Worksheet Section 3.2: More Product Rule and Quotient Rule
Introduction to Section 3.3

**Solutions**

1. For each of the following, compute the derivative using the product or quotient rule, and then compute the derivative without using the product or quotient rule, after the indicated algebraic pre-processing. Do you get the same answer? Which one is easier?

(a) $F(x) = \dfrac{e^x+1}{\sqrt{2}+1} = \underbrace{\dfrac{1}{\sqrt{2}+1}}_{f}\underbrace{(e^x+1)}_{g}$

with product/quotient

$F'(x) = f' \cdot g + g' \cdot f =$

$= 0 \cdot (e^x + 1) + (e^x+1)' \cdot \dfrac{1}{\sqrt{2}+1} =$

$= 0 + e^x \cdot \dfrac{1}{\sqrt{2}+1} = \dfrac{e^x}{\sqrt{2}+1}$

without product/quotient

$F(x) = \dfrac{1}{\sqrt{2}+1} e^x + \dfrac{1}{\sqrt{2}+1}$

$F'(x) = \left(\dfrac{1}{\sqrt{2}+1} e^x\right)' + \left(\dfrac{1}{\sqrt{2}+1}\right)' =$

$= \dfrac{1}{\sqrt{2}+1} e^x + 0 = \dfrac{e^x}{\sqrt{2}+1}$

(b) $H(s) = \dfrac{A}{B\sqrt{s}} = \underbrace{\dfrac{A}{B}}_{f}\underbrace{s^{-1/2}}_{g}$

with product/quotient

$H'(s) = f' \cdot g + g' \cdot f =$

$= \left(\dfrac{A}{B}\right)' \cdot s^{-1/2} + (s^{-1/2})' \cdot \dfrac{A}{B}$

$= 0 + \left(-\dfrac{1}{2}\right) s^{-3/2} \cdot \dfrac{A}{B} = -\dfrac{A}{2B} s^{-3/2}$

without product/quotient

$H'(s) = \left(\dfrac{A}{B} s^{-1/2}\right)' =$

$= \dfrac{A}{B} (s^{-1/2})' =$

$= \dfrac{A}{B} \cdot \left(-\dfrac{1}{2}\right) \cdot s^{-3/2} =$

$= -\dfrac{A}{2B} \cdot s^{-3/2}$

(c) $k(t) = \dfrac{t^3+6}{t} = t^2 + 6t^{-1}$

with product/quotient

$t^3 + 6 = f$
$t = g$

$k'(t) = \dfrac{f' \cdot g - g' \cdot f}{g^2} =$

$= \dfrac{3t^2 \cdot t - 1 \cdot (t^3+6)}{t^2} = \dfrac{3t^3 - t^3 - 6}{t^2}$

$= \dfrac{2t^3 - 6}{t^2}$

without product/quotient

$k'(t) = (t^2 + 6t^{-1})' =$

$= 2 \cdot t + (-6) t^{-2} =$

$= \dfrac{2t}{1} - \dfrac{6}{t^2} = \dfrac{2t^3 - 6}{t^2}$

(d) $y = \underbrace{(x-\pi)}_{f}\underbrace{e^x}_{g} = xe^x - \pi e^x$

with product/quotient

$y' = f' \cdot g + g' \cdot f =$

$= (x-\pi)' \cdot e^x + (e^x)' \cdot (x-\pi)$

$= 1 \cdot e^x + e^x (x - \pi) = e^x (1 + x - \pi)$

$(x^1)' = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$

without product/quotient

$y' = (xe^x - \pi e^x)' =$

$= (xe^x)' - (\pi e^x)' =$

$= (xe^x)' - \pi e^x$

product rule

## Tangent line equation

for the given $y = f(x)$ at $x = a$ is

$$\boxed{y = f'(a)(x-a) + f(a)}$$

**Example**

$f(x) = e^x + x, \quad x = 1$

$$y = f'(1)(x-1) + f(1)$$

$f(1) = e + 1$

$f'(x) = e^x + 1$

$f'(1) = e + 1$

$y = (e+1)(x-1) + (e+1)$ — Tangent line equation

Let $S(t)$ be a distance function.

$\boxed{\text{Rate of change}}$ of $S(t)$ at $t=1$ is:

$$S'(1) = \lim_{h \to 0} \frac{S(h+1) - S(1)}{h}$$

Let

$$S(t) = t^2 - t^3 + 1$$

Then

$$S'(t) = 2t - 3t^2$$

$$S'(1) = 2 - 3 = -1 \text{ (meters/s)}$$

$S(t)$ – distance function (meters)

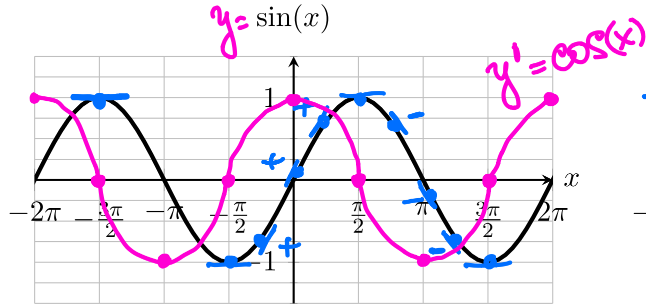
$v(t) = S'(t)$ – velocity function (meters/s)

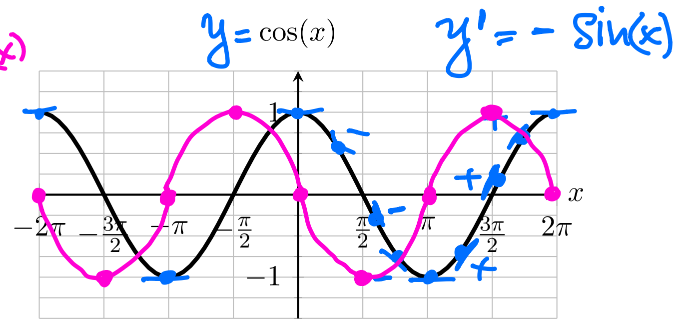
$a(t) = S''(t)$ – acceleration function (meters/s2)

For a given $S(t)$:

$v(t) = S'(t)$ and $a(t) = S''(t) = v'(t)$

2. Use the graphs of $y = \sin x$ and $y = \cos x$ to sketch a graph of $y'$ (on the same set of axes).




What do you notice?

$y = \sin(x)$ $\quad y' = \cos(x)$

$y = \cos(x)$ $\quad y' = -\sin(x)$

3. Use the fact that $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$ to find the derivative of

$$y = \underbrace{3x^4}_{f}\,\underbrace{\cos(x)}_{g}.$$

$$y'(x) = (3x^4)' \cdot \cos(x) + (\cos(x))' \cdot 3x^4 =$$

$$= 12x^3 \cdot \cos(x) + (-\sin(x)) \cdot 3x^4$$

4. Use the fact that $\tan(x) = \frac{\sin(x)}{\cos(x)}$ (f: $\sin(x)$, g: $\cos(x)$) and the quotient rule to determine the derivative of $y = \tan(x)$. Simplify your answer as much as possible.

$$(\tan(x))' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{f'(x)\cdot g(x) - g'(x) f(x)}{g^2(x)} =$$

$$= \frac{\cos(x)\cdot\cos(x) + \sin(x)\cdot\sin(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} =$$

$$= \sec^2(x)$$

$$\frac{1}{\cos(x)} = \sec(x)$$

$$\boxed{(\tan(x))' = \sec^2(x)}$$

5. The equation of motion of a particle is $s(t) = t^3 - 3t$, where $s$ is in meters and $t$ is in seconds. Find:

   (a) the velocity and acceleration as functions of $t$,

   (b) the acceleration after 2 s,

   (c) the acceleration when the velocity is 0.

$$\left(\cot(x)\right)' = \left(\frac{\cos(x)}{\sin(x)}\right)' = \frac{(\cos(x))' \cdot \sin(x) - (\sin(x))' \cdot \cos(x)}{\sin^2(x)} =$$

$$= \frac{-\sin(x) \cdot \sin(x) - \cos(x) \cdot \cos(x)}{\sin^2(x)} =$$

$$= \frac{-\left(\sin^2(x) + \cos^2(x)\right)}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\csc^2(x)$$

$$\boxed{\left(\cot(x)\right)' = -\csc^2(x)}$$

$$\left(\sec(x)\right)' = \left(\frac{1}{\cos(x)}\right)' = \frac{0 \cdot \cos(x) - (-\sin(x)) \cdot 1}{\cos^2(x)} =$$

$$= \frac{\sin(x)}{\cos^2(x)} = \tan(x) \cdot \sec(x)$$

$$\frac{\sin(x)}{\cos(x) \cdot \cos(x)} = \underbrace{\frac{\sin(x)}{\cos(x)}}_{\tan(x)} \cdot \underbrace{\frac{1}{\cos(x)}}_{\sec(x)}$$

$$\boxed{\left(\sec(x)\right)' = \tan(x) \cdot \sec(x)}$$

? $\boxed{\left(\csc(x)\right)' = -\cot(x) \cdot \csc(x)}$ HW