Section 4.5 Curve Sketching & Section 4.7 Applied Optimization (Day 1)

1. Follow the guidelines from the previous worksheet to sketch the graph of

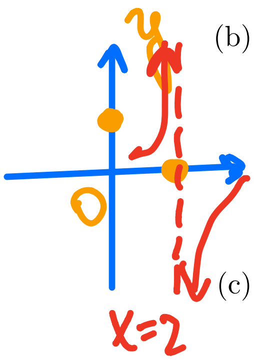
$$f(x) = \frac{2}{x} + \ln(x). \qquad \text{(Note: } f'(x) = \frac{x-2}{x^2} \text{ and } f''(x) = \frac{4-x}{x^3}\text{)}$$

(a) What is the function's domain?

Domain: $x > 0$ or $(0, \infty)$

0

$f_1(x) = \frac{2}{x}$; $\mathrm{Dom}(f_1) = \mathbb{R} \setminus \{0\}$ $\quad f_2(x) = \ln(x)$; $\mathrm{Dom}(f_2) = (0, \infty)$



(b) (if defined) Determine the y-intercept. Determine the x-intercepts if it's not too hard.

y-intercept: $x = 0$ None

x-intercept: $y = 0$ $\quad \frac{2}{x} + \ln(x) = 0$ None

(c) (if defined) What behavior occurs for this function as $x \to \pm\infty$?

$$\lim_{x\to\infty}\left(\frac{2}{x} + \ln(x)\right) = \underbrace{\lim_{x\to\infty}\frac{2}{x}}_{\to 0} + \underbrace{\lim_{x\to\infty}\ln(x)}_{\to\infty} = 0 + \infty = \infty$$

$$\cancel{\lim_{x\to-\infty}\left(\frac{2}{x} + \ln(x)\right)}$$

(d) Does the function have any vertical asymptotes? Where?

VA: $\lim_{x\to a} f(x) = \pm\infty$, then $x = a$ is VA

$x = 0$

$$\lim_{x\to0^+}\left(\frac{2}{x} + \ln(x)\right) = \lim_{x\to0^+}\frac{2}{x}\left(1 + \frac{\ln x}{\frac{2}{x}}\right) = \boxed{+\infty}$$

(e) Find intervals where f is increasing/decreasing and identify critical points.

CP: $f'(x) = 0$ or $f'(x)$ DNE

$$\lim_{x\to0^+}\frac{\ln x}{\frac{2}{x}} = \lim_{x\to0^+}\frac{\frac{1}{\cancel{x}}}{-\frac{2}{x^{\cancel{2}}}} = \lim_{x\to0^+}\frac{1}{-\frac{2}{x}} = \lim_{x\to0^+} -\frac{x}{2} \to 0$$

$$f'(x) = -\frac{2}{x^2} + \frac{1}{x} = 0$$

$$\frac{-2+x}{x^2} = 0$$

$$-2 + x = 0$$

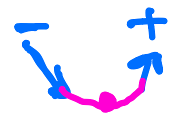
$x = 2$ CP

$x = 0$ is not a CP since it is not in

D(f).

$f'(x)$	$-$		$+$
$f(x)$	$\searrow$	2	$\nearrow$
	$(-\infty, 2)$		$(2, \infty)$

(f) Classify each critical point as a local min/max/neither.



1st Derivative Test:

$x = 2$ f': $-$ to $+$, then at $x = 2$ f attains its loc. min value

(g) Find intervals where f is concave up/concave down and identify points of inflection

$$f''(x) = \left(-\frac{2}{x^2} + \frac{1}{x}\right)' = \left(+\frac{4}{x^3} - \frac{1}{x^2}\right) = 0$$

$$\frac{4-x}{x^3} = 0 \qquad 4 - x = 0 \Rightarrow x = 4$$

f'': $+$ on the left of 4 (CU), $-$ on the right of 4 (CD)

$x = 4$ is an inflection point

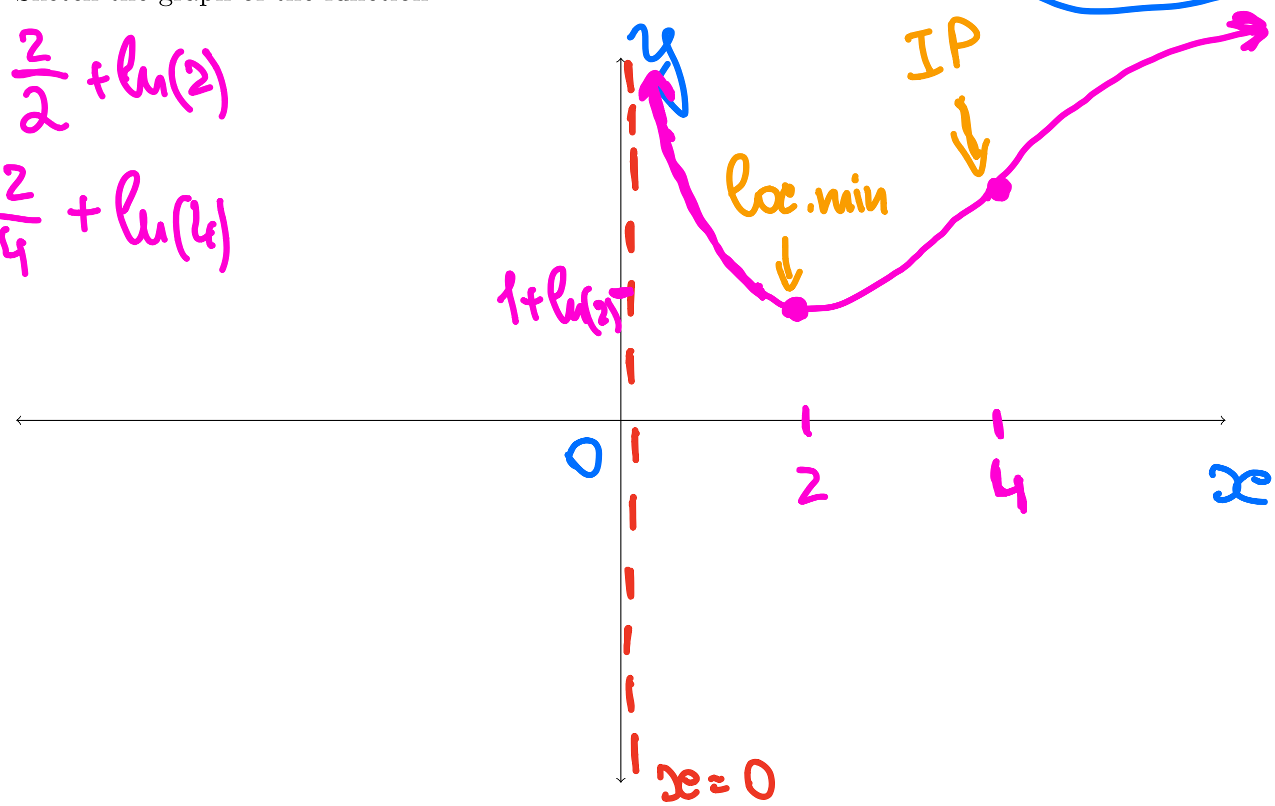
(h) Collect all the information you have determined into a handy list.

1. $\mathrm{Dom}(f) = (0, \infty)$
2. $x = 0$ is VA
3. $f \nearrow$ on $(2, \infty)$ and $f \searrow$ on $(-\infty, 2)$
4. $f(2)$ has loc. min
5. f is CU on $(-\infty, 4)$ f is CD on $(4, \infty)$ $x = 4$ is an IP

(i) Sketch the graph of the function

$$f(2) = \frac{2}{2} + \ln(2)$$

$$f(4) = \frac{2}{4} + \ln(4)$$



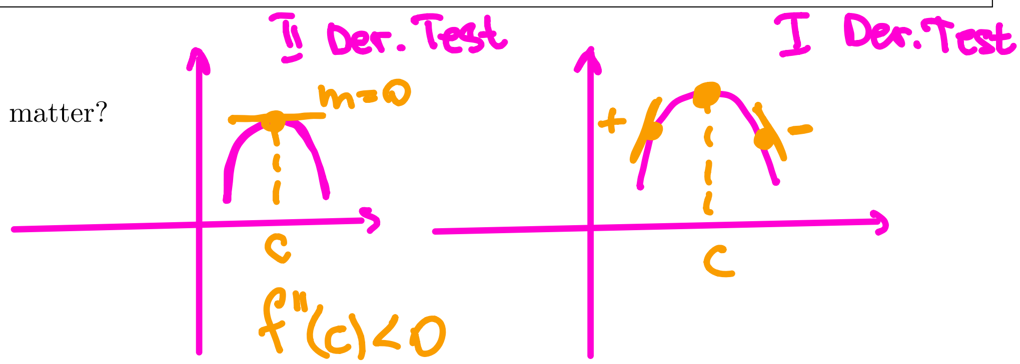
max $A(x)$
min $A(x)$

Section 4.7 Applied Optimization (Day 1)

Algorithm for Approaching Optimization

1. Read the problem two or three times. Draw pictures. Label them. Pick specific numerical examples, to make the problem concrete. Be creative. Try more than just one approach.
2. Identify the quantity to be minimized or maximized (and which one... min or max).
3. Chose notation and explain what it means.
4. Write the thing you want to maximize or minimize as a function of one variable, including a reasonable domain.
5. Use calculus to answer the question and justify that your answer is correct.

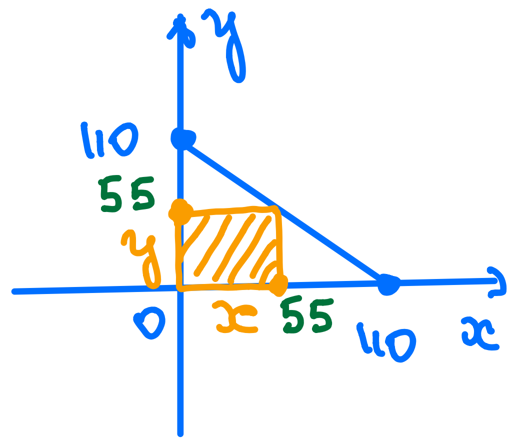
(a) Why does justification matter?



(b) Find two positive numbers whose sum is 110 and whose product is a maximum.

Let $x > 0$, $y > 0$ be two positive numbers.

$x + y = 110 \Rightarrow y = 110 - x$



max xy = max P

$xy = P$

We want to represent the function P in one variable.

$P = x \cdot y = x(110 - x) \Rightarrow P(x) = x(110 - x)$

$y = 110 - x$

$\max\limits_{0 \le x \le 110} P(x) = \max\limits_{0 \le x \le 110} x(110 - x)$

Domain for $P(x)$ is $[0, 110]$.

Max or min is finding extreme values of the function $f(x)$.

$P(x) = x(110-x)$ on $[0, 110]$.

1. find CP. $\Big\}$ + justification
   $f(CP)$
2. $f(0)$, $f(110)$
3. max $\{1, 2\}$

1. $P'(x) = 110 - 2x = 0$

$$110 = 2x$$

$$x = 55 \text{ - CP}$$

$$P(55) = 55(110-55) = 55 \cdot 55 = 55^2$$

2. $P(0) = 0$

$$P(110) = 110(110-110) = 0$$

3. Justification.

II Derivative Test

$x = 55$

(a) $P'(55) = 0$

(b) $P''(55) < 0$ holds

$P''(x) = -2 < 0$

Therefore, at $x = 55$ our function $P(x)$ attains its loc. max value.

$P(55) = 55^2$.

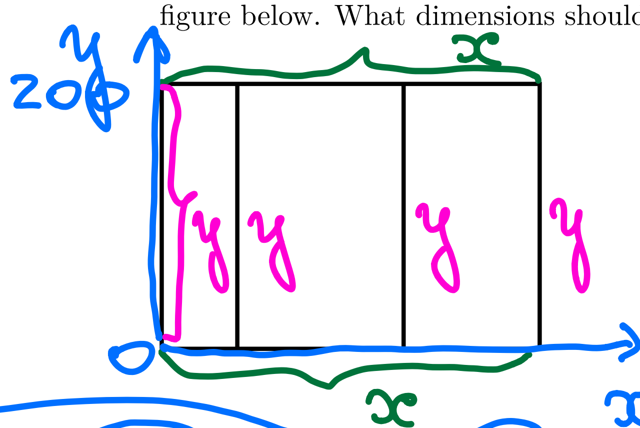
Furthermore, $P(x)$ has its global max value at $\underline{\underline{x = 55}}$,

$$x + y = 110$$

$$y = 110 - x = 110 - 55 = 55$$

Answer: $P_{max} = \underbrace{55}_{x} \cdot \underbrace{55}_{y} = \boxed{55^2}$

(c) A rancher has 800 feet of fencing with which to enclose three adjacent rectangular corrals. See figure below. What dimensions should be used so that the enclosed area will be a maximum?



$P = 2x + 4y = 800$ feet

$A = x \cdot y \to \max$

$2x + 4y = 800$

$x + 2y = 400$

$x = 400 - 2y$

$x > 0$

$y > 0$

y in $[0, 200]$

$A(y) = (400 - 2y)y \to \max$

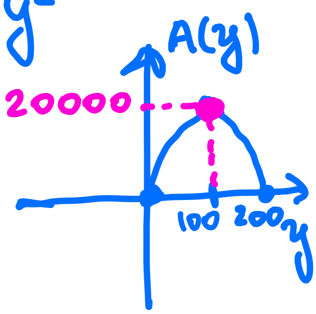
$A(y) = 400y - 2y^2$

1. Find CP of $A(y)$.

$A'(y) = 400 - 4y = 0$

$400 - 4y = 0$

$y = \frac{400}{4} = 100$ CP



$A(y) = 0$

$400y - 2y^2 = 0$

$y(400 - 2y) = 0$

$y = 0$ or

$400 - 2y = 0 \Rightarrow y = 200$

$A(100) = (400 - 2 \cdot 100) \cdot 100 = 200 \cdot 100 = 20000$

$A(0) = 0$, $A(200) = 0$

Justification:

II Der. Test

(a) $A'(100) = 0$ ✓

(b) $A''(100) < 0$ ✓

$A''(y) = -4 < 0$ for all y values

Therefore, at $y = 100$ our function $A(y)$ attains its loc. max value.

Furthermore, at $y = 100$ $A(y)$ attains its global max,

$\boxed{y = \underline{100}}$ (ft)

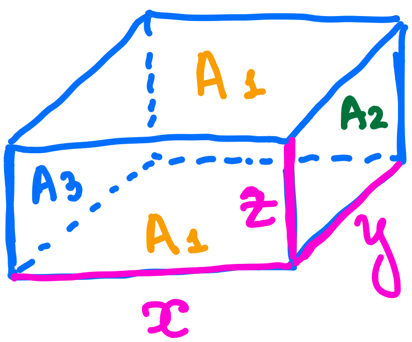
$x = 400 - 2y = 400 - 2 \cdot 100 = \underline{\underline{200}}$ (ft)

Answer: $x = 200$, $y = 100$

$A_{max} = 200 \cdot 100 = 20\,000\ (ft)^2$

SECTION 4.7 APPLIED OPTIMIZATION (DAY 2)

1. A rectangular storage container with lid is to have a volume of 36 cubic inches. The length of the base is three times the width. Material for the base and lid costs \$4 per square inch. Material for the sides costs \$1 per square inch. Find the cost of materials for the least expensive container.



$V = 36\ (\text{inch.})^3$

$x \cdot y \cdot z = 36$

$x = 3y$

A_1 is an area of lid and base

A_2 is an area of the yz side of the box

A_3 is an area of the xz side of the box

$A_1 = x \cdot y$

$A_2 = y \cdot z$

$\Rightarrow$ Total cost function is

$$C = 4 \cdot 2 \cdot xy + 1 \cdot 2 \cdot yz + 1 \cdot 2 \cdot xz$$

top and bottom

2 sides

$$C = 8 \cdot 3y \cdot y + 2 \cdot \frac{36}{3y} + \cancel{3} \cdot \frac{36}{\cancel{3}y} \cdot 2$$

$$y \cdot z = \frac{36}{x} = \frac{36}{3y}$$

Therefore,

$$C(y) = 24y^2 + \frac{24}{y} + \frac{72}{y}$$

$$\boxed{C(y) = 24y^2 + \frac{96}{y}} \rightarrow \text{min}$$

Domain of $C(y)$ is $(0,\infty)$.

1. Find CP of $C(y)$.

$$C'(y) = 48y - \frac{96}{y^2} = 0$$

$$48y^3 = 96$$

$$y^3 = 2 \Rightarrow \boxed{y = \sqrt[3]{2}} \text{ CP}$$

is this a min?

Justification:

$$C''(y) = 48 + \frac{96 \cdot 2}{y^3} > 0$$

$C''(y)$ is positive when y is positive $(y > 0)$.

Since the domain of $C(y)$ is $(0,\infty)$, we have that $C''(y) > 0$. Therefore, at $y = \sqrt[3]{2}$ the cost function $C(y)$ attains loc. min.

But $C(y)$ is concave upward on

$(0,\infty)$. Hence, $C(y)$ has a global maximum at $y = \sqrt[3]{2}$.

Answer: $y = \sqrt[3]{2}$, $x = 3\sqrt[3]{2}$,

$$z = \frac{36}{xy} = \frac{36}{3\sqrt[3]{2}\cdot\sqrt[3]{2}}$$

are dimensions of the box that minimize the cost.