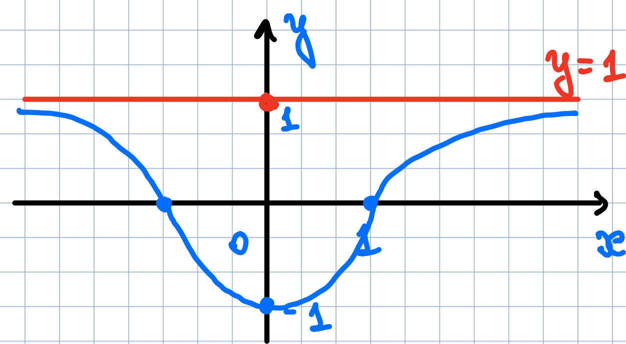
Section 2.6. Limits at Infinity; Horizontal Asymptotes

Example

$$y = \frac{x^2-1}{x^2+1}$$



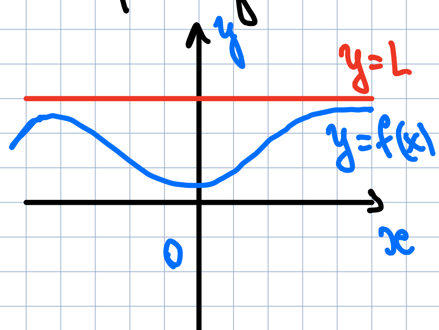
$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1$$

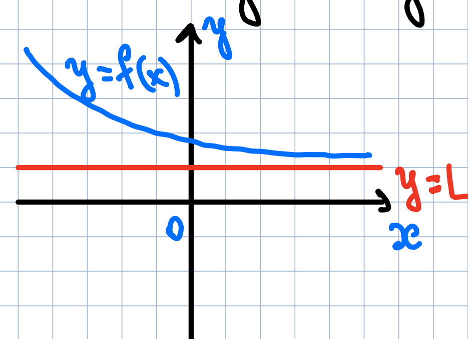
Def. (Intuitive limit at infinity)

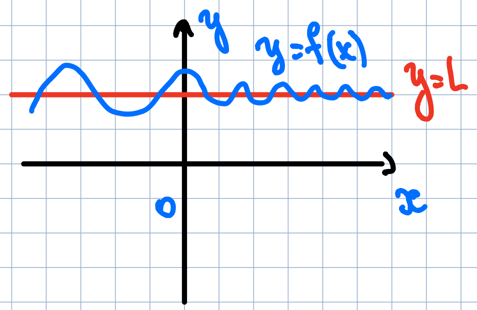
Let f be a function defined on some interval (a,∞). Then

$$\boxed{\lim_{x\to\infty} f(x) = L}$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.





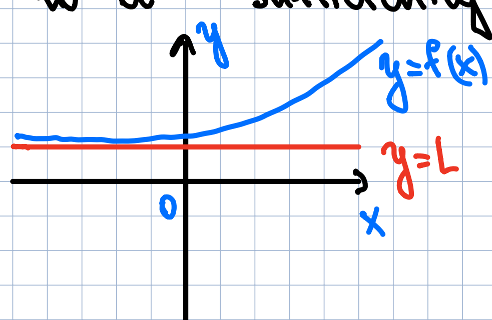


Def.

Let f be a function defined on some interval $(-\infty, a)$. Then

$$\boxed{\lim_{x\to -\infty} f(x) = L}$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

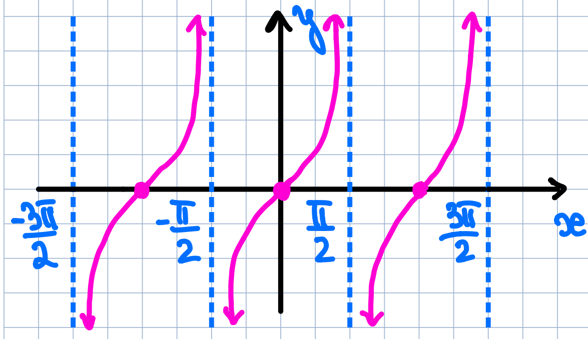


Def. The line $y=L$ is called a horizontal asymptote of the curve $y=f(x)$ if either

$$\boxed{\lim_{x\to\infty} f(x) = L} \quad \text{or} \quad \boxed{\lim_{x\to -\infty} f(x) = L}$$

Example

$$\lim_{x\to -\infty} \arctan(x) = -\frac{\pi}{2} \qquad \lim_{x\to \infty} \arctan(x) = \frac{\pi}{2}$$



Theorem If $r>0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

If $r>0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to -\infty} \frac{1}{x^r} = 0$$

In particular, $\boxed{\lim_{x\to -\infty} e^x = 0}$

- Infinite Limits at Infinity:

$$\lim_{x\to\infty} f(x) = \infty \qquad \lim_{x\to -\infty} f(x) = \infty$$

$$\lim_{x\to -\infty} f(x) = -\infty \qquad \lim_{x\to \infty} f(x) = -\infty$$

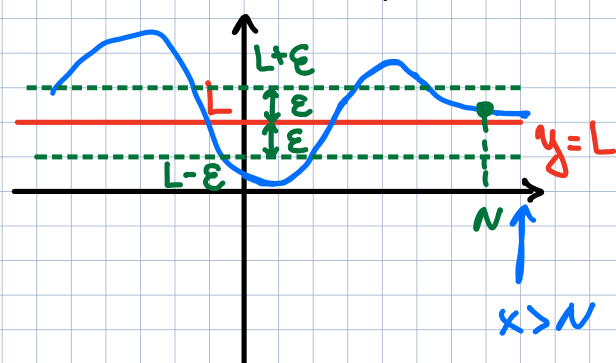
Def. (Precise definition of a limit at infinity)

Let f be a function defined on some interval (a,∞). Then

$$\boxed{\lim_{x\to\infty} f(x) = L}$$

means that for every $\varepsilon > 0$ there exists N in $\mathbb{N}$:

if $x > N$, then $|f(x) - L| < \varepsilon$

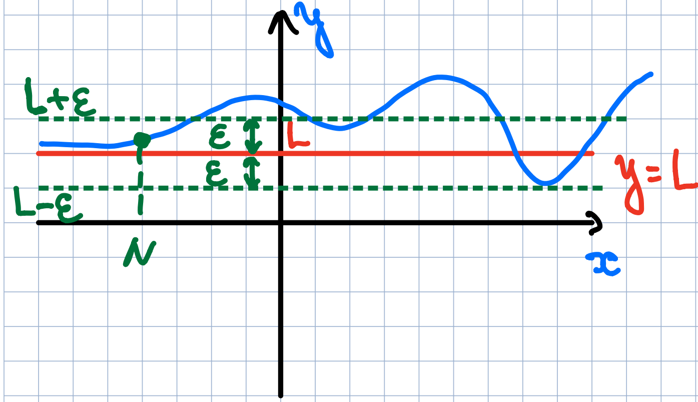


Def. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\boxed{\lim_{x\to-\infty} f(x) = L}$$

means that for every $\varepsilon > 0$ there exists N in $\mathbb{N}$ such that

if $x < N$, then $|f(x) - L| < \varepsilon$



Def. Let f be a function defined on (a,∞).
Then

$$\boxed{\lim_{x\to\infty} f(x)=\infty}$$

means that for every $M>0$ there exists
$N\in\mathbb{N}$ such that
if $x>N$, then $f(x)>M$.