Mathematical and Numerical Modeling of Natural Convection in an Enclosure Region with Heat-conducting Walls by the R-functions and Galerkin Method

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Abstract—**This paper is dedicated to the investigation of the natural convection in an enclosed region. The mathematical model has been formulated using the dimensionless variables for the stream function and temperature. The numerical results have been obtained by means of the R-functions and Galerkin methods.**

Index Terms—**natural convection, stream function, temperature, R-functions method, Galerkin method.**

I. Introduction

The problem of the natural convection in an enclosed region has vital importance in many technical applications such as microelectronics, radio electronics, energetics etc. Obviously, such problem has a lot of important implications which makes the corresponding investigation actual.

Such problems are mainly resolved using the finite difference and finite element methods. They are easy to program, but they are not universal since a new grid generation is required every time a transition to a new area is made. The R-functions method developed by the academician of the Ukrainian Academy of Sciences V. L. Rvachev allows considering the geometry of the problem accurately [5].

The objective of this work is the mathematical simulation of the natural convection in an enclosed region by means of the R-functions method and Galerkin method.

II. Problem Statement

The mathematical model of the natural convection in an enclosed region with heat-conducting walls in an arbitrary closed region is shown in Fig.1.

Let's consider the $\Omega = \Omega_s \bigcup \Omega_f$ area, where Ω_f is the gas cavity, Ω_s – solid walls, $\partial\Omega_{sf}$ – impermeable and fixed bound between Ω_f and Ω_s. It is assumed that the fluid is Newtonian, incompressible, and viscous.

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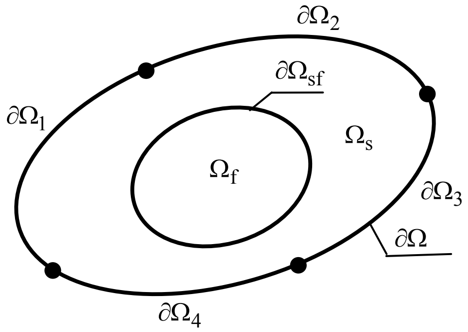

Fig. 1. Problem Solution Region

The mathematical model using the dimensionless variables takes the following form [3]:

in the cavity:

$$\frac{\partial\Delta\psi}{\partial\tau} + \frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y} = \sqrt{\frac{\mathbf{Pr}}{\mathbf{Ra}}}\Delta^2\psi + \frac{\partial\theta}{\partial x}, \quad (1)$$

$$\frac{\partial\theta}{\partial\tau} + \frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} = \frac{1}{\sqrt{\mathbf{Ra}\cdot\mathbf{Pr}}}\Delta\theta, \quad (2)$$

in the solid walls:

$$\frac{\partial\theta}{\partial\tau} = \frac{a_{sf}}{\sqrt{\mathbf{Ra}\cdot\mathbf{Pr}}}\Delta\theta, \quad (3)$$

Where x, y are the dimensionless coordinates,

τ – dimensionless time,

Δ – Laplace operator,

ψ – dimensionless stream function,

θ – dimensionless temperature,

$\mathbf{Ra} = \frac{g\beta TL^3}{\nu a_f}$ – Rayleigh number,

$\mathbf{Pr} = \frac{\nu}{a_f}$ – Prandtl number,

g – acceleration of gravity,

β – coefficient of volumetric thermal expansion,

ν – kinematic coefficient of viscosity,

a_f – temperature diffusivity coefficient of the gas,

$a_{sf} = \frac{a_{solid}}{a_{fluid}}$ – relative temperature diffusivity coefficient,

$\lambda_{sf} = \frac{\lambda_{solid}}{\lambda_{fluid}}$ – relative heat conduction coefficient,

L – length of the gas cavity.

Equation (1) is considered for Ω_f, and equations (2) – (3) are considered for Ω_f and Ω_s respectively.

Initial conditions for the problem (1) – (3) are set as follows:

$$\psi|_{\tau=0} = \psi_0(x,y), \tag{4}$$

$$\theta|_{\tau=0} = \theta_0(x,y). \tag{5}$$

The boundary conditions have the following form:

at external borders:

$$\theta|_{\partial\Omega_1} = \theta_1, \quad \theta|_{\partial\Omega_3} = \theta_2, \tag{6}$$

$$\left.\frac{\partial\theta}{\partial\vec{n}}\right|_{\partial\Omega_2} = 0, \quad \left.\frac{\partial\theta}{\partial\vec{n}}\right|_{\partial\Omega_4} = 0, \tag{7}$$

at internal borders:

$$\psi|_{\partial\Omega_{sf}} = 0, \quad \left.\frac{\partial\psi}{\partial\vec{n}}\right|_{\partial\Omega_{sf}} = 0, \tag{8}$$

$$\theta_s = \theta_f, \quad \frac{\partial\theta_f}{\partial\vec{n}} = \lambda_{sf}\frac{\partial\theta_s}{\partial\vec{n}} \text{ on } \partial\Omega_{sf}, \tag{9}$$

where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3 \cup \partial\Omega_4$, θ_s – temperature in the solid wall, θ_f – temperature in the gas cavity, $\vec{n}$ is a normal vector to the boundary.

III. The R-functions Method

Consider the inverse problem of analytical geometry. Let's consider a geometric object Ω in space R^2 with a piecewise smooth bound $\partial\Omega$. It is required to construct a function $\omega(x,y)$ that would be positive inside Ω, negative outside of Ω and equal to zero at $\partial\Omega$. The equation The equation $\omega(x,y)=0$ determines an implicit form of the locus for the points that belong to the boundary $\partial\Omega$ of the region Ω.

Definition 1. The function with the sign entirely determined by the signs of its arguments is called the R-function corresponding to the partition of the numerical axis within the $(-\infty,0)$ and $[0,+\infty)$ intervals, i.e. the function $z = f(x,y)$ is called the R-function if the Boolean function F exists and $S[z(x,y)] = F[S(x), S(y)]$, where $S(x)$ is a double-valued predicate $S(x) = \begin{cases} 0, & x<0, \\ 1, & x \ge 0. \end{cases}$

The $\Re_\alpha$ is the most widespread R-function system:

$$x \wedge_\alpha y \equiv \frac{1}{1+\alpha}\left(x+y-\sqrt{x^2+y^2-2\alpha xy}\right),$$

$$x \vee_\alpha y \equiv \frac{1}{1+\alpha}\left(x+y+\sqrt{x^2+y^2-2\alpha xy}\right),$$

$$\bar{x} \equiv -x,$$

where

$-1 < \alpha(x,y) \le 1$, $\alpha(x,y) \equiv \alpha(y,x) \equiv \alpha(-x,y) \equiv \alpha(x,-y)$.

Let's consider the Ω region that can be created based on simpler regions $\Omega_1 = \{\omega_1(x,y) \ge 0\}, \ldots, \Omega_m = \{\omega_m(x,y) \ge 0\}$, by means of the of set-theoretic operations such as union, intersection and complement. Therefore, let's assume that the predicate

$$\Omega = F(\Omega_1, \Omega_2, \ldots, \Omega_m) \tag{10}$$

corresponding to the region Ω is equal to 1 if $(x,y) \in \bar{\Omega}$ and is equal to 0 if $(x,y) \notin \bar{\Omega}$.

The transition from the predicate-based form of the region defining (10) to an ordinary analytical geometry equation is made using the formal substitution of Ω with $\omega(x,y)$, Ω_i with $\omega_i(x,y)$ $(i = 1, 2, \ldots, m)$, and the $\{\cap, \cup, \neg\}$ are substituted with the R-operations symbols $\{\wedge_\alpha, \vee_\alpha, -\}$ respectively. As a result, an analytic expression for $\omega(x,y)$ is derived. This expression defines the required equation $\omega(x,y)=0$ of the bound $\partial\Omega$ for the elementary functions. Note that $\omega(x,y)>0$ for the interior points and $\omega(x,y)<0$ for the exterior points of Ω.

Definition 2. The equation $\omega(x,y)=0$ for the bound $\partial\Omega$ of $\Omega \subset R^2$ is normalized to the order n if the function $\omega(x,y)$ satisfies these conditions: $\omega|_{\partial\Omega} = 0$, $\left.\frac{\partial\omega}{\partial\vec{n}}\right|_{\partial\Omega} = -1$, $\left.\frac{\partial^k\omega}{\partial\vec{n}^k}\right|_{\partial\Omega} = 0$ $(k = 2, 3, \ldots, n)$, where $\vec{n}$ is an outer normal vector to $\partial\Omega$, that is defined for all regular points of Ω.

The equation $\omega(x,y)=0$ normalized to the first order can be obtained from the equation $\omega(x,y)=0$ as described below.

Theorem 1. If $\omega(x,y) \in C^m(R^2)$ satisfies the conditions $\omega|_{\partial\Omega} = 0$ and $\left.\frac{\partial\omega}{\partial\vec{n}}\right|_{\partial\Omega} > 0$, then the function

$$\omega_1 \equiv \frac{\omega}{\sqrt{\omega^2 + |\nabla\omega|^2}} \in C^{m-1}(R^2), \quad |\nabla\omega| \equiv \sqrt{\left(\frac{\partial\omega}{\partial x}\right)^2 + \left(\frac{\partial\omega}{\partial y}\right)^2},$$

satisfies the conditions $\omega_1|_{\partial\Omega} = 0$ and $\left.\frac{\partial\omega_1}{\partial\vec{n}}\right|_{\partial\Omega} = -1$ for all regular points of the bound $\partial\Omega$.

We can use this simplified formula: $\omega \equiv \frac{\omega_1}{|\nabla\omega_1|}$ for the equation normalized to the first order if $|\nabla\omega_1| \neq 0$ in $\overline{\Omega} = \Omega \bigcup \partial\Omega$.

Let's consider the R-function application scheme for the boundary problems solving. The problem of the physical field calculation can be reduced to finding the solution u of the equation $Au = f$ within the region Ω under the following conditions on the bound $\partial\Omega$ of Ω: $L_i u = \varphi_i$, $i = 1, ..., m$, where A and L_i are known differential operators; f and φ_i – functions defined inside Ω and in the areas of its boundary $\partial\Omega$. The areas $\partial\Omega_i$ are not necessarily all different, and may coincide with the whole bound $\partial\Omega$. The functions u, f, φ_i and operators A and L_i mentioned in the boundary problem statement are called analytic components of the boundary problem, the area Ω, its boundary $\partial\Omega$, border areas $\partial\Omega_i$ are called geometric components.

The existence of two different types of information (analytical and geometrical) is a major obstacle for the solution finding. Not only the look of the formulas included into the problem statement should be considered, but the geometrical information should be transferred to the analytical look to so that it can be involved into the solution algorithm. The R-functions method allows this procedure implementation.

The sheaves of functions can be built by means of the normalized equations. The normal derivatives of such functions or an arbitrary linear combination of the normal derivative and the function itself take the given values on the region bounds.

In order to achieve this, let's consider the following operator

$$D_1 \equiv \frac{\partial\omega}{\partial x}\frac{\partial}{\partial x} + \frac{\partial\omega}{\partial y}\frac{\partial}{\partial y},$$

where $\omega(x, y)$ is a normalized equation of the region bound. Moreover, for any sufficiently smooth function f on the bound $\partial\Omega$ this statement will be valid:

$$D_1 f\big|_{\partial\Omega} = -\frac{\partial f}{\partial \vec{n}}\bigg|_{\partial\Omega},$$

where $\vec{n}$ is an outer normal vector to $\partial\Omega$.

Let

$$D_1^{(i)} \equiv \frac{\partial\omega_i}{\partial x}\frac{\partial}{\partial x} + \frac{\partial\omega_i}{\partial y}\frac{\partial}{\partial y}$$

denote the analog of D_1 corresponding to the areas $\partial\Omega_i$ of the bound $\partial\Omega$, where $\omega_i(x, y)$ are normalized equations of for the areas $\partial\Omega_i$.

One can prove that

$$D_1\omega = 1 + O(\omega),$$

$$D_1(\omega\Phi) = \ = (D_1\omega)\Phi + \omega D_1\Phi = \Phi + O(\omega),$$

where $\omega(x, y)$ is the normalized equation of the region bound.

Definition 3. The expression

$$u = B(\Phi,\ \omega,\ \{\omega_i\}_{i=1}^{m},\ \{\varphi_j\}_{j=1}^{m})$$

is called the general boundary problem solution structure if that expression exactly satisfies all boundary conditions of the problem for any undetermined component Φ chosen. B is the operator dependent on the geometry of the region and parts of its border, as well as on the operators of the boundary conditions, but is not dependent on the type of operator A and function f.

Let's consider the expression $u = B_i(\Phi,\ \omega,\ \omega_i,\ \varphi_j)$ as a partial solution structure that exactly satisfies the boundary condition only on $\partial\Omega_i$ for any undetermined component.

Thus, the solution structure provides extension of the boundary conditions into the region.

The task of the equation creation for the complex geometric object is a specific case of a more general problem where the unknown function φ takes the given values on different parts of the bound $\partial\Omega_i$, i.e.

$$\varphi = \varphi_i \text{ on } \partial\Omega_i,\ i = 1,\ ...,\ m. \tag{11}$$

For simplicity, let's assume that φ_i are elementary functions defined everywhere in the region $\Omega \bigcup \partial\Omega$. After the methodology described above is applied, the functions ω_i^0 equal to zero everywhere, except for the area $\partial\Omega_i$ are constructed. Thus, the function

$$\varphi = \left(\sum_{i=1}^{m} \varphi_i \omega_i^0\right)\left(\sum_{j=1}^{m} \omega_j^0\right)^{-1} \tag{12}$$

satisfies (11) and is defined everywhere in the region, with the exception of the points that are common to the different sections. Instead of (12) we can also apply the formula

$$\varphi = \left(\sum_{i=1}^{m} \varphi_i \omega_i^{-1}\right)\left(\sum_{j=1}^{m} \omega_j^{-1}\right)^{-1}, \tag{13}$$

where $\omega_i = 0$ are equations of $\partial\Omega_i$ of the bound $\partial\Omega$, and $\omega_i > 0$ outside $\partial\Omega_i$. The function $\omega_i \to 0$ when approaching the area $\partial\Omega_i$ and the limit values of the function φ match the values of the corresponding function φ_i.

Let's denote the bonding operator for the boundary values defined by any of the above formulas (12) and (13) as EC $(EC\varphi_i = \varphi)$.

Practically all of the approximate methods for the boundary problems solving for the partial differential equations are based on the infinite-dimensional problem to a finite-dimensional one reducing. The method of R-functions provides the corresponding result achieving by means of the undetermined component of the solution structure representation as the sum:

$$\Phi(x,y) \approx \Phi_n(x,y) = \sum_{k=1}^{n} c_k \varphi_k(x,y),$$

where $\varphi_k(x,y)$ are known elements of the complete functional sequence, and c_k $(k = 1,2,...,n)$ are unknown expansion coefficients.

The undefined functions included into the structural formulas should be chosen so that the basic differential equation of the problem is satisfied with the best results. The methods of the undefined function approximations search can be very different. For example, one can use the variational (Ritz, least squares, etc.), projection (Galerkin, collocation, etc.), grid and other methods.

IV. Solution Method

The R-functions and Galerkin methods are used for the initial-boundary problem (1) – (9) solving.

Let's consider the boundaries $\partial\Omega$ and $\partial\Omega_{sf}$ that are are piecewise smooth and that can be described by means of the elementary functions $\omega(x,y)$ and $\omega_{sf}(x,y)$. According to the R-functions method, $\omega(x,y)$ and $\omega_{sf}(x,y)$ satisfy the below conditions:

1) $\omega(x,y) > 0$ in Ω;
2) $\omega(x,y) = 0$ on $\partial\Omega$;
3) $\frac{\partial\omega}{\partial\vec{n}} = -1$ on $\partial\Omega$, $\vec{n}$ is an outer normal vector to $\partial\Omega$,

and

1) $\omega_{sf}(x,y) > 0$ in Ω_f;
2) $\omega_{sf}(x,y) = 0$ on $\partial\Omega_{sf}$;
3) $\frac{\partial\omega_{sf}(x,y)}{\partial\vec{n}} = -1$ on $\partial\Omega_{sf}$, $\vec{n}$ is a normal vector pointing into Ω_f.

The investigation in [5] shows that the boundary conditions (7) – (8) are satisfied by the sheaf of functions

$$\psi = \omega_{sf}^2 \Phi,$$

where $\Phi = \Phi(x,y,\tau)$ is an undefined component.

The solution structure of (2) – (3), i.e. the sheaf of functions which satisfies the boundary conditions (5), (6), (9), was built by means of the region-structure Rvachev-Slesarenko method [6]. Hence

$$\theta = \begin{cases} B(\Upsilon) \text{ in } \Omega_s, \\ B(\Upsilon) - (1-\lambda_{sf})\omega_{sf} D_1 B(\Upsilon) \text{ in } \Omega_f, \end{cases} \tag{14}$$

where $\Upsilon = \Upsilon(x,y,t)$ is an undefined component, $B(\Upsilon)$ satisfies the boundary conditions on external borders.

The undefined components Φ and Υ were found by means of the Galerkin method. Therefore, we will obtain an approximate solution of the problem (1) – (9).

V. Numerical Results

Let's consider the mathematical model of natural convection (1) – (3) in a closed region (fig. 2) [3]. It is assumed that the fluid is Newtonian, incompressible and viscous.

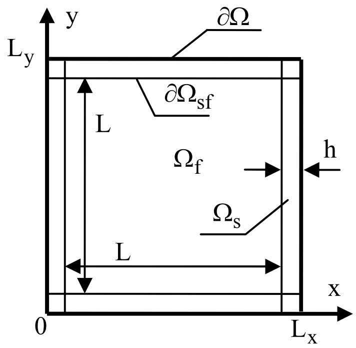

Fig. 2. Problem Solution Region

The initial conditions for the problem (1) – (3) have the below form:

$$\psi\big|_{\tau=0} = \theta\big|_{\tau=0} = 0. \tag{15}$$

The boundary conditions are set as follows:

on external borders:

$$\theta\big|_{x=0} = \theta_1, \quad \theta\big|_{x=L_x} = \theta_2, \text{ where } 0 \le y \le L_y, \tag{16}$$

$$\left.\frac{\partial\theta}{\partial\vec{n}}\right|_{y=0} = 0, \quad \left.\frac{\partial\theta}{\partial\vec{n}}\right|_{L_y} = 0, \text{ where } 0 \le x \le L_x, \tag{17}$$

on internal borders:

$$\psi\big|_{x=h} = \psi\big|_{x=L_x-h} = \psi\big|_{y=h} = \psi\big|_{y=L_y-h} = 0, \tag{18}$$

$$\left.\frac{\partial\psi}{\partial\vec{n}}\right|_{x=h} = \left.\frac{\partial\psi}{\partial\vec{n}}\right|_{x=L_x-h} = \left.\frac{\partial\psi}{\partial\vec{n}}\right|_{y=h} = \left.\frac{\partial\psi}{\partial\vec{n}}\right|_{y=L_y-h} = 0, \tag{19}$$

$$\theta_s = \theta_f, \quad \frac{\partial\theta_f}{\partial\vec{n}} = \lambda_{sf}\frac{\partial\theta_s}{\partial\vec{n}}. \tag{20}$$

where θ_s is the temperature in the solid wall, θ_f – temperature in the gas cavity, $\vec{n}$ – normal vector to the boundary, L_x and L_y are normalized by the length of the gas cavity L.

The functions $\omega(x,y)$ and $\omega_{sf}(x,y)$ have the following form:

$$\omega(x,y) = \frac{1}{L_x}x(L_x - x) \wedge_0 \frac{1}{L_y}y(L_y - y),$$

$$\omega_{sf}(x,y) =$$

$$= \frac{1}{L_x - 2h}(x-h)(L_x - x) \wedge_0 \frac{1}{L_y - 2h}(y-h)(L_y - y).$$

After (14) is applied, $B(\Upsilon)$ satisfies the boundary conditions on external borders (16) – (17), i.e.

$$B(\Upsilon)\Big|_{\frac{x}{L_x}(L_x-x)=0} = \frac{\theta_1(L_x-x)+\theta_2 x}{L_x},$$

$$\frac{\partial B(\Upsilon)}{\partial \vec{n}}\Bigg|_{\frac{y}{L_y}(L_y-y)=0} = 0 .$$

The basic functions used are power polynomials, trigonometric polynomials and Legendre polynomials. The Gauss formula with 16 knots was used for evaluation of integrals in the Galerkin method.

The stream lines, temperature field and vorticity field for $L_X = L_Y = L = 1$, $h = 0.05$, $\mathbf{Ra} = 10^3$, $\mathbf{Pr} = 0.7$, $\lambda_{sf} = 10$, $\theta_1 = 0.5$, $\theta_2 = -0.5$, $a_{sf} = 1$, $T = 5$ are given in figures 1, 2; 3, 4, and 5, 6 for different time respectively.

The results of numerical experiment well correspond to those obtained by the other authors [3].

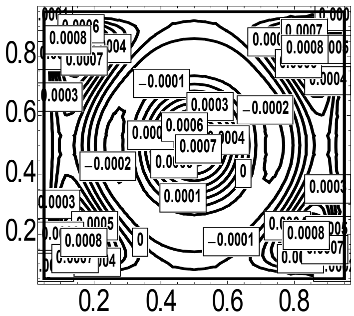

Fig. 3. Stream Lines $t = 0.02$

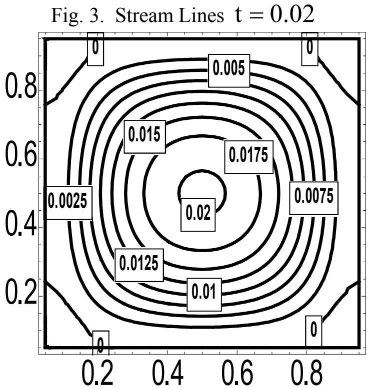

Fig. 4. Stream Lines $t = 3$

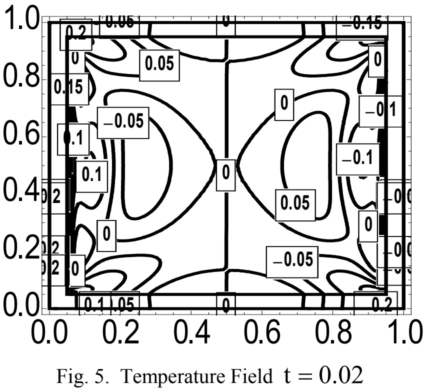

Fig. 5. Temperature Field $t = 0.02$

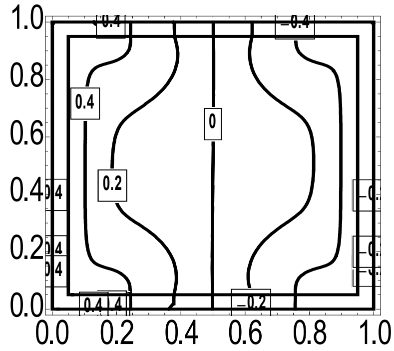

Fig. 6. Temperature Field $t = 3$

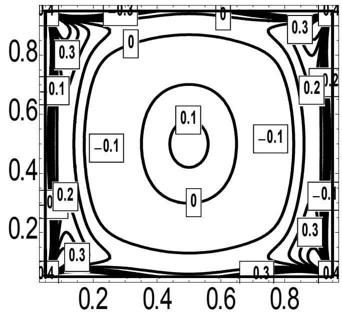

Fig. 7. Vorticity Field $t = 0.02$

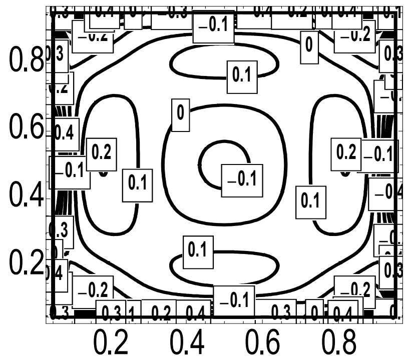

Fig. 8. Vorticity Field $t = 3$

VI. Conclusion

The natural convection in an enclosed region with the presence of local heat is investigated. The solution structures of unknown function were built by means of the R-functions method, and the Galerkin method was used for the approximate undefined components. Thus, the stream function and the temperature were represented in analytical way.

The algorithm for solving the problem of mathematical modeling and numerical analysis of non-stationary natural convection in an enclosed region based on the R-functions method and the Galerkin method is used. The advantage of the suggested algorithm is that it does not have to be modified for different geometries of the regions being reviewed which illustrates the scientific innovation of the results obtained. As a result, the approximate solution for such streams investigation problems is obtained in the non-classic geometry field.

The methods developed for analysis of natural convection in an enclosed region are easy to use for the program algorithms and are more versatile than those used at the present time, as one only needs to change the boundary equation in order to make the transition from one region to another. The obtained results allow us to carry out computational experiments in mathematical modeling of various physical, mechanical, and biological streams.

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