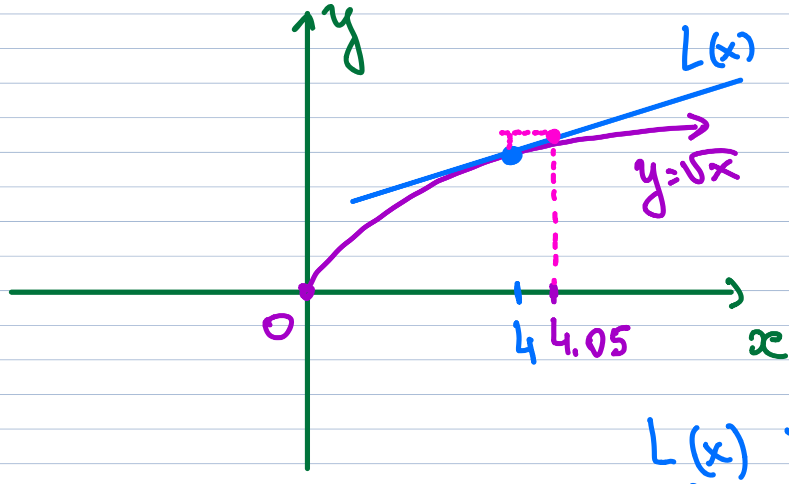
Section 3.10. Linear approximation and differentials



$y=\sqrt{x}$

$\sqrt{4.05} \approx ?$

$\sqrt{4}$

$L(x)$ is a liniarization function

$$\sqrt{4.05} \approx L(4.05)$$

$L(x)$ is a tangent line to $y=\sqrt{x}$ at $x=4$.

$$L(x) = f'(a)(x-a) + f(a)$$

$a=4 \qquad f(x)=\sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f'(4) = \frac{1}{4}$$

$$f(4) = 2$$

$$L(x) = \frac{1}{4}(x-4)+2$$

$$\underbrace{L(x) = \frac{1}{4}x + 1}$$

$$\sqrt{4.05} \approx L(4.05) = \frac{1}{4}(4+0.05)+1 =$$

$$= 1 + \frac{\cancel{5}}{4\cdot 1\cancel{00}_{20}} + 1 = 2 + \frac{1}{80}$$

$$\sqrt{4} = 2$$

Example

$$x^2y + \cos(y) = e^{xy}$$

$$\frac{dy}{dx} = y'(x) = y' \;?$$

$$\frac{d}{dx}\left(x^2y + \cos(y)\right) = 2x\cdot y + y'\cdot x^2 +$$

$$+ (-\sin(y))\cdot y'$$

$$\frac{d}{dx}\left(e^{xy}\right) = e^{xy}\cdot(y + y'\cdot x)$$

$$2xy + y' \cdot x^2 - \sin(y) \cdot y' = e^{xy}(y + y'x)$$

$$y'(x^2 - \sin y - e^{xy} \cdot x) = e^{xy} \cdot y - 2xy$$

$$y' = \frac{e^{xy} \cdot y - 2xy}{x^2 - \sin(y) - e^{xy} \cdot x}$$

1. Use the linear approximation of $f(x) = \sqrt{x}$ at $x = 4$ to approxmiate $\sqrt{4.1}$ and compare your result to its approximation computed by your calculator.

$L(x) = f'(a)(x-a) + f(a)$ $a=4$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f'(a) = f'(4) = \frac{1}{4}$

$f(a) = \sqrt{4} = 2$

$L(x) = \frac{1}{4}(x-4)+2$

$\sqrt{4.1} \approx L(4.1) = \frac{1}{4}(4.1-4)+2$
$= 2.025$

By calculator:

$\sqrt{4.1} \approx 2.02484$

error is $\underline{0.00016}$

2. Use the linear approximation to approximate the cosine of $29^\circ = \frac{29}{30}\frac{\pi}{6}$ radians.

$L(x)$

$f(x) = \cos(x)$

$\cos(29^\circ) \approx ?$

$L(x) = f'(a)(x-a) + f(a)$

$a = 30^\circ = \frac{\pi}{6}$

$f(a) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$f'(x) = (\cos(x))' = -\sin(x)$

$f'(a) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

$\underline{L(x) = -\frac{1}{2}\left(x - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}}$

$\cos(29^\circ) \approx L(29^\circ) = -\frac{1}{2}\left(\frac{29}{30}\frac{\pi}{6} - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2} = -\frac{1}{2}\cdot\left(-\frac{1}{30}\right)\frac{\pi}{6} + \frac{\sqrt{3}}{2}$

3. Find the linear approximation of $f(x) = \ln(x)$ at $a = 1$ and use it to approximate $\ln(0.5)$ and $\ln(0.9)$. Compare your approximation with your calculator's. Sketch both the curve $y = \ln(x)$ and $y = L(x)$ and label the points $A = (0.5, \ln(0.5))$ and $B = (0.5, L(0.5))$

$L(x) = f'(1)(x-1) + f(1)$

$f'(x) = \frac{1}{x}$

$f'(1) = 1$

$f(1) = 0$

$L(x) = 1\cdot(x-1) = \underline{\underline{x-1}}$

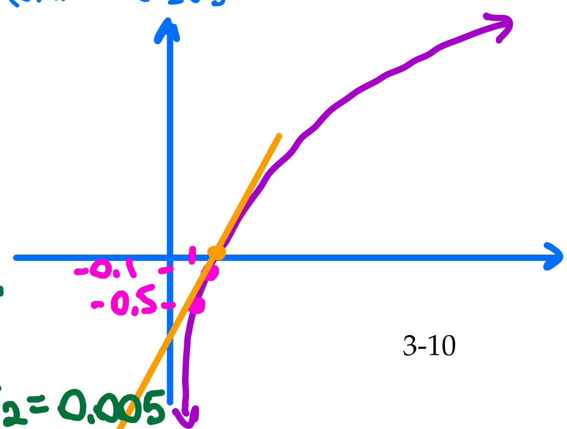
$\ln(0.5) \approx L(0.5) = -0.5$

$\ln(0.9) \approx L(0.9) = -0.1$

By calculator:

$\ln(0.5) \approx -0.6931$

$\ln(0.9) \approx -0.105$



We see that at $x=0.5$ we have not such a good appr. as at $x=0.9$.

$\text{error}_1 = 0.1931$ $\text{error}_2 = 0.005$

4. A tree is growing and the radius of its trunk in centimeters is $r(t) = 2\sqrt{t}$ where t is measured in years. Use the differential to estimate the change in radius of the tree from 4 years to 4 years and one month.

$r(t) = 2\sqrt{t}$

Δr is a change in radius

$\Delta r = r(t_2) - r(t_1)$

Δt is a change in time

$\Delta t = \frac{1}{12}$ year $= 1$ month

$$\boxed{\frac{\Delta r}{\Delta t} \approx \frac{dr}{dt} = r'(t)}$$

$t_1 = 4$ years

? $t_2 = 4$ years + 1 mon.

$\Delta r = \boxed{\frac{dr}{dt}} \cdot \Delta t$

$\frac{dr}{dt} = \frac{1}{\sqrt{t}}$

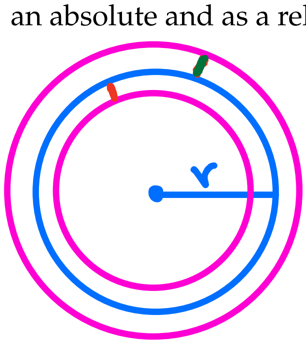
$\left.\frac{dr}{dt}\right|_{t=4\text{ years}} = \frac{1}{2}$

5. A coat of paint of thickness 0.05cm is being added to a hemispherical dome of radius 25m. Estimate the volume of paint needed to accomplish this task. [Challenge: will this be an underestimate or an overestimate? Thinking geometrically or thinking algebraically will both give you the same answer.]

$\Delta r = \frac{1}{2} \cdot \frac{1}{12} = \frac{1}{24}$ (centimet)

$r(4) = 2 \cdot \sqrt{4} = 4$ cen.

6. The radius of a disc is 24cm with an error of ± 0.5cm. Estimate the error in the area of the disc as an absolute and as a relative error.



$+0.5$cm

-0.5 cm

$A = \pi r^2$

$\Delta r = \pm 0.5$ cm

$r = 24$ cm

$\frac{\Delta A}{\Delta r} \approx \frac{dA}{dr}$

$$\boxed{\Delta A \approx \frac{dA}{dr} \cdot \Delta r}$$

?

$\frac{dA}{dr} = 2\pi r$

$\left.\frac{dA}{dr}\right|_{r=24} = 2\pi \cdot (24) =$

$= 48\pi$

$$\Delta A = 48\pi \cdot (\pm 0.5) \approx \pm 24\pi \ (cm^2)$$

absolute error

$$\frac{\Delta A}{A} = \frac{\pm \cancel{24}\,\cancel{\pi}}{\cancel{\pi} \cdot 24^{\cancel{2}}} = \pm \frac{1}{24} = \pm 0.0416$$

$$\approx \pm 4\,\%$$

relative error

$$A = \pi r^2$$

$$A = \pi \cdot 24^2$$