Section 5.5. The Substitution Rule

Examplee 1

$$\int \left(e^{x} + \sin(x)\right) dx = \int e^{x}\, dx + \int \sin(x)\, dx =$$

$$= \boxed{e^{x} - \cos(x) + C}$$

Examplee 2

$$\int 2x \cdot \sqrt{\underbrace{1+x^2}_{u}}\, dx =$$

Introducing something extra:

- $\boxed{u = 1 + x^2}$
- $dx \to du$

$$\frac{du}{dx} = 2x \quad \Big| \cdot dx$$

$$du = 2x \cdot dx \Rightarrow \boxed{dx = \frac{du}{2x}}$$

The Substitution Rule

$$= \int \cancel{2x} \sqrt{u}\, \frac{du}{\cancel{2x}} = \int \sqrt{u}\, du =$$

$$= \int u^{1/2}\, du = \frac{u^{3/2}}{3/2} + C = \boxed{\frac{(1+x^2)^{3/2}}{3/2} + C}$$

$u = g(x) = 1+x^2$ $\quad \sqrt{x} = f(x)$

$\sqrt{1+x^2} = f(g(x))$

$$\int f(g(x)) \cdot g'(x)\, dx =$$

$$= \int f(u)\, du$$

The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du$$

$$\int 2x\sqrt{1+x^2}\, dx = \int \sqrt{u}\, du$$

$f = \sqrt{\underbrace{1+x^2}_{g(x)}}$ $\quad g'(x) = 2x$ $\quad u = g(x)$

$du = g'(x)dx$

Example 3

Substitution Rule

$$\int \sqrt{2x+1}\, dx = \underbrace{\int \sqrt{u}\,\frac{1}{2}\,du}_{} =$$

- $\boxed{u = g(x) = 2x+1}$
- $du = g'(x)\,dx = 2 \cdot dx$

$du = 2 \cdot dx$

$dx = \frac{1}{2}\,du$

$$= \frac{1}{2}\int \sqrt{u}\, du = \frac{1}{2}\,\frac{u^{3/2}}{3/2} + C =$$

$$= \boxed{\frac{1}{2}\,\frac{(2x+1)^{3/2}}{3/2} + C}$$

1. Compute $\int t\sin(t^2+1)\,dt$ $\boxed{=}$ $\int \cancel{t}\cdot \sin(u)\cdot \frac{1}{2\cancel{t}}\,du = \frac{1}{2}\int \sin(u)\,du =$

- $u = t^2+1$
- $du = 2t \cdot dt$

$g'(t)$

$dt = \frac{1}{2t}\,du$

$= \frac{1}{2}(-\cos(u)) + C =$

$= \boxed{\frac{1}{2}(-\cos(t^2+1)) + C}$

2. Compute $\int e^{4x-9}\,dx$ $= \int e^u \cdot \frac{1}{4}\,du = \frac{1}{4}\cdot e^{4x-9} + C$

- $u = 4x-9$
- $du = 4\cdot dx$

$dx = \frac{1}{4}\,du$

3. Compute $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$ $= \int \frac{e^u}{\cancel{\sqrt{x}}}\cdot 2\cancel{\sqrt{x}}\,du = 2\int e^u\,du = 2e^u + C =$

- $u = \sqrt{x}$
- $du = \frac{1}{2\sqrt{x}}\cdot dx$

$dx = 2\sqrt{x}\,du$

$= \boxed{2e^{\sqrt{x}} + C}$

4. Compute $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$ $= \int_1^2 \frac{e^u}{\cancel{\sqrt{x}}}\cdot 2\cancel{\sqrt{x}}\,du = 2\int_1^2 e^u\,du \underset{\text{part 2}}{\overset{\text{FTC}}{=}} 2e^u\Big|_1^2 =$

- $u = \sqrt{x}$
- $du = \frac{1}{2\sqrt{x}}\,dx$

$dx = 2\sqrt{x}\,du$

- $1 \le x \le 4$

$\boxed{1 \le u \le 2}$

$= 2e^2 - 2e^1 = 2(e^2 - e) =$

$= \boxed{2e(e-1)}$

5. Compute $\int \frac{\arctan(x)}{1+x^2}\,dx = \int u \cdot du = \frac{u^2}{2} + C = \boxed{\frac{(\arctan(x))^2}{2} + C}$

- $u = \arctan(x)$
- $du = \frac{1}{1+x^2} \cdot dx$

6. Compute $\int \frac{x^3}{\sqrt{1-x^4}}\,dx$

7. Compute $\int \frac{x}{\sqrt{1-x^4}}\,dx$.

8. Compute $\int_0^{\pi/6} \frac{\sin(t)}{(cos(t))^2}\,dt$ two ways: (1) by computing the antiderivative using substitution and then using FTC2 to evaluate using the original bounds; (2) by substituting and changing the bounds to match the substitution.