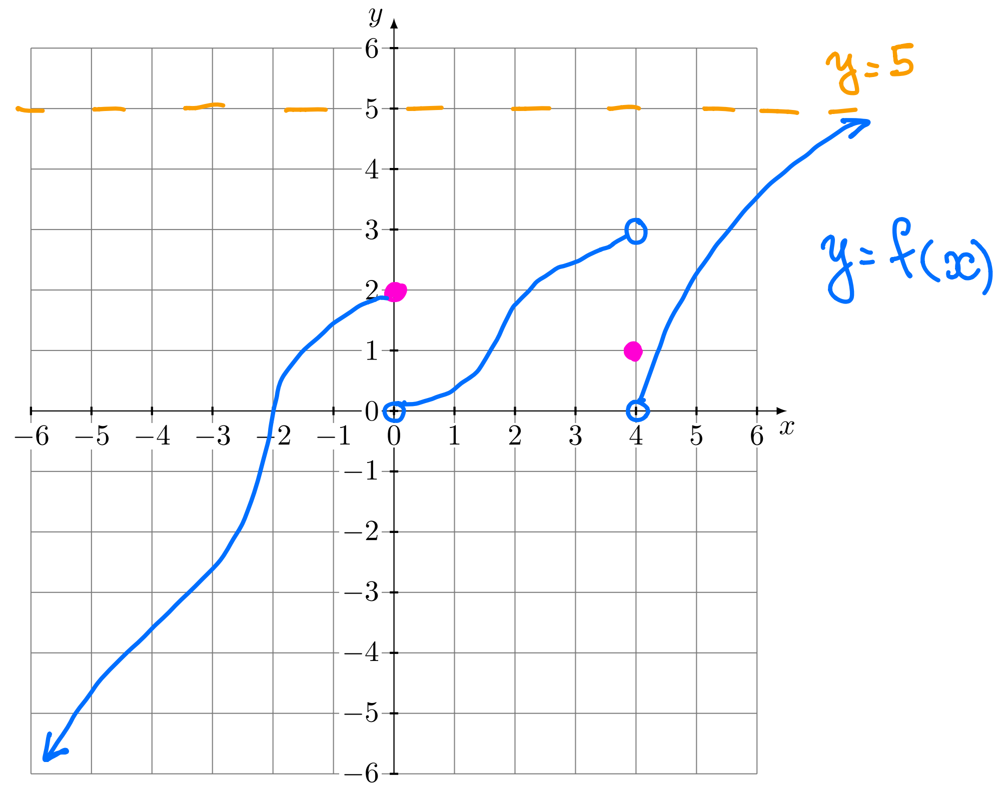
Midterm Exam 1 Review

Solutions

Exercise 1. Sketch the graph of an example of a function f that satisfies all of the given conditions:

$$\lim_{x\to 0^-} f(x) = 2, \quad \lim_{x\to 0^+} f(x) = 0, \quad \lim_{x\to 4^-} f(x) = 3, \quad \lim_{x\to +\infty} f(x) = 5,$$
$$\lim_{x\to 4^+} f(x) = 0, \quad f(0) = 2, \quad f(4) = 1, \quad \lim_{x\to -\infty} f(x) = -\infty.$$



Exercise 2. Evaluate the limit, if it exists.

(a) $\lim_{t\to 0} \frac{\sqrt{t^2+16}-4}{t^2} = \lim_{t\to 0} \frac{(\sqrt{t^2+16}-4)(\sqrt{t^2+16}+4)}{t^2(\sqrt{t^2+16}+4)} = \lim_{t\to 0} \frac{t^2+\cancel{16}-\cancel{16}}{t^2(\sqrt{t^2+16}+4)} =$

$= \lim_{t\to 0} \frac{\cancel{t^2}}{\cancel{t^2}(\sqrt{t^2+16}+4)} = \boxed{\frac{1}{8}}$

(b) $\lim_{x\to 3^-} \frac{\sqrt{x}}{\underbrace{(x-3)^5}_{<0}} = \frac{\sqrt{3}}{0^-} = \boxed{-\infty}$

powering a negative number will result in a negative number

(c) $\lim_{x\to\infty} \frac{1+e^x}{1+2e^x}$ $= \lim_{x\to\infty} \frac{\frac{1}{e^x}(1+e^x)}{\frac{1}{e^x}(1+2e^x)} = \lim_{x\to\infty} \frac{\frac{1}{e^x}+1}{\frac{1}{e^x}+2} = \boxed{\frac{1}{2}}$

$\frac{1}{e^x} \to 0$

$e^x \to \infty$ when $x \to \infty$

$\frac{1}{e^x} \to 0$ when $x \to \infty$

(d) $\lim_{x\to\infty} \ln\left(\frac{1+e^x}{1+2e^x}\right)$ $= \ln\left(\lim_{x\to\infty} \frac{1+e^x}{1+2e^x}\right) = \boxed{\ln\left(\frac{1}{2}\right).}$

cont. function

cont. functions $\begin{cases} y = \sin(x) \\ y = \cos(x) \\ y = \arctan(x) \end{cases}$

We can switch the order of $\ln(x)$ and $\lim_{x\to\infty}$ since $y = \ln(x)$ is a continuous function.

(e) $\lim_{x\to\infty} \frac{x^2}{\sqrt{x^4+1}}$

$$\lim_{x\to\infty} \frac{x^2}{\sqrt{x^4+1}} = \lim_{x\to\infty} \frac{\frac{1}{x^2}\cdot x^2}{\frac{1}{x^2}\sqrt{x^4+1}} = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{1}{x^4}(x^4+1)}} =$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^4}}} = \boxed{1}$$

$\frac{1}{x^4} \to 0$

Exercise 3. In the first few years after a coal mine's operation, the total deposit of coal (in millions of tons) t years after opening is approximately

$$C(t) = 300 - \frac{t^{3/2}}{2}.$$

(a) Find the average rate of change of the amount of coal in the deposit from the opening of the mine to year 4. Include correct units in your answer.

$t_2 = 4$ year $\quad t_1 = 0$ year

$$\frac{\Delta C}{\Delta t} = \frac{C(t_2)-C(t_1)}{t_2-t_1} = \frac{C(4)-C(0)}{4-0} = \frac{\cancel{300} - \frac{4^{3/2}}{2} - \cancel{300} + \cancel{\frac{0}{2}}}{4} = -\frac{4^{3/2}}{8} =$$

$$= -\frac{8}{8} = -1 \text{ (millions of tons / year)}$$

(b) It is a fact that $C'(t) = -\frac{3}{4}\sqrt{t}$. Compute $C'(4)$ and indicate what this quantity tells us about the mine. Write your answer in a sentence. Again, include correct units in your description.

decreasing

$$C'(4) = -\frac{3}{4}\sqrt{4} = -\frac{3}{4}\cdot 2 = -\frac{3}{2} \text{ millions of tons / year}$$

means that the amount of coal is decreasing at rate $\boxed{\frac{3}{2}}$ after the $t=4$ year.

Exercise 4. Consider the function

$$f(x) = \frac{1}{x+2}.$$

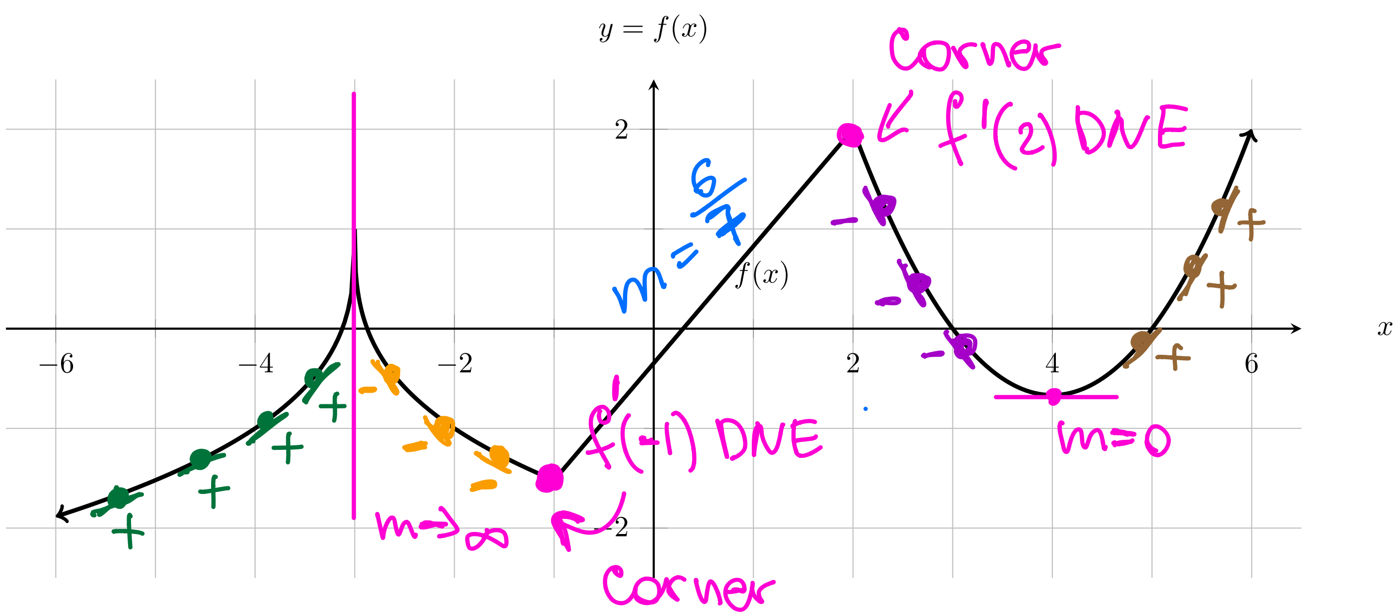
Using the $\boxed{\textbf{definition of the derivative}}$, find $f'(5)$.

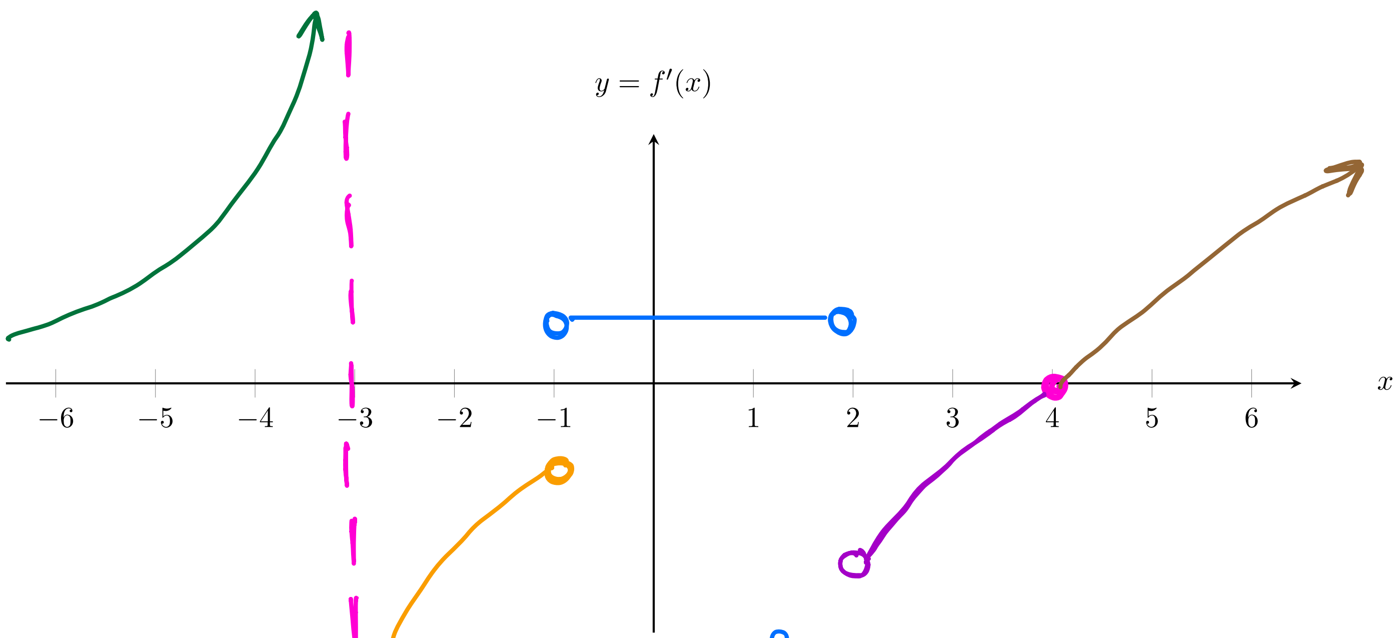
$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

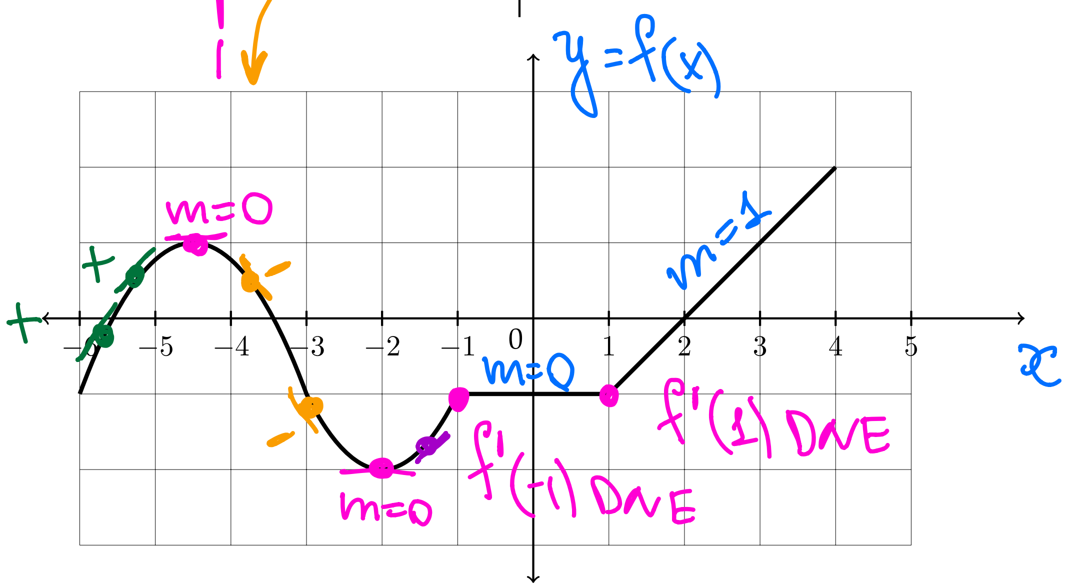
$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h} = \lim_{h \to 0} \frac{\cancel{x}+\cancel{2} - \cancel{x} - h - \cancel{2}}{h(x+h+2)(x+2)} =$$

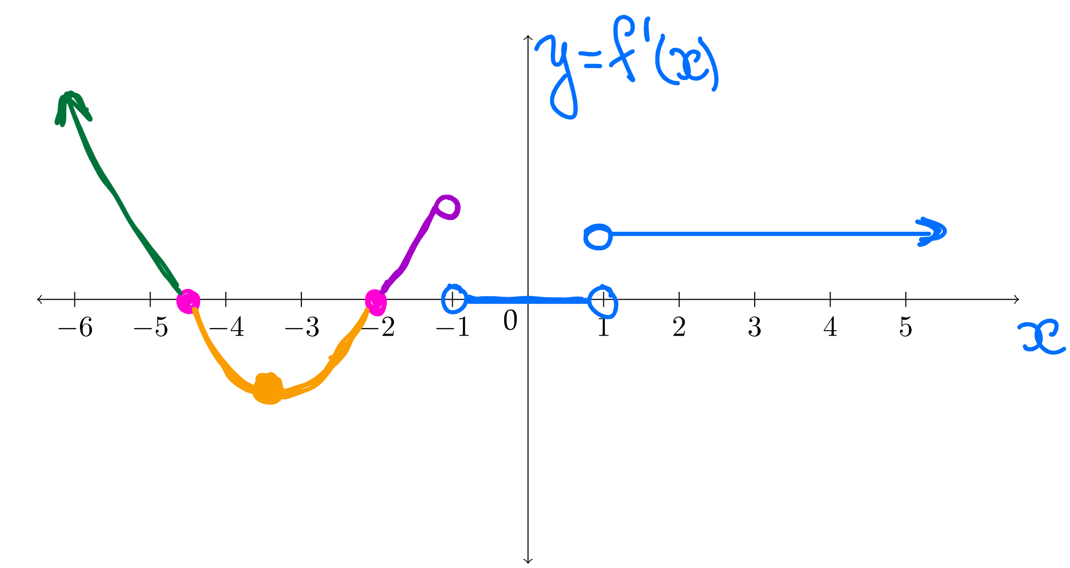
$$= \lim_{h \to 0} \frac{-1}{(x+2)(x+h+2)} = \boxed{\frac{-1}{(x+2)^2}}$$

Exercise 5. The graph of $f(x)$ is shown on the top set of axes. Sketch the derivative of $f(x)$ on the second set of axes.




$y = f'(x)$
x
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6


y = f(x)
m=0
+
+
−6 −5 −4 −3 −2 −1 0 1 2 3 4 5
m=0
m=1
f'(1) DNE
m=0
f'(-1) DNE
x


y = f'(x)
−6 −5 −4 −3 −2 −1 0 1 2 3 4 5
x

Exercise 6. Find the equation of the tangent line to the curve at the given point:

$$y = 4x - 3x^2, \quad P = (2, -4).$$

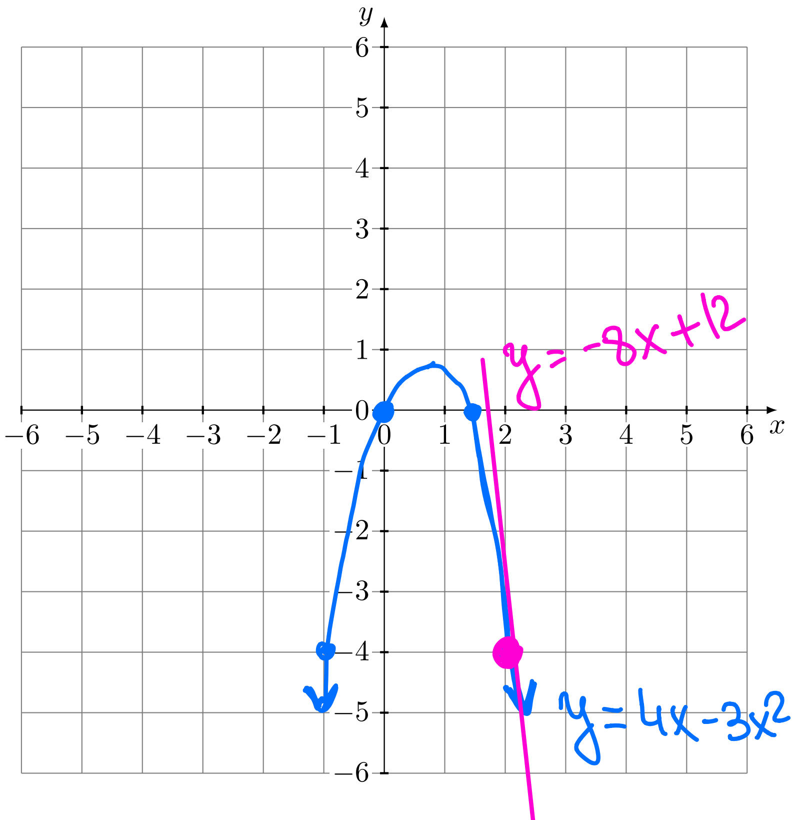
$a = 2$

$f(a) = -4$

$y = f'(a)(x-a) + f(a)$

$f'(a) = \lim_{h \to 0} \frac{f(h+a) - f(a)}{h} = \lim_{h \to 0} \frac{4(h+a) - 3(h+a)^2 - 4a + 3a^2}{h} =$

Sketch both your function $f(x)$ and the tangent line to it at point P on axes below.



$= \lim_{h \to 0} \frac{4h + \cancel{4a} - 3h^2 - 6ha - \cancel{3a^2} - \cancel{4a} + \cancel{3a^2}}{h} =$

$= \lim_{h \to 0} \frac{\cancel{h}(4 - 3h - 6a)}{\cancel{h}} = 4 - 6a$

$f'(2) = 4 - 12 = -8$

Therefore, $\boxed{y = -8(x-2) - 4 = -8x + 12}$ TL

Exercise 7. The limit represents the derivative of some function f at some point a. State such a function f and a point a.

$$\lim_{h \to 0} \frac{e^{-2+h} - e^{-2}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$f(a) = e^{\underbrace{-2}_{a}}$$

$$f(a+h) = e^{\underbrace{-2}_{a}+h}$$

So $\boxed{a = -2}$ and $\boxed{f(x) = e^x}$