Section 1.5. Inverse functions and logarithms

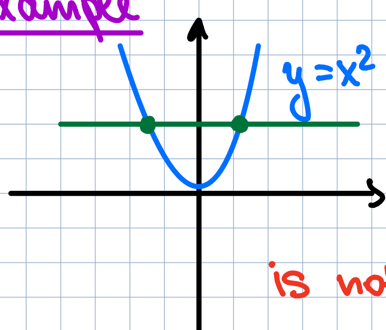
Def. A function f is called a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2$$

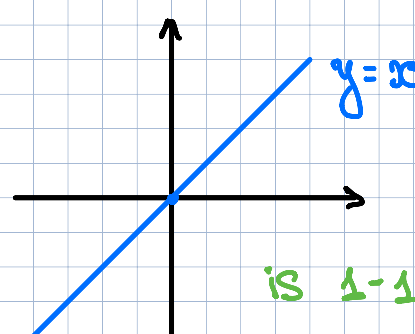
Horizontal Line Test

A function is 1-1 if and only if no horizontal line intersects its graph more than once.

Example



is not 1-1



is 1-1

Remark

1-1 functions are important because they are precisely the functions that possess inverse functions.

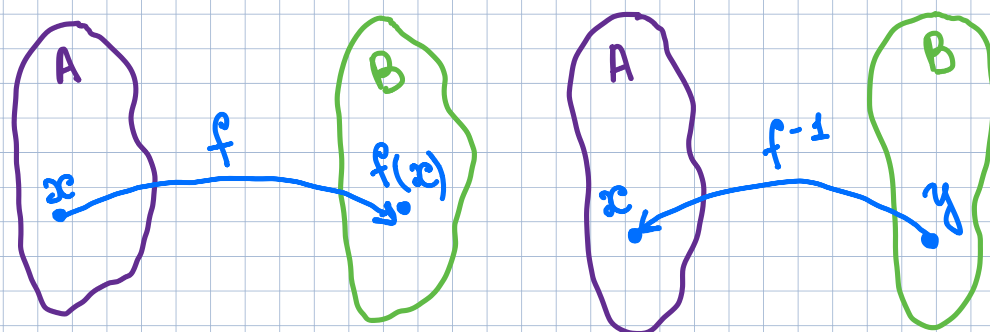
Def. Let f be a 1-1 function.

$\text{Domain}(f) = A$ and $\text{Range}(f) = B$.

Then its inverse function f^{-1} has domain $\text{Dom}(f^{-1}) = B$ and range $\text{Range}(f^{-1}) = A$ and is defined by

$$f^{-1}(y) = x \quad \text{iff} \quad y = f(x)$$

for any $y \in B$.



Cancellation equations:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

How to find the inverse function of 1-1 function f:

1. Write $y = f(x)$
2. Solve this equation for x in terms of y (if possible)

3. To express f^{-1} as a function of x do

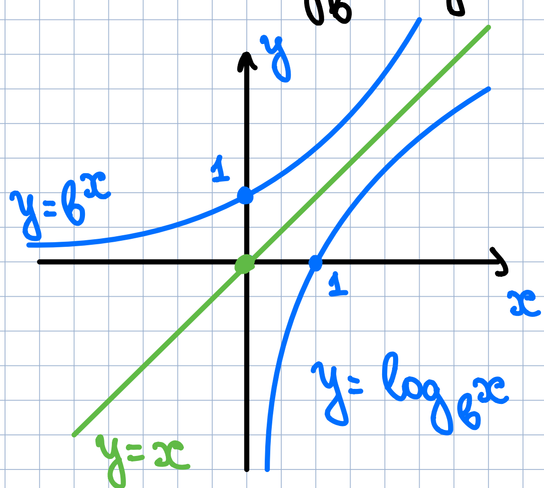
$y \longleftrightarrow x$

$$y = f^{-1}(x)$$

Remark The graph of f^{-1} is obtained by reflecting the graph of f about $y = x$.

Logarithmic Functions

$\log_b x = y$ if and only if $x = b^y$ $b > 0$, $b \neq 1$



$$\boxed{\begin{array}{ll} \log_b b^x = x & \text{for every } x \text{ in } \mathbb{R} \\ b^{\log_b x} = x & \text{for every } x > 0 \end{array}}$$

Laws of Logarithms:

If $x, y > 0$, then

1. $\log_b(xy) = \log_b x + \log_b y$

2. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

3. $\log_b(x^r) = r \log_b x \quad (r \text{ in } \mathbb{R})$

Natural Logarithms:

$$\ln x = \log_e x$$

$\ln x = y$ if and only if $x = e^y$

$$\boxed{\begin{aligned} &\ln e^x = x, \quad x \text{ in } \mathbb{R} \\ &e^{\ln x} = x, \quad x > 0 \\ &\ln e = 1 \end{aligned}}$$

Change of Base formula:

$$\log_b x = \frac{\ln x}{\ln b}$$

$y = \log_b x \Rightarrow b^y = b^{\log_b x} = x \Rightarrow \ln x = \ln b^{\log_b x} =$

$= \log_b x \cdot \ln b$

Inverse Trigonometric Functions:

- $f(x) = \sin(x)$

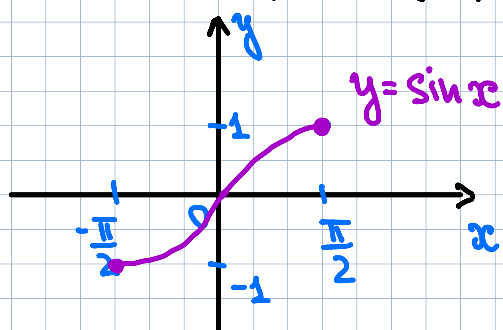
  $f^{-1}(x) = \arcsin(x)$, x in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

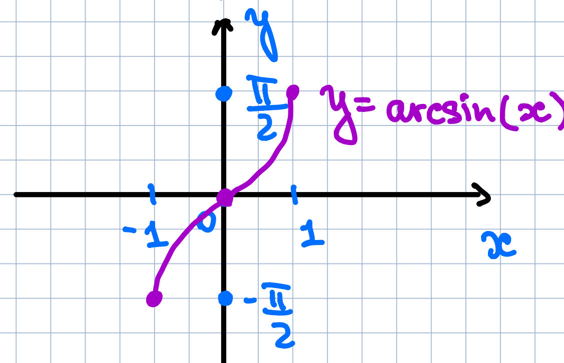
  $f^{-1}(x) = y$ iff $x = f(y)$

  $\arcsin(x) = y$ iff $x = \sin y$, y in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

  $\arcsin(\sin(x)) = x$, x in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

  $\sin(\arcsin(x)) = x$, x in $[-1, 1]$



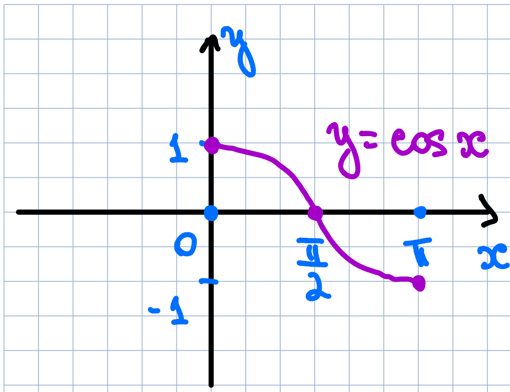


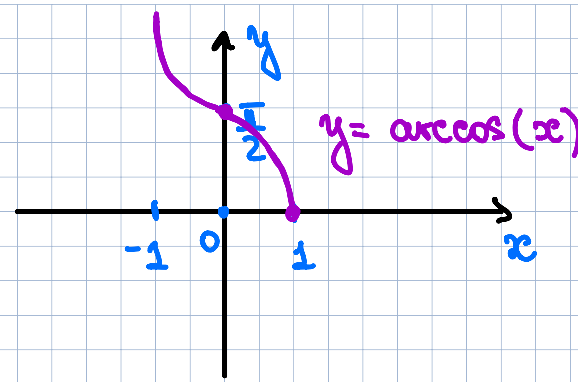
- $f(x) = \cos(x)$, x in $[0, \pi]$

  $\arccos(x) = y$ iff $x = \cos(y)$, y in $[0, \pi]$

  $\arccos(\cos(x)) = x$, x in $[0, \pi]$

  $\cos(\arccos(x)) = x$, x in $[-1, 1]$





- $y = \tan(x)$, x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

  $\arctan(x) = y$ iff $x = \tan(y)$, y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$