Section 3.1. Derivatives of Polynomials and Exponential Functions

- $f(x) = c$ (constant function)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0$$

$$\boxed{\frac{d}{dx}(c) = 0}$$

- $f(x) = x^n$ (power functions)

1) $n=1$, $f(x)=x$

$$\boxed{\frac{d}{dx}(x) = 1}$$

2) $n>0$ is a positive integer

$$\boxed{\frac{d}{dx}(x^n) = n x^{n-1}}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left((x+h)^n - x^n\right) =$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\cancel{x^n} + n x^{n-1} h + \underbrace{\frac{n(n-1)}{2} x^{n-2} h^2}_{\to 0} + \dots + \underbrace{h^n}_{\to 0}\right) - \cancel{\frac{1}{h} x^n} = n x^{n-1}$$

The Power Rule (General case):

If n is any real number, then

$$\boxed{\frac{d}{dx}(x^n) = n x^{n-1}}$$

Def. The normal line to a curve C at a point P is the line through P that is perpendicular to the tangent line at P.

Example

$$y = x\sqrt{x} = x^{3/2}, \quad P(1,1)$$

$x_0 = 1 \qquad f(x_0) = 1$

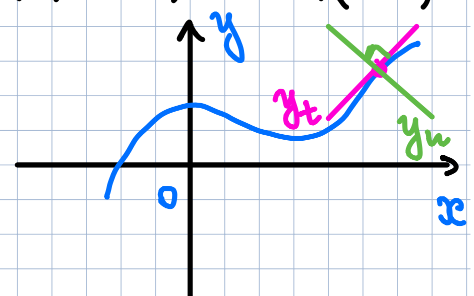
- find a tangent line?
- find a normal line?

Solution:

Tangent line equation: $f(x) - f(x_0) = f'(x_0)(x - x_0)$

$$f'(x) = \frac{3}{2} x^{1/2}$$

$$y = \frac{3}{2} 1^{1/2}(x-1) + 1$$



$y = \frac{3}{2}(x-1)+1$ is a tangent line

$m_1 = \frac{3}{2}$ is a slope of a tangent line

Let m_2 be a slope of a normal line.
Then

$y = m_2(x-1)+1$ is a normal line

Now we find m_2.
We know that tangent line is $\perp$ to a normal line. Then

$$m_1 \cdot m_2 = -1$$
$$\frac{3}{2} m_2 = -1 \Rightarrow m_2 = -\frac{2}{3}$$

$y = -\frac{2}{3}(x-1)+1$ is a normal line

- New derivatives from old:

1. The constant multiple rule:
   if c is a constant and f is a differentiable function, then

$$\boxed{\frac{d}{dx}\left(cf(x)\right) = c\frac{d}{dx}f(x)}$$

2. The sum rule:

if f and g are both differentiable, then

$$\boxed{\frac{d}{dx}\left(f(x)+g(x)\right)=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)}$$

3. The difference rule:

if f and g are both differentiable, then

$$\boxed{\frac{d}{dx}\left(f(x)-g(x)\right)=\frac{d}{dx}f(x)-\frac{d}{dx}g(x)}$$

- Exponential functions:

$f(x)=b^x$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=b^x\lim_{h\to 0}\frac{b^h-1}{h}$$

$$\lim_{h\to 0}\frac{b^h-1}{h}=f'(0)$$

If f is differentiable at 0, then

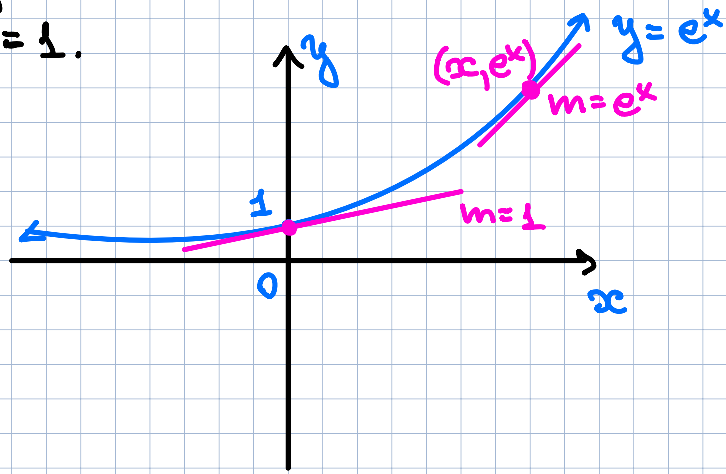
$$f'(x)=f'(0)\,b^x$$

Definition of a number "e":

e is a number such that $\lim_{h\to 0}\frac{e^h-1}{h}=1$

Geometrical meaning:

The function $f(x) = e^x$ is one whose tangent line at $(0,1)$ has a slope $f'(0) = 1$.



Derivative of the Natural Exponential function:

$$\boxed{\frac{d}{dx}\left(e^x\right) = e^x}$$