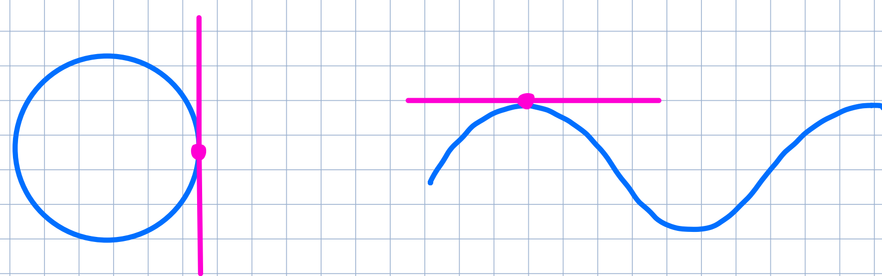
Section 2.1. The tangent and velocity problems

• The Tangent Problem

tangent (from Latin) means "touching"

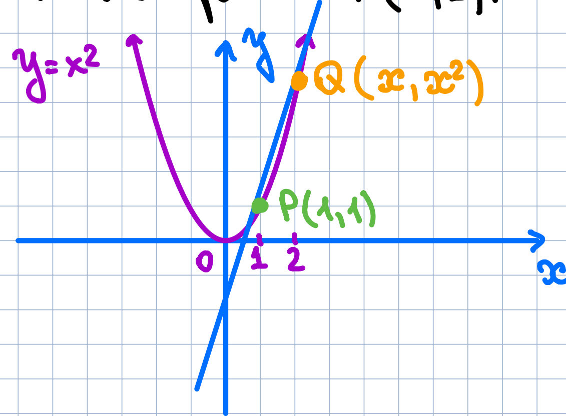


We begin with considering an example.

Example

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1,1)$.

Solution



- To find a tangent line we need to know its slope m.

- to find a tangent line slope m we need two points.

We have point $P(1,1)$.

We also can consider any point on parabola. Let us denote it by Q.

Point Q has coordinates $Q(x,y)=Q(x,x^2)$.

- First, we compute a slope of a secant line (a line that intersects a curve more than once).

We denote this slope by m_{PQ}.

m_{PQ} is a good approximation of the slope of tangent line m.

- How can we compute m_{PQ}?

Since we are given two points P and Q, we can use the point-slope formula for writing the equation of a line which goes through P and Q.

$P(1,1)$
$Q(x,x^2)$

$$m_{PQ}=\frac{y_2-y_1}{x_2-x_1}=\frac{x^2-1}{x-1}$$

$$m_{PQ}=\frac{x^2-1}{x-1}$$

- Since m_{PQ} is a good approximation of m, we have

$$m = \lim_{Q \to P} m_{PQ}$$ // this is what "good approximation" means

Hence,

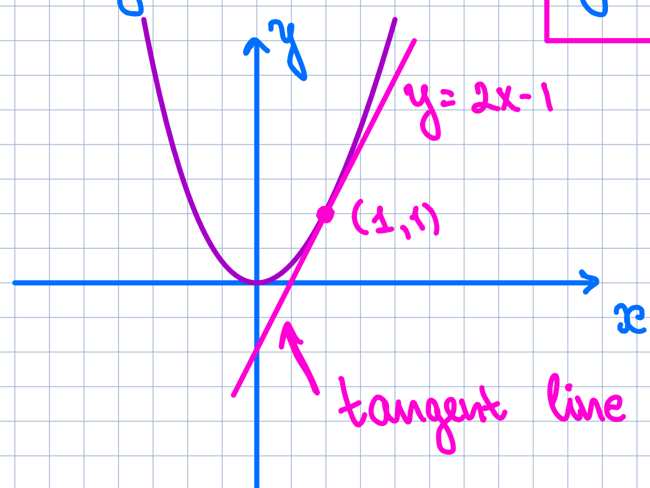
$$m = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x+1) = 2.$$

- Now, since $m = 2$, we use the point-slope form of the equation of a line

$$y - y_1 = m(x - x_1)$$

to write the equation of the tangent line through $(1, 1)$

$$y - 1 = 2(x-1) \Rightarrow \boxed{y = 2x - 1}$$ tangent line equation to $y = x^2$



• The Velocity Problem

We begin with considering the following example.

Example

Suppose a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

Solutions

We denote by $s(t)$ the distance fallen after t seconds measured in meters.

Then one obtains

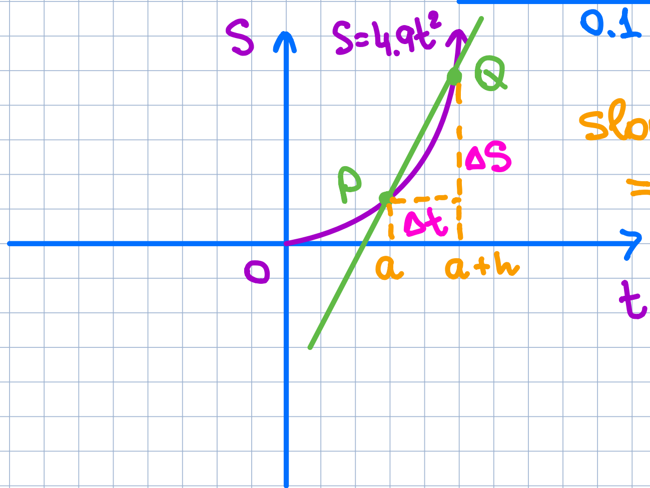
$$s(t) = 4.9t^2$$

$$\boxed{\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}}$$

Hence, we can compute the $v_{average}$ over the brief time interval of a tenth of a second from $t=5$ to $t=5.1$:

$$v_{av.} = \frac{S(5.1)-S(5)}{0.1}$$

$$= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49\,(m/s)$$

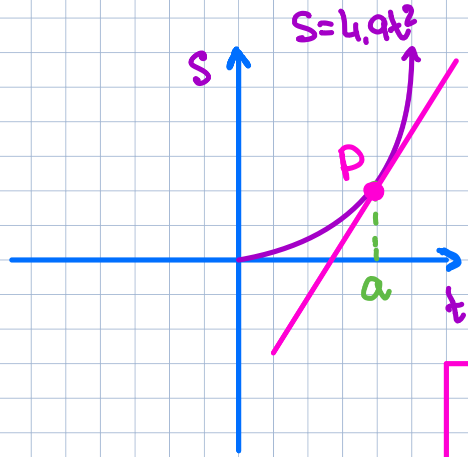


slope of secant line PQ = average velocity

$$\boxed{v_{average} = \frac{\Delta S}{\Delta t}}$$

It appears that as we shorten the time period, the average velocity is becoming closer to $49\,(m/s)$.

The instantaneous velocity when $t=5$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t=5$.



$$v_{inst.} = \lim_{\Delta t \to 0} v_{av.}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t}$$

$$\boxed{\text{Slope of a tangent line} = \text{instantaneous velocity}}$$