Statistical Inference II

Data Sciences Institute
Linear Regression, Classification, and Resampling

Learning objectives

- Define and interpret null and alternative hypotheses for linear regression.
- Differentiate between the t-test (for individual coefficients) and the F-test (for overall model fit).
- Perform hypothesis testing in Python to evaluate the significance of individual regression beta coefficients (including intercept) and assess the overall fit.

Hypothesis testing

- Hypothesis testing is a method used to make decisions or inferences about a population based on sample data.
- It helps determine if the observed data provides enough evidence to support a specific claim or reject it.
- For example, *is there a relationship between two variables?*
    - Between the size of a house in Sacramento and its sale price.

Null and Alternative Hypotheses

- When conducting statistical inference, we evaluate two competing hypotheses:
    i. Null hypothesis (H_0): The hypothesis we formally test. It typically represents "no effect" or "no difference". The null hypothesis is what we assume to be true.
    ii. Alternative hypothesis (H_1): The hypothesis that is contradictory to H_0 and of particular interest in our research.
- **Important**: We cannot prove any hypothesis (H_0 or H_1) is **true**. Instead:
    i. If our data are inconsistent with the null hypothesis (H_0), we reject H_0 in favor of the alternative hypothesis (H_1). However, this does not imply that H_1 is true; rather, it indicates that we have evidence against H_0.
    ii. If our data are consistent with the null hypothesis (H_0), we cannot reject H_0. **This is not the same as accepting H_0.**

Recap of Linear Regression Equation

- The equation for the straight line in simple linear regression is:

$$\mathbf{y} \approx \beta_0 + \beta_1 \mathbf{x}$$

where:

- $\mathbf{y}$ is the response variable.
- β_0 is the vertical intercept.
- β_1 is the slope of the predictor.
- $\mathbf{x}$ is the predictor (e.g., house size).

Finding the line of best fit involves determining coefficients β_0 and β_1 that define the line.

Example dataset

932 real estate transactions in Sacramento, California is the dataset we will be using, specifically for predicting whether the size of a house in Sacramento can be used to predict its sale price.

- *Key features:*
    - 932 observations (rows)
    - predictor of interest (sqft; house size, in livable square feet)
    - response variable of interest (house sale price, in USD)

Hypothesis test for coefficients

One question we may ask: *Is there a relationship between the size of the house and the house price?*
We can address this question by testing whether the regression coefficient β_1 is sufficiently far enough from 0.

- $H_0 : \beta_1 = 0$ meaning there is no relationship between house size and the price.
- $H_1 : \beta_1 \neq 0$ meaning there is a relationship between the house size and price.

Essentially, we have two competing hypotheses:

- $H_0 : \beta_1 = 0$ vs. $H_1 : \beta_1 \neq 0$

Hypothesis test for coefficients when predictors are categorical

Note: When using categorical predictors in a regression model, the interpretation of coefficients differs from that of continuous predictors.

- Here, β coefficients tell us the impact of each category (relative to the reference category) on the outcome.
- This is done by testing if the β coefficients for the categories are significantly different from zero.

Let's assume we have a regression model where house type (Residential, Condo, Multi-Family) is our predictor variable and house price is the outcome variable.

- Residential (our reference category) will not have a coefficient because it is the baseline.
- Condo and Multi-Family will have their own coefficients.
- For example, we have:
  - β_0: Intercept (average house price for Residential). This represents the average house price for the Residential category (since it is the reference group).
  - β_1: Coefficient for Condo. This represents how much the average house price for a Condo differs from the Residential group.
  - β_2: Coefficient for Multi-Family. This represents how much the average house price for a Multi-Family house differs from the Residential group.

Hypothesis test for coefficients

In order to test the null hypothesis, we need to determine whether β_1 is sufficiently *far* enough from 0.

To test this, we use the **t-statistic**:

$$t = \frac{\hat{\beta}_1 - 0}{SE(\hat{\beta}_1)}$$

- $\hat{\beta}_1$: Estimated coefficient for the predictor.
- $SE(\hat{\beta}_1)$: Standard error of the estimated coefficient.
- 0: The value under the null hypothesis (*recall* $H_0 : \beta_1 = 0$).

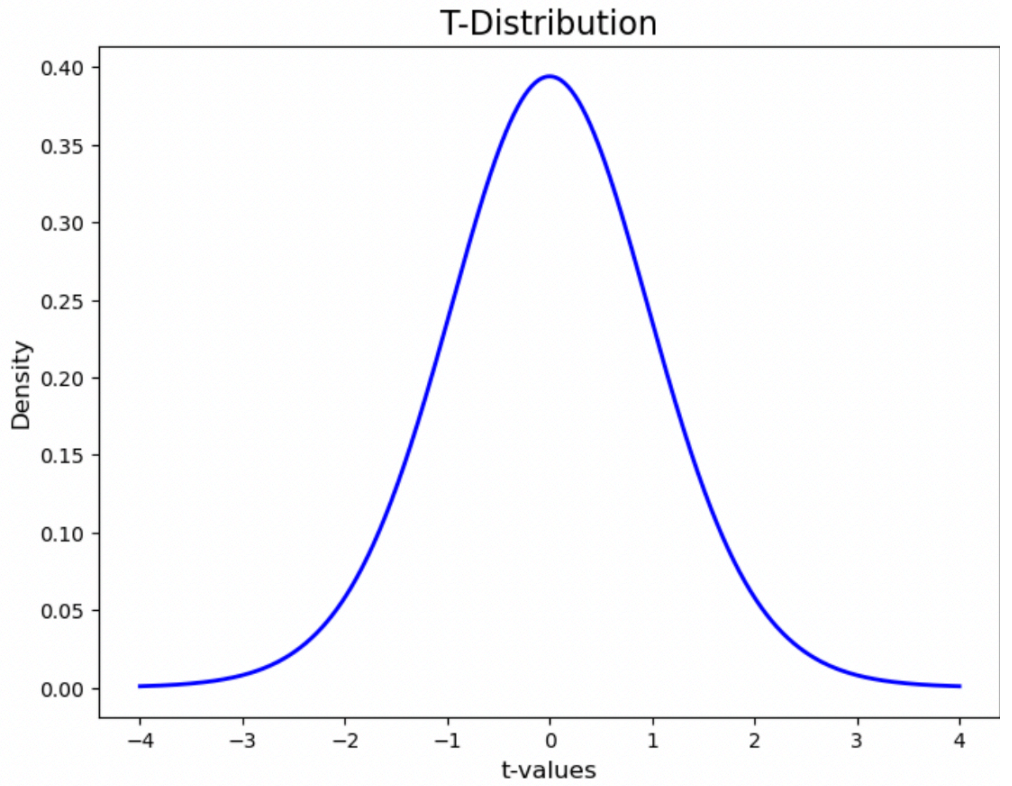
Note: The hat represents our estimate of the parameter.

What Does "Sufficiently Far from 0" Mean?

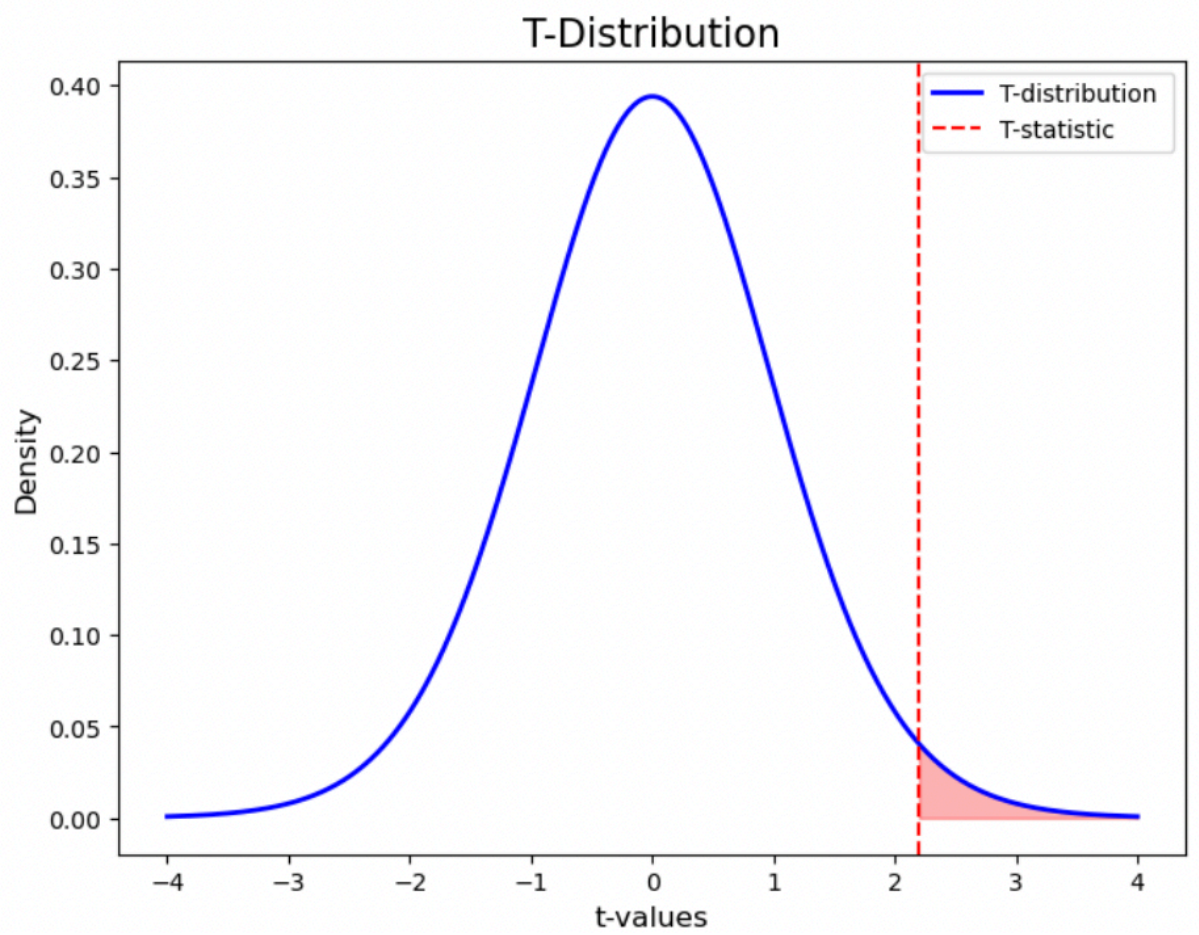
- Here, 0 represents the value under the null hypothesis.
- The t statistic measures how *far* β_1 is from 0, expressed in terms of standard errors:
  - Large $|\mathrm{t}|$: β_1 is far from 0. Evidence against H_0.
  - Small $|\mathrm{t}|$: β_1 is close to 0. Data are consistent with H_0.
- The exact threshold for what is considered "far" enough depends on the **sample size** and the **significance level** α of the test (typically set to 0.05).
  - We compare $|\mathrm{t}|$ to the critical value corresponding to these levels of significance (this critical value is calculated under the t-distribution table, dependent on our sample size).

What is the t-distribution?

- The t-distribution is a type of curve used in statistics, just like the bell-shaped 🔔 normal distribution.
- The t-distribution helps determine how "far" from zero is sufficiently far to reject the null hypothesis.
- By comparing t to critical values from the t-distribution, we can determine whether the beta coefficient is likely due to chance or if there is a statistically significant relationship between the predictor and the outcome.



- The t-distribution is used to determine if the t-statistic (t) is large enough to reject the null hypothesis.
- The red dashed line represents the t value calculated for the estimated beta coefficient.
- The area shaded in red represents the rejection region, where we would reject the null hypothesis if our t-statistic lies there.



Multivariable linear regression

- We can extend our model to include multiple predictors! Each predictor variable *may* give us new information to help create our model.
- For example, let's say we now want to include both house size and number of bedrooms as predictors in our model

$$\mathrm{y} = \beta_0 + \beta_1(x_1) + \beta_2(x_2)$$

where:

y is the response variable.

β_0 is the vertical intercept.

β_1 is the slope for predictor 1 (e.g., (x_1) house size)

β_2 is the slope for predictor 2 (e.g., (x_2) number of bedrooms)

Hypothesis test for multiple coefficients

For multiple predictors, the process is the same!
We perform separate t-tests for each β coefficient to determine if it differs significantly from 0.

1. Null hypothesis (H_0): The predictor has no effect on the outcome.
2. Alternative hypothesis (H_1): The hypothesis that is contradictory to H_0. The predictor has an effect on the outcome.

- $H_0 : \beta_1 = 0, \beta_2 = 0 \ldots$
- $H_1 : \beta_1 \neq 0, \beta_2 \neq 0 \ldots$

The F-Statistic for Testing Multiple Predictors in Linear Regression

- While the t-test is used to test significance of **individual** β coefficients, the **F-statistic** is used to test the **overall significance** of the regression model with multiple predictors.
- The F-statistic test tells us if the model *as a whole* is significantly better than a model with no predictors at all (intercept model only).

Null and Alternative Hypotheses for the F-test

1. Null hypothesis (H_0) : **ALL** predictors have no effect on the outcome
2. Alternative hypothesis (H_1): At least **ONE** of the predictors significantly affects the outcome.

- $H_0 : \beta_1 = 0, \beta_2 = 0 \ldots$
- H_1 : At least one $\beta_i \neq 0$

How the F-test works

- The F-statistic is calculated by comparing two models:
  - A full model with the predictors in the regression.

$$\mathrm{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$

  - A reduced model with no predictors (just the intercept).

$$\mathrm{y} = \beta_0$$

- The reduced model simply predicts the outcome using the mean of y, whereas the full model incorporates the predictors x_1, x_2, and x_3 to potentially explain the variance in y.
- The F-statistic tests if the full model significantly improves our ability to predict y compared to the reduced model.

How the F-test works

- It compares the explained variance in the full model to the unexplained variance.
    - Explained Variance: Variability in y is explained by the predictors in the full model.
    - Unexplained Variance: Variability in y that is not explained by the predictors, representing the error or residuals

$$F = \frac{\text{Explained Variance}}{\text{Unexplained Variance}}$$

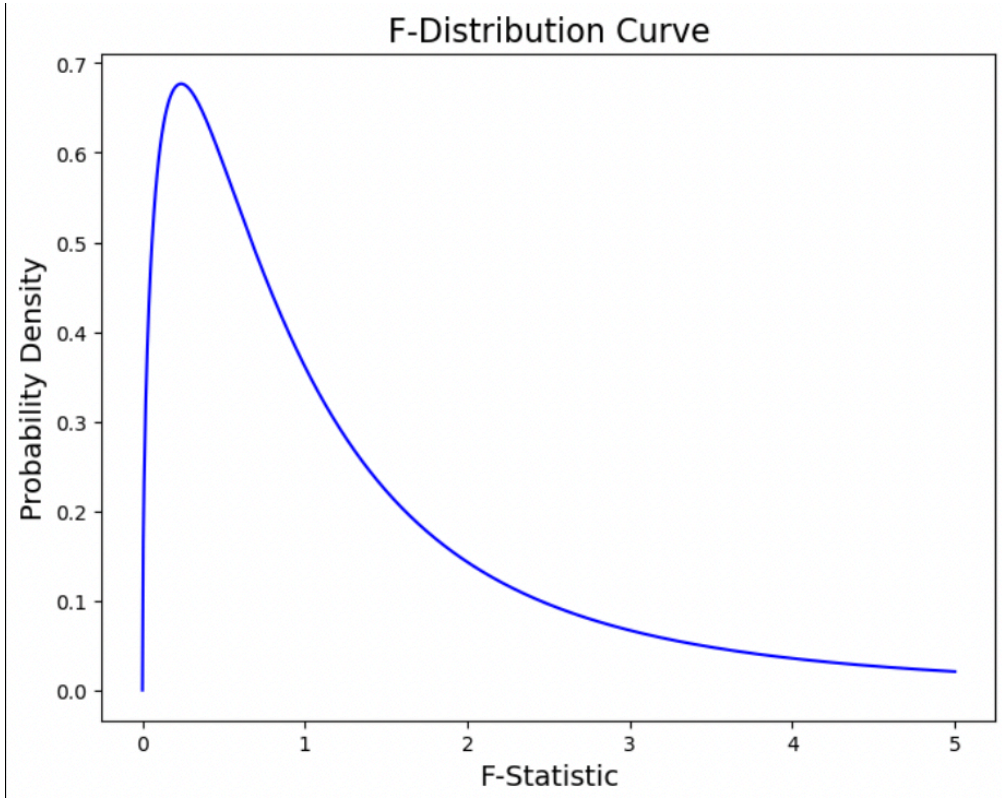
    - Large F: indicates the full model significantly improves the prediction of y compared to the reduced model. Evidence against H_0
    - Small F: indicates the model does not explain much more variance than just using the mean of y. Data are consistent with H_0

What Does "Sufficiently Large" Mean for F?

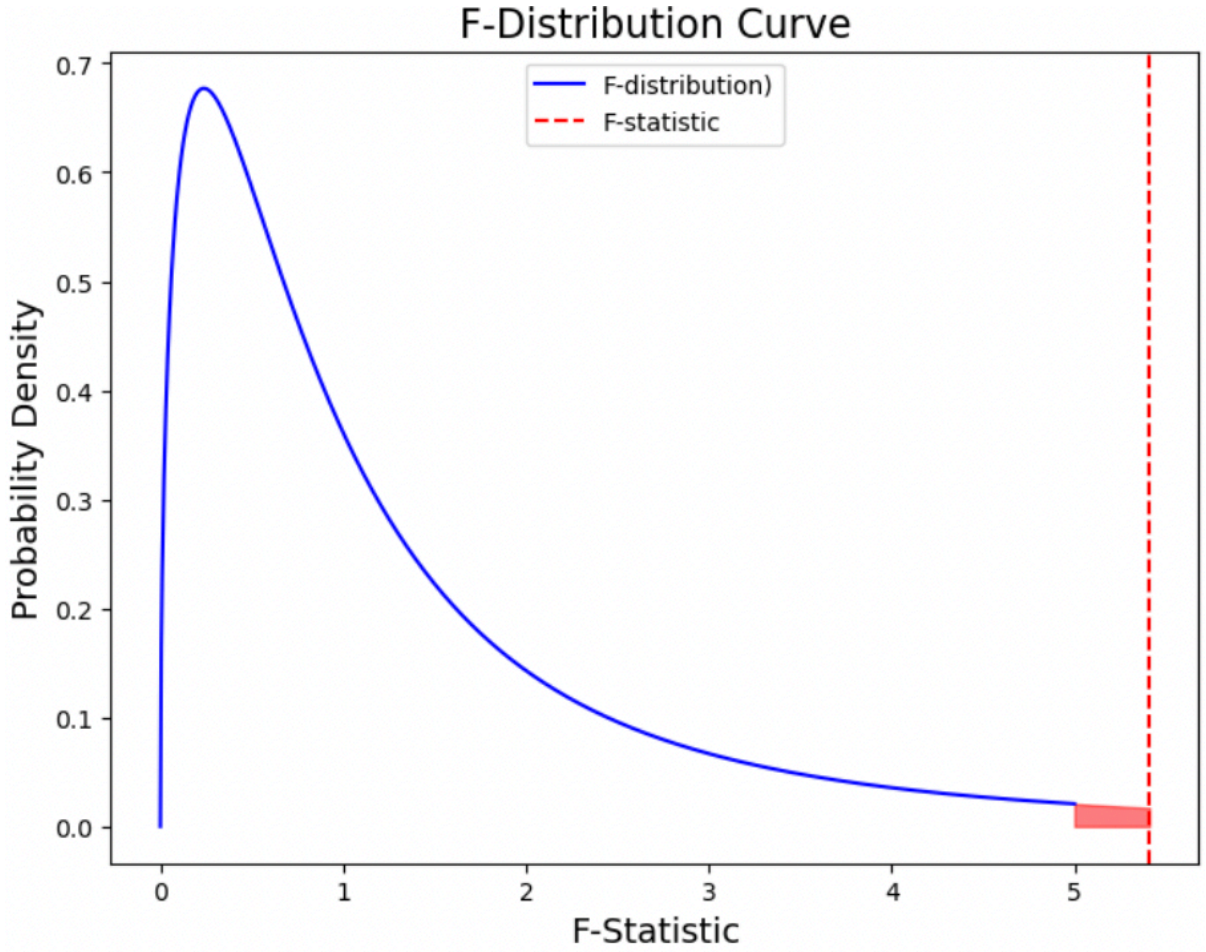
- The exact threshold for what is considered "large" depends on the **sample size** and the **significance level**, typically set to 0.05.
- If the **F** exceeds the critical value from the **F-distribution** for the given significance level, we **reject the null hypothesis** (H_0) and conclude that the full model is a better fit for the data.
- In other words, a "large" F-statistic suggests that the model explains a significant amount of variance in **y**, beyond what would be expected by chance.

What is the F-distribution?

- The F-distribution is commonly used to compare how much better a full model fits the data compared to a simpler model.
- The F-distribution is **right-skewed**, meaning it has a long tail on the right side, and the value of F is always positive.



- We use the F-statistic, which is calculated from the model, to compare it against critical values from the F-distribution to decide whether the full model is a better fit for the data.



Key Assumptions in Linear Regression

You can test these assumptions **before** running the model.

1. **Linearity**: There's a straight-line relationship between the predictor (X) and response (Y) variables.
    - You can visually check by plotting the data (X vs. Y) to see if the relationship appears to be roughly linear.
2. **Independence**: Data points should be independent of each other, meaning no correlation between errors from different observations.
    - This assumption is usually inherent in how the data are collected.
3. **No or Little Multicollinearity**: In multiple regression, predictors should not be highly correlated with each other.
    - You can check using techniques like Variance Inflation Factor (VIF).

Key Assumptions in Linear Regression

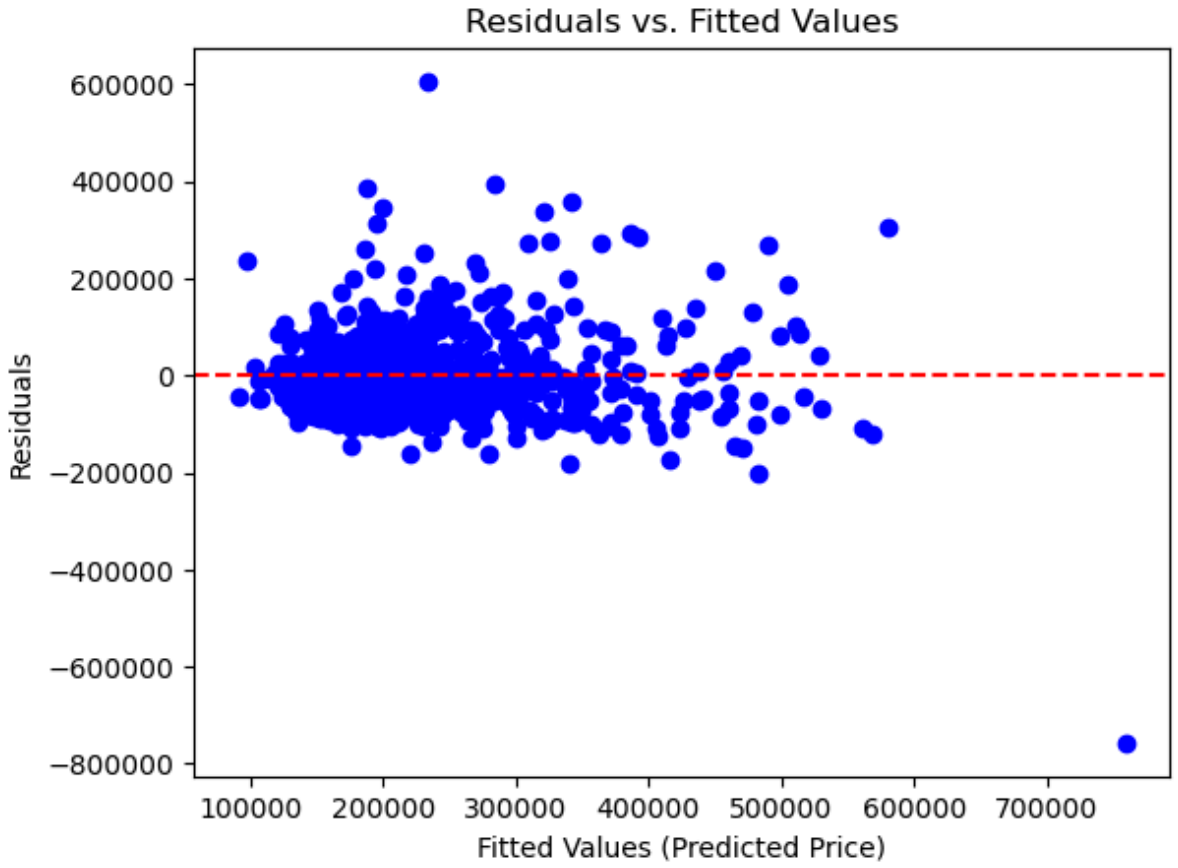
You can test these assumptions **after** running the model.

4. **Homoscedasticity**: The spread of the errors should be constant across all values of (X). In other words, the error variation should not change as (X) changes.
    - After fitting the model, you can plot the residuals (errors) against the fitted values (predicted Y values). If the spread of residuals is constant, the assumption holds.
5. **Normality of Errors**: The errors should be roughly normally distributed, especially for hypothesis testing.
    - After fitting the model, you can use a Quantile-Quantile (Q-Q) plot to check if the residuals are approximately normally distributed. If the residuals deviate significantly from a straight line on the Q-Q plot, the assumption might be violated, although a violation is usually not a major concern.

Residuals vs. Fitted Values Plot

- **Fitted Values:** Predicted values of Y from the regression model.
- **Residuals:** Differences between actual values of Y and the fitted values.

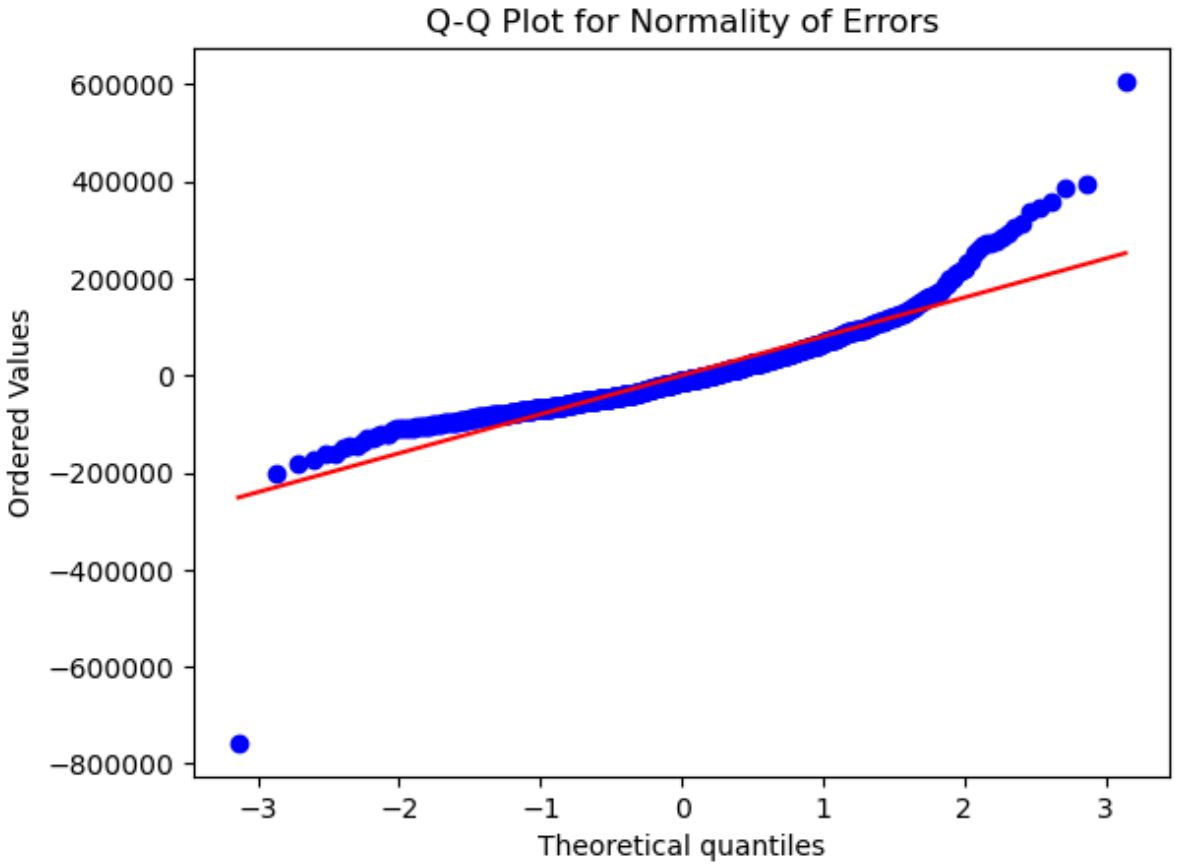
The horizontal line at zero shows where residuals should center. If the points are randomly scattered around it, homoscedasticity is met. A funnel shape suggests a violation of homoscedasticity.



Q-Q Plot

A **Q-Q plot** checks if the residuals follow a normal distribution.

- **Straight line**: Residuals are approximately normal, satisfying the normality assumption.
- **Bend away from the line**: Residuals are skewed or have heavy tails, suggesting a violation of normality.



Putting it all together

Statistical testing in Python