Section 3.4. The Chain Rule

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2+1}$$

Let $g(x) = x^2+1$, $u = g(x)$, $f = \sqrt{u}$. Then

$$F = f(g(x)) = f \circ g$$

$$\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The Chain Rule:

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F(x) = f \circ g$ is differentiable at x and F' can be computed by:

$$F'(x) = f'(g(x)) \cdot g'(x)$$

or

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example

$$F(x) = \sqrt{x^2+1}$$

$u = g(x) = x^2 + 1$

$f(u) = \sqrt{u}$

$f'(u) = \dfrac{1}{2\sqrt{u}}$

$g'(x) = 2x$

$$F'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$

$$F'(x) = \frac{x}{\sqrt{x^2+1}}$$

$$\boxed{\frac{d}{dx} \underbrace{f}_{\text{outer function}}(\underbrace{g(x)}_{\text{inner function}}) = f'(g(x)) \cdot g'(x)}$$

The Power Rule Combined with the Chain Rule:

If $y = (g(x))^n$, $y = f(u) = u^n$, $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = n \cdot u^{n-1} \cdot g'(x) = n(g(x))^{n-1} \cdot g'(x)$$

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = n \cdot u^{n-1} \frac{du}{dx}$$

$$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} \cdot g'(x)$$

Let $b > 0$; then $b^x = e^{x \ln b}$.

By the chain rule:

$$\frac{d}{dx} b^x = \frac{d}{dx}\left(e^{x \ln b}\right) = e^{x \ln b} \cdot \ln b = \ln b \cdot b^x$$

$$\boxed{\frac{d}{dx} b^x = \ln b \cdot b^x}$$