Sum of Squares

Labs in Mathematical Optimization

Quick reminder on PSD matrices

- A matrix M is called Positive-Semidefinite (PSD), if it is:
  1. Symmetric: $\mathrm{M}^{\mathrm{T}} = M$
  2. Non-negative: for any vector x: $\mathrm{x}^{\mathrm{T}}\mathrm{M}\mathrm{x} \geq 0$

Other equivalent formulations:

1. $\mathrm{M} = \mathrm{A}^{\mathrm{T}}\mathrm{A}$ for some matrix A
2. $\mathrm{M} = \sum_{\mathrm{i}=1}^{\mathrm{k}} \lambda_{\mathrm{i}}\mathrm{x}_{\mathrm{i}}\mathrm{x}_{\mathrm{i}}^{\mathrm{T}}$ for some $\lambda_{\mathrm{i}} \geq 0$ and x_{i} orthonormal vectors (spectral decomposition)

Remember that any symmetric matrix is diagonalizable with an orthonormal basis and real eigenvalues (do you remeber what that means? :)

Sum of squares

- Definition: A polynomial $p(x) \in R[x]$ is called a Sum of Squares (SOS) if it can be written as $\mathrm{p(x)} = \sum_{\mathrm{j}} \mathrm{p}_{\mathrm{j}}^{2}(\mathrm{x})$ where $\mathrm{p_j(x)}$ are some polynomials. We denote it as $p(x) \in \Sigma[x]$
- Only polynomials of even degree can be SOS. (Show it)
- Other results:
    1. The set of SOS is closed, convex (Show it) and non-empty
    2. If a polynomial is SOS, then it's non-negative. Converse is not true in general

Excercises

- Only polynomials of even degree can be SOS

Assume, that $p(x)$ is SOS. Then there exist polynomilas $p_j(x)$,such that: $p(x) = \sum_j p_j(x)^2$. Since the coefficients are non-zero, they can't cancell out in a way that they reduce the degree. So that the degree of the polynomial p is $\max\{\, 2\deg\left(p_j\right) | j \text{ is the coefficient enumerator}\}$.

- SOS is a convex set

Let $p, q \in \Sigma[x]$, and let $r \in [0,1]$.Then $p(x) = \sum_j p_j(x)^2$and $q(x) = \sum_{j'} q_{j'}(x)^2$. Consider the convex combination $rp(x) + (1 - r)q(x) = \sum_j \left[\sqrt{r} p_j(x)\right]^2 + \sum_{j'} \left[\sqrt{1 - r} q_{j'}(x)\right]^2$ so that any convex combination is also SOS.

Sum of squares and PSD matrices

- A polynomial $\mathrm{p}(\mathrm{x}) \in \mathrm{R}[\mathrm{x}]$ of degree $2\mathrm{d}$ is SOS if and only if it can be written as $v_x^T \mathrm{M} \mathrm{v}_\mathrm{x}$ where M is some PSD matrix and $v_x = (1, x, \dots, x^d)$

⇒Let us assume that $p(x)$ is SOS. Let us first notice that one can always write $p(x)$ as $v_x^T M v_x$,where M is symmetric matrix (check it, hint: if M is not symmetric then $M' = (M + M^T)/2$ is). M has to be PSD, because for the negative eigenvalues, p is not SOS.

Crucial part: $v_x^T M v_x = \sum_{i,j} m_{ij}\, x^i x^j = \sum_{i,j} \frac{1}{2} m_{ij}\, x^i x^j + \sum_{i,j} \frac{1}{2} m_{ij}\, x^i x^j = \sum_{i,j} \frac{1}{2} m_{ij}\, x^i x^j + \sum_{i,j} \frac{1}{2} m_{ji}\, x^i x^j = \sum_{i,j} \frac{1}{2} (m_{ij} + m_{ji}) x^i\, x^j = v_x^T \frac{1}{2} (M + M^T) v_x$

⇐Let us assume that $p(x) = v_x^T M v_x$ for some PSD . Then by eigen decomposition: $M = \sum_j \lambda_j\, \mathrm{z}_j \mathrm{z}_j^T$ for some non-negative $\lambda_j$and $p(x) = \sum_j \left[\sqrt{\lambda_j} z_j^T v_x\right]^2$ and $z_j^T v_j \;=\; \sum_{a=0}^{d} (z_j)_a\, x^a$, then p(x) is SOS

Task (obligatory)

- Check whether polynomial $\mathrm{p(x)} = 5x^4 - 4x^3 - x^2 + 2x + 2$ is SOS and if yes, then find the decomposition of it. Write a test, to check the result.

More challenging task*

- Write a function *sum_of_squares* that takes a list representing the coefficients of the polynomial, and then returns the list of polynomials form the SOS decomposition. Remeber to handle the exceptions properly if the solution is infeasible.

The ultimate challenge**

- Write a function *sum_of_squares_2* that takes a list, representing the multivariate polynomial and then returns its decomposition into SOS
- *Hint: just figure out what should be the dimension and the form of the vectors of monomials. Conceptually, the rest is pretty much the same.*