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I agree to the honor code pledge

Question 1

$(x^2 - y^2)$ is the more accurate way

$fl(x \cdot x) = (x^2)(1+\delta)$

$fl(y \cdot y) = (y^2)(1+\delta)$

$fl(x \cdot x - y \cdot y) = ((x^2)(1+\delta) - y^2(1+\delta))(1+\delta) \leq (x^2(1+\epsilon_m) - y^2(1+\epsilon_m))(1+\epsilon_m)$

$\leq (x^2 + x^2\epsilon_m - y^2 - y^2\epsilon_m)(1+\epsilon_m)$

$fl(x+y) = (x+y)(1+\delta)$

$fl(x-y) = (x-y)(1+\delta)$

$fl((x+y)(x-y)) = ((x+y)(1+\delta)(x-y)(1+\delta))(1+\delta)$

$\leq ((x+y)(1+\epsilon_m)(x-y)(1+\epsilon_m))(1+\epsilon_m)$

$= (x + x\epsilon_m + y + y\epsilon_m)(x + x\epsilon_m - y - y\epsilon_m)(1+\epsilon_m)$

$= x^2 + \underline{x^2\epsilon_m} - yx - xy\epsilon_m + \underline{x^2\epsilon_m} - xy\epsilon_m - xy\epsilon_m \ldots$

$(x+y)(x-y)$ has more "extra" terms not included in the original equation & will produce a less accurate result.

Question 2

$(a,b) = (-1/3, 1)$

a) $\frac{b-a}{2^{N_{max}}} < tol \Rightarrow \frac{(1+1/3)}{2^{N_{max}}} < 10^{-3}$

$\frac{4/3}{2^{N_{max}}} < 10^{-3}$

$(4/3)(10^3) < 2^{n_{max}}$

$\boxed{\log_2((4/3)10^3)} < N_{max}$

b) $f(x) = \sin(x)$

$\sin(-1/3) < 0$

$\sin(1) > 0$

step 1

$mid = \frac{-1/3+1}{2} = 1/3 \nless 10^{-3}$

$\sin(mid) = \sin(1/3) \neq 0$

$\sin(-1/3) \times \sin(1/3) < 0$

$b = 1/3$

step 2

$mid = \frac{-1/3 + 1/3}{2} = 0$

$\sin(0) = 0$

c) $2 < \log_2((4/3)(10^3)) < N_{max}$

Question 3

a) $f(x) = (x-\alpha)^m c(x)$ where $g(\alpha) \neq 0$

b) $g(x) = x - \frac{f(x)}{f'(x)}$ $\quad x_{n+1} = g(x_n)$

Substitute a) Let α be the root

$$g(x) = x - \frac{(x-\alpha)^m c(x)}{m(x-\alpha)^{m-1}c(x) + (x-a)^m c'(x)} = x - \frac{(x-\alpha)c(x)}{mc(x) + (x-\alpha)c'(x)}$$

$g'(\alpha) = 1 - \frac{1}{m}$ if $m > 1$ $\neq 0$

$\alpha - x_{n+1} \approx \left(1 - \frac{1}{m}\right)(A - x_n)$ $\therefore$ linearly convergent

c) $g(x) = x - m\frac{f(x)}{f'(x)}$

Using b)

$g'(\alpha) = \left(1 - \frac{1}{m}\right)m = 0$ $\quad\therefore$ quadratically convergent

d) Makes $g'(\alpha) = 0$ so the series is guaranteed to converge at least quadratically.

Question 4

a) $F(x,y) = \begin{bmatrix} 2(x_n-2)^2+4(y_n-1)^2-16 \\ x_ny_n-2y_n-4 \end{bmatrix}$

$J(x,y) = \begin{bmatrix} 4(x_n-2) & 8(y_n-1) \\ y_n & x_n-2 \end{bmatrix}$

b) $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 4(4-2) & 8(4-1) \\ 3 & 3-2 \end{bmatrix}^{-1} \begin{bmatrix} 2(4-2)^2+4(4-1)^2-16 \\ (4)(3)-2(3)-4 \end{bmatrix}$

c) No, this would result in a singular Jacobian matrix, + the inverse could not be taken.

Question 5

a) $\lim_{x \to 0} \frac{|f(x)|}{|g(x)|}$ $\lim_{x\to 0} \left| \frac{\frac{\sqrt{\epsilon}}{\sin(\epsilon)}}{1/\sqrt{\epsilon}} \right| = \lim_{x \to 0} \left| \frac{\sqrt{\epsilon}}{\sin(\epsilon)} \cdot \frac{\sqrt{\epsilon}}{1} \right| = \frac{0}{0}$

L'Hopitals Rule

$\lim_{x \to 0} \frac{1}{\cos(\epsilon)} = 1 \neq 0$ No, $\frac{\sqrt{\epsilon}}{\sin(\epsilon)} \neq O(\epsilon^{-1/2})$

b) $\lim_{h \to 0} \frac{|f(h)|}{|h^{\alpha}|} = M$ $\left| \frac{h^{\beta} f(h)}{h^{\alpha+\beta}} \right| = \left| \frac{\cancel{h^{\beta}} f(h)}{h^{\alpha} \cancel{h^{\beta}}} \right| = \left| \frac{f(h)}{h^{\alpha}} \right|$

so $\lim_{h \to 0} \frac{h^{\beta} f(h)}{h^{\alpha+\beta}} = M \therefore h^{\beta} f(h) = O(h^{\alpha+\beta})$

c) Method 1 → superlinearly $e_{n+1} \approx c(e_n^2)$
Newton's Method, order of convergence = 2

Method 2 → ~~superlinearly~~ $e_{n+1} \not\approx c e_n^2$
$e_{n+1} \not\approx c e_n$

Method 3 → superlinearly convergen $e_{n+1} \not\approx c(e_n^2)$
$e_{n+1} \not\approx c e_n$

Secant method → Order of convergence
is similar to $\approx 1.6 \neq 2, 1$
methods 2 + 3

d) False