Not all differential equations can be solved analytically. Luckily, a there exist a wide variety of methods that enable us to obtain very close approximations through iterative computer algorithms. Euler and Runge-Kutta methods take advantage of the fact that we can estimate the function as a sum of the derivatives of the function over small time steps multiplied by the time steps raised to the appropriate power. Such algorithms basically harness the power of the Taylor approximation of a function. The first order approximation when only the first two terms in the Taylor series are taken is called the Euler method. This method provides a good and quantitative estimate; however, it has multiple drawbacks and produces high error over large time steps. The comparison between the analytical solution of a radioactive decay problem and Euler method estimates for both the small- and big-time steps can be seen in the first and second graphs respectively. Despite this, Euler method remains a very useful tool as problems in physics often include finding the function from the expression of its first derivative.

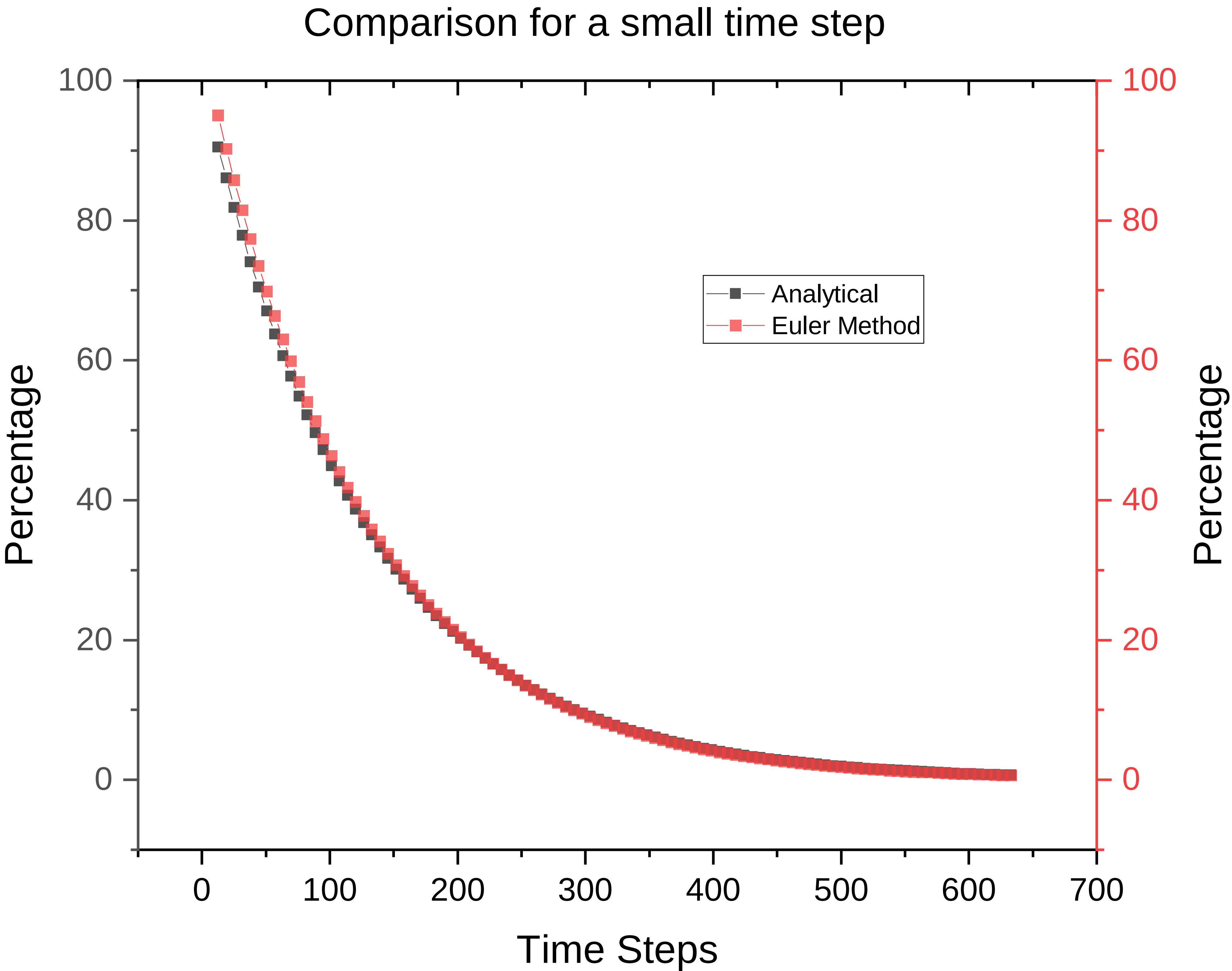
A way of reducing the error for small time steps can be to include more terms form the Taylor series into the approximation, however, this method requires a long evaluation of higher order derivatives from the known first derivative at each of the time steps, which adds computational complexity. One way of improving the accuracy of approximation is known as the Runge-Kutta method.

Instead of dealing with one value for slope of a function at the beginning or the end of the interval, Runge-Kutta method uses whited average for the values of the slope taken the beginning of the time interval, two values in the middle of the time interval, and a value at the end. Other Runge-Kutta methods can include more terms in the estimation of the slope, however, four terms are often satisfactory for a wide range of applications. The Runge-Kutta method with four terms is often abbreviated as RK4 and is the method we used in this project.

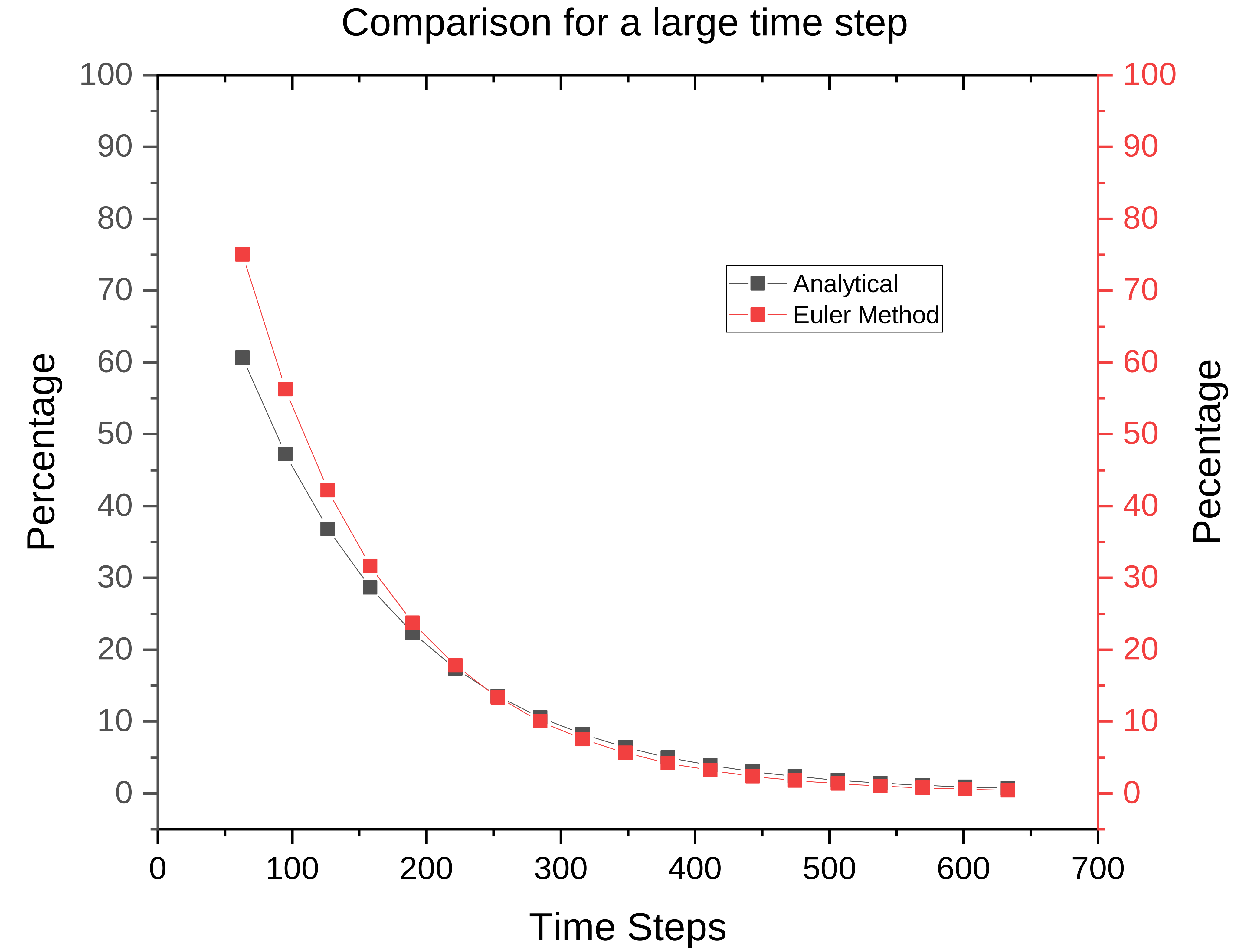
$$x(t+\Delta t) \equiv x(t) + \frac{1}{6}[f(x_1', t_1') + 2f(x_2', t_2') + 2f(x_3', t_3') + f(x_4', t_4')]\Delta t$$

$$\begin{aligned}
x_1' &= x(t)\ , & t_1' &= t \\
x_2' &= x(t) + \frac{1}{2}f(x_1', t_1')\Delta t\ , & t_2' &= t + \frac{1}{2}\Delta t \\
x_3' &= x(t) + \frac{1}{2}f(x_2', t_2')\Delta t\ , & t_3' &= t + \frac{1}{2}\Delta t \\
x_4' &= x(t) + f(x_3', t_3')\Delta t\ , & t_4' &= t + \Delta t\ .
\end{aligned}$$

The comparison between an analytical solution and a RK4 method for a large time step can be seen in the graph3 below. We can see that RK4 offers a better convergence with the analytical solution than Euler method for a large time step.



Comparison for a small time step
Percentage
Percentage
Time Steps
Analytical
Euler Method
0
20
40
60
80
100
0
100
200
300
400
500
600
700


Comparison for a large time step
Percentage
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Time Steps
Analytical
Euler Method
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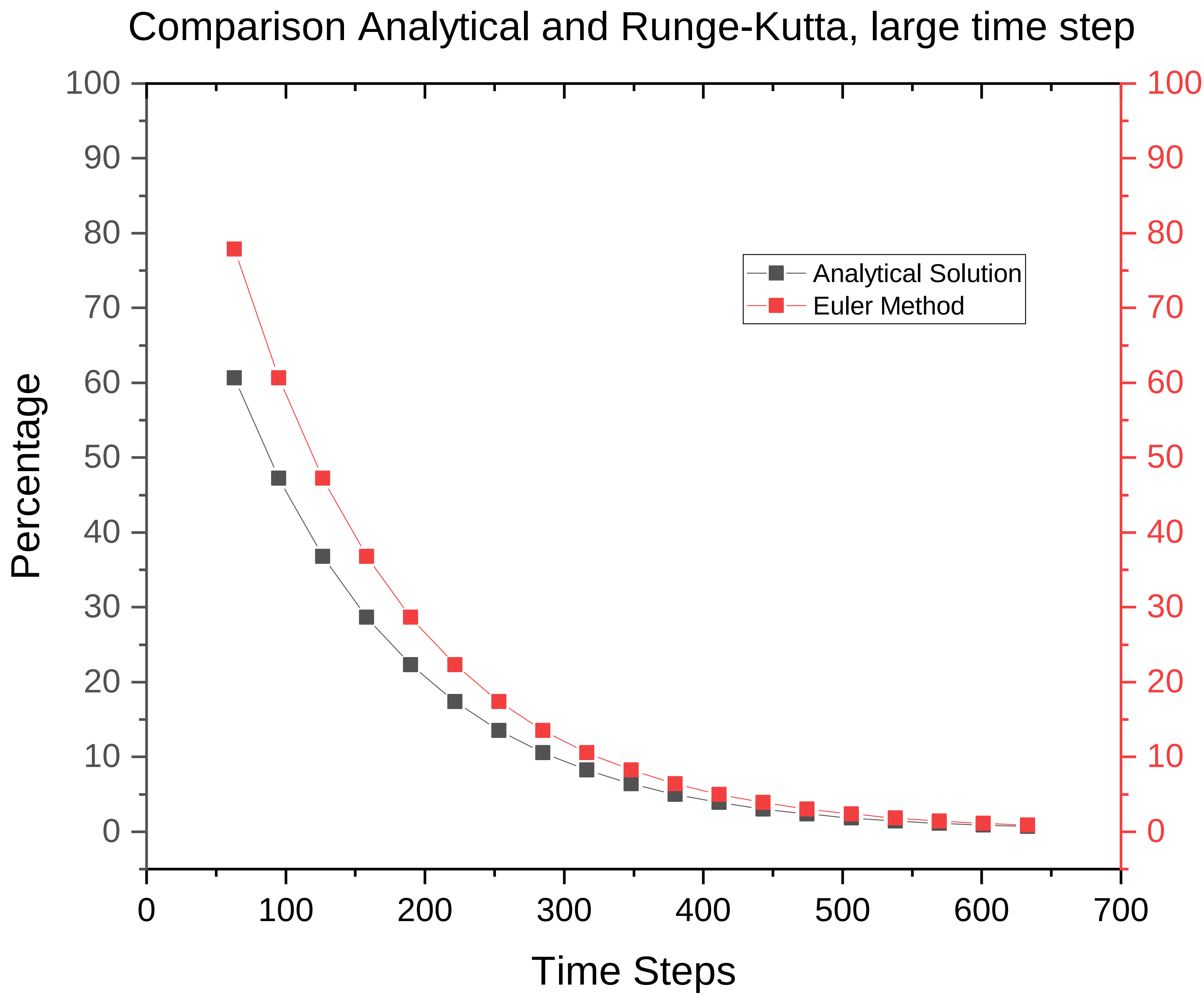

Comparison Analytical and Runge-Kutta, large time step
Analytical Solution
Euler Method
Percentage
Percentage
Time Steps
0
100
200
300
400
500
600
700
0
10
20
30
40
50
60
70
80
90
100