Section 5.4. Indefinite Integrals. Net change theorem.

$$\boxed{\int f(x)\,dx} = \underbrace{F(x) + C}_{\text{set of functions}}$$

$$\int_a^b f(x)\,dx = \underbrace{F(b) - F(a)}_{\text{number}}$$

Properties of the indefinite integral:

1. $\int c\, f(x)\,dx = c \int f(x)\,dx$

2. $\int (f \pm g)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

3. $\int \underbrace{A}_{\text{constant}} \cdot \overset{f(x)}{\overset{\shortparallel}{1}}\,dx = A \int 1\,dx = A \cdot x + C$

The table of antiderivatives:

Def. The function $F(x)$ is the antiderivative of the function $f(x)$ on I, if

$$F'(x) = f(x)$$

$$\int F'(x)\,dx = \int f(x)\,dx$$

$$F(x) + C = \int f(x)\,dx$$

$$\boxed{\int f(x)\,dx = F(x) + C}$$

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
2. $\int e^x\,dx = e^x + C$
3. $\int \frac{1}{x}\,dx = \ln|x| + C$
4. $\int \sin(x)\,dx = -\cos(x) + C$

5. $\int \cos(x)\, dx = \sin(x) + C$

6. $\int \sec^2(x)\, dx = \tan(x) + C$

7. $\int \csc^2(x)\, dx = -\cot(x) + C$

8. $\int \sec(x)\tan(x)\, dx = \sec(x) + C$

9. $\int \csc(x)\cot(x)\, dx = -\csc(x) + C$

10. $\int \frac{1}{1+x^2}\, dx = \arctan(x) + C$

11. $\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin(x) + C$

Problem #3

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \cot(\theta)\cdot\csc(\theta)\,d\theta =$$

$$\frac{\cos\theta}{\sin^2\theta} = \underbrace{\frac{\cos\theta}{\sin\theta}}_{\cot\theta}\cdot\underbrace{\frac{1}{\sin\theta}}_{\csc\theta}$$

$$= -\csc(\theta) + C$$

Problem #4

$$\int\left(\frac{1+r}{r}\right)^2 dr = \int \frac{(1+r)^2}{r^2}\,dr = \int \frac{1+2\cdot1\cdot r+r^2}{r^2}\,dr =$$

$$= \int\left(\frac{1}{r^2} + \frac{2\cancel{r}}{r^{\cancel{2}}} + \frac{\cancel{r^2}}{\cancel{r^2}}\right)dr = \int\left(\frac{1}{r^2}+\frac{2}{r}+1\right)dr =$$

$$= \frac{r^{-1}}{-1} + 2\cdot\ln|r| + r + C$$

$$\frac{r^{-1}}{-1} = -\frac{1}{r}$$

Net Change Theorem

Application of <u>definite</u> $\left(\int_a^b f(x)\,dx\right)$ integrals.

By the FTC part two:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b =$$

$$= F(b) - F(a)$$

Based on the definition of the antiderivative function:

$$\boxed{F'(x) = f(x)}$$

$$\boxed{\int_a^b F'(x)\,dx = F(b) - F(a)}$$

Theorem (Net Change Theorem)

The integral of the rate of change is the net change

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Examples

1. $$\int_a^b v(t)\,dt = \int_a^b S'(t)\,dt =$$

$$= S(b) - S(a)$$

2. $C(x)$ will be the cost of producing x units of some commodity.

$$\int_a^b C'(x)\,dx = C(b) - C(a)$$

1. Compute $\int x^2(3-x)\,dx$ $= \int (3x^2 - x^3)\,dx =$

$= \int 3x^2\,dx - \int x^3\,dx = x^3 - \frac{x^4}{4} + C$

2. Compute $\int \left(9\sqrt{x} - 3\sec(x)\tan(x)\right) dx$ $= \int 9\sqrt{x}\,dx - \int 3\sec(x)\tan(x)\,dx =$

$= 9\int \sqrt{x}\,dx - 3\int \sec(x)\tan(x)\,dx = 9 \cdot \frac{x^{3/2}}{3/2} - 3\sec(x) + C$

3. Snow is falling on my garden at a rate of

$$m'(t) = A(t) = 10e^{-2t}$$

kilograms per hour for $0 \leq t \leq 2$, where t is measured in hours.

(a) Find $A(1)$ and interpret in the context of the problem.

$A(1) = 10e^{-2\cdot 1} = \frac{10}{e^2}$ kg/h

(b) If $m(t)$ is the total mass of snow on my garden, how are $m(t)$ and $A(t)$ related to each other?

total mass: $m(t) =$ initial mass $+ \underbrace{\int_0^t A(x)\,dx}_{\text{net change}}$, $0 \leq t \leq 2$

(c) What does $m(2) - m(0)$ represent?

$m(2) - m(0) = \int_0^2 m'(t)\,dt$ net change of snow mass on $[0,2]$.

(d) Find an antiderivative of $A(t)$.

$\int A(t)\,dt = \int 10e^{-2t}\,dt = 10\int e^{-2t}\,dt = 10\frac{e^{-2t}}{-2} + C$

(e) Compute the total amount of snow accumulation from $t = 0$ to $t = 1$.

$m(1) - m(0) = \int_0^1 A(t)\,dt = \int_0^1 10e^{-2t}\,dt = \frac{10}{-2}e^{-2t}\Big|_0^1 =$

$= -5e^{-2} + 5$ (kg)

(f) Compute the total amount of snow accumulation from $t = 0$ to $t = 2$.

$m(2) - m(0) = \int_0^2 A(t)\,dt$

(g) From the information given so far, can you compute $m(2)$?

$m(2) = m(0) + \int_0^2 A(t)\,dt$

unknown

known

We cannot compute m(2)

(h) Suppose $m(0) = 9$. Compute $m(1)$ and $m(2)$.

$m(1) = \underbrace{m(0)}_{} + \underbrace{\int_0^1 A(t)\,dt}_{}$ (kg) known

$m(2) = m(0) + \int_0^2 A(t)\,dt$ (kg)

4. A airplane is descending. Its **rate of change** of height is $r(t) = -4t + \dfrac{t^2}{10}$ meters per second.

   (a) if $A(t)$ is the altitude of the airplane in meters, how are $A(t)$ and $r(t)$ related?

   (b) What physical quantity does $\displaystyle\int_1^3 r(t)\,dt$ represent?

   (c) Compute $A(3) - A(1)$.

   (d) Can we determine the height of the plane when $t = 3$? If so, determine it; if not, explain why.

5. Gravel is being added to a pile at a rate of rate of $1 + t^2$ tons per minute for $0 \leq t \leq 10$ minutes. If $G(t)$ is the amount of gravel (in tons) in the pile at time t, compute $G(10) - G(0)$.