SINUSOIDAL SIGNALS

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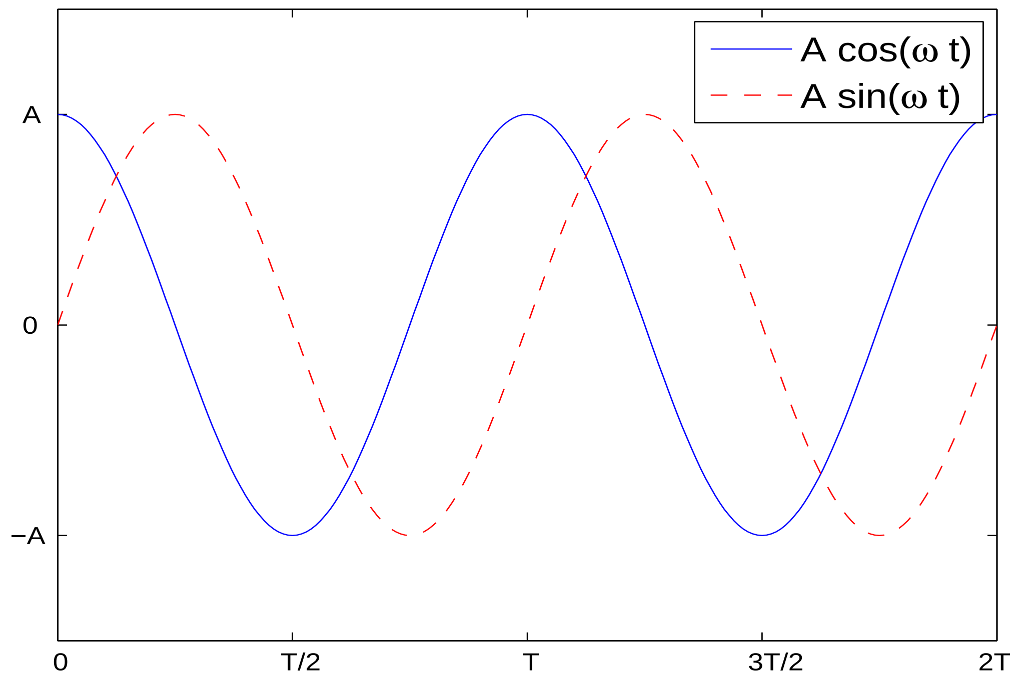
$$
\begin{aligned}
x(t) &= \cos(2\pi f t + \varphi) \\
&= \cos(\omega t + \varphi) \quad \text{(continuous time)}
\end{aligned}
$$

$$
\begin{aligned}
x[n] &= \cos(2\pi f n + \varphi) \\
&= \cos(\omega n + \varphi) \quad \text{(discrete time)}
\end{aligned}
$$

f : frequency (s^{-1} (Hz))
ω : angular frequency (radians/s)
φ : phase (radians)

Sinusoidal signals



$$x_c(t) = A\cos(2\pi f t)$$

- amplitude A
- period $T = 1/f$
- phase: 0

$$x_s(t) = A\sin(2\pi f t) \\ = A\cos(2\pi f t - \pi/2)$$

-amplitude A

-period $T = 1/f$

-phase: $-\pi/2$

WHY SINUSOIDAL SIGNALS?

- Physical reasons:
  - harmonic oscillators generate sinusoids, e.g., vibrating structures
  - waves consist of sinusoidals, e.g., acoustic waves or electromagnetic waves used in wireless transmission

- Psychophysical reason:
  - speech consists of superposition of sinusoids
  - human ear detects frequencies
  - human eye senses light of various frequencies

- Mathematical (and physical) reason:
  - Linear systems, both physical systems and man-made filters, affect a signal frequency by frequency (hence low-pass, high-pass etc filters)

EXAMPLE:

TRANSMISSION OF A LOW-FREQUENCY SIGNAL USING HIGH-FREQUENCY ELECTROMAGNETIC (RADIO) SIGNAL

- A POSSIBLE (CONVENTIONAL) METHOD: AMPLITUDE MODULATION (AM)

Example: low-frequency signal

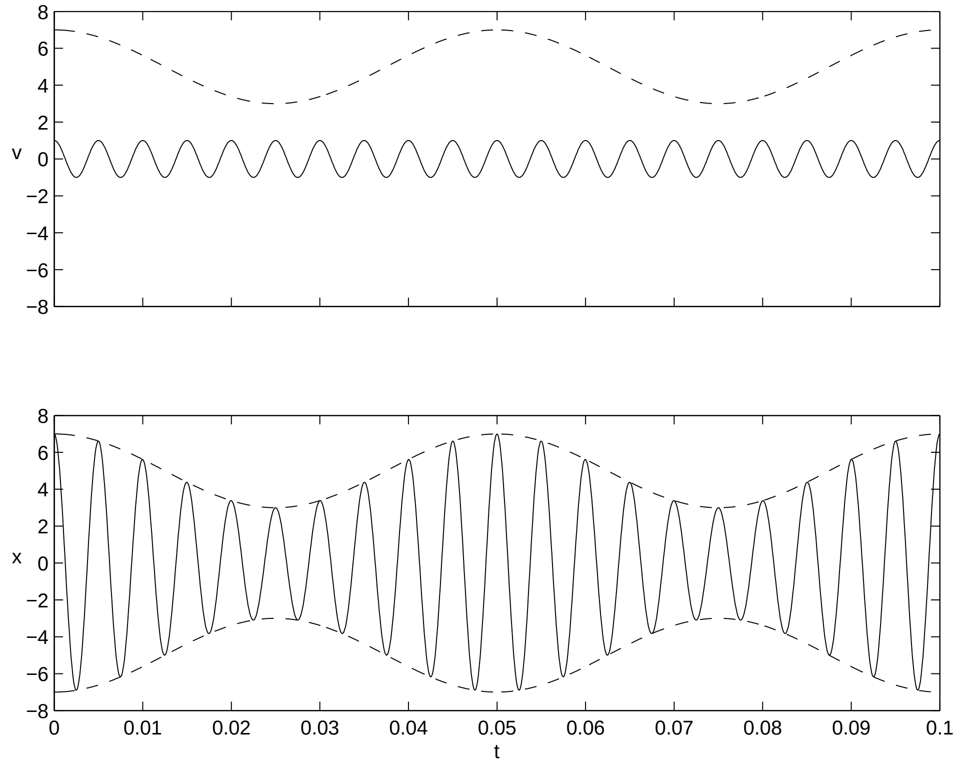
$$v(t) = 5 + 2\cos(2\pi f_\Delta t), \quad f_\Delta = 20 \text{ Hz}$$

High-frequency carrier wave

$$v_c(t) = \cos(2\pi f_c t), \quad f_c = 200 \text{ Hz}$$

Amplitude modulation (AM) of carrier (electromagnetic) wave:

$$x(t) = v(t)\cos(2\pi f_c t)$$



Top: $v(t)$ (dashed) and $v_c(t) = \cos(2\pi f_c t)$.
Bottom: transmitted signal $x(t) = v(t)\cos(2\pi f_c t)$.

Frequency contents of transmitted signal

$$\begin{aligned} x(t) &= v(t)\cos(2\pi f_c t) \\ &= (5 + 2\cos(2\pi f_\Delta t))\cos(2\pi f_c t) \\ &= 5\cos(2\pi f_c t) + 2\cos(2\pi f_\Delta t)\cos(2\pi f_c t) \end{aligned}$$

Trigonometric identity:

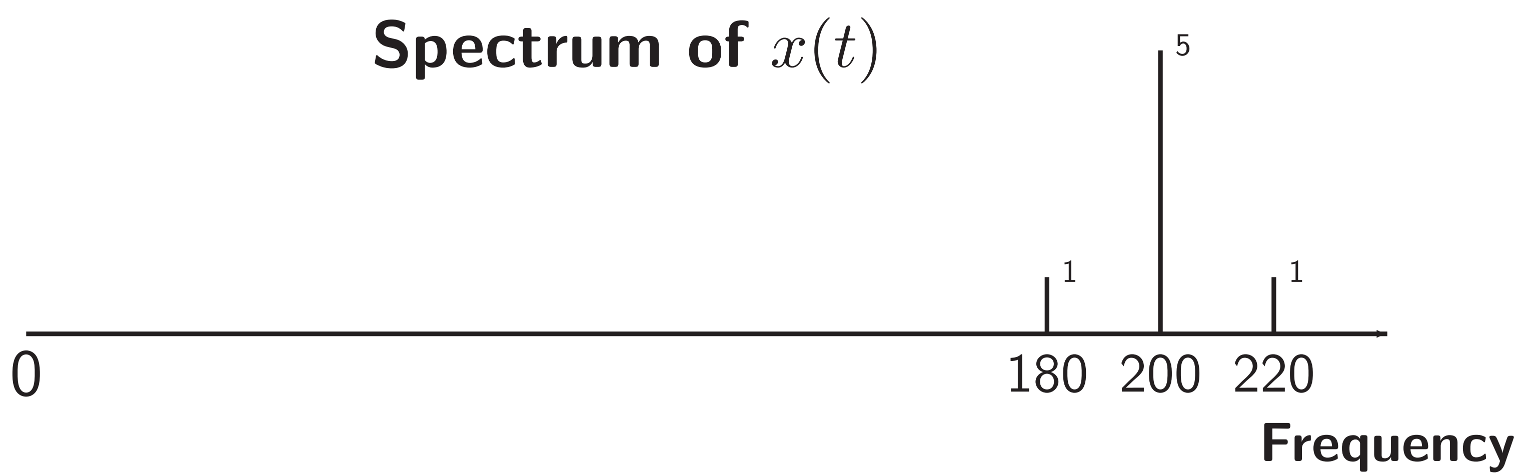
$$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha - \beta) + \frac{1}{2}\cos(\alpha + \beta)$$

$\Rightarrow$

$$2\cos(2\pi f_\Delta t)\cos(2\pi f_c t) = \cos\left(2\pi(f_c - f_\Delta)t\right) + \cos\left(2\pi(f_c + f_\Delta)t\right)$$

$\Rightarrow$

$$x(t) = 5\cos(2\pi f_c t) + \cos\left(2\pi(f_c - f_\Delta)t\right) + \cos\left(2\pi(f_c + f_\Delta)t\right)$$



We see that a low-frequency signal in frequency range
$0 \leq f_s \leq f_{\max}$ (*baseband signal*)
can be transmitted as a signal in the frequency range
$f_c - f_{\max} \leq f \leq f_c - f_{\max}$ (*"RF" (radio frequency) signal*).

Another analog modulation technique is frequency modulation (FM)

Representation of sinusoidal signal

A. Amplitude and phase

B. Sine and cosine components

C. Complex exponentials

A. Amplitude and phase

$$x(t) = A\cos(2\pi f t + \varphi)$$

or

$$x(t) = A\cos(\omega t + \varphi)$$

where $\omega = 2\pi f$ is the angular frequency (radians/second)

The phase φ describes translation in time compared to $\cos(\omega t)$:

$$x(t) = A\cos(\omega t + \varphi) = A\cos(\omega(t + \varphi/\omega))$$

B. Sine and cosine components

$$x(t) = a\cos(\omega t) + b\sin(\omega t)$$

Follows from trigonometric identity:

$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
$\Rightarrow$

$$\begin{aligned} x(t) &= A\cos(\omega t + \varphi) \\ &= A\cos(\varphi)\cos(\omega t) - A\sin(\varphi)\sin(\omega t) \end{aligned}$$

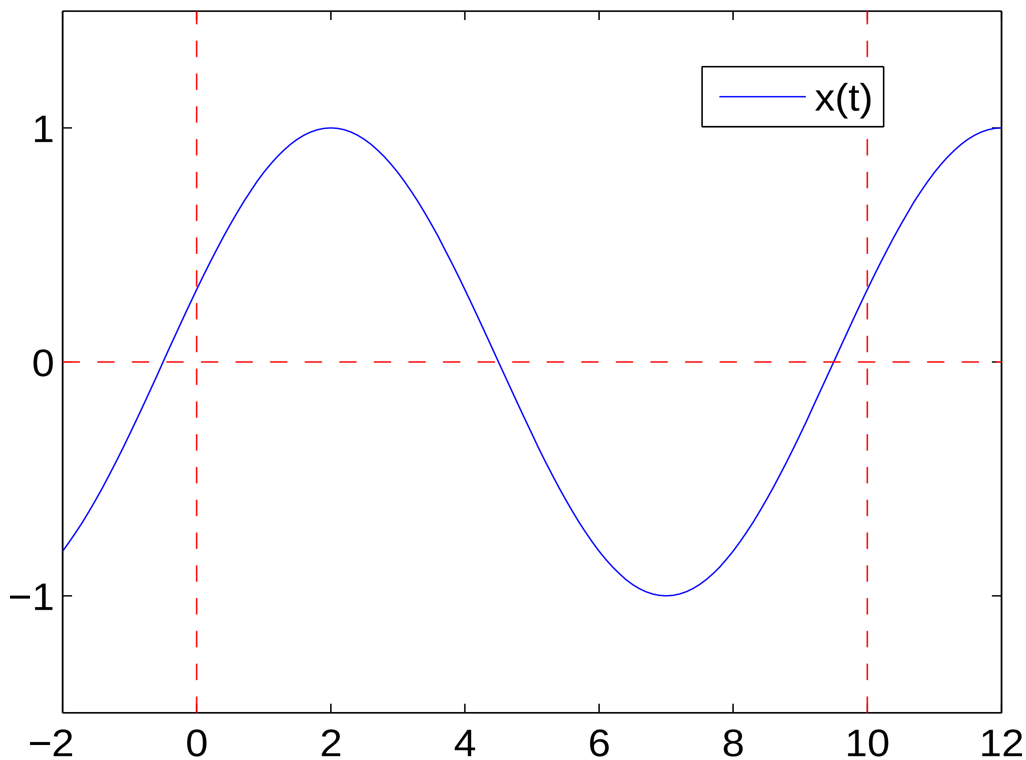
Hence:

$$a = A\cos(\varphi), \;\; b = -A\sin(\varphi)$$

and, conversely,

$$A = \sqrt{a^2 + b^2}, \;\; \varphi = -\arctan(b/a)$$

EXAMPLE



Signal $x(t) = \cos(\omega t + \varphi)$
Period: $T = 10$ (s)
Frequency: $f = 1/T = 0.1$ (s^{-1} =Hz (periods/second))
Angular frequency: $\omega = 2\pi f = 0.628$ (radians/s)

A. Representation using amplitude and phase

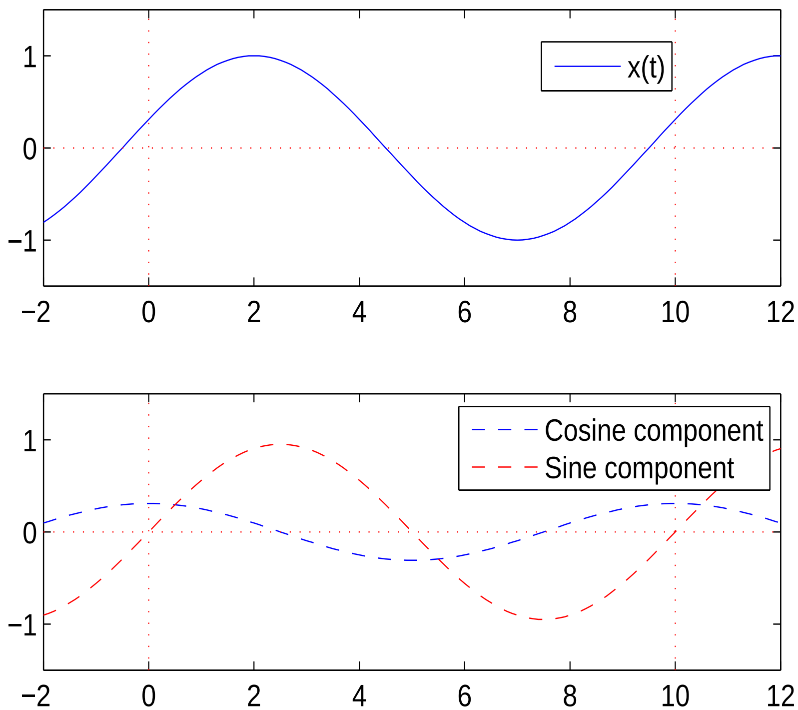
Amplitude: $A = 1$
From figure we see that $x(t) = \cos(\omega(t-2))$.
But $x(t) = \cos(\omega t + \varphi) = \cos(\omega(t + \varphi/\omega)$
$\Rightarrow$
Phase: $\varphi = -2\omega = -0.4\pi$ radians or $-0.4 \cdot 180/\pi = -72^\circ$

B. Sine and cosine components

$x(t) = a\cos(\omega t) + b\sin(\omega t)$ where

$$a = \cos(\varphi) = \cos(-0.4\pi) = 0.3090$$

$$b = -\sin(\varphi) = -\sin(-0.4\pi) = 0.9511$$



$$x(t) = 0.3090\cos(\omega t) + 0.9511\sin(\omega t)$$

C. Representation using complex exponentials

Background. A common operation is to combine sinusoidal components with different phases:

$$x(t) = A_1 \cos(\omega t + \varphi_1) + A_2 \cos(\omega t + \varphi_2)$$

One way: first write

$$\begin{aligned} A_1 \cos(\omega t + \varphi_1) &= a_1 \cos(\omega t) + b_1 \sin(\omega t) \\ A_2 \cos(\omega t + \varphi_2) &= a_2 \cos(\omega t) + b_2 \sin(\omega t) \end{aligned}$$

Then

$$x(t) = (a_1 + a_2) \cos(\omega t) + (b_1 + b_2) \sin(\omega t)$$

Streamlining the above procedure

We have:

$$\cos(\omega t + \varphi) = \cos(\varphi)\cos(\omega t) - \sin(\varphi)\sin(\omega t)$$

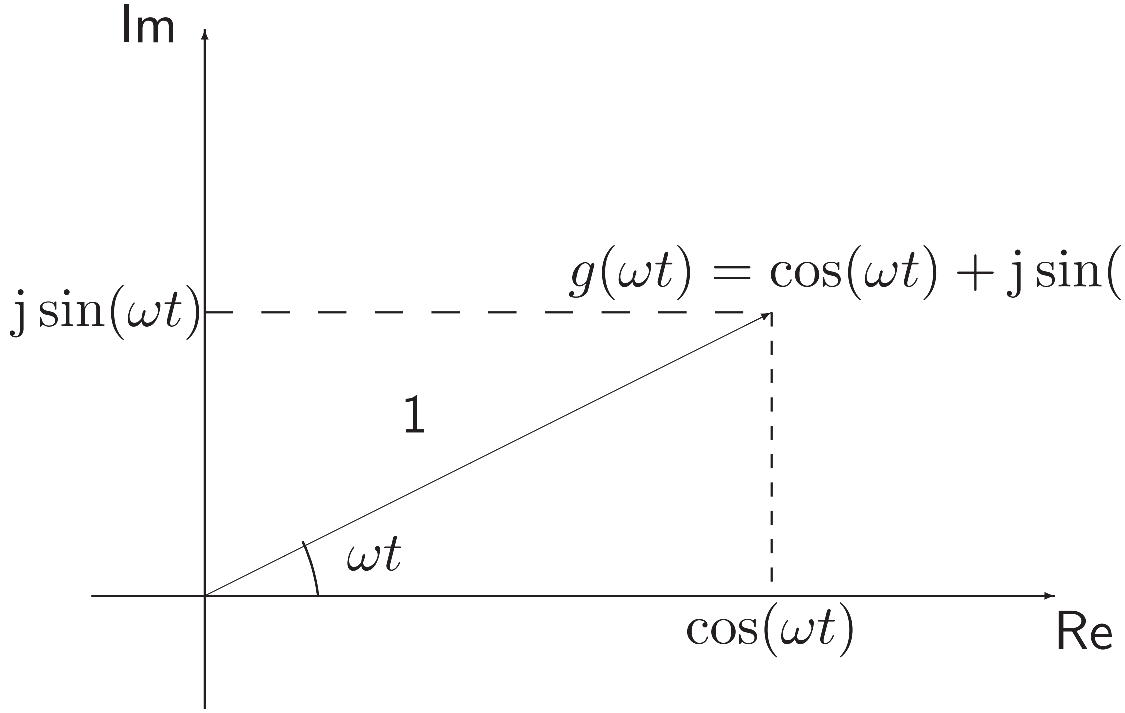
and

$$\sin(\omega t + \varphi) = \sin(\varphi)\cos(\omega t) + \cos(\varphi)\sin(\omega t)$$

Instead of working with the two signals $\cos(\omega t)$ and $\sin(\omega t)$ it is convenient to working with the single *complex-valued* signal

$$g(\omega t) = \cos(\omega t) + \mathrm{j}\sin(\omega t)$$

where $\mathrm{j} = \sqrt{-1}$ (imaginary unit number)



Then:

$$
\begin{aligned}
g(\omega t + \varphi) &= \cos(\omega t + \varphi) + \mathrm{j}\sin(\omega t + \varphi) \\
&= \cos(\varphi)\cos(\omega t) - \sin(\varphi)\sin(\omega t) \\
&\quad + \mathrm{j}\Big(\sin(\varphi)\cos(\omega t) + \cos(\varphi)\sin(\omega t)\Big) \\
&= \Big(\cos(\varphi) + \mathrm{j}\sin(\varphi)\Big)\Big(\cos(\omega t) + \mathrm{j}\sin(\omega t)\Big) \\
&= g(\varphi)g(\omega t)
\end{aligned}
$$

In terms of the complex-valued signal, phase shift with angle φ is equivalent to multiplication by the complex number $g(\varphi)$:

$$g(\omega t + \varphi) = g(\varphi)g(\omega t) \qquad (A)$$

$\Rightarrow$

$$\begin{aligned} \cos(\omega t + \varphi) &= \mathrm{Re}\,(g(\varphi)g(\omega t)) \\ \sin(\omega t + \varphi) &= \mathrm{Im}\,(g(\varphi)g(\omega t)) \end{aligned}$$

We will see that using the multiplication (A) to describe phase shifts results in significant simplification of formulae and calculations

Properties of the complex function $g(\theta) = \cos(\theta) + \mathrm{j}\sin(\theta)$:

- $g(0) = 1$

- $g(\theta_1 + \theta_2) = g(\theta_1)g(\theta_2)$

- $\frac{\mathrm{d}g(\theta)}{\mathrm{d}\theta} = -\sin(\theta) + \mathrm{j}\cos(\theta) = \mathrm{j}g(\theta)$

The only function which satisfies these relations is

$$g(\theta) = \mathrm{e}^{\mathrm{j}\theta}$$

$\mathrm{e} = 2.71828182845904523536\ldots$ (Euler's number)

Hence we have:

Euler's relation

$$\mathrm{e}^{\mathrm{j}\theta} = \cos(\theta) + \mathrm{j}\sin(\theta)$$

$\mathrm{e} = 2.71828182845904523536\ldots$ (Euler's number)

As $\mathrm{e}^{-\mathrm{j}\theta} = \cos(\theta) - \mathrm{j}\sin(\theta)$ we have

$$\begin{aligned} \cos(\theta) &= \frac{1}{2}\left(\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta}\right) \\ \sin(\theta) &= \frac{1}{2\mathrm{j}}\left(\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^{-\mathrm{j}\theta}\right) \end{aligned}$$

It follows that

$$
\begin{aligned}
x(t) &= a\cos(\omega t) + b\sin(\omega t) \\
&= \left(\frac{a}{2} + \frac{b}{2\mathrm{j}}\right)\mathrm{e}^{\mathrm{j}\omega t} + \left(\frac{a}{2} - \frac{b}{2\mathrm{j}}\right)\mathrm{e}^{-\mathrm{j}\omega t} \\
&= \frac{1}{2}(a - \mathrm{j}b)\,\mathrm{e}^{\mathrm{j}\omega t} + \frac{1}{2}(a + \mathrm{j}b)\,\mathrm{e}^{-\mathrm{j}\omega t}
\end{aligned}
$$

or

$$x(t) = c\ \mathrm{e}^{\mathrm{j}\omega t} + c^*\mathrm{e}^{-\mathrm{j}\omega t}$$

where $c = \frac{1}{2}(a - \mathrm{j}b)$ and $c^* = \frac{1}{2}(a + \mathrm{j}b)$ (complex conjugate)

Thus the cosine and sine components are parameterized as the real and imaginary components of the complex number c

PREVIOUS EXAMPLE:

The signal

$$\begin{aligned} x(t) &= \cos(\omega t - 0.4\pi) \\ &= a\cos(\omega t) + b\sin(\omega t) \\ &= 0.3090\cos(\omega t) + 0.9511\sin(\omega t) \end{aligned}$$

can be represented using complex exponentials as
$x(t) = c\ \mathrm{e}^{\mathrm{j}\omega t} + c^*\ \mathrm{e}^{-\mathrm{j}\omega t}$
where

$$\begin{aligned} c &= \frac{1}{2}(a - \mathrm{j}b) = 0.1545 - \mathrm{j}0.4755 \\ c^* &= \frac{1}{2}(a + \mathrm{j}b) = 0.1545 + \mathrm{j}0.4755 \end{aligned}$$

Hence:

$$x(t) = (0.1545 - \mathrm{j}0.4755)\,\mathrm{e}^{\mathrm{j}\omega t} + (0.1545 + \mathrm{j}0.4755)\,\mathrm{e}^{-\mathrm{j}\omega t}$$

SUMMARY

Let us summarize the results so far. We have the following alternative representations of a sinusoidal signal:

- Amplitude and phase:

$$x(t) = A\cos(\omega t + \varphi)$$

- Sine and cosine components:

$$x(t) = a\cos(\omega t) + b\sin(\omega t)$$

  where

$$a = A\cos(\varphi), \;\; b = -A\sin(\varphi)$$

  or

$$A = \sqrt{a^2 + b^2}, \;\; \varphi = -\arctan(b/a)$$

- Complex exponentials:

$$x(t) = c\ \mathrm{e}^{\mathrm{j}\omega t} + c^*\ \mathrm{e}^{-\mathrm{j}\omega t}$$

where

$$c = \frac{1}{2}\,(a - \mathrm{j}b)\,,\ \ c^* = \frac{1}{2}\,(a + \mathrm{j}b)$$

or

$$a = 2\mathrm{Re}(c),\ \ b = -2\mathrm{Im}(c)$$

EXAMPLE - Signal transmission with reflected component

Transmitted signal: $x(t) = \cos(\omega t)$

Received signal: $y(t) = x(t) + rx(t - \tau)$

PROBLEM: Determine amplitude and phase of $y(t)$

We have

$$y(t) = \cos(\omega t) + r\cos(\omega(t - \tau))$$

The problem can be solved in a straightforward way by introducing complex exponential:

$$x(t) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{j}\omega t} + \mathrm{e}^{-\mathrm{j}\omega t}\right)$$

Then:

$$
\begin{aligned}
y(t) &= \cos(\omega t) + r\cos(\omega(t-\tau)) \\
&= \frac{1}{2}\left(\mathrm{e}^{\mathrm{j}\omega t} + \mathrm{e}^{-\mathrm{j}\omega t}\right) + r\frac{1}{2}\left(\mathrm{e}^{\mathrm{j}\omega(t-\tau)} + \mathrm{e}^{-\mathrm{j}\omega(t-\tau)}\right) \\
&= \frac{1}{2}\left(1 + r\mathrm{e}^{-\mathrm{j}\omega\tau}\right)\mathrm{e}^{\mathrm{j}\omega t} + \frac{1}{2}\left(1 + r\mathrm{e}^{\mathrm{j}\omega\tau}\right)\mathrm{e}^{-\mathrm{j}\omega t}
\end{aligned}
$$

Here,

$$
\frac{1}{2}\left(1 + r\mathrm{e}^{-\mathrm{j}\omega\tau}\right) = \frac{1}{2}\Big(1 + r\cos(\omega\tau) - \mathrm{j}r\sin(\omega\tau)\Big)
$$

and

$$
\frac{1}{2}\left(1 + r\mathrm{e}^{\mathrm{j}\omega\tau}\right) = \frac{1}{2}\Big(1 + r\cos(\omega\tau) + \mathrm{j}r\sin(\omega\tau)\Big)
$$

Then, defining

$$a = 1 + r\cos(\omega\tau), \quad b = r\sin(\omega\tau)$$

we have (cf. above)

$$\begin{aligned} y(t) &= \frac{1}{2}(a - \mathrm{j}b)\,\mathrm{e}^{\mathrm{j}\omega t} + \frac{1}{2}(a + \mathrm{j}b)\,\mathrm{e}^{-\mathrm{j}\omega t} \\ &= a\cos(\omega t) + b\sin(\omega t) \\ &= A_y\cos(\omega t + \varphi_y) \end{aligned}$$

where

$$A_y = \sqrt{a^2 + b^2}, \;\; \varphi_y = -\arctan(b/a)$$

SOME SIMPLIFICATIONS

As

$$a\cos(\omega t) + b\sin(\omega t) = \frac{1}{2}\,(a - \mathrm{j}b)\,\mathrm{e}^{\mathrm{j}\omega t} + \frac{1}{2}\,(a + \mathrm{j}b)\,\mathrm{e}^{-\mathrm{j}\omega t}$$

it follows that it is sufficient to consider the complex-valued signal

$$\frac{1}{2}\,(a - \mathrm{j}b)\,\mathrm{e}^{\mathrm{j}\omega t}$$

Here, the complex number

$$c = \frac{1}{2}\,(a - \mathrm{j}b)$$

determines the signal's amplitude and phase.

To see how this works, consider a linear dynamical system with input $x(t)$ and output $y(t)$

$$x(t) \longrightarrow \boxed{G} \longrightarrow y(t)$$

For the sinusoidal input

$$x(t) = \cos(\omega t)$$

the output takes the form

$$y(t) = A\cos(\omega t + \varphi) = a\cos(\omega t) + b\sin(\omega t)$$

and we want to determine the amplitude A and phase φ.

As the phase shift introduces a combination of cosines and sines, the problem can be simplified by embedding the input signal into a larger class of signals involving both a cosine and a sine component.

It turns out that a convenient way to do this is to consider the *complex* input signal

$$\begin{aligned} x_{\mathrm{c}}(t) &= \cos(\omega t) + \mathrm{j}\sin(\omega t) \\ &= \mathrm{e}^{\mathrm{j}\omega t} \end{aligned}$$

The output then also has real and imaginary components:

$$\begin{aligned} y_{\mathrm{c}}(t) &= A\cos(\omega t + \varphi) + \mathrm{j}A\sin(\omega t + \varphi) \\ &= A\mathrm{e}^{\mathrm{j}(\omega t + \varphi)} \\ &= A\mathrm{e}^{\mathrm{j}\varphi}\mathrm{e}^{\mathrm{j}\omega t} \end{aligned}$$

or

$$
\begin{aligned}
y_{\mathrm{c}}(t) &= c\, \mathrm{e}^{\mathrm{j}\omega t} \\
&= c\, x_{\mathrm{c}}(t)
\end{aligned}
$$

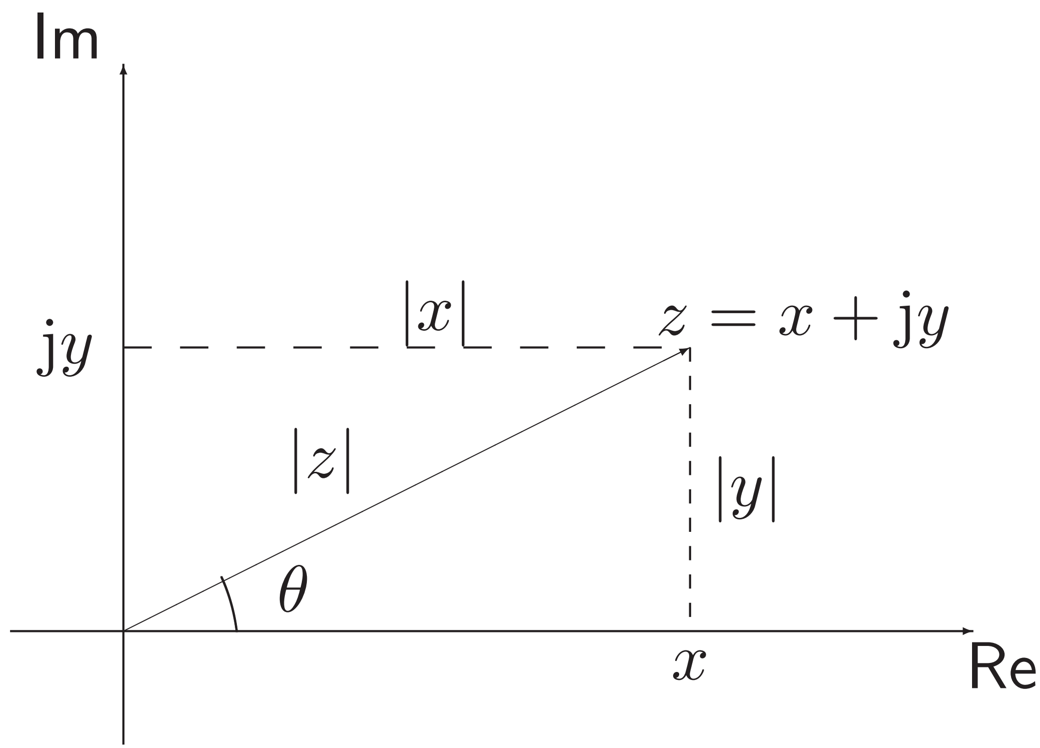
where $c = A\mathrm{e}^{\mathrm{j}\varphi}$.

$\Rightarrow$

The effect of a linear dynamical system on a complex-valued sinusoidal can be characterized in terms of a multiplication with a complex number; $y_{\mathrm{c}}(t) = c\, x_{\mathrm{c}}(t)$.

If we can write $c = A\mathrm{e}^{\mathrm{j}\varphi}$, then A is the amplitude and φ is the phase of the output relative to the amplitude and phase of the input.

Complex number $z = x + \mathrm{j}y$



We have: $x = |z|\cos\theta$, $y = |z|\sin\theta$

$\Rightarrow$

$$z = |z|\left(\cos\theta + \mathrm{j}\sin\theta\right) = |z|\mathrm{e}^{\mathrm{j}\theta}$$

The representation

$$z = |z|\mathrm{e}^{\mathrm{j}\theta}$$

is called the *polar form* of the complex number z.

$|z|$: absolute value (magnitude) of z

$\theta = \arg z$: argument of z

We have:

$$|z| = \sqrt{x^2 + y^2}$$

and

$$\tan\theta = \frac{y}{x}$$

As $\frac{y}{x} = \frac{-y}{-x}$, the relation $\tan\theta = \frac{y}{x}$ does not determine θ uniquely.

A unique characterization of θ n terms of x and y is obtained as

$$\theta = \begin{cases} \arctan\frac{y}{x}, & \text{if } x \geq 0 \\ \\ \arctan\frac{y}{x} + \pi, & \text{if } x < 0 \end{cases}$$

The *complex conjugate* of $z = x + \mathrm{j}y$ is

$$z^* = x - \mathrm{j}y$$

or

$$z^* = |z|\mathrm{e}^{-\mathrm{j}\theta}$$

$$|z^*| = |z|, \quad \arg z^* = -\arg z$$

`MATLAB:`

For $x = x + \mathrm{j}y$, we have:

x: `real(z)`

y: `imag(z)`

$|z|$: `abs(z)`

$\arg z$: `angle(z)` (in radians)

z^*: `conj(z)`

EXAMPLE

Consider the simple discrete-time filter

$$y(n) = 0.5y(n-1) + x(n)$$

with input

$$x(n) = \cos(\omega n)$$

the output is then

$$\begin{aligned} y(n) &= 0.5y(n-1) + x(n) \\ &= 0.5^2 y(n-2) + 0.5x(n-1) + x(n) \\ &\vdots \\ &= x(n) + 0.5x(n-1) + \cdots + 0.5^k x(n-k) + \cdots \end{aligned}$$

or, with $x(n) = \cos(\omega t)$,

$$y(n) = \cos(\omega n) + 0.5\cos(\omega(n-1)) + \cdots + 0.5^k \cos(\omega(n-k)) + \cdots$$

It is not straightforward to determine the amplitude and phase of $y(n)$ from this expression.

Instead we introduce the complex signal

$$\begin{aligned} x_{\mathrm{c}}(n) &= \cos(\omega n) + \mathrm{j}\sin(\omega n) \\ &= \mathrm{e}^{\mathrm{j}\omega n} \end{aligned}$$

Then

$$
\begin{aligned}
y_c(n) &= x_c(n) + 0.5x_c(n-1) + \cdots + 0.5^k x_c(n-k) + \cdots \\
&= e^{j\omega n} + 0.5e^{j\omega(n-1)} + \cdots + 0.5^k e^{j\omega(n-k)} + \cdots \\
&= \left(1 + 0.5e^{-j\omega} + \cdots + 0.5^k e^{-j\omega k} + \cdots\right) e^{j\omega n} \\
&= \left(1 + 0.5e^{-j\omega} + \cdots + (0.5e^{-j\omega})^k + \cdots\right) e^{j\omega n} \\
&= \frac{1}{1 - 0.5e^{-j\omega}}\, e^{j\omega n} \qquad (1)
\end{aligned}
$$

where we have used the formula for a geometric sum.

Here, the complex number

$$
c(\omega) = \frac{1}{1 - 0.5e^{-j\omega}}
$$

is the *frequency response* of the filter.

Writing $c(\omega)$ in polar form,

$$c(\omega) = A(\omega)e^{\mathrm{j}\varphi(\omega)}$$

gives the amplitude and phase of the output, as

$$\begin{aligned}
y_{\mathrm{c}}(n) &= c(\omega)\mathrm{e}^{\mathrm{j}\omega n} \\
&= A(\omega)e^{\mathrm{j}\varphi(\omega)}\mathrm{e}^{\mathrm{j}\omega n} \\
&= A(\omega)e^{\mathrm{j}(\omega n+\varphi(\omega))} \\
&= A(\omega)\cos(\omega n+\varphi(\omega)) + \mathrm{j}A(\omega)\sin(\omega n+\varphi(\omega))
\end{aligned}$$

The frequency response $c(\omega)$ thus determines how the filter affects the amplitude and phase of different frequency components