USE OF THE R FUNCTION METHOD TO CALCULATE ELECTRODYNAMIC CHARACTERISTICS OF WAVEGUIDES WITH COMPLEX SHAPES

V. F. Kravchenko, V. V. Kravchenko,
G. P. Man'ko, V. L. Rvachev,
and T. I. Sheiko

UDC 621.372.81:537.86

Modern radio physics has in recent years been characterized by rapid development of theory and engineering practice in the millimeter and submillimeter wave ranges. The dimensions of the systems, which are usually quite arbitrary in shape, are commensurate with or substantially in excess of the wavelength. This necessitates determination of the eigenvalues and eigenfunctions of regions with boundaries of complex form (waveguides, microstrip structures, resonators, etc.). One important problem is to lay the electrodynamic foundation for computer-aided design of microwave devices over a broad wavelength range [1]. A procedure for analytic description of the complex geometric shapes of real objects is especially important for development of the appropriate algorithm and their numerical implementation in computers. The theory of R functions, which is utilized in solving interior and exterior electrodynamic boundary value problems [2-11], is promising for this purpose.

The present paper describes a method based on R function theory and variational/difference methods that can be used to calculate the eigenvalues and eigenfunctions of waveguides with complicated cross sections [12, 13]. The error level for the calculation is estimated, the field patterns are constructed, and the eigenvalues are determined. It should be remarked that, e.g., this approach and the overall formulation of the problem permit consideration of a broad class of interior electrodynamic problems for arbitrary regions, unlike the result of [14, 15].

We will illustrate the foregoing by considering the problem of calculating the TE modes of a rectangular waveguide with two symmetrically positioned dual ridges (Fig. 1).

The mathematical model of the problem of calculating the eigenvalues and eigenfunctions has the form

$$\left.\begin{array}{l}\Delta u+k^2u=0;\\ \partial u/\partial n|_{\partial\Omega}=0\end{array}\right\}. \qquad (1)$$

The operator of this problem is positive-definite [16], so that it can be solved by Ritz's method, minimizing the functional

$$J=\int_{\Omega}(\nabla u)^2d\Omega/\int_{\Omega}u^2d\Omega$$

on the class of functions from the domain of the boundary value problem operator (with solution structure $u = P - \omega D_1 P$ [10]) or from the energy space (solution structure $u = P$). According to [16], the smallest eigenvalue remains unchanged in this case.

It is important in practical application of Ritz's method to identify the main natural boundary conditions, since it is sufficient to choose them solely from the energy space and not necessarily from the boundary value problem operator domain. This considerably facilitates selection of basis functions in solving many problems of practical importance, especially for multidimensional domains with boundaries having complicated forms. Use of the R function theory permits construction of a sheaf of functions both in the boundary value problem operator domain and in the energy space. The concepts of general (satisfying all the boundary conditions) and partial structures were introduced in [11]. The general structure utilized in solving practical problems for operation of POLE series systems were described in [6-11]. Partial structures that do not satisfy the natural boundary conditions (for a second-order differential operator) are given below. Their effectiveness was tested with a number of model and real problems:

Translated from Izmeritel'naya Tekhnika, No. 1, pp. 55-57, January, 1993.

0543-1972/93/3601-0081$12.50

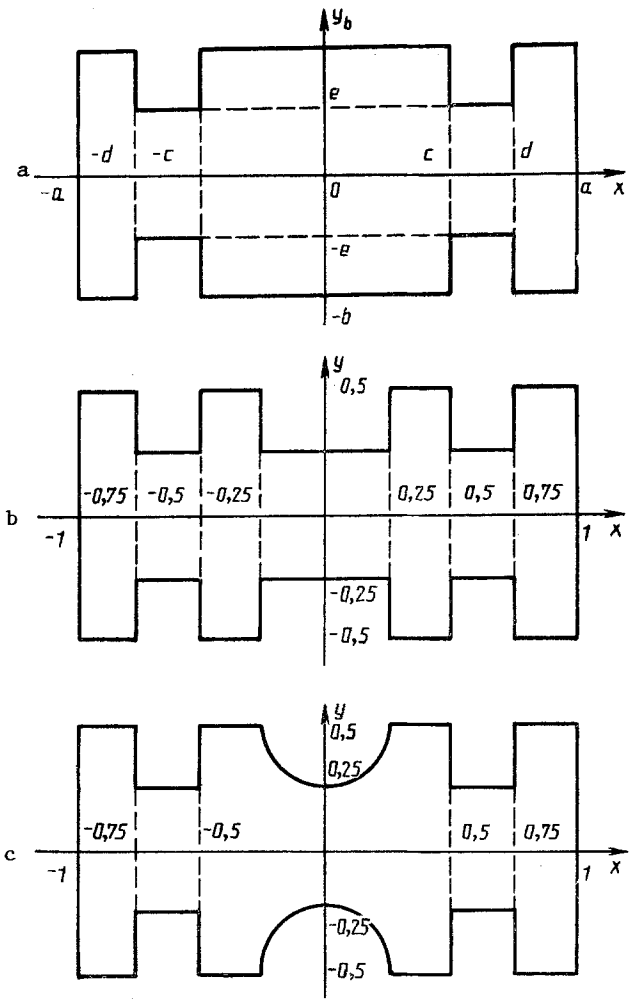

Fig. 1

$$1)\ \frac{\partial u}{\partial \nu}+hu\Big|_{\partial\Omega_i}=\varphi_i;\quad u=u_1+u_0=P+\omega\varphi_0;$$

$$2)\ \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega_i}=\varphi_i;\quad u=u_1+u_0=P+\omega\varphi_0;$$

$$3)\ u|_{\partial\Omega_{1i}}=\varphi_i;\quad \frac{\partial u}{\partial \nu}+h_j u\Big|_{\partial\Omega_{2j}}=\psi_j;$$

$$u=u_1+u_0=\omega_1 P+\varphi_0+\frac{\omega_1\omega_2}{\omega_1+\omega_2}(\psi_0-D_1\varphi_0+h_0\varphi_0),$$

where φ_0, ψ_0, and h_0 are the continuation of the loading functions within domain Ω, P is the indeterminate component, and $\omega = 0$ ($\omega_i = 0$) is the equation of the boundary $\partial(\Omega)$ (boundary segment $\partial\Omega_i$). The loading functions are continued into the domain on the basis of a generalized Lagrangian ("gluing" formula) [10]; the corresponding functionals are given in [10, 16]. Here we also employed the partial structure u = P, where

$$P(x,y)=\sum_{i=1}^{n} c_i\chi_i.$$

Chebyshev and spline polynomials were subsequently used for the basis $\{\chi_i\}$. The domain geometry was described by logical operations over the support domains.

All the calculations were made under the operating conditions for the POLE system [11]. The problem was solved for 10-55 coordinate functions with polynomials (the best result was obtained with 45 coordinate functions) and 121-357 coordinate functions with B1 splines.

TABLE 1

Version	N_i	k_i	$k_i - k_{i+1}$
1	6	3,25103	0,0155
2	8	3,23552	0,0395
3	10	3,1960	0,0462
4	12	3,1498	0,0295
5	14	3,1203	0,0429
6	16	3,0774	0,0279

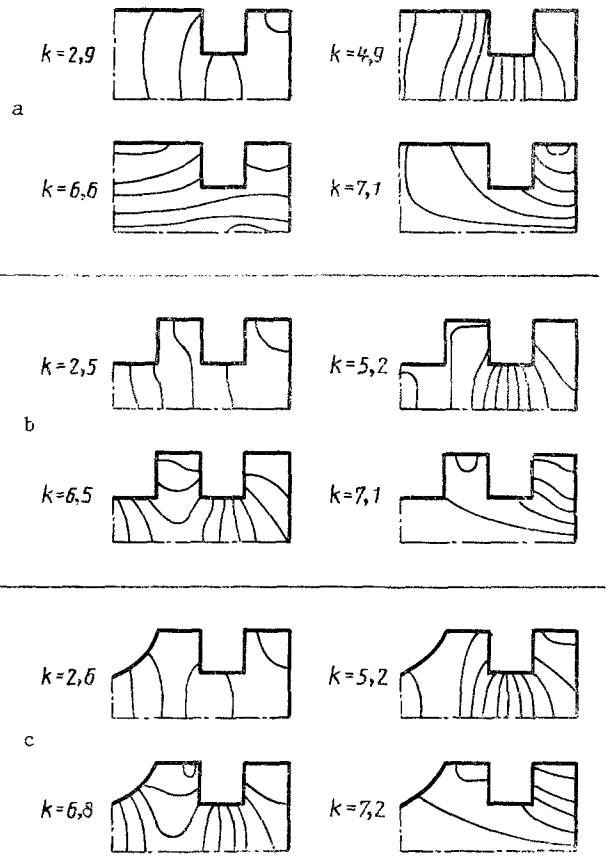

Fig. 2

Integration was with a 20-point Gauss quadrature formula in each of the basis rectangles. The average solution time for the problem was 15 min (BÉSM-6 computer), including printout of the equipotential surfaces.

The reliability of the results was verified by testing the general structure $u = P - \omega D_1 P$, whose use proved to be less effective for this problem; more time and hardware resources were required. The error amounted to 5-8%, as against 0.5-3% for the partial structure (all other conditions being equal).

The equation for the boundary of the domain shown in Fig. 1a had the form

$$W=((F1\wedge F2)\overline{\wedge}((F3\wedge F4)\wedge F5)),$$
$$\text{where } F1=(a^2-x^2)/2a,\ F2=(b^2-y^2)/2b,$$
$$F3=(x^2-c^2)/2c,\ F4=(d^2-x^2)/2d,$$
$$F5=(y^2-l^2)/2l,$$

with a > d > c and b > ℓ. Changing the parameters ℓ, c, and d led to the changes in cutout width, depth, and position. Changing the parameters a and b changed the basic characteristic dimensions of the domain.

TABLE 2

$k \cdot 2a$	λ/a	$k \cdot 2a$	λ/a	$k \cdot 2a$	λ/a
2,964	2,119	2,508	2,505	2,673	2,350
4,921	1,276	5,284	1,189	5,214	1,205
6,641	0,946	6,487	0,968	6,799	0,924
7,123	0,882	7,052	0,891	7,184	0,874
7,218	0,870	7,067	0,889	7,229	0,869
7,570	0,830	7,860	0,799	8,188	0,767
9,483	0,663	7,986	0,870	8,481	0,740

Numerical calculations were made for a = 1, b = 0.5, c = 0.5, d = 0.75, and ℓ = 0.25, corresponding to the ratios b/a = 0.5, c/a = 0.5, and (d − c)/(2a) = 0.125 from [15].

Available on-line memory and computer speed restrict attainable approximate solution accuracy in many cases. Construction of highly approximate solutions is a very pressing problem in computational mathematics. Various approaches have been taken to construction of such solutions. Wise use has been made of difference and variational/difference schemes with higher-order accuracy and of net sequence to improve the exactness of approximate solutions.

The notation underlying another approach stems from Richardson [12, 13]; he called it extrapolation to the limit. The method consists of utilizing sequencies of nets and of monotypic approximations corresponding to them for constructing approximate solutions with specified order of accuracy. Only standard difference approximation can be employed for problems of first and second orders of accuracy in this approach.

We will utilize solution of problem (1) for Fig. 2a as our model in a numerical experiment on application of Richardson's idea in variational methods. This enables us to improve accuracy by employing solutions obtained with a small number of coordinate functions.

Suppose that u_1, u_2, and u_3 are approximate solutions obtained by approximation with polynomials of degrees N1, N2, and N3 respectively. The exact solution of the problem is represented as

$$u=u_1+kN_1^{-\alpha};\ u=u_2+kN_2^{-\alpha};\ u=u_3+kN_3^{-\alpha}\ ,$$

where the exponent α and the constant k are unknown. Obviously,

$$k=\frac{u_1-u_2}{N_1^{\alpha}-N_2^{\alpha}}\,N_1^{\alpha}\,N_2^{\alpha};$$

$$\frac{u_1-u_3}{u_1-u_2}=\frac{N_2^{\alpha}}{N_3^{\alpha}}\cdot\frac{N_3^{\alpha}-N_1^{\alpha}}{N_2^{\alpha}-N_1^{\alpha}}\ . \tag{2}$$

Hence

$$u=u_1-\frac{u_1-u_2}{N_2^{\alpha*}-N_1^{\alpha*}}\,N_2^{\alpha*}\ , \tag{3}$$

where α^* is the solution of Eq. (2).

Put $N_2 = 2N_1$ and $N_3 = 4N_1$ for definiteness; then

$$\frac{u_1-u_3}{u_1-u_2}=\frac{2^{\alpha}}{4^{\alpha}}\cdot\frac{4^{\alpha}-1}{2^{\alpha}-1}=\frac{2^{\alpha}+1}{2^{\alpha}}=1+2^{-\alpha}\ ; \tag{4}$$

$$\alpha=-\ln\frac{u_2-u_3}{u_1-u_2}\Big/\ln 2.$$

Since $\alpha > 0$, it follows from (4) that

$$u_2 - u_3 < u_1 - u_2. \tag{5}$$

This condition is necessary for realization of process (3).

Solution of (1) for Fig. 2a yields the results in Table 1. It follows from this table that condition (5) is satisfied by versions 3, 4, and 5 for N_i = 10, 12, 14 and versions 5 and 6 for N_i = 14, 16.

The first three versions give α = 1.6 and k = 3.0134. The second two versions give α = 2.5 and k = 2.9691, while k = 2.9684 in [13], i.e., the error is 0.02%, as against 2.7% prior to application of our scheme. The initial versions provide an error of 1.5%, as against 3.8%. Since our procedure calls for use of different approximation techniques in conducting numerical experiments, calculations were made with spline approximations; B1 splines were used. We obtained the following results for various partitions of the original domain and different approximation space dimensions:

$m \times n$	11×11	13×13	17×17	21×17
k	2,9792	2,9688	2,9642	2,9623

Here the error was 0.2%, but the results cast doubt on the accuracy with which the first eigenvalue was determined. Actually, since the approximation to the first eigenvalue is from above, the eigenvalue k = 2.9623 is more exact than the k = 2.9684 found in [12].

The two left-hand columns in Table 2 contain the results yielded by calculation of the first seven eigenvalues of a waveguide with the cross-sectional configuration depicted in Fig. 2a; this figure also shows the level lines for the four eigenfunctions corresponding to the first four eigenvalues.

A similar method was used to make calculations for the domain depicted in Fig. 2b. Here

$$W=((F1 \wedge F2)\,\overline{\wedge}\,(((F3 \wedge F4) \vee F6) \wedge F5));$$
$$F1=(1-x^2)/2;\ F2=0{,}25-y^2;\ F4=(0{,}75-x^2)/1{,}5;$$
$$F5=(y^2-0{,}25^2)/0{,}5;\ F6=(0{,}25^2-x^2)/0{,}5.$$

The two middle columns in Table 2 contain eigenvalues, and Fig. 2b shows the level lines of the eigenfunctions corresponding to the first four eigenvalues. Here the presence of additional ridge structures and the greater part played by approximation tools with local bias caused preference to be given to the use of the spline approximation.

For the domain depicted in Fig. 2c,

$$W=(((F1 \wedge F2)\,\overline{\wedge}\,((F3 \wedge F4) \wedge F5)) \wedge (F6 \wedge F7));$$
$$F1=(1-x^2)/2;\ F2=0{,}25-y^2;\ F3=x^2-0{,}25;$$
$$F4=(0{,}75^2-x^2)/1{,}5;\ F5=(y^2-0{,}25^2)/0{,}5;$$
$$F6=(x^2+(y-0{,}5)^2-0{,}25^2)/0{,}5;\ F7=(x^2+(y+0{,}5)^2-$$
$$-0{,}25^2)/0{,}5.$$

The two right-hand columns of Table 2 give the eigenvalues, and Fig. 2c shows the level lines of the eigenfunctions corresponding to the first four eigenvalues, as above. It should be remarked that the problem cannot be solved by the partial subdomain method for Fig. 2c, since the semicircular cutouts make it impossible to partition the original domain into rectangles.

We might conclude by noting that the following criteria have been established for assessing the effectiveness of the method elaborated and substantiated here for computing the eigenvalues and eigenfunctions of waveguides of arbitrary cross-sectional shape: suitability for design of waveguides with arbitrary cross-sectional shape (flexible program); automatic

determination of solution for broad class of cross-sectional establishment of parameters and characteristics for both fundamental and higher wave types; accuracy; simplicity of algorithm and its programmatic implementation; computational efficiency of algorithm.

The above recommendations can be helpful in developing microwave radio systems and used to assess parameter error levels for waveguides of complex cross section.

LITERATURE CITED

1. K. Gupta, R. Gardzh, and R. Chadkha, Computer Modeling of Microwave Systems [in Russian], Radio i Svyaz, Moscow (1987).
2. V. F. Kravchenko, V. I. Polevoy, and V. L. Rvachov, Intern. Symp. Electrodynamics and Electromagnetic Wave Theory [in Russian], Tbilisi (1971), p. 9.
3. V. F. Kravchenko, G. P. Man'ko, V. L. Rvachev, and V. V. Ivakhov, Tr. Metrol. Inst. SSSR, No. 158 (218), 55 (1974).
4. V. F. Kravchenko and E. I. Nefedov, Dokl. Akad. Nauk SSSR, 248, No. 1, 74 (1979).
5. V. F. Kravchenko and E. I. Nefedov, ibid., 256, No. 5, 1097 (1981).
6. V. F. Kravchenko and A. M. Bondarenko, Radioélektronika, 29, No. 6, 1196 (1984).
7. V. F. Kravchenko and V. L. Ravchev, in: Lectures at Workshop on DICs [in Russian], Izd. GPI, Moscow (1988), p. 36.
8. V. F. Kravchenko, V. L. Ravchev, and I. V. Taldykin, Dokl. Akad. Nauk SSSR, 302, No. 1, 72 (1988).
9. V. F. Kravchenko, V. L. Rvachev, and Yu. V. Shkvarko, Dokl. Akad. Nauk SSSR, 304, No. 2, 345 (1989).
10. V. L. Rvachev, Theory of R Functions and Some Applications [in Russian], Naukova Dumka, Kiev (1982).
11. V. L. Rvachev and G. P. Man'ko, Computer-Aided Programming for Boundary Value Problems [in Russian], Naukova Dumka, Kiev (1983).
12. L. F. Richardson, Philos. Trans. R. Soc., London, Ser. A, 210 (1910).
13. G. I. Marchuk, Methods in Computational Mathematics [in Russian], Nauka, Moscow (1980).
14. G. F. Zargano et al., Waveguides of Complex Cross Section [in Russian], Radio i Svyaz, Moscow (1986).
15. D. Dasgupta and P. K. Sa Ha, Trans. IEEE., 29, No. 1, 47 (1981).
16. S. G. Mikhlin, Variational Methods in Mathematical Physics [in Russian], Nauka, Moscow (1970).