsto (\ell_0, \mathsf{some}(\tau_1, \tau_2))], \ell_1 \\
\quad \text{if } \ell_1 \notin \mathit{dom}(\Sigma) \\
\quad \text{and } \mathsf{pointsto}(\Sigma, \ell_0) = v \\
\quad \text{and } v = \lambda(x_0{:}\tau_1).\, e_0 \\
\quad \text{and } v \propto^L_{\mathit{mon}} \tau_1 \wedge v \propto^L_{\mathit{mon}} \tau_2
\end{array}
$$

$$
\begin{array}{l}
\Sigma, \mathsf{mon}\ \{\tau_1 \Leftarrow \tau_2\}\ \ell_0 \\
\hookrightarrow_L \Sigma, \mathsf{TypeErr}(\tau_1,\ \ell_0) \\
\quad \text{if } \neg\mathsf{pointsto}(\Sigma, \ell_0) \propto^L_{\mathit{mon}} \tau_1
\end{array}
$$

$$
\begin{array}{l}
\Sigma, \ell_0\ \ell_1 \\
\hookrightarrow_L \quad \mathsf{mon}\ \{\mathit{cod}(\tau_1) \Leftarrow \mathit{cod}(\tau_2)\} \\
\qquad (\ell_2\ \mathsf{mon}\ \{\mathit{dom}(\tau_2) \\
\qquad\qquad \Leftarrow \mathit{dom}(\tau_1)\}\ \ell_1) \\
\quad \text{if } \Sigma(\ell_0) = (\ell_2, \mathsf{some}(\tau_1, \tau_2))
\end{array}
$$

$\boxed{\propto\colon v \times K \longrightarrow \mathbb{B}}$

$$
\begin{array}{lll}
n \propto \mathsf{Nat} & & = \mathsf{True} \\
i \propto \mathsf{Int} & & = \mathsf{True} \\
b \propto \mathsf{Bool} & & = \mathsf{True} \\
\langle v_1, v_2 \rangle & \propto * \times * & = \mathsf{True} \\
(\lambda(x{:}\tau).\, e) & \propto * \to * & = \mathsf{True} \\
(\mathsf{mon}\ \{\tau \Leftarrow \tau'\}\ v) & \propto * \to * & = \mathsf{True} \\
v & \propto * & = \mathsf{True} \\
v & \propto K & = \mathsf{False} \\
 & & \text{otherwise}
\end{array}
$$

$\boxed{\propto^L_c\colon v \times K \longrightarrow \mathbb{B}}$

c	$v \propto^{\mathsf{N}}_c \tau$	$v \propto^{\mathsf{T}}_c \tau$
cast	$v \propto \lfloor \tau \rfloor$	$v \propto \lfloor \tau \rfloor$
mon	$v \propto \lfloor \tau \rfloor$	True
assert	True	$v \propto \lfloor \tau \rfloor$

Fig. 8. Evaluation Syntax and Reduction Semantics for λ^{IGTL}

each value relation is also indexed by a type history Ψ that, intuitively, records the run-time typing histories for all values in Σ, as we explain in detail later.

For base types, ℓ belongs to the relation $\mathcal{V}^L[\![B]\!]$ if $\mathsf{pointsto}(\Sigma, \ell)$ is a value of the expected form. Since pairs are evaluated eagerly, they are never wrapped by extra types in the store, so the relation for pairs, $\mathcal{V}^L[\![\tau_1 \times \tau_2]\!]$ contains only labels with label pairs, and as usual, the components of the pair must belong to τ_1 and τ_2, respectively.

For function types, a function usually belongs to $\mathcal{V}^L[\![\tau_1 \to \tau_2]\!]$ if, when applied in some future world — when there are fewer steps left and the value log and type history potentially contain more labels — to a value that behaves like τ_1, it produces a result that behaves like τ_2. Our definition is slightly different: since we support subsumption and since applications in our language are annotated with type assertions, we consider applications in which the assertion τ_0 is any supertype of the result type τ_2, and require that the result behave like τ_0.

Finally, for the dynamic type, $\mathcal{V}^L[\![*]\!]$ is an untagged union over base types $\mathcal{V}^L[\![B]\!]$, pairs of dynamic types $\mathcal{V}^L[\![* \times *]\!]$, and functions between dynamic types $\mathcal{V}^L[\![* \to *]\!]$. Since these types

$$
\begin{array}{ll}
& \emptyset, \mathsf{app}\{*\}\ (\lambda f : \mathsf{Nat} \to \mathsf{Nat}.\ \mathsf{cast}\ \{* \Leftarrow \mathsf{Nat}\}\ \mathsf{app}\{\mathsf{Nat}\}\ f\ 42)\ (\mathsf{cast}\ \{\mathsf{Nat} \to \mathsf{Nat} \Leftarrow * \to *\}\ \lambda x : *.\ x) \\
\longrightarrow_L^* & \{\ell_1 \mapsto (v_1, \mathsf{none}), \ell_2 \mapsto (v_2, \mathsf{none})\}, \mathsf{app}\{*\}\ \ell_1\ (\mathsf{cast}\ \{\mathsf{Nat} \to \mathsf{Nat} \Leftarrow * \to *\}\ \ell_2) \\
\longrightarrow_L^* & \{\ell_1 \mapsto (v_1, \mathsf{none}), \ell_2 \mapsto (v_2, \mathsf{none})\}, \mathsf{app}\{*\}\ \ell_1\ (\mathsf{mon}\ \{\mathsf{Nat} \to \mathsf{Nat} \Leftarrow * \to *\}\ \ell_2) \\
\longrightarrow_L^* & \{\ell_1 \mapsto (v_1, \mathsf{none}), \ell_2 \mapsto (v_2, \mathsf{none}), \ell_3 \mapsto (l_2, \mathsf{some}(\mathsf{Nat} \to \mathsf{Nat}, * \to *))\}, \mathsf{app}\{*\}\ \ell_1\ \ell_3 \\
\longrightarrow_L^* & \{\ell_1 \mapsto (v_1, \mathsf{none}), \ell_2 \mapsto (v_2, \mathsf{none}), \ell_3 \mapsto (l_2, \mathsf{some}(\mathsf{Nat} \to \mathsf{Nat}, * \to *))\}, \mathsf{assert}\ *\ (\ell_1\ \ell_3) \\
\longrightarrow_L & \{\ldots\}, \mathsf{assert}\ *\ \mathsf{cast}\ \{* \Leftarrow \mathsf{Nat}\}\ \mathsf{app}\{\mathsf{Nat}\}\ \ell_3\ 42 \quad (\dagger) \\
\longrightarrow_L^* & \{\ldots, \ell_4 \mapsto (42, \mathsf{none})\}, \mathsf{assert}\ *\ \mathsf{cast}\ \{* \Leftarrow \mathsf{Nat}\}\ \mathsf{app}\{\mathsf{Nat}\}\ \ell_3\ \ell_4 \\
\longrightarrow_L^* & \{\ldots\}, \mathsf{assert}\ *\ \mathsf{cast}\ \{* \Leftarrow \mathsf{Nat}\}\ \mathsf{assert}\ \mathsf{Nat}\ \mathsf{mon}\ \{\mathsf{Nat} \Leftarrow *\}\ (\ell_3\ (\mathsf{mon}\ \{* \Leftarrow \mathsf{Nat}\}\ \ell_4))
\end{array}
$$

Fig. 9. Example of log-based reduction.

$$
\begin{array}{rl}
\mathcal{V}^L[\![C]\!] & \triangleq \{(k, \Psi, \Sigma, \ell) \mid \mathsf{pointsto}(\Sigma, \ell) \in C\} \\
\mathcal{V}^L[\![\tau_1 \times \tau_2]\!] & \triangleq \{(k, \Psi, \Sigma, \ell) \mid \Sigma(\ell) = (\langle \ell_1, \ell_2 \rangle, \_) \wedge\ (k, \Psi, \Sigma, \ell_1) \in \mathcal{V}^L[\![\tau_1]\!] \wedge\ (k, \Psi, \Sigma, \ell_2) \in \mathcal{V}^L[\![\tau_2]\!]\} \\
\mathcal{V}^L[\![\tau_1 \to \tau_2]\!] & \triangleq \{(k, \Psi, \Sigma, \ell) \mid \forall\ (j, \Psi') \sqsupseteq (k, \Psi),\ \Sigma' \supseteq \Sigma,\ \ell_v,\ \tau_0 \geqslant: \tau_2. \\
& \qquad \Sigma' : (j, \Psi') \wedge (j, \Psi', \Sigma', \ell_v) \in \mathcal{V}^L[\![\tau_1]\!].\ \Rightarrow (j, \Psi', \Sigma', \mathsf{app}\{\tau_o\}\ \ell\ \ell_v) \in \mathcal{E}^L[\![\tau_0]\!]\} \\
\mathcal{V}^L[\![*]\!] & \triangleq \{(k, \Psi, \Sigma, \ell) \mid (k-1, \Psi, \Sigma, \ell) \in \mathcal{V}^L[\![C]\!] \cup \mathcal{V}^L[\![* \times *]\!] \cup \mathcal{V}^L[\![* \to *]\!]\} \\
\mathcal{E}^L[\![\tau]\!] & \triangleq \{(k, \Psi, \Sigma, e) \mid \forall j \leq k.\ \forall \Sigma' \supseteq \Sigma, e'.\ (\Sigma, e) \longrightarrow_L^j (\Sigma', e') \wedge \mathtt{irred}(e') \Rightarrow (e' = \mathsf{Err}^\bullet \vee \\
& \qquad (\exists (k-j, \Psi') \sqsupseteq (k, \Psi).\ \Sigma' : (k-j, \Psi') \wedge (k-j, \Psi', \Sigma', e') \in \mathcal{V}^L[\![\tau]\!]))\}
\end{array}
$$

Fig. 10. Vigilance: Value and Expression Relations

are not structurally smaller than $*$, step-indexing becomes crucial. For well-foundedness, a term that behaves like $*$ for k steps is only required to behave like one of the types in the union for $k-1$.

To extend this characterization to expressions, we define the expression relation $\mathcal{E}^L[\![\tau]\!]$. An expression e behaves like type τ if it does not terminate within the step-index budget, if it runs to an expected error, or if it produces a value that belongs to $\mathcal{V}^L[\![\tau]\!]$.

The logical relation defined thus far is a mostly standard type soundness relation. We now consider how to ensure that every value also satisfies all the types from casts that were evaluated to produce that value. Note that all values that flow through casts are entered into the value log Σ. Thus Σ is analogous to a dynamically allocated (immutable) store and we can take inspiration from models of dynamically allocated references[2] to (1) keep track of the run-time typing histories of values in a type history Ψ, just as models of references keep track of the types of references in a store typing, and (2) ensure that values in Σ satisfy the run-time typing histories in Ψ, just as models of references ensure that the store S satisfies the store typing.

Thus, as the highlighted parts in Figure 10 show, we set up a Kripke logical relation[19] indexed by worlds comprised of a step-index k and a type history Ψ, which is a mapping from labels ℓ to run-time typing histories h that are essentially lists of types. We define a world accessibility relation $(j, \Psi') \sqsupseteq (k, \Psi)$, which says that (j, Ψ') is a future world accessible from (k, Ψ') if $j \leq k$ (we may have potentially fewer steps available in the future) and the future type history Ψ' may have more entries than Ψ. Whenever we consider future logs Σ', we require that there is a future world $(j, \Psi') \sqsupseteq (k, \Psi)$ such that the value log satisfies the typing history $\Sigma' : (j, \Psi')$. Where our relation differs from the standard treatment of state is in the constraints placed on Σ by Ψ via the value-log type-satisfaction relation $\Sigma : (k, \Psi)$, defined in Figure 11.

$$\vdash \Psi \triangleq \forall \ell.\, \Psi(\ell) = \tau, \tau', h \Rightarrow \tau' \geqslant: \mathrm{head}(h)$$

$$\vdash \Sigma \triangleq \forall \ell \in dom(\Sigma).\, (\Sigma(\ell) = (v, \mathsf{none}) \wedge v \notin \mathbb{L})$$
$$\vee\ (\Sigma(\ell) = (\ell', \mathsf{some}(\tau', \tau)) \wedge \exists v.\, v = \mathsf{pointsto}(\Sigma, \ell) \wedge \neg(v \propto *\times *) \vee \wedge v \propto \tau' \wedge v \propto \tau)$$

$$\Psi \vdash^{\ell} (v, \mathsf{none}) \triangleq \exists \tau.\, \Psi(\ell) = [\tau]$$

$$\Psi \vdash^{\ell} (\ell', \mathsf{some}(\tau, \tau')) \triangleq \Psi(\ell) = [\tau, \tau', \Psi(\ell')] \qquad h ::= \tau \mid \tau, \tau, h$$

$$\Psi \vdash \Sigma \triangleq \vdash \Sigma \wedge \forall \ell \in dom(\Sigma) \cup dom(\Psi), \Psi \vdash^{\ell} \Sigma(\ell) \qquad \Psi : \ell \to h$$

$$\Sigma : (k, \Psi) \triangleq \vdash \Psi \wedge \Psi \vdash \Sigma \wedge \forall \ell \in dom(\Sigma).\, (j, \Psi, \Sigma, \ell) \in \mathcal{VH}^{L}[\![\Psi(\ell)]\!]$$

Fig. 11. Vigilance: Value-Log Type Satisfaction

In more detail, as Figure 11 shows, our typing history Ψ associates with each label in the value log a run-time typing history h, where h is either a single type, indicating that the value was produced at that type, or h is two types appended onto another typing history h', indicating type obligations added to the value by a cast expression. We say that a value log Σ satisfies a world $\Sigma : (k, \Psi)$ when three things are true:

1. The type history must be syntactically well-formed: $\vdash \Psi$. The well-formedness constraint $\vdash \Psi$ ensures that each value-log entry is well formed ($\vdash h$). Because casts in our model may be coercive, they can only be expected to function appropriately when the value passed to the cast is of the appropriate (semantic) type. Since casts add types to the run-time typing history of a value, this gives rise to a syntactic constraint that the "from" type of such an entry matches (up to subsumption) the type that the casted value previously held in the history.

2. The value log must be well-formed given the type history: $\Psi \vdash \Sigma$. This requires certain syntactic constraints $\vdash \Sigma$ that are independent of Ψ, and that for each location ℓ, Σ should provide some value-log entry that is itself consistent with Ψ. The former constraint $\vdash \Sigma$ corresponds to the basic syntactic invariants preserved by the operational semantics: casted values are always compatible (due to the $v \propto_L^{cast} \tau$ checks performed by the cast evaluation rules) and are never pairs because our pairs are evaluated eagerly. For the latter, $\Psi \vdash^{\ell} (v, -)$ specifies that if the entry does not record a cast around a value, it is consistent with Ψ when its run-time typing history $\Psi(\ell)$ does not include any type obligations added by a cast. If the entry does record a cast, then the recorded types must match those in $\Psi(\ell)$, and the casted location must itself be well formed with respect to the remaining entries in the run-time typing history.

3. The values in the log must satisfy their run-time typing history. The core semantic condition of value-log type satisfaction is that $\Sigma(\ell)$ must behave like each type τ in its run-time typing history $\Psi(\ell)$. But we cannot simply ask that $\Sigma(\ell) \in \mathcal{V}^{L}[\![\tau]\!]$ for each $\tau \in \Psi(\ell)$. Since casts in our model may be coercive, they can only be expected to function appropriately when the value passed to the cast is of the appropriate (semantic) type. Because $\mathcal{V}^{L}[\![\tau_1 \to \tau_2]\!]$ quantifies over values $v \in \mathcal{V}^{L}[\![\tau_1]\!]$, if we were to take the above approach, a function cast from $\tau_1 \to \tau_2$ to τ' would need to behave well when applied to an argument $v \in \mathcal{V}^{L}[\![\tau_1]\!]$. Since a cast on a function must ensure that the function's actual argument v belongs to the type expected by the original function, it must semantically perform a cast equivalent to cast $\{\tau_1 \Leftarrow dom(\tau')\}\ v$[5] to be well formed, which one would not expect to be true in general.

[5]In our reduction semantics, this constraint is ensured by a term of the form mon $\{\tau_1 \Leftarrow dom(\tau')\}\ v$ for the sake of Transient, which does not perform the expected checks here; see §5.

$$\mathcal{VH}^L[\![C, \tau_2, \ldots \tau_n]\!] \triangleq \{(k, \Psi, \Sigma, \ell) \mid \forall \tau \in [C, \tau_2, \ldots \tau_n].\ (k, \Psi, \Sigma, \ell) \in \mathcal{V}^L[\![\tau]\!]\}$$

$$\mathcal{VH}^L[\![\tau_1' \times \tau_1'', \tau_2, \ldots \tau_n]\!] \triangleq \{(k, \Psi, \Sigma, \ell) \mid \Sigma(\ell) = (\langle \ell_1, \ell_2 \rangle, \_) \wedge\ (k, \Psi, \Sigma, \ell_1) \in \mathcal{VH}^L[\![\tau_1', fst(\tau_2), \ldots fst(\tau_n)]\!] \wedge\ (k, \Psi, \Sigma, \ell_2) \in \mathcal{VH}^L[\![\tau_1'', snd(\tau_2), \ldots snd(\tau_n)]\!]\}$$

$$\mathcal{VH}^L[\![\tau_1' \to \tau_1'', \tau_2, \ldots \tau_n]\!] \triangleq \{(k, \Psi, \Sigma, \ell) \mid \forall (j, \Psi') \sqsupseteq (k, \Psi),\ \Sigma' \supseteq \Sigma,\ \ell_v,\ \tau_0 \geqslant: \tau_2.\ (j, \Psi', \Sigma', \ell_v) \in \mathcal{V}^L[\![\tau_1']\!] \wedge \Sigma' : (j, \Psi') \Rightarrow (j, \Psi', \Sigma', \mathsf{app}\{\tau_0\}\ \ell\ \ell_v) \in \mathcal{EH}^L[\![\tau_0, cod(\tau_2), \ldots cod(\tau_n)]\!]\}$$

$$\mathcal{VH}^L[\![*, \tau_2, \ldots \tau_n]\!] \triangleq \{(k, \Psi, \Sigma, \ell) \mid (k-1, \Psi, \Sigma, \ell) \in \mathcal{VH}^L[\![C, \tau_2, \ldots \tau_n]\!] \cup \mathcal{VH}^L[\![* \times *, \tau_2, \ldots, \tau_n]\!] \cup \mathcal{VH}^L[\![* \to *, \tau_2, \ldots, \tau_n]\!]\}$$

$$\mathcal{EH}^L[\![\overline{\tau}]\!] \triangleq \{(k, \Psi, \Sigma, e) \mid \forall j \leq k.\ \forall \Sigma' \supseteq \Sigma, e'.\ (\Sigma, e) \longrightarrow_L^j (\Sigma', e') \wedge \mathtt{irred}(e') \Rightarrow (e' = \mathsf{Err}^\bullet \vee (\exists (k-j, \Psi') \sqsupseteq (k, \Psi).\ \Sigma' : (k-j, \Psi') \wedge (k-j, \Psi', \Sigma', e') \in \mathcal{VH}^L[\![\overline{\tau}]\!]))\}$$

Fig. 12. Vigilance: Typing History Relations

$$\mathcal{G}^L[\![\Gamma]\!] \triangleq \{(k, \Psi, \Sigma, \gamma) \mid dom(\Gamma) = dom(\gamma) \wedge \forall x (k, \Psi, \Sigma, \gamma(x)) \in \mathcal{V}^L[\![\Gamma(x)]\!]\}$$

$$[\![\Gamma \vdash_{\mathsf{t}} e : \tau]\!]^L \triangleq\ \forall (k, \Psi, \Sigma, \gamma) \in \mathcal{G}^L[\![\Gamma]\!].\ \Sigma : (k, \Psi)\ \Rightarrow (k, \Psi, \Sigma, \gamma(e)) \in \mathcal{E}^L[\![\tau]\!]$$

Fig. 13. Vigilance: Top-Level Relation

To properly incorporate this constraint, we define typing-history relations that specify when a value or expression behaves like multiple types at once[6]. These relations, $\mathcal{VH}^L[\![\overline{\tau}]\!]$ and $\mathcal{EH}^L[\![\overline{\tau}]\!]$ are given in Figure 12. For a nonempty list of types $\tau, \overline{\tau}$, the relation is defined inductively over the first type in the list, following a similar structure to $\mathcal{V}^L[\![\tau]\!]$. When τ is a base type B, the value typing-history relation $\mathcal{VH}^L[\![B, \overline{\tau}]\!]$ contains any ℓ such that pointsto(Σ, ℓ) is in $\mathcal{V}^L[\![\tau]\!]$ for each $\tau \in [B, \overline{\tau}]$. Just as in the value relation, because casts on pairs are evaluated eagerly, $\mathcal{VH}^L[\![\tau_1 \times \tau_2, \overline{\tau}]\!]$ contains only literal pairs $\langle \ell_1, \ell_2 \rangle$ whose components inductively satisfy all the appropriate types.

As discussed above, $\mathcal{VH}^L[\![\tau_1 \to \tau_2, \overline{\tau}]\!]$ requires a function to behave well only when it is given an argument $v \in \mathcal{V}^L[\![\tau_1]\!]$. As in the $\mathcal{V}$ relation, it must also behave well when the application is annotated with any $\tau_0 \geqslant: \tau_2$. Behaving "well" means that an application, evaluated with a future store $\Sigma' : (j, \Psi') \sqsupseteq (k, \Psi)$ should behave like all the types $\tau_0, cod(\overline{\tau})$. Since this is an expression, we define the $\mathcal{EH}^L[\![\overline{\tau}]\!]$ relation to characterize expressions as behaving like several types at once; since this only matters when the expression reduces to a value, it is precisely the same as the $\mathcal{E}$ relation, except that it is indexed by $\overline{\tau}$ rather than τ, and it requires the eventual value to be in $\mathcal{VH}^L[\![\overline{\tau}]\!]$ rather than $\mathcal{V}^L[\![\tau]\!]$.

Finally, as in the value relation, $\mathcal{VH}^L[\![*, \overline{\tau}]\!]$ is an untagged union over base types $\mathcal{VH}^L[\![B, \overline{\tau}]\!]$, pairs of dynamic types $\mathcal{VH}^L[\![* \times *, \overline{\tau}]\!]$, and functions between dynamic types $\mathcal{VH}^L[\![* \to *, \overline{\tau}]\!]$, at step index $k-1$.

Vigilance: Top-level Relation. We now define vigilance for open terms in the standard way, see Figure 13. We extend the characterization of vigilance for closed terms and types to open terms by defining $[\![\Gamma \vdash_{\mathsf{t}} e : \tau]\!]^L$ which says that an expression e that type checks in context Γ behaves like τ when, given a substitution γ that behaves like Γ, $\gamma(e)$ behaves like τ. And a substitution γ, mapping free variables x to labels ℓ in Σ, behaves like Γ when for each $x : \tau$ in Γ, $\gamma(x)$ behaves like

[6]An arbitrary list of types used as this index is more general than the grammar of h, but we will freely interconvert them, since the syntax of h is a subset of that of $\overline{\tau}$.

$$
\begin{array}{rcl}
K & ::= & \mathsf{Nat} \mid \mathsf{Int} \mid \mathsf{Bool} \mid * \times * \mid * \to * \mid * \\
\Gamma & ::= & \cdot \mid \Gamma, (x{:}K_0) \\
e & ::= & x \mid n \mid i \mid \mathsf{True} \mid \mathsf{False} \mid \lambda(x{:}K).\, e \mid \langle e, e \rangle \mid \mathsf{app}\{K\}\, e\, e \\
 & \mid & \mathsf{fst}\{K\}\, e \mid \mathsf{snd}\{K\}\, e \mid \mathit{binop}\, e\, e \\
 & \mid & \mathsf{if}\ e\ \mathsf{then}\ e\ \mathsf{else}\ e \mid \mathsf{cast}\ \{K \Leftarrow K\}\ e
\end{array}
$$

$\boxed{\Gamma \vdash_{\mathsf{tag}} e : K}$ (selected rules)

$$
\textsc{T-App}\ \frac{\Gamma_0 \vdash_{\mathsf{tag}} e_0 : * \to * \qquad \Gamma_0 \vdash_{\mathsf{tag}} e_1 : K_0}{\Gamma_0 \vdash_{\mathsf{tag}} \mathsf{app}\{K_1\}\, e_0\, e_1 : K_1}
$$

Fig. 14. Tag Typing for λ^{IGTL}

$\Gamma(x)$. We parameterize the typing judgment with type system t which ranges (so far) over sim for simple typing, and tag for tag typing.

With our vigilance logical relation in place, we formally define vigilance as the fundamental property of the vigilance relation.

Theorem 4.1 (Vigilance Fundamental Property). *If* $\Gamma \vdash_{\mathsf{t}} e : \tau$ *then* $[\![\Gamma \vdash_{\mathsf{t}} e : \tau]\!]^L$.

Vigilance has two components: values satisfy the types ascribed to them by the type system, and the types of all casts that were evaluated to produce them. The theorem says that values satisfy the types ascribed to them by the type system when we ask that well typed terms $\vdash e : \tau$ are in the expression relation $\mathcal{E}^L[\![\tau]\!]$, which says that if e runs to a value, then that value must behave like τ. Moreover, the theorem states values satisfy the types of all casts that were evaluated to produce them in the requirement that when e runs to a value with value log Σ', we also ask that there is a future world $(k - j, \Psi') \sqsupseteq (k, \Psi)$, where $\Sigma' : (k - j, \Psi')$.

Natural is vigilant for simple typing. Since Natural employs proxies to enforce the types of every cast it encounters during the evaluation of a term, we can show that it is vigilant for simple typing.

Theorem 4.2 (Natural is Vigilant for λ^{IGTL} Simple Typing). *If* $\Gamma \vdash_{\mathsf{sim}} e : \tau$ *then* $[\![\Gamma \vdash_{\mathsf{sim}} e : \tau]\!]^N$.

As described in §3, the type-preserving translation for simple typing can be composed with the vigilance theorem to get a vigilance result for λ^{GTL} expressions.

Theorem 4.3 (Natural is Vigilant for λ^{GTL} Simple Typing).
If $\Gamma \vdash_{\mathsf{sim}} t : \tau \rightsquigarrow e$ *then* $[\![\Gamma \vdash_{\mathsf{sim}} e : \tau]\!]^N$.

5 TRANSIENT IS STRONGER THAN YOU THOUGHT

Transient does not use proxies and, as a result, it does not enforce all the types from casts it encounters during the evaluation of an IGTL program. For instance, consider $p = \langle -1, -1 \rangle$, $f = \lambda(x{:}\mathsf{Int} \times \mathsf{Int}) \to (\mathsf{Nat} \times \mathsf{Nat}).\, x$, and $t = f\ p$. The translation of t is

$$e = \mathsf{app}\{* \times *\}\ (\lambda(x{:}\mathsf{Int} \times \mathsf{Int}).\, \mathsf{cast}\ \{\mathsf{Nat} \times \mathsf{Nat} \Leftarrow \mathsf{Int} \times \mathsf{Int}\}\ x)\ \langle -1, -1 \rangle$$

Interestingly, since Transient only checks the top-level constructor of types on casts, $e \hookrightarrow_T \langle -1, -1 \rangle$. However, the final result of e is also the result of f which is supposed to be a $\mathsf{Nat} \times \mathsf{Nat}$ according to f's casts. Hence, Transient is not vigilant for simple typing:

Theorem 5.1 (Transient is Not Vigilant for λ^{GTL} Simple Typing).
There are Γ, t, e, τ *such that* $\Gamma \vdash_{\mathsf{sim}} t : \tau \rightsquigarrow e$ *and* $\neg[\![\Gamma \vdash_{\mathsf{sim}} e : \tau]\!]^T$.

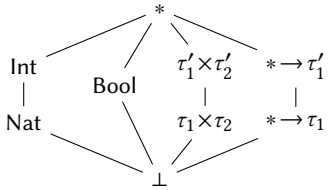
However, as previous work hints at [10, 11, 27], Transient does enforce the tags of all the types it encounters during the evaluation of an IGTL program. To formalize the relation between Transient and the enforcement of tags as a vigilance property, we first define a so-called tag type system for λ^{IGTL} (Figure 14). The rules of a tag type system relate a term with a tag K that corresponds to the top-level type constructor of the type. The typing rules for the tag type system are entirely

$\boxed{\Gamma \vdash_{\mathsf{tru}} e : \tau}$ (selected rules)

$$\frac{\Gamma,\ (x{:}K) \vdash_{\mathsf{tru}} e : \tau}{\Gamma \vdash_{\mathsf{tru}} \lambda(x{:}K).\, e : * \to \tau} \qquad \frac{\Gamma \vdash_{\mathsf{tru}} e_1 : * \to \tau_1 \quad \Gamma \vdash_{\mathsf{tru}} e_2 : \tau_2}{\Gamma \vdash_{\mathsf{tru}} \mathsf{app}\{K\}\ e_1\ e_2 : K \sqcap \tau_1}$$

$$\frac{\Gamma \vdash_{\mathsf{tru}} e_1 : \bot \quad \Gamma \vdash_{\mathsf{tru}} e_2 : \tau_2}{\Gamma \vdash_{\mathsf{tru}} \mathsf{app}\{K\}\ e_1\ e_2 : \bot} \qquad \frac{\Gamma \vdash_{\mathsf{tru}} e_b : \mathsf{Bool} \quad \Gamma \vdash_{\mathsf{tru}} e_1 : \tau_1 \quad \Gamma \vdash_{\mathsf{tru}} e_2 : \tau_2}{\Gamma \vdash_{\mathsf{tru}} \mathsf{if}\ e_b\ \mathsf{then}\ e_1\ \mathsf{else}\ e_2 : \tau_1 \sqcup \tau_2}$$

$\tau ::= \mathsf{Nat} \mid \mathsf{Int} \mid \mathsf{Bool} \mid \tau \times \tau \mid * \to \tau \mid * \mid \bot$



$$\frac{\Gamma \vdash_{\mathsf{tru}} e : \tau}{\Gamma \vdash_{\mathsf{tru}} \mathsf{cast}\,\{K_2 \Leftarrow K_1\}\ e_0 : K_2 \sqcap K_1 \sqcap \tau}$$

Fig. 15. Truer Typing for λ^{IGTL}

unsurprising; the most interesting one is that for applications, which, as Figure 14 shows, allows arguments at any type, since the type of a function is simply the procedure tag that does not contain any information about the function's domain. Another important restriction of this setting is that the types of casts and assertions are also restricted to tags K. Since this is exactly the precision of types that Transient can enforce, the type ascriptions of an IGTL program should reflect that.

Overall, we can obtain a λ^{GTL} and λ^{IGTL} with tag typing, and a type-preserving translation between the two by lifting the metafunction $\lfloor\cdot\rfloor$, which given a type τ returns the corresponding tag K, to the syntax definitions, the translation, and the typing rules from §3.

$$\Gamma \vdash_{\mathsf{tag}} t : K \rightsquigarrow e \text{ iff } \exists \tau.\ \lfloor\tau\rfloor = K \wedge \Gamma \vdash_{\mathsf{sim}} t : \tau \rightsquigarrow e' \wedge e = \lfloor e' \rfloor$$

As a final note, every program that simple type checks also tag type checks:

Theorem 5.2 (Simple Typing Implies Tag Typing). *If* $\Gamma \vdash_{\mathsf{sim}} e : \tau$, *then* $\lfloor\Gamma\rfloor \vdash_{\mathsf{tag}} \lfloor e\rfloor : \lfloor\tau\rfloor$.

With tag typing in hand, we can show that Transient is indeed vigilant for it:[7]

Theorem 5.3 (Transient is Vigilant for λ^{IGTL} Tag Typing). *If* $\Gamma \vdash_{\mathsf{tag}} e : K$ *then* $[\![\Gamma \vdash_{\mathsf{tag}} e : K]\!]^{\mathsf{T}}$.

Despite Theorem 5.2, tag types are not "weaker" than simple types: a function with type $* \to *$ can be used safely in more contexts than a function with type $\tau \to *$. Because of this difference tag typing is unsound for Natural. For instance, the GTL expression $(\lambda(f{:}*) \to *.\, f\ 42)\ (\lambda(x{:}*\times*) \to *.\, \mathsf{fst}\, x)$ translates to

$$e = \mathsf{app}\{*\}\ (\lambda(f{:}*).\, \mathsf{app}\{*\}\ f\ \mathsf{cast}\,\{* \Leftarrow \mathsf{Nat}\}\ 42)\ (\mathsf{cast}\,\{* \Leftarrow * \to *\}\ \lambda(x{:}*\times*).\, \mathsf{fst}\{*\}\ x)$$

which has tag type $*$ but $e \hookrightarrow_N \mathsf{Wrong}$. In contrast, the same expression produces a type error TypeErr in Transient since Transient uses a type assertion to check the tag of function arguments. Consequently, Natural is not vigilant for tag typing:

Theorem 5.4 (Natural is Not Vigilant for λ^{GTL} Tag Typing).
There are Γ, t, K, e *such that* $\Gamma \vdash_{\mathsf{tag}} t : K \rightsquigarrow e$ *and* $\neg[\![\Gamma \vdash_{\mathsf{tag}} e : K]\!]^{\mathsf{N}}$.

5.1 A Truer Type System for λ^{IGTL}

While each individual Transient cast checks only a tag, because Transient is vigilant for tag typing extra information about a value is available to developers. For example, consider the function $f = \lambda(x{:}*\times*).\, x$. Under the tag type system, f type checks at $* \to *$. From this type, a developer

[7] The relational interpretation of tag types is slightly different from that of simple types; it is modified in the same ways as the relation discussed in §5.1.

can deduce only that f is well-behaved when given any argument, and that it makes no promises about its result. However, vigilance implies that Transient also checks that the argument of f is a pair before returning it. Consequently, a developer should be able to conclude that the return type of f is really $*\times*$. In general, each time Transient performs a check, a developer can deduce that the checked value has both the tag that the check confirms, and the tag that the tag type system deduces for the checked expression.

To make this extra type information manifest statically, we design a *truer* type system for λ^{IGTL}. In the remainder of this section, we describe this type system, show that Transient is vigilant for it, and demonstrate how it enables the provably correct elision of some of the tag checks that Transient performs.

Figure 15 presents the truer type system. Just as for tag typing, its rules assume a restricted λ^{IGTL} syntax where type ascriptions are tag K. Similarly, type environments Γ map variables to tags. However, truer typing deduces more precise types τ than tags. These differ from simple types in two major ways. First, the domain of Transient function types is always $*$. After all, in Transient, the internal checks of a function – including the tag check of its argument — guarantee that the function can handle any argument. Second, truer typing can deduce that some expressions raise a run-time type error due to incompatible tag checks, and hence, truer types include $\bot$. This inclusion of $\bot$ allows us to define a full subtyping lattice $\leq$ on truer transient types, as shown in the upper right portion of Figure 15. Like other systems with subsumption rules for subtyping, the truer type system includes a subsumption rule, but for the subtyping lattice $\leq$.

The typing rule for anonymous functions type checks the body of a function under the assumption that the function's argument has the ascribed tag. But, as discussed above, the domain of the function is $*$ because applications implicitly check the argument of a lambda against this tag annotation. Dually, the rule for applications admits function arguments that typecheck at any tag.

Because applications perform a tag check on the result of the application, rather than typing the entire expression at the codomain of the function τ_1, the truer transient type system seeks to take advantage of the fact that the result of the application satisfies *both* τ_1 *and* K. For that, the typing rule calculates the result type as the greatest lower bound $K \sqcap \tau_1$. If τ_1 is $\bot$, a special application rule propagates it to the result type of the application. Similar to the non-$\bot$ application rule, the rule for cast expressions refines the type of its result type with the tag check from the cast. Finally, conditionals typecheck at the least upper bound, $\tau_1 \sqcup \tau_2$, of *both* branches.

This type system accepts just as many programs as the tag type system, but calculates more precise types for them:

Theorem 5.5 (Tag Typing Implies Truer Typing). *Suppose that* $\Gamma \vdash_{\mathsf{tag}} e : K$. *Then there exists some* $\tau \leq K$ *such that* $\Gamma \vdash_{\mathsf{tru}} e : \tau$.

Similarly, since simple typing implies tag typing, it also implies truer typing:

Corollary 5.6 (Simple Typing Implies Truer Transient Typing).
If $\Gamma \vdash_{\mathsf{sim}} e : \tau$, *then* $\lfloor \Gamma \rfloor \vdash_{\mathsf{tru}} \lfloor e \rfloor : \tau'$ *where* $\tau' \leq \lfloor \tau \rfloor$.

The typing rule for conditionals is revealing of the limitations of truer typing. Even though the truer type system aims to reflect statically as much information as possible from the tag checks performed during the evaluation of a λ^{IGTL} expression, as all type systems, it fails to do so accurately. As a result, vigilance for truer typing is a stronger property than type soundness for truer typing. If truer typing deduces that the result of a conditional has type $*$, then a semantics can ignore some of the checks the branches of the conditional are supposed to perform and still truer typing would be sound. However, vigilance requires that the semantics performs all the checks from the evaluated branch nevertheless.

$\boxed{\Gamma \vdash_{\mathsf{sim}} t \Rightarrow \tau \rightsquigarrow e}$ (selected rules)

$$\frac{\Gamma,\ (x{:}K) \vdash_{\mathsf{tru}} t \Leftarrow^{+} \tau \rightsquigarrow e : \tau'}{\Gamma \vdash_{\mathsf{tru}} \lambda(x{:}K) \to \tau.\, t \Rightarrow * \to \tau \rightsquigarrow \lambda(x{:}K).\, e : * \to \tau'} \qquad \frac{\Gamma \vdash_{\mathsf{tru}} t \Leftarrow \tau \rightsquigarrow e : \tau'}{\Gamma \vdash_{\mathsf{tru}} t \Leftarrow^{+} \tau \rightsquigarrow e : \tau'} \qquad \frac{\neg(\exists e, \tau'.\ \Gamma \vdash_{\mathsf{tru}} t \Leftarrow \tau \rightsquigarrow e : \tau') \quad \Gamma \vdash_{\mathsf{tru}} t \Leftarrow^{\Rightarrow} \tau \rightsquigarrow e : \tau'}{\Gamma \vdash_{\mathsf{tru}} t \Leftarrow^{+} \tau \rightsquigarrow e : \tau'}$$

$$\frac{\Gamma \vdash_{\mathsf{tru}} t \Leftarrow^{+} (\tau \setminus \lfloor \tau \rfloor) \times * \rightsquigarrow e : \tau_1 \times \tau_2}{\Gamma \vdash_{\mathsf{tru}} \mathsf{fst}\, t \Leftarrow \tau \rightsquigarrow \mathsf{fst}\{\lfloor \tau \rfloor\}\, e : \tau_1 \sqcap \lfloor \tau \rfloor} \qquad \frac{\Gamma \vdash_{\mathsf{tru}} t \Rightarrow \tau' \rightsquigarrow e : \tau'' \quad \tau' \leq \tau}{\Gamma \vdash_{\mathsf{tru}} t \Leftarrow^{\Rightarrow} \tau \rightsquigarrow e : \tau''} \qquad \frac{\Gamma \vdash_{\mathsf{tru}} t \Rightarrow \tau' \rightsquigarrow e : \tau'' \quad \tau' \not\leq K}{\Gamma \vdash_{\mathsf{tru}} t \Leftarrow^{\Rightarrow} K \rightsquigarrow \mathsf{cast}\,\{K \Leftarrow \lfloor \tau' \rfloor\}\, e : K \sqcap \lfloor \tau' \rfloor \sqcap \tau''}$$

Fig. 16. Truer Translation from λ^{GTL} to λ^{IGTL}

Transient *is vigilant for truer typing.* The specifics of the truer type system require a few changes to the vigilance logical relation from §4. These changes are akin to the necessary changes to the logical relation for type soundness when one modifies a type system. Namely, we obtain a relational interpretation of truer types by (1) removing the restriction that an application annotation has to be a super type of the expected codomain type in the function case of the $\mathcal{V}$ and $\mathcal{VH}$ — unlike simple typing, truer typing accepts the type ascription of an application as it is anticipating that the semantics of the IGTL checks it (see Figure 15); (2) removing the constraint $\vdash \Psi$ from $\Sigma : (k, \Psi)$ — unlike simple typing, truer typing does not impose the constraint that an expression can be cast only from its deduced type to a compatible type, since the semantics is expected to check that a casted value matches the tag of both the "source" and "destination" type ascription of a cast; and (3) adding trivial cases for $\bot$ to the $\mathcal{V}$ and $\mathcal{VH}$ relations. (The full relation is shown in the supplemental material.) We can now use this logical relation for truer types to show that Transient is vigilant for truer typing:

Theorem 5.7 (Transient is Vigilant for λ^{IGTL} Truer Typing). *If* $\Gamma \vdash_{\mathsf{tru}} e : \tau$ *then* $[\![\Gamma \vdash_{\mathsf{tru}} e : \tau]\!]^{\mathsf{T}}$.

5.2 From λ^{IGTL} with Truer Typing to λ^{GTL}

Adjusting λ^{GTL} to truer typing is not as straightforward as for tag typing. Recall that for tag typing, we obtain a restricted syntax for λ^{GTL}, a type system and a type-preserving translation simply by using the lifted metafunction $\lfloor \cdot \rfloor$ on the corresponding definitions from §3. However, a λ^{GTL} with truer typing calls for a bidirectional type system and translation to λ^{IGTL} in order to capitalize on truer typing's ability to take advantage of the tag checks from casts to refine the types of expressions.

Figure 16 presents a few salient rules of the judgments that together define the type checker for λ^{GTL} expressions and their translation to λ^{IGTL}. Unlike the translations for simple and tag, the truer typing translation produces a type τ' for the translated term e. The translation is defined so that $\tau' \leq \tau$ in the subtyping lattice, which implies that the evaluation of e in Transient must at least enforce the types of t (by the subsumption rule), but might possibly also enforce even "stronger" types. From the perspective of a programmer, the λ^{GTL} τ-based reasoning is preserved, while from the perspective of a compiler implementer, the λ^{IGTL} τ'-based reasoning is preserved (an example is given in § 5.3).

The "infers" judgment of the bidirectional translation ($\Gamma \vdash_{\mathsf{tru}} t \Rightarrow \tau \rightsquigarrow e' : \tau'$) is similar to that of the translation from §3; given a type environment and a λ^{GTL} expression, it type checks t at type τ, translates it to the λ^{IGTL} expression e, and calculates the λ^{IGTL} type τ'. However, when t contains

a type annotation—such as in the return type annotation of a function—rather than inserting a cast expression that is supposed to enforce the annotation, the judgment appeals to the "checks" judgment $\Gamma \vdash_{\mathsf{tru}} t \Leftarrow^{+} \tau \rightsquigarrow e : \tau'$ that attempts to construct a translated function body e' that has type τ, and calculate its type τ'. After all, the truer type system is supposed to reflect the types that Transient performs, and Transient only uses casts that perform tag checks and do not enforce a type τ otherwise.

The "checks" judgment itself employs two other judgments. $\Gamma \vdash_{\mathsf{tru}} t \Leftarrow \tau \rightsquigarrow e : \tau'$ inserts an appropriate type ascription to an application or a pair projection, and delegates back to the "checks" judgment to construct a term that has the portion of τ that the ascription does not cover (see Figure 18 for the definition of $\cdot \setminus \cdot$). $\Gamma \vdash_{\mathsf{tru}} t \Leftarrow^{\Rightarrow} \tau \rightsquigarrow e : \tau'$ applies to any expression where the previous judgment does not apply. It attempts to recursively use the "infers" judgment to infer a type τ' for t that is either a subtype of τ, or a type whose tag it can cast to τ (if τ is merely a tag).

As an example of this translation, consider the λ^{GTL} expression

$$t = \lambda(x : * \times *) \to \mathsf{Nat} \times \mathsf{Nat}.\ \langle \mathsf{fst}\, x, \mathsf{snd}\, x \rangle$$

The "infers" judgments translates the body of the function , but doesn't simply try to insert a single cast for the to enforce Nat×Nat, like the simple translation judgment from §3 would. Instead, it delegates to the "checks" judgment which attempts to find a way to insert type ascriptions into the body of the function in order to construct a body that has the desired result type. Hence, with the help of the $\Leftarrow$ judgment, it ascribes the pair projections in the body of the function with Nat, the tag of the required type, leading to the following translated λ^{IGTL} expression:

$$e = \lambda x : * \times *.\ \langle \mathsf{fst}\{\mathsf{Nat}\}\, x, \mathsf{snd}\{\mathsf{Nat}\}\, x \rangle$$

Importantly, however, just like the translation from §3, this translation is type-preserving:

Theorem 5.8 (The Translation From λ^{GTL} to λ^{IGTL} Preserves Truer Types).
If $\Gamma \vdash_{\mathsf{tru}} t \Rightarrow \tau \rightsquigarrow e : \tau'$ *then* $\tau' \leq \tau$ *and* $\Gamma \vdash_{\mathsf{tru}} e : \tau'$.

And as described, this result can be composed with subsumption to get a theorem for λ^{GTL} types:

Theorem 5.9 (The Translation From λ^{GTL} to λ^{IGTL} Preserves λ^{GTL} Truer Types).
If $\Gamma \vdash_{\mathsf{tru}} t \Rightarrow \tau \rightsquigarrow e : \tau'$ *then* $\Gamma \vdash_{\mathsf{tru}} e : \tau$.

With translation in hand, the type preserving theorem for truer typing can be composed with the vigilance theorem as described in §3, to get a vigilance result for λ^{GTL} expressions.

Theorem 5.10 (Transient is Vigilant for λ^{GTL} Truer Typing).
If $\Gamma \vdash_{\mathsf{tru}} t \Rightarrow \tau \rightsquigarrow e$ *then* $[\![\Gamma \vdash_{\mathsf{tru}} e : K]\!]^{\mathsf{T}}$.

And for the same reasons that Natural is not vigilant for tag typing, neither is it vigilant for truer typing:

Theorem 5.11 (Natural is Not Vigilant for λ^{GTL} Truer Typing). *There are* Γ, t, e, τ *such that* $\Gamma \vdash_{\mathsf{tru}} t \Rightarrow \tau \rightsquigarrow e$ *and* $\neg[\![\Gamma \vdash_{\mathsf{tru}} e : \tau]\!]^{\mathsf{N}}$.

5.3 When are Transient Checks Truly Needed?

Since in Transient *all* elimination forms perform tag checks, even those in code with precise types, some of these checks are redundant. Vitousek et al. [26] use a sophisticated constraint system to infer when Transient's tag checks may be elided due to static information that the type system

$\boxed{\Gamma \vdash_{\text{opt}} e : \tau \rightsquigarrow e}$ (selected rules)

$$\frac{\Gamma_0 \vdash_{\text{opt}} e_0 : * \to \tau_1 \rightsquigarrow e_0' \quad \Gamma_0 \vdash_{\text{opt}} e_1 : \tau_0' \rightsquigarrow e_1'}{\Gamma_0 \vdash_{\text{opt}} \text{app}\{K_1\}\, e_0\, e_1 : K_1 \sqcap \tau_1 \rightsquigarrow \text{app}\{K_1 \setminus \tau_1\}\, e_0'\, e_1'} \qquad \frac{\Gamma_0 \vdash_{\text{opt}} e_0 : \bot \rightsquigarrow e_0' \quad \Gamma_0 \vdash_{\text{opt}} e_1 : \tau_0' \rightsquigarrow e_1'}{\Gamma_0 \vdash_{\text{opt}} \text{app}\{K_1\}\, e_0\, e_1 : \bot \rightsquigarrow \text{app}\{K_1 \setminus \bot\}\, e_0'\, e_1'}$$

$$\frac{\Gamma_0 \vdash_{\text{opt}} e_0 : \tau_0 \rightsquigarrow e_0'}{\Gamma_0 \vdash_{\text{opt}} \text{cast}\,\{K_1 \Leftarrow K_0\}\, e_0 : K_1 \sqcap K_0 \sqcap \tau_0 \rightsquigarrow \text{cast}\,\{K_1 \setminus (K_0 \sqcap \tau_0) \Leftarrow K_0 \setminus \tau_0\}\, e_0'} \qquad K \setminus \tau = \begin{cases} * & \text{if } \tau \leq K \\ K & \text{otherwise} \end{cases}$$

Fig. 18. Truer Typing: Check-elision optimization for λ^{IGTL}

computes. The static information the truer type system provides may very naturally be used to implement and prove correct a similar elision pass for Transient tag checks.[8]

For example, consider the variant of the example from §1 in Figure 17. Here, the developer creates two different type adapters for the library function `segment`. The truer types of `segment_png_small` and `segment_png` are different. Since the calls to `png_crop`'s ensure a tag check on each result, the first is will produce `PNG`s, while the second is not.

```
def segment_png_small(img : PNG) -> (PNG, PNG):
  let (a, b) = segment(img)
  in png_crop(a), png_crop(b)
def segment_png(img : PNG) -> (Dyn, Dyn):
  return segment(img)
png1: PNG = segment_png(...)[0]
png2: PNG = segment_png_small(...)[0]
```

Fig. 17. Two Type Adapters for an Image Library

When the developer attempts to project from the results of each function, Transient checks that the result is a `PNG`. This tag check is however only necessary in the case of `segment_png`, where it is not statically known that (due to other checks) a `PNG` would be produced. Precisely this difference between `segment_png` and `segment_png_small`, which allows eliding a check in one case but not the other, is reflected in their truer types!

In terms of the rules of the truer type system from Figure 15, all rules that involve an expressions that performs a check of a tag k strengthen the type of the expression to $K \sqcap \tau$. Hence, the tag check improves what can be statically known about the behavior of the expression in hand — rather than only knowing that it behaves according to τ, we also know that it behaves according to K. As a result, such a tag check is useful only when the strengthened type ($K \sqcap \tau \neq \tau$) is more precise than τ — that is, when it is not already known that the value in question would behave like a K ($\tau \not\leq K$).

Figure 18 provides an overview of a elision pass for redundant tag checks. The judgment $\Gamma \vdash_{\text{opt}} e : \tau \rightsquigarrow e$ consumes a typing environment and a λ^{IGTL} expression e, type checks e at τ the same way as the truer type system for λ^{IGTL}, and uses the deduced types to translate e to an equivalent expression e' without some redundant tag checks. In essence, the translation replaces a type ascription τ with $K \setminus \tau$ where K is a tag that the translated expression checks. In general, $K \setminus \tau$ denotes corresponds the tag check that is necessary to enforce K given that τ is already known — in the subtyping lattice, this is $*$ if $\tau \leqslant: K$, and K otherwise. The elision pass preserves contextual equivalence:

Theorem 5.12 (Check-elision soundness). *If* $\Gamma \vdash_{\text{opt}} e : \tau \rightsquigarrow e'$, *then* $\Gamma \vdash_{\text{tru}} e \approx^{\text{ctx}} e' : \tau$.

[8] Most notably, since our type system is not a whole-program (or whole-module) analysis, it assumes that any lambda may eventually be cast to $*$ and thereby confronted with arguments about which nothing is known statically; consequently, it does not seek to optimise the checks a function performs on its arguments.

6 RELATED WORK

Type soundness, the mainstay of statically typed languages, has seen numerous interpretations in the gradually typed world. §2 discusses two different type soundness theorems and their shortcomings in the context of Reticulated Python: the standard type soundness and tag soundness. But the heterogeneity of how the literature uses the term type soundness goes well beyond that. Chaudhuri et al. [4] prove standard type soundness but only for fully annotated GTL programs. Muehlboeck and Tate [17] prove a type soundness theorem for a restrictive nominal gradual type system rather than a typical structural gradual type system. Tobin-Hochstadt and Felleisen [22]'s type soundness concerns a migratory setting where the components of a GTL program have either all their annotations or no annotations. Furthermore, they strengthen type soundness to a *blame theorem* [16, 22, 23, 28] that describes what kinds of run-time type errors a well-typed program can raise. Vitousek et al. [27] establish an "open-world" type soundness theorem for a GTL with Transient semantics that makes no predictions about the evaluation of a well-typed program except what run-time errors can occur. These properties are all variants of syntactic type soundness. Vigilance is a semantic property that goes beyond (semantic) type soundness.

As a standard for gradually typed languages, Siek et al. [21] propose the *gradual guarantees*. Even though the gradual guarantees are useful guidelines for language designers, they are orthogonal to the question of whether the semantics of an IGTL match the type system of a GTL. The static gradual guarantee concerns only the type system. The dynamic one can be true for a semantics that enforces no types at all. We conjecture that the simple and tag type systems satisfy the static gradual guarantee, and with the Natural and Transient semantics respectively, they satisfy the dynamic gradual guarantee. However, the truer type system violates the static gradual guarantee.

As an example of how truer typing violates the static guarantee, consider the Reticulated Python code in Figure 19. The function `foo` uses the return type annotation to insert assertions on the first and second projections, meaning the constructed pair is checked to have natural numbers as components. The function `bar` simply returns the result of applying `foo` to its argument. If we were to change the return annotation on foo to `Dyn`, the projections would no longer have assertions for `Nat`. Since Transient only performs checks for tags, there is no check or assertion on the result of `bar` that would be inserted by the translation that can guarantee the type `(Nat, Nat)`, and therefore this program cannot typecheck with a return annotation of `Dyn` on `foo`. Fundamentally, truer typing is flow sensitive and specific to the checks and assertions that Transient inserts, and therefore seems incompatible with the static gradual guarantee. That said, as the rest of the paper shows, truer typing is a useful instrument for exploring what Transient offers to programmers, despite not satisfying the gradual guarantees. We leave uniting truer typing with the gradual guarantees as future work.

```
def foo(x: (Dyn, Dyn)) -> (Nat, Nat):
  return (fst x, snd x)

def bar(x: (Dyn, Dyn)) -> (Nat, Nat):
  return foo(x)
```

Fig. 19. Truer Typing: Counterexample to static gradual guarantee

Gradual Type Theory [18] axiomatizes the dynamic gradual guarantee and a set of contextual equivalence properties as the essential properties of a well-designed gradually typed language. In particular, the equivalence properties entail that developers can reason in the gradually typed setting using the same basic principles as in the typed setting. They show that only a GTL with a simple type system and Natural semantics lives up to this standard. Our study of vigilance shows that even outside this combination, developers can still rely on type annotations to make decisions about how to structure their code, if the type system and the semantics are in sync.

Jacobs et al. [13] propose an alternative to the gradual guarantees. Specifically, they focus on preserving equivalences, requiring that the embedding of a fully statically typed subset of the GTL

into the GTL be fully abstract. We leave it to future work to investigate if the design of truer typing can be tuned so we have a fully abstract embedding of fully typed terms into the truer GTL.

Abstracting Gradual Typing (AGT) [6] does not propose a new property but is a method for obtaining a well-designed gradually typed language from a typed one. Their system produces IGTLs that manage "evidence objects", which act as "proofs" that the semantics enforce a value's typing history. Hence, we conjecture that AGT is a recipe for creating semantics for an IGTL that are by construction vigilant for the type system of the GTL. A novelty of this paper is that we also do the converse: start with the semantics of an IGTL (Transient) and arrive at a type system for a GTL (truer) for which it is vigilant.

There is a significant body of work on equipping gradual type systems with blame in a correct manner [1, 9, 11, 27, 29]. Vigilance currently says nothing about blame. Vigilance is concerned with a semantics of types, both static in the form of the typing system and dynamic in the form of boundary annotations. Blame is instead concerned with the mechanisms for providing accurate errors that developers can use in debugging. We conjecture that with additional instrumentation in our logical relation, we can incorporate properties about blame, but leave that as future work.

7 REFLECTIONS ON VIGILANCE AND LOOKING FORWARD

There are two possible directions for future work. The first concerns the improvement and evaluation of the truer type system. For instance, equipping the type system with occurrence types [24] or other path- and flow-sensitive features can help improve its precision. At the same time, implementing truer typing for a production language, and evaluating its performance and ability to detect issues in real codebases will provide guidance for how to grow the formal kernel of this paper to a useful tool for developers. The second future direction is theoretical and involves developing a framework for reasoning about contextual equivalence, such as an equational theory, based on vigilance.

Vigilance is a semantic property that describes a gradual type system as two components, a static and a dynamic, that work together to help developers organize their code in a correct manner. When the static component relies on the dynamic, but the latter does not deliver, the meaning of types becomes misleading, and hence, developers may discover that their type-based assumptions about their code are false. When the dynamic component offers more than what the static component can capture, there is a missed opportunity for language designers. Vigilance is a compass for exploring the design space and finding a balance between the static and dynamic components of gradual typing. For instance, in this paper we use vigilance to show that neither the combination of Transient and simple typing, nor the combination of Transient and tag typing is balanced. The first fails vigilance which entails that the dynamic component does not live up to the expectations of the static one. The second satisfies vigilance but the dynamic component performs more checks than needed. In response, with vigilance as a guide, we propose a balance point for Transient. Specifically, we transfer some of the reasoning power due to the dynamics of Transient to the statics of our truer type system while making sure the combination is still vigilant. We hope that vigilance can more generally be a tool for identifying opportunities to further incorporate reasoning patterns that developers routinely use in dynamically typed languages into the design of gradual type systems [8, 24].

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