Programming Languages for Feasible Programs

Datalogisk institut, Copenhagen University (DIKU)
Master's Thesis

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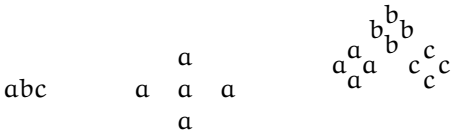
Chapter 0

Preface

0.1 Formal Systems

Notion 1. *A formal system, $\mathcal{F}$, is a system of symbols and rules governing their manipulation.*

A symbol of $\mathcal{F}$ is distinguished from all other symbols of $\mathcal{F}$. An object of $\mathcal{F}$ is a combination of some symbols of $\mathcal{F}$. For instance, given the symbols a, b, and c, we may consider, among others, the following objects:



The rules of $\mathcal{F}$ enable us to make judgements about the objects of $\mathcal{F}$. A judgement is a statement about some objects of $\mathcal{F}$, which holds if it can be inferred from the rules of $\mathcal{F}$. Judgements come in various forms, judging over a number of objects. For instance, given the objects n, n_1, and n_2, we may consider, among others, the following forms of judgement:

Judgement form	**Reading**
n nat	n is a natural number.
$n = n_1 + n_2$	n is the sum of n_1 and n_2.

The rules of $\mathcal{F}$ are rules of inference, dealing in judgements. A rule is a relation between some judgements, called its premises, and a judgement, called its conclusion[1]. The rule, provided evidence that its premises hold, permits to infer that the conclusion holds.

Let us adopt the common "bar-notation" for denoting rules, with the premises of a rule presented above a horizontal line, and the conclusion below. For instance, given the symbols $\mathtt{z}$ (read: zero) and $\mathtt{s}$ (read: successor), we may consider the judgement n nat to be defined as follows:

[1] Although a set of premises may permit us to infer multiple conclusions, we assume that they are inferred by different rules.

$$\textsc{NatZ-Zero}: \frac{}{\mathtt{z}\ \mathrm{nat}} \quad \textsc{NatZ-Rec}: \frac{\mathtt{n}\ \mathrm{nat}}{\mathtt{sn}\ \mathrm{nat}} \tag{1}$$

Informally, these rules state that a natural number is either zero, or a successor of a natural number.

Rules without premises are typically called axioms in that they hold unconditionally. For instance, zero is always a natural number by the NAT-ZERO axiom. The conclusion of one rule may form a premise of another, perhaps the same, rule. This permits us to combine rule applications into proofs of judgements. A proof is a witness that the judgement holds.

For instance, 1 permits us to make the judgement sssz nat:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathtt{z}\ \mathrm{nat}}}{\mathtt{sz}\ \mathrm{nat}}}{\mathtt{ssz}\ \mathrm{nat}}}{\mathtt{sssz}\ \mathrm{nat}}$$

An object may be empty, that is, not consist of any symbols what-so-ever. For instance, given some symbol • (read: star), we may consider the judgement $\mathtt{n}$ nat to be defined as follows:

$$\textsc{NatE-Zero}: \frac{}{\mathrm{nat}} \quad \textsc{NatE-Rec}: \frac{\mathtt{n}\ \mathrm{nat}}{\bullet\mathtt{n}\ \mathrm{nat}} \tag{2}$$

Informally, these rules state that a natural number is either the empty object, or a • followed by a natural number. Meaning that we can deduce which natural number (in the classical sense) a natural number in $\mathcal{F}$ represents by counting the number of •s. For instance, 2 permits us to make the judgement • • • nat:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{nat}}}{\bullet\ \mathrm{nat}}}{\bullet\ \bullet\ \mathrm{nat}}}{\bullet\ \bullet\ \bullet\ \mathrm{nat}}$$

Empty objects can be a bit cryptic however, as they are not clearly distinguished from other objects. It is convenient to adopt the following notation:

Definition 1. *Let ε (read: epsilon) denote the empty object.*

2 can now be restated as follows:

$$\textsc{Nat-Zero}: \frac{}{\varepsilon\ \mathrm{nat}} \quad \textsc{Nat-Rec}: \frac{\mathtt{n}\ \mathrm{nat}}{\bullet\mathtt{n}\ \mathrm{nat}} \tag{3}$$

Another informal reading of these rules is that a natural number is either the empty sequence, or a sequence of •s. Such a representation is convenient as we can determine which natural number (in the classical sense) $\mathtt{n}$ nat represents by counting the number of •s in $\mathtt{n}$.

As another example, let the judgement s bitseq, reading s is a binary digit (bit) sequence, be defined as follows:

$$\textsc{BitSeq-Zero}: \frac{}{\varepsilon \text{ bitseq}} \qquad \textsc{BitSeq-Rec0}: \frac{s \text{ bitseq}}{0s \text{ bitseq}} \quad \textsc{BitSeq-Rec1}: \frac{s \text{ bitseq}}{1s \text{ bitseq}} \tag{4}$$

Informally, these rules state that a bit sequence is either the empty sequence, or a sequence of bits (0 or 1).

The rules of 3 and 4 bear a lot of resemblance. Both permit the sequencing of symbols chosen from an alphabet. Such sequencing will be frequent throughout the thesis and so we define a generalized recursor for it:

Definition 2. *Given a judgement $w\ \Sigma$, let the judgement $s\ \Sigma^*$ be defined as follows:*

$$\textsc{Star-Zero}: \frac{}{\varepsilon\ \Sigma^*} \quad \textsc{Star-Rec}: \frac{w\ \Sigma \quad s\ \Sigma^*}{ws\ \Sigma^*} \tag{5}$$

For instance, let w bullet be defined as follows:

$$\frac{}{\bullet \text{ bullet}} \tag{6}$$

Theorem 1. *s bullet* $\Leftrightarrow$ s nat.*

Proof. Each direction by induction on the structure of s.. □

Similarly, let w bit be defined as follows:

$$\frac{}{0 \text{ bit}} \quad \frac{}{1 \text{ bit}} \tag{7}$$

Theorem 2. *s bit* $\Leftrightarrow$ s bitseq.*

Proof. Each direction by induction on the structure of s.. □

Next: for each $s\ \Sigma^*$, there is a unique n nat (recursive enumerability).

Formal systems are a purely mental construct. To be used to any effect in the material world, a formal system must be realised by a physical system.

Notion 2. *A physical system, $\mathcal{P}$, is a system of physical symbols and processes manipulating them.*

$\mathcal{P}$ can be a reasonable realisation of formal system $\mathcal{F}$, provided that (1) there is a reasonable correspondence between the symbols of $\mathcal{F}$ and the physical symbols of $\mathcal{P}$, and (2) $\mathcal{P}$ manipulates the physical symbols in reasonable accordance with the rules of $\mathcal{F}$. For instance, this report is (hopefully) a reasonable realisation of the formal systems that the author had in mind.

If $\mathcal{P}$ is a reasonable realisation of $\mathcal{F}$, $\mathcal{F}$ can serve to specify the desired (or perceived) behaviour of $\mathcal{P}$. Such a specification becomes useful in communicating how a physical system is to be employed to solve a real-world problem.

Notion 3. *$\mathcal{F}$ **solves** a problem P, if P can be stated as formula f in $\mathcal{F}$, and after a finite sequence of symbolic manipulations legal for $\mathcal{F}$, we arrive at a formula f', which can be deemed a statement of a solution to P.*

Formal systems, and so their realisations, bear little intrinsic purpose — symbolic manipulation is a means to an end for more purposeful beings. In how far a realisation is reasonable, is therefore a matter of the intended **purpose** of the system.

Notion 4. *A formal system $\mathcal{F}$ can solve a **problem** P, if P can be expressed with the symbols of $\mathcal{F}$, in accordance to the rules of $\mathcal{F}$,*

A system specification is useful to more purposeful beings for communicating how a system can be used to solve a particular problem. The class of problems which can be solved by rote symbolic manipulation is the class of **computable** problems.

Definition 3. *A problem is **computable** if it can be solved by performing a sequence of symbolic manipulations.*

Definition 4. *A **derivation** is a sequence of rule applications.*

Formal systems in this regard, become useful in communicating how a physical system is to be used to solve a particular problem.

Physical systems realising formal systems are often referred to as **computers**, and the activity of symbolic manipulation as **computation**.

Before the mid-20th century, computation was generally the faculty of human beings, with the help of paper and pencil, and perhaps, an abacus. With the whirlwind of world history, human realisation of formal systems became unreasonable, and we turned to computation by machines, less prone to err, defect, or betray, and much faster at it.

This has allowed us to deal in a range of formal systems which we can reasonably assume to be **absolutely realisable**, i.e. having physical realisations which do not deviate from their formal specification. For instance, we can reasonably assume that the Intel(R) Core(TM) i7-4600U processor performs computation in accordance with its data sheet.

Dealing in absolutely realisable formal systems directly has proven a challenge however.

As formal systems are a means to an end for more purposeful beings, we are not only concerned with employing a formal system,

This turn allowed for an elevation from much concern about physical realisation, and today we generally assume that modern computers absolutely realise the formal systems that specify them. As a result, the field of Computer Science is generally concerned with (1) what can be solved using formal systems, with **reasonable elegance** (2) , while (3) staying within the realm of reasonable realisability.

To comply with (3), ...

Much like a physical system can realise a formal system, a formal system $\mathcal{G}$ can realise a formal system $\mathcal{F}$. This is usually referred to as a **simulation**.

Definition 5. *A formal system $\mathcal{F} = \langle \mathcal{F}_S, \mathcal{F}_R \rangle$ can be simulated by a formal system $\mathcal{G} = \langle \mathcal{G}_S, \mathcal{G}_R \rangle$, provided that (1) there is a 1-1 correspondence between the $\mathcal{F}_S$ and $\mathcal{G}'_S$, where $\mathcal{G}'_S \subseteq \mathcal{G}_S$, and (2) $\mathcal{G}_R$ allow to manipulate $\mathcal{G}'_S$ in accordance with $\mathcal{F}_R$.*

0.2 Audience

The audience of this thesis is anyone interested in the connection of computability and complexity to the theory of programming languages. In particular, what the admittance of particular programming language constructs implies for the complexity of the programs that you can write.

The thesis is directed towards the level of a Computer Science graduate student at the time of writing: The reader is assumed to be familiar with the basics of discrete mathematics, as in [Graham et al. (1998)]. The analysis of time and space complexity of algorithms, as in [Cormen et al. (2009)], §§1–17 and §§21–24. Regular and context-free languages, as in [Sipser (2013)] §§1–2, and their use for programming language design, as in [Mogensen (2010)]. The reader should also be familiar with Logic in Computer Science, as in [Huth & Ryan (2004)], §§1–4.

0.3 Preliminaries

Definition 6. *Let* $\mathbb{N}$ *denote the type of natural numbers, including* 0.

Definition 7. *Given a type* Σ, *let* Σ^* *be the type of* ***symbolic strings*** *over* Σ, *formed using either the* ***empty string*** *or the* ***string concatenation*** *operator:*

$$\frac{}{\varepsilon_\Sigma : \Sigma^*} \qquad \frac{s_0 : \Sigma \quad s : \Sigma^*}{s_0 \cdot s : \Sigma^*}$$

We refer to Σ *as an* ***alphabet****, and to the terms of* Σ *as* ***symbols****.*

We omit Σ in ε_Σ, when it is clear from context, e.g. $\varepsilon : \Sigma^*$.

Definition 8. *We say that a term* $s \equiv s_0 \cdot s_1 \cdots s_{n-1} \cdot \varepsilon : \Sigma^*$ *is a string of length* $n : \mathbb{N}$ *over the alphabet* Σ.

Definition 9. *Let* $|\Sigma|$ *be the number of symbols in alphabet* Σ, *called it's* ***cardinality****.*

If the cardinality of an alphabet is finite, strings over the alphabet are **recursively enumerable**, i.e. there exists a bijection $f : \Sigma^* \to \mathbb{N}$.

Theorem 3. *If* $|\Sigma| = n$ *for some* $n \in \mathbb{N}$, *then* $|\Sigma^*| = |\mathbb{N}|$.

Proof. We have $|\Sigma| = n$. The terms of Σ can be arranged in a sequence such that for each $s : \Sigma$ we assign a unique natural number $g(s)$, such that $1 \leq g(s) \leq |\Sigma|$. We now inductively define the function $f : \Sigma^* \to \mathbb{N}$.

$$f(\varepsilon) = 0 \tag{8}$$
$$f(s_0 \cdot s) = g(s_0) + f(s) \cdot |\Sigma| \tag{9}$$

To show that the function is a bijection, we also define its inverse by induction:

$$f^{-1}(0) = \varepsilon \tag{10}$$
$$f^{-1}(n) = g^{-1}(n \mod |\Sigma|) \cdot f^{-1}(\lfloor n / |\Sigma| \rfloor) \qquad \text{for } n > 0 \tag{11}$$

□

This means that strings over any finite alphabet can be used to represent strings over any other finite alphabet. One such basic alphabet that has proven useful in practice is the binary alphabet, consisting of e.g. the symbols `0` and `1`.

Definition 10. *Let* $\mathbb{B}$ *denote the type of booleans.*

Define the set $\mathbb{B}$, and the usual boolean connectives.

Definition 11. *An infix function* $\leq : A \times A \to \mathbb{B}$ *defines a* ***total order*** *on the type* A *iff for all* $x, y, z : A$*:*

$$x \leq y \wedge y \leq x \Rightarrow y = x \quad \textit{(antisymmetry)} \tag{12}$$
$$x \leq y \wedge y \leq z \Rightarrow x \leq z \quad \textit{(transitivity)} \tag{13}$$
$$x \leq y \vee y \leq x \quad \textit{(totality)} \tag{14}$$

Definition 12. *Given a total order on Σ, we define the **lexicographic order** on Σ^* as follows:*

$$\frac{}{\varepsilon \leq s_0 \cdot s} \quad \frac{s_0 = t_0 \quad s \leq t}{s_0 \cdot s \leq t_0 \cdot t} \quad \frac{s_0 \neq t_0 \quad s_0 \leq t_0}{s_0 \cdot s \leq t_0 \cdot t} \tag{15}$$

Theorem 4. *A lexicographic order is a total order.*

Proof. □

Definition 13. *An infix function $<\colon A \times A \to \mathbb{B}$ defines a **strict total order** on the type A, iff*

$$\text{F-LESS:} \frac{\neg\,(y \leq x)}{x < y} \tag{16}$$

We say that y has a higher **value** than x, whenever $x < y$, and equal in value whenever $x = y$.

Definition 14. *A function f is defined by **primitive recursion** from the functions $h, g_1, g_2, \ldots, g_n$ for $n : \mathbb{N}$, iff*

$$f(\varepsilon, \overline{y}) = h(\overline{y}) \tag{17}$$
$$f(s_i(x), \overline{y}) = g_i(x, \overline{y}, f(x, \overline{y})) \tag{18}$$
$$x < s_i(x) \tag{19}$$

That is, a function is defined by primitive recursion, if on every invocation of the function, we recurse on at most one formal parameter, and only recurse after the actual parameter has been decreased in value. It follows that primitive recursion demands a strict total order on the value type in question.

Definition 15. *A function is **primitive recursive** if it is non-recursive, or defined by primitive recursion from non-recursive, or primitive recursive functions.*

Primitive recursive functions are not necessarily polytime functions.

Example 1. *Unary addition over binary notation is primitive recursive, but not polytime.*

Let $\leq$ be the lexicographic order on binary notation.

We now define unary addition over binary notation using primitive recursion:

$$\mathtt{add}'(0,0,0) = (0,0) \tag{20}$$
$$\mathtt{add}'(1,0,0) = (1,0) \tag{21}$$
$$\mathtt{add}'(0,1,0) = (1,0) \tag{22}$$
$$\mathtt{add}'(0,0,1) = (1,0) \tag{23}$$
$$\mathtt{add}'(0,0,1) = (1,0) \tag{24}$$
$$\mathtt{add}'(1,1,0) = (0,1) \tag{25}$$
$$\mathtt{add}'(1,1,1) = (1,1) \tag{26}$$

Part I

Background

Chapter 1

Computability

Notion 5. *A problem is "computable" if it can be solved by transforming a mathematical object over a finite amount of time, without ingenuity.*

Any attempt at a more definite notion of computability seems to arrive at a philosophical impasse, where the notions of "transformation", "mathematical object", and "ingenuity" form a philosophical conundrum. The indefinite notion however, is sufficient to state the following theorem:

Theorem 5. *The class of computable problems is closed under concatenation.*

That is, if a problem P can be solved by solving a computable problem Q, followed by solving a computable problem R, then P itself is computable.

Proof. Since both Q and R are computable, and no transformations are performed, other than to solve the problems Q and R, P itself is computable. □

Thus we arrive at the folklore notion of an algorithm:

Notion 6. *An "algorithm" is a specification of how a problem can be solved by solving a finite sequence of computable problems.*

Such indefinite notions are useful for little else. We are left to take a philosophical leap of faith and define some notion of mathematical object, and transformation without ingenuity.

1.1 Function Algebras

A formal system is a system of mathematical symbols and rules for employing them. We'll refer to formal systems as algebras.

An algebra is a set of symbols and rules for manipulating them. In this sense, and algebra

Definition 16. *A **function algebra** $\mathcal{A}$ is a set of functions, including a set of basic functions $\mathcal{A}_{\mathrm{B}}$ and a set of operations $\mathcal{A}_{\mathrm{O}}$.*

1.2 Machines

TODO:

- The above definition is useful for little else - provide some historical perspective on defining computability beyond this notion, and provide a characterization using function algebras (similar to Church) and Turing machines. Draw parallels to type theory an constructivism.
- The classical result that primitive recursive functions are computable. To argue for this, we probably need to argue for a type theoretic approach, in that, what we can construct, we can compute. Function algebras should also be introduced here.
- General recursion (due to Kleene wrt. definition) and it's undecidability (due to Church).
- A different approach to computability: Post and Turing machines. Prove their equivalence to general recursion above.

Chapter 2

Complexity

This may be a good point to mention that, although I have so far been tacitly equating computational difficulty with time and storage requirements, I don't mean to commit myself to either of these measures. It may turn out that some measure related to the physical notion of work will lead to the most satisfactory analysis; or we may ultimately find that no single measure adequately reflects our intuitive concept of difficulty.

— ALAN COBHAM, *Logic, Methodology and Philosophy of Science* (1964)

In practice, the length of computer computations must be restricted, otherwise the cost in time and money would be prohibitive.

— H. E. ROSE, *Subrecursion: functions and hierarchies* (1984)

Definition 17. *The computational complexity of a function* f*, wrt. a particular resource, quantifies the use of that resource as a function of the length of the input string.*

2.1 Time

2.1.1 Polynomial Time

- Recursive characterization of polytime functions in [Rose (1984)], proving certain claims by [Cobham (1965)]. Both question the relation to the Grzegorczyk hierarchy [Grzegorczyk (1953)].
- Leivant's paper - A Foundational Delineation of Computational Feasibility.
- Bellantoni and Cook paper - A NEW RECURSION-THEORETIC CHARACTERIZATION OF THE POLYTIME FUNCTIONS
- Niel Jones paper.
- Caporaso
- Upper bounds (algorithms) can be produced by expressing the property of interest in one of our languages. Lower bounds proven elsewhere can be used as a proof that the language is expressive enough.

2.1.2 Subpolynomial Time

In what follows, we delineate a hierarchy of complexity classes, strictly under polynomial time. That is, we present a sequence $C_1(n)$, $t(n) = o\left(n^{O(1)}\right)$. For each complexity class C, we present a representative problem P_C. We aim to find problems which are known to be $t(n) = \Theta(C(n))$.

- Some problems, although computable in polynomial time, are still hard to compute in practice (ICALP'2014, Amir Abboud).
- Remind of the definitions of O, Ω, etc.
- For each of the below show that every subsequent class is distinct from the proceeding, and exhibit some "complete" problems for these classes.

$O(1)$ **— Constant**

$O(\alpha(n))$ **— Inverse Ackermann**

$O(\log^*(n))$ **— Log star**

$O(\log\log(n))$ **— Log-log**

$O(\log(n))$ **— Log**

$O\left(\log(n)^{O(1)}\right)$ **— Polylog**

$O(n^c)$**, for** $0 < c < 1$ **— Fractional power**

$O(n)$ **— Linear time**

$O(n\log^*(n))$ **— n log star**

$O(n\log\log(n))$ **— n log-log**

$O(n\log(n))$ **— n log n** Comparison-based sorting $o(n\log n)$.

$O\left(n^2\right)$ **— quadratic**

$O\left(n^3\right)$ **— cubic**

$O\left(n^{O(1)}\right)$ **— polynomial**

2.1.3 Space

Chapter 3

Implicit Characterizations of P

3.1 Primitive Recursion on Notation

When dealing with the manipulation of symbolic strings, there is a natural total order on the values of the formal parameters — the length of their representing string. Primitive recursion on notation utilizes this order, requiring that the length of the input string be decreased before a recursive call.

Definition 18. *A function* f *is defined by primitive recursion on notation from functions* $g, h_1, h_2, \ldots, h_{|\Sigma|}$ *iff*

$$f(\varepsilon, \bar{t}) = g(\bar{t}) \tag{3.1}$$

$$f(s_i \cdot s, \bar{t}) = h_i(s, \bar{t}, f(s, \bar{t})) \qquad \forall s_i : \Sigma \tag{3.2}$$

We say that a function is primitive recursive on notation (PRN), if it is defined by primitive recursion on notation from non-recursive or PRN functions.

Unfortunately, not all PRN functions take polynomial time.

Theorem 6. *There exist PRN functions which do a superpolynomial amount of work.*

Proof. Consider a function g, which duplicates every symbol in the input string:

$$g(\varepsilon) = \varepsilon \tag{3.3}$$

$$g(0 \cdot s) = 0 \cdot 0 \cdot g(s) \tag{3.4}$$

Consider furthermore a function h, which calls g iteratively, the same number of times as the length of its input string:

$$h(\varepsilon) = 0 \tag{3.5}$$

$$h(0 \cdot s) = g(h(s)) \tag{3.6}$$

Calling g with a string of length n, we obtain a string of length 2^n due to iterated duplication. It follows that g does a superpolynomial amount of work. □

Example 2. *We illustrate the above proof with an example:*

$$\begin{aligned}
h(0 \cdot 0) &\rightsquigarrow g(h(0)) && (3.7)\\
&\rightsquigarrow g(g(h(\varepsilon))) && (3.8)\\
&\rightsquigarrow g(g(0)) && (3.9)\\
&\rightsquigarrow g(0 \cdot 0 \cdot g(\varepsilon)) && (3.10)\\
&\rightsquigarrow g(0 \cdot 0) && (3.11)\\
&\rightsquigarrow 0 \cdot 0 \cdot g(0) && (3.12)\\
&\rightsquigarrow 0 \cdot 0 \cdot 0 \cdot 0 \cdot g(\varepsilon) && (3.13)\\
&\rightsquigarrow 0 \cdot 0 \cdot 0 \cdot 0 && (3.14)
\end{aligned}$$

3.2 Bounded Primitive Recursion on Notation

With reference to the "extended rudimentary functions" of [Bennett (1962)], [Cobham (1965)] defined bounded primitive recursion on integers in decimal notation. For a similar definition on binary notation, see [Rose (1984), p. 127].

We generalize this to an arbitrary alphabet Σ, building upon PRN above:

Definition 19. *A function f is defined by bounded primitive recursion on notation from functions $g, h_1, h_2, \ldots, h_{|\Sigma|}$, and k iff*

$$\begin{aligned}
f(\varepsilon, \overline{y}) &= g(\overline{y}) && && (3.15)\\
f(s_i \cdot x, \overline{y}) &= h_i(x, \overline{y}, f(x, \overline{y})) && \forall s_i : \Sigma && (3.16)\\
f(x, \overline{y}) &\leq k(x, \overline{y}) && && (3.17)
\end{aligned}$$

We say that a function is bounded primitive recursive on notation (BPRN) if it is defined by bounded primitive recursion on notation from non-recursive, or BPRN functions. The scheme is also known as limited primitive recursion on notation.

The addition that we make to PRN is that the function f must be bounded from above by function i. Letting i characterize polynomial time functions, we obtain perhaps the earliest implicit characterization of P due to [Cobham (1965)].

This straight-forward approach has an obvious limitation: it requires defining an ordering relation on our functions — a problem that is undecidable in general. Furthermore

3.3 Finite Model Theory

Finite model theory is

3.4 Ramification

3.4.1 Safe Recursion

3.4.2 Tiering

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