

Mathematical background

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Outline

Elementary background

- Logic and Set theory

- Functions, sequences and sums

Probability background

- Probability facts

- Conditional probability and independence

- Random variables, expectation and variance

Linear algebra

- Vectors

- Linear operators and matrices

Calculus

- Univariate calculus

- Multivariate calculus

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Logic

Statements

- ▶ A statement A may be true or false

Unary operators

- ▶ negation: $\neg A$ is true if A is false (and vice-versa).

Binary operators

- ▶ or: $A \vee B$ (A or B) is true if either A or B are true.
- ▶ and: $A \wedge B$ is true if both A and B are true.
- ▶ implies: $A \Rightarrow B$: is false if A is true and B is false.
- ▶ iff: $A \Leftrightarrow B$: is true if A, B have equal truth values.

Operator precedence

$\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$

Set theory

- ▶ First, consider some universal set Ω .
- ▶ A set A is a collection of points x in Ω .
- ▶ $\{x \in \Omega : f(x)\}$: the points in Ω for which $f(x)$ is true.
- ▶ Empty set: \emptyset . No point ever belong in the empty set.

Unary operators

- ▶ Negation: $\neg A = \{x \in \Omega : x \notin A\} = \Omega \setminus A$.

Binary operators

- ▶ Union: $A \cup B = \{x \in \Omega : x \in A \vee x \in B\}$ - (c.f. $A \vee B$)
- ▶ Intersection: $A \cap B = \{x \in \Omega : x \in A \wedge x \in B\}$ - (c.f. $A \wedge B$)
- ▶ Difference: $A \setminus B = \{x \in \Omega : x \in A \wedge x \notin B\}$ (c.f. $A \wedge \neg B$)

Binary relations

- ▶ $A \subset B$ iff $x \in A \Rightarrow x \in B$ - (c.f. $A \Rightarrow B$)
- ▶ $A = B$ iff $x \in A \Leftrightarrow x \in B$ - (c.f. $A \Leftrightarrow B$)

Functions

Definition

A function $f : X \rightarrow Y$ is a relation from a set X to a set Y so that, for any $x \in X$, there exists a unique $y \in Y$ so that $y = f(x)$.

Inverse of a function

A function $f : X \rightarrow Y$ has an inverse $f^{-1} : Y \rightarrow X$ if, for any $y \in Y$, there exists a **unique** $x \in X$ such that $f(x) = y$, and so that $f^{-1}(y) = x$.

Sequences

Sequences

A sequence x_1, \dots, x_t is written $\{x_i | i = 1, \dots, t\}$ or $\{x_i\}$ for brevity

Sums

A sum of a sequence of variables $\{x_i\}$ is defined as:

$$\sum_{i=1}^t x_i = x_1 + x_2 + \dots + x_t.$$

The python notation `sum([f(x) for x in A])` corresponds to:

$$\sum_{x \in A} f(x)$$

Products

A product is similarly defined at

$$\prod_{i=1}^t x_i = x_1 \times x_2 \times \dots \times x_t.$$

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Events as sets

The universe and random outcomes

- ▶ The Ω contains all events that can happen.
- ▶ When something happens, we observe an element $\omega \in \Omega$.

Events in the universe

- ▶ An event is true if $\omega \in A$, and false if $\omega \notin A$.
- ▶ The negative event $\neg A = \Omega \setminus A$ is the set
- ▶ The possible events are a collection of subsets Σ of Ω so that

(i) $\Omega \in \Sigma$, (ii) $A, B \in \Sigma \Rightarrow A \cup B \in \Sigma$ (iii) $A \in \Sigma \Rightarrow \neg A \in \Sigma$

Example: Traffic violation

- ▶ A car is moving with speed $\omega \in [0, \infty)$ in front of the speed camera.
- ▶ $A_0 = [0, 50]$: below the speed limit
- ▶ $A_1 = (50, 60]$: low fine
- ▶ $A_2 = (60, \infty]$: high fine
- ▶ $A_3 = (100, \infty)$: Suspension of license
- ▶ All combinations of the above events are interesting.

Probability fundamentals

Probability measure P

Probability can be seen as an area-like function assigning a likelihood to sets.

- ▶ $P : \Sigma \rightarrow [0, 1]$ gives the likelihood $P(A)$ of an event $A \in \Sigma$.
- ▶ $P(\Omega) = 1$
- ▶ For $A, B \subset \Omega$, if $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$.

Marginalisation

If $A_1, \dots, A_n \subset \Omega$ are a partition of Ω

$$P(B) = \sum_{i=1}^n P(B \cap A_i).$$

Conditional probability

Definition (Conditional probability)

The conditional probability of an event A given an event B is defined as

$$P(A|B) \triangleq \frac{P(A \cap B)}{P(B)}$$

The above definition requires $P(B)$ to exist and be positive.

Conditional probabilities as a collection of probabilities

More generally, we can define conditional probabilities as simply a collection of probability distributions:

$$\{P_\theta(A) \mid \theta \in \Theta\},$$

where Θ is an arbitrary set.

The theorem of Bayes

Theorem (Bayes's theorem)

$$P(A|B) = \frac{P(B|A)}{P(B)}$$

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The general case

If A_1, \dots, A_n are a partition of Ω , meaning that they are mutually exclusive events (i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$) such that one of them must be true (i.e. $\bigcup_{i=1}^n A_i = \Omega$), then

$$P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$$

and

$$P(A_j|B) = \frac{P(B|A_j)}{\sum_{i=1}^n P(B|A_i)P(A_i)}$$

Independence

Independent events

A, B are independent iff $P(A \cap B) = P(A)P(B)$.

Conditional independence

A, B are conditionally independent given C iff
 $P(A \cap B|C) = P(A|C)P(B|C)$.

Random variables

A random variable $f : \Omega \rightarrow \mathbb{R}$ is a real-value function measurable with respect to the underlying probability measure P , and we write $f \sim P$.

The distribution of f

The probability that f lies in some subset $A \subset \mathbb{R}$ is

$$P_f(A) \triangleq P(\{\omega \in \Omega : f(\omega) \in A\}).$$

Independence

Two RVs f, g are independent in the same way that events are independent:

$$P(f \in A \wedge g \in B) = P(f \in A)P(g \in B) = P_f(A)P_g(B).$$

In that sense, $f \sim P_f$ and $g \sim P_g$.

IID (Independent and Identically Distributed) random variables

A sequence x_t of r.v.s is IID if $x_t \sim P$ ($x_1, \dots, x_t, \dots, x_T$) $\sim P^T$.

Expectation

For any real-valued random variable $f : \Omega \rightarrow \mathbb{R}$, the expectation with respect to a probability measure P is

$$\mathbb{E}_P(f) = \sum_{\omega \in \Omega} f(\omega) P(\omega).$$

When Ω is continuous, we can use a density p

$$\mathbb{E}_P(f) = \int_{\Omega} f(\omega) p(\omega) d\omega.$$

Linearity of expectations

For any RVs x, y :

$$\mathbb{E}_P(x + y) = \mathbb{E}_P(x) + \mathbb{E}_P(y)$$

Independence

If x, y are independent RVs then $\mathbb{E}_P(xy) = \mathbb{E}(x) \mathbb{E}(y)$.

Correlation

If x, y are **not** correlated then $\mathbb{E}_P(xy) = \mathbb{E}(x) \mathbb{E}(y)$.

Conditional expectation

The conditional expectation of a random variable $f : \Omega \rightarrow \mathbb{R}$, with respect to a probability measure P conditioned on some event B is simply

$$\mathbb{E}_P(f|B) = \sum_{\omega \in \Omega} f(\omega)P(\omega|B).$$

Variance

For any real-valued random variable $f : \Omega \rightarrow \mathbb{R}$, the variance with respect to a probability measure P is

$$\mathbb{V}_P(f) = \sum_{\omega \in \Omega} [f(\omega) - \mathbb{E}_P(f(\omega))]^2 P(\omega).$$

Linearity of variance

If f, g are uncorrelated RVs

$$\mathbb{V}_P(f + g) = \mathbb{V}_P(f) + \mathbb{V}_P(g).$$

Variance products

If f, g are independent RVs

$$\mathbb{V}_P(f + g) = \mathbb{E}_P(f)^2 \mathbb{V}_P(g) + \mathbb{E}_P(g)^2 \mathbb{V}_P(f) + \mathbb{V}_P(f) \mathbb{V}_P(g).$$

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Vector space F

- ▶ The Euclidean space \mathbb{R}^d is a vector space.
- ▶ Vector spaces have the following properties.

Axioms of F

- ▶ $(x + y) + z = x + (y + z)$, for all $x, y, z \in F$.
- ▶ $x + y = y + x$, for all $x, y \in F$.
- ▶ There is a zero element $0 \in F$ such that $x + 0 = x$ for all $x \in F$.
- ▶ For all $x \in F$, there is an element $-x \in F$ so that $x + (-x) = 0$.
- ▶ $a(x + y) = ax + ay$, For any $a \in \mathbb{R}$, $x, y \in F$.
- ▶ $(a + b)x = ax + bx$, For any $a, b \in \mathbb{R}$, $x \in F$.

The real vector space $F = \mathbb{R}^d$

For $a \in \mathbb{R}$ and $x, y \in F$,

- ▶ Elements: $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$
- ▶ Addition: $x + y = (x_1 + y_1, \dots, x_d + y_d)$.
- ▶ Scalar multiplication: $ax = (ax_1, \dots, ax_d)$.
- ▶ Negation: $-x = (-1)x$.
- ▶ Zero element: $0 = (0, \dots, 0)$

Linear operators

Linear operator $A : F \rightarrow G$.

A linear operator A from one vector space F to another, G has the following properties:

- ▶ $A(x + y) = Ax + Ay$ for all $x, y \in F$.
- ▶ $A(ax) = a(Ax)$, for all $a \in \mathbb{R}$, $x \in F$.

Matrices in $\mathbb{R}^{n \times m}$.

A matrix $A \in \mathbb{R}^{n \times m}$ is a tabular array

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,m} \end{bmatrix}$$

Matrices can be seen as linear operators $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$, mapping vectors in \mathbb{R}^n to vectors in \mathbb{R}^m .

Multiplication operators

Matrix multiplication

For $A \in \mathbb{R}^{n \times d}$, $B \in \mathbb{R}^{d \times m}$, the ij -th element of the result of the multiplication AB is

$$(AB)_{i,j} = \sum_{k=1}^d A_{i,k} B_{k,j}.$$

so that $AB \in \mathbb{R}^{n \times m}$.

Matrix-vector multiplication

A matrix $A \in \mathbb{R}^{n \times m}$ defines the following linear operator $A : \mathbb{R}^m \rightarrow \mathbb{R}^n$.

$$Ax = \left(\sum_{j=1}^m A_{i,j} x_j : i = 1, \dots, n \right)$$

All vectors $x \in \mathbb{R}^m$ are equivalent to matrices in $\mathbb{R}^{m \times 1}$.

Matrix inverses

The identity matrix $I \in \mathbb{R}^{n \times n}$

- ▶ For this matrix, $I_{i,i} = 1$ and $I_{i,j} = 0$ when $j \neq i$.
- ▶ $Ix = x$ and $IA = A$.

The inverse of a matrix $A \in \mathbb{R}^{n \times n}$

A^{-1} is called the inverse of A if

- ▶ $AA^{-1} = I$.
- ▶ or equivalently $A^{-1}A = I$.

The pseudo-inverse of a matrix $A \in \mathbb{R}^{n \times m}$

- ▶ \tilde{A}^{-1} is called the **left pseudoinverse** of A if $\tilde{A}^{-1}A = I$.
- ▶ \tilde{A}^{-1} is called the **right pseudoinverse** of A if $A\tilde{A}^{-1} = I$.

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Derivatives

Derivative

The derivative of a single-argument function is defined as:

$$\frac{d}{dx}f(x) = \lim_{\epsilon \rightarrow 0} \frac{f(x + \epsilon) - f(x)}{\epsilon}.$$

f must be absolutely continuous at x for the derivative to exist.

Subdifferential

For non-differential functions, we can sometimes define the set of all subderivatives:

$$\partial f(x) = \left[\lim_{\epsilon \rightarrow 0} \frac{f(x) - f(x - \epsilon)}{\epsilon}, \lim_{\epsilon \rightarrow 0} \frac{f(x + \epsilon) - f(x)}{\epsilon} \right]$$

Integrals

Riemann integral

The Riemann integral is obtained by taking a **horizontal** discretisation of a function to the limit:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{t=1}^n f(x_t) \frac{b-a}{n}, \quad x_t = a + (t-1) \cdot \frac{b-a}{n}$$

Lebesgue integral

This integral is obtained by taking a **vertical** discretisation of a function to the limit. Let λ be the Lebesgue measure (i.e. area) of a set. Then:

$$\int_X f(x) d\lambda(x) = \lim_{n \rightarrow \infty} \sum_{t=1}^n y_t \lambda(S_t),$$

$$S_t = \{x : f(x) \in (y_{t-1}, y_t]\}, \quad y_0 = -\infty, \quad y_n = \sup_x f(x).$$

Fundamental theorem of calculus

$$f(x) = \frac{d}{dx} \int_a^x f(t) dt$$

If $\frac{d}{dx} F = f$ then its integral from a to b is:

$$\int_a^b f(x) dx = F(b) - F(a),$$

Multivariate Functions

We consider functions operating in multi-dimensional Euclidean spaces.

$$f : \mathbb{R}^n \rightarrow \mathbb{R}.$$

- ▶ Any $x \in \mathbb{R}^n$ is $x = (x_1, \dots, x_n)$, with $x_i \in \mathbb{R}$.
- ▶ We write $f(x)$ instead of $f(x_1, \dots, x_n)$.

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m.$$

- ▶ If $y = f(x)$ then y_i is the i -th component of $y \in \mathbb{R}^m$.
- ▶ Can be seen as m functions $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$, with $y_i = f_i(x)$.

Derivatives in many dimensions

Partial derivative

The partial derivative of f with respect to its i -th argument is: $\frac{\partial}{\partial x_i} f(x)$, where we see all x_j with $j \neq i$ as fixed.

Gradient of f

This is the vector of all its partial derivatives:

$$\nabla_x f(x) = \left(\frac{\partial}{\partial x_1} f(x) \cdots \frac{\partial}{\partial x_i} f(x) \cdots \frac{\partial}{\partial x_n} f(x) \right)^\top$$

Directional derivative

We can also look at how much the function increases in a specific direction $\delta \in \mathbb{R}^n$.

$$D_\delta f(x) = \lim_{\epsilon \rightarrow 0} \frac{f(x + \epsilon \delta) - f(x)}{\epsilon}.$$