Olivia Gadden

I agree to the honor code pledge

Question 1

(2,0,1)
(0,2,2)
(0,1,4)

$r^2 = ax^2 + by^2$

$1^2 = a(4) + b(0) \Rightarrow 4a = 1$
$2^2 = a(0) + b(4) \quad 4b = 4$
$16 = a(0) + b(1) \quad b = 16$

$$\underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 4 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 4 \\ 16 \end{bmatrix}}_{b}$$

$A^TAx = A^Tb$

$x = (A^TA)^{-1}A^Tb$

$$A^TA = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 0 \\ 0 & 17 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 16 \end{bmatrix} = \begin{bmatrix} 4 \\ 32 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 16 & 0 & 1 & 0 \\ 0 & 17 & 0 & 1 \end{array}\right] \begin{array}{l} R_1/16 \\ R_2/17 \end{array} \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/16 & 0 \\ 0 & 1 & 0 & 1/17 \end{array}\right]$$

$$\begin{bmatrix} 1/16 & 0 \\ 0 & 1/17 \end{bmatrix} \begin{bmatrix} 4 \\ 32 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 32/17 \end{bmatrix}$$

$$\boxed{r^2 = 1/4\, a^2 + 32/17\, y^2}$$

Question 2

a)

$$f(x) = \begin{cases} y - 1 = \dfrac{(e^{.5} - 1)}{.5}(x) & x \in [0, .5) \\ y - e^{.5} = \dfrac{e - e^{.5}}{.5}(x - .5) \end{cases}$$

$e^0 = 1$ $(0,1)$ $1$ $e^{.5}$

$e^{.5}$ $(.5, e^{.5})$

$e^1$ $(1, e)$

b)

$$\frac{f^{(2)}(x)}{(2)}(x)(x - .5) = \frac{e^x}{2}(x)(x - .5)$$

$$\max_{0 \le x \le .5} \left|\frac{e^x}{2}\right| |(x)(x - .5)| = \left|\frac{e^{.5}}{2}\right| |(.5)(-.5)| \approx .206$$

$$\max_{.5 \le x \le 1} \left|\frac{e^x}{2}\right| |(x)(x - .5)| = \left|\frac{e}{2}\right| |(1)(.5)| \approx .680$$

c)

$$\max_{0 \le x \le 1} \left|\frac{e^x}{2}\right| |(x)(x - .5)| = \left|\frac{e}{2}\right| |(1)(.5)| \approx .680$$

Question 3

a) $\int_{-1}^{1} f(x)\,dx = a f(-.5) + b f'(1) + c f(0)$

$f(x) = x^0 = 1 \quad f'(x) = 0$

$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} 1\,dx = 2 = a + c$

$f(x) = x^1 \quad f'(x) = 1$

$\int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = 0 = -.5a + b$

$f(x) = x^2 \quad f'(x) = 2x$

$\int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3} = .25a + 2b$

$-2(-.5a + b) = 0$

$(.25a + 2b) = 2/3$

$a - 2b = 0$

$+ .25a + 2b = 2/3$

$1.25a = 2/3$

$$\boxed{\begin{aligned} a &\approx .533 \\ b &\approx .267 \\ c &\approx 1.467 \end{aligned}}$$

Question 4

a) $\int_a^b f(x)g(x)w(x) = 0$

$P_0 = 1$

$P_1 = x$

$2P_2 = 3x(x) - 1(1)$

$P_2 = \frac{3x^2-1}{2}$

$\int_{-1}^{1} 1(x)(1)\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = 0$ ✓

$\int_{-1}^{1} \left(\frac{3x^2-1}{2}\right) x(1)\,dx = \frac{(3x^2-1)^2}{24}\Big|_{-1}^{1} = 0$ ✓

$\int_{-1}^{1} \frac{3x^2-1}{2}(1)(1)\,dx = \frac{x^3-x}{2}\Big|_{-1}^{1} = 0$ ✓

b) $f(x) = e^x$ on $[-1, 1]$

$P_1(x) = a_0 + a_1 x$

$a_0 \int_{-1}^{1} dx + a_1 \int_{-1}^{1} x\,dx = \int_{-1}^{1} e^x\,dx$

$a_0 \int_{-1}^{1} x\,dx + a_1 \int_{-1}^{1} x^2\,dx = \int_{-1}^{1} xe^x\,dx$

$2a_0 + \cancel{a_1(0)}^{0} = e - \frac{1}{e}$

$$\boxed{P_1(x) = \frac{e - 1/e}{2} + \frac{3}{e}x}$$

$a_0 = \frac{e - 1/e}{2}$

$\cancel{a_0(0)}^{0} + a_1(2/3) = 2/e$

$a_1 = \frac{2}{e} \cdot \frac{3}{2} = 3/e$

Question 5

a) $\int_0^1 (1+xe^{-x})dx$ $\quad$ $Error = \frac{-h^3}{6} f''(\xi)$

$\frac{h^3}{6} f''(\xi) = 1.2E-3$ $\quad$ $\left| \frac{(1/16)^3}{6} \right| |3.868| \approx \boxed{1.5E-4}$

$\frac{(1/8)^3}{6} f''(\xi) = 1.2E-3$

$f''(\xi) \approx 3.686$

b) Two advantages Broyden has is not having to calculate an inverse each iteration, instead do a rank 1 update. As a consequence of this it has a lower cost $O(n^2)$ v. Newtons $O(n^3)$. Both of these benefits make Broyden a good + cost effective alternative to Newtons method

c) Chebyshev points are better for interpolation since they focus the points at the ends to address Ranges phenomenon.

d) In order to use Hermite, $f(x_j)$ + $f'(x_j)$ for each $x_j$ must be known. It can evaluate up to $2n+1$ deg.