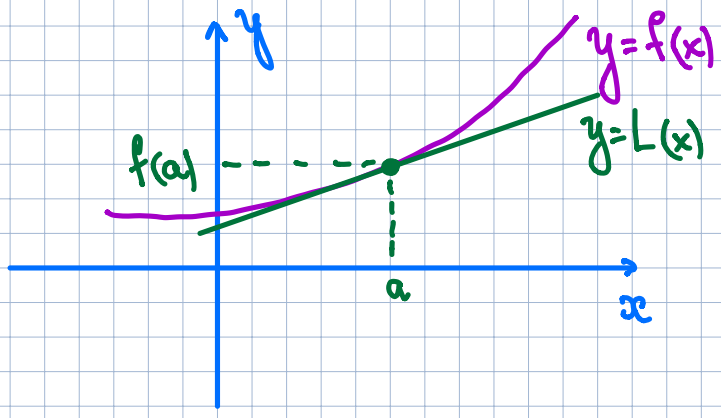
Section 3.10 Linear Approximations and Differentials



Linear approximation: we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a.

$$y = f(a) + f'(a)(x-a)$$

and the approximation

$$f(x) \approx f(a) + f'(a)(x-a)$$

is called the **linear approximation** or **tangent line approximation** of f at a.

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$ — linearization of f at a

Example 1

- find the linearization of the function

  $f(x) = \sqrt{x+3}$ at $x = 1$

- approximate $\sqrt{3.98}$ and $\sqrt{4.05}$

Solution

$$f'(x) = \frac{1}{2\sqrt{x+3}} \qquad f(1) = 2$$

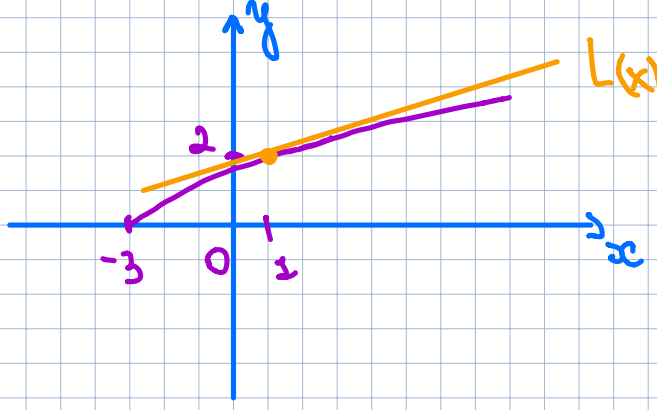
$$f'(1) = \frac{1}{4}$$

$$L(x) = 2 + \frac{1}{4}(x-1) = \frac{1}{4}x + \frac{7}{4}$$

$$\sqrt{x+3} \approx \frac{1}{4}x + \frac{7}{4}$$

$$\sqrt{3.98} \approx \frac{1}{4} \cdot 0.98 + \frac{7}{4} = 1.995 \quad \text{and}$$

$$\sqrt{4.05} \approx \frac{1}{4} \cdot 1.05 + \frac{7}{4} = 2.0125$$



$L(x) = \frac{1}{4}x + \frac{7}{4}$

Approximations are overestimates because the tangent line lies above the curve

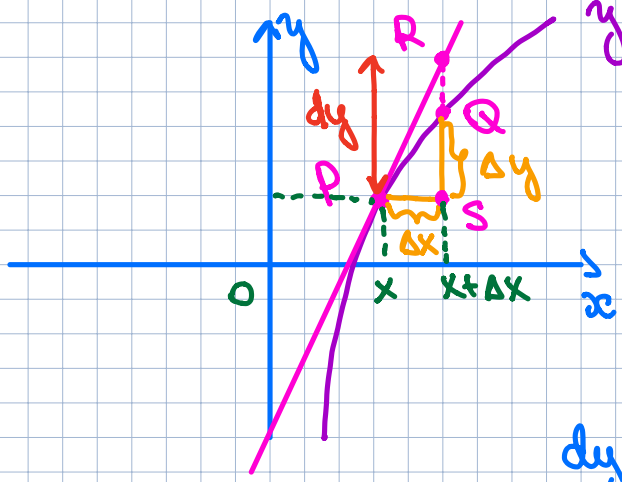
- **Differentials**

If $y = f(x)$ where f is differentiable, then the differential dx is an independent variable (can be given the value of any real number).

Then

$$dy = f'(x) \cdot dx$$

dy is dependent variable (depends on x and dx).



$dx = \Delta x$

$\Delta y = f(x+\Delta x) - f(x)$

slope of $PR = f'(x)$

$RS = f'(x) \cdot \Delta x = f'(x)\, dx$

$=$

dy

dy – how tangent line rises or falls

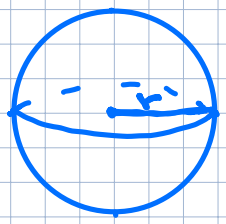
Δy – how $y = f(x)$ rises or falls when x changes by dx.

Example

$r = 21$ cm

$dr = 0.05$ cm

$dr = \Delta r$



ΔV can be approximated by dV

$dV - ?$

$V = \frac{4}{3}\pi R^3$

$dV = 4\pi R^2\, dr$

$dV \approx 277\ cm^3$ – maximum error.

Relative error: $\frac{\Delta V}{V} \approx \underbrace{\frac{dV}{V}}_{\text{relative error in volume}} = \frac{4\pi R^2\, dr}{\frac{4}{3}\pi R^3} = \underbrace{\frac{3\,dR}{R}}_{\text{relative error in radius}}$

Percentage error: $\frac{dR}{R} = \frac{0.05}{21} \approx 0.0024 \approx 0.24\,\%.$