



# Digital Arithmetic

## Computer Systems 2016

Oleks

DIKU

September 14, 2016

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# Truth Tables

NOT

$A$	$\overline{A}$
0	1
1	0

# Truth Tables

## NOT

$A$	$\overline{A}$
0	1
1	0

## AND

$A$	$B$	$A \cdot B$
0	0	0
0	1	0
1	0	0
1	1	1

## OR

$A$	$B$	$A + B$
0	0	0
0	1	1
1	0	1
1	1	1

An **algebra** is a collection of functions adhering to **laws**.

The functions **NOT**, **AND**, and **OR** form a **boolean algebra**:

$$\mathbf{NOT} : \mathbb{B}^1 \rightarrow \mathbb{B}$$

$$\mathbf{AND} : \mathbb{B}^2 \rightarrow \mathbb{B}$$

$$\mathbf{OR} : \mathbb{B}^2 \rightarrow \mathbb{B}$$

# Boolean Algebra Laws - AND form

Name	AND Form
Identity Law	$1 \cdot A = A$
Null Law	$0 \cdot A = 0$
Idempotent Law	$A \cdot A = A$
Inverse Law	$A \cdot \bar{A} = 0$
Commutative Law	$A \cdot B = B \cdot A$
Associative Law	$(A \cdot B) \cdot C = A \cdot (B \cdot C)$
Distributive Law	$A + B \cdot C = (A + B) \cdot (A + C)$
Absorption Law	$A \cdot (A + B) = A$
De Morgan's Law	$\overline{A \cdot B} = \bar{A} + \bar{B}$



# Boolean Algebra Laws - OR form

Name	OR Form
Identity Law	$0 + A = A$
Null Law	$1 + A = 1$
Idempotent Law	$A + A = A$
Inverse Law	$A + \overline{A} = 1$
Commutative Law	$A + B = B + A$
Associative Law	$(A + B) + C = A + (B + C)$
Distributive Law	$A \cdot (B + C) = A \cdot B + A \cdot C$
Absorption Law	$A + A \cdot B = A$
De Morgan's Law	$\overline{A + B} = \overline{A} \cdot \overline{B}$

# Truth Tables

## NOT

$A$	$\overline{A}$
0	1
1	0

## AND

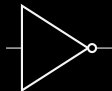
$A$	$B$	$A \cdot B$
0	0	0
0	1	0
1	0	0
1	1	1

## OR

$A$	$B$	$A + B$
0	0	0
0	1	1
1	0	1
1	1	1

# Gates

NOT



AND



OR



# More Truth Tables

## XOR

$A$	$B$	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

## NAND

$A$	$B$	$A \uparrow B$
0	0	1
0	1	1
1	0	1
1	1	0

## NOR

$A$	$B$	$A \downarrow B$
0	0	1
0	1	0
1	0	0
1	1	0

# More Truth Tables

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Do we add these as basic functions to our algebra?

# More Truth Tables

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0	1	0
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Do we add these as basic functions to our algebra?

Do we need fancy new hardware?

# Admissibility

A function is **admissible** if

it can be derived from other functions

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A function is **admissible** if

it can be derived from other functions

$\equiv$

it does not contradict the laws of the algebra



# XOR is admissible

— Proof by reading the truth table.

$A$	$B$	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

$$A \oplus B \equiv (\overline{A} \cdot B) + (A \cdot \overline{B}) \quad \text{(read 1s)}$$

$$\equiv \overline{(\overline{A} \cdot \overline{B})} + (A \cdot B) \quad \text{(read 0s)}$$

“ $A$  or  $B$ , but not both  $A$  and  $B$ ”

# NAND is admissible

— Proof by reading the truth table.

$A$	$B$	$A \uparrow B$
0	0	1
0	1	1
1	0	1
1	1	0

$$\begin{aligned} A \uparrow B &\equiv (\overline{A} \cdot \overline{B}) + (A \cdot \overline{B}) + (\overline{A} \cdot B) && \text{(read 1s)} \\ &\equiv \overline{(A \cdot B)} && \text{(read 0s)} \end{aligned}$$

“not both  $A$  and  $B$ ”

# NOR is admissible

— Proof by reading the truth table.

$A$	$B$	$A \downarrow B$
0	0	1
0	1	0
1	0	0
1	1	0

$$A \downarrow B \equiv (\overline{A} \cdot \overline{B}) \quad \text{(read 1s)}$$

$$\equiv \overline{(A \cdot \overline{B}) + (\overline{A} \cdot B) + (A \cdot B)} \quad \text{(read 0s)}$$

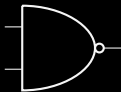
“neither  $A$  nor  $B$ ”

# More Gates

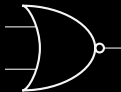
**XOR**



**NAND**



**NOR**



We like fancy new hardware!

{ AND, OR, NOT }

is not the **only** possible set of basic functions  
for a boolean algebra.

**AND is admissible in { OR, NOT }**

$$A \cdot B \equiv \overline{\overline{A} + \overline{B}}$$

— Proof by truth table.

# AND is admissible in { OR, NOT }

$$A \cdot B \equiv \overline{\overline{A} + \overline{B}}$$

— Proof by truth table.

$A$	$B$	$A \cdot B$	$\overline{A}$	$\overline{B}$	$\overline{A} + \overline{B}$	$\overline{\overline{A} + \overline{B}}$
0	0	0	1	1	1	0
0	1	0	1	0	1	0
1	0	0	0	1	1	0
1	1	1	0	0	0	1

**OR is admissible in { AND, NOT }**

$$A + B \equiv \overline{\overline{A} \cdot \overline{B}}$$

— Proof by truth table.



# OR is admissible in { AND, NOT }

$$A + B \equiv \overline{\overline{A} \cdot \overline{B}}$$

— Proof by truth table.

$A$	$B$	$A + B$	$\overline{A}$	$\overline{B}$	$\overline{A} \cdot \overline{B}$	$\overline{\overline{A} \cdot \overline{B}}$
0	0	0	1	1	1	0
0	1	1	1	0	0	1
1	0	1	0	1	0	1
1	1	1	0	0	0	1

# Optional Exercises

**NOT** is admissible in { **NAND** }

**AND** is admissible in { **NAND** }

**OR** is admissible in { **NAND** }

# Optional Exercises

**NOT** is admissible in { **NAND** }

**AND** is admissible in { **NAND** }

**OR** is admissible in { **NAND** }

**NOT** is admissible in { **NOR** }

**AND** is admissible in { **NOR** }

**OR** is admissible in { **NOR** }

# Optional Exercises

**NOT** is admissible in  $\{ \text{NAND} \}$

**AND** is admissible in  $\{ \text{NAND} \}$

**OR** is admissible in  $\{ \text{NAND} \}$

**NOT** is admissible in  $\{ \text{NOR} \}$

**AND** is admissible in  $\{ \text{NOR} \}$

**OR** is admissible in  $\{ \text{NOR} \}$

**NAND** is admissible in  $\{ \text{NOR} \}$

**NOR** is admissible in  $\{ \text{NAND} \}$

# Optional Exercises

**NOT** is admissible in { **NAND** }

**AND** is admissible in { **NAND** }

**OR** is admissible in { **NAND** }

**NOT** is admissible in { **NOR** }

**AND** is admissible in { **NOR** }

**OR** is admissible in { **NOR** }

**NAND** is admissible in { **NOR** }

**NOR** is admissible in { **NAND** }

**Conclusion:** We can make due with different **basic** gates.