# On a selection of Basis Functions in Numerical Analyses of Engineering Problems

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**SUMMARY**

This paper presents basic characteristics of the existing numerical procedures from the aspect of selection of approximate solution basis functions for different technical problems. Given are the advantages of algebraic and trigonometric polynomials due to universality of vector spaces they are forming. It is also shown that spline functions have extreme advantages for the practice as a result of finiteness. At the same time, they show disadvantages because of the loss of universality due to limited smoothness. Basis functions, which maintain properties of universality and infinite differentiability, and at the same time retain the characteristics of practical application of splines, are $R_{bf}$ - Rvachev's basis functions. The properties of these functions classify them between classical polynomials and spline functions; hence, they complete a set of elementary functions. Procedures for calculation of $R_{bf}$ functions are given together with their distribution for the forming of numerical solutions and an illustration of basic possibilities for their application in practice. The biggest consideration is given to finite functions $Fup_n(x)$ of $C^\infty$ class which are the elements of an universal space $UP^n$, containing also algebraic polynomials to the n-th degree. An illustration is given of determination and application of finite basis functions, also pertaining to $C^\infty$ class, the linear combination of which can be used for exact description of exponential and trigonometric functions.

## 1. MAIN TYPES OF BASIS FUNCTIONS

### 1.1 Introduction

Solution of an engineering problem includes description of equilibrium and compatibility conditions as well as of the geometry of the area, applied load and boundary conditions. Numerical solutions of real physical processes in the continuum are mostly the results of the methods that can be obtained from the definition of the scalar product of functions within the Hilbert space:

$$(v,w) = \int_\Omega v(x)\cdot w(x)\ d\Omega \quad ; \quad x \in \Omega \tag{1}$$

A basic lemma of variation calculus is derived from the properties of scalar product of functions (1) as follows:

If for a continuous function $v:\Omega \to R$ and each continuous function $w:\Omega \to R$; $\Omega \subset R^n$ :

$$\int_\Omega v(x)\cdot w(x)\ d\Omega = 0 \ ; \quad x \in \Omega \tag{2}$$

then $v(x) \equiv 0$ for each $x \in \Omega$.

A basic lemma (2) is used as a starting point for many approximate and exact procedures that are applied in numerical analyses. Generally, a solution of different technical problems can be defined as obtaining a solution $u \in X$ of the equation:

$$A\ u = f \tag{3}$$

over the area $\Omega \subset R^n$, for the given conditions at the boundary Γ:

$$L_i u = g_i \quad \text{na} \ \ \Gamma_i \quad ; \qquad i = 1,\dots,m$$

where $\Gamma_i \subset \Gamma$, $\Gamma = \bigcup_{i=1}^{n} \Gamma_i$, $\Gamma_i \cap \Gamma_j = 0 \ (i \neq j)$; $A: X \to Y$ and $L_i: X_i \to Y_i$ are known differential operators; $X$, $Y$, $X_i$, $Y_i$ are functional spaces, while $f$ and $g_i$ are known elements of the spaces $Y$ and $Y_i$, respectively. To solve the equation (3), using a basic lemma of variation calculus, means satisfying the equation (2) so the vector $v(x) = Au - f$ becomes a zero vector.

Approximate procedures in numerical analyses require approximation of the function sought no matter if it was set directly or by a differential equation [1]. Approximation of the function $u(x): \Omega \to R$ is sought as an n-dimensional vector $\tilde{u}(x)$ in the form of the following linear combination:

$$\tilde{u}(x) = \sum_{i=1}^{n} c_i \varphi_i(x) \quad ; \quad \varphi_i(x): \Omega \to R \tag{4}$$

where:

$\varphi_i$ – basis or coordinate vectors

$c_i$ – coefficients of linear combination

The difference between the function $u(x)$ and its approximation $\tilde{u}(x)$ gives the following deviation:

$$\varepsilon(x) = u(x) - \tilde{u}(x) = u(x) - \sum_{i=1}^{n} c_i \varphi_i(x)$$

The aim of approximation is to minimise the deviation ε(x). Unknown parameters of linear combination (4) are calculated by application of the basic lemma:

$$\int_\Omega \left( u(x) - \sum_{i=1}^{n} c_i \varphi_i(x) \right) w_j \; d\Omega = 0 \; ; \quad j = 1,2,3,\ldots,n \qquad (5)$$

If (5) is valid for every continuous function $w_j(x)$, then a sought function $u(x)$ will be developed in a convergent series (4) over a base $\varphi_i$. For calculation of *n* unknown parameters of linear combination, it is enough to set *n* independent conditions which is obtained by selection of *n* linearly independent functions $w$. When *n* is a finite number, basic lemma is fulfilled in an approximate sense; and function $u$ is developed over *n* members of a convergent series.

Vectors $w_j(x)$ in the equation (5) represent *n* linearly independent functions and form a base of a vector space which is called a test space. Vector $w_j(x)$ is called a test vector.

Therefore, unknown parameters of approximation $c_i$ are obtained by developing the equation (5):

$$\sum_{j=1}^{n} c_j \int_\Omega \varphi_j w_i d\Omega = \int_\Omega u w_i \; d\Omega \quad ; \quad i = 1,2,3,\ldots,n$$

which, following the calculation of the integral, gives the following equation system:

$$a_{ji} \; c_j = b_i \quad ; \quad i,j = 1,2,3,...,n$$

where: $a_{ji} = \int_\Omega \varphi_j \; w_i \; d\Omega$ ; $b_i = \int_\Omega u \; w_i \; d\Omega$.

A short overview of functions used in numerical methods as approximate solution base and test space base is given in the following sections with a special regard to their advantages and disadvantages in solution of engineering problems.

## 1.2 Classic basis functions

### 1.2.1 Algebraic polynomials

In the procedure of determining an approximate solution using algebraic polynomials, an approximation base is selected out of set of power functions:

$$1, x, x^2, x^3, \ldots, x^n$$

where *n* is the degree of algebraic polynomial i.e. degree of approximation.

Obtained functional approximation is a vector with *n+1* members:

$$\tilde{u}(x) = P_n(x) = \sum_{i=0}^{n} c_i x^i$$

**EXAMPLE**: Approximate solution of the equation $-\dfrac{d^2u(x)}{dx^2} + f(x) = 0$ is sought on the interval $(0,1)$ for boundary conditions: $u(0) = u(1) = 0.0$ ; where $f(x)$ is a given function: $f(x) = -16.0$ for $0 \le x < 0.5$ and $f(x) = 0.0$ for $0.5 < x \le 1.0$. The exact solution is:

$$u(x) = \begin{cases} 2x(3-4x) & \text{za} \quad 0 \le x < 0.5 \\ 2(1-x) & \text{za} \quad 0.5 < x \le 1 \end{cases}$$

For the basis functions of an approximate solution, the following series can be selected:

$$\varphi_i(x) = x^i (1-x) \quad , \quad i = 1,2,3,\ldots$$

When two basis functions: $\varphi_1 = x(1-x)$, $\varphi_2 = x^2(1-x)$ are selected, an approximate solution is sought in the following form:

$$\tilde{u}(x) = c_1 x(1-x) + c_2 x^2 (1-x)$$

Therefore, approximate solution exactly satisfies the boundary conditions.

Adopting the same functions as the test space base ($w_j = \varphi_j$), and using the Galjerkin method to calculate the integrals which give the coefficients, the following system of linear equations is obtained:

$$\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 2/15 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 1/4 \end{bmatrix} \qquad (6)$$

and an approximate solution: $\tilde{u}_2(x) = x\left(49 - 99x + 50x^2\right)/6$

Selecting the first three members of a series $\varphi_i(x)$ the following system of equations is obtained:

$$\left[\begin{array}{cc|c} 1/3 & 1/6 & 1/10 \\ 1/6 & 2/15 & 1/10 \\ \hline 1/10 & 1/10 & 3/35 \end{array}\right] \cdot \left[\begin{array}{c} d_1 \\ d_2 \\ \hline d_3 \end{array}\right] = \left[\begin{array}{c} 4/3 \\ 1/4 \\ \hline 3/20 \end{array}\right] \qquad (7)$$

and an approximate solution:

$$\tilde{u}_3(x) = x\left(1449 - 2724x + 1450x^2 - 175x^3\right)/216$$

It can be observed than (7) contains a matrix of the system and the right side vector of the system (6), which is a very significant property of universality of the selected vector space.

### 1.2.2 Trigonometric polynomials

Numerical procedure can be simplified significantly if orthonormal basis functions are selected as approximation base and test space base. The following can be applied to orthonormal basis functions:

$$\int_\Omega \varphi_i w_j d\Omega = \delta_{ij}$$

where $\delta_{ij}$ is the Kronecker delta. In that case, the equation system becomes explicit:

$$a_{ji} \; c_j = b_i \quad \Rightarrow \quad \delta_{ij} \; c_j = b_i$$

Significant representative of orthogonal functions, which can be used for building of different bases, is the system of trigonometric functions:

$$1, \sin(x), \cos(x), \ldots, \sin(nx), \cos(nx), \ldots$$

**EXAMPLE**: The same problem is analysed as in the example given in Section 1.2.1. A series of orthogonal trigonometric functions is selected as basis functions:

$$\varphi_i(x) = \sin(i\pi x) \quad ; \quad i = 1, 2, \ldots \tag{8}$$

For the first two members of the series (8) the following equation system is obtained:

$$\begin{bmatrix} \pi^2/2 & 0 \\ 0 & 2\pi^2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 16/\pi \\ 16/\pi \end{bmatrix} \tag{9}$$

therefore, an approximate solution has the following form:

$$\tilde{u}_2(x) = (32 \sin\pi x + 8 \sin 2\pi x)/\pi^3$$

By analogy, for the first three members of the series (8) the following system is obtained:

$$\left[\begin{array}{cc|c} \pi^2/2 & 0 & 0 \\ 0 & 2\pi^2 & 0 \\ \hline 0 & 0 & 9\pi^2/2 \end{array}\right] \cdot \left[\begin{array}{c} c_1 \\ c_2 \\ \hline c_3 \end{array}\right] = \left[\begin{array}{c} 16/\pi \\ 16/\pi \\ \hline 16/3\pi \end{array}\right] \tag{10}$$

and appurtenant approximate solution:

$$\tilde{u}_3(x) = (32 \sin\pi x + 8 \sin 2\pi x + 32/27 \sin 3\pi x)/\pi^3$$

It can be observed that system (10) contains the matrix and right side vector of the system (9) which means that vector space of trigonometric polynomials also has a property of universality.

## 1.3 Spline functions; B-splines

Finite basis functions, which are often used in approximation procedures are the splines. Spline functions are smooth, piecewise polynomial functions with the same structure i.e. consisted of polynomials of the same degree. The following sections present basic postulates and some properties of the spline functions connected with $R_{bf}$ basis functions described in Section 2.

Let the set of nodes $\Delta$ from the interval $[a, b]$ be expanded with additional nodes outside the interval $[a, b]$, e.g..:

$$x_0 < a \quad ; \quad b < x_{N+2}$$

Natural spline of $n$-th degree, which is identical to zero at both end intervals $[x_0, x_1]$ and $[x_{N+1}, x_{N+2}]$ is called B-spline (basis spline). Basis splines are finite functions, the displacement and simultaneous compression of which can be used to express all other natural splines of an uniform net. It can be shown [2] that every natural spline $S_n(x)$ can be expressed in the following form:

$$S_n(x) = \sum_{i=0}^{N+2} b_i \cdot B_n^i(x)$$

where $b_i$ are the coefficients and $B_n^i(x)$ are the basis splines.

For an unit distance of the nodes, distributed according to $x_p = p - (n+1)/2$, $p = 0, 1, \ldots, n+1$, B-spline can be expressed in the following form:

$$B_n(x) = \sum_{j=0}^{n+1} \frac{(-1)^j}{(n-j)!\, j!} \cdot \left(x + \frac{n+1}{2} - j\right)_+^n \tag{11}$$

According to (11) basis splines of the zero, first and second degree are:

$$B_0(x) = 1 \cdot (x + 1/2)_+^0 - 1 \cdot (x - 1/2)_+^0$$

$$B_1(x) = (x+1)_+ - 2(x)_+ + (x-1)_+$$

$$B_2(x) = \left[(2x+3)_+^2 - 3(2x+1)_+^2 + 3(2x-1)_+^2 - (2x-3)_+^2\right]/8$$

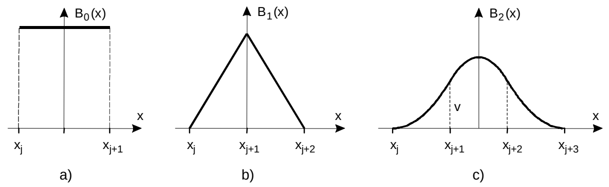


Fig. 1 B-splines: a) $B_0(x)$; b) $B_1(x)$; c) $B_2(x)$

The Paley-Wiener theorem [3] can be applied to basis $B_n(x)$ splines i.e.:

$$\int_{-\infty}^{\infty} B_n(x)\, dx = 1 < \infty \tag{12}$$

Since $B_n(x)$ is $n$-th degree polynomial on characteristic interval $[x_i, x_{i+1}]$, it is necessary to satisfy $n+1$ conditions on every characteristic interval. A support of $B_n(x)$ function consists of *(n+1)* characteristic intervals, therefore *(n+2)* nodes are required, which is sufficient to satisfy all the conditions.

Spline function $B_n(x)$ is defined over the entire real axis, therefore, starting form any of the points denoted by $x_i$, node order is the following:

$$a = x_i < \ldots\ldots < x_{i+n+1} = b$$

Connection between the derivative of higher order and a derivative of zero order i.e. by the function $B_n(x)$ is obvious. The first derivative of a spline $B_n(x)$ can be expressed as linear combination of displaced spline functions $B_{n-1}(x)$. Due to that fact, using the convolution theorem [4] the basis splines $B_n(x)$ can be expressed in the following form:

$$B_n(x) = \int_{-\infty}^{\infty} B_{n-1}(x - \xi) B_0(\xi)\, d\xi$$

i.e.

$$B_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \left(\frac{\sin t/2}{t/2}\right)^{n+1} dt \tag{13}$$

As mentioned before, a support of basis spline function $B_n(x)$ can be defined as a union of *(n+1)* characteristic intervals.

The consequence of the fact that a base, the elements of which are the splines of a selected degree, does not contain lower degree splines is non-universality. As shown, classic bases are universal. In that sense, this is a disadvantage of splines because a matrix of the system of $n$-th order is not contained in a matrix of the system of $(n+1)$-th order.

**EXAMPLE**: A problem given in section 1.2.1 is solved again. This time a linear combination of displaced and compressed splines $B_1(x)$ is used for obtaining approximate solutions.

The following equation system is obtained for two basis functions:

$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 14/3 \\ 2/3 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 10/9 \\ 6/9 \end{bmatrix} \tag{14}$$

as well as appurtenant approximate solution:

$$\tilde{u}_2(x) = \frac{10}{9} B_1(x - 1/3) + \frac{6}{9} B_1(x - 2/3) .$$

For three basis functions, the following equation system is obtained:

$$\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.0 \\ 0.5 \end{bmatrix} \tag{15}$$

i.e. an approximate solution.

$$\tilde{u}_3(x) = B_1(x - 1/4) + B_1(x - 1/2) + 0.5\, B_1(x - 3/4)$$

By comparison of equations (14) and (15) it can be observed that system matrix and the right side vector from equation (14) are not contained in (15). Therefore, the splines do not have the property of universality. Matrix given in (15) is banded and symmetrical one, diagonally dominant, and therefore well conditioned.

Fig. 2 gives a comparison between the exact and approximate solutions obtained by algebraic and trigonometric polynomials and spline functions.

It is well known that application of algebraic and trigonometric polynomials as basis functions in solution of problems on irregular areas is almost impossible. Besides, (7) shows that the system matrix for a base made of algebraic polynomials is full and ill-conditioned and thus not appropriate for numerical elaboration. From the numerical aspect, the splines, due to their property of finiteness have several advantages over classic basis functions.

Trigonometric polynomials provide the best approximate solution. However, they can not be applied to 2D and 3D areas. Therefore, their exceptional properties of orthogonality and universality become useless. Spline functions, although adopted with the minimum smoothness, provide exact solution in vortexes of singular basis spline. Between the vortexes of two adjacent basis splines, an approximate solution has the smoothness of a spline. Therefore, better approaching can be obtained by condensing the basis splines or by selection of basis splines of higher smoothness.

The question is if it is possible to construct basis functions which will maintain properties of algebraic and trigonometric polynomials i.e. universality and infinitive derivability, and at the same time retain finiteness as a good property for practical application of splines?

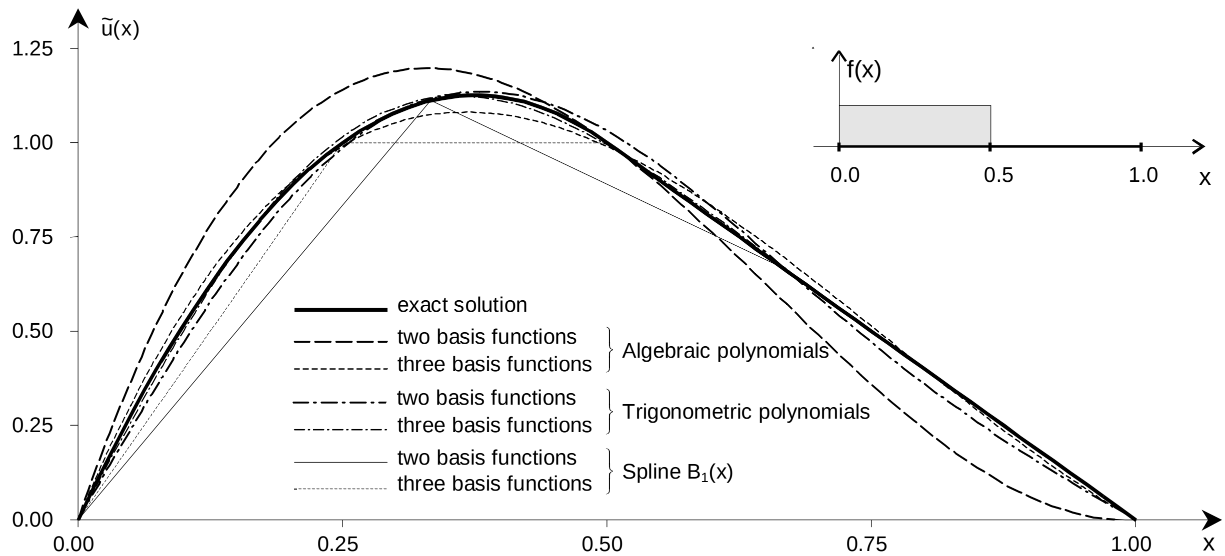


Fig. 2 Comparison of solutions obtained by different basis functions

# 2. FORMULATION AND BASIC PROPERTIES OF $R_{bf}$ FUNCTIONS

## 2.1 General

Basis functions which maintain the property of universality of algebraic and trigonometric polynomials, as well as finiteness, which as a property of the spline functions, obviously must be sought among finite functions with a compact support of $C^{\infty}$ class. A function having the aforementioned property is a well known ″Chebyshev's hat″ function shown in Fig. 3.

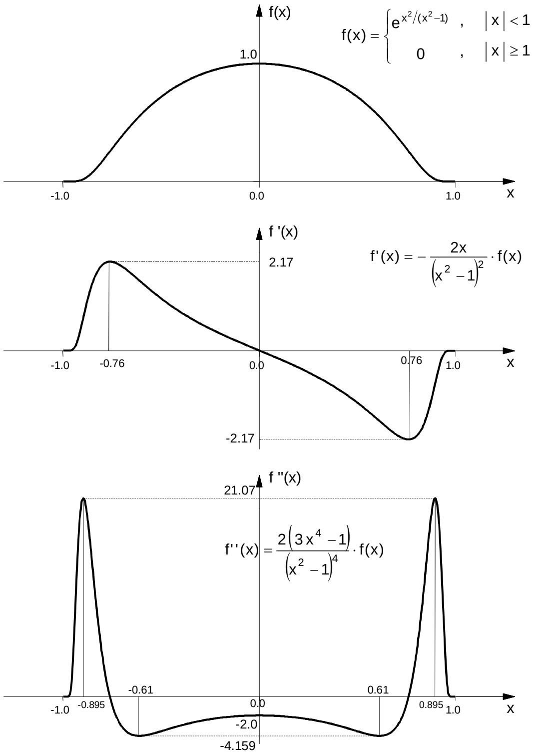


Fig. 3 ″Chebyshev's hat″

However, in order to be applied in practice, basis functions must be "simple". It refers to rapid calculation of the function values and derivatives in selected points as well as of a scalar product of the function with itself, its derivatives and elementary functions. Function given in Fig. 3 obviously does not fulfil the aforementioned requirements. Rvachev's basis functions – $R_{bf}$, described in this section, are finite functions of $C^{\infty}$ class [5], [6], [7], [8], [9]. They are classified between classic polynomials and spline functions. However, in practice, their application as basis functions is still closer to splines. Therefore, it makes sense to describe the class of $R_{bf}$ functions as splines of an infinitely high degree.

An overview of elementary basis functions of $R_{bf}$ class is given in reference [4], with regard to the following functions: up(x), $Fup_n(x)$, $\Xi_n(x)$, $y_{\omega}(x)$ and $y_{\omega,h}(x)$. Basic properties of the aforementioned functions are given in detail in Ref. [4] complete with the possibility of their implementation in numerical methods. The first three function types are polynomial basis functions suitable for physical problems, the solutions of which are algebraic polynomials. Functions $y_{\omega}(x)$ are suitable for the problems, the solutions of which belong to a class of exponential functions while functions $y_{\omega,h}(x)$ are suitable for approximate solutions of trigonometric functions class or the ones which contain trigonometric functions.

## 2.2 Determination of finite functions of $R_{bf}$ class

Rvachev's basis functions $R_{bf}$ are defined as finite solutions of differential-functional equations of the following type:

$$Ly(x) = \lambda \sum_{k=1}^{M} C_k\, y\,(ax - b_k) \tag{16}$$

where $L$ is a common linear differential operator with constant coefficients, $\lambda$ is a scalar different than zero, $C_k$ are solution coefficients, $a>1$ is a parameter of the length of finite function support, $b_k$ are coefficients which determine displacements of finite basis functions.

# 3. FUNCTION up(x)

Type of finite function of $R_{bf}$ class is determined by the selection of operator $L$ in equation (16). Function up(x) is a solution of differential-functional equation in which the differential operator of the first order, according to equation (16) has the following form:

$$y'(x) = \lambda\left[C_1 y(ax - b_1) + C_2 y(ax - b_2)\right] \tag{17}$$

Support of the function up(x) is the interval $[-1,1]$.

Parameter of ″compression″ i.e. ″extension″ of the support of function up(x) is $a = 2$, characteristic displacements of the function on the abscise are $b_1 = -1$ and $b_2 = 1$, and according to [4] value $\lambda = -2$ and coefficients $C_1 = -1, C_2 = 1$. Therefore, basic equation for the function up(x), according to (17) is:

$$up'(x) = 2up(2x+1) - 2up(2x-1) \tag{18}$$

If the length of up(x) function support is described as an union of lengths $2^{-k}$, $k = 0, 1, \ldots, \infty$, the Fourier transform of the function up(x), using the procedure given in Fig. 4, is obtained as a product of the Fourier transforms of zero degree splines condensed to a support length $2^{-k}$ with the ordinates $2^{k}$:

$$\hat{up}(t) = \prod_{j=1}^{\infty} \frac{\sin(t\, 2^{-j})}{t\, 2^{-j}} \tag{19}$$

Finite solution of equation (18), with the fulfilment of normed condition (12), has the following form:

$$up(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx}\, \hat{up}(t)\, dt \tag{20}$$

Based on (20), i.e. the fact that the function up(x) is expressed by its Fourier transform (19), function up(x) can be generated using the convolution theorem. A procedure of generation of up(x) function is given in Fig. 4.

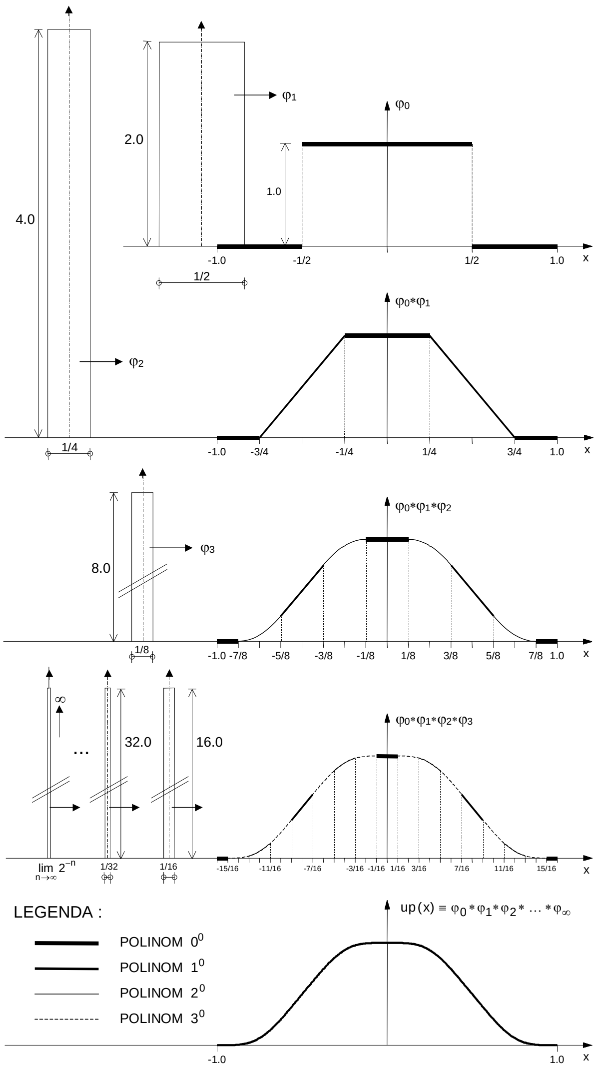


Fig. 4 Generating of function up(x)

### 3.1 Derivatives and integrals of function up(x)

As it can be observed in equation (18), the first derivative can be expressed as linear combination of displaced and compressed function up(x). By differentiating of the basic equation (18) and replacement of the first derivative of function up(x) with the right side of the initial equation (18), second derivative can also be expressed as linear combination of compressed and displaced function up(x).

If the procedure of differentiating and replacement of the first derivative from the basic equation continues, general expression for the derivative of the m-th degree is obtained:

$$\mathrm{up}^{(m)}(x) = 2^{C^2_{m+1}} \sum_{k=1}^{2^m} \delta_k \mathrm{up}(2^m x + 2^m + 1 - 2k)\,,\ m \in N \quad (21)$$

where $C^2_{m+1} = m(m+1)/2$ are the binomial coefficients and $\delta_k$ are the coefficients of value $\pm 1$ which determine the sign of each term. They change according to the following recursive formulas:

$$\delta_{2k-1} = \delta_k\ ,\ \delta_{2k} = -\delta_k\ ,\ k \in N\ ,\ \delta_1 = 1 \quad (22)$$

Fig. 5 shows the function up(x) and its derivatives. It can be observed that the derivatives consist of the function up(x) ″compressed″ to the interval of length $2^{-m+1}$ and with ordinates ″exteneded″ with the factor $2^{C^2_{m+1}}$. High degree derivative of the function up(x) when $m \to \infty$, becomes a series in which every single member corresponds to Dirac’s function.

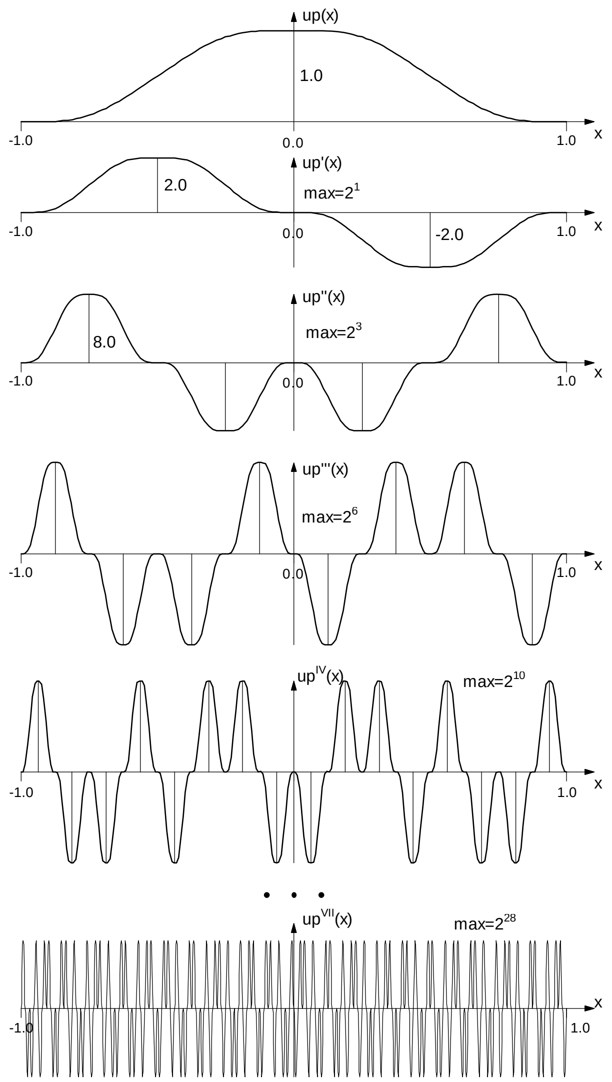


Fig. 5 Function up(x), its first four and the seventh derivative

Successive integration of function up(x) within the limits -∞ and $x \in (-\infty, 1]$ i.e. from +1 to $x \in (1, \infty)$ the following formula is obtained:

$$I_m(x) = \begin{cases} 2^{C^2_m} \mathrm{up}(2^{-m} x - 1 + 2^{-m})\,, & x \le 1 \\ 2^{C^2_m} \mathrm{up}(2^{-m+1} - 1) + \dfrac{(x-1)^{m-1}}{(m-1)!}\,, & x > 1 \end{cases} \quad , m \in N$$

Therefore, an integral of the function up(x) of any order within the limits -∞ and $x \in (-\infty, 1]$ corresponds to the function up(x), which is extended along the abscise to $2^m$ times greater length, and displaced in a way that $x = -1$ is the first point of the support. Within the limits +1 and $x \in (1, \infty)$, *m*-th integral of function up(x) corresponds to algebraic polynomial of *(m-1)*-th degree.

### 3.2 Moments of function up(x)

Expression (20) is numerically inadequate for calculation of function up(x) values. Ref. [4] shows that function up(x) values can be expressed as linear combination of function up(x) moments.

Function up(x) moments with an even index (odd ones are equal zero because up(x) is an even function):

$$a_{2k} = \int_{-1}^{1} x^{2k} up(x) dx \qquad (23)$$

can be calculated, based on [4], according to formula:

$$a_{2k} = \frac{(2k)!}{2^{2k}-1} \sum_{\ell=1}^{k} \frac{a_{2k-2\ell}}{(2k-2\ell)!(2\ell+1)!}, \ k \in N; a_0 = 1 \qquad (24)$$

Scalar product of a polynomial and function up(x) on even half of the support is:

$$b_n = \int_{0}^{1} x^n up(x)\, dx, \ n = -1, 0, 1, \ldots \qquad (25)$$

Since function up(x) is even, comparison of expressions (23) and (25) gives:

$$b_{2k} = \frac{1}{2} a_{2k}, \ k = 0, 1, \ldots \qquad (26)$$

For odd indexes, according to (25) using (24), the following is obtained:

$$b_{2k+1} = \frac{1}{(k+1)\, 2^{2k+3}} \sum_{\ell=0}^{k+1} a_{2\ell}\, C_{2(k+1)}^{2\ell}; \ k = 0, 1, 2, 3, \ldots$$

$$b_{-1} = 1 \ \text{(by definition)} \qquad (27)$$

Scalar products of function up(x) and algebraic polynomials are easily calculate using (26) and (27).

### 3.3 Characteristic points of function up(x)

Characteristic points $x_m^{(n)}$ are the points in which the function up(x) values and values of the first *n* derivatives are calculated exactly in the form of a rational number. In other points of the support, the values are calculated with a computer precision i.e. accuracy depends on the possibility to describe selected point coordinate in the base used by the computer.

A set of characteristic points of given density on the function up(x) support can be described in a simpler manner as:

$$x_k = -1 + k2^{-n}, \ n \in N, \ 1 \le k \le 2^{n+1} \qquad (28)$$

*n* determines the distance between characteristic points on the function up(x) support:

$$\Delta x_n = 2^{-n} \qquad (29)$$

### 3.4 Function up(x) value in a characteristic point

Ref. [7], [4] provide numerically more adequate expression for calculation of function up(x) values. Function up(x) value in a characteristic point $x_k = -1 + k2^{-n}, \ n \in N, \ 1 \le k \le 2^{n+1}$ can be expressed in the following form:

$$up(x_k) = \frac{2^{-n(n+1)/2}}{n!} \sum_{j=1}^{k} \delta_j \sum_{\ell=0}^{[n/2]} C_n^{2\ell} (2(k-j)+1)^{n-2\ell} \cdot a_{2\ell} \qquad (30)$$

where $\delta_j$ are the coefficients in the role of sign according to expression (22), $C_n^{2\ell}$ are binomial coefficients, $a_{2\ell}$ are even moments of function up(x) while square brackets in expression $[n/2]$ denote the maximum integer of the fraction within the brackets.

In characteristic point $x_1 = -1 + 2^{-n}$, expression (30) can be written as:

$$up(-1+2^{-n}) = \frac{b_{n-1}}{2^{n(n-1)/2}(n-1)!}, \ n = 0, 1, \ldots \qquad (31)$$

Introducing (31) in a general expression of function up(x) derivative (21), the following value of function up(x) derivative in characteristic point $x_1 = -1 + 2^{-n}$ is obtained:

$$up^{(\ell)}(-1+2^{-n}) = \frac{2^{-n(n-2\ell-1)/2}}{(n-\ell-1)!} b_{n-\ell-1} \qquad (32)$$

### 3.5 Function up(x) value in an arbitrary point

Based on the fact that development of function up(x) in a Taylor series, in characteristic points $x_k$, is a polynomial of *n*-th degree (see section 3.3), a special series [6] for calculation of function up(x) values in an arbitrary point $x \in [0,1]$ is proposed as:

$$up(x) = 1 - \sum_{k=1}^{\infty} (-1)^{1+p_1+\ldots+p_k} \ p_k \sum_{j=0}^{k} C_{jk} (x - 0, p_1 \ldots p_k)^j \qquad (33)$$

where coefficients $C_{jk}$ are rational numbers determined according to the following expression:

$$C_{jk} = \frac{1}{j!} 2^{j(j+1)/2}\, up(-1+2^{-(k-j)}); \ j = 0, 1, \ldots, k \quad k = 1, 2, \ldots, \infty \qquad (34)$$

Expression $(x-0, p_1 \ldots p_k)$ in (33) is the difference between the real value of coordinate *x* and its binary form with *k* bits, where $p_1 \ldots p_k$ are the digits 0 or 1 of the binary development of the coordinate *x* value. Therefore, the accuracy of coordinate *x* computation, and thus the accuracy of function up(x) in an arbitrary point, depends upon the accuracy of a computer. For *n*, an error of calculated function up(x) value in an arbitrary point *x*, i.e. the residue of a series given in (33) when $k = 1, \ldots, n$, does not exceed the function $up(-1+2^{-n})$ value obtained from (31). Using the functional subprogram, see section 8, function up(x) value for $x \in [0,1]$ is calculated with an error smaller than $10^{-21}$.

## 3.6 Polynomial as linear combination of displaced up(x) functions

Linear combination of displaced up(x) functions can be expressed as:

$$\varphi(x) = \sum_{k=-\infty}^{\infty} C_k \mathrm{up}(x - k2^{-n}) \ , \ \ k \in Z \qquad (35)$$

Especially, if coefficient $C_k$ is *n*-th degree polynomial of index *k*, then the function φ(x) from (35) is the polynomial of *n*–th degree. In that case, an arbitrary polynomial can be expresses as:

$$x^n = \sum_{k=-\infty}^{\infty} \sum_{i=0}^{n} A_i^{(n)} k^i \mathrm{up}(x - k2^{-n}) \qquad (36)$$

where coefficients $A_i^{(n)}$ are calculated using the folowing formulas:

$$A_n^{(n)} = 2^{-(n^2+n)}; A_{n-2}^{(n)} = 2^{1-n} C_n^2 \sum_{j=1}^{2^n-1} \alpha_j^{(n)} j^2 A_n^{(n)}; A_{n-2i+1}^{(n)} = 0;$$

$$A_{n-2i}^{(n)} = 2^{1-n}\left( C_n^{2i} \sum_{j=1}^{2^n-1} \alpha_j^{(n)} j^{2i} A_n^{(n)} + C_n^{2(i-1)} \sum_{j=1}^{2^n-1} \alpha_j^{(n)} j^{2(n-1)} A_{n-2}^{(n)} + \right.$$

$$\left. \ldots + C_n^2 \sum_{j=1}^{2^n-1} \alpha_j^{(n)} j^2 A_{n-2(i-1)}^{(n)} \right) \ ; \ \ \alpha_j^{(n)} = -\mathrm{up}(j2^{-n}) \qquad (37)$$

For example, for the polynomial of the second degree, coefficients $C_k$ can be expressed in the following form:

$$C_k = A_0^{(2)} k^0 + A_1^{(2)} k^1 + A_2^{(2)} k^2$$

According to (37) values of coefficients $A_i^{(2)}$ are: $A_2^{(2)} = 1/64$ ; $A_1^{(2)} = 0$ ; $A_0^{(2)} = -1/36$ . Therefore, coefficient $C_k$ for the polynomial of second degree is a squared function of index *k*:

$$C_k = \frac{1}{4}\left( \frac{k^2}{16} - \frac{1}{9} \right) , \ \ k \in Z$$

Coefficients for the polynomial of any degree n = 0,1,... can be calculated by analogy. For example the following monomials are expressed as:

$$n = 0 \ \rightarrow \ 1 = \sum_{k=-\infty}^{\infty} \mathrm{up}(x - k)$$

$$n = 1 \ \rightarrow \ x = 2^{-2} \sum_{k=-\infty}^{\infty} k \, \mathrm{up}(x - k/2) \qquad (38)$$

$$n = 2 \ \rightarrow \ x^2 = \frac{2^{-6}}{9} \sum_{k=-\infty}^{\infty} (9k^2 - 16) \, \mathrm{up}(x - k/4)$$

Fig. 6 shows distribution of basis functions obtained by displacement of function up(x) by $k \cdot 2^{-n}$, $k \in Z$. According to (38), polynomials of $n = 0{,}1$ and 2 degrees can be expressed exactly as a linear combination of those basis functions on the interval $\Delta x_n = 2^{-n}$.

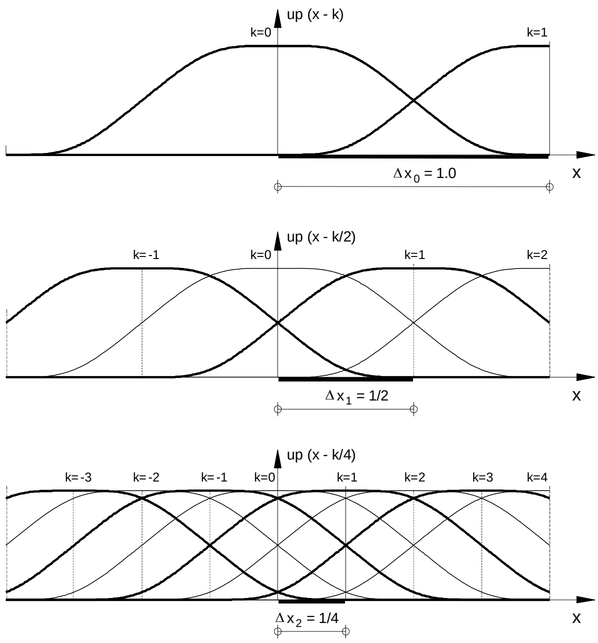


Fig. 6 Distribution of basis functions for an exact description of 0,1 and 2 degree polynomials

## 3.7 Scalar product of mutually displaced functions up(x)

When functions up(x) are selected as basis and test functions, scalar product of the functions shall be calculated as:

$$\int_{k2^{-n}}^{(k+1)2^{-n}} \mathrm{up}(x - \alpha) \, \mathrm{up}(x) dx \ , \ \ k \in Z \ ; \ n = 0,1,... \qquad (39)$$

Using the connection between function up(x) and polynomials of *n*-th degree, according to section 3.6, it can be observed that the integral value (39) over different integration areas $\Delta_n$ depends on coefficient γ only:

$$\gamma = \int_{-1}^{1} \mathrm{up}^2(t) dt$$

Value of coefficient γ can be calculated only numerically. According to [4], coefficient γ value is:

$$\gamma = 0.8088735359678874110416 60 \qquad (40)$$

## 3.8 Vector space of functions $UP_n$

Polynomial of *n*–th degree, as shown in Section 3.6, can be expressed as linear combination of basis functions obtained by displacement of function up(x). Each basis function $\varphi_k(x)$ is obtained by displacement of function up(x) along the abscise by value $k \cdot 2^{-n}$, therefore:

$$\varphi_k(x) = \mathrm{up}(x - k2^{-n}) \ , \ k \in Z \ , \ n = 0{,}1, \ldots$$

Power *n* determines the highest degree of a polynomial which can be expressed exactly as a linear combination of basis functions $\varphi_k(x)$ according to (36). Coefficient *k* measures the displacement of function up(x) in reference to the origin of a global coordinate system with a step $2^{-n}$, which gives a basis function $\varphi_k(x)$ (Fig. 7). Therefore, *k* is a global index of basis function.

As it can be observed in Fig. 7, for en exact description of the monomial $x^n$ on the interval of length $2^{-n}$, $2^{n+1}$ basis functions are required. In that case, dimension of the vector space is:

$$\dim UP_n = 2^{n+1}$$

For an exact description of the monomial $x^{n+1}$, $2^{n+2}$ basis function are required i.e. dimension of a vector space $UP_{n+1}$ is $2^{n+2}$. Therefore, linear vector space of functions $UP_{n+1}$ contains the space $UP_n$ because it is obtained by expansion of $UP_n$ space by $2^{n+1}$ linearly independent vectors i.e. displaced functions up(x). Therefore, in difference from the space built of basis splines, space of functions $UP_n$ is **universal**, i.e.:

$$UP_0 \subset UP_1 \subset \ldots \subset UP_n \subset UP_{n+1}$$

# 4. FUNCTION $\mathbf{Fup_n(x)}$

Functions $Fup_n(x)$, which also belong to a class of $R_{bf}$ functions, maintain all the properties of function up(x). For the development of a function, less basis functions $Fup_n(x)$ are needed when compared to basis functions obtained by displacement of a function up(x). Functions $Fup_n(x)$ are finite functions of $C^\infty$ class with a compact support. They are the elements of linear vector space $UP_n$. Index *n* denotes the highest degree of the polynomial which can be exactly expressed as linear combination of basis functions obtained by displacement of function $Fup_n(x)$ by a characteristic interval $2^{-n}$.

The function $Fup_n(x)$ support is:

$$\text{supp}\, Fup_n(x) = \left[-(n+2)\,2^{-n-1}\,;\,(n+2)\,2^{-n-1}\right] \tag{41}$$

For an *n* high enough, the function $Fup_n(x)$ support is very short. Therefore, each function $Fup_k(x), k < n$, including the function up(x), can be expressed using the function $Fup_n(x)$.

When n = 0:

$$Fup_0(x) = up(x)$$

Functions $Fup_n(x)$ can be defined as finite solutions of differential functional equations, similarly as function up(x) (Section 3.). Using an analogue procedure as for the function up(x), a general form of the Fourier transform $F_n(t)$ for the function $Fup_n(x)$ is obtained:

$$F_n(t) = \left(\frac{\sin t2^{-n-1}}{t2^{-n-1}}\right)^{n+1} \prod_{j=n+2}^{\infty} \frac{\sin t2^{-j}}{t2^{-j}} \tag{42}$$

Function $Fup_n(x)$ can be written as an integral:

$$Fup_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx}\, F_n(t)\, dt \tag{43}$$

Also, functions $Fup_n(x)$ can be expresses using the spline function $B_n(x)$ and compresses function up(x) if convolution theorem is applied:

$$\begin{aligned} Fup_n(x) &= B_n(2^{n+1}x) * up(2^{n+1}x) = \\ &= B_{n+1}(2^{n+1}x) * up(2^n x) \end{aligned} \tag{44}$$

Expressions (43) and (44) are numerically unsuitable for calculation of function $Fup_n(x)$ values. The best would be to construct the functions $Fup_n(x)$ in the form of linear combination of displaced up(x) functions.

## 4.1 $\mathbf{Fup_n(x)}$ as liner combination of displaced up(x) functions

According to (41), function $Fup_n(x)$ support is a interval consisted of (n+2) subintervals with the length of $2^{-n}$. Based on general form of Fourier transform (42) and a possibility to express the function $Fup_n(x)$ using the convolution theorem (44), the following expression for the function $Fup_n(x)$ values on the first subinterval is obtained:

$$Fup_n(x) = 2^{C_{n+1}^2}\, up\left(x - 1 + (n+2)2^{-n-1}\right) \tag{45}$$

The formula shows that on the first subinterval, the function $Fup_n(x)$ is proportional to the function up(x) values on the interval $\left[-1, -1+2^{-n}\right]$. On other subintervals, function $Fup_n(x)$ is a linear combination of functions up(x), displaced in reference to each other by a characteristic interval $2^{-n}$.

$$Fup_n(x) = \sum_{k=0}^{\infty} C_k(n)\, up\left(x - 1 - \frac{k}{2^n} + \frac{n+2}{2^{n+1}}\right) \tag{46}$$

Coefficient $C_0(n)$ is derived from (45), so:

$$C_0(n) = 2^{C_{n+1}^2} = 2^{n(n+1)/2} \tag{47}$$

Other coefficients are calculated in [4], in the form $C_k(n) = C_0(n) \cdot C'_k(n)$, where a recursive formula is used for calculation of auxiliary coefficients $C'_k(n)$:

$$\begin{aligned} &C'_0(n) = 1, \text{when } k = 0; \;\; \text{i.e. when } k > 0 \\ &C'_k(n) = (-1)^k C_{n+1}^k - \sum_{j=1}^{\min\{k;2^{n+1}-1\}} C'_{k-j}(n) \cdot \delta_{j+1} \end{aligned} \tag{48}$$

Table 1. Coefficients $C'_k(n)$ when $n \le 6$ and $k \le 9$

| n \ $C'_k(n)$ | $C'_0$ | $C'_1$ | $C'_2$ | $C'_3$ | $C'_4$ | $C'_5$ | $C'_6$ | $C'_7$ | $C'_8$ | $C'_9$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 |
| 2 | 1 | -2 | 2 | -2 | 3 | -4 | 4 | -4 | 5 | -6 |
| 3 | 1 | -3 | 4 | -4 | 5 | -7 | 8 | -8 | 10 | -14 |
| 4 | 1 | -4 | 7 | -8 | 9 | -12 | 15 | -16 | 18 | -24 |
| 5 | 1 | -5 | 11 | -15 | 17 | -21 | 27 | -31 | 34 | -42 |
| 6 | 1 | -6 | 16 | -26 | 32 | -38 | 48 | -58 | 65 | -76 |

Derivatives of the function $Fup_n(x)$ are also obtained by liner combination of derivatives of displaced functions up(x) using the coefficients given in Table 1.

Fig. 7 shows function $Fup_2(x)$ and its first three derivatives. It can be observed that intervals of the third derivative correspond to compressed function up(x).

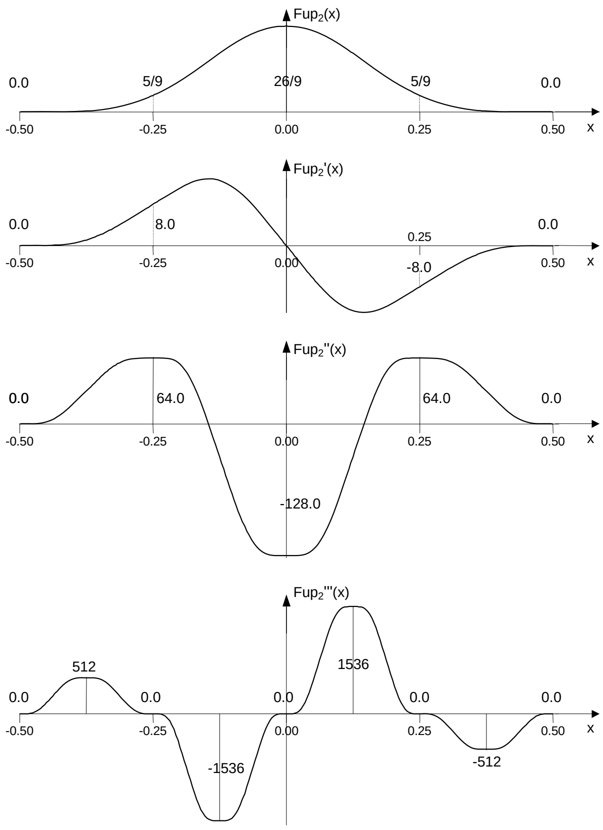


Fig. 7 Function $Fup_2(x)$ and its first three derivatives

### 4.2 Development of polynomials over a base of displaced $Fup_n(x)$ functions

Linear combination of displaced $Fup_n(x)$ functions in the form:

$$\varphi(x) = \sum_{k=-\infty}^{\infty} D_k \, Fup_n(x - k2^{-n})$$

is a polynomial of n-th degree if coefficients $D_k$ are the polynomials of degree *n* of index *k*. Coefficients $D_k$ are determined similarly as coefficients for development of polynomials over a base of displaced up(x) functions (see Section 3.6).

Therefore, a polynomial of n-th degree can be exactly expressed on $2^{-n}$ long interval, as a linear combination of basis functions obtained by function $Fup_n(x)$ displacement over a characteristic interval $\Delta x_n = 2^{-n}$. Fig. 8 shows distributions of displaced functions $Fup_n(x)$, $n = 0,1,\ldots,4$ for an exact development of a polynomials to *n*–th degree. For example, using the function $Fup_0(x)$, a polynomial of 0 degree can be exactly developed over the interval [0,1] (Fig. 8a):

$$x^0 = \sum_{k=0}^{1} Fup_0(x - k)$$

Arranging the basis functions $Fup_n\left(\frac{x}{2^n \Delta x}\right)$ at equidistances of an arbitrary length $\Delta x$, a polynomial of $m = 0,1, \ldots , n$ degree can be exactly developed over the entire real axis in the following form:

$$x^m = 2^{-n} \cdot \Delta x^m \sum_{k=-\infty}^{\infty} D_k(m) \, Fup_n\left(\frac{x}{2^n \Delta x} - \frac{k}{2^n}\right) \qquad (49)$$

Coefficients $D_k(m)$ for a base of mutually displaced and compressed functions $Fup_n\left(\frac{x}{2^n \Delta x}\right)$ are determined using the following formulas given in [4]:

$n = 0 \rightarrow D_k(0) = 1$

$n = 1 \rightarrow D_k(0) = 1$

$$D_k(1) = k_* , \quad k_* = \frac{2k+1}{2}$$

$n = 2 \rightarrow D_k(0) = 1$ , $D_k(1) = k$ , $D_k(2) = k^2 - 5/18$

$n = 3 \rightarrow D_k(0) = 1$ , $D_k(1) = k_*$ ,

$$D_k(2) = k_*^2 - 13/36 , \; D_k(3) = k_*^3 - 13k_*/12$$

$n = 4 \rightarrow D_k(0) = 1$ , $D_k(1) = k$ , $D_k(2) = k^2 - 4/9$ ,

$$D_k(3) = k^3 - 4k/3 , \; D_k(4) = k^4 - 8k^2/3 + 857/1350$$

It can be observed that (n+2) basis function are needed to exactly develop a polynomial of *n*–th degree on the interval of length $\Delta x$.

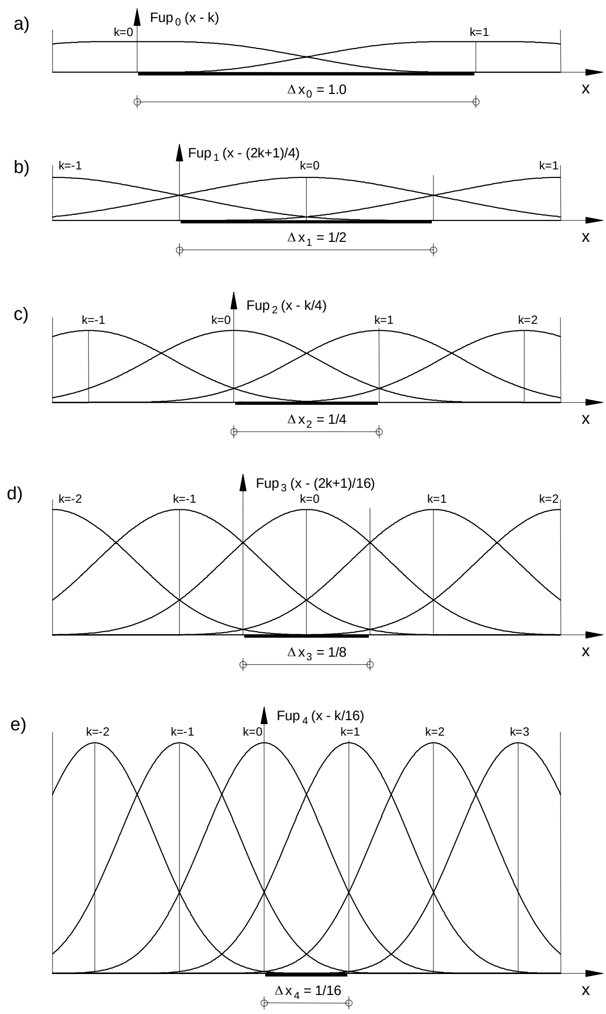


Fig. 8 Development of polynomials over a base of displaced functions $Fup_n(x)$

### 4.3 Application of functions $Fup_n(x)$

Functions $Fup_n(x)$ can be applied in practice for approximation of a function with a higher degree smoothness regardless of if it is given directly or it is a solution of a certain physical problem.

Approximation properties of basis functions $Fup_2(x)$ are illustrated in the following example. Function $f(x) = 1 + \cos(\pi x)$, $x \in [0,1]$ and its second derivative are approximated, using two different bases; one constructed of the functions $Fup_2(x)$ displaced in respect to each other, and the other formed by Hermite polynomials $H_3^1(x)$. Arrangements of basis functions are given in Figures 9a. and 9b. It can be observed that for each of the bases, four basis functions are used over a given domain. A base formed by functions $Fup_2(x)$ consists of second degree polynomials while base formed by Hermite polynomials consists of third degree polynomials.

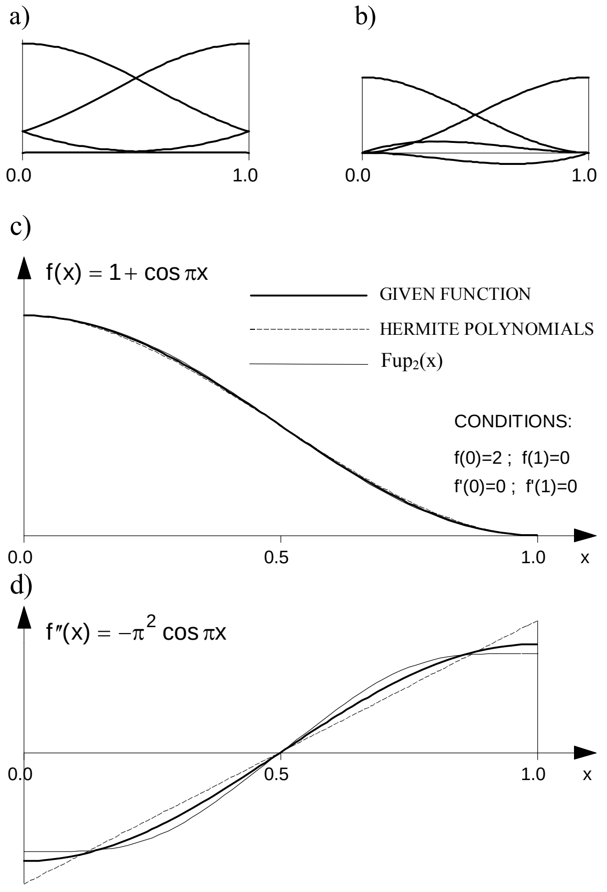


Fig. 9 a) Base $Fup_2(x)$; b) Base $H_3^1(x)$;
c) Approximation of a function f(x);
d) Approximation of the second derivative f ''(x)

Fig. 9c shows that both bases provide good approximation of a given function. However, base formed of functions $Fup_2(x)$, although consists of polynomials of lower degree than functions $H_3^1(x)$, provides significantly better approximation of the given function second derivative than the one obtained using Hermite polynomials, as illustrated in Fig. 9d. This good approximation properties of basis functions $Fup_2(x)$ are especially valuable for solving practical engineering problems where good approximation of higher order derivatives of solution function is far more important than the approximation of solution function itself.

## 5. FUNCTION $y_\omega(x)$

Numerical solutions of physical problems having exponential functional law can not be adequately described by basis functions from algebraic polynomials space e.g. by functions $Fup_n(x)$. For that class of problems, basis functions $y_\omega(x)$ are developed, the linear combination of which can describe exactly the exponential function $f(x) = e^{\omega x}$.

Starting from the observation of finite solutions of equation (16), similar as for the function up(x), differential equation for determination of basis function $y_\omega(x)$ is obtained as:

$$y'_\omega(x) - \omega\, y_\omega(x) = a_\omega\, y_\omega(2x+1) - b_\omega\, y_\omega(2x-1) \quad (50)$$

where coefficients are:

$$a_\omega = \frac{\omega e^{-\omega/2}}{sh(\omega/2)} \;;\; b_\omega = a_\omega \cdot e^{\omega} \quad (50a)$$

For the frequency $\omega = 0$, the following is obtained $a_0 = 2$, $b_0 = 2$. In that case equation (50) is equivalent to equation (18).

For the function $y_\omega(x)$, as it was for function up(x), a support $\mathrm{supp}\; y_\omega(x) = [-1,1]$ is selected together with normed condition $\int_{-\infty}^{\infty} y_\omega(x)\,dx = 1$. Therefore, a solution of the equation (50) can be obtained in the following form:

$$y_\omega(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx}\, \hat{y}_\omega(t)\, dt$$

where $\hat{y}_\omega(t)$ is the Fourier transform of the function $y_\omega(x)$, constructed using similar procedure as for the function up(x):

$$\hat{y}_\omega(t) = \prod_{j=1}^{\infty} \frac{\omega}{2 sh(\omega/2)} \frac{sh(\omega/2 + it/2^j)}{\omega/2 + it/2^j} \quad (51)$$

Also, using similar procedure as for the function up(x), the following important relations can be calculated for the function $y_\omega(x)$:

- **Basis function value in the point $x=0$**:

$$y_\omega(0) = \frac{\omega}{2 sh(\omega/2)} \cdot \hat{y}_\omega(i\omega/2) \quad (52)$$

- **Function $y_\omega(x)$ value in characteristic points** $x = -1 + k \cdot 2^{-m}$, $k = 1, \ldots, 2^{m+1}$, where $\Delta x = 2^{-m}$, $m \in N$:

$\rightarrow$ when $x \in (-1, 0]$:

$$y_\omega(-1 + k \cdot 2^{-m}) = \sum_{p=0}^{m} \alpha_p 2^{\frac{m(m-1)}{2}-1} \cdot \hat{y}_\omega\left(i\omega 2^{p-m-1}\right) \cdot \sum_{r=1}^{k} D_r^{(m)} e^{2^{p-m-1}\omega(2k-2r+1)} \quad (53)$$

where:

$$\alpha_p = \frac{\omega^{-m}}{\prod\limits_{\substack{j=0\\ p\neq j}}^{m}(2^p - 2^j)} \quad ; \quad \begin{array}{l} \text{za} \;\; m = 0 \to \alpha_0 = 1 \\ m \in N \end{array}$$

$$D_r^{(m)} = D_k^{(0)}\, D_e^{(m-1)} \quad ; \quad \begin{array}{l} r = 1, \ldots, 2^{m+1} \\ k = 1,2 \\ e = 1, \ldots, 2^m \end{array}$$

Coefficients $D_k^{(0)}$ correspond to coefficients in equation (50):

$$D_1^{(0)} = a_\omega \;\; ; \;\; D_2^{(0)} = -b_\omega$$

→ when $x \in (0,1]$:

$$y_\omega(x) = y_\omega(0)\cdot e^{\omega x} - e^{\omega} y_\omega(x-1) \qquad (54)$$

- **Function $y_\omega(x)$ derivatives** are obtained by applying the differential operator:

$$L_m = \prod_{j=0}^{m}\left(\frac{d}{dx} - 2^j\omega\right)$$

to equation (50):

$$L_m y_\omega(x) = 2^{m(m+1)/2} \sum_{j=1}^{2^{m+1}} D_j^{(m)} y_\omega\left(2^{m+1}x - 2j + 2^{m+1} + 1\right) \qquad (55)$$

Required derivative is calculated from the left side of equation (55), e.g..:

$$y'_\omega(x) = \omega\, y_\omega(x) + a_\omega y_\omega(2x+1) - b_\omega y_\omega(2x-1)$$

$$y''_\omega(x) = 3\,\omega\, y'_\omega(x) - 2\,\omega^2 y_\omega(x) + 2\sum_{j=1}^{4} D_j^{(1)} y_\omega(4x - 2j + 5)$$

… … … … … … … … … … …

Fig. 10 shows function $y_\omega(x)$ and its first two derivatives for the frequency value $\omega = 1$.

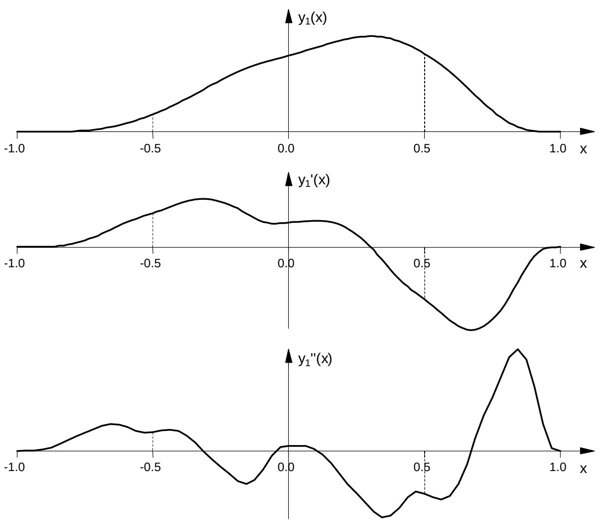


Fig. 10 Function $y_1(x)$ and its first two derivatives

Numerical values of the graphs are also given in table form in Appendix - Table A2., in characteristic points $x_k = -1 + k\cdot 2^{-3}$, $k = 0,1,2,\ldots,16$.

## 5.1 Illustrative example

Equation $y' = y + x$, complete with initial condition $y(0) = 1$, is often used for testing of different numerical integration procedures of common differential equations. Equation can also be written in the following form:

$$y'(x) - y(x) = x \;\; ; \;\; y(0) = 1 \qquad (56)$$

suitable for determination of basis functions frequencies (by comparison with equation (50), $\omega = 1$), and required derivatives from given initial conditions, e.g. $y'(0) = 1$, $y''(0) = 2$, $y'''(0) = 2$, etc.

Since the required solution must belong to a class of exponential functions, which results from the left side of the equation (56), and algebraic polynomials, which results from the right side of equation (56), numerical solution is sought in the form of linear combination of basis functions $y_\omega(x)$ and $Fup_1(x)$:

$$\tilde{y}(x(= \sum_{k=0}^{\infty} C_k y_\omega(x-k) + \sum_{\ell=-1}^{\infty} D_\ell Fup_1\left(\frac{x}{2} - \frac{2\ell+1}{2}\right) \qquad (57)$$

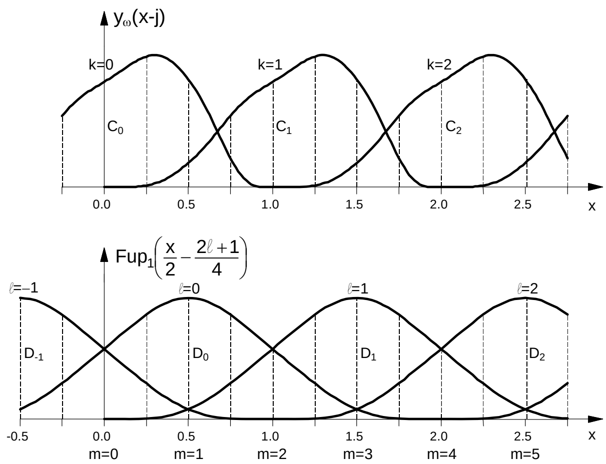


Fig. 11 Distribution of basis functions in conformity with (57)

It is enough to obtain particular values of basis functions and their derivatives in points $x_m = 1/2\cdot m$, $m = 0,1,\ldots$ , which are given in Appendix, see Tables A2. and A3. Values of coefficients $C_0$, $D_{-1}$ and $D_0$ are determined from the given initial conditions as a solution of the following equation system:

$$\left.\begin{array}{r} D_{-1} + D_0 + y_1(0)\cdot C_0 = 1 \\ -2\,D_{-1} + 2\,D_0 + y'_1(0)\cdot C_0 = 1 \\ y''_1(0)\cdot C_0 = 2 \end{array}\right\} \quad \left.\begin{array}{l} D_{-1} = -1/4 \\ D_0 = -3/4 \\ C_0 = 2/y_1(0) \end{array}\right\} \qquad (58)$$

For determination of subsequent coefficients, conditional equations $y'(x_m) = y(x_m) + x_m$ are written in points $x_1 = 0.5$ and $x_2 = 1.0$. The following values of coefficients $D_1$ and $C_1$ are obtained:

$$D_1 = x_2 + 3D_0 = 1 + 3\cdot(-3/4) = -5/4$$
$$C_1 = \frac{y_1(0.5) - y_1'(0.5)}{y_1'(-0.5) - y_1(-0.5)} C_0 = 2.71828\ldots \cdot C_0 = e\cdot\frac{2}{y_1(0)} \tag{59}$$

Expressions (59) are the recursive formulas for calculation of all subsequent coefficients:

$$D_\ell = x_{2\ell} + 3D_{\ell-1}$$
$$C_k = e\cdot C_{k-1} \tag{60}$$

It is easy to observe that coefficients $C_k$ from (60) introduced into the first sum of (57) give exactly $2e^x$, and coefficients $D_\ell$ from (60) introduced in the second sum of (57) according to (49) give algebraic polynomial $-x-1$. Therefore, approximate solution (57) is obtained in a closed form:

$$\tilde{y}(x) = 2e^x - x - 1 \tag{61}$$

Therefore, solution (61), obtained numerically, is also an exact solution of problem (56).

## 6. FUNCTION $y_{\omega,h}(x)$

For approximate solutions belonging to a class of trigonometric functions or containing trigonometric functions, finite basis functions $y_{\omega,h}(x)$ are developed. They can be obtained as a solution of equation (16) written in the following form:

$$y''_{\omega,h}(x) + \omega^2 y_{\omega,h}(x) = a\, y_{\omega,h}(3x+2h) - b\, y_{\omega,h}(3x) + a\, y_{\omega,h}(3x-2h) \tag{62}$$

where $\omega$ is a frequency, $h$ is the length of the half of function $y_{\omega,h}(x)$ support, while coefficients $a$ and $b$ are:

$$a = \frac{3}{2}\cdot\frac{\omega^2}{1-\cos(2\omega h/3)}\quad,\quad b = 2a\cos(2\omega h/3)$$

Function $y_{\omega,h}(x)$ support is selected in dependence of the value of frequency $\omega$:

$$\operatorname{supp}\ y_{\omega,h}(x) = [-h,h]$$

Finite solution of equation (62) must satisfy the normed condition (12):

$$\int_{-h}^{h} y_{\omega,h}(x)\,dx = 1$$

and in that case has the following form:

$$y_{\omega,h}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx}\,\hat{y}_{\omega,h}(t)\,dt$$

where $\hat{y}_{\omega,h}(t)$ is the Fourier transform of function $y_{\omega,h}(x)$:

$$\hat{y}_{\omega,h}(t) = \prod_{j=1}^{\infty}\left\{\frac{2}{3}a\frac{\cos(2th/3^j) - \cos(2\omega h/3)}{\omega^2 - t^2/9^{j-1}}\right\} \tag{63}$$

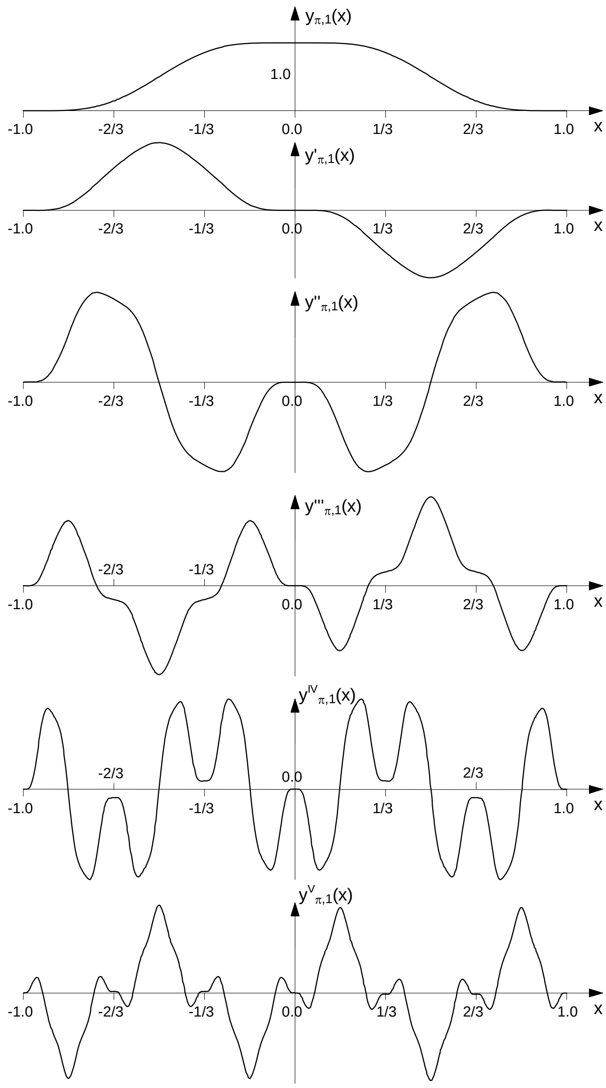


Fig. 12 Function $y_{\pi,1}(x)$ and its first five derivatives

### 6.1 Characteristic points of $y_{\omega,h}(x)$ function

Length of the finite function $y_{\omega,h}(x)$ support can be expressed as a sum of infinite number of intervals with the length $2h3^{-j}$, j = 1,2, … , ∞. The support is compact, and in points $x_k = -h + k\cdot(2h3^{-j-1})$, $k = 1,\ldots,3^{j+1}$, $j = 0,1,2,\ldots$, similar as for the function up(x) in binary-rational points, en exact solution of equation (62) can be obtained in trinary-rational points. Distance between characteristic points determines a displacement of a basis function in order to obtain a suitable base i.e. for $j = 0$ distance between characteristic points is $2h/3$. In such base, a function $\varphi(x)$ can be developed as:

$$\varphi(x) = \sum_{k=-\infty}^{\infty} C_k y_{\omega,h}\left(x - \frac{2hk}{3}\right)$$

## 6.2 Connection between trigonometric and basis $y_{\omega,h}(x)$ functions

Trigonometric functions of the given frequency $\omega$, can be described exactly in a base of displaced $y_{\omega,h}(x)$ functions according to [4] as follows:

$$\sin(\omega x)=\frac{1}{2\omega\sin\frac{2\omega h}{3}\,y'_{\omega,h}\left(-\frac{2h}{3}\right)}\cdot\sum_{k=-\infty}^{\infty}\sin\left(\frac{2\omega hk}{3}\right)\cdot y_{\omega,h}\left(x-\frac{2hk}{3}\right)$$

$$\cos(\omega x)=\frac{1}{y_{\omega,h}(0)+2\cos\frac{2\omega h}{3}\,y_{\omega,h}\left(-\frac{2h}{3}\right)}\cdot\sum_{k=-\infty}^{\infty}\cos\left(\frac{2\omega hk}{3}\right)\cdot y_{\omega,h}\left(x-\frac{2hk}{3}\right)$$

## 6.3 Values of the function $y_{\omega,h}(x)$ and its first derivative in characteristic points

Ref. [4] gives an expressions for calculation of the first derivative and function $y_{\omega,h}(x_k)$ in characteristic points $x_k=-h+2k3^{-m-1}$, $k=1,\ldots,3^{m+1}$, $m$ = 0,1,2, … .

$$y_{\omega,h}(x_k)=\sum_{p=0}^{m}\frac{\alpha_p}{\omega}3^{m^2-p-1}\cdot\sum_{r=1}^{k}C_r^{(m)}\sin(\gamma_m)\hat{y}_{\omega,h}\left(\frac{\omega}{3^{m-p+1}}\right)$$

$$y'_{\omega,h}(x_k)=\sum_{p=0}^{m}\alpha_p 3^{m^2-1}\cdot\sum_{r=1}^{k}C_r^{(m)}\cos(\gamma_m)\hat{y}_{\omega,h}\left(\frac{\omega}{3^{m-p+1}}\right) \tag{64}$$

where:

$$\alpha_p=(-1)^m\cdot\left(\omega^{2m}\prod_{\substack{j=0\\p\neq j}}^{m}(3^{2p}-3^{2j})\right)^{-1}$$

$$\gamma_m=(2(k-r)+1)\omega h/3^{m-p+1}$$

$$\left.\begin{aligned}C_r^{(m)}=C_s^{(0)}\,C_q^{(m-1)}\quad;\quad & r=1,\ldots,3^{m+1}\\ & s=1,2,3\\ & q=1,\ldots,3^m\end{aligned}\right\}\text{ for } m>0$$

$$C_1^{(0)}=a\;,\;C_2^{(0)}=-b\;,\;C_3^{(0)}=a\;\Big\}\text{ for } m=0$$

In halves of the distances between characteristic points $x_k^*=-h+(2k-1)h3^{-m-1}$, $k=1,\ldots,3^{m+1}$, $m$ = 0,1,2, …, values of the function and its first derivative are calculated using the following formulas:

$$y_{\omega,h}(x_1^*)=y_{\omega,h}(-h+h/3^{m+1})\;;\;y'_{\omega,h}(x_1^*)=y'_{\omega,h}(-h+h/3^{m+1})$$

$$y_{\omega,h}(x_k^*)=\sum_{p=0}^{m}\frac{\alpha_p}{\omega}3^{m^2-p-1}\cdot\sum_{r=2}^{k}C_r^{(m)}\sin(\gamma_m^*)\,\hat{y}_{\omega,h}\left(\frac{\omega}{3^{m-p+1}}\right)+y_{\omega,h}(x_1^*)$$

$$y'_{\omega,h}(x_k^*)=\sum_{p=0}^{m}\alpha_p 3^{m^2-1}\cdot\sum_{r=2}^{k}C_r^{(m)}\cos(\gamma_m^*)\,\hat{y}_{\omega,h}\left(\frac{\omega}{3^{m-p+1}}\right)-y'_{\omega,h}(x_1^*) \tag{65}$$

where: $\gamma_m^*=2(k-r)\,\omega h/3^{m-p+1}$,

$$y_{\omega,h}(-h+h/3^{m+1})=\sum_{p=0}^{m}\frac{\alpha_p}{\omega}3^{m^2-p-1}a^{m+1}\cdot\Big\{-\frac{a}{b}\left(\frac{1}{3^{m-p}}\sin\frac{2\omega h}{3}\hat{y}\left(\frac{\omega}{3}\right)-\right.$$
$$\left.-\sin\frac{2\omega h}{3^{m-p+1}}\hat{y}\left(\frac{\omega}{3^{m-p+1}}\right)\right)+\frac{\omega}{a3^{m-p-1}}y\left(-\frac{2h}{3}\right)-\sum_{i=1}^{m-p}\prod_{j=i}^{m-p}\left(\frac{3^{2j}-1}{3^{2j-1}}\cdot\frac{\omega^2}{b}\right)\cdot$$
$$\cdot\left[\frac{a}{b}\left(\frac{1}{3^{i-1}}\sin\frac{2\omega h}{3}\hat{y}\left(\frac{\omega}{3}\right)-\sin\frac{2\omega h}{3^i}\hat{y}\left(\frac{\omega}{3^i}\right)\right)-\frac{\omega}{a3^{i-2}}y\left(-\frac{2h}{3}\right)\right]\Big\}\quad,$$

$$y'_{\omega,h}\left(-h+\frac{h}{3^{m+1}}\right)=\sum_{p=0}^{m}\alpha_p a^{m+1}3^{m^2-1}\hat{y}\left(\frac{\omega}{3^{m-p+1}}\right)$$

Values of the first derivative $y'_{\omega,h}(-2h/3)$ and the function itself $y_{\omega,h}(-2h/3)$ are:

$$y'_{\omega,h}(-2h/3)=\frac{a}{6}\,\hat{y}\left(\frac{\omega}{3}\right)$$

$$y_{\omega,h}(-2h/3)=\frac{a^2}{3\omega b}\left\{\sum_{n=1}^{\infty}A_n\left[\sin\frac{2\omega h}{3}\,\hat{y}\left(\frac{\omega}{3}\right)-\right.\right.$$
$$\left.\left.-3^n\sin\frac{2\omega h}{3^{n+1}}\,\hat{y}\left(\frac{\omega}{3^{n+1}}\right)\right]\right\}\Big/\left(1+\sum_{n=1}^{\infty}A_n\right)$$

where: $A_n=\left[3^{n(n-1)}\cdot b^n\right]\Big/\left[\omega^{2n}\prod_{k=1}^{n}(1-3^{2k})\right]$

## 6.4 Function $y_{\omega,h}(x)$ derivatives of higher order

Following the calculation of function $y_{\omega,h}(x)$ and first derivative values in characteristic points and halves of the distances between characteristic points, derivatives of higher degree are calculated using differential operators in the following form:

$$L_m y_{\omega,h}(x)=3^{m(m+1)}\sum_{j=1}^{3^{m+1}}C_j^{(m)}y_{\omega,h}(3^{m+1}x-2jh+3^{m+1}h+h)$$

$$L_m y'_{\omega,h}(x)=3^{(m+1)^2}\sum_{j=1}^{3^{m+1}}C_j^{(m)}y'_{\omega,h}(3^{m+1}x-2jh+3^{m+1}h+h)$$

Thus, the following is obtained e.g.:

$$y_{\omega,h}^{(2)}(x)=-\omega^2\,y_{\omega,h}(x)+L_0\,y_{\omega,h}(x)$$

$$y_{\omega,h}^{(3)}(x)=-\omega^2\,y'_{\omega,h}(x)+L_0\,y'_{\omega,h}(x)$$

$$y_{\omega,h}^{(4)}(x)=-10\omega^2\,y''_{\omega,h}(x)-9\omega^4\,y_{\omega,h}(x)+L_1\,y_{\omega,h}(x)$$

$$y_{\omega,h}^{(5)}(x)=-10\omega^2\,y'''_{\omega,h}(x)-9\omega^4\,y'_{\omega,h}(x)+L_1\,y'_{\omega,h}(x)\text{ , itd.}$$

## 6.5 Numerical example

A problem of free oscillations of one-degree dynamic system is analysed. Free oscillations of one-degree system of given characteristics are described by a differential equation:

$$\ddot{x}(t)+\pi^2 x(t)=0 \tag{66}$$

and given initial conditions:

$$x_0=1\;;\;\dot{x}_0=\pi\;\rightarrow\;\ddot{x}_0=-\pi^2 \tag{66a}$$

Numerical solution is sought in the following form:

$$x(t) = \sum_{k=-1}^{\infty} C_k \; y_{\pi,1}(t - 2k/3) \tag{67}$$

for which the distribution of basis functions is given in Fig. 13. By satisfying the initial conditions (66a), and using values of function $y_{\pi,1}(t)$ given in Table A3., the values of first three coefficients of the solution can be calculated as:

$$C_{-1} = C_1 = \frac{\pi}{2y'(2/3)} - \frac{\pi^2}{2y''(2/3)} \;;\; C_0 = y(0) + \frac{\pi^2}{y''(2/3)} \cdot y(2/3)$$

Subsequent coefficients are obtained from the condition that problem equation (66) is satisfied in the moments $k\,\Delta t$, $k \in N$:

$$C_{k-1}\left[y''(2/3) + \pi^2 y(2/3)\right] + C_k \pi^2 y(0) + C_{k+1}\left[y''(-2/3) + \pi^2 y(-2/3)\right] = 0 \tag{68}$$

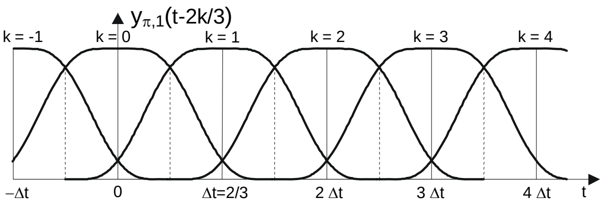


Fig. 13 Distribution of basis functions $y_{\pi,1}(t)$

Introducing the values given in Table A3. into (68), correlation between coefficients is obtained:

$$C_{k+1} = -C_{k-1} - C_k \tag{69}$$

Introducing calculated coefficients into general numerical solution (67), an approximate solution is obtained, which corresponds to an exact solution in every moment:

$$x(t) = \cos(\pi t) + \sin(\pi t)$$

Simple derivation of equation (67) provides numerical solutions for velocity and acceleration of a material point that oscillates. Those solutions also correspond to exact solutions:

$$\dot{x}(t) = \sum_{k=-1}^{\infty} C_k \; y'_{\pi,1}(t - 2k/3)$$

$$\ddot{x}(t) = \sum_{k=-1}^{\infty} C_k \; y''_{\pi,1}(t - 2k/3)$$

## 7. CONCLUSION

The paper presents a new class of basis functions $R_{bf}$. It is shown that functions $R_{bf}$ maintain good properties of algebraic and trigonometric polynomials i.e. universality of the space they form. At the same time, they are finite functions i.e. close to spline functions with regard to good properties for numerical applications.

Due to the aforementioned properties, functions $R_{bf}$ can be applied efficiently to 1D and 2D problems using the collocation method in a point and hierarchic expansion of the base, see [10], [11]. A property of all previously described $R_{bf}$ functions is that an universal vector space can be formed using only one basis function. Also, any order of a derivative can be calculate using the values of function itself. In Appendix is given a FORTRAN code for the calculation of function up(x) values. It shall be mentioned that for application in practice it is more economical to calculate the values of several first derivatives of function up(x) in binary-rational points at distances e.g. $\Delta x = 2^{-10}$ and keep them as a database. This enables very fast and accurate calculation of function $Fup_n(x)$ values and all required derivatives. An orthonormalized system of Wavelet-type [9] can be formed using functions $Fup_n(x)$ which enables efficient numerical solution of a wide range of technical problems.

In this paper, for the first time, exact numerical solutions of the problems described by differential equations (56) and (66), are determined using $R_{bf}$ basis functions.

## 8. APPENDIX

FORTRAN Code for the function up(x) values

```
C.........................................................
C
C    Functional subprogram for the calculation of function
C    up(x) values in an arbitrary point x ∈ (-∞,∞)
C
C.........................................................
     REAL*8 FUNCTION UPX(X)
     IMPLICIT REAL*8 (A-H, O-Z)
     DIMENSION UN(0:10),XK(10),FAK(0:10),DIV(0:10),UNN(0:10)
     INTEGER*4 PK(10),SPK(10)
     DVA(M) = 2.0D0**M
C
     DATA DIV/ 1.0D0,  1.0D0,  5.0D0,  1.0D0, 143.0D0, 19.0D0,
    + 1153.0D0, 583.0D0,1616353.0D0,132809.0D0, 134926369.0D0/
     DATA UNN/ 1.0D0,  2.0D0,  72.0D0,  288.0D0,  2073600.0D0,
    + 33177600.0D0,   561842749440.0D0,   179789679820800.0D0,
    + 7042002179221094400000.0D0, 18027525578806001664000.0D0,
    + 1246394851358539387238350848000.0D0/,        ZERO/0.0D0/
     DATA FAK/ 1.0D0,1.0D0,2.0D0,6.0D0,24.0D0,120.0D0,720.0D0,
    + 5040.0D0,40320.0D0,362880.0D0,3628800.0D0/,   ONE/1.0D0/
C
      XX = DABS(X)
      IF(XX .GE. ONE )  THEN
      UPX = ZERO
      ELSE
             DO K = 1,10
                PK(K) = 0
                SPK(K) = 0
                XK(K) = ZERO
             END DO
                DO I = 0,10
                   UN(I) = DIV(I)/UNN(I)
                END DO
                  XK(1) = XX
                  IF(XX .GE. 0.5D0) XK(1) = XX-0.5D0
                  IF(XX .GE. 0.5D0) PK(1) = 1
                  SPK(1) = 1+PK(1)
                       DO K = 2,10
                            DVAMK = ONE/DVA(K)
                            IF(XK(K-1) .GE. DVAMK) THEN
                                  XK(K) = XK(K-1)-DVAMK
                                  PK(K) = 1
                                  SPK(K)= 1+SPK(K-1)
                            ELSE
                                  XK(K) = XK(K-1)
                                  SPK(K)= SPK(K-1)
                            END IF
                       END DO
                SUMAK = ZERO
                DO K = 1,10
                PRED = (-ONE)**SPK(K)
                            SUMA = ZERO
                            IF(PK(K) .EQ. 1) THEN
                            DO J = 0,K
     PR = DVA(J*(J+1)/2)/FAK(J)*UN(K-J) * XK(K)**J
                            SUMA = SUMA + PR
                            END DO
                            ELSE
                            CYCLE
                            END IF
                SUMAK = SUMAK + PRED * SUMA
                END DO
     UPX = ONE - SUMAK
     END IF
     END
```

Table A1. Function $Fup_1(x_k)$ values and its first two derivatives

| $x_k = -0.75 + k/16$, $k = 0,1,\dots,24$ | $Fup_1(x_k)$ | $Fup_1'(x_k)$ | $Fup_1''(x_k)$ |
|---|---|---|---|
| -.7500 | .000000000 | .000000000 | .000000000 |
| -.6875 | .000137924 | .013888889 | 1.111111111 |
| -.6250 | .006944444 | .277777778 | 8.000000000 |
| -.5625 | .045000965 | 1.013888889 | 14.888888889 |
| -.5000 | .138888889 | 2.000000000 | 16.000000000 |
| -.4375 | .295000965 | 2.986111111 | 14.888888889 |
| -.3750 | .506944444 | 3.722222222 | 8.000000000 |
| -.3125 | .750137924 | 3.986111111 | 1.111111111 |
| -.2500 | 1.000000000 | 4.000000000 | .000000000 |
| -.1875 | 1.249724151 | 3.972222222 | -2.222222222 |
| -.1250 | 1.486111111 | 3.444444444 | -16.000000000 |
| -.0625 | 1.659998071 | 1.972222222 | -29.777777778 |
| .0000 | 1.722222222 | .000000000 | -32.000000000 |
| .0625 | 1.659998071 | -1.972222222 | -29.777777778 |
| .1250 | 1.486111111 | -3.444444444 | -16.000000000 |
| .1875 | 1.249724151 | -3.972222222 | -2.222222222 |
| .2500 | 1.000000000 | -4.000000000 | .000000000 |
| .3125 | .750137924 | -3.986111111 | 1.111111111 |
| .3750 | .506944444 | -3.722222222 | 8.000000000 |
| .4375 | .295000965 | -2.986111111 | 14.888888889 |
| .5000 | .138888889 | -2.000000000 | 16.000000000 |
| .5625 | .045000965 | -1.013888889 | 14.888888889 |
| .6250 | .006944444 | -.277777778 | 8.000000000 |
| .6875 | .000137924 | -.013888889 | 1.111111111 |
| .7500 | .000000000 | .000000000 | .000000000 |

Table A2. Function $y_1(x_k)$ values and its first two derivatives

| $x_k = -1 + k/8$, $k = 0,1,\dots,16$ | $y_1(x_k)$ | $y_1'(x_k)$ | $y_1''(x_k)$ |
|---|---|---|---|
| -1.000 | .0000000000 | .0000000000 | .0000000000 |
| -.875 | .0003379933 | .0162116846 | .6038400194 |
| -.750 | .0136377376 | .2524277724 | 3.1575416837 |
| -.625 | .0760283452 | .7821474164 | 4.6856711472 |
| -.500 | .2051542888 | 1.2479511108 | 3.3335447548 |
| -.375 | .3888855221 | 1.6847139795 | 2.7653298028 |
| -.250 | .6066557844 | 1.6768384734 | -2.7815172781 |
| -.125 | .7821717260 | 1.0703419756 | -5.1253338699 |
| .000 | .8959094150 | .8959094150 | .8959094150 |
| .125 | 1.0142796067 | .9711304402 | -.6262089844 |
| .250 | 1.1132992456 | .4642006332 | -7.4327177214 |
| .375 | 1.0968740379 | -.8225566020 | -11.4334342266 |
| .500 | .9194377339 | -1.9151779182 | -7.5844092223 |
| .625 | .6166786946 | -2.9057482539 | -5.8431666098 |
| .750 | .2475788517 | -2.6614793051 | 9.4575881192 |
| .875 | .0230117719 | -.7603161812 | 16.0812768847 |
| 1.000 | .0000000000 | .0000000000 | .0000000000 |

Table A3. Function $y_{\pi,1}(x_k)$ values and its first two derivatives

| $x_k = -1 + k/9$, $k = 0,1,\dots,2\cdot 9$ | $y_{\pi,1}(x_k)$ | $y'_{\pi,1}(x_k)$ | $y''_{\pi,1}(x_k)$ |
|---|---|---|---|
| -1.0000000 | .000000000 | .000000000 | .000000000 |
| -.8888889 | .000529392 | .030931029 | 1.419777536 |
| -.7777778 | .026498778 | .569906388 | 8.183069521 |
| -.6666667 | .144382932 | 1.551917694 | 8.444601979 |
| -.5555556 | .364408349 | 2.346744822 | 4.848035737 |
| -.4444444 | .635591651 | 2.346744822 | -4.848035737 |
| -.3333333 | .855617068 | 1.551917694 | -8.444601979 |
| -.2222222 | .973501222 | .569906388 | -8.183069521 |
| -.1111111 | .999470608 | .030931029 | -1.419777536 |
| .0000000 | 1.000000000 | .000000000 | .000000000 |
| .1111111 | .999470608 | -.030931029 | -1.419777536 |
| .2222222 | .973501222 | -.569906388 | -8.183069521 |
| .3333333 | .855617068 | -1.551917694 | -8.444601979 |
| .4444444 | .635591651 | -2.346744822 | -4.848035737 |
| .5555556 | .364408349 | -2.346744822 | 4.848035737 |
| .6666667 | .144382932 | -1.551917694 | 8.444601979 |
| .7777778 | .026498778 | -.569906388 | 8.183069521 |
| .8888889 | .000529392 | -.030931029 | 1.419777536 |
| 1.0000000 | .000000000 | .000000000 | .000000000 |

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## O IZBORU BAZNIH FUNKCIJA U NUMERIČKOJ ANALIZI INŽENJERSKIH PROBLEMA

### SAŽETAK

U ovome radu prikazane su osnovne značajke postojećih numeričkih postupaka s aspekta izbora baznih funkcija približnih rješenja različitih tehničkih zadataka. Navedene su prednosti koje posjeduju algebarski i trigonometrijski polinomi zbog svojstva univerzalnosti vektorskih prostora koje tvore. Pokazano je također da spline funkcije imaju veoma važne praktične prednosti kao posljedice svojstva finitnosti, ali istovremeno i nedostatke zbog izgubljenog svojstva univerzalnosti, uslijed ograničene glatkosti. Bazne funkcije koje zadržavaju svojstva univerzalnosti i beskonačne diferencijabilnosti, a istovremeno čuvaju odlike praktične primjene splineova, su $R_{bf}$ - Rvačevljeve bazne funkcije. Svojstva ovih funkcija svrstavaju ih između klasičnih polinoma i spline funkcija, pa u tom smislu popunjavaju skup elementarnih funkcija. Izloženi su postupci izračunavanja vrijednosti $R_{bf}$ funkcija, njihov raspored za formiranje numeričkih rješenja te ilustracija osnovnih mogućnosti praktične primjene. Najdetaljnije su obrađene finitne funkcije $Fup_n(x)$ iz klase $C^\infty$ koje su elementi univerzalnog prostora $UP^n$, a koji ujedno sadrži i algebarske polinome do uključivo n-tog stupnja. Zatim je ilustriran postupak određivanja i primjene finitnih baznih funkcija, također iz klase $C^\infty$, s čijom linearnom kombinacijom se mogu točno prikazati eksponencijalne i trigonometrijske funkcije.