Termination analysis of first order programs

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$$F(M) = \begin{cases} \mathit{true} & H(M,M) \rightsquigarrow \mathit{false}, \\ \mathit{false} & H(M,M) \rightsquigarrow \mathit{true}. \end{cases}$$

Consider $F(F)$.

$$H(M,x) = \begin{cases} \textit{true} & M \text{ halts on } x, \\ \textit{false} & M \text{ does not halt on } x, \\ \textit{unknown} & M \text{ may or may not halt on } x. \end{cases}$$

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“Unfortunately, many have drawn too strong of a conclusion about the prospects of automatic program termination proving and falsely believe we are always unable to prove termination, rather than the more benign consequence that we are unable to always prove termination.”

[Cook et al., 2011]

"The **size-change termination principle** for a first-order functional language with well-founded data is: a program terminates on all inputs if *every infinite call sequence* (following program control flow) would cause an infinite descent in some data values."

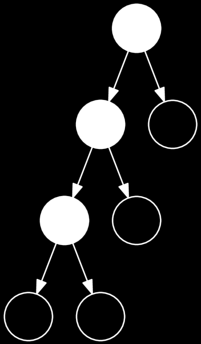
[Lee et al., 2001]

Δ

An untyped, call-by-value, functional first-order language.

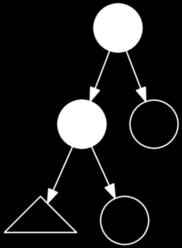
Δ, values and shapes

$b \in \mathbb{B}$



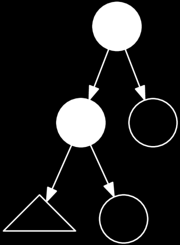
$\succ$

$s \in \mathbb{S}$



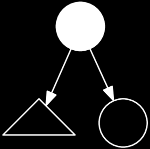
Δ, shapes and shapes

$s_1 \in \mathbb{S}$



$\succ$

$s_2 \in \mathbb{S}$



Disjoint shapes

$$s_1 \cap s_2 = \emptyset \quad \text{iff} \quad B_1 \cap B_2 = \emptyset$$

where

$$s_1, s_2 \in \mathbb{S} \wedge B_1 = \{b \mid b \in \mathbb{B} \wedge b \succ s_1\} \wedge B_2 = \{b \mid b \in \mathbb{B} \wedge b \succ s_2\}$$

Given a shape $s_i \in \mathbb{S}$, we define the **sibling set** S_i^d, to be the pairwise disjoint set of shapes disjoint with s_i.

```haskell
data Pattern
  = PNil
  | PVariable String
  | PNode Pattern Pattern

getSiblings :: Pattern -> [Pattern]

getSiblings PNil =
  [PNode (PVariable "_") (PVariable "_")]

getSiblings (PVariable _) = []

getSiblings (PNode leftP rightP) =
  let
    leftS = getSiblings leftP
    rightS = getSiblings rightP
    leftInit = map (\s -> PNode leftP s) rightS
    rightInit = map (\s -> PNode s rightP) leftS
  in
    [PNil] ++
      leftInit ++ rightInit ++
      interleaveSiblings name leftS rightS
```

$$T(1) = 4$$
$$T(n) = 1 + T(n-1) + T(n-1) + T(n-1) \cdot T(n-1)$$

$$2^{\lceil log(n) \rceil} \cdot \left(4 + \frac{25}{2} + \frac{676}{4} + \frac{458239}{8} + \ldots\right)$$

$$S_1 \Cup S_2 = \left\{ s \,\middle|\, \begin{array}{ll} & \big(s \in S_1 \wedge \big(\exists\, s' \in S_2\ s \cap s' \neq \emptyset \longrightarrow s \succ s'\big)\big) \\ \vee & \big(s \in S_2 \wedge \big(\exists\, s' \in S_1\ s \cap s' \neq \emptyset \longrightarrow s \succ s'\big)\big) \end{array} \right\}.$$

Δ, syntax

```
        <program> ::= <clause>+ <expression>
     <expression> ::= <element> ( ‘.’ <expression> ) ?
        <element> ::= ‘0’ | ‘(’ <element> ‘)’ | <name> | <application>
    <application> ::= <name> <expression>*
         <clause> ::= <name> <pattern>* ‘:=’ <expression>‘;’
        <pattern> ::= <pattern-element> ( ‘.’ <pattern> ) ?
<pattern-element> ::= ‘0’ | ‘_’ | ‘(’ <pattern> ‘)’ | <name>
           <name> ::= [‘a’-‘z’] ( [‘-’ ‘a’-‘z’]* [‘a’-‘z’] ) ?
```

Δ, sample programs

```
reverse 0 := 0
reverse left.right := (reverse right).(reverse left)

reverse input
```

```
fibonacci n = fibonacci-aux (normalize n) 0 0

fibonacci-aux 0 x y := 0
fibonacci-aux 0.0 x y := y
fibonacci-aux 0.n x y := fibonacci-aux n y (add x y)

fibonacci input
```

```
ackermann 0 n := 0.n
ackermann a.b 0 := ackermann (decrease a.b) 0.0
ackermann a.b c.d :=
  ackermann (decrease a.b) (ackermann a.b (decrease c.d))

ackermann input input
```

Description	**Instance**	**Finite list**	**Space**
Expression	x	X	$\mathbb{X}$
Element (of an expression)	e	E	$\mathbb{E}$
Function	f	F	$\mathbb{F}$
Clause	c	C	$\mathbb{C}$
Pattern	p	P	$\mathbb{P}$
Value (think "binary")	b	B	$\mathbb{B}$
Name (think "variable")	v	V	$\mathbb{V}$
Program (*p* was taken)	r	R	$\mathbb{R}$
Shape	s	S	$\mathbb{S}$

Δ, syntax

```
          <program> ::= <clause>+ <expression>
       <expression> ::= <element> ( ‘.’ <expression> ) ?                                x
          <element> ::= ‘0’ | ‘(’ <element> ‘)’ | <name> | <application>                e
      <application> ::= <name> <expression>*                                        ⟨v,X⟩
           <clause> ::= <name> <pattern>* ‘:=’ <expression>‘;’                 c = ⟨v,P,x⟩
          <pattern> ::= <pattern-element> ( ‘.’ <pattern> ) ?                           p
  <pattern-element> ::= ‘0’ | ‘_’ | ‘(’ <pattern> ‘)’ | <name>                          p
             <name> ::= [‘a’-‘z’] ( [‘-’ ‘a’-‘z’]* [‘a’-‘z’] ) ?                        v
```

Δ, syntax

```
        <program> ::= <clause>+ <expression>
     <expression> ::= <element> ( ‘.’ <expression> ) ?                                x
        <element> ::= ‘0’ | ‘(’ <element> ‘)’ | <name> | <application>                 e
    <application> ::= <name> <expression>*                                        ⟨v,x⟩
         <clause> ::= <name> <pattern>* ‘:=’ <expression>‘;’               c = ⟨v,p,x⟩
        <pattern> ::= <pattern-element> ( ‘.’ <pattern> ) ?                            p
<pattern-element> ::= ‘0’ | ‘_’ | ‘(’ <pattern> ‘)’ | <name>                           p
           <name> ::= [‘a’-‘z’] ( [‘-’ ‘a’-‘z’]* [‘a’-‘z’] ) ?                         v
```

Functions in Δ

$(15, 36)$

$$f = \langle v, C \rangle \quad \text{s.t.} \quad \forall \ \langle v', \_, \_ \rangle \in C \ \left(v' = v \right)$$

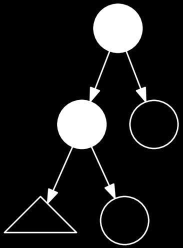
Pattern matching is ensured **exhaustive** at compile time, i.e.

$$\forall \ b \in \mathbb{B} \ \exists \ c \in C \ c \succ b.$$

$$\text{WLOG, } c = \langle p, x \rangle.$$

Patterns and shapes

```
f (_.0).0 := ...
```



```
Ensure-Exhaustive($c : C$)
$P_{siblings} = $ Get-Siblings($c$)
$C' = [c]$
for $c' \in C$
    $(P_{success}, P_{fail}) = $ Match-Clause-To-Siblings($c, P_s$)
    for $p \in P_{success}$
        $c'' = $ Clone($c'$)
        Merge-Pattern($c'', p$)$^*$
        $C' = c' : C'$
    $P_{siblings} = P_{fail}$
return $C'$
```

Invariants:

- $P_{siblings}$ is always a list of siblings that wasn't matched by any forthcoming clause.
- $P_{success}$ and P_{fail} are always sibling lists.

$$S_1 \Cup S_2 = \left\{ s \,\middle|\, \begin{array}{ll} & (s \in S_1 \wedge (\exists\, s_2 \in S_2\ s \cap s_2 \neq \emptyset \longrightarrow s \succ s_2)) \\ \vee & (s \in S_2 \wedge (\exists\, s_1 \in S_1\ s \cap s_1 \neq \emptyset \longrightarrow s \succ s_1)) \end{array} \right\}$$

There’s a small detail missing…

Demo

Programs in Δ

$$r = \langle F, x \rangle$$

WLOG, $r = \langle C, x \rangle$

“The **size-change termination principle** for a first-order functional language with well-founded data is: a program terminates on all inputs if *every infinite call sequence* (following program control flow) would cause an infinite descent in some data values.”

[Lee et al., 2001]

Call graph for $r = \langle C, x \rangle$

$$G = \langle C, E \rangle$$

$$E = \left\{ \langle v_s^c, v_t^c, v_s, v_t, x \rangle \;\middle|\; \begin{array}{ll} & \langle v_s^c, \_, x_s^c \rangle, \langle v_t^c, p_t, \_ \rangle \in C \\ \wedge & \langle v_t^c, x \rangle \Subset x_s^c \\ \wedge & v_s \Subset x_s \\ \wedge & v_t \Subset p_t \end{array} \right\}$$

$$\Phi = \left\{ \langle e, \rho \rangle \;\middle|\; \begin{array}{cl} & e \in E \\ \wedge & \rho \in \{\bot, <, \leq\} \end{array} \right\}$$

Initially, let $\forall\, e \in E\ \Phi(e) = \bot$.

For each $e = \langle v_s^c, v_t^c, v_s, v_t, x \rangle \in E$, let p_x represent x as a pattern where all calls have been replaced by _, and let p_t be the pattern of the clause v_t^c ..

$$\frac{(p_x = 0 \lor (p_x = v \land v \neq v_s)) \land \Phi \to \Phi_1}{\langle \mathrm{A}, p_x, p_t, \Phi \rangle \to \Phi_1}$$

```
f a := f 0
f a := f _
```

$$\frac{(p_t = 0 \lor p_t = \_) \land \Phi \to \Phi_1}{\langle \mathrm{B}, p_x, p_t, \Phi \rangle \to \Phi_1}$$

```
f 0 := f ...
f _ := f ...
```

$$\frac{p_t = v_t \wedge p_x = v_s \wedge \langle \Phi\,(e) \mapsto \leq \rangle \to \Phi_1}{\langle \mathrm{C}, p_x, p_t, \Phi \rangle \to \Phi_1}$$

```
f a := f a
```

$$\frac{p_t = p_{t_1} \cdot p_{t_2} \wedge p_x = v_s \wedge \langle \Phi\left(e\right) \mapsto < \rangle \rightarrow \Phi_1}{\langle \mathrm{D}, p_x, p_t, \Phi \rangle \rightarrow \Phi_1}$$

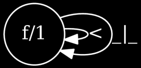
```
f a.b := f a
```

$$\frac{p_t = p_{t_1} \cdot p_{t_2} \wedge p_s = p_{x_1} \cdot p_{x_2} \wedge \langle p_{t_1}, p_{x_1}, \Phi \rangle \to \Phi_2 \wedge \langle p_{t_2}, p_{x_2}, \Phi_2 \rangle \to \Phi_1}{\langle \mathrm{E}, p_x, p_t, \Phi \rangle \to \Phi_1}$$

```
f a.b := f (...).(...)
```

Size-change termination

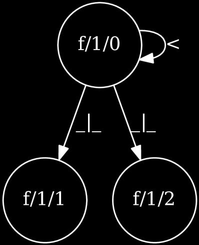
```
f 0.a := f a;
f a := a;
(f (0.0).0).0
```



Shape-change termination

```
f 0.a := f a;
f a := a;
(f (0.0).0).0
```

```
f/1
0: [(0.a)] := (f [(a [])]) / ["f/1"]
1: [a@((_._)._)] := (a []) / ["f/1"]
2: [a@0] := (a []) / ["f/1"]
((f [((0.0).0)]).0)
```



Demo

Observed mistakes

List indexing

- Lists are sometimes 0-indexed rather than 1-indexed.
- $\forall \; \{i \mid 0 \geq i < |P|\}$ should obviously be $\forall \; \{i \mid 0 < i \leq |P|\}$.

A few type errors, like e.g. $S \Subset p$, where $S \subset \mathbb{S} \wedge p \in \mathbb{P}$.

The $\Cup$ relation is flawed.

Superflous conditions.

References

[Cook et al., 2011] B. Cook, A. Podelski & A. Rybalchenko, *Proving program termination*, Communications ACM Vol. 54(5), 2011, 88–98.

[Lee et al., 2001] Chin Soon Lee, Neil D. Jones & Amir M. Ben-Amram, *The size-change principle for program termination*, POPL '01, 81–92.