SECTION 5-3: THE FUNDAMENTAL THEOREM OF CALCULUS, PART 1

1. Suppose $f$ is the function whose graph is shown and that $g(x) = \int_0^x f(t)dt$.

(a) Find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, $g(5)$, and $g(6)$. Then, sketch a rough graph of $g$.

(a) $g(0) =$ $\int_0^0 f(t)\,dt = 0$

(b) $g(1) =$ $\int_0^1 f(t)\,dt = 1$

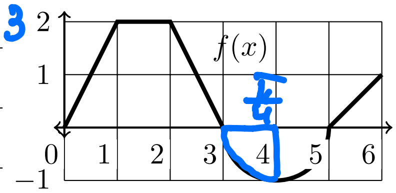
(c) $g(2) =$ $\int_0^2 f(t)\,dt = 3$

(d) $g(3) =$ $\int_0^3 f(t)\,dt = 4$

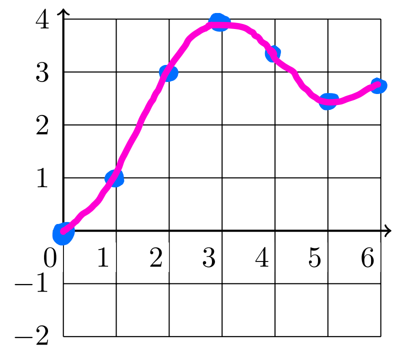
(e) $g(4) =$ $4 - \frac{\pi}{4}$

(f) $g(5) =$ $4 - \frac{\pi}{2}$

(g) $g(6) =$ $4\frac{1}{2} - \frac{\pi}{2}$




Sketch of $g(x)$




(i) Where is $g(x)$ increasing? $(0,3) \cup (5,6)$

(ii) Describe $f$ when $g(x)$ is increasing. positive

(iii) Where is $g(x)$ decreasing? $(3,5)$

(iv) Describe $f$ when $g(x)$ is decreasing. negative

(v) Where does $g(x)$ have a local maximum? $x = 3$ $\quad g_{max} = 4$

(vi) Describe $f$ when $g(x)$ has a local max. $f$ is changing its sign from + to −

(vii) Where does $g(x)$ have a local minimum? $x = 5$

(viii) Describe $f$ when $g(x)$ has a local min. $f$ is changing its sign from − to +

(b) Make a guess: what is the relationship between $g(x)$ and $f(x)$?

$$\boxed{f(x) = g'(x)} \longleftarrow g(x) = \int_0^x f(t)\,dt$$

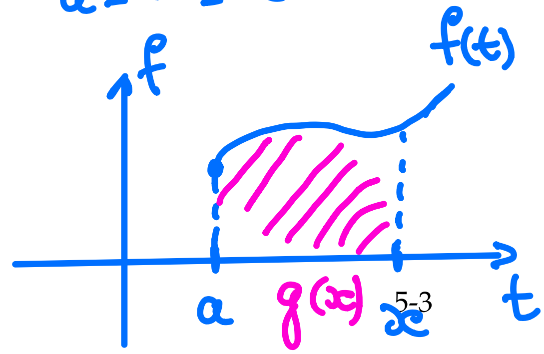
> **The Fundamental Theorem of Calculus, Part 1** If $f$ is continuous on $[a, b]$, the function $g$ defined by
>
> $$g(x) = \int_a^x f(t)\,dt \quad a \le x \le b$$
>
> is continuous on $[a, b]$ and differentiable on $(a, b)$ and $g'(x) = f(x)$.

a $\quad$ ~~b~~

2. Find the derivative of $g(x) = \int_2^x t^2\,dt$.

$f(t) = t^2$ $\qquad$ $a \le t \le x$

$g'(x) = x^2$



$g(x) = \int_0^x f(t)\,dt \Rightarrow g'(x) = f(x)$

3. The Fresnel function $S(x) = \int_0^x \sin(\pi t^2/2)\,dt$ first appeared in Fresnel's theory of the diffraction of light waves. Recently it was be applied to the design of highways. Find the derivative of the Fresnel function.

$S'(x) = \sin\left(\frac{\pi x^2}{2}\right)$

$g(x) = \int_0^x f(t)\,dt$

$g(x) = \int_a^x f(t)\,dt$

4. Consider $g(x) = \displaystyle\int_1^{x^4} \sec t\,dt$.

   Let $u = x^4$ and $h(x) = \int_1^x \sec t\,dt$.

   (a) Write $g(x)$ as a composition.

   $x^4 = u$

   $g(x) = h(u(x))$

   $g(x) = h(u) = \int_1^u \sec t\,dt$

   (b) Use FTC1 and the chain rule to differentiate $g(x)$.

5. Consider $g(x) = \displaystyle\int_{2x+1}^{2} \sqrt{t}\,dt$.

   (a) Write $g(x)$ as a composition.

   $g(x) = -\int_2^{2x+1} \sqrt{t}\,dt$

   $g'(x) = -\sqrt{2x+1} \cdot 2$

   (b) Use FTC1 and the chain rule to differentiate $g(x)$.

$g'(x) = \underbrace{h'(u)}\cdot \underbrace{\frac{du}{dx}} = \sec(u)\cdot 4x^3 = \boxed{\sec(x^4)\cdot 4x^3}$

By FTC1 : $h'(u) = \sec(u)$

$\frac{du}{dx} = 4x^3$

6. Consider the function $g(x) = \displaystyle\int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt$. Observe that

$$\int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt = \int_{\tan x}^{0} \frac{1}{\sqrt{2+t^4}}\,dt + \int_0^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt.$$

Use properties of definite integrals, FTC1, and the chain rule to determine $g'(x)$.

$g(x) = \int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt = \int_{\tan x}^{1} \frac{1}{\sqrt{2+t^4}}\,dt + \int_1^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt =$

$= -\int_1^{\tan x} \frac{1}{\sqrt{2+t^4}}\,dt + \int_1^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt$

$$g'(x) = \boxed{\frac{-1}{\sqrt{2+(\tan x)^4}} \cdot \sec^2(x) + \frac{1}{\sqrt{2+x^8}} \cdot 2x}$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

# SECTION 5-3: THE FUNDAMENTAL THEOREM OF CALCULUS, PART 2

**The Fundamental Theorem of Calculus (Part 2)** If $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F$ is **any antiderivative** of $f$, that is, is a function such that $F' = f$. To evaluate, we write

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

1. Evaluate the following integrals.

   (a) $\displaystyle\int_0^1 x^2\,dx$

   (b) $\displaystyle\int_1^4 (1 + 3y - y^2)\,dy$

2. Review from §4.9: To compute integrals effectively you **must** have your basic antidifferentiation formulas down. You should know that antiderivatives to the following functions. We are using the $\int$ symbol to mean "find the antiderivative" of the function right after the symbol.

**Antiderivatives of common functions:**

- $\displaystyle\int x^n\,dx =$ ______________
- $\displaystyle\int \sin x\,dx =$ ______________
- $\displaystyle\int \cos x\,dx =$ ______________
- $\displaystyle\int \sec^2 x\,dx =$ ______________
- $\displaystyle\int \sec x \tan x\,dx =$ ______________
- $\displaystyle\int \csc^2 x\,dx =$ ______________
- $\displaystyle\int \csc x \cot x\,dx =$ ______________
- $\displaystyle\int e^x\,dx =$ ______________
- $\displaystyle\int a^x\,dx =$ ______________
- $\displaystyle\int \frac{1}{1+x^2}\,dx =$ ______________
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx =$ ______________
- $\displaystyle\int \frac{1}{x}\,dx =$ ______________

3. Evaluate the following integrals.

   (a) $\displaystyle\int_2^5 \frac{3}{x}\,dx$

   (b) $\displaystyle\int_0^{\pi/2} \cos x\,dx$

4. Evaluate the following integrals.

(a) $\displaystyle\int_1^8 \sqrt[3]{x}\,dx$ (b) $\displaystyle\int_{\pi/6}^{\pi/2} \csc x \cot x\,dx$ (c) $\displaystyle\int_0^1 \frac{9}{1+x^2}\,dx$

5. We do not have any product or quotient rules for antidifferentiation. To evaluate an integral that is expressed as a product or quotient you must try to manipulate the integrand (the stuff inside the $\int$ sign) to look like something you know how to antidifferentiate. The following integrals are examples of this. Evaluate the following integrals.

(a) $\displaystyle\int_1^3 \frac{x^3+3x^6}{x^4}\,dx$ (b) $\displaystyle\int_0^1 x(3+\sqrt{x})\,dx$

6. Evaluate the following integrals.

(a) $\displaystyle\int_0^2 (5^x + x^5)\,dx$ (b) $\displaystyle\int_{1/2}^{\sqrt{2}/2} \frac{1}{\sqrt{1-x^2}}\,dx$

7. What is wrong with the following calculation? (Hint: draw a picture!)

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$