KALMAN FILTER

Given

$$x_n = F x_{n-1} + B u_{n-1} + w_{n-1}$$
$$z_n = H x_n + v_n$$

$x_0 : N(\bar{x}_0, P_0)$
$w_n : N(0, Q_n)$
$v_n : N(0, R_n)$

Derive Kalman Filter

$$\hat{x}_{a|b} \triangleq E[x_a \mid \text{given } x_b, x_{b-1} \ldots x_0]$$

thus, from above

$$\hat{x}_{n|n-1} = F \hat{x}_{n-1|n-1} + B u_{n-1} \quad (1)$$

state innovation Process $= \tilde{x}_{n|n-1} = x_n - E[x_n \mid x_{n-1} \; x_{n-2} \ldots x_0]$

$$= x_n - \hat{x}_{n|n-1} \quad (2)$$

innovation process $= \tilde{z}_{n|n-1} = H x_n + v_n - H \hat{x}_{n|n-1}$

$$= H(x_n - \hat{x}_{n|n-1}) + v_n \quad (3)$$
$$= H \tilde{x}_{n|n-1} + v_n$$

Kalman filter : $\tilde{x}_{n|n-1} = K_n \tilde{z}_{n|n-1}$ (4)

To find K_n, Kalman gain matrix,

$$E[\tilde{x}_{n|n-1} \tilde{z}^*_{n|n-1}] = K_n E[\tilde{z}_{n|n-1} \tilde{z}^*_{n|n-1}] \quad (5)$$

Let's find $E[\tilde{x}_{n|n-1} \tilde{z}^*_{n|n-1}]$ in (5) first

$$E[\tilde{x}_{n|n-1} \tilde{z}^*_{n|n-1}] = E\left[(x_n - \hat{x}_{n|n-1})\left[(x_n - \hat{x}_{n|n-1})^* H^* + v_n^*\right]\right]$$

where $(x_n - \hat{x}_{n|n-1}) = F(x_{n-1} - \hat{x}_{n-1|n-1}) + w_{n-1}$

$$E[\tilde{x}_{n|n-1}\tilde{z}^*_{n|n-1}] = E[(F(x_{n-1}-\hat{x}_{n-1|n-1})+w_{n-1})(((x_{n-1}-\hat{x}_{n-1|n-1})^*F^*+w^*_{n-1})H^*+v^*_n)]$$

$$= F P_{n-1|n-1} F^* H^* + Q_{n-1} H^*$$

$$= (F P_{n-1|n-1} F^* + Q_{n-1}) H^*$$

$$E[\tilde{x}_{n|n-1}\tilde{z}^*_{n|n-1}] = P_{n|n-1} H^* \quad (6)$$

where

$$P_{n|n-1} = E[(x_n - \hat{x}_{n|n-1})(x_n - \hat{x}_{n|n-1})^*] = E[\tilde{x}_{n|n-1}\tilde{x}^*_{n|n-1}]$$

$$= E[(F x_{n-1} + \cancel{B u_{n-1}} + w_{n-1} - F\hat{x}_{n-1|n-1} - \cancel{B u_{n-1}})(\quad)^*]$$

$$= E[(F(x_{n-1} - \hat{x}_{n-1|n-1}) + w_{n-1})(\quad)^*]$$

$$P_{n|n-1} = F P_{n-1|n-1} F^* + Q_{n-1} = E[\tilde{x}_{n|n-1}\tilde{x}^*_{n|n-1}] \quad (7)$$

Now find $E[\tilde{z}_{n|n-1}\tilde{z}^*_{n|n-1}]$ in (5) by using (3)

$$E[\tilde{z}_{n|n-1}\tilde{z}^*_{n|n-1}] = E[(H(x_n - \hat{x}_{n|n-1}) + v_n)((x_n - \hat{x}_{n|n-1})^* H^* + v_n^*)]$$

$$= H P_{n|n-1} H^* + R_n \quad (8)$$

Now we can calculate K_n by using (5), (6), and (8)

$$E[\tilde{x}_{n|n-1}\tilde{z}^*_{n|n-1}] = K_n[\tilde{z}_{n|n-1}\tilde{z}^*_{n|n-1}]$$

$$P_{n|n-1}\cdot H^* = K_n(H P_{n|n-1} H^* + R_n)$$

assuming $P_{n|n-1} \geq 0$ and $R_n > 0$

$$K_n = P_{n|n-1} H^* (H P_{n|n-1} H^* + R_n)^{-1} \quad (9)$$

Theorem: (P46 of Kalmon Filter)

Let x and y be jointly (independent) Gaussion

$$E[x|y] = E[x] + A\bar{y} \quad (10)$$

where

$$E[\bar{x}\bar{y}^*] = A\,E[\bar{y}\bar{y}^*] \quad (11) \text{ (wiener-Hopf Equation)}$$

$$\bar{x} = x - E[x] \quad \text{and} \quad \bar{y} = y - E[y]$$

Also, the error $\tilde{x} = x - \hat{x}$ is independent of y

and hence of $\hat{x}$ $\Rightarrow$

$$E[\tilde{x}y^*] = E[\tilde{x}\hat{x}^*] = 0$$

From (10) the error is $\tilde{x} = x - E[x] - A\bar{y} = \bar{x} - A\bar{y}$

thus $E[\tilde{x}] = E[\bar{x} - A\bar{y}] = 0$

Also $$E[\tilde{x}y^*] = E[\bar{x}(\bar{y}+m_y)^*] - A\,E[\bar{y}(\bar{y}+m_y)^*]$$

$$= E[\bar{x}\bar{y}^*] + \overbrace{E[\bar{x}]}^{0}m_y - A\,E[\bar{y}\bar{y}^*] - A\,\overbrace{E[\bar{y}]}^{0}m_y^*$$

$$= E[\bar{x}\bar{y}^*] - A\,E[\bar{y}\bar{y}^*] = 0$$

shows that $\tilde{x}\,y^*$ are independent

Similarly;

$$E[\tilde{x}\hat{x}^*] = E[(\bar{x} - A\bar{y})(m_x + A\bar{y})^*] = E[\bar{x}\bar{y}^*A^*] - A\,E[\bar{y}\bar{y}^*]A^*$$

shows that $\tilde{x}\,\hat{x}$ are independent $= \left[E[\bar{x}\bar{y}^*] - A\,E[\bar{y}\bar{y}^*]\right]A^*$

$= 0$

Now from (10)

$$\hat{x}_{n|n} = E[x_n] + \sum_{i=0}^{n} E[x_n \tilde{z}^*_{i|i-1}]\, E[\tilde{z}_{i|i-1} \tilde{z}^*_{i|i-1}]^{-1}\, \tilde{z}_{i|i-1} \qquad (11)$$

for $i=n$, $E[x_n \tilde{z}^*_{n|n-1}]\, E[\tilde{z}_{n|n-1} \tilde{z}^*_{n|n-1}]^{-1} = K_n = P_{n|n-1} H^* (H P_{n|n-1} H^* + R_n)^{-1}$

$$\hat{x}_{n|n} = E[x_n] + \sum_{i=0}^{n-1} [\quad] + \sum_{n} [\quad]$$

now

$$\hat{x}_{n|n} = \left[E[x_n] + \sum_{i=0}^{n-1} E[x_n \tilde{z}^*_{i|i-1}]\, E[\tilde{z}_{i|i-1} \tilde{z}^*_{i|i-1}]^{-1} \tilde{z}_{i|i-1} \right] + K_n \tilde{z}_{n|n-1}$$

$$\hat{x}_{n|n} = \hat{x}_{n|n-1} + K_n \tilde{z}_{n|n-1} \qquad (12)$$

$$(x_n - \hat{x}_{n|n}) = (x_n - \hat{x}_{n|n-1}) - K_n \tilde{z}_{n|n-1}$$

$$= \tilde{x}_{n|n-1} - K_n \tilde{z}_{n|n-1} \qquad \text{where } \tilde{z}_{n|n-1} = H \tilde{x}_{n|n-1} + v_n$$

$$= (I - K_n H) \tilde{x}_{n|n-1} + v_n$$

now take covariance of each side, $E[\tilde{x}_{n|n-1}] = 0$

$$P_{n|n} = (I - K_n H) P_{n|n-1} (I - K_n H)^* + K_n R K_n^*$$

$$= P_{n|n-1} - P_{n|n-1} H^* K_n^* - K_n H P_{n|n-1} + K_n H P_{n|n-1} H^* K_n^* + K_n R_n K_n^*$$

$$= P_{n|n-1} - P_{n|n-1} H^* K_n^* - K_n H P_{n|n-1} + K_n (H P_{n|n-1} H^* + R_n) K_n^*$$

we know that $P_{n|n-1} H^* = K_n (H P_{n|n-1} H^* + R_n)$

$$P_{n|n} = P_{n|n-1} - K_n H P_{n|n-1} = (I - K_n H) P_{n|n-1} \qquad (13)$$