Proofs 1.7 Week 5

(1) Prove e is irrational? — easiest prove: proof by contradiction

- attack problem by separating prop into several cases, proof by cases
- what definitions, terms, axioms might be relevant
- prove line of argument is sound

proposition under discussion $\rightarrow T$

Proof by contradiction

let p be the proposition of e, we suppose $\neg p$ is T: "It is NOT the case that e is irrational", which means e is RATIONAL ($\neg p$)

(1) If e is rational, then there exists a and b ints w/ $\boxed{e = \frac{a}{b}}$, $b \neq 0$ $a, b > 0$ (a/b is in lowest terms)

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \ldots$$

$$\boxed{x} = b!\left(\boxed{e} - \sum_{n=0}^{b} \frac{1}{n!}\right)$$

summation notation

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$x = b!\left(e - \sum_{n=0}^{b} \frac{1}{n!}\right) = b!\left(\frac{a}{b} - \sum_{n=0}^{b} \frac{1}{n!}\right) =$$

$$= b!\left(\frac{a}{b}\right) - \sum_{n=0}^{b} \frac{b!}{n!} = \underbrace{a(b-1)!}_{\text{integer}} - \underbrace{\sum_{n=0}^{b} \frac{b!}{n!}}_{\text{integer}}$$

$b \geq n$ for all n in this sum Σ

② $X = b!\left(\sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^{b} \frac{1}{n!}\right)$

both are summing same thing 1st is up to ∞ and is up to b

$$= \underbrace{\sum_{n=b+1}^{\infty} \frac{b!}{n!}}_{\text{positive}} < \sum_{k=1}^{\infty} \frac{1}{(b+1)^k} = \frac{\frac{1}{b+1}}{1-\frac{1}{b+1}} = \frac{1}{b} < 1$$

$$\frac{b!}{n!} = \frac{b(b-1)(b-2)\cdot\ldots}{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(b+1)\cdot b\cdot(b-1)\cdot\ldots}$$

$n \geq b+1$

$$= \frac{1}{n(n-1)(n-2)\cdot\ldots\cdot(b+1)} \leq \frac{1}{(b+1)^{n-b}}$$

there is NO int >0 and <1 it's a contradiction

⑦ Use direct proof that every odd int is the diff of 2 squares.

if $p \to$ then q

if odd int $\to$ then $a^2 - b^2$

$\underset{\text{odd}}{5} = \underset{\text{even}}{3^2} - \underset{\text{even}}{2^2}$

p - hypothesis is True, then conclusion q - can't be false

①. $a^2 - b^2 = \cancel{(a-b)(a+b)} = (a+b)(a-b)$.

$a+b =$ odd $\quad a, b = n/2$

$a-b = 1 \quad n = 11, a = 6, b = 5$

$n = 2k+1$ (odd)

"Google data analyst cert"

$a = k+1$, $b = k$ $\boxed{n = a^2 - b^2}$

- Since n = odd, we can write

$n = 2k+1$ for some int k

Then $(k+1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k+1 = n$

This expresses n as the difference of 2 squares

(9) Use proof by contradiction to prove sum of irrational number and rational number is irrational.

There exists 'c' that is irrational, when

$\underbrace{a\ (\cancel{\text{irrational}}) + b\ (\cancel{\text{rational}})}$.

rational q irrational

- p = irrational
$a = \sqrt{2}$ $b = \frac{1}{2}$

$\sqrt{2} + \frac{1}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

if $p \to q$, $\neg p$ is True (contradiction)

If 'a' is a rational number and 'b' is irrational number, then $c = a + b$ is an irrational number. $\neg p = \neg c$

1. Suppose that a = rational, b = irrational) q

$\neg c$ = rational) $\neg p$

2. Sum of rational nums a and c must be rational. If $c = \frac{a}{b}$ and $a = \frac{e}{d}$

- $b = \cancel{a(+b+a)} = \cancel{(a+b)} \left(\frac{a}{a} - \frac{b}{e}\right) / \frac{e}{d}$,
$c - a$

so $c - a$ is rational

But $c - a = a + b - a = b$, means b is rational too. This contradicts hypothesis that b is irrational. Therefore, c was rational is Not True, so c is irrational.

$$\frac{a}{b} + \left(-\frac{c}{d}\right) = \frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd} \text{ ; (rational)}$$

$$c + (-a) = \underbrace{a + b}_{c} - a = b \text{ (rational ?)}$$

Contradiction of "sum of irrational num and rational $\overset{a}{\text{is}}$ irrational^{b} (not rational)"

(11) Product of 2 irrational nums is irrational.

If a is irrational and b is irrational, then $c = a \cdot b$ is an irrational number.

a = irrat , b = irrat , $\neg p = \neg c$ (ration)

$\sqrt{2} \times \sqrt{3} = \sqrt{6} \rightarrow$ irrational since 6 is not perfect square

$\sqrt{2} \times \frac{1}{\sqrt{2}} = 1 \rightarrow$ rational num

$\sqrt{2} \times \sqrt{2} = 2 \rightarrow$ rational num

We disproved that $\sqrt{k} \cdot \sqrt{k} = \sqrt{k}$ in all cases.

(13) Use a proof by contraposition to show if $x + y \geq 2$, where x, y are real numbers, then $x \geq 1$ OR $y \geq 1$

Contraposition: $\overset{p}{(x)} \to \overset{q}{(y)}$

$(\neg y) \to (\neg x)$

$\neg(q) \to \neg(p)$!

if $\overset{p}{(x+y \geq 2)}$ then $\overset{q}{(x \geq 1)}$ OR $\overset{R}{(y \geq 1)}$

~~$(\neg q\ \ (\neg R)\ (\to \neg p)$~~

$p \to (q \vee R)$ Morgan's Law

$\neg(q \vee R) \to \neg p$

$(\neg q \wedge \neg R) \to \neg p$ if $x < 1$ and $y < 1$, then $x + y < 2$

(contraposit.)

$x+y < 2$ is the negation of $x+y \geq 2$, so our proof is complete

17. If n is odd, then n^3+5 is even

If $\underbrace{n}$ is int and $\underbrace{n^3+5}_{p}$ is odd, then $\underbrace{n}_{q}$ is even

1. Contraposition

$\neg q \to \neg p$, $\neg p = n^3+5$ is even

$\neg q = n$ is odd

$n = 2k+1$

$n^3+5 = (2k+1)^3+5 = 8k^3 + 12k^2 + 6k + 1 + 5$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$= 8k^3 + 12k^2 + 6k + 6$

$= 2(4k^3 + 6k^2 + 3k + 3)$

Thus, n^3+5 is even

2. a proof by contradiction

$p \to q$, $p \to \neg q$

$n^3+5 \to n$ is odd

p $\neg q$

$n = (2k+1)$ odd

$(2k+1)^3+5 = \cancel{12k^3 + 12k^2 + 6k + 5}$

$= 8k^3 + 12k^2 + 6k + 1 + 5$

$= \underbrace{2(k^3 + 6k^2) + 3k + 3}_{\text{even, so } n \text{ is NOT odd}}$

If n is an int and n^3+5 is odd, then n is even.

(1) Problem:

the barber is one who shaves all men who don't ~~have~~ shave themselves.

Does the barber shave himself?

U - set of all men in community (universal)

$S : U \to \{T; F\}$ propos function, shaves or not

$\forall x \in U : B(x) \leftrightarrow x$ is shaved by barber

all men such that

$\underbrace{\forall x \in U : (\neg S(x))}_{\text{all men that dont shave themselves}} \leftrightarrow \underbrace{B(x)}_{\text{... are shaved by barber}}$

they are shaved by barber

$B(b) \leftrightarrow S(b)$ barber (b) is shaved so he is one of who shaves

$$\underbrace{S(8)}_{p} \leftrightarrow \underbrace{B(8)}_{q} \leftrightarrow \underbrace{(\neg S(8))}_{\neg p}$$

$p \leftrightarrow q \leftrightarrow \neg p$

$p \leftrightarrow \neg p \longrightarrow$ Contradiction

p	q	R	$p \leftrightarrow q$	$q \leftrightarrow R$
0	0	0	1	1
0	0	1	1	0
0	1	0	0	0
0	1	1	0	1
1	0	0	0	1
1	0	1	0	0
1	1	0	1	0
1	1	1	1	1

(8) Problem: Show if x, y ints and both xy and $x+y$ are even the x and y are even

Ex 7

Contraposition

$p \longrightarrow q$, $\neg q \longrightarrow \neg p$

$x = 2k+1 \rightarrow$ odd x or y is odd

$xy \rightarrow$ odd xy or $x+y$ is odd

(i) y even $\rightarrow 2k$, $x+y = (2k+1) + 2n = \underbrace{2(k+n)+1}_{\text{odd}}$

(ii) y odd $\rightarrow 2k+1$, $xy = (2k+1)(2n+1)$

$= 4kn + 2k + 2n + 1$

$= \underbrace{2(kn+k+n)+1}_{\text{odd}}$

less of generality