APPM4605-Homework3

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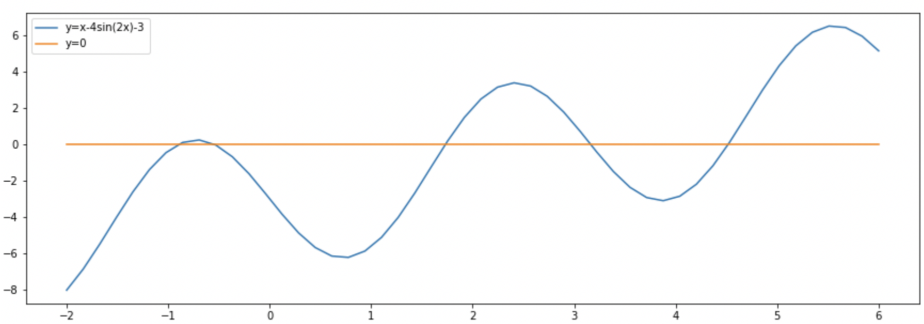
1. (a) $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}$
$g(x) = -16 + 6x + \frac{12}{x}$
$g(2) = -16 + 12 + 6 = 2$ which is on y=x, so 2 is a fixed point
$g^{'}(x) = 6 - 12x^{-2}$
$g^{'}(2) = 6 - 3 = 3$
$g^{'}(2) = 3 > 1$ so x_{n+1} does not necessarily converge.
Let $x_n = x^* + \epsilon$ where $\epsilon > 0$ and x_n is close to x^*
$e_n = |\epsilon|$, $e_{n+1} = |g(x^* + \epsilon) - x^*|$
$e_{n+1} = |-16 + 6(x^* + \epsilon) + \frac{12}{(x^*+\epsilon)}| = |6\epsilon + \frac{12}{2+\epsilon} + 12 - 16 - 2| = |6\epsilon + \frac{12}{2+\epsilon} - 6|$
$\frac{1}{2+\epsilon} = 1/2 - 1/4\epsilon + O(\epsilon^2)$
$e_{n+1} = |6\epsilon + 12(1/2 - 1/4\epsilon + O(\epsilon^2)) - 6| = |3\epsilon + O(\epsilon^2)| > |\epsilon| = e_n$
Therefore, the series is divergent.

   (b) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$
$g(x) = \frac{2}{3}x + \frac{1}{x^2}$
$g(3^{1/3}) = \frac{2}{3^{2/3}} + \frac{1}{3^{2/3}} = \frac{3}{3^{2/3}} = 3^{1/3}$ which is on y=x, so $3^{1/3}$ is a fixed point.
$g^{'}(x) = \frac{2}{3} - \frac{2}{x^3}$
$g^{'}(3^{1/3}) = \frac{2}{3} - \frac{2}{3} = 0$
$g^{'}(3^{1/3}) = 0 < 1$ so x_{n+1} converges to $3^{1/3}$ by the Fixed Point Thm.
$\lim_{n\to\infty} \frac{|\frac{2}{3}x_n + \frac{1}{x_n^2} - 3^{1/3}|}{|x_n - 3^{1/3}|^2} = \lim_{n\to\infty} \frac{\frac{2}{3}x_n + \frac{1}{x_n^2} - 3^{1/3}}{(x_n - 3^{1/3})^2}$
Since both the numerator is positive as x approaches $3^{1/3}$ and the denominator is squared

$$\lim_{n\to\infty} \frac{\frac{2}{3}x_n + \frac{1}{x_n^2} - 3^{1/3}}{(x_n - 3^{1/3})^2} = \frac{|\frac{2}{3}3^{1/3} + \frac{1}{3^{2/3}} - 3^{1/3}|}{|3^{1/3} - 3^{1/3}|^2} = \frac{0}{0}$$

Using L'Hopitals Rule
$\lim_{n\to\infty} \frac{\frac{2}{3} - 2x_n^{-3}}{2(x_n - 3^{1/3})} = \frac{\frac{2}{3} - \frac{2}{3}}{0} = \frac{0}{0}$
Using L'Hopitals Rule
$lim_{n\to\infty} \frac{6x_n^{-4}}{2} = \frac{6}{3^{4/3} * 2} = 3^{-1/3} < 1$
Therefore, the series is quadratically convergent.

(c) $x_{n+1} = \frac{12}{1+x_n}$
$g(x) = \frac{12}{1+x}$
$g(3) = \frac{12}{4} = 3$ which is on y=x, so 3 is a fixed point
$g^{'}(x) = \frac{-12}{(1+x)^2}$
$g^{'}(3) = \frac{-12}{16} = \frac{-3}{4} = \lambda$
$g^{'}(3) = |\frac{-3}{4}| < 1$ so x_{n+1} converges to 3 by the Fixed Point Thm.
$\lim_{n\to\infty} \frac{|\frac{12}{1+x_n}-3|}{|x_n-3|} => \lim_{n\to-\infty} \frac{\frac{-12}{1+x_n}+3}{x_n-3}, \lim_{n\to+\infty} \frac{\frac{12}{1+x_n}-3}{-x_n+3}$
Since $|\frac{12}{1+x_n} - 3|$ is negative when $n \to +\infty$ and positive when $n \to -\infty$ and $|x_n - 3|$ is positive as $n \to +\infty$ and negative as $n \to -\infty$.
$\lim_{n\to-\infty} \frac{\frac{-12}{1+x_n}+3}{x_n-3} = \frac{-3+3}{3-3} = \frac{0}{0}$
Using L'Hoptials Rule $\lim_{n\to-\infty} = \frac{12}{(1+x_n)^2} = \frac{12}{16} = \frac{3}{4}$
$\lim_{n\to+\infty} \frac{\frac{12}{1+x_n}-3}{-x_n+3} = \frac{3-3}{-3+3} = \frac{0}{0}$
Using L'Hoptials Rule $\lim_{n\to+\infty} \frac{12}{(1+x_n)^2} = \frac{3}{4}$
$lim_{n\to-\infty} \frac{\frac{-12}{1+x_n}+3}{x_n-3} = \lim_{n\to+\infty} \frac{\frac{12}{1+x_n}-3}{-x_n+3} = \frac{3}{4} < 1$
Therefore, the series is linearly convergent at a rate of $|\lambda| = \frac{3}{4}$.

2. (a) 

There are five zeros.

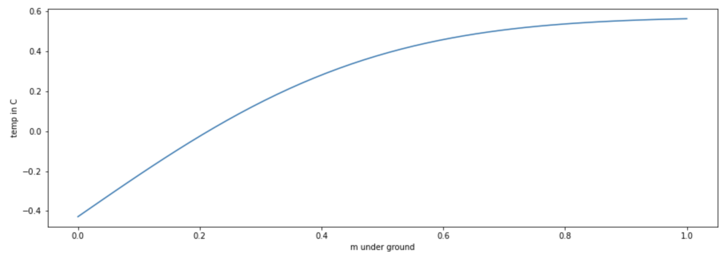
(b)
```
def fixed_point_iteration(func, p0, tol):
    i=1
    while i<10000000:
        p=func(p0)
        if abs(p-p0)<tol:
            return[0,p0]
        i=i+1
        p0=p
    return [1, None]

p3=lambda x: -np.sin(2*x)+5*x/4-3/4
```

The only two roots that can be found are (-.544,0) and (3.161,0). The absolute value of the derivatives of the other roots are too large

(> 1) for the fixed point iteration to converge.

3. (a) $f(x) = erf(\frac{x}{2\sqrt{(.138e^{-6}*518400)}}) - \frac{15}{35}$

$f^{'}(x) = \frac{1}{\sqrt{(\pi * .138 * 10^{-6} * 518400)}} e^{-(\frac{x}{2\sqrt{(.138e^{-6}*518400)}})^2}$



(b) Using the bisect function from previous homework, the depth is 0.21407413492141814.

(c) Using a python implementation of Newton's method, the depth is 0.21407413492143315. If the initial guess is changed to 1, the result becomes 0.21407413492144375. Starting at 1 requires more iterations than starting at .01, since the .01 is closer to the root. Otherwise, the results are the same up to 13 digits.