Theorem L10.1. Deciding membership in the cone of nonnegative polynomials $\mathbb{R}[x]_{\geq 0}$ for any polynomial of degree ≥ 4 with $n \geq 2$ variables is computationally intractable.

Proof. By reduction from the problem of deciding if a symmetric matrix $M \in \mathbb{R}^{n \times n}$ is copositive (see Lecture 9), that is,

$$z^\top M z \geq 0 \qquad \forall z \in \mathbb{R}^n_{\geq 0}.$$

Indeed, consider the n-variate polynomial of degree 4 defined as

$$p(x_1, ..., x_n) := \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix}^\top M \begin{pmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{pmatrix}.$$

If M is copositive, this polynomial is nonnegative on $\mathbb{R}^n$, since the vector of squares $(x_1^2, ..., x_n^2)$ is always nonnegative. Conversely, if p is nonnegative on $\mathbb{R}^n$, then since $(x_1^2, ..., x_n^2)$ spans all of $\mathbb{R}^n_{\geq 0}$, we have that M is copositive. □

As a side remark, this is the second *convex* set we see for which membership—let alone optimization—is intractable (the first being the copositive cone). This should serve as a reminder that equating convexity with tractability can be a dangerous oversimplification.

L10.3 Sum-of-squares polynomials

Theorem L10.1 shows that even deciding *membership* in the cone of nonnegative polynomials is hard in general, let alone optimizing over it. This begs the question of what is the largest set of polynomials for which deciding nonnegativity is easy. The following cone will provide an extremely important step in this direction.

Definition L10.2 (Sum-of-squares polynomials, $\Sigma[x]$)**.** An n-variate polynomial $p \in \mathbb{R}[x]$ is said to be a *sum-of-squares* polynomial (or SOS for short) if it can be written as

$$p(x) = \sum_k p_k(x)^2, \qquad x \in \mathbb{R}^n$$

for appropriate polynomials $p_k \in \mathbb{R}[x]$.

Remark L10.1. Only polynomials of even degree can be sum-of-squares. Furthermore, if p has degree $2d$, then each of the p_k's can only have degree at most d.

It is immediate to check [▷ you should check!] that the set of sum-of-squares polynomials is a closed, convex, nonempty cone. Furthermore, every sum-of-squares polynomial is clearly a nonnegative polynomial, so $\Sigma[x] \subseteq \mathbb{R}[x]_{\geq 0}$. In general, the inclusion is strict, as there exist polynomials that are nonnegative but not sum-of-squares. We will see a well-known example, called the Motzkin polynomial, later in this lecture in Example L10.2.

L10.3.1 The connection to the positive semidefinite cone

Unlike the cone of nonnegative polynomials, the cone of sum-of-squares polynomials is easy to characterize. The following theorem provides a very useful criterion for deciding whether a polynomial is sum-of-squares.

Theorem L10.2. Membership in the cone of sum-of-squares polynomials can be determined by checking the feasibility of a semidefinite program. In particular, let $p(x)$ be an arbitrary polynomial of degree $2d$ in n variables, and let v_d be the vector of all monomials of degree up to d that can be constructed using the variables x.

Then, $p(x) \in \Sigma[x]$ if and only if there exists a positive semidefinite matrix Q such that

$$p(x) = (v_d(x))^\top Q v_d(x),$$

Proof. ($\Longrightarrow$) Any polynomial of degree up to d can be written as a linear combination of the monomials in v_d. Hence, any sum-of-squares polynomial can be written in the form

$$p(x) = \sum_k \left(v_d(x)^\top \alpha_k\right)^2$$

for some appropriate coefficient vectors α_k. But then, we can write

$$p(x) = \left\| \underbrace{\begin{pmatrix} -\alpha_1^\top- \\ -\alpha_2^\top- \\ \vdots \end{pmatrix}}_{=: C} v_d(x) \right\|^2 = (v_d(x))^\top C^\top C v_d(x). \tag{1}$$

Since a matrix of the form $C^\top C$ is positive semidefinite, this direction holds.

($\Longleftarrow$) Conversely, if $p(x) = (v_d(x))^\top Q v_d(x)$ for some positive semidefinite matrix Q, we can write $Q = C^\top C$ for some C. But then, we can use (1) from right to left and extract from the rows of C the coefficients α_k for the sum-of-squares representation of $p(x)$. $\square$

Example L10.1. A bivariate polynomial $p(x_1, x_2)$ of degree 4 is a sum-of-squares polynomial if and only if there exists $Q \succeq 0$ such that

$$\begin{aligned} p(x_1, x_2) &= \begin{pmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \\ x_1 \\ x_2 \\ 1 \end{pmatrix}^\top \begin{pmatrix} q_{00} & q_{01} & q_{02} & q_{03} & q_{04} & q_{05} \\ q_{01} & q_{11} & q_{12} & q_{13} & q_{14} & q_{15} \\ q_{02} & q_{12} & q_{22} & q_{23} & q_{24} & q_{25} \\ q_{03} & q_{13} & q_{23} & q_{33} & q_{34} & q_{35} \\ q_{04} & q_{14} & q_{24} & q_{34} & q_{44} & q_{45} \\ q_{05} & q_{15} & q_{25} & q_{35} & q_{45} & q_{55} \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \\ x_1 \\ x_2 \\ 1 \end{pmatrix} \\ &= (q_{00})x_1^4 + (2q_{01})x_1^3x_2 + (q_{11} + 2q_{02})x_1^2x_2^2 + (2q_{12})x_1x_2^3 + (q_{22})x_2^4 \\ &\quad + (2q_{03})x_1^3 + (2q_{04} + 2q_{13})x_1^2x_2 + (2q_{23} + 2q_{14})x_1x_2^2 + (2q_{24})x_2^3 \\ &\quad + (q_{33} + 2q_{05})x_1^2 + (2q_{15} + 2q_{34})x_1x_2 + (q_{44} + 2q_{25})x_2^2 \\ &\quad + (2q_{35})x_1 + (2q_{45})x_2 + (q_{55}). \end{aligned}$$