Section 3.6 Derivatives of Logarithmic Functions

$$y = \log_b x$$

$$y' = (\log_b x)' = \frac{1}{x \cdot \ln b}$$

$$b^{y} = b^{\log_b x} = x$$

$$\boxed{b^y = x} \qquad y = y(x)$$

Implicit differentiation Rule:

$$\frac{d}{dx}\left(b^y\right) = \frac{d}{dx}(x) \qquad (b^x)' = b^x \cdot \ln b$$

$$b^y \cdot \ln b \cdot y'(x) = 1$$

$$y'(x) = y' = \frac{1}{\underbrace{b^y}_{=x} \cdot \ln b}$$

$$\boxed{y' = \frac{1}{x \ln b}}$$

$$(\log_b x)' = \frac{1}{x \cdot \ln b} \qquad (1)$$

$b = e$

$$(\ln x)' = \frac{1}{x \cdot \underbrace{\ln e}_{=1}} = \frac{1}{x}$$

$\log_a a = 1$

$$(\ln x)' = \frac{1}{x} \qquad (2)$$

Example 1

$y = \ln(x^3 + 1)$

$y'(x) - ?$

Chain Rule:

$$y' = \frac{1}{x^3+1} \cdot 3x^2$$

Example 2

$$y = \sqrt{\ln x}$$

Chain Rule:

$$y' = \frac{1}{2}(\ln x)^{-1/2} \cdot \frac{1}{x}$$

Example 3

$$y = \ln|x|$$

$$y' = ?$$

$$y = \ln|x| = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$$

$$y' = (\ln|x|)' = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{-x} \cdot (-1), & x < 0 \end{cases} =$$

$$= \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{x}, & x < 0 \end{cases} = \frac{1}{x}$$

$$(\ln|x|)' = \frac{1}{x}$$

▼

Logarithmic Differentiation Rule

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$

1. Take the natural logarithm of the left and right hand side + apply natural log. properties to it.

$$\ln(y) = \ln\left(\frac{x^{3/4}\cdot\sqrt{x^2+1}}{(3x+2)^5}\right)$$

$$\ln(a\cdot b) = \ln a + \ln b$$

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

$$\ln(y) = \ln\left(x^{3/4}\right) + \ln\left(\sqrt{x^2+1}\right) -$$

$$- \ln\left((3x+2)^5\right)$$

2. Apply an implicit differentiation rule.

$$\frac{d}{dx}\left(\ln(y)\right) = \frac{1}{y} \cdot y'$$

$$\frac{d}{dx}(\ldots) = \frac{1}{x^{3/4}} \cdot \frac{3}{4} x^{-1/4} +$$

A

$$+ \frac{1}{\sqrt{x^2+1}} \cdot \frac{1}{2} (x^2+1)^{-1/2} \cdot 2x -$$

$$- \frac{1}{(3x+2)^5} \cdot 5(3x+2)^4 \cdot 3$$

$$\frac{1}{y} \cdot y' = A$$

3. Solve for y'.

$$y' = A \cdot y$$

$$y' = A \cdot \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}.$$

Four cases for exponents and bases:

1. $\frac{d}{dx}\left(b^n\right) = 0$ (b and n are constants)

2. $\frac{d}{dx}\left(\left(f(x)\right)^n\right) = n \cdot \left(f(x)\right)^{n-1} \cdot f'(x)$

3. $\frac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)} \cdot \ln(b) \cdot g'(x)$

4. $\frac{d}{dx}\left(f(x)^{g(x)}\right)$ Use Logarithmic Differentiation Rule

Example $y = x^{\sqrt{x}}$

$f(x) = x$

$g(x) = \sqrt{x}$

1. $\ln(y) = \ln(x^{\sqrt{x}})$

$\log_a x^b = b \cdot \log_a x$

$\ln(y) = \sqrt{x} \cdot \ln(x)$

Product Rule

2.

$\frac{1}{y} \cdot y' = \frac{1}{2\sqrt{x}} \cdot \ln(x) + \sqrt{x} \cdot \frac{1}{x}$

3. $y' = y\left(\frac{1}{2\sqrt{x}} \ln(x) + \sqrt{x} \cdot \frac{1}{x}\right)$

$$y = x^{\sqrt{x}}$$

$$\underbrace{y' = x^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\ln(x) + \sqrt{x}\cdot\frac{1}{x}\right).}$$

The number e as a limit

$$e \approx 2.7\ldots\ldots$$

$$\boxed{e = \lim_{x\to 0}(1+x)^{1/x}}$$

$$\frac{1}{x} = n \Rightarrow x = \frac{1}{n}$$

$$\boxed{e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}}$$

1. Fill in the derivative rules below:

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arccos(x)] = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}[\log_b(x)] = \frac{1}{x\ln b}$$

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

2. Find the derivative of each function below:

(a) $y = \ln(x^5)$

$$y' = \frac{1}{x^5} \cdot 5x^4$$

(b) $y = (\ln x)^5$

$$y' = 5(\ln x)^4 \cdot \frac{1}{x}$$

(c) $y = \ln(5x)$

$$y' = \frac{1}{5x} \cdot 5$$

3. Find the derivative of each function below:

(a) $f(x) = x^2 \log_2(5x^3 + x)$

$$f'(x) = 2x \cdot \log_2(5x^3+x) + x^2 \cdot \frac{1}{(5x^3+x)\ln 2} \cdot (15x^2+1)$$

(b) $g(x) = \ln(x^2 \tan^2 x)$

$$g'(x) = \frac{1}{x^2\tan^2 x} \cdot \left(2x \cdot \tan^2 x + x^2 \cdot 2\tan x \cdot \sec^2(x)\right)$$

4. Find $\frac{dy}{dx}$ for $y = \ln\sqrt{\frac{x+\sin x}{x^2-e^x}}$. $= \ln\left(\frac{x+\sin x}{x^2-e^x}\right)^{1/2} = \frac{1}{2}\ln\left(\frac{x+\sin x}{x^2-e^x}\right)$

$$y' = \frac{1}{2}\cdot\frac{1}{\frac{x+\sin x}{x^2-e^x}}\cdot\frac{(1+\cos x)(x^2-e^x)-(2x-e^x)(x+\sin x)}{(x^2-e^x)^2}$$

5. Find y' for each of the following:

(a) $y = \ln|x|$

$$y = \ln|x| = \begin{cases}\ln x, & x>0\\ \ln(-x), & x<0\end{cases}$$

$$y' = \begin{cases}\frac{1}{x}, & x>0\\ \frac{-1}{-x}, & x<0\end{cases} \Rightarrow y' = \frac{1}{x}.$$

(b) $y = \frac{e^{-x}\sin x}{\sqrt{1-x^2}}$ (Logarithmic differentiation makes this easier.)

1. $\ln(y) = \ln\left(\frac{e^{-x}\sin x}{\sqrt{1-x^2}}\right) = \ln e^{-x} + \ln\sin x - \ln\sqrt{1-x^2}$

2. $\frac{d}{dx}(\ln y) = \frac{d}{dx}(\ln e^{-x}) + \frac{d}{dx}(\ln\sin x) - \frac{d}{dx}\left(\ln\sqrt{1-x^2}\right)$

$$\frac{1}{y}\cdot y' = \underbrace{\frac{1}{e^{-x}}\cdot e^{-x}\cdot(-1) + \frac{1}{\sin x}\cdot\cos x - \frac{1}{\sqrt{1-x^2}}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)}_{A}$$

3. $y' = A\cdot y = A\cdot\frac{e^{-x}\sin x}{\sqrt{1-x^2}}$

(c) $y = x^{\sqrt[3]{x}}$ (Logarithmic differentiation is required.)

1. $\ln(y) = \ln x^{\sqrt[3]{x}} = \sqrt[3]{x}\,\ln x$

2. $\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\sqrt[3]{x}\,\ln x\right)$

$$\frac{1}{y}\cdot y' = \frac{1}{3}x^{-2/3}\ln x + \sqrt[3]{x}\cdot\frac{1}{x}$$

3. $y' = y\left(\frac{1}{3}x^{-2/3}\ln x + \sqrt[3]{x}\cdot\frac{1}{x}\right)$

$$y' = x^{\sqrt[3]{x}}\left(\frac{1}{3}x^{-2/3}\ln x + \sqrt[3]{x}\cdot\frac{1}{x}\right)$$