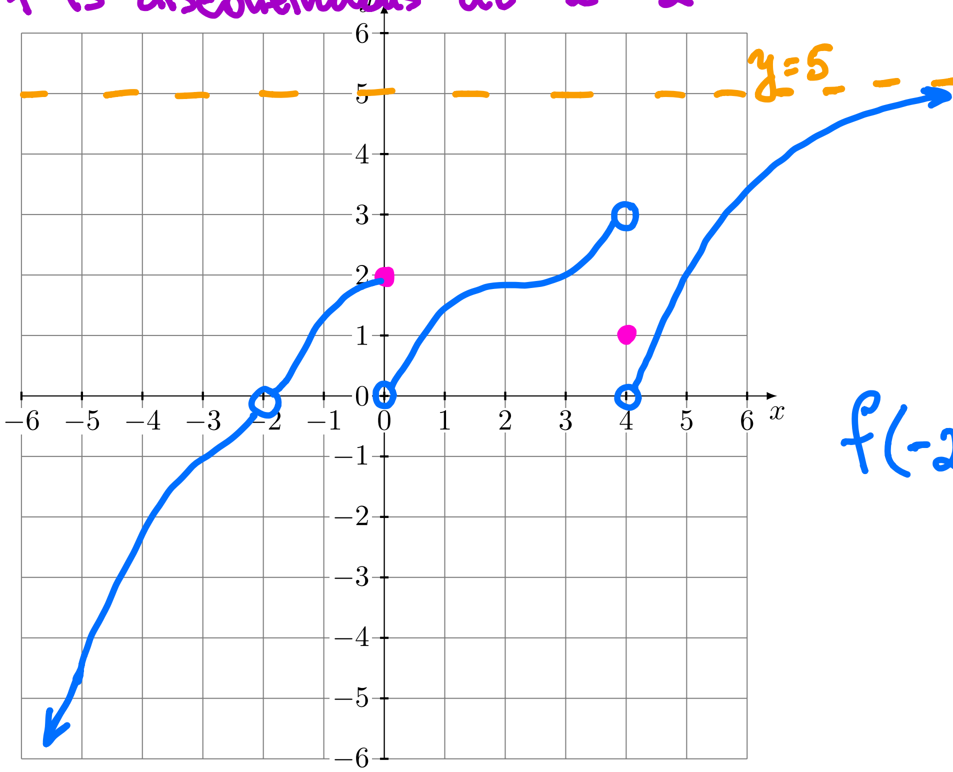
Solutions

Exercise 1. Sketch the graph of an example of a function f that satisfies all of the given conditions:

$$\lim_{x\to 0^-} f(x) = 2, \quad \lim_{x\to 0^+} f(x) = 0, \quad \lim_{x\to 4^-} f(x) = 3, \quad \lim_{x\to +\infty} f(x) = 5,$$
$$\lim_{x\to 4^+} f(x) = 0, \quad f(0) = 2, \quad f(4) = 1, \quad \lim_{x\to -\infty} f(x) = -\infty.$$

f is discontinuous at $x=-2$



$f(-2)$ DNE

Exercise 2. Evaluate the limit, if it exists.

(a) $\displaystyle\lim_{t\to 0} \frac{\sqrt{t^2+16}-4}{t^2}$ = "$\frac{0}{0}$" type Needs algebra

$$= \lim_{t\to 0} \frac{(\sqrt{t^2+16}-4)(\sqrt{t^2+16}+4)}{t^2(\sqrt{t^2+16}+4)} = \lim_{t\to 0} \frac{(\sqrt{t^2+16})^2-16}{t^2(\sqrt{t^2+16}+4)} =$$
$$= \lim_{t\to 0} \frac{\cancel{t^2}+\cancel{16}-\cancel{16}}{\cancel{t^2}(\sqrt{t^2+16}+4)} = \lim_{t\to 0} \frac{1}{\sqrt{t^2+16}+4} = \frac{1}{8}.$$

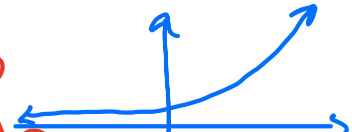
(b) $\displaystyle\lim_{x\to 3^-} \frac{\sqrt{x}}{(x-3)^5}$ =

$$= \frac{\sqrt{3}}{0^-} = -\infty$$

2.9

3

$2.9-3=-0.1$



(c) $\displaystyle \lim_{x\to\infty}\frac{1+e^x}{1+2e^x} = \lim_{x\to\infty}\frac{\frac{1}{e^x}(1+e^x)}{\frac{1}{e^x}(1+2e^x)} = \lim_{x\to\infty}\frac{\overset{0}{\frac{1}{e^x}}+1}{\underset{0}{\frac{1}{e^x}}+2} = \frac{1}{2}$

$x\to\infty \qquad \frac{1}{x}\to 0$

$x\to\infty \qquad e^x\to\infty \qquad \frac{1}{e^x}\to 0$

(d) $\displaystyle \lim_{x\to\infty}\ln\left(\frac{1+e^x}{1+2e^x}\right) =$

contin. $\displaystyle = \ln\left(\lim_{x\to\infty}\frac{1+e^x}{1+2e^x}\right) = \ln\left(\frac{1}{2}\right).$

$y=\ln(x)$ is a continuous function

$y=\ln(x^2)$

$\displaystyle \lim_{x\to a}\ln(x^2) = \ln\left(\lim_{x\to a} x^2\right)$

(e) $\displaystyle \lim_{x\to\infty}\frac{x^2}{\sqrt{x^4+1}} =$

$\displaystyle = \lim_{x\to\infty}\frac{\frac{1}{x^2}(x^2)}{\frac{1}{x^2}\sqrt{x^4+1}} = \lim_{x\to\infty}\frac{1}{\sqrt{\frac{1}{x^4}(x^4+1)}} = \lim_{x\to\infty}\frac{1}{\sqrt{1+\underbrace{\frac{1}{x^4}}_{0}}} = 1$

$\displaystyle \frac{1}{x^4}\xrightarrow[x\to\infty]{} 0$

Exercise 3. In the first few years after a coal mine's operation, the total deposit of coal (in millions of tons) t years after opening is approximately

$$C(t) = 300 - \frac{t^{3/2}}{2}.$$

(a) Find the average rate of change of the amount of coal in the deposit from the opening of the mine to year 4. Include correct units in your answer.

$t_1 = 0$ year

$t_2 = 4$ year

$\displaystyle \frac{\Delta C}{\Delta t} = \frac{C(t_2)-C(t_1)}{t_2-t_1} = \frac{\cancel{300} - \frac{4^{3/2}}{2} - \cancel{300} + \cancel{\frac{0}{2}}}{4-0} = -\frac{4^{3/2}}{8}$ (tons/years)

(b) It is a fact that $C'(t) = -\frac{3}{4}\sqrt{t}$. Compute $C'(4)$ and indicate what this quantity tells us about the mine. Write your answer in a sentence. Again, include correct units in your description.

derivative

$\displaystyle C'(t) = \lim_{t\to a}\frac{C(t)-C(a)}{t-a}$

$\displaystyle C'(4) = -\frac{3}{4}\sqrt{4} = -\frac{3}{4}\cdot 2 = -\frac{3}{2}$ (tons/years)

the amount of coal is decreasing

at rate $\frac{3}{2}$ after $t=4$ years.

Exercise 4. Consider the function
$$f(x) = \frac{1}{x+2}.$$
Using the **definition of the derivative**, find $f'(5)$.

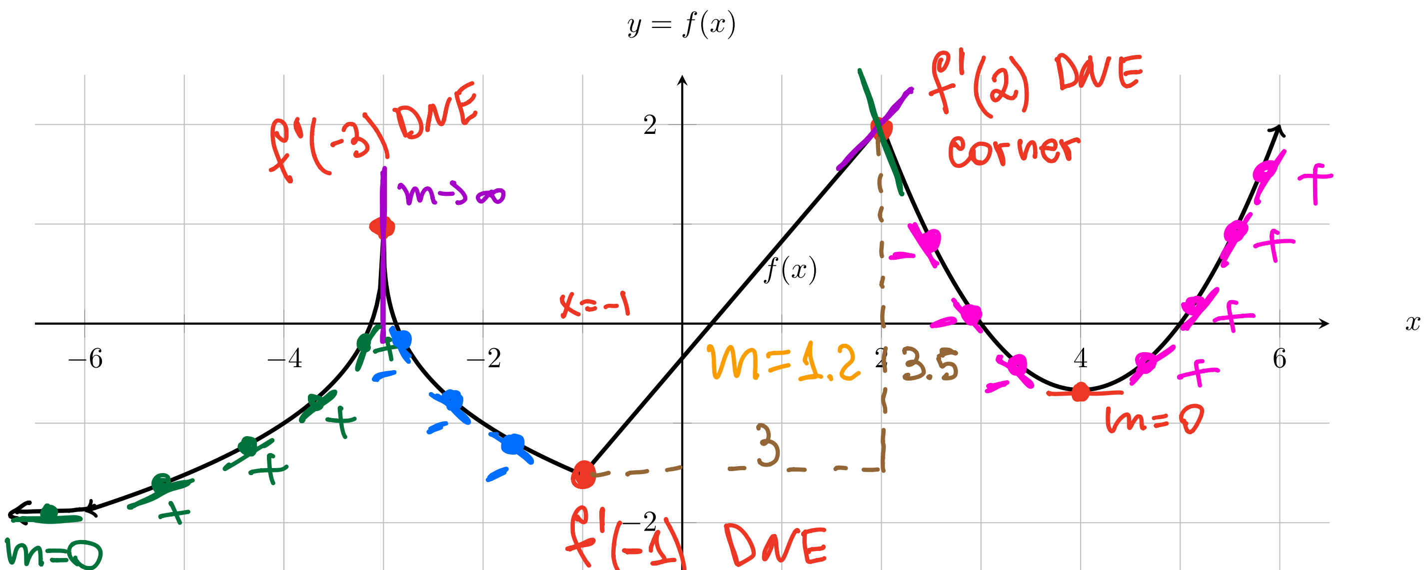
$$\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h} =$$

$$= \lim_{h\to 0} \frac{\frac{\cancel{x}+\cancel{2}-\cancel{x}-h-\cancel{2}}{(x+2)(x+h+2)}}{h} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(x+2)(x+h+2)} =$$

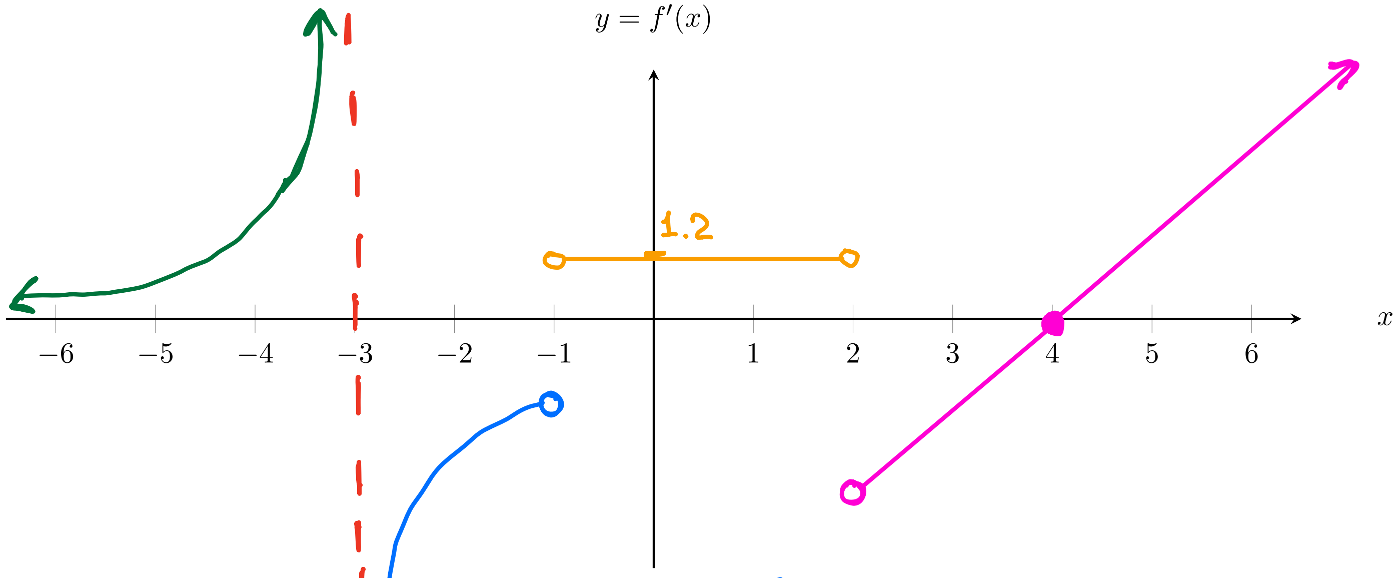
$$= \lim_{h\to 0} \frac{1}{(x+2)(x+h+2)} = \frac{1}{(x+2)(x+2)} = \boxed{\frac{1}{(x+2)^2}}$$

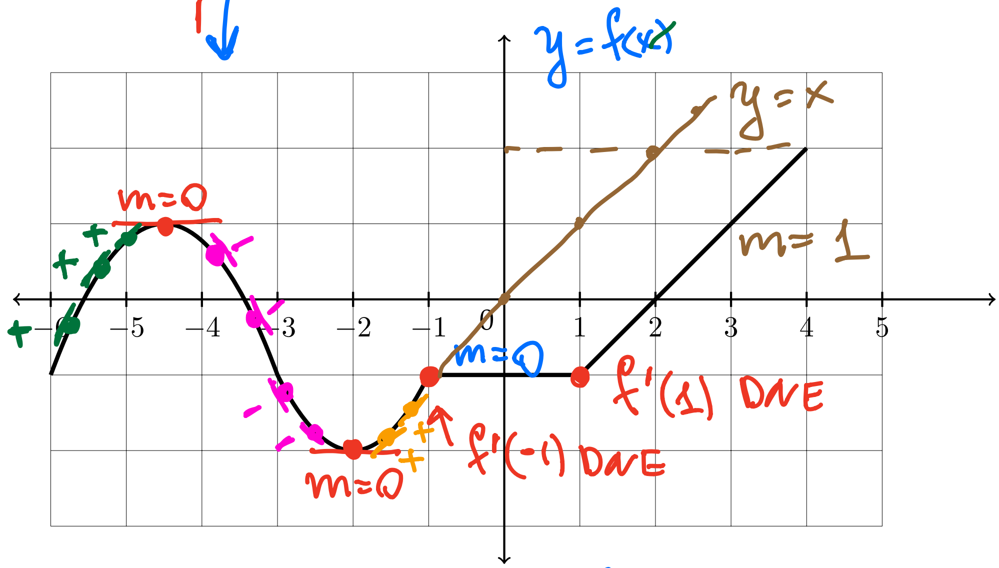
Exercise 5. The graph of $f(x)$ is shown on the top set of axes. Sketch the derivative of $f(x)$ on the second set of axes.

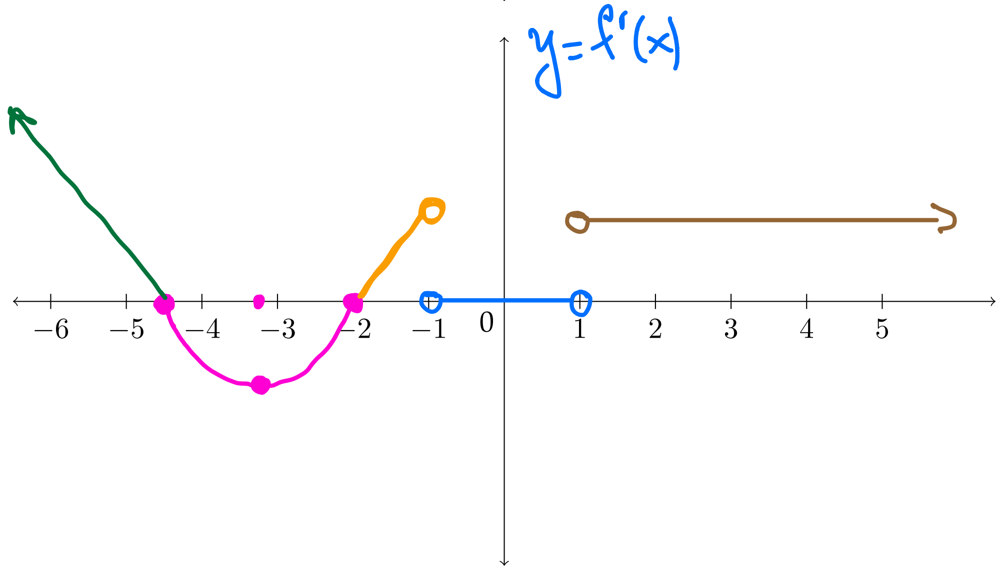


$x=-3$, $x=-1$, $x=2$ } $f'(x)$ DNE

≈ 1.5







$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Exercise 6. Find the equation of the tangent line to the curve at the given point:

$$y = 4x - 3x^2, \quad P = (2, -4).$$

$y = f'(a)(x-a) + f(a)$ $\qquad a = 2$, $f(a) = -4$

tangent line equation

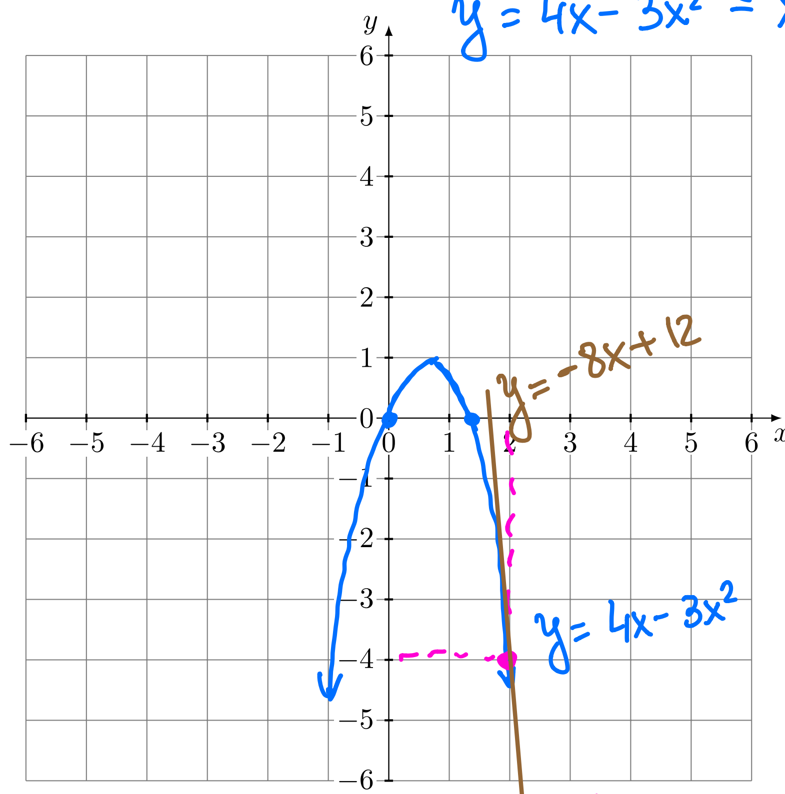
$y = \underline{f'(2)}(x-2) - 4 = -8(x-2) - 4 = \boxed{-8x + 12}$

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{4(2+h) - 3(2+h)^2 - 8 + 12}{h} =$$

Sketch both your function $f(x)$ and the tangent line to it at point P on axes below.

$y = 4x - 3x^2 = x(4-3x) = 0$

$x = 0$ or $x = \frac{4}{3}$



$$= \lim_{h\to 0} \frac{\cancel{8} + 4h - \cancel{12} - 12h - 3h^2 - \cancel{8} + \cancel{12}}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{h}(4 - 12 - 3h)}{\cancel{h}} = 4 - 12 = \boxed{-8}$$

$m = f'(2) = -8$

Exercise 7. The limit represents the derivative of some function f at some point a. State such a function f and a point a.

$$\lim_{h\to 0} \frac{e^{-2+h} - e^{-2}}{h} = f'(a)$$

$f(x) - ?$

$x = a - ?$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$f(a+h) = e^{-2+h}$

$f(a) = e^{-2}$

$a = -2$ $\qquad$ $f(x) = e^{x}$

$f(-2) = e^{-2}$

$f(a+h) = e^{-2+h}$

Example.

$$\lim_{h\to 0} \frac{\sqrt{9+h} - 3}{h}$$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$f(x) = \sqrt{x}$

$a = 9$

$$\lim_{h\to 0} \frac{\sqrt{9+h} - \sqrt{9}}{h} = \lim_{h\to 0} \frac{\sqrt{9+h} - 3}{h}$$

Example

$f(0) = 0$

$f'(0) = 3$

$f'(1) = 0$

$f'(2) = -1$