Section 4.7. Optimization Problems

Steps in solving Optimization Problems:

$\boxed{1.}$ Understand the problem

$\boxed{2.}$ Draw a diagram

$\boxed{3.}$ Introduce notations

$\boxed{4.}$ Express f in some of other symbols.

$\boxed{5.}$ If f has been expressed as a function of more than one variable in step 4, use the given information to find relationships among these variables.
Then as a final result you should get $f = f(x)$.

$\boxed{6.}$ Use the already known methods to find absolute max or min value of f.

If the domain of f is a closed interval, then the Closed Interval Method can be used.

Example:

We are given 2400 ft of fencing.

We want to maximize the area:

$A \to \max$

$$A = x \cdot y \to \max$$

$$P = 2x + y = 2400$$

$$y = 2400 - 2x$$

$$A = x(2400 - 2x) = 2400x - 2x^2$$

$$A'(x) = 2400 - 4x = 0$$

$$\underline{x = 600 (ft)}$$

$A''(x) = -4 < 0$ (concave downward for all x)

$$y = 2400 - 1200 = \underline{1200 \text{ ft}}$$

Alternative Approach:

First Derivative Test for Absolute Extreme Values:

Suppose that c is a critical number of a continuous function f defined on an interval.

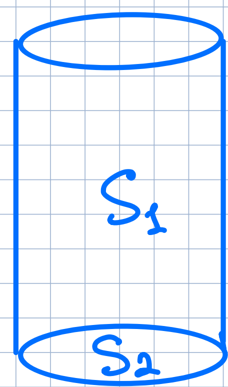
- if $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f
- if $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

Alternative Approach:

Implicit Differentiation:

Example 2

A cylindrical can is to be made to hold 1L of oil. Find the dimensions that will maximize the cost of the metal to manufacture the can.



$V = 1\ L$

$V = \pi R^2 h = 1000$

$A = 2S_2 + S_1$

$A = 2\pi R^2 + 2\pi R h$

$A' = 4\pi R + 2\pi R h' + 2\pi h$

$V' = 0 = 2\pi R h + \pi R^2 h'$

$A' = 0$

$2R + h'R + h = 0$

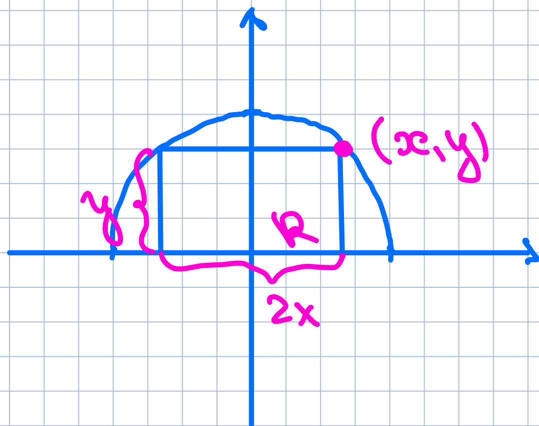
$Rh' + 2h = 0$

$\Downarrow$

$2R - h = 0$ or $\underline{h = 2R}$

Example 3

Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.



$x^2 + y^2 = r^2$

$A = 2x \cdot y$

$y = \sqrt{R^2 - x^2}$

$A = 2x\sqrt{R^2 - x^2}$

$0 \le x \le R$

$$A' = \frac{2(R^2 - 2x^2)}{\sqrt{R^2 - x^2}} = 0 \quad \text{when} \quad R^2 = 2x^2$$

$$x = \frac{R}{\sqrt{2}} \quad (x \geq 0)$$

The largest area is:

$$A\left(\frac{R}{\sqrt{2}}\right) = 2\frac{R}{\sqrt{2}}\sqrt{R^2 - \frac{R^2}{2}} = R^2.$$

Application to Business and Economics:

$C(x)$ - cost function, x - # of units

$C'(x)$ - rate of change of cost

$P(x)$ - price function (demand function)

$R(x)$ - quantity $\times$ price $= x\,p(x)$ - revenue function

$R'(x)$ – marginal revenue function

$P(x) = R(x) - C(x)$ – profit function

$P'(x)$ – marginal profit function.