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NEW COMPLEX WAVELETS FAMILY BASED ON ATOMIC FUNCTIONS

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New Kravchenko-Rvachev complex wavelets built on the base of atomic functions (AF) are proposed and validated. Analytical statements are obtained for the wavelets and their Fourier transforms. The numerical implementation of the AF computation procedures forming the basis of proposed wavelets is considered. The main properties of the Kravchenko-Rvachev complex wavelets are researched and illustrated by numerical experiment.

KEY WORDS: *wavelet, atomic function, mathematical approximation, magnitude and phase of signal*

1. INTRODUCTION

Wavelet analysis that was developed in the late 80s of the last century has become now a powerful mean for analysis of different physical processes. It mainly concerns the analysis of processes (signals) which frequency content changes in time. The most efficient kind of wavelet analysis for such signals turns out to be a continuous wavelet analysis based on complex wavelets which allow to decouple information about a magnitude and phase of the investigated signals. The efficiency of such analysis depends on the characteristics of the mother wavelet. This leads to the necessity of development of complex wavelets that have certain useful properties for physical applications. The development of wavelet families which parameters, e.g. frequency-time resolution of the mother wavelet, can be changed is of particular interest.

On the other hand the integration (combination) of various mathematical approximation methods has become one of the recent tendencies. The well-known example is applying methods based on spline wavelets that use both the spline and wavelet functions. Integration of wavelets and atomic functions (AF) [1-4] that were presented and investigated in [1] long time before wavelets appearance is very promising part of this general tendency. Currently a work on efficient combination of these two approximation instruments is carried out. The main results in this field are presented in [4] where one can find a pioneering approach for constructing wavelets based on AF.

In this work new wavelet functions built on the base of atomic functions (AF) are proposed and validated. These wavelets possess some useful characteristics such as «zero mean» and compact support properties, the possibility to vary frequency-time resolution by changing the wavelet order, etc. The obtained wavelets and their Fourier transforms are described in analytical form.

2. ATOMIC FUNCTIONS. THE KEY PROPERTIES AND CALCULATION PROCEDURE

It is known [1-4] that AF are compactly supported infinitely differentiable functions representing solutions of the differential equations with shifted argument:

$$Lf\left(t\right)=\lambda\sum_{k=1}^{M}c\left(k\right)f\left(at-b\left(k\right)\right),\left|a\right|>1, \tag{1}$$

where L is a linear differential operator with constant coefficients. Let us consider the AF $up\left(t\right)$, $up_m\left(t\right)$, $fup_n\left(t\right)$, $h_a\left(t\right)$, $eup_a\left(t\right)$ and their key properties (the detailed information about these properties are presented in [2,3]).

Properties of atomic functions. *The atomic function* $up(t)$ has the following representation based on the Fourier transform:

$$\mathrm{up}\left(t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{jut}\prod_{k=1}^{\infty}\frac{\sin\left(u\cdot 2^{-k}\right)}{u\cdot 2^{-k}}du\,, \tag{2}$$

where $up(t)$ is an even function with the support [-1, 1].

The atomic function $up_m(t)$ is determined by the following expression:

$$\mathrm{up}_m(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{iut}\prod_{k=1}^{\infty}\frac{\sin^2\left(\frac{mu}{(2m)^k}\right)}{\frac{mu}{(2m)^k}\,m\sin\left(\frac{u}{(2m)^k}\right)}du\,. \tag{3}$$

Here $up_m(t)$ is an even function with the support [-1, 1]. If m=1 $up_m(t)=up(t)$.

The atomic function $fup_n(t)$ is determined as:

$$fup_n\left(t\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{jut}\left(\frac{\sin\left(u/2\right)}{u/2}\right)^{n}\prod_{k=1}^{\infty}\frac{\sin\left(u\cdot 2^{-k}\right)}{u\cdot 2^{-k}}du\,, \tag{4}$$

$fup_n(t)$ is an even function with the support $\left[-(n+2)/2,\,(n+2)/2\right]$. If n=0 $fup_n(t)=up(t)$.

The atomic function $h_a(t)$ is represented as follows:

$$h_a(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{jut}\prod_{k=1}^{\infty}\frac{\sin\left(u\cdot a^{-k}\right)}{u\cdot a^{-k}}du\,, \tag{5}$$

$h_a(t)$ is an even function with the support $\left[-1/(a-1),\,1/(a-1)\right]$. If $a=2$ $h_a(t)=up(t)$.

The exponential atomic function $eup_a(t)$ can be written as:

$$eup_a(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{jut}\prod_{k=1}^{\infty}\frac{shc\left(\ln(a)/2-ju/2^k\right)}{shc\left(\ln(a)/2\right)}du\,, \tag{6}$$

where $shc(t)=sh(t)/t.$ $eup_a(t)$ is neither an even nor an odd function. Its support is [-1, 1].

For the AF computation (2)-(6) we will use their expansions in a Fourier series. The Fourier coefficients can be calculated with the help of AF Fourier transformation expressions. Let us firstly consider the calculation procedure for even AF and then for a general case.

2.1. A finite even function with a symmetrical support computation procedure based on its Fourier transform

Let $\hat{f}(\omega)$ be the Fourier transform of an even compactly supported function with the support $[-l,l]$. Then it is represented by the Fourier series having the following form

$$f(t)=\left(\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi t}{l}\right)\right)\left(\mathbf{1}(t+l)-\mathbf{1}(t-l)\right), \tag{7}$$

where $\mathbf{1}(t)=\begin{cases}1, & t\geq 0;\\ 0, & t<0\end{cases}$ is used for the elimination of the *f(t)* periodic continuation beyond its support bounds. The Fourier coefficients a_n are determined by the expression

$$a_n = \frac{2}{l}\int_0^l f(t)\cos\left(\frac{n\pi}{l}t\right)dt. \tag{8}$$

Fourier transform of an even function has the form

$$\widehat{f}(\omega) = 2\int_0^\infty f(t)\cos(\omega t)dx\,. \tag{9}$$

Considering the function's support compactness property (9) can be written as

$$\widehat{f}(\omega) = 2\int_0^l f(t)\cos(\omega t)dt. \tag{10}$$

Having compared (8) and (10) we get

$$a_n = \frac{1}{l}\widehat{f}\left(\frac{n\pi}{l}\right). \tag{11}$$

Thus, an even compactly supported function with a symmetric support ($up(t)$, $up_m(t)$, $fup_n(t)$, $h_a(t)$ belong to this class of functions) can be calculated with a series (7) and the expression for Fourier coefficients computation (11).

2.2. Calculation procedure for any compactly supported function with a symmetric support based on its Fourier transform by the example of $\mathrm{eup}_a(\mathrm{t})$

The considered above calculation procedure can not be applied to the AF $eup_a(t)$. Let $\widehat{eup}_a(\omega)$ be the Fourier transform of $eup_a(t)$ which is known to have the support $[-l,l]$ ($l=1$). Then it can be represented by the following Fourier series

$$\begin{aligned} eup_a(t) &= \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi t}{l}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi t}{l}\right)\right)\left(\mathbf{1}(t+l)-\mathbf{1}(t-l)\right) = \\ &= \left(eup_a^1(t) + eup_a^2(t)\right)\left(\mathbf{1}(t+l)-\mathbf{1}(t-l)\right), \end{aligned} \tag{12}$$

where $eup_a^1(t)$ and $eup_a^2(t)$ are an even and odd parts of the $eup_a(t)$ function respectively. Fourier coefficients a_n and b_n are defined as follows

$$a_n = \frac{2}{l}\int_0^l eup_a^1(t)\cos\left(\frac{n\pi}{l}t\right)dt, \tag{13}$$

$$b_n = \frac{2}{l}\int_0^l eup_a^2(t)\sin\left(\frac{n\pi}{l}t\right)dt. \tag{14}$$

Taking into account the $eup_a(t)$ support compactness property we can represent $\widehat{eup}_a(\omega)$ in the form

$$\widehat{eup}_a(\omega) = 2\int_0^l eup_a^1(t)\cos(\omega t)\,dt - 2j\int_0^l eup_a^2(t)\sin(\omega t)\,dt =$$
$$= \mathrm{Re}\left[\widehat{eup}_a(\omega)\right] - j\,\mathrm{Im}\left[\widehat{eup}_a(\omega)\right], \tag{15}$$

where $\mathrm{Re}\left[\widehat{eup}_a(\omega)\right]$ is an even real function, $\mathrm{Im}\left[\widehat{eup}_a(\omega)\right]$ is an odd real function. Having compared (13), (14) with (15), we get

$$a_n = \frac{1}{l}\mathrm{Re}\left[\widehat{eup}_a\left(\frac{n\pi}{l}\right)\right], \tag{16}$$

$$b_n = -\frac{1}{l}\mathrm{Im}\left[\widehat{eup}_a\left(\frac{n\pi}{l}\right)\right]. \tag{17}$$

Thus, the function $eup_a(t)$ can be calculated according to the series (12) where the Fourier coefficients are determined by (16) and (17).

3. THE CONSTRUCTION AND VALIDATION OF THE COMPLEX KRAVCHENKO-RVACHEV WAVELETS

The complex Kravchenko-Rvachev wavelets [4] are based on the convolution of AF and $\cos^{p}\left(\frac{\omega}{2}\right)$ multiplier , $p \in \mathbb{N}$:

$$\hat{h}_\theta^p(\omega) = \left(\cos\left(\frac{\omega}{2}\right)\right)^p \hat{\theta}(\omega), \tag{18}$$

where $\hat{\theta}(\omega)$ is the AF spectrum.

The complex wavelet family spectrum has the form (disregarding normalization)

$$\hat{\tilde{\psi}}_\theta^p = \hat{h}_\theta^p(\omega - \pi). \tag{19}$$

We should firstly calculate the inverse Fourier transform of (19) to find the analytic representation of the wavelets. Let us consider for this purpose the inverse Fourier transform of $\cos^{\mathrm{p}}\left(\frac{\omega}{2}\right)$. We will use the following formula for an odd p:

$$\cos^p\left(\frac{\omega}{2}\right) = \frac{1}{2^{p-1}}\left[\cos\left(p\frac{\omega}{2}\right) + C_p^1\cos\left((p-2)\frac{\omega}{2}\right) + \right. \\ \left. + C_p^2\cos\left((p-4)\frac{\omega}{2}\right) + \ldots + C_p^{\frac{p-1}{2}}\cos\left(\frac{\omega}{2}\right)\right], \tag{20}$$

where $C_p^k = \dfrac{p!}{k!(p-k)!}$.

If a p is even then

$$\cos^p\left(\frac{\omega}{2}\right) = \frac{1}{2^{p-1}}\left[\cos\left(p\frac{\omega}{2}\right) + C_p^1\cos\left((p-2)\frac{\omega}{2}\right) + \right. \\ \left. + C_p^2\cos\left((p-4)\frac{\omega}{2}\right) + \ldots + C_p^{\frac{p-2}{2}}\cos\left(\frac{\omega}{2}\right)\right] + \frac{1}{2^p}C_p^{\frac{p}{2}}. \tag{21}$$

It is known that

$$F^{-1}\left\{\cos(a\omega)\right\} = \frac{\delta(t+a) + \delta(t-a)}{2}. \tag{22}$$

Thus, according to (20)-(22) for an odd p:

$$F^{-1}\left\{\cos^p\left(\frac{\omega}{2}\right)\right\}=\frac{1}{2^{p-1}}\left[\frac{\delta\left(t+\frac{p}{2}\right)+\delta\left(t-\frac{p}{2}\right)}{2}+\right.$$

$$+C_p^1\frac{\delta\left(t+\frac{p-2}{2}\right)+\delta\left(t-\frac{p-2}{2}\right)}{2}+C_p^2\frac{\delta\left(t+\frac{p-4}{2}\right)+\delta\left(t-\frac{p-4}{2}\right)}{2}+ \tag{23}$$

$$\left.+\ldots+C_p^{\frac{p-1}{2}}\frac{\delta\left(t+\frac{p-4}{2}\right)+\delta\left(t-\frac{p-4}{2}\right)}{2}\right]=$$

$$=\frac{1}{2^p}\sum_{k=0}^{(p-2)/2}C_p^k\left[\delta\left(t+\frac{p-2k}{2}\right)+\delta\left(t-\frac{p-2k}{2}\right)\right],$$

for an even p:

$$F^{-1}\left\{\cos^p\left(\frac{\omega}{2}\right)\right\}=\frac{1}{2^{p-1}}\left[\frac{\delta\left(t+\frac{p}{2}\right)+\delta\left(t-\frac{p}{2}\right)}{2}+C_p^1\frac{\delta\left(t+\frac{p-2}{2}\right)+\delta\left(t-\frac{p-2}{2}\right)}{2}+\right.$$

$$\left.+C_p^2\frac{\delta\left(t+\frac{p-4}{2}\right)+\delta\left(t-\frac{p-4}{2}\right)}{2}+\ldots+C_p^{\frac{p-2}{2}}\frac{\delta\left(t+\frac{p-4}{2}\right)+\delta\left(t-\frac{p-4}{2}\right)}{2}\right]+ \tag{24}$$

$$+\frac{1}{2^p}C_p^{\frac{p}{2}}\delta(t)=\frac{1}{2^p}\left[\sum_{k=0}^{(p-2)/2}C_p^k\left[\delta\left(t+\frac{p-2k}{2}\right)+\delta\left(t-\frac{p-2k}{2}\right)\right]+C_p^{\frac{p}{2}}\delta(t)\right].$$

Let us find out the inverse Fourier transform of (18) using (23)-(25). For an even p we get

$$\bar{h}_\theta^p(t)=\frac{1}{2^p}\int_{-\infty}^{\infty}\left[\sum_{k=0}^{(p-2)/2}C_p^k\left(\delta\left(t+\frac{p-2k}{2}-\tau\right)+\delta\left(t-\frac{p-2k}{2}-\tau\right)\right)+\right.$$

$$\left.+C_p^{\frac{p}{2}}\delta(t-\tau)\right]\theta(\tau)\,d\tau= \tag{25}$$

$$=\frac{1}{2^p}\left[\sum_{k=0}^{(p-2)/2}C_p^k\left(\theta\left(t+\frac{p-2k}{2}\right)+\theta\left(t-\frac{p-2k}{2}\right)\right)+C_p^{\frac{p}{2}}\theta(t)\right].$$

Similarly for an odd p we have

$$\tilde{h}_\theta^p(t) = \frac{1}{2^p}\sum_{k=0}^{(p-1)/2} C_p^k \left(\theta\left(t + \frac{p-2k}{2}\right) + \theta\left(t - \frac{p-2k}{2}\right)\right). \tag{26}$$

It follows from (2.19)

$$\tilde{\psi}_\theta^p(t) = e^{i\pi t} h_\theta^p(t), \tag{27}$$

$$h_\theta^p(t) = \begin{cases} \bar{h}_\theta^p(t), & p \text{ is even}, \\ \tilde{h}_\theta^p(t), & p \text{ is odd}. \end{cases} \tag{28}$$

Thus, disregarding the normalization multiplier (25)-(28) defines the complex Kravchenko-Rvachev wavelets. However in practice we need the wavelets with unit norm in L_2. Let us consider the norm of the function (27). For the purpose we will calculate $\left\|h_\theta^p(t)\right\|$ taking into account that the shifts of a function do not influence its norm in L_2. For an even p

$$\begin{aligned} \left\|\bar{h}_\theta^p(t)\right\| &= \sqrt{\int_{-\infty}^{\infty}\left|\bar{h}_\theta^p(t)\right|^2 dt} = \\ &= \frac{1}{2^p}\left(\int_{-\infty}^{\infty}\left(\left[\sum_{k=0}^{(p-2)/2} C_p^k\left(\theta\left(t+\frac{p-2k}{2}\right)+\theta\left(t-\frac{p-2k}{2}\right)\right)\right]^2 + \left(C_p^{p/2}\theta(t)\right)^2 + \right.\right. \\ &\left.\left. + 2\sum_{k=0}^{(p-2)/2} C_p^k\left(\theta\left(t+\frac{p-2k}{2}\right)+\theta\left(t-\frac{p-2k}{2}\right)\right)C_p^{p/2}\theta(t)\right)dt\right)^{1/2}. \end{aligned} \tag{29}$$

Since

$$\begin{aligned} \left[\sum_{k=0}^{(p-2)/2} C_p^k\left(\theta\left(t+\frac{p-2k}{2}\right)+\theta\left(t-\frac{p-2k}{2}\right)\right)\right]^2 &= \sum_{k=0}^{(p-2)/2}\left(C_p^k\theta\left(t+\frac{p-2k}{2}\right)\right)^2 + \\ + \sum_{k=0}^{(p-2)/2}\left(C_p^k\theta\left(t-\frac{p-2k}{2}\right)\right)^2 &+ 2\sum_{i=0}^{(p-2)/2}\sum_{j=0}^{(p-2)/2} C_p^i C_p^j \theta\left(t+\frac{p-2i}{2}\right)\theta\left(t-\frac{p-2j}{2}\right), \end{aligned} \tag{30}$$

then finally we get

$$\left\|\bar{h}_\theta^p(t)\right\| = \frac{1}{2^p}\left(2\sum_{k=0}^{(p-2)/2}\left(C_p^k\right)^2\left\|\theta(t)\right\|^2 + \left(C_p^{p/2}\right)^2\left\|\theta(t)\right\|^2 + \right.$$

$$+2\int_{-\infty}^{\infty}\left[\sum_{k=0}^{(p-2)/2}\left(C_p^k C_p^{p/2}\theta\left(t+\frac{p-2k}{2}\right)\theta(t)\right) + \sum_{k=0}^{(p-2)/2}\left(C_p^k C_p^{p/2}\theta\left(t-\frac{p-2k}{2}\right)\theta(t)\right) + \right. \tag{31}$$

$$\left.\left. + \sum_{i=0}^{(p-2)/2}\sum_{j=0}^{(p-2)/2} C_p^i C_p^j \theta\left(t+\frac{p-2i}{2}\right)\theta\left(t-\frac{p-2j}{2}\right)\right]^2 dt\right)^{1/2}.$$

Similarly for an odd p

$$\left\|\tilde{h}_\theta^p(t)\right\| = \frac{1}{2^p}\left(2\sum_{k=0}^{(p-1)/2}\left(C_p^k\right)^2\left\|\theta(t)\right\|^2 + \right.$$

$$\left. +2\sum_{i=0}^{(p-1)/2}\sum_{j=0}^{(p-1)/2} C_p^i C_p^j \theta\left(t+\frac{p-2i}{2}\right)\theta\left(t-\frac{p-2j}{2}\right)\right]^2 dt\right)^{1/2}. \tag{32}$$

Shifts in frequency domain also do not change the L_2-norm of a function, so

$$\left\|\tilde{\psi}_\theta^p(t)\right\| = \left\|e^{i\pi t}h_\theta^p(t)\right\| = \left\|h_\theta^p(t)\right\|. \tag{33}$$

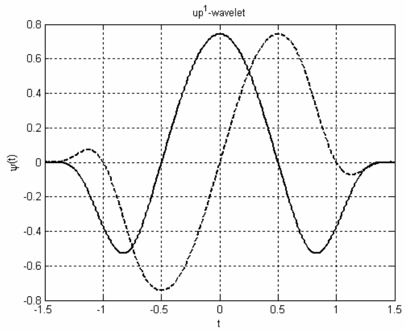
Then the expression for the considered family of Kravchenko-Rvachev wavelets with a unit norm has the form

$$\psi_\theta^p(t) = \frac{1}{\left\|h_\theta^p(t)\right\|}e^{i\pi t}h_\theta^p(t), \tag{34}$$

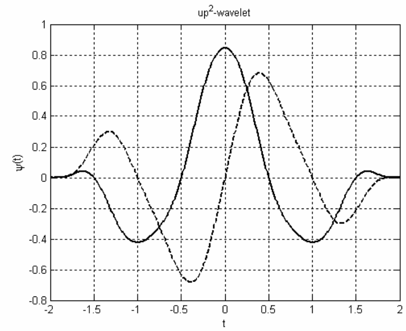
where $h_\theta^p(t)$ is defined by (28). Consequently Fourier transform of the wavelets (34) will be as follows

$$\widehat{\psi}_\theta^p(\omega) = \frac{\sqrt{2\pi}}{\left\|\hat{h}_\theta^p(\omega)\right\|}\hat{h}_\theta^p(\omega-\pi),\ \ p\in\mathbb{N}, \tag{35}$$

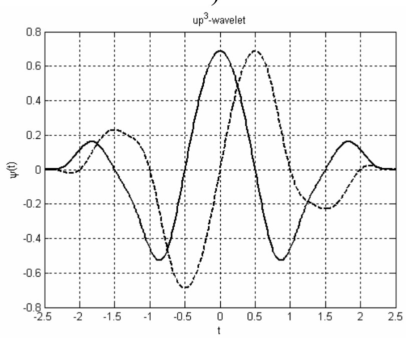
where $\hat{h}_\theta^p(\omega)$ is defined by (18). The graphs of some normalized complex Kravchenko-Rvachev wavelets and their Fourier transforms are pictured on the Figs. 1-5.


up^{1}-wavelet
ψ(t)
t

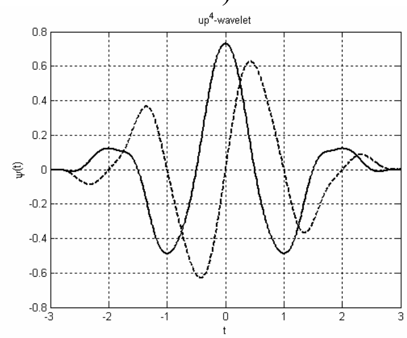
a)


up^{2}-wavelet
ψ(t)
t

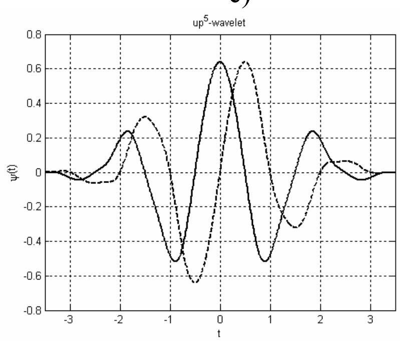
b)


up^{3}-wavelet
ψ(t)
t

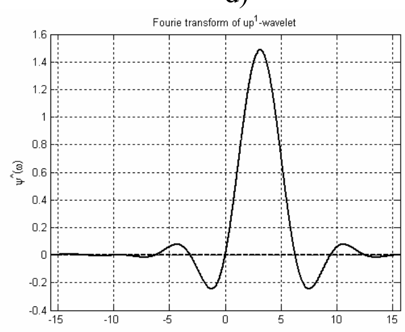
c)


up^{4}-wavelet
ψ(t)
t

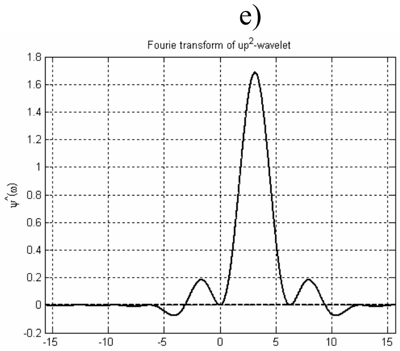
d)


up^{5}-wavelet
ψ(t)
t

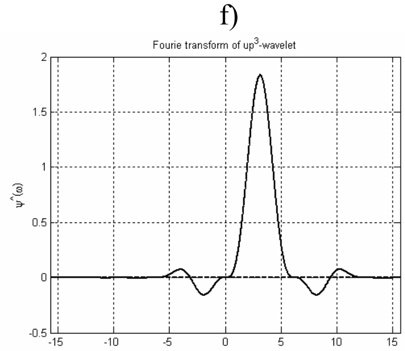
e)


Fourie transform of up^{1}-wavelet
ψ^(ω)
ω

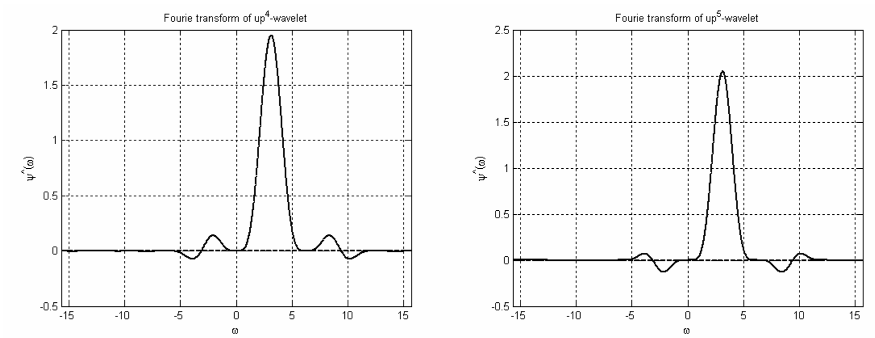
f)


Fourie transform of up^{2}-wavelet
ψ^(ω)
ω

g)


Fourie transform of up^{3}-wavelet
ψ^(ω)
ω

h)



i) j)

FIG. 1: (a)-(e): the complex Kravchenko-Rvachev wavelets based on the AF $up(t)$ of the orders 1-5 (dashed line represents the imaginary part)); (f)-(j): the corresponding Fourier transform of the wavelets

Let us show that the obtained functions (34) are the wavelets indeed. If so they have to meet the conditions [6]:

$$\psi_\theta^p \in L_2, \quad \left\|\psi_\theta^p\right\| = 1, \tag{36}$$

$$2\pi \int_{-\infty}^{\infty} \frac{\left|\widehat{\psi}_\theta^p(a)\right|^2}{|a|} da = C_\psi < \infty. \tag{37}$$

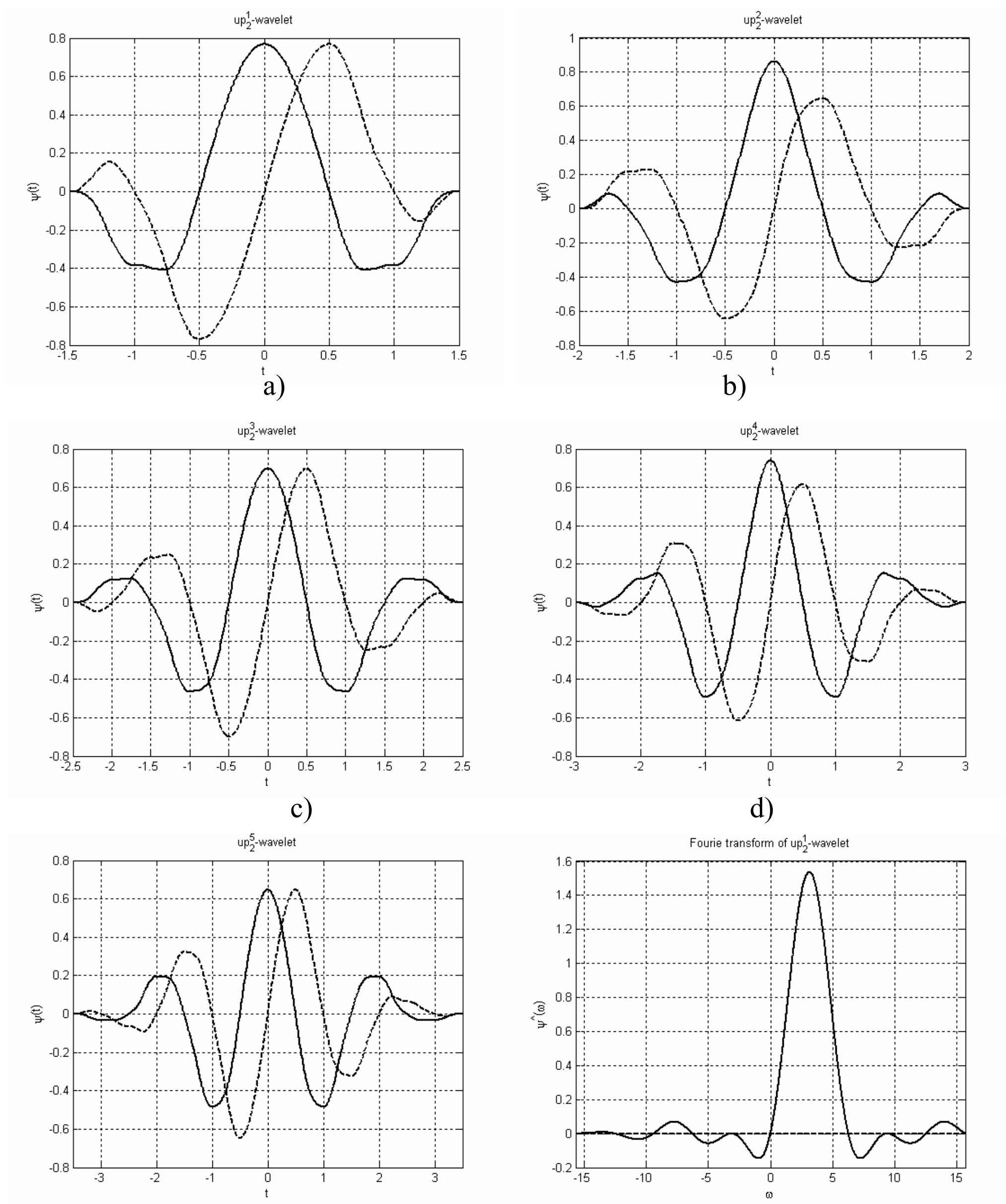
If (36) is true then the condition (37) can be modified as

$$\int_{-\infty}^{\infty} |t|\left|\psi_\theta^p(t)\right| dt < \infty, \tag{38}$$

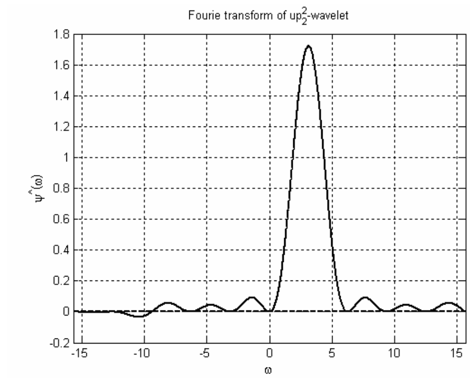
$$\int_{-\infty}^{\infty} \psi_\theta^p(t)\,dt = 0 \text{ or equivalently } \widehat{\psi}_\theta^p(0) = 0. \tag{39}$$

It follows from (25)-(28) that the discussed above wavelets are represented by the linear combination of finite number of shifted AF and the continuous bounded function $e^{i\pi t}$. Consequently they bear the following AF properties which they are based on: continuity, support compactness and roundedness. These properties provide the validity of (36) and (38) (the unity norm is provided by the above normalization procedure of the function (27)). Thus, it remains to show that for the functions (34) the property of zero mean (39) is fulfilled. This is true if the Fourier transform (35) at zero point equals zero:

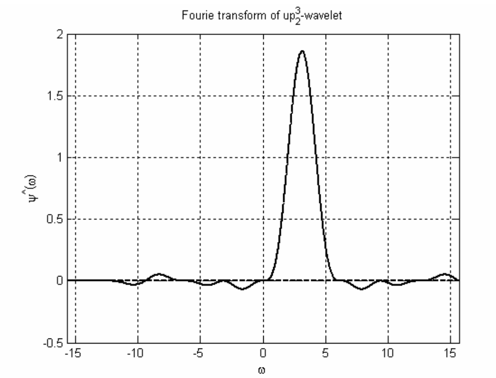
$$\widehat{\psi}_{\theta}^{\,p}(0)=\frac{1}{\left\|h_{\theta}^{p}(t)\right\|}\hat{h}(-\pi)=\frac{1}{\left\|h_{\theta}^{p}(t)\right\|}\left(\cos\left(\frac{\pi}{2}\right)\right)^{p}\hat{\theta}(-\pi)=0. \qquad (40)$$



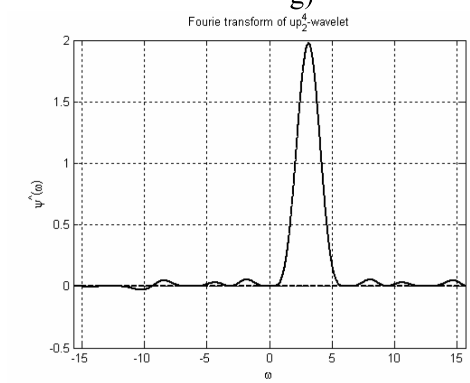
a) b) c) d) e) f)



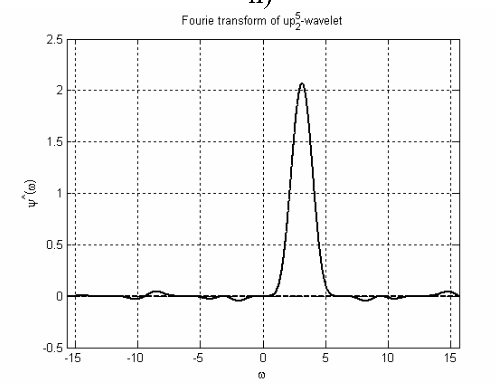
g)



h)

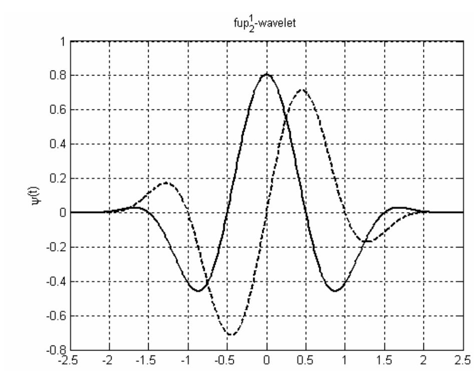


i)

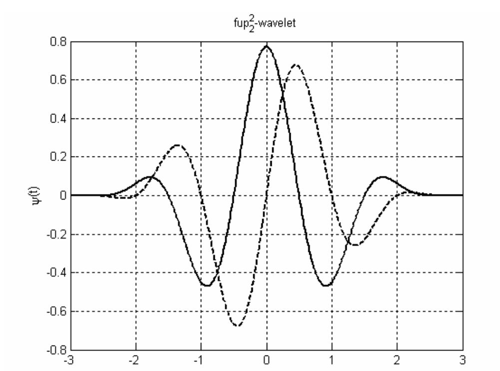


j)

FIG. 2: (a)-(e): the complex Kravchenko-Rvachev wavelets based on the AF $up_2(t)$ of the orders 1-5 (dashed line represents the imaginary part)); (f)-(j): the corresponding Fourier transform of the wavelets



a)



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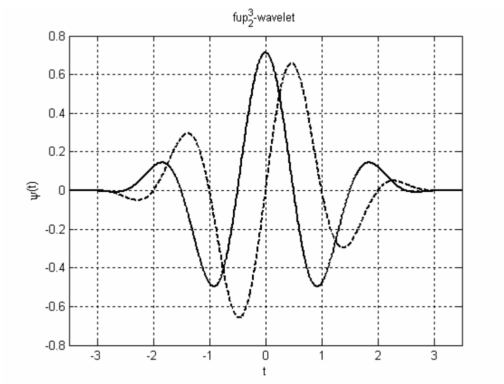
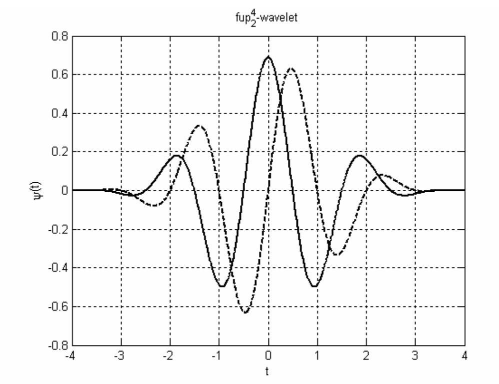
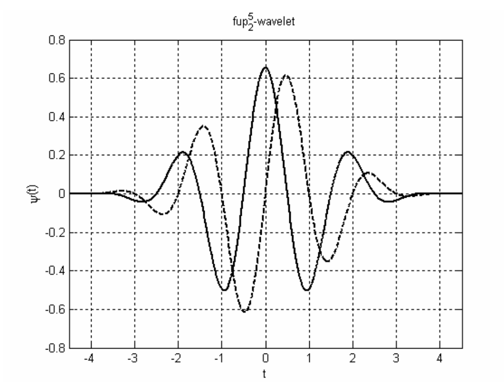

fup$_2^3$-wavelet
ψ(t)
t

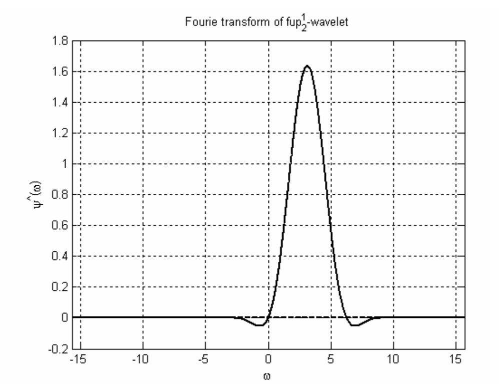
c)


fup$_2^4$-wavelet
ψ(t)
t

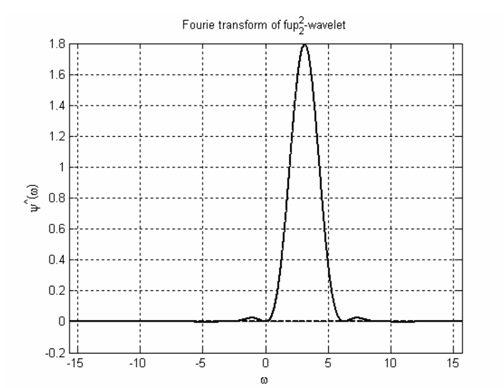
d)


fup$_2^5$-wavelet
ψ(t)
t

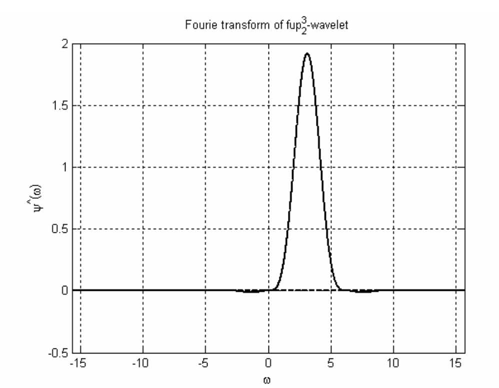
e)


Fourie transform of fup$_2^1$-wavelet
ψ^(ω)
ω

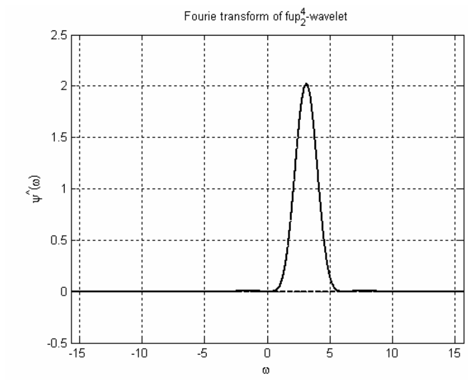
f)


Fourie transform of fup$_2^2$-wavelet
ψ^(ω)
ω

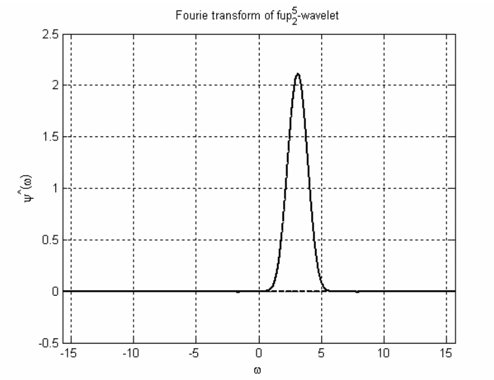
g)


Fourie transform of fup$_2^3$-wavelet
ψ^(ω)
ω

h)

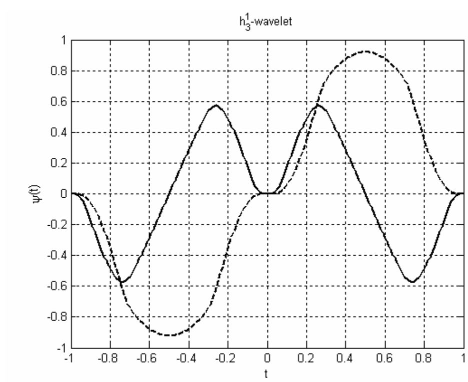


i)

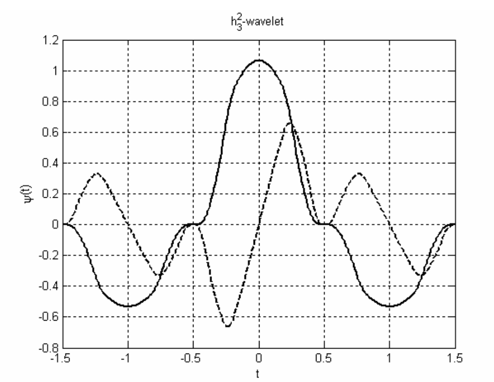


j)

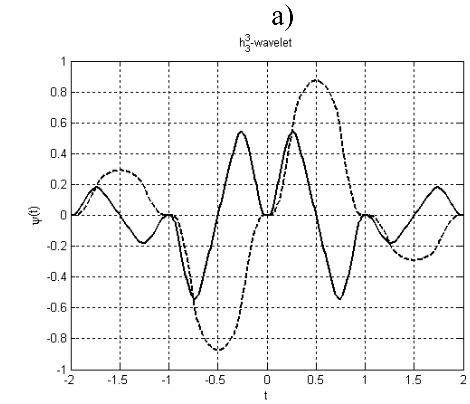
FIG. 3: (a)-(e): the complex Kravchenko-Rvachev wavelets based on the AF $fup_2(t)$ of the orders 1-5 (dashed line represents the imaginary part)); (f)-(j): the corresponding Fourier transform of the wavelets



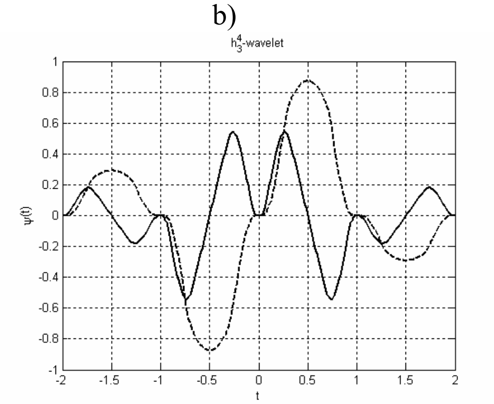
a)



b)



c)



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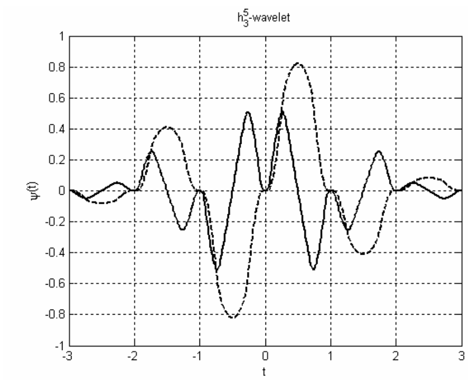

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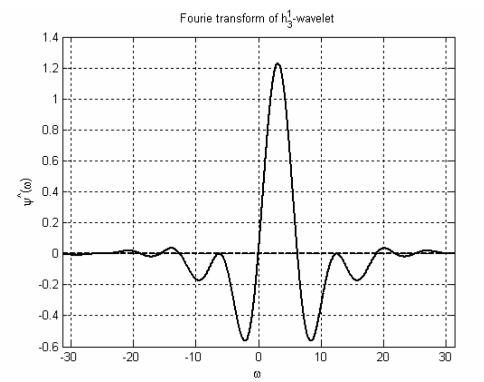

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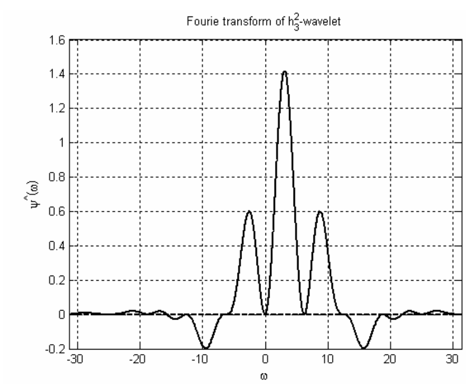

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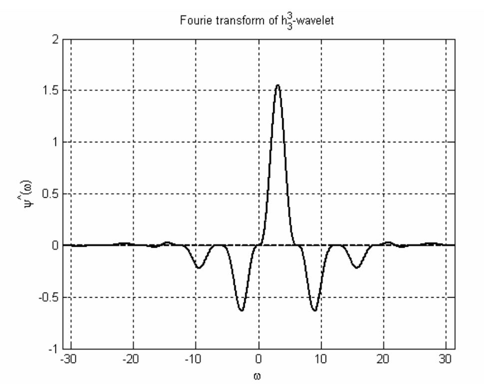

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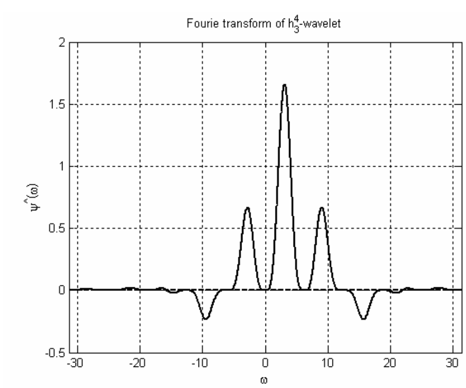

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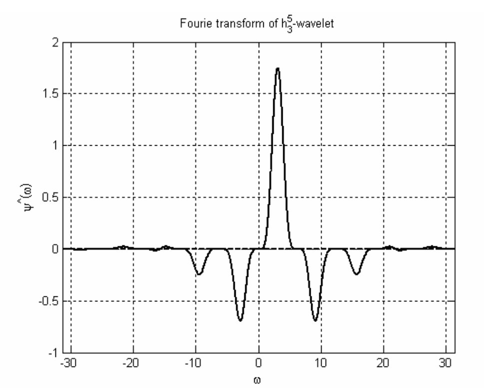
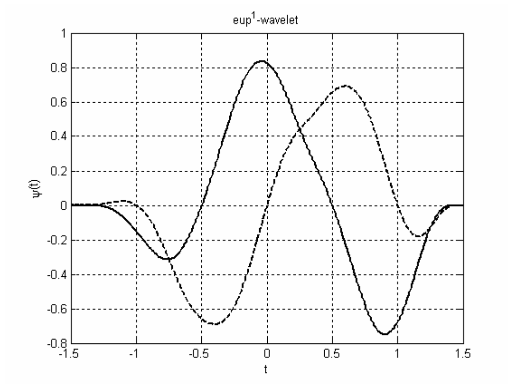
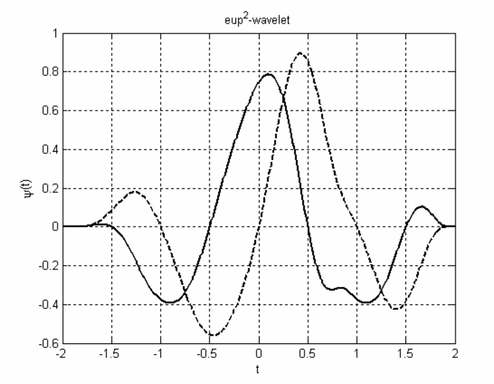


j)

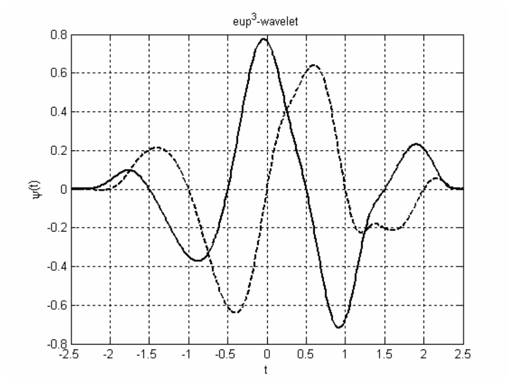
FIG. 4: (a)-(e): the complex Kravchenko-Rvachev wavelets based on the AF $h_3(t)$ of the orders 1-5 (dashed line represents the imaginary part)); (f)-(j): the corresponding Fourier transform of the wavelets


eup^1-wavelet
ψ(t)
t

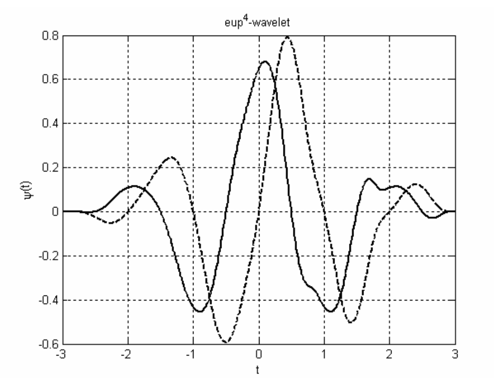
a)


eup^2-wavelet
ψ(t)
t

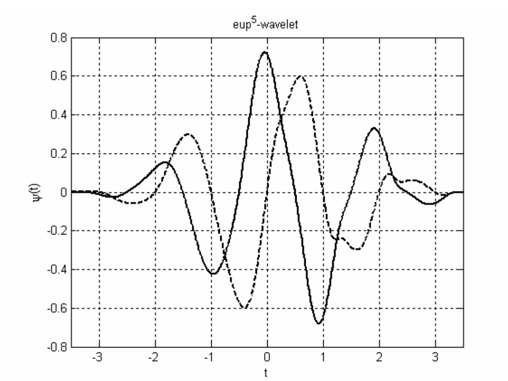
b)


eup^3-wavelet
ψ(t)
t

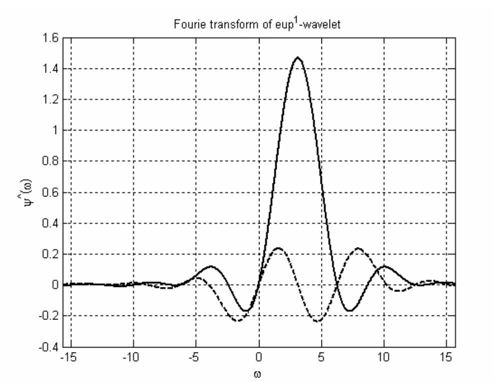
c)


eup^4-wavelet
ψ(t)
t

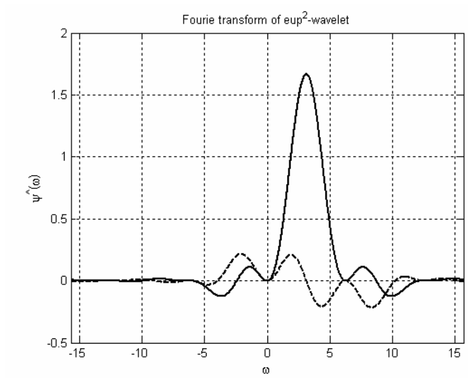
d)


eup^5-wavelet
ψ(t)
t

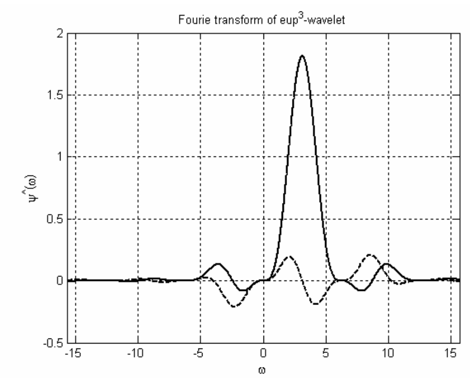
e)


Fourie transform of eup^1-wavelet
ψ^(ω)
ω

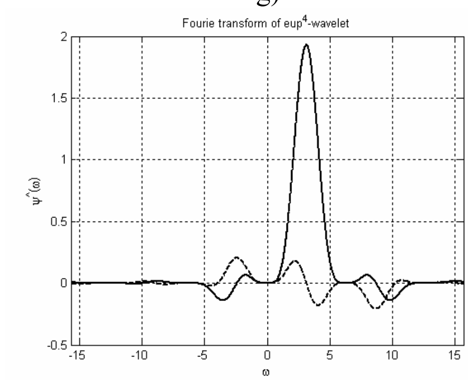
f)



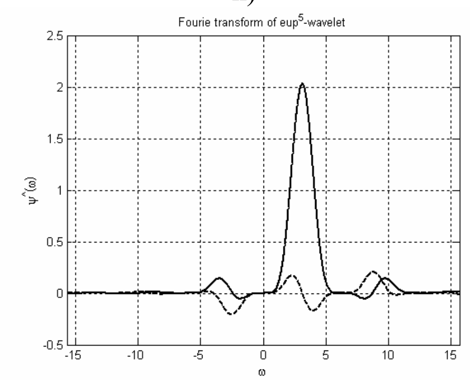
g)



h)



i)



j)

FIG. 5: (a)-(e): the complex Kravchenko-Rvachev wavelets based on the AF $eup(t)$ of the orders 1-5 (dashed line represents the imaginary part)); (f)-(j): the corresponding Fourier transform of the wavelets

4. THE PROPERTIES OF THE KRAVCHENKO-RVACHEV WAVELETS

In this section we will consider the properties of the new family of the Kravchenko-Rvachev wavelets.

Zero mean. It was showed in the above section (see (39)-(40)) that the complex Kravchenko-Rvachev wavelets have the zero mean property:

$$\int_{-\infty}^{+\infty} \psi_{\theta}^{p}(t)\,dt = 0. \tag{41}$$

The smoothness. The complex Kravchenko-Rvachev wavelets keep the AF smoothness order which they are based on. Indeed, the wavelets (34) are represented (disregarding constants) by the multiplication of $e^{i\pi t}$ infinitely differentiable function and the $h_\theta^p(t)$ function that is a sum of argument weighted shifts of a $\theta(t)$ AF. Consequently for the infinitely differentiable AF functions the corresponding complex wavelets (34) are also infinitely differentiable.

The support size. The complex Kravchenko-Rvachev wavelets have a compact support that follows the support of AF. Supports of the wavelets (34) coincide with the ones of the $h_\theta^p(t)$ that in its turn depends on supports of the corresponding AF. This relation can be easily found with the help of (25)-(28). Let the $\theta(t)$ AF have the support

$$\mathrm{supp}\,\theta(t) = [a,b],$$

then the support of the corresponding wavelet

$$\mathrm{supp}\,\psi_\theta^p(t) = [a - p/2,\ b + p/2]. \tag{42}$$

TABLE 1: Frequency spread around the centre frequency and the support of the complex Kravchenko-Rvachev wavelets $\psi_\theta^p(t)$ based on the AF $up_m(t)$

$\theta(t)$ / p	$up_1(t)$		$up_2(t)$		$up_3(t)$		$up_4(t)$		$up_5(t)$	
	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp
1	1.78	[-1.5,1.5]	1.56	[-1.5, 1.5]	1.33	[-1.5, 1.5]	1.19	[-1.5, 1.5]	1.17	[-1.5, 1.5]
2	1.15	[-2, 2]	0.97	[-2, 2]	0.83	[-2, 2]	0.74	[-2, 2]	0.74	[-2, 2]
3	0.85	[-2.5, 2.5]	0.71	[-2.5, 2.5]	0.60	[-2.5, 2.5]	0.54	[-2.5, 2.5]	0.54	[-2.5, 2.5]
4	0.67	[-3, 3]	0.56	[-3, 3]	0.47	[-3, 3]	0.43	[-3, 3]	0.42	[-3, 3]
5	0.56	[-3.5, 3.5]	0.46	[-3.5, 3.5]	0.39	[-3.5, 3.5]	0.35	[-3.5, 3.5]	0.35	[-3.5, 3.5]

TABLE 2: Frequency spread around the centre frequency and the support of the complex Kravchenko-Rvachev wavelets $\psi_\theta^p(t)$ based on the AF $fup_n(t)$

$\theta(t)$ / p	$fup_1(t)$		$fup_2(t)$		$fup_3(t)$		$fup_4(t)$		$fup_5(t)$	
	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp
1	1.00	[-2, 2]	0.83	[-2.5, 2.5]	0.73	[-3, 3]	0.65	[-3.5, 3.5]	0.59	[-4, 4]
2	0.65	[-2.5, 2.5]	0.57	[-3, 3]	0.53	[-3.5, 3.5]	0.49	[-4, 4]	0.45	[-4.5, 4.5]
3	0.49	[-3, 3]	0.44	[-3.5, 3.5]	0.41	[-4, 4]	0.39	[-4.5, 4.5]	0.37	[-5, 5]
4	0.39	[-3.5, 3.5]	0.36	[-4, 4]	0.34	[-4.5, 4.5]	0.32	[-5, 5]	0.31	[-5.5, 5.5]
5	0.33	[-4, 4]	0.31	[-4.5, 4.5]	0.29	[-5, 5]	0.28	[-5.5, 5.5]	0.27	[-6, 6]

TABLE 3: Frequency spread around the centre frequency and the support of the complex Kravchenko-Rvachev wavelets $\psi_{\theta}^{p}(t)$ based on the AF $h_a(t)$

$\theta(t)/p$	$h_2(t)$		$h_3(t)$		$h_4(t)$		$h_5(t)$		$h_6(t)$	
	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp
1	1.789	[-1.5, 1.5]	13.327	[-1, 1]	30.570	[-5/6, 5/6]	53.401	[-0.75, 0.75]	68.129	[-0.7, 0.7]
2	1.152	[-2, 2]	13.322	[-1.5, 1.5]	30.565	[-4/3, 4/3]	53.467	[-1.25, 1.25]	68.024	[-1.2, 1.2]
3	0.849	[-2.5, 2.5]	13.320	[-2, 2]	30.563	[-11/6, 11/6]	53.497	[-1.75, 1.75]	67.977	[-1.7, 1.7]
4	0.673	[-3, 3]	13.319	[-2.5, 2.5]	30.563	[-7/3, 7/3]	53.514	[-2.25, 2.25]	67.951	[-2.2, 2.2]

TABLE 4: Frequency spread around the centre frequency and the support of the complex Kravchenko-Rvachev wavelets $\psi_{\theta}^{p}(t)$ based on the AF $eup_a(t)$

$\theta(t)/p$	$eup_1(t)$		$eup_2(t)$		$eup_3(t)$		$eup_4(t)$		$eup_5(t)$	
	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp
1	2.11	[-1.5, 1.5]	2.62	[-1.5, 1.5]	3.15	[-1.5, 1.5]	3.66	[-1.5, 1.5]	4.14	[-1.5, 1.5]
2	1.47	[-2, 2]	1.98	[-2, 2]	2.50	[-2, 2]	3.02	[-2, 2]	3.51	[-2, 2]
3	1.16	[-2.5, 2.5]	1.67	[-2.5, 2.5]	2.20	[-2.5, 2.5]	2.71	[-2.5, 2.5]	3.20	[-2.5, 2.5]
4	0.99	[-3, 3]	1.49	[-3, 3]	2.01	[-3, 3]	2.53	[-3, 3]	3.02	[-3, 3]
5	0.87	[-3.5, 3.5]	1.37	[-3.5, 3.5]	1.89	[-3.5, 3.5]	2.41	[-3.5, 3.5]	2.90	[-3.5, 3.5]

It follows the (42) that the increasing of the p wavelet order tends to the growing of its support, i.e. when using these wavelets for the signal processing the time resolution of the wavelets with higher order would be worse comparing to the ones with less order. The tables 1-4 show the support sizes of some Kravchenko-Rvachev wavelets.

Frequency resolution. The frequency resolution of a $\psi(t)$ wavelet is the inverse value to its frequency spread around the centre frequency [5]:

$$\sigma_\omega^2 = \frac{1}{2\pi}\int_0^\infty (\omega-\eta)^2 \left|\widehat{\psi}(\omega)\right|^2 d\omega, \tag{43}$$

where $\eta = \frac{1}{2\pi}\int_0^{+\infty} \omega\left|\widehat{\psi}(\omega)\right|^2 d\omega$ is the centre frequency of $\widehat{\psi}(\omega)$. For all the wavelets considered in this work the centre frequency $\eta = \pi$. This value is a result of the frequency shift (19) of the corresponding AF which frequency centre is at 0 point. Frequency spread around the centre frequency of some complex Kravchenko-Rvachev wavelets is represented in Tables 1-4. These data show that increasing the wavelet order p tends to the increasing of its frequency resolution. Thus, the obtained results correspond to the Heisenberg uncertainty principle which says that frequency resolution increasing (in this case when the wavelet order increases) brings to decreasing of time resolution (i.e. increasing of support size) of the wavelet.

The parity of real and imaginary parts. Considering complex functions it makes sense to investigate the parity of their imaginary and real parts separately. The construction procedure of the complex Kravchenko-Rvachev wavelets results in the following - the real part of the wavelets (34) for an even AF is also even while the one for an odd AF is odd

$$\mathrm{Re}\left(\psi_\theta^p(-t)\right)=\mathrm{Re}\left(\psi_\theta^p(t)\right),\ \mathrm{Im}\left(\psi_\theta^p(-t)\right)=-\mathrm{Im}\left(\psi_\theta^p(t)\right), \tag{44}$$

when $\theta(t)=\theta(-t)$.

5. CONCLUSION

The new family of the complex Kravchenko-Rvachev wavelets was obtained in this work. The wavelets and their Fourier transforms are described by analytical expressions. The calculation procedure of AF that underlies the computation scheme for the obtained functions was considered. The main properties of the complex Kravchenko-Rvachev wavelets were investigated. It was shown that these properties (the support size, smoothness, etc.) are inherited by the wavelets from the underlying AF. It should be noted that the presented technique of complex wavelets construction can be based not only on the AF. In (18) $\hat{\theta}(\omega)$ may be the spectrum of an any window compactly supported function while the frequency shift of (35) may be done by an any odd number multiplied by π. The higher frequency shifts could be useful when using an analytical wavelet transform.

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