

Functional Forkshop: Part 1

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About the Forkshop

- Basic forkshop is split into several parts:
 1. Type classes, Semigroups and Monoids.
 2. Functors and Applicative Functors.
 3. Monads.
 4. Readers.
 5. Comonads.
- Theory and practice. Make sure that you are ready to write the code.
- Forkshop is duplicated in Haskell.

Whys

- Functional programming roams (a bit).
 - More projects are using functional programming techniques and idioms (at different scale).
- Some people are still confused by all these functional talks (`OptionT`, type lambdas etc).
- Having a common language and understanding of some fundamental stuff is important.

Agenda

- Type classes
- Semigroups
- Monoids
- 3 interesting™ tasks

Application definition

- We are writing a game.
- With multiple different creatures.
- Everyone introduces themselves.
- Introduction consists of animations and text showing in a bubble.

Meet the hero

```
case class Hero(name: String, job: String, level: Int) {  
  def introduce(): String = s"Hi! My name is $name. I am $level level $job."  
}  
  
object Game extends App {  
  val player = Hero("Valik", "Black Mage", 20)  
  
  someRealShitSounds()  
  drawBubble(player.introduce())  
  someRealShitAnimations()  
}  
  
// Hi! My name is Valik. I am 20 level Black Mage.
```

Every hero needs a monster

```
case class Orc(name: String, level: Int) {  
  def introduce(): String =  
    s"Lok-tar ogar! Me be $name. Me be strong. Level $level strong!"  
}  
  
case class Ooze(level: Int) {  
  def introduce(): String = 1.to(level).map(_=>"brlup").mkString("-")  
}
```

Game

```
object Game extends App {  
  val player = Hero("Valik", "Black Mage", 20)  
  val orc = Orc("Garrosh", 105)  
  val ooze = Ooze(2)  
  
  // Introduce player  
  someRealShitSounds()  
  drawBubble(player.introduce())  
  someRealShitAnimations()  
  
  // Introduce orc  
  someRealShitSounds()  
  drawBubble(orc.introduce())  
  someRealShitAnimations()  
  
  // Introduce ooze  
  someRealShitSounds()  
  drawBubble(ooze.introduce())  
  someRealShitAnimations()  
}  
  
// Hi! My name is Valik. I am 20 level Black Mage.  
// Lok-tar ogar! Me be Garrosh. Me be strong. Level 105 strong!  
// brlup-brlup
```


Game

```
object Game extends App {  
  val player = Hero("Valik", "Black Mage", 20)  
  val orc = Orc("Garrosh", 105)  
  val ooze = Ooze(2)
```

```
  // Introduce player  
  someRealShitSounds()  
  drawBubble(player.introduce())  
  someRealShitAnimations()
```

```
  // Introduce orc  
  someRealShitSounds()  
  drawBubble(orc.introduce())  
  someRealShitAnimations()
```

```
  // Introduce ooze  
  someRealShitSounds()  
  drawBubble(ooze.introduce())  
  someRealShitAnimations()
```

```
}
```

```
// Hi! My name is Valik. I am 20 level Black Mage.  
// Lok-tar ogar! Me be Garrosh. Me be strong. Level 105 strong!  
// brlup-brlup
```

Issues with this code:

1. Repetition
2. Noise

Introducing abstractions

```
//
```

```
case class Hero(...) {  
  def introduce(): String = s"..."  
}
```

```
case class Orc(...) {  
  def introduce(): String = s"..."  
}
```

```
case class Ooze(...) {  
  def introduce(): String = s"..."  
}
```

Introducing abstractions

```
//
```

```
case class Hero(...) {  
  def introduce(): String = s"..."  
}
```

```
case class Orc(...) {  
  def introduce(): String = s"..."  
}
```

```
case class Ooze(...) {  
  def introduce(): String = s"..."  
}
```

```
trait Introdutable {  
  def introduce(): String  
}
```

```
case class Hero(...) extends Introdutable {  
  override def introduce(): String = s"..."  
}
```

```
case class Orc(...) extends Introdutable {  
  override def introduce(): String = s"..."  
}
```

```
case class Ooze(...) extends Introdutable {  
  override def introduce(): String = s"..."  
}
```

Game with trait

```
def introduce(phrase: String): Unit = {  
  someRealShitSounds()  
  drawBubble(phrase)  
  someRealShitAnimations()  
}
```

```
object Game extends App {  
  /* ... */  
  
  introduce(player.introduce())  
  introduce(orc.introduce())  
  introduce(ooze.introduce())  
}
```

```
def introduce(creature: Introdutable): Unit = {  
  someRealShitSounds()  
  drawBubble(creature.introduce())  
  someRealShitAnimations()  
}
```

```
object Game extends App {  
  /* ... */  
  
  introduce(player)  
  introduce(orc)  
  introduce(ooze)  
}
```

Game with trait

```
def introduce(phrase: String): Unit = {  
  someRealShitSounds()  
  drawBubble(phrase)  
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}
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```
object Game extends App {  
  /* ... */  
  
  introduce(player.introduce())  
  introduce(orc.introduce())  
  introduce(ooze.introduce())  
}
```

```
def introduce(creature: Introducible): Unit = {  
  someRealShitSounds()  
  drawBubble(creature.introduce())  
  someRealShitAnimations()  
}
```

```
object Game extends App {  
  /* ... */  
  
  introduce(player)  
  introduce(orc)  
  introduce(ooze)  
}
```

- No more `introduce(_.introduce())`.
- We are adaptive. Less code needs to be changed if we need something new in the `introduce` function (e.g. sound name) - just add new 'method' to the trait.
- Refactoring becomes easier.

Here comes the cockatrice

```
import io.proprietary.monsters.cockatrice._

/* ... */

object Game extends App {
  /* ... */

  val cockatrice = Cockatrice(level = 666, element = Element.Fire)

  introduce(cockatrice) // ???
                        // ain't gonna work
}
```

Shawarma to the rescue



```
import io.proprietary.monsters.cockatrice._

/* ... */

case class CockatriceWrapper(cockatrice: Cockatrice) extends Introducible {
  override def introduce(): String = {
    import cockatrice._
    s"Haha. I am a ${element.shortName} cockatrice of level ${level}."
  }
}

object Game extends App {
  /* ... */

  val cockatrice = Cockatrice(level = 666, element = Element.Fire)
  val cockatriceW = CockatriceWrapper(cockatrice)

  introduce(cockatriceW)

  /* ... */
}

// Haha. I am a fire cockatrice of level 666.
```

Calm down and reevaluate our goal

- Abstraction - caring about what you can do and not what you are. E.g. separation of data and behaviour.

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- Abstraction - caring about what you can do and not what you are. E.g. separation of data and behaviour.
- Composition - having a way to express something that can do several things at once.
- Extensibility - extending all kind of types:
 - types we own
 - types we don't own
 - even built-in types

trait + wrapper: abstraction

Abstraction holds. Proof is the introduce function itself.

```
def introduce(creature: Introducible): Unit = {  
  someRealShitSounds()  
  drawBubble(creature.introduce())  
  someRealShitAnimations()  
}
```

trait + wrapper: composition

Composition holds thanks to `with` keyword.

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```
trait CanAttack {  
  def attack(): Unit  
}  
  
def patheticAttack[A <: Introdutable with CanAttack](creature: A): Unit
```

trait + wrapper: composition

Composition holds thanks to with keyword.

```
trait CanAttack {  
  def attack(): Unit  
}  
  
def patheticAttack[A <: Introdutable with CanAttack](creature: A): Unit
```

with keyword is not commutative

Introdutable with CanAttack != CanAttack with Introdutable.

trait + wrapper: extensibility

Extensibility holds, but with several caveats:

1. No consistency - we wrap only types we don't own.
2. Wrappers don't compose very well. You might even wrap your wrappers.
3. Bad usability:
 - 3.1 You can't interchangeably use wrapper and the underlying value.
 - 3.2 You can't plug in different behaviour implementations.

You know where it's going to, right?

You know where it's going to, right?



Dividing data and behaviour

```
trait Introdutable {  
  def introduce(): String  
}  
  
def introduce(creatute: Introdutable): Unit = {  
  /* ... */  
  drawBubble(creatute.introduce())  
  /* ... */  
}
```

Dividing data and behaviour

```
trait Introdutable {  
  def introduce(): String  
}
```

```
def introduce(creature: Introdutable): Unit = {  
  /* ... */  
  drawBubble(creature.introduce())  
  /* ... */  
}
```

```
trait Introdutable[A] {  
  def introduce(a: A): String  
}
```

```
def introduce[A](creature: A,  
                 impl: Introdutable[A]): Unit = {  
  /* ... */  
  drawBubble(impl.introduce(creature))  
  /* ... */  
}
```

Usage

```
// Define new trait  
trait Introdutable[A] {  
  def introduce(a: A): String  
}
```

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trait Introdutable[A] {  
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```
// Remove behaviour from data  
case class Hero(name: String, job: String, level: Int)
```

Usage

```
// Define new trait
trait Introdutable[A] {
  def introduce(a: A): String
}

// Remove behaviour from data
case class Hero(name: String, job: String, level: Int)

// Implement behaviour as a value in companion object
object Hero {
  val introdutableHero: Introdutable[Hero] = new Introdutable[Hero] {
    override def introduce(a: Hero): String =
      s"..."
  }
}
```

Usage

```
// Define new trait
trait Introdutable[A] {
  def introduce(a: A): String
}

// Remove behaviour from data
case class Hero(name: String, job: String, level: Int)

// Implement behaviour as a value in companion object
object Hero {
  val introdutableHero: Introdutable[Hero] = new Introdutable[Hero] {
    override def introduce(a: Hero): String =
      s"..."
  }
}

// Pass data and behaviour separately
object Game extends App {
  /* ... */
  introduce(
    creature = hero,
    impl = Hero.introdutableHero
  )
}
```

External types? Pff...

```
import io.proprietary.monsters.cockatrice._

// Implement behaviour as a value in companion object
object CockatriceInstances {
  val introducibleCockatrice: Introducible[Cockatrice] = new Introducible[Cockatrice] {
    override def introduce(a: Cockatrice): String =
      s"... "
  }
}
```


External types? Pff...

```
import io.proprietary.monsters.cockatrice._

// Implement behaviour as a value in companion object
object CockatriceInstances {
  val introducibleCockatrice: Introducible[Cockatrice] = new Introducible[Cockatrice] {
    override def introduce(a: Cockatrice): String =
      s"..."
  }
}

// Pass data and behaviour separately
object Game extends App {
  /* ... */
  introduce(
    creature = cockatrice,
    impl = CockatriceInstances.introducibleCockatrice
  )
}
```

But passing implementation around is...



Cucumbersome

So implicits :(

```
object Hero {
  val introducibleHero:
    Introducible[Hero] = ???
}

object CockatriceInstances {
  val introducibleCockatrice:
    Introducible[Cockatrice] = ???
}

def introduce[A](creature: A,
  impl: Introducible[A]): Unit = {
  /* ... */
  drawBubble(impl.introduce(creature))
  /* ... */
}

object Game extends App {
  /* ... */
  introduce(hero, introducibleHero)
  introduce(cockatrice, introducibleCockatrice)
}
```

So implicits :(

```
object Hero {
  val introducibleHero:
    Introducible[Hero] = ???
}

object CockatriceInstances {
  val introducibleCockatrice:
    Introducible[Cockatrice] = ???
}

def introduce[A](creature: A,
  impl: Introducible[A]): Unit = {
  /* ... */
  drawBubble(impl.introduce(creature))
  /* ... */
}

object Game extends App {
  /* ... */
  introduce(hero, introducibleHero)
  introduce(cockatrice, introducibleCockatrice)
}
```

```
object Hero {
  implicit val introducibleHero:
    Introducible[Hero] = ???
}

object CockatriceInstances {
  implicit val introducibleCockatrice:
    Introducible[Cockatrice] = ???
}

def introduce[A](creature: A)
  (implicit impl: Introducible[A]): Unit = {
  /* ... */
  drawBubble(impl.introduce(creature))
  /* ... */
}

object Game extends App {
  /* ... */
  introduce(hero)
  introduce(cockatrice)
}
```

Summoning the summoner

```
trait Introdutable[A] {  
  def introduce(a: A): String  
}
```

```
def introduce[A](creature: A)  
  (implicit impl: Introdutable[A]): Unit = {  
  /* ... */  
  drawBubble(impl.introduce(creature))  
  /* ... */  
}
```

Summoning the summoner

```
trait Introdutable[A] {  
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```

```
def introduce[A](creature: A)  
  (implicit impl: Introdutable[A]): Unit = {  
  /* ... */  
  drawBubble(impl.introduce(creature))  
  /* ... */  
}
```

```
trait Introdutable[A] {  
  def introduce(a: A): String  
}
```

```
object Introdutable {  
  def apply[A: Introdutable]: Introdutable[A] =  
    implicitly[Introdutable[A]]  
}
```

```
def introduce[A: Introdutable](creature: A): Unit = {  
  /* ... */  
  drawBubble(Introdutable[A].introduce(creature))  
  /* ... */  
}
```

What have we done?

Type class is just a construct that supports **ad hoc polymorphism**. E.g. allows one to define polymorphic functions that can be applied to arguments of different types and behave differently based the type of the arguments.

In other words, **type classes** are solution for supporting **function overloading**.

What have we done?

Type class is just a construct that supports **ad hoc polymorphism**. E.g. allows one to define polymorphic functions that can be applied to arguments of different types and behave differently based the type of the arguments.

In other words, **type classes** are solution for supporting **function overloading**.

In Scala this can be achieved in several ways:

- Class inheritance or traits.
- Type classes (traits + implicits).

Type classes: abstraction

Abstraction holds. Proof is the `introduce` function itself.

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Abstraction holds. Proof is the introduce function itself.

```
def introduce(creature: Introducible): Unit = {  
  /* ... */  
  drawBubble(creature.introduce())  
  /* ... */  
}
```

```
def introduce[A: Introducible](creature: A): Unit = {  
  /* ... */  
  drawBubble(Introducible[A].introduce(creature))  
  /* ... */  
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```

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Abstraction holds. Proof is the introduce function itself.

```
def introduce(creature: Introducible): Unit = {  
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def introduce[A: Introducible](creature: A): Unit = {  
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  drawBubble(Introducible[A].introduce(creature))  
  /* ... */  
}
```

We gain literal data and behaviour separation.

Type classes: composition

Composition holds. We just pass two different behaviours.

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```
def patheticAttack[A <: Introdutable with CanAttack](creature: A): Unit
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```
def patheticAttack[A <: Introdutable with CanAttack](creature: A): Unit
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```
def patheticAttack[A : Introdutable : CanAttack](creature: A): Unit
```

```
def patheticAttack[A](creature: A)  
  (implicit introducibleImpl: Introdutable[A],  
   canAttackImpl: CanAttack[A]): Unit
```

Type classes: composition

Composition holds. We just pass two different behaviours.

```
def patheticAttack[A <: Introdutable with CanAttack](creature: A): Unit
```

```
def patheticAttack[A : Introdutable : CanAttack](creature: A): Unit
```

```
def patheticAttack[A](creature: A)  
  (implicit introducibleImpl: Introdutable[A],  
   canAttackImpl: CanAttack[A]): Unit
```

But with type classes we don't care about the order.

Type classes: extensibility

Extensibility holds with some gains:

1. Consistency - we treat our own type the same way we treat external types.
2. Usability - no wrappers, no interchangeability problem.

Type classes: final thoughts

1. Simple idea giving us good properties.
2. Found a good use for controversial implicits feature.
3. Literal separation of data and behaviour.
4. Good for overloading.
5. + more abstraction, - more code

Time for a quiz!

What is common between:

1. Int
2. String
3. List
4. PartialFunction
5. HttpMapping

Time for a quiz!

What is common between:

1. Int
2. String
3. List
4. PartialFunction
5. HttpMapping

They can be composed!

1. $\text{Int} + \text{Int} = \text{Int}$
2. $\text{String} + \text{String} = \text{String}$
3. $\text{List} :: \text{List} = \text{List}$
4. $\text{PartialFunction} \text{ orElse } \text{PartialFunction} = \text{PartialFunction}$
5. $\text{HttpMapping} + \text{HttpMapping} = \text{HttpMapping}$

Associativity

1. $Int + Int + Int = Int + (Int + Int) = (Int + Int) + Int$
2. $String + String + String = String + (String + String) = (String + String) + String$
3. etc. . .

Semigroup

Semigroup is a set S with binary closed operation $\cdot : S \times S \rightarrow S$ that satisfies associativity property:

$$\forall a, b, c \in S : (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

Operation is closed when $\forall a, b \in S : a \cdot b \in S$.

But it's not that scary

```
package object typeclass {  
  
  //  
  // Laws:  
  // 1.  $\forall a, b, c \in A : (a \cdot b) \cdot c = a \cdot (b \cdot c)$   
  //  
  trait Semigroup[A] {  
    def combine(x: A, y: A): A  
  }  
  
  object Semigroup {  
    def apply[A: Semigroup]: Semigroup[A] =  
      implicitly[Semigroup[A]]  
  }  
}
```

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```
package object typeclass {  
  
  //  
  // Laws:  
  // 1.  $\forall a, b, c \in A : (a \cdot b) \cdot c = a \cdot (b \cdot c)$   
  //  
  trait Semigroup[A] {  
    def combine(x: A, y: A): A  
  }  
  
  object Semigroup {  
    def apply[A: Semigroup]: Semigroup[A] =  
      implicitly[Semigroup[A]]  
  }  
}
```

In simple words, semigroup is a set with means of combining elements of that set.

But it's not that scary

```
package object typeclass {  
  
  //  
  // Laws:  
  // 1.  $\forall a, b, c \in A : (a \cdot b) \cdot c = a \cdot (b \cdot c)$   
  //  
  trait Semigroup[A] {  
    def combine(x: A, y: A): A  
  }  
  
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    def apply[A: Semigroup]: Semigroup[A] =  
      implicitly[Semigroup[A]]  
  }  
}
```

In simple words, semigroup is a set with means of combining elements of that set.



Important!

Semigroup is a pair of the set and the operation.

You can't say that string is a semigroup, you must provide an operation.

And in many cases there is more than one operation for a set to form a semigroup.

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def combine(x: A, y: A): A
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- Laws describe properties of these operations and connection between operations in one type class.

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- In programming world it's just a contract.
- Operations in the type classes are very generic.
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- Laws describe properties of these operations and connection between operations in one type class.
- Contract of the interface gives us confidence when we write generic code.

What is law?

- In programming world it's just a contract.
- Operations in the type classes are very generic.
`def combine(x: A, y: A): A`
- So type classes should have some associated laws.
- Laws describe properties of these operations and connection between operations in one type class.
- Contract of the interface gives us confidence when we write generic code.
- And as you will see, we really care about these laws.

Instance example

```
package object implicits {  
  implicit val stringSemigroup: Semigroup[String] = new Semigroup[String] {  
    override def combine(x: String, y: String): String = x + y  
  }  
}
```


Checking laws - pen and paper in comments

```
package object implicits {  
  implicit val stringSemigroup: Semigroup[String] = new Semigroup[String] {  
    override def combine(x: String, y: String): String = x + y  
  }  
}  
  
/*  
combine(a, combine(b, c))  
  = combine(a, b + c)  
  = a + (b + c)  
  = (associativity of +)  
  = (a + b) + c = combine(a + b, c)  
  = combine(combine(a, b), c)  
*/
```

You're programmer after all

ТЫ Ж ПРОГРАММИСТ



Question on the interview: property based testing

```
object SemigroupSpecification extends Properties("Semigroup") with SemigroupSpecificationSupport {
  include(semigroup[String](stringSemigroup))
}

trait SemigroupSpecificationSupport {
  def semigroup[A](sg: Semigroup[A])(implicit ar: Arbitrary[A], tag: ClassTag[A]): Properties =
    new Properties(s"Semigroup[${tag.toString}]") {

      //  $\forall a, b, c \in A : (a \cdot b) \cdot c = a \cdot (b \cdot c)$ 
      property("associativity") = forAll { (a: A, b: A, c: A) =>
        sg.combine(sg.combine(a, b), c) == sg.combine(a, sg.combine(b, c))
      }

    }
}

/*
+ Semigroup.Semigroup[java.lang.String].associativity: OK, passed 100 tests
*/
```

More than one valid instance

```
package object implicits {  
  implicit val stringSemigroup: Semigroup[String] = new Semigroup[String] {  
    override def combine(x: String, y: String): String = x  
  }  
}
```

More examples

- Numbers with $+$, $*$, \min , \max
- Booleans with conjunction, disjunction, implication etc.
- Square nonnegative matrices with multiplication.
- Lists, Strings, Maps etc. with concatenation/union
- We will see even more examples during practical part.

Contra-examples

- $\{\mathbb{N}, /\}$ is not a Semigroup, because $/$ is not associative.
- The same goes for $\{\mathbb{N}, a^b\}$.
- $\{\mathbb{N}, -\}$ is not a Semigroup, because $-$ is not a closed operation, e.g.
 $\exists a, b \in \mathbb{N} : a - b \notin \mathbb{N}$, for example $10 - 15 = -5 \notin \mathbb{N}$.

Coding time

1. Clone `git@github.com:d12frosted/wax.git`
2. Import it as sbt project.
3. Go to `scala/src/main`

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4. Task 1
 - 4.1 Implement missing `Semigroup` instances in `wax.typeclass.semigroup.cats.implicit`s
 - 4.2 Run `wax.typeclass.semigroup.laws.cats.SemigroupSpec`

Coding time

1. Clone `git@github.com:d12frosted/wax.git`
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4. Task 1
 - 4.1 Implement missing Semigroup instances in `wax.typeclass.semigroup.cats.implicit`s
 - 4.2 Run `wax.typeclass.semigroup.laws.cats.SemigroupSpec`
5. Task 2
 - 5.1 Implement missing Semigroup instances in `wax.typeclass.semigroup.manual.implicit`s
 - 5.2 Run `wax.typeclass.semigroup.laws.manual.SemigroupSpec`

Monoid

- Sometimes you want to compose n elements where $n \geq 0$.
- Semigroup works only for $n > 0$.
- We need a default element to use if $n = 0$.

Monoid

One does not simply become a default element:

- $Int + 0 = 0 + Int = Int$
- $String + "" = "" + String = String$
- etc. . .

Monoid

Back to fancy words.

Monoid

Back to fancy words.

A monoid is a set S with binary closed operation $\cdot : S \times S \rightarrow S$ that satisfies associativity property:

$$\forall a, b, c \in S : (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

and identity element e that satisfies

$$\forall a \in S : e \cdot a = a \cdot e = a$$

Operation is closed when $\forall a, b \in S : a \cdot b \in S$.

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In other words, monoid is just a semigroup with identity element.

Again, it's not that scary

```
package object typeclass {  
  
  //  
  // Laws:  
  // 1.  $\forall a, b, c \in S : (a \cdot b) \cdot c = a \cdot (b \cdot c)$   
  // 2.  $\forall a \in S : e \cdot a = a \cdot e = a$   
  //  
  trait Monoid[A] extends Semigroup[A] {  
    def empty: A  
  }  
  
  object Monoid {  
    def apply[A: Monoid]: Monoid[A] = implicitly[Monoid[A]]  
  }  
}
```

Examples

- $\{\mathbb{N}_0, +\}$, where 0 is the identity element.
- $\{\mathbb{N}, *\}$, where 1 is the identity element.
- Boolean with XOR, XNOR, OR, AND.
- String with concatenation (empty string is identity element).

Examples

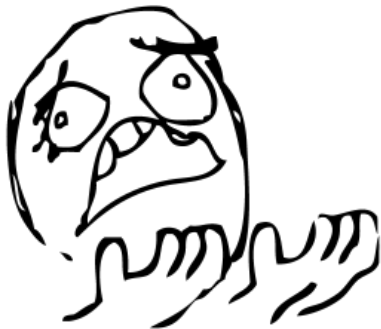
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But not every Semigroup forms a Monoid (we are not talking about free monoids here):

- `BigInteger` practically doesn't have identity element for `min`.

The most important question

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Why did we learn this?

The Fibonacci numbers

The Fibonacci numbers

On the interview we ask people to write a function that returns the n th Fibonacci number.

The Fibonacci numbers

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$$F_0 = 0$$

$$F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2}, \forall n > 1$$

Solution

What we expect

```
def fib(n: Int): Int = {  
  def fibTail(n: Int, a: Int, b: Int): Int = n match {  
    case 0 => a  
    case _ => fibTail(n - 1, b, a + b)  
  }  
  
  fibTail(n, 0, 1)  
}
```

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Ideal solution

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$$
$$= \frac{\phi^n - (-\phi)^{-n}}{2\phi - 1}$$

$$\phi = \frac{1 + \sqrt{5}}{2}$$

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As they say, truth is somewhere in the logarithm.

Two folds (mod)

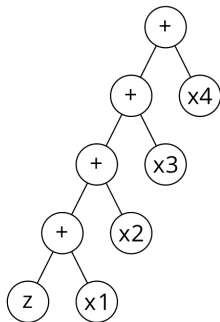
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$$+ : B \rightarrow A \rightarrow B$$

$$(((z + x1) + x2) + x3) + x4$$



Two folds

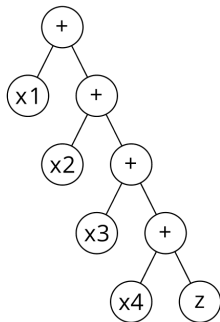
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$+: A \rightarrow B \rightarrow B$

$x1 + (x2 + (x3 + (x4 + z)))$



Two folds

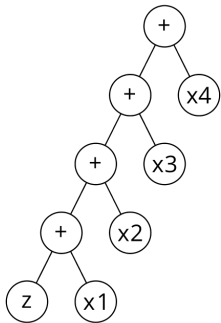
- `def foldl[A, B](xs: Seq[A])(z: B)(op: B => A => B): B`
- `def foldr[A, B](xs: Seq[A])(z: B)(op: A => B => B): B`
- Since combining function is asymmetrical in its types:
 - It's impossible to place parentheses in the arbitrary fashion or even just change the direction of the fold
 - It's impossible to implement a total fold without default value of type B

Two folds

foldl

$$+ : B \rightarrow A \rightarrow B$$

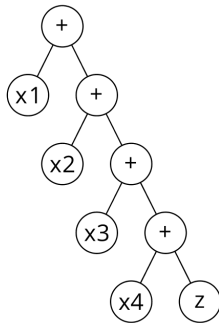
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foldr

$$+ : A \rightarrow B \rightarrow B$$

$$x1 + (x2 + (x3 + (x4 + z)))$$



What Monoid gives us

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- Associativity law says that we can put parentheses in an arbitrary fashion.

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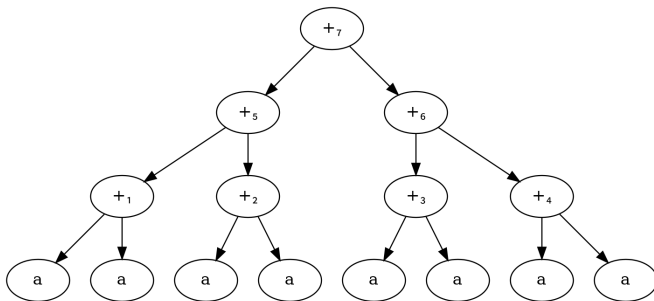
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Power in terms of Monoid

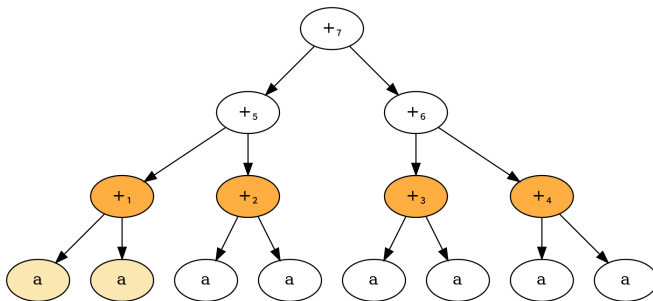
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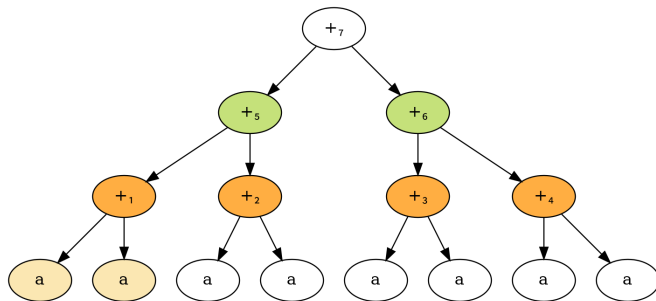


Power in terms of Monoid



Evaluating $a + a$ always yields the same result. So there is no point in repeating this calculation 4 times.

Power in terms of Monoid



The same thing with the upper level. In this particular example, we can avoid 4 operations out of 7. In general, this optimisation leads to the result in $\log n$ operations.

Power in terms of Monoid

All this means that we can define a function `exp`:

```
def exp[A: Monoid](a: A, n: Int): A = {  
  ???  
}
```

Back to Fibonacci

Fibonacci number can be defined in a different way.

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$$

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$$\begin{pmatrix} F_4 & F_3 \\ F_3 & F_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

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- The Fibonacci number can be calculated using square nonnegative matrix multiplication.
- Square nonnegative matrices form Monoid with multiplication.
- So we can put parentheses in a way we like it.

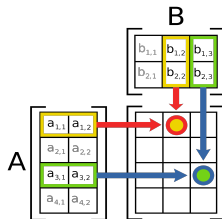
Coding time

- Open `wax.exercise.fibonacci.Main` object.
 - `Main` runs two implementations and profiles them.
 - `Fib` contains implementation of tailrec and matrix approaches.
 - `ExpUtils` implements generic `exp` function.
- Task is to implement monoid for `Matrix2x2` in the `Fib` object.
- Run `MatrixSpec` to test your instance.
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$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} =$$
$$\begin{pmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} \end{pmatrix}$$



Profiling results

N	Matrix	Tailrec
10	60	0
100	0	0
1000	1	1
10000	5	6
100000	46	168
1000000	888	15211
10000000	11266	-

Outcome

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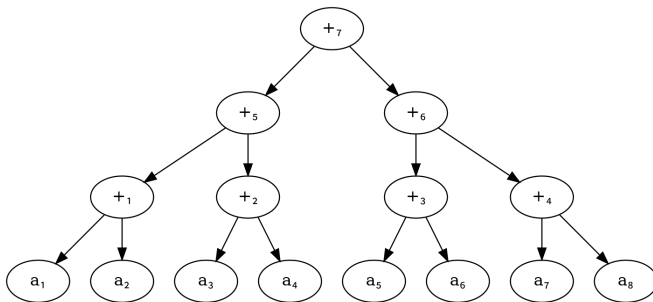
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- Monoids are everywhere around us. We deal with them every day, without even noticing it. Did you expect us to solve Fibonacci using `Monoid`?
- You forgot how matrix multiplication works, but now you remember, right?

Folds with Monoids

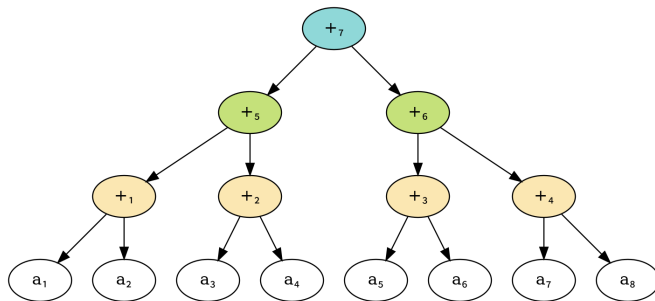
- We already know that Monoids give us an ability to place parentheses in any fashion.
- We already saw that when it comes to folding the list of the same elements we gain performance.
- But what if the elements are not equal? Do we gain anything?

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Folds with Monoids



Every expression on each level does not depend on other expressions from the same level, which means that we can evaluate them in parallel.

MapReduce

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- There is a strange accent, where people pronounce 'fold' as 'reduce'.
- This is how we get the `mapReduce`.
- Just think about it, `mapReduce` is possible thanks to Monoid and its *laws*.

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- Use helpers from `FileUtils`:
 - `readTokens` to get the list of words from the file.
 - `authorBooks` to get the list of books (files) by author (e.g. `authorBooks("boris")`).
 - `allBooks` to get the list of all book among all available authors.

Benchmarks

Par

duration = 65633 ms

result = List(..., (people,37798), ...)

Seq

duration = 396530 ms

result = List(..., (people,37798), ...)

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- This is what makes `mapReduce` possible.
- Many applications: inverted index, document clustering, machine learning.
- Google used it to regenerate index of World Wide Web.

Homework

mapReduce is really interesting!

Homework

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So play with it after the forkshop.

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- Strings and lists with concatenation.
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Can function be monoid?

Let's start with some wrappers (pun intended)

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- Can it be a monoid?
- Well, generally speaking, not! Because we know nothing about the type `A`.
- But what if `A` is a monoid?
- Hell, yeah!

```
case class Wrapper[A](value: A)

object Wrapper {
  implicit def wrapperMonoid[A: Monoid]: Monoid[Wrapper[A]] = new Monoid[Wrapper[A]] {
    override def empty: Wrapper[A] = Wrapper(Monoid[A].empty)

    override def combine(x: Wrapper[A], y: Wrapper[A]): Wrapper[A] =
      Wrapper(Monoid[A].combine(x.value, y.value))
  }
}
```

Wrappers of monoids are monoids

- Since IO is a wrapper (in some sense), it IO can also be a monoid.

```
def ioMonoid[A: Monoid]: Monoid[IO[A]] = ???
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- Which means that we can combine IO actions (in some new sense).
- Functions are wrappers (in some sense), so they also can be monoids

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```
def functionMonoid[A, B: Monoid]: Monoid[Function[A, B]] = ???
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Logger

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- So `Logger` forms a monoid.

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`type Logger = String => IO[Unit]`
- `Unit` forms a monoid.
- So `IO[Unit]` forms a monoid.
- So `String => IO[Unit]` forms a monoid.
- So `Logger` forms a monoid.
- So we can combine loggers
 - `fileLogger |+| consoleLogger` - logs both into file and to console

Logger

```
def consoleLogger: IO[Logger] = IO { input =>
  IO {
    print(input)
  }
}
```

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def consoleLogger: IO[Logger] = IO { input =>
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def fileLogger(filePath: String): IO[Logger] = IO {
  val stream = new FileOutputStream(filePath)
  input => IO(stream.write(input.getBytes))
}
```

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}

val program: IO[Unit] = for {
  logger <- consoleLogger |+| fileLogger("logging.log")
  _ <- logger("I am the log")
} yield ()
```

Logger

- Open `wax.exercise.logging.Logging` object.

Logger

- Open `wax.exercise.logging.Logging` object.
- Fill missing implementations.

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- Have fun!

Bonus questions

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- Some of you probably gonna write new `co1og` lib (but for Scala).

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- We want some rest after a long session of forkshop.

Questions?

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Thank you very much!

We hope you enjoyed this session.