Final Exam Review Spring 2021

One page sheet of paper
(8 ½ in. × 11 in.)

5. (10 points)

Differentiate the following functions.

a. $y = \cos\left(e^{\sin x}\right)$ Chain Rule:

$$y' = -\sin\left(e^{\sin x}\right) \cdot e^{\sin(x)} \cdot \cos(x)$$

$$y' = -\sin\left(e^{\sin(x)}\right) \cdot \left(e^{\sin(x)}\right)' =$$

$$= -\sin\left(e^{\sin(x)}\right) \cdot e^{\sin(x)} \cdot \cos(x)$$

b. $g(t) = \dfrac{1 - 2t^5}{1 + \tan t}$ Quotient Rule:

$$g'(t) = \frac{-10t^4 \cdot (1+\tan t) - \sec^2 t \cdot (1-2t^5)}{(1+\tan t)^2}$$

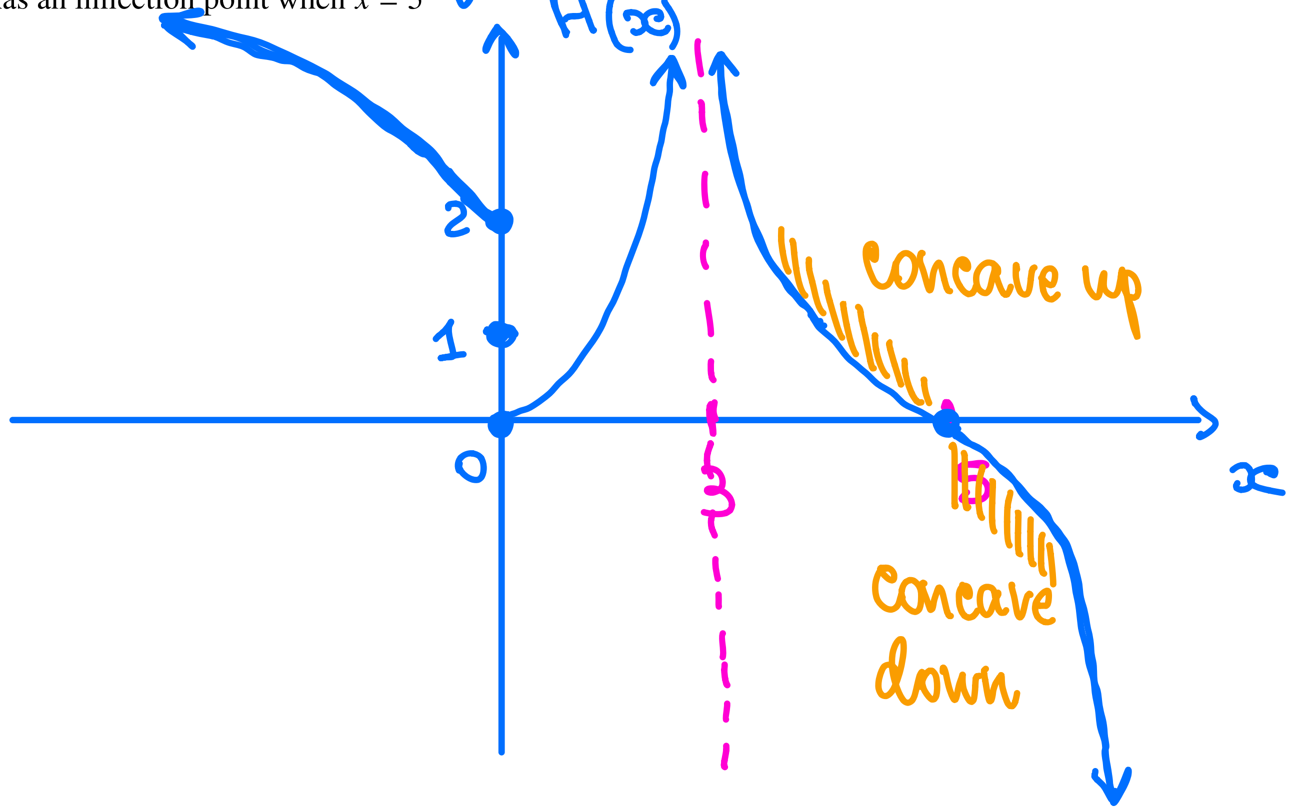
$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - g' \cdot f}{g^2}$$

~~6. (5 points)~~

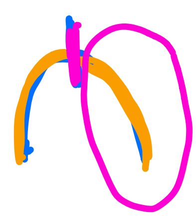
Find $\dfrac{dy}{dx}$ by implicit differentiation: $e^y + \dfrac{1}{x} = xy + 2y.$

1. **(10 points)**
Sketch a graph $H(x)$ with all of the properties below. Label your graph.

- The domain of $H(x)$ is $(-\infty, 3) \cup (3, \infty)$. $x = 3$ ✓
- $H(0) = 1$ ✓
- $\lim_{x \to 0^-} H(x) = 2$ ✓
- $\lim_{x \to 0^+} H(x) = 0$ ✓
- $\lim_{x \to 3^{\pm}} H(x) = \infty$ ✓
- $H'(x) < 0$ and $H''(x) < 0$ on the interval $(-\infty, 0)$ ✓
- H has an inflection point when $x = 5$ ✓



$H'(x) < 0$ – decreasing
$H'(x) > 0$ – increasing



$H''(x) < 0$ – concave down
$H''(x) > 0$ – concave up

$x = a$ is an inflection point if f is changing its concavity at $x=a$.

7. (10 points)

Find the limit or show that it does not exist.

a. $\lim_{x\to\infty} \frac{\sqrt{x^2+3}}{7x}$

b. $\lim_{t\to 1} \frac{t^8-1}{t^5-1}$

8. (10 points)

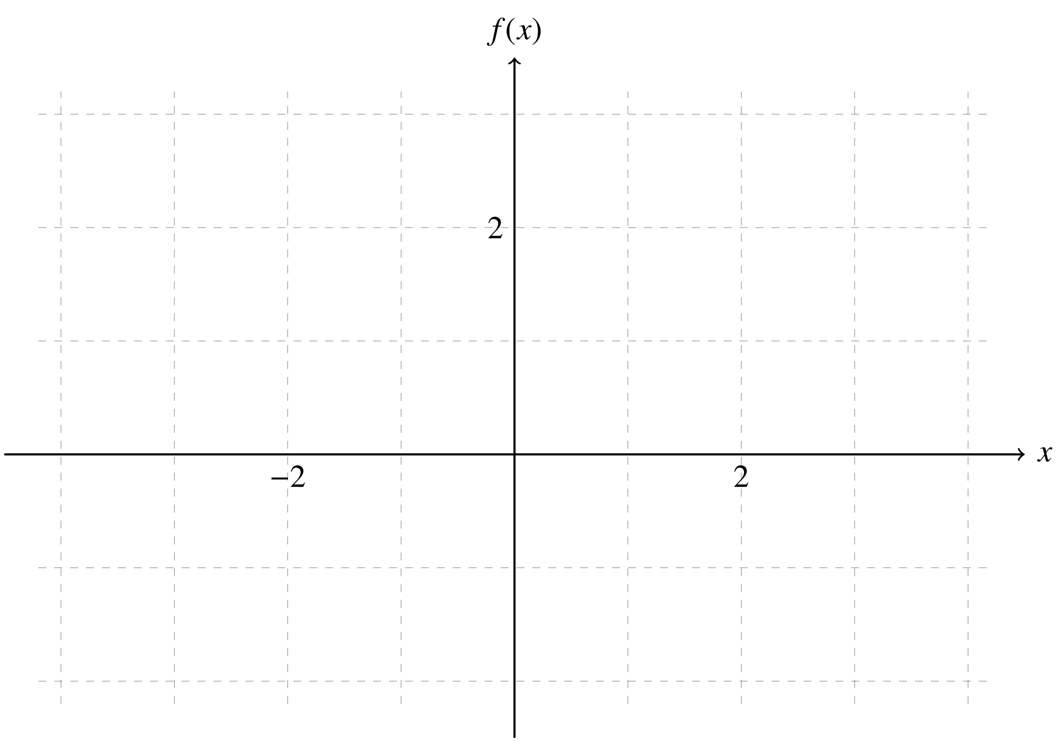
On the axes below, sketch the graph of a function that satisfies **all** of the given conditions:

a. $f(1) = 0$,

b. $f'(x) > 0$ if $x < -2$ and $f'(x) < 0$ if $x > -2$,

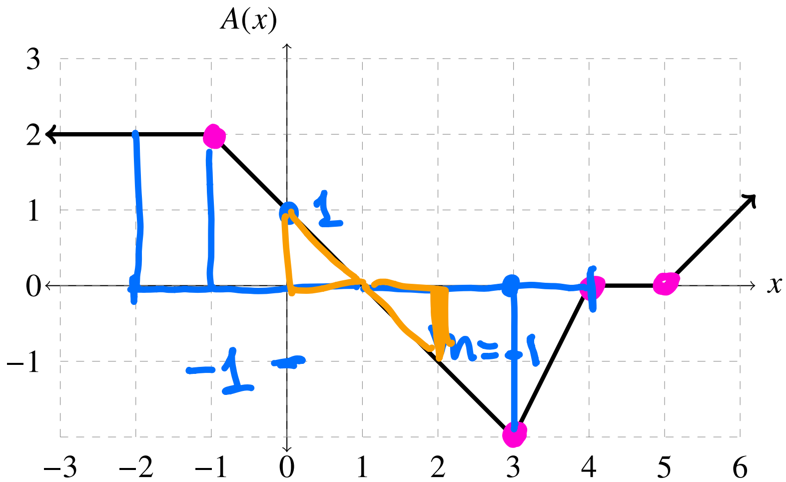
c. $f''(x) < 0$ if $x < 0$ and $f''(x) > 0$ if $x > 0$,

d. there is a vertical asymptote at $x = 0$.



3. (10 points)

The function $A(x)$ is graphed below.



a. $A(0) =$ 1

b. $A'(0) =$ -1

c. At what x values, if any, does $A'(x)$ not exist?

$x = -1, 3, 4, 5$

d. By using your knowledge of areas, evaluate $\int_{-2}^{4} A(x)\,dx.$ $= 1$

$2 + \cancel{2} - \cancel{2} - 1 = 1$

For parts (e)-(g), let $H(x) = \int_0^x A(s)\,ds.$ H(x) is a varied area

e. What is the value of $H(2)$?

$H(2) = \int_0^2 A(s)\,ds = 0$

f. What is the value of $H'(2)$?

FTC part 1

$H'(x) = A(x)$, $H'(2) = A(2) = -1$

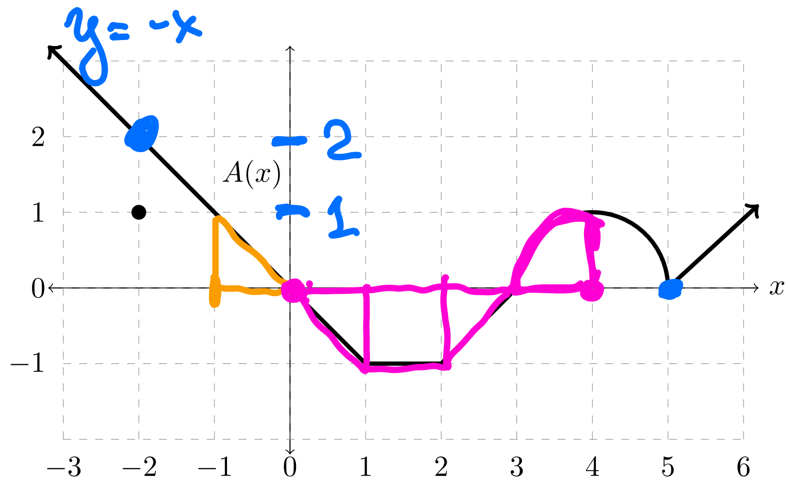
g. Where on the interval $[0, 6]$ is $H(x)$ decreasing?

$H(x)$ is ↘ on $[0,6]$, if $H'(x) < 0$ on $[0,6]$

$H'(x) = A(x) < 0$ on $(1, 4)$

FTC part 1 : if $f(x) = \int_0^x g(s)\, ds$, then $f'(x) = g(x)$

4. (10 points) Consider the function $A(x)$ graphed below. Between $x = 3$ and $x = 5$, the graph is of a semicircle of radius 1.



(a) $\lim_{x \to -2} A(x) =$ $\boxed{2}$

(b) $A(-2) =$ 1

(c) $A'(-1) =$ -1

(d) At what x values, if any, does $A'(x)$ not exist?

$x = -2, 1, 2, 3, 5$

(e) Evaluate $\int_{-1}^{2} A(x)\, dx$. $= \frac{1}{2} - \frac{1}{2} - 1 = -1$

(f) Let $H(x) = \int_0^x A(s)\, ds$. What is the value of $H(4)$?

$H(4) = \int_0^4 A(s)\, ds = -2 + \frac{1}{4}\pi \cdot 1^2$

(g) For $H(x)$ from part **f.**, what is the value of $H'(4)$.

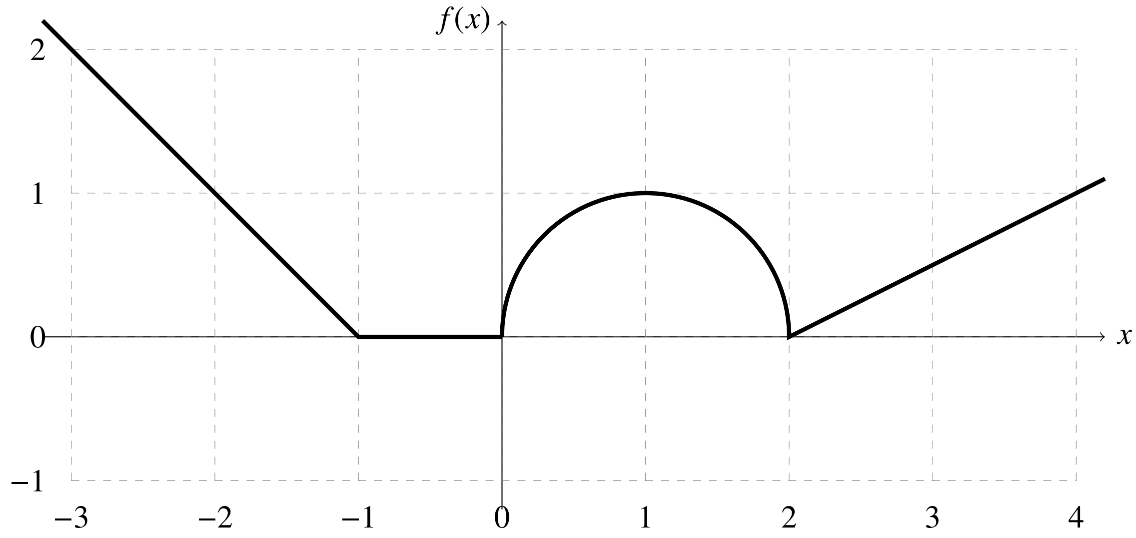
FTC part 1

$H'(x) = A(x)$

$H'(4) = A(4) = 1$

4. (15 points)

Consider the function $f(x)$ graphed below. Between $x = 0$ and 2, the graph is of a semicircle of radius 1.



a. At what x values, if any, does $f'(x)$ not exist?

b. What is the value of $f'(-2)$?

c. Evaluate $\int_{-1}^{4} f(x)\,dx$.

d. Let $g(x) = \int_{1}^{x} f(s)\,ds$. What is the value of $g(0)$?

e. For $g(x)$ from part **d.**, what is the value of $g'(4)$.

1. (10 points)

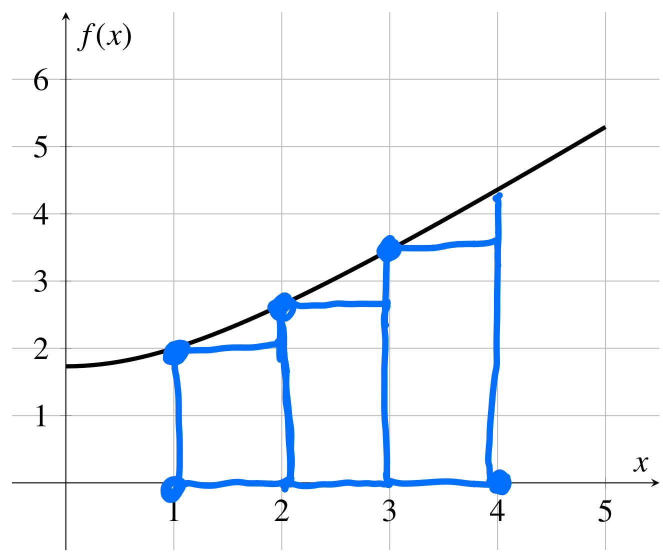
~~Find an equation of the tangent line to the curve at $x = e$:~~ $\quad y = x^2 \ln x$

2. (10 points)

The graph of the function $f(x) = \sqrt{x^2+3}$ is shown.

a. On the graph sketch 3 rectangles, using left endpoints, that would be used to approximate

$$\int_1^4 \sqrt{x^2+3}\,dx.$$



(width) $\Delta x = \dfrac{b-a}{3}$

$\Delta x = \dfrac{4-1}{3} = 1$

b. Compute the approximation in part **(a)**. You do not need to simplify, but your answer should be in a form where a calculator would compute a numerical value.

Riemann Sum $A = 1 \cdot f(1) + 1 \cdot f(2) + 1 \cdot f(3) =$

$= 1 \cdot \sqrt{1^2+3} + 1 \cdot \sqrt{2^2+3} + 1 \cdot \sqrt{3^2+3} =$

$= \sqrt{4} + \sqrt{7} + \sqrt{12}$

7. (15 points)

A particle moves so that its velocity (in m/sec) at time t sec is

$$v(t) = t^2 + 7.$$

a. What is the average rate of change of the velocity from time $t = 2$ to $t = 3$? Simplify, and give units.

$$\boxed{\frac{f(b)-f(a)}{b-a}}$$

$$\frac{\Delta v}{\Delta t} = \frac{v(3)-v(2)}{3-2} = \frac{(3^2+7)-(2^2+7)}{3-2} = 5\ (m/sec)$$

~~**b.**~~ Using the limit definition of the derivative, compute $v'(2)$. (No credit will be given for using a different method to compute the derivative.)

At what rate the $v(t)$ is changing at $t=3$?

$v'(t) = 2t$ $\quad$ $v'(3) = 6\ (m/sec^2)$

5. (15 points)

The temperature of an oven in °F is

$$T(t) = 150 + 30t^2$$

for $t = 0$ to $t = 3$ minutes.

a. Find the average rate of change of temperature in the oven from time $t = 0$ to $t = 3$. Include units in your answer.

$$\frac{\Delta T}{\Delta t} = \frac{T(3)-T(0)}{3-0} = \frac{(150+30\cdot 3^2)-(150+\cancel{30\cdot 0^2})}{3} = \frac{270}{3} = 90 \left({}^\circ F/_{min}\right)$$

b. It is easy to compute that $T'(2) = 120$. What does this mean in everyday language? (Be sure to include units in your answer.)

c. Using the limit definition of the derivative, compute $T'(1)$. (No credit will be granted for using other methods to compute the derivative.)

b. At what rate is the temperature changing at time $t=0$?

~~c. At what time is the temperature at a maximum?~~

b. $T'(t) = 60t$

$T'(0) = 0 \ (°F/min)$

10. **(15 points)**

Water flows from a tank at a rate of $r(t) = 3t^2 - t^3$ liters per minute from $t = 0$ to $t = 3$ minutes.

a. Compute $r(0)$, $r(1)$ and $r(3)$, and explain what these quantities mean in everyday language. Your answer should include units.

$r(0) = 0$ (litirs/min) no rate (~~~~)

$r(1) = 2$ (liters/min) at rate 2 l/min

$r(3) = 3\cdot 9 - 27 = 0$ (liters/min) no rate ~~~~

b. Compute the total amount of water that drains from the tank from time $t = 0$ to $t = 3$.

Net change theorem

$$A = \int_0^3 (3t^2 - t^3)\,dt = \left(\frac{3t^3}{3} - \frac{t^4}{4}\right)\Big|_0^3 =$$

$$\int_a^b r(t)\,dt = \underbrace{R(b) - R(a)}_{\text{net change}}$$

$$= \left(\frac{3\cdot 3^3}{3} - \frac{3^4}{4}\right) = 27 - \frac{81}{4} \text{ (liters)}$$

~~**c.** At what time is the rate of flow at a maximum? (Only consider t in the interval $[0, 3]$.)~~

c. If at time $t=0$ the amount of water was 5 liters. What will be the amount of water at $t=3$?

$$A(3) = 5 + \int_0^3 (3t^2 - t^3)\,dt =$$

$$= 5 + 27 - \frac{81}{4}$$

Net change theorem:

$$\int_0^3 r(t)\,dt = \underbrace{R(3)}_{\text{amount of water at } t=3} - \underbrace{R(0)}_{\text{amount of water at } t=0}$$

$$R(3) = R(0) + \int_0^3 r(t)\,dt$$

2. (10 points) During a storm, snow is falling on a mountain at a rate of

$$r\ \cancel{M}(t) = t^2 - \frac{t^3}{3}$$

feet per hour for a three hour period starting at time $t = 0$.

(a) Determine the *net change* in the height of snow during the first two hours of the storm. Include units with your answer.

Net change Theorem

$$\int_a^b r(t)\,dt = \underbrace{R(b) - R(a)}_{\text{net change}}$$

$$\int_0^2 \left(t^2 - \frac{t^3}{3}\right) dt = \left(\frac{t^3}{3} - \frac{t^4}{12}\right)\Big|_0^2 = \frac{2^3}{3} - \frac{2^4}{12}$$

(b) Determine the height of the snow on the mountain or explain why this is not possible with the present information.

height at $t = 2$?

$$h(2) = h(0) + \int_0^2 \left(t^2 - \frac{t^3}{3}\right) dt$$

~~(c) Observe that $M(2.5) > 0$ and $M'(2.5) < 0$. Explain what these two facts indicate about the snow falling when $t = 2.5$.~~

What is the $h(3)$ - ?

$$h(3) = h(0) + \int_0^3 r(t)\,dt$$

Let $h(0) = 2$ ft

$$\int_0^3 \left(t^2 - \frac{t^3}{3}\right) dt = \boxed{\frac{3^3}{3} - \frac{1}{3} \cdot \frac{3^4}{4} + 2}$$

7. (10 points)

Evaluate the integrals below. Note that these problems will be graded **largely** by the quality of the work written. So make sure to include proper notation and compete steps.

a. $\int \sin(2x) + \frac{(1+\ln x)^2}{x} dx$ $= \int \sin(2x)\,dx + \int \frac{(1+\ln x)^2}{x}dx =$

$2x = u$

$du = 2\,dx$

$dx = \frac{du}{2}$

$= \frac{1}{2}\int \sin(u)\,du + \int u^2\,du =$

$= -\frac{1}{2}\cos(u) + \frac{u^3}{3} + C =$

$u = 1+\ln x$

$du = \frac{1}{x}dx$

$dx = du \cdot x$

$= -\frac{1}{2}\cos(2x) + \frac{(1+\ln x)^3}{3} + C$

b. $\int_0^2 (1 + xe^{\pi x^2})dx$ $= \int_0^2 1\,dx + \int_0^2 xe^{\pi x^2}dx =$

$u = \pi x^2$

$du = 2\pi x\,dx$

$dx = \frac{1}{2\pi x}du$

$0 \le x \le 2$

$0 \le u \le 4\pi$

$= 2 + \int_0^{4\pi} \cancel{x} \cdot e^u \cdot \frac{du}{2\pi \cancel{x}} =$

$= 2 + \frac{1}{2\pi}\int_0^{4\pi} e^u\,du =$

$= 2 + \frac{1}{2\pi} e^u \Big|_0^{4\pi} =$

8. (15 points)

Evaluate the integrals. For full credit, include a constant of integration whenever one would be justified.

a. $\displaystyle\int \sin^5(x)\cos x\,dx =$

b. $\displaystyle\int_1^3 2e^x + \frac{1}{x}\,dx =$

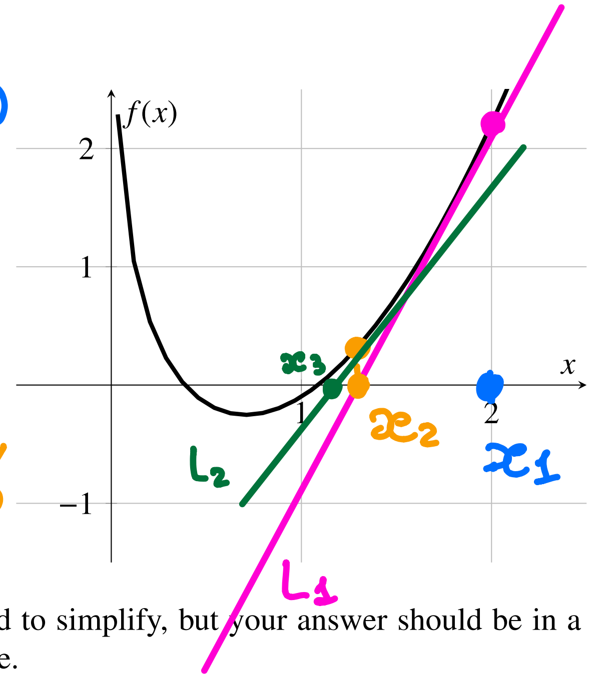
c. $\displaystyle\int \sqrt{x}\left(x^2 - x^{1/4} + \pi^2\right)\,dx =$

V

$x^2 - \ln(3x) = 0$

11. **(10 points)**

The graph of the function $f(x) = x^2 - \ln(3x)$ is shown.



a. Suppose Newton's method is used to find an approximate solution to $f(x) = 0$ from an initial guess of $x_1 = 2$. Sketch on the graph how the next approximation x_2 will be found, labeling its location on the x-axis.

$f'(x) = 2x - \frac{1}{\cancel{3}x} \cdot \cancel{3}$

b. For $x_1 = 2$, give a formula for x_2. You do not need to simplify, but your answer should be in a form where a calculator would compute a numerical value.

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$

$x_2 = 2 - \frac{2^2 - \ln(6)}{4 - \frac{1}{2}} \approx \ldots$

X **c.** What value of x_1 might you use if you wanted to find the **smaller** solution of $f(x) = 0$?

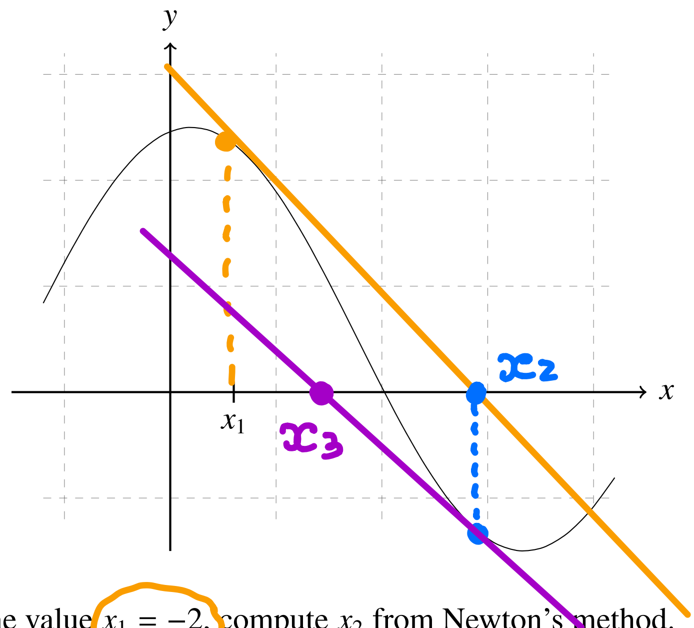
Extra Credit. **(3 points)**

Compute the following integral by interpreting it as an area:

$$\int_0^4 \sqrt{4 - (x-2)^2}\, dx$$

11. (10 points)

a. A generic graph $y = f(x)$ is shown and a first approximation x_1 is indicated. Show, by adding to the sketch, how Newton's method would find the next approximation x_2.



b. For the equation $x^3 - 4x + 2 = 0$ and the value $x_1 = -2$, compute x_2 from Newton's method.

12. (Extra Credit: 5 points)

Find **and simplify** the derivative of the function:

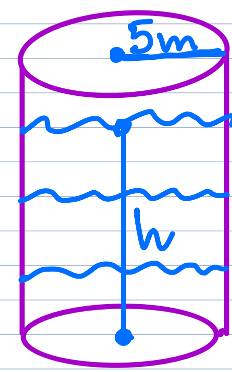
$$h(x) = \int_1^{e^x} \ln t \, dt$$

Explain your steps.

Related Rates Problem

A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/min. How fast is the height of the water $\uparrow$?

Solution:



$r = 5m$

$\frac{dV}{dt} = 3 \ m^3/min$

$\frac{dh}{dt} = \ ?$

$V(t), \ h(t)$

constant

$V = \pi r^2 \cdot h$

$$\frac{dV}{dt} = \pi r^2 \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{1}{\pi r^2}$$

$$\frac{dh}{dt} = 3 \cdot \frac{1}{\pi \cdot 5^2} = \frac{3}{25\pi} \left(m/min \right)$$

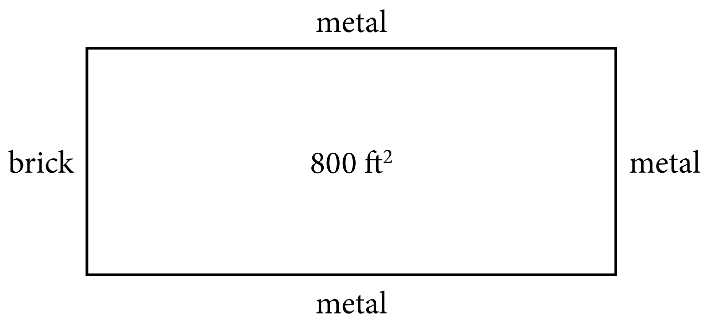
10. (10 points) Newton's method can be used to find an approximate solution to the equation $x^2 = 8$. To apply Newton's method to find these roots, let $f(x) = x^2 - 8$.

(a) Use Newton's method with initial approximation $x_0 = 2$ to find x_1, a better estimate of a root of the given equation.

(b) Apply one more iteration of Newton's method to find x_2.

(c) Notice the equation $x^2 = 8$ has two roots. What value of x_0 would make a good choice to find the **other** root?

6. (10 points) A landscape architect wishes to enclose a rectangular garden on one side by a brick wall costing \$30 per foot and on the other three sides with a metal fence costing \$10 per foot. The area of the garden is to be 800ft^2. What are the dimensions of the garden that minimize the cost of the fencing? (For full credit, you must justify your answer.)



- Related Rates Problems:

  Textbook: $\boxed{\text{Section 3.9 (\# 5,6).}}$

- Optimization (p. 23)
- Newton's method (p. 18)

Optimization Problem

2 (8 points) A rectangular field adjacent to a river is to be enclosed. Fencing along the river costs \$ 5 per meter, and the fencing for the other sides costs \$ 3 per meter. The area of the field is to be 1200 square meters. Find the dimensions of the field that is the least expensive to enclose. What is the minimum cost?

(a) Write a formula that determines the **cost** of the fencing as a function of x and y.

$$A = 1200\ m^2$$

$$C \to \min$$

(b) Solve the problem. Indicate **units** in your answer. Justify that the solution you found really is a solution.

$$\boxed{x, y > 0}$$

$$C = 3 \cdot 2y + 5 \cdot x = 6y + 5x \to \min$$

$$A = x \cdot y = 1200$$

$$y = \frac{1200}{x}$$

$$C(x) = 6 \cdot \frac{1200}{x} + 5x \to \min$$

$$C'(x) = -6 \cdot \frac{1200}{x^2} + 5 = 0$$

$$\frac{6 \cdot 1200}{x^2} = 5 \Rightarrow x^2 = \frac{\cancel{5}}{6 \cdot 1200}$$

2

$$\boxed{x_* = \frac{1}{12\sqrt{10}}}$$

(m)

Dimensions: $x =$ ________ $y =$ ________

Total minimum cost: $6 \cdot 1200 \frac{12\sqrt{10}}{1} + 5 \cdot \frac{1}{12\sqrt{10}}$

Justification: (Second Derivative Test)

if $C''(x_*) > 0$, then at x_* we have a loc. min

$$C''(x) = 2 \cdot 6 \cdot \frac{1200}{x^3} > 0$$

$$x \to 0$$

At $\boxed{x_* = \frac{1}{12\sqrt{10}}}$ we have a global min.

$$y = 1200 \cdot \frac{12\sqrt{10}}{1} \quad (m)$$

Problem

$$\boxed{g(x) = \int_{x^2}^{x} \sin(t)\, dt}$$

$g'(x) - ?$

FTC Part 1

$$g(x) = \int_a^x f(s)\, ds$$

$$g'(x) = f(x)$$

$$\int_a^c + \int_c^b$$

a c b

$$\int_c^b$$

u

$$g(x) = \int_{x^2}^{x} \sin(t)\,dt = \int_{x^2}^{a} \sin(t)\,dt + $$

$$+ \int_{a}^{x} \sin(t)\,dt = -\int_{a}^{x^2} \sin(t)\,dt + $$

$$+ \int_{a}^{x} \sin(t)\,dt$$

$$\boxed{g'(x) = -\sin(x^2)\cdot 2x + \sin(x)}$$