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Decisions and randomness

Christos Dimitrakakis

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1 Statistical Decision Theory

1.1 Elementary Decision Theory

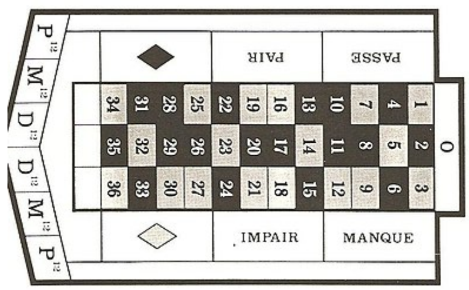
1.1.1 Preferences

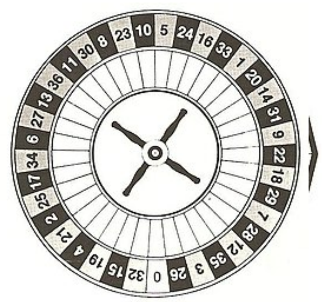
1. Types of rewards EXAMPLE
   - For e.g. a student: Tickets to concerts.
   - For e.g. an investor: A basket of stocks, bonds and currency.
   - For everybody: Money.
2. Preferences among rewards For any rewards $x, y \in R$, we either
   - (a) Prefer x at least as much as y and write $x \preceq^* y$.
   - (b) Prefer x not more than y and write $x \succeq^* y$.
   - (c) Prefer x about the same as y and write $x \eqsim^* y$.
   - (d) Similarly define $\succ^*$ and $\prec^*$

1.1.2 Utility and Cost

1. Utility function To make it easy, assign a utility $U(x)$ to every reward through a utility function $U : R \to \mathbb{R}$.
2. Utility-derived preferences We prefer items with higher utility, i.e.
   - (a) $U(x) \geq U(y) \Leftrightarrow x \succeq^* y$
   - (b) $U(x) \leq U(y) \Leftrightarrow y \succeq^* x$
3. Cost It is sometimes more convenient to define a cost function $C : R \to \mathbb{R}$ so that we prefer items with lower cost, i.e.
   - $C(x) \geq C(y) \Leftrightarrow y \succeq^* x$

1.1.3 Random outcomes





1. Choosing among rewards: Roulette
   - [A] Bet 10 CHF on black
   - [B] Bet 10 CHF on 0
   - [C] Bet nothing
   - What is the reward here?
   - What is the outcome?

1.1.4 Uncertain outcomes

- [A] Taking the car to Zurich (50'-80' with delays)
- [B] Taking the train to Zurich (60' without delays)

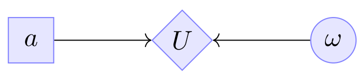
What is the reward here?

1. Car BMCOL

2. Train BMCOL

1.1.5 Independent outcomes

1. Graphical model



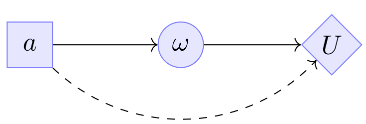
2. Random rewards

   - We **select** our action.
   - Outcomes are **random**, with $\omega \sim P$, but **independent** of our action
   - We then obtain a random **utility** with distribution depending on a.

$$\mathbb{P}(U = u \mid a) = P(\{\omega : U(\omega, a) = u\})$$

1.1.6 General case

1. Graphical model



2. Random rewards

   - We **select** our action.
   - The action determines the **outcome** distribution.
   - The utility may depend on **both** the outcome and reward.

1.1.7 Route selection

1. Utility B_EXAMPLE

$U(\omega, a)$	30'	40'	50'	60'	70'	80'	90'
Train	-1	-2	-5	-10	-15	-20	-30
Car	-10	-20	-30	-40	-50	-60	-70

2. Probability B_EXAMPLE

$P(\omega \mid a)$	30'	40'	50'	60'	70'	80'	90'
Train	0%	0%	50%	45%	4%	1%	0%
Car	0	40%	30%	15%	10%	3%	2%

3. Expected utility

Train	-7.8
Car	-20.82

1.1.8 Calculation in python

For discrete variables, the implemenation is easy.

1. Expected utility of action a: $\mathbb{E}_P[U|a] = \sum_{\omega \in \Omega} U(\omega, a)$.

```
# U: A matrix U[a, w]
# P: A matrix P[w, a]
# a: The action taken
def expected_utility(U, P, a):
  return np.dot(U[a, :],  P[:, a])
```

2. Finding the optimal action: $a^* = \arg\max_{a \in A} \mathbb{E}_P[U \mid a]$.

```
# A: set of actions
def best_action(U, P, A):
  return np.argmax([expected_utility(U, P, a) for a in A])
```

1.2 Statistical Decision Theory

1.2.1 Expected utility

1. Actions, outcomes and utility In this setting, we obtain random outcomes that depend on our actions.
   - Actions $a \in A$
   - Outcomes $\omega \in \Omega$.
   - Probability of outcomes $P(\omega \mid a)$
   - Utility $U : \Omega \times A \to \mathbb{R}$

2. Expected utility The expected utility of an action is:

$$\mathbb{E}_P[U \mid a] = \sum_{\omega \in \Omega} U(\omega, a) P(\omega| \, a).$$

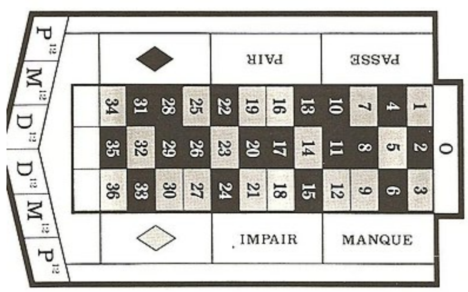
3. The expected utility hypothesis We prefer a to a' if and only if

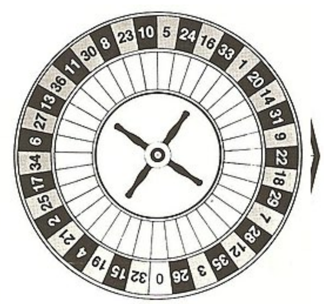
$$\mathbb{E}_P[U \mid a] \geq \mathbb{E}_P[U \mid a']$$

1.2.2 Example: Betting

In this example, probabilities reflect actual randomness

Choice	Win Probability p	Payout w	Expected gain
Don't play	0	0	0
Black	18/37	2	
Red	18/37	2	
0	1/37	36	
1	1/37	36	





What are the expected gains for these bets?

1.2.3 The St-Petersburg Paradox

1. The game If you give me x CHF, then I promise to:
   - (a) Throw a fair coin until it comes heads.
   - (b) If it does so after T throws, then I will give you 2^T CHF.
2. The question
   - How much x are you willing to pay to play?
   - Given that the expected amount of money is infinite, why are you only willing to pay a small x?

1.2.4 Example: Route selection

- In this example, probabilities reflect subjective beliefs

Choice	Best time	Chance of delay	Delay amount	Expected time
Train	80	5%	5	
Car, route A	60	50%	30	
Car, route B	70	10%	10	

1.2.5 Example: Noisy optimisation

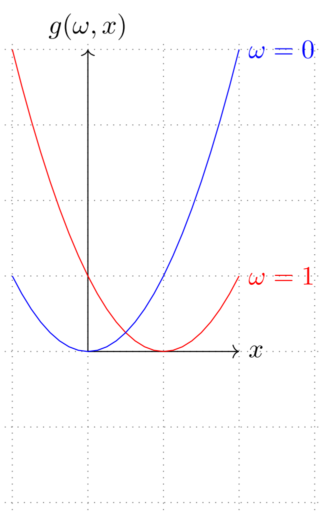
1. Simple maximisation For a function $f : \mathbb{R} \to \mathbb{R}$, find a maximum x^* i.e. $f(x^*) \geq f(x) \forall x$.
2. Necessary conditions B_THEOREM If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, a maximum point x^* satisfies:

$$\frac{d}{dx} f(x^*) = 0, \qquad \frac{d}{dx^2} f(x^*) < 0.$$

3. Noisy optimisation
   - We select x but **do not** observe $f(x)$.
   - We observe a **random** g with $\mathbb{E}[g|x] = f(x)$.

$$f(x) \triangleq \mathbb{E}[g|x], \qquad \mathbb{E}[g|x] = \int_{-\infty}^{\infty} g(\omega, x) p(\omega) d\omega \tag{1}$$

1.2.6 Mean-squared error cost function



This example is for a quadratic loss: $g(\omega, x) = (\omega - x)^2$.

1.2.7 Example: Estimation

- θ: **parameter** (random)
- $\hat{\theta}$: **estimate** (our action)
- $(\theta - \hat{\theta})^2$: **cost** function

1. Mean-squared error minimiser If we want to guess $\hat{\theta}$, and we knew that $\theta \sim P$, then the guess

   $$\hat{\theta} = \mathbb{E}_P(\theta) = \arg\min_{\hat{\theta}} \mathbb{E}_P[(\theta - \hat{\theta})^2]$$

   minimises the squared error. This is because

   $$\frac{d}{d\hat{\theta}} \mathbb{E}_P[(\theta - \hat{\theta})^2] = \frac{d}{d\hat{\theta}} \sum_{\omega} [\theta(\omega) - \hat{\theta}]^2 P(\omega) \tag{2}$$

   $$= \sum_{\omega} \frac{d}{d\hat{\theta}} [\theta(\omega) - \hat{\theta}]^2 P(\omega) \tag{3}$$

   $$= \sum_{\omega} 2[\theta(\omega) - \hat{\theta}](-1)P(\omega) \qquad = 2(\hat{\theta} - \mathbb{E}_P[\theta]). \tag{4}$$

   Setting this to 0 gives $\hat{\theta} = \mathbb{E}_P[\theta]$

2 Gradient methods

2.1 Gradients for optimisation

2.1.1 The gradient descent method: one dimension

- Function to minimise $f : \mathbb{R} \to \mathbb{R}$.
- Derivative $\frac{d}{d\theta} f(\theta)$

1. Gradient descent algorithm
   - Input: initial value θ^0, **learning rate** schedule α_t
   - For $t = 1, \ldots, T$
     - $\theta^{t+1} = \theta^t - \alpha_t \frac{d}{d\theta} f(\theta^t)$
   - Return θ^T
2. Properties
   - If $\sum_t \alpha_t = \infty$ and $\sum_t \alpha_t^2 < \infty$, it finds a local minimum θ^T, i.e. there is $\epsilon > 0$ so that

     $$f(\theta^T) < f(\theta), \forall \theta : \|\theta^T - \theta\| < \epsilon.$$

2.1.2 One-dimensional minimisation example

2.1.3 Gradient methods for expected value EXAMPLE

1. Estimate the expected value $x_t \sim P$ with $\mathbb{E}_P[x_t] = \mu$.
2. Objective: mean squared error Here $\ell(x, \theta) = (x - \theta)^2$.
$$\min_\theta \mathbb{E}_P[(x_t - \theta)^2].$$
3. Exact derivative update If we know P, then we can calculate
$$\theta^{t+1} = \theta^t - \alpha_t \frac{d}{d\theta} \mathbb{E}_P[(x - \theta^t)^2] \tag{5}$$
$$\frac{d}{d\theta} \mathbb{E}_P[(x - \theta^t)^2] = 2(\mathbb{E}_P[x] - \theta^t) \tag{6}$$

2.1.4 Stochastic derivative

- Function to minimise $f : \mathbb{R} \to \mathbb{R}$.
- Derivative $\frac{d}{d\theta} f(\theta)$
- $f(\theta) = \mathbb{E}[g|\theta]$
- $\frac{d}{d\theta} f = \mathbb{E}[\frac{d}{d\theta} g|\theta]$

1. Stochastic derivative algorithm
   - Input: initial value θ^0, **learning rate** schedule α_t
   - For $t = 1, \ldots, T$
     - Observe $g(\omega_t, \theta^t)$, where $\omega_t \sim P$.
     - $\theta^{t+1} = \theta^t - \alpha_t \frac{d}{d\theta} g(\omega_t, \theta^t)$
   - Return θ^T

2.1.5 Stochastic gradient for mean estimation

1. Sampling B_THEOREM For any bounded random variable f,
$$\mathbb{E}_P[f] = \int_X dP(x) f(x) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^T f(x_t) = \mathbb{E}_P \left[\frac{1}{T} \sum_{t=1}^T f(x_t) \right], \qquad x_t \sim P$$
2. Derivative ampling B_EXAMPLE We can also approximate the gradient through sampling:
$$\frac{d}{d\theta} \mathbb{E}_P[(x - \theta)^2] = \int_{-\infty}^{\infty} dP(x) \frac{d}{d\theta} (x - \theta)^2$$
$$\approx \frac{1}{T} \sum_{t=1}^T \frac{d}{d\theta} (x_t - \theta)^2 = \frac{1}{T} \sum_{t=1}^T 2(x_t - \theta)$$
   - Wen can even update θ after **each sample** x_t:
$$\theta^{t+1} = \theta^t + 2\alpha_t (x_t - \theta^t)$$

2.1.6 The gradient method

- Function to minimise $f : \mathbb{R}^n \to \mathbb{R}$.
- **Gradient** $\nabla_\theta f(\theta) = \left(\frac{\partial f(\theta)}{\partial \theta_1}, \ldots, \frac{\partial f(\theta)}{\partial \theta_n}\right)$,
- **Partial** derivative $\frac{\partial f}{\partial \theta_n}$

1. Gradient descent algorithm
   - Input: initial value θ^0, learning rate schedule α_t
   - For $t = 1, \ldots, T$
     - $\theta^{t+1} = \theta^t - \alpha_t \nabla_\theta f(\theta^t)$
   - Return θ^T
2. Properties
   - If $\sum_t \alpha_t = \infty$ and $\sum_t \alpha_t^2 < \infty$, it finds a local minimum θ^T, i.e. there is $\epsilon > 0$ so that

$$f(\theta^T) < f(\theta), \forall \theta : \|\theta^T - \theta\| < \epsilon.$$

2.1.7 When the cost is an expectation B_EXAMPLE

In machine learning, we sometimes want to minimise the **expectation** of a **cost** ℓ,

$$f(\theta) \triangleq \mathbb{E}[\ell|\theta] = \int_\Omega dP(\omega)\ell(\omega, \theta)$$

This can be approximated with a sample

$$f(\theta) \approx \frac{1}{T} \sum_t \ell(\omega_t, \theta)$$

The same holds for the gradient:

$$\nabla_\theta f(\theta) = \int_\Omega dP(\omega) \nabla_\theta \ell(\omega, \theta) \approx \frac{1}{T} \sum_t \nabla_\theta \ell(\omega_t, \theta)$$

2.1.8 Stochastic gradient method

- Function to **minimise** $f : \mathbb{R}^n \to \mathbb{R}$.
- **Gradient** $\nabla f(\theta)$
- $f(\theta) = \mathbb{E}[\ell|\theta]$
- $\nabla_\theta f = \mathbb{E}[\nabla_\theta \ell|\theta]$

1. Algorithm

- Input: initial value θ^0, **learning rate** schedule α_t
- For $t = 1, \ldots, T$
  - Observe $\ell(\omega_t, \theta^t)$, where $\omega_t \sim P$.
  - $\theta^{t+1} = \theta^t - \alpha_t \nabla_\theta g(\omega_t, \theta^t)$
- Return θ^T

2. Alternative view: Noisy gradients
   - $\theta^{t+1} = \theta^t - \alpha_t [\nabla_\theta f(\theta^t) + \epsilon_t]$
   - $\mathbb{E}[\epsilon_t] = 0$ is sufficient for convergence.