# Multi-Agent Systems

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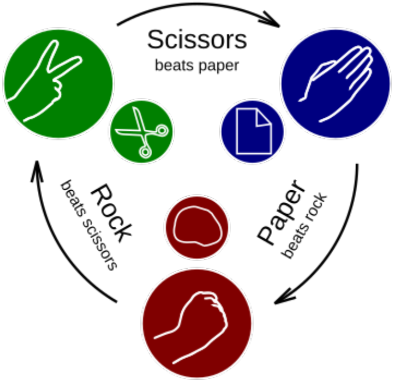
# Outline

# Multi-agent decision making

- Two versus n-player games
- Co-operative games
- Zero-sum games
- General-sum games
- Stochastic games
- Partial information games

# Rock/Paper/Scissors

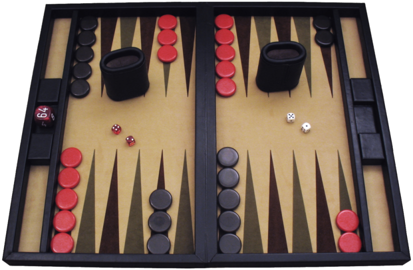



- Number of players: 2
- Zero-sum.
- Deterministic.
- Simultaneous move.

# Chess/Go/Checkers/Othello

- Number of players: 2
- Zero-sum
- Deterministic
- Alternating, Full information

# Backgammon



- Number of players: 2
- Zero-sum
- Stochastic
- Alernating, Full information

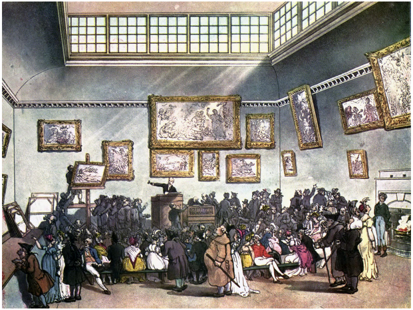
# Poker/Blackjack

- Number of players: $n$
- Zero-sum
- Stochastic [Partially]
- Alternating, Partial information

# Doom/Quake/CoD


- Number of players: $n$
- General sum
- Stochastic
- Simultaneous, Sequential, Partial information

# Auctions



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- General sum
- Deterministic
- Simultaneous move

# Humans and AI

Any system involving interaction between multiple agent can be describe through game theory. One question is how to define the preferences of each agent.

## Human preferences

- These are typically unknown.
- They might not be expressible in mathematical form.
- Nevertheless, we make the utility assumption.

## AI preferences

- These are typically specified by humans as utilities.
- However, it is hard to fully specify them.

# Normal form

In the table below, we see how much reward each player obtains for every combination of actions

| $\rho^1, \rho^2$ | b = 0 | b = 1 |
|---|---|---|
| a = 0 | 2, 1 | 4, 0 |
| a = 1 | 1, 0 | 3, 1 |

## Simultaneous moves

We assume that each player is playing without seeing the move of the other player.

## Commitment

However, we can also look at commitment or Stackleberg games, where one player either *commits* to playing a move, or plays before the other player.

## Information structure

For other types of move sequencing, we have to encode the information structure of a game as a graph.

# Co-operative, adversarial and general games

More generally, we can say that every player $i$ in the game:

- Takes an action $a^i \in A_i$.
- Obtains a reward $\rho^i(x)$ for each possible outcome/choice $x$.

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- $\rho^1 = -\rho^2$
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- $\rho^i = \rho^j$ for all players $i, j$.
- If the players can co-ordinate, then it reduces to a single-agent problem with action-space $A = A_1 \times \cdots A_n$.

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## n-player General-sum games

- $\rho^i$ can be anything.
- Finding solutions for these games is harder.

# Zero-Sum: Rock Paper Scissors

| $\rho^1, \rho^2$ | Rock | Paper | Scissors |
|---|---|---|---|
| Rock | 0, 0 | -1, 1 | 1, -1 |
| Paper | 1, -1 | 0, 0 | -1, 1 |
| Scissors | -1, 1 | 1, -1 | 0, 0 |

# Co-operative: Party

People want to bring something to the party. Ideally, one brings food, and the other drinks. But if they do not co-ordinate, then there is only food, or only drink.

| $\rho^1, \rho^2$ | food | drink |
|---|---|---|
| food | 2, 2 | 10, 10 |
| drink | 10, 10 | 1, 1 |

Here, co-ordination makes the outcomes better for everybody.

# General-Sum: Prisoner's dilemma

| $\rho^1, \rho^2$ | cooperate | defect |
|---|---|---|
| cooperate | -1, -1 | -5, 0 |
| defect | 0, -5 | -3, -3 |

# Basic concepts in normal form games

| $\rho^1, \rho^2$ | b = 0 | b = 1 |
|---|---|---|
| a = 0 | 2, 1 | 4, 0 |
| a = 1 | 1, 0 | 3, 1 |

## Domination and best response

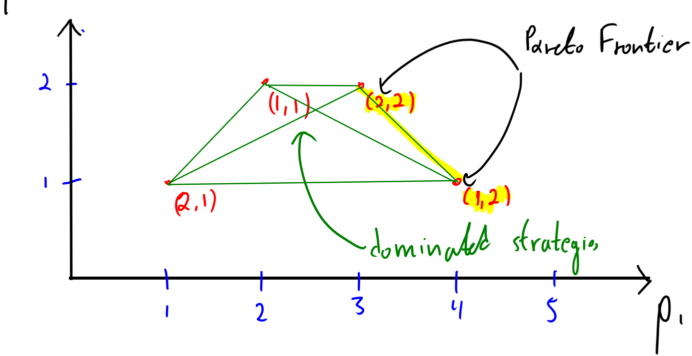
- $b = 1$ is a best response to $a = 1$, i.e. $\rho^2(1,1) > \rho^2(1,0)$
- $a = 0$ is a strictly dominant strategy. Given any $b$, it is strictly better to play $a = 0$, i.e. $\rho^1(0, b) > \rho^1(1, b)$.
- If a pair $(a, b)$ is *not dominated*, then it is Pareto-efficient.

## Questions

- How much reward can $a$ obtain?
- Does $b$ have a dominant strategy?
- Does this take into account what $b$ likes?

# Pareto-Optimality

| $p_1, p_2$ | | Player 2: 1 | Player 2: 2 |
|---|---|---|---|
| play. | 1 | 2, 2 | 4, 1 |
| play. | 2 | 1, 1 | 3, 2 |



# Commitment

Let us see what happens when one player commits to a move

| $\rho^1, \rho^2$ | b = 0 | b = 1 |
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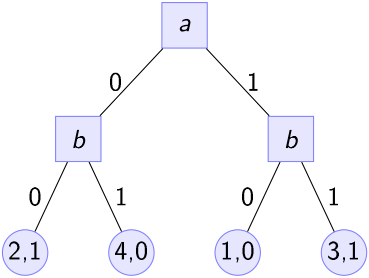
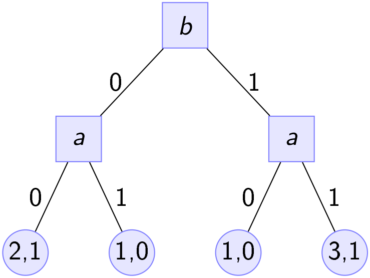
## Player *a* is first

- What should *b* play?
- What is *a*'s best move?




## Player *b* is first

- What should *a* play in each case?



# Extensive-form alternating-move game

- The state $s_t \in S$.
- The actions $a_t^i \in A$.
- The rewards $r_t^i \in \mathbb{R}$, $r_t = (r_t^1, r_t^2)$.
- The transition probabilities

  $$\mathbb{P}(s_{t+1} \mid s_t, a_{t-1}^i)$$

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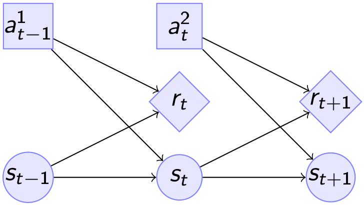

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- The state changes to $s_{t+2}$.
- Player 1 receives $\rho(s_t)$ and 2 receives $-\rho(s_t)$.

The utility for player 1 is

$$U^1 = \sum_t \rho(s_t),$$

while for 2 it is

$$U^2 = -\sum_t \rho(s_t)$$

# Backwards induction for Alternating Zero Sum Games

Let $\pi_1$ and $\pi_2$ be the policies of each player and $\pi$ the joint policy.

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Let $\pi_1$ and $\pi_2$ be the policies of each player and $\pi$ the joint policy.

## The value function of a policy $\pi = (\pi_1, \pi_2)$

For the utility of player 1, we get:

$$V_t^{1,\pi}(s) \triangleq \mathbb{E}_\pi[U_t^1 \mid s_t = s] = \rho(s) + \mathbb{E}[U_{t+1}^1 \mid s_t = s] \tag{3}$$

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$$= \rho(s) + \sum_{a^1} \pi(a^1 \mid s) \sum_j V_{t+1}^{1,\pi}(j) P(j \mid s, a^1)$$

$$\tag{3}$$

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$$= \rho(s) + \sum_{a^1} \pi(a^1 \mid s) \sum_j V_{t+1}^{1,\pi}(j) P(j \mid s, a^1) \tag{2}$$

$$V_{t+1}^{1,\pi}(j) = \rho(j) + \sum_{a^2} \pi(a^2 \mid j) \sum_j V_{t+2}^{1,\pi}(j) P(k \mid j, a^2) \tag{3}$$

# The optimal value function

We can define the optimal value function analogously to MDPs, but player 2 is minimising.
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$$
\begin{aligned}
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&= \rho(s) + \max_{a^1} \sum_j V_{t+1}^{1,*}(j) P(j \mid s, a^1) && \\
& && (6)
\end{aligned}
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$$= \rho(s) + \max_{a^1} \sum_j V_{t+1}^{1,*}(j) P(j \mid s, a^1) \tag{5}$$

$$V_{t+1}^{1,*}(j) = \rho(j) + \min_{a^2} \sum_j V_{t+1}^{1,*}(j) P(k \mid j, a^2) \tag{6}$$

# Normal-form simultaneous-move zero-sum games

(Also called minimax games)

- Player $a$ chooses action $a$ in secret.
- Player $b$ chooses action $b$ in secret.
- Players observe both actions
- Player $a$ receives $\rho(a, b)$, and $b$ receives $-\rho(a, b)$.

## Mixed strategies

Each player chooses an action randomly, independently of one another:

$$\pi(a, b) = \pi_1(a)\pi_2(b)$$

$\pi_i$ is called a mixed strategy.

# Optimal strategies for zero-sum games

## The value of a strategy pair

The expected value of the game for the first player is

$$V(\pi_1, \pi_2) \triangleq \sum_{a,b} \pi_1(a)\rho(a,b)\pi_2(b) = \boldsymbol{\pi}_1^\top \boldsymbol{R}\boldsymbol{\pi}_2,$$

where $\boldsymbol{\pi}_i$ is the vector form representation of $i$'s strategy.

## The value of the game

Any zero-sum game has at least one solution $\pi^*$ over mixed strategies so that

$$V(\pi_1^*, \pi_2^*) = \max_{\pi_1} \min_{\pi_2} V(\pi_1, \pi_2) = \min_{\pi_2} \max_{\pi_1} V(\pi_1, \pi_2)$$

The problem can be solved through linear programming
The idea is to set find a policy corresponding to the greatest lower bound (or lowest upper bound) on the value.

# Linear programming solution for ZSG

## linear programming problem

This is an optimisation problem with linear objective and constraints. In canonical form it is written as:

$$\min_{x} \; \theta^\top x, \qquad \text{s.t. } c^\top x \geq 0.$$

## Primal formulation

Find the higest lower bound for player 1

$$\max_{v} \; v, \qquad \text{s.t. } (\boldsymbol{R}\boldsymbol{\pi}_2)_j \geq v \; \forall j, \; \sum_j \pi_2(j) = 1, \pi_2(j) \geq 0$$

## Dual formulation

Find the lowest upper bound for player 2

$$\min_{v} \; v, \qquad \text{s.t. } (\boldsymbol{\pi}_1^\top \boldsymbol{R})_j \leq v \; \forall j, \; \sum_j \pi_1(j) = 1, \pi_1(j) \geq 0$$

# Normal-form general sum games

## Game structure

- Each player $i$ chooses action $a^i \in A_i$ in secret.
- The joint action is $\boldsymbol{a} = (a^1, \ldots, a^n)$.
- The players then receive a reward $\rho^i(\boldsymbol{a})$

## Mixed strategies

Players can independently draw actions $a^i$ from $\pi(a^i)$ The expected utility of the strateg

# Example: penalty shot

| $\rho^1, \rho^2$ | kick left | kick right |
|---|---|---|
| dive left | 1, -1 | -1, 1 |
| dive right | -1 1 | 1, -1 |

# Example: Chicken

| $\rho^1, \rho^2$ | turn | dare |
|---|---|---|
| turn | 0, 0 | -1, +1 |
| dare | +1, -1 | -10, -10 |

# Example: Prisoner's dilemma

| $\rho^1, \rho^2$ | cooperate | defect |
|---|---|---|
| cooperate | -1, -1 | -5, 0 |
| defect | 0, -5 | -3, -3 |

# Computing Nash equlibria

- A Nash equilibrium always exists (Nash, 1950)
- Nash is PPAD, with $\mathtt{P} \subseteq \mathtt{PPAD} \subseteq \mathtt{NP}$

## The Brouwer problem (PPAD)

Input:

- a function $F : [0,1]^m \to [0,1]^m$
- $L \in (0,1)$ is a Lipschitz constant such that $\|F(x) - F(x')\| \leq L\|x - x'\|$
- An $\epsilon > 0$

Output:

- $x^*$ such that $\|F(x^*) - x^*\| \leq \epsilon$.

## The connection with Nash

- Given by Nash himself in his 1950 proof.
- The fixed point of $F$ is the Nash equlibrium

# The Linear Complementarity Problem

- $\sum_b \rho^1(a,b)\pi_2(b) + s_1(a) = v_1$ for all $a$
- $\sum_a \rho^2(j,b)\pi_1(a) + s_2(b) = v_1$ for all $b$
- $\$\backslash|\pi_{\mathrm{i}}\backslash|_1 = \$,\ \pi_i \geq 0$
- $s \geq 0$
- $\pi_i \cdot s_i = 0$: assigns zero to slack variables corresponding to actions with probability $> 0$

# Optimistic hedge

## Hedge

$$w_{t+1} \propto w_t * exp(\eta r_t)$$

## Optimistic hedge

$$x_{t+1} \propto x_t * \exp(\eta r_{t-1} - 2 r_t)$$

# Extensive-form general sum games

- At time $t$:
- The state is $s_t$, players receive rewards $\rho^i(s_t)$.
- Player $i = I(s_t)$ chooses an action.
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The utility for each player is

$$U^i = \sum_t \rho^i(s_t)$$

# Backwards induction for Alternating General Sum Games

Let $\pi_i$ be the policy of the $i$-th player and $\pi$ the joint policy.

## The value function of a policy $\pi = (\pi_i)_{i=1}^n$

For any player $i$, we can define their value at time $t$ as:

$$V_t^{i,\pi}(s) \triangleq \mathbb{E}_\pi[U_t^i \mid s_t = s] \tag{7}$$

$$= \rho^i(s) + \sum_{a \in A} \pi_{I(s)}(a \mid s) \sum_j V_{t+1}^{1,\pi}(j) P(j \mid s, a) \tag{8}$$

## Optimal policies

For perfect information games, we can use this recursion:

$$a_t^*(s) = \arg\max_{a \in A} \sum_j V_{t+1}^{I(s),*}(j) P(j \mid s, a) \tag{9}$$

$$V_t^{i,*} = \rho^i(s) + \sum_j V_{t+1}^{i,\pi}(j) P(j \mid s, a_t^*(s)) \qquad \forall i \tag{10}$$

This ensures that we update the values of all players at each step.