Section 4.1. Maximum and Minimum Values

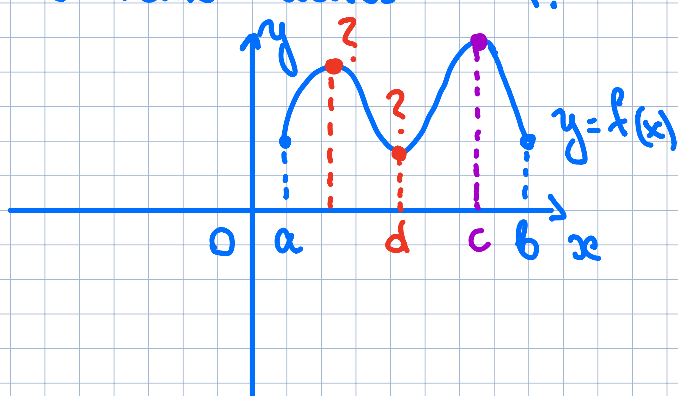
Motivation: Optimization problems

Def. Let c be a number in the domain D of a function f. Then $f(c)$ is the

- abs max value of f on D if
  $f(c) \geq f(x)$ for all x in D
- abs min value of f on D if
  $f(c) \leq f(x)$ for all x in D

Abs max or min = global max or min

Max and Min values of f are called extreme values of f.



abs min $f(d)$

abs max $f(c)$

Def The number $f(c)$ is a

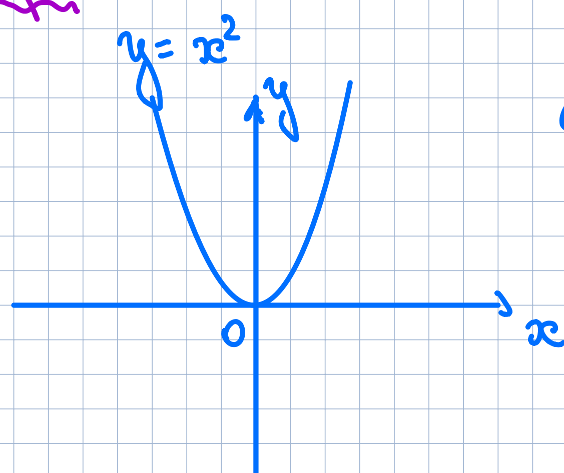
- local maximum value of f if

$$f(c) \geq f(x) \text{ when } x \text{ is near } c$$

- local minimum value of f if

$$f(c) \leq f(x) \text{ when } x \text{ is near } c$$

Example

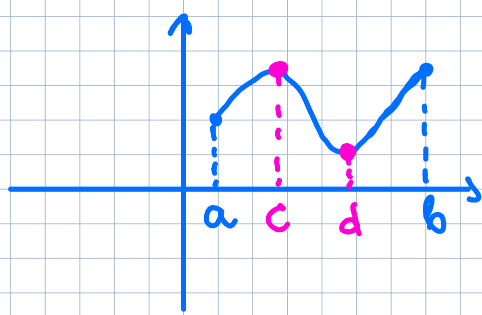


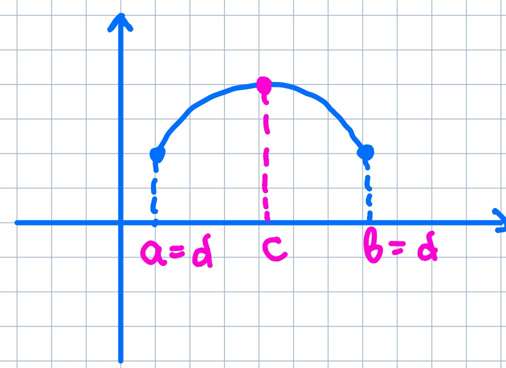
abs min = loc min = $f(0)$

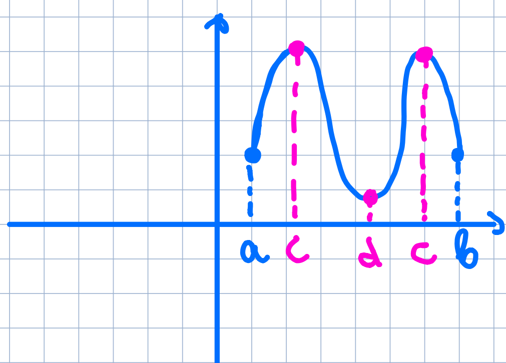
y has no max value

The Extreme Value Theorem

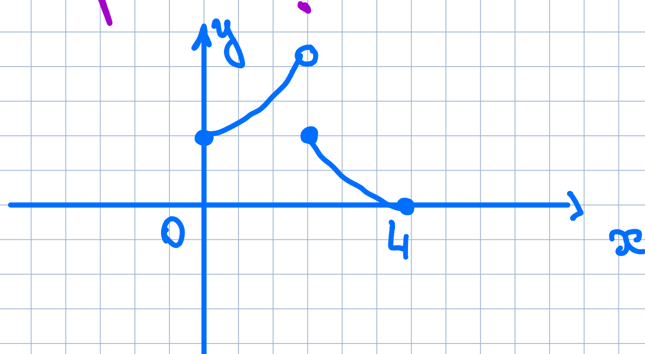
If f is continuous on a closed interval $[a,b]$, then f attains an abs max value $f(c)$ and an abs min value $f(d)$ at some numbers c and d in $[a, b]$.



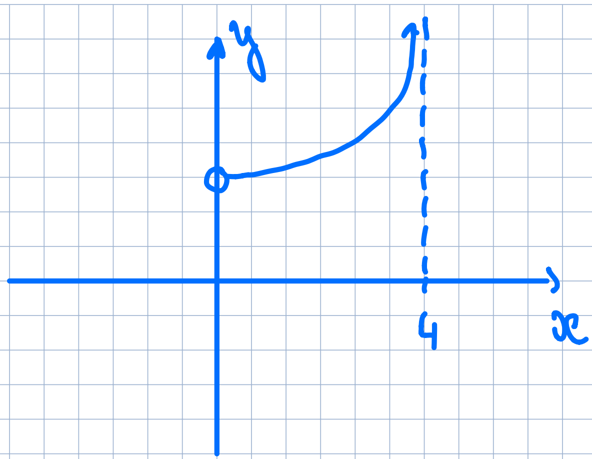




Why continuity and closeness are important?

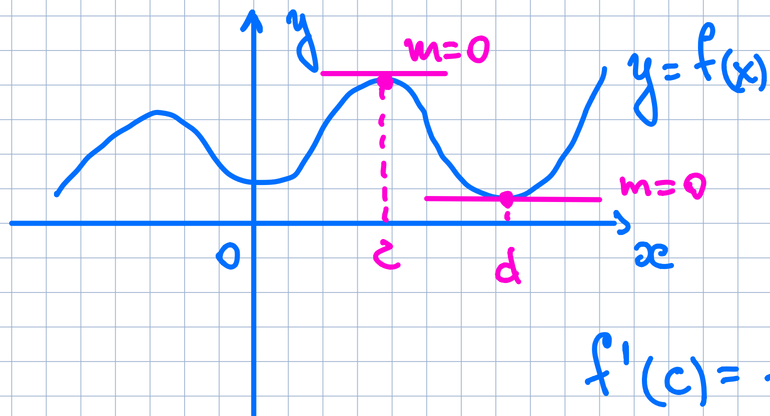


$f(4)$ has a min value but it has no max value.



$y = f(x)$ has no max or min

How to find local max or min values?



$$f'(c) = f'(d) = 0$$

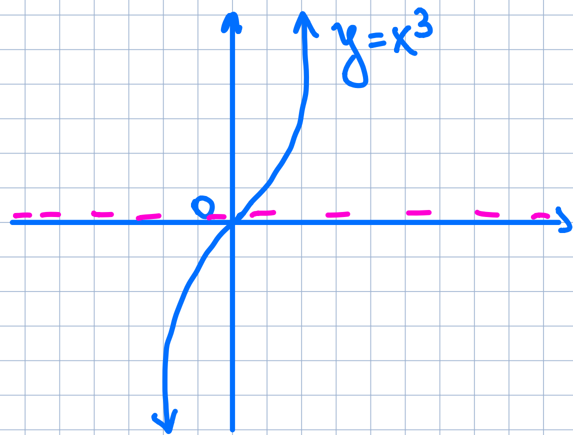
Fermat's Theorem

If f has a local max or min at c, and if $f'(c)$ exists, then

$$f'(c) = 0$$

WARNING:

1) Let us consider a function $y = x^3$.



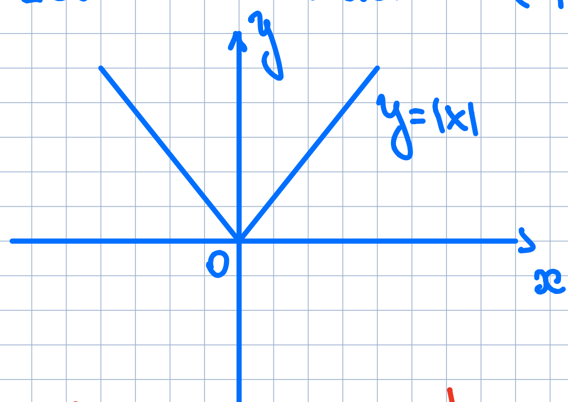
$y' = 3x^2 = 0 \iff x = 0$

$f'(0) = 0$

- f has no max or min at $x=0$
- f has a horizontal tangent at $(0,0)$ and crosses it

Conclusion: even if $f'(c)=0$ there need not to be a max or min at c.

2) Let us consider $f(x) = |x|$



Conclusion: we have an abs min at $x=0$, but $f'(0)$ DNE.

Fermat's Theorem suggestions:

start looking for extreme values of f at c where $f'(c)=0$ or $f'(c)$ DNE.

Def. A critical number of a function f is a number c in the domain of f such that either

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ DNE}$$

Proposition

If f has a local max or min at c, then c is a critical number of f.

Abs max or min appears at:

- loc max or min
- end points of a closed interval

The closed interval method:

To find the absolute max and min values of a continuous function f on a closed interval $[a, b]$:

1. find the values of f at the critical points of f in (a, b)
2. find the values of f at the endpoints of the interval
3. take max $\{$ step 1, step 2 $\}$ = abs max value

   take min $\{$ step 1, step 2 $\}$ = abs min value

Example Find abs max and min?

$$f(x) = x^3 - 3x^2 + 1, \quad x \text{ in } [-\tfrac{1}{2}, 4]$$

Solution

f is continuous on $[-\frac{1}{2}, 4]$.

$f'(x) = 3x^2 - 6x = 0 \Rightarrow x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0$ or $x = 2$

$f(0) = 1$

$f(2) = -3$

$f(-\frac{1}{2}) = \frac{1}{8}$

$f(4) = 17$

$\Longrightarrow$ Answer:

Abs max value is $f(4) = 17$

Abs min value is $f(2) = -3$.

$\blacksquare$