LECTURE NOTES: REVIEW OF CHAPTERS 3 & 4

Summary of Topics

Chapter 3

- Sections 1-6 primarily involve derivative rules. You will *not* be explicitly tested on "can you take a derivative", but you will need to be able to accurately compute derivatives to answer all the problems.
- Section 5 involves implicilty defined functions. You are expected to be able to use implicit differentiation to, for example, find the equation of a tangent line to an implicitly defined curve at a given point.
- Sections 7 and 8 focus on applications of the derivative in science and particularly to exponential growth and decay. Position, velocity and acceleration were again discussed. The overall emphasis is on *interpretation* of the derivative in the context of an applied problem.
- Section 9 Related Rate Problems. In these problems you are always taking the derivative implicitly with respect to time and almost always seeking of find a rate of change at a particular instant.
- Section 10 Linear Approximations and Differentials. The crucial idea here is that the derivative can be used to estimate function-values or changes in function-values.
- Section 11 we did not cover.

Chapter 4

- Section 1 makes a careful study of the ideas of local/absolute maximum/minimum and the difference between an extreme value (y-value) and where it occurs (x-value).
- Section 4.2 The Mean Value Theorem. Know, roughly, what it says and be able to draw a picture.
- Section 4.3 discussed how the sign of f' and f'' can tell us things about f such as intervals on which f is increasing, decreasing, concave up, concave down, local/absolute extreme values.
- Section 4.4 involved L'Hôpital's Rule. Recall that before using this rule one should make sure it applies.
- Section 4.5 put a whole bunch of Calculus together to sketch a graph. In addition to topics from Section 1 and 2, we also included things like x- and y-intercepts, vertical and horizontal asymptotes, and the function's domain.
- Section 4.6 was not discussed.
- Section 4.7 involved Optimization. Recall that by this time we have a clear understanding of how the domain of the function may determine the techniques we use to determine the answer.
- Section 4.8 will be discussed at the end of the semester and will not appear on this midterm.
- Section 4.9 involves antiderivatives.

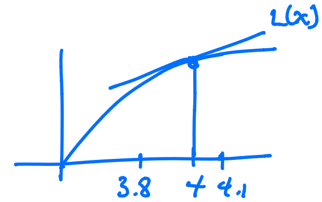
Note that the problems provided below are not necessarily comprehensive; they are intended to remind you of the sorts of problems we have discussed, but there may be other problems on the Midterm that don't look just like these!

1. Find the linearization of $f(x) = \sqrt{x}$ at $a = 4$ and use it to estimate $\sqrt{4.1}$ and $\sqrt{3.8}$.

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$. $\quad f'(4) = \frac{1}{4}$. So $L(x) = \frac{1}{4}(x-4) + 2$

$\sqrt{4.1} \approx \frac{1}{4}(4 + \frac{1}{10} - 4) + 2 = \frac{1}{4}(\frac{1}{10}) + 2 = 2 + \frac{1}{40}$

$\sqrt{3.8} \approx \frac{1}{4}(4 - \frac{2}{10} - 4) + 2 = \frac{1}{4}(\frac{-2}{10}) + 2 = 2 - \frac{1}{20}$



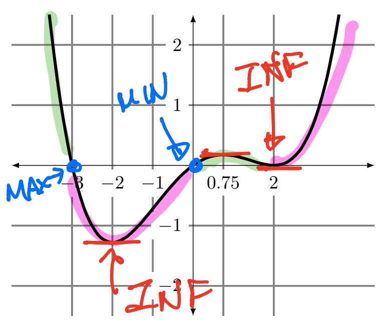
2. Find the differential of $y = \sqrt{x}$ and use it to estimate how much y will change as x changes from $x = 4$ to $x = 4.1$.

$\Delta y \approx f'(4)\,\Delta x \quad$ where $f'(x) =$

$\Delta y \approx \frac{1}{4}(\frac{1}{10}) = \frac{1}{40}$

As x changes from 4 to 4.1, y increases by $\frac{1}{40}$

3. If the $\boxed{\textbf{derivative}}$ of a function is shown below, identify all local maxima, minima, intervals of increase and decrease, intervals of concave up and concave down, and inflection points.



Critical pts at $x = -3, 0, 2$

$\boxed{\begin{array}{lll} x = -3 & \nearrow \searrow & \text{Local max} \\ x = 0 & \searrow \nearrow & \text{Local min} \end{array}}$

Incr $\quad$ Decr

$(-\infty, -3) \cup (0, 2)$

$(-3, 0) \cup (2, \infty)$

$x = -2, 3/4, 2$ possible infl. pts, since $f''(x) = 0$

CD on $(-\infty, -2) \cup (\frac{3}{4}, 2)$ (where $f' \searrow$)

CU on $(-2, 3/4) \cup (2, \infty)$ (where $f' \nearrow$)

$\boxed{x = -2, 2 \text{ are infl. pts}}$

4. Evaluate the following limits. Show your work.

(a) $\displaystyle\lim_{x\to 0} \frac{1 + x - e^x}{\sin x}$

type $\frac{1 + 0 - 1}{\sin(0)} = \frac{0}{0}$

$\overset{L'H}{=} \displaystyle\lim_{x\to 0} \frac{1 - e^x}{\cos(x)}$

$= \frac{1 - 1}{1}$

$= 0$

(b) $\displaystyle\lim_{x\to\infty} x \ln(1 + \frac{2}{x})$ $\quad$ type $\infty(1 + 0)$

$= \displaystyle\lim_{x\to\infty} (x) \cdot \lim_{x\to\infty} (1 + \frac{2}{x})$

$= \displaystyle\lim_{x\to\infty} (x)(1)$

$= \infty$

5. (a) What are critical numbers of a function f?

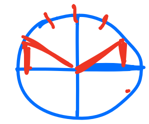
places where $f'(x) = 0$ or $f'(x)$ DNE

(b) How do you find the absolute maximum and minimum of a function f on a closed interval? (Assume f is continuous on the interval.)

Evaluate f at critical points & endpoints, and take the largest / smallest quantities.

(c) Find the critical numbers of $f(x) = \sin(x) + \cancel{\sec(x)}\ (\cos(x))^2$ in $[-2\pi, \pi]$.

$f'(x) = \cos(x) + \qquad = \cos(x) - 2\cos(x)\sin(x)$



$f'(x)$ is always defined

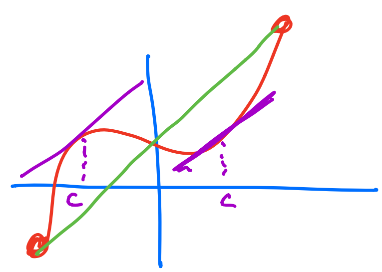
$f'(x) = 0 \Rightarrow \cos(x)\,(1 - 2\sin(x)) = 0 \Rightarrow \cos(x) = 0$ which happens at $\frac{\pi}{2}, \frac{-\pi}{2}, \frac{-3\pi}{2}$ or $\sin(x) = \frac{1}{2}$ which happens at $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{-\pi}{6}, \frac{-5\pi}{6}$

6. (a) State the Mean Value Theorem and draw a picture to illustrate it.

If f is continuous on $[a,b]$ & d'ble on (a,b) then there exists $\boxed{c \in (a,b)}$ s.t. $\boxed{\frac{f(b)-f(a)}{b-a}} = f'(c)$

there is at least one parallel tangent line

slope of green secant line



(b) Determine whether the Mean Value Theorem applies to $f(x) = x(x^2 - x - 2)$ $= x^3 - x^2 - 2$ on $[-1, 1]$. If it can be applied find all numbers that satisfy the conclusion of the Mean Value Theorem.

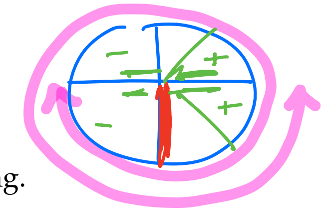
We know Polynomials are continuous & d'ble everywhere, so they certainly are on $[-1,1]$, so MVT applies.

$$\frac{f(1) - f(-1)}{1-(-1)} = \frac{1(1-1-2) - ((-1)(1+1-2))}{2} = \frac{-2}{2} = -1.$$

$f'(x) = 3x^2 - 2x$. Solve $f'(x) = -1 \Rightarrow 3x^2 - 2x - 1 = 0 \Rightarrow (3x+1)(x-1) = 0$

$\Rightarrow x = \frac{1}{3}$ or $x = 1$. However, $x = 1$ is NOT in the domain $(-1, 1)$

So only $x = \frac{1}{3}$ is the value we are asked to find!



7. Consider $f(x) = 2x - 2\cos x$ on the interval $[-\pi, 2\pi]$

(a) Find the open intervals on which the function is increasing or decreasing.

$f'(x) = 2 + 2\sin(x)$. Note $-2 \le 2\sin(x) \le 2$ so

$f'(x) \ge 0$ for all x. Therefore f is <u>always</u> increasing!

$f'(x) = 0 \Rightarrow \sin(x) = -1 \Rightarrow x = \frac{-\pi}{2}, \frac{3\pi}{2}$

x		$-\pi/2$		$3\pi/2$	
f'	$+$	0	$+$	0	$+$
f	↗	—	↗	—	↗

(b) Apply the first derivative test to identify all relative extrema. Classify each as a local maximum or local minimum.

There are no local extrema because f' never changes sign!

(c) Find the open intervals on which the function is concave up or concave down.

$f''(x) = 2\cos(x)$ $\quad f''(x) = 0 \Rightarrow \cos(x) = 0 \Rightarrow x = \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

x		$-\frac{\pi}{2}$		$\frac{\pi}{2}$		$\frac{3\pi}{2}$	
f''	$-$	0	$+$	0	$-$	0	$+$
f	CD		CU		CD		CU

(d) Find the inflection points.

$x = \pi/2, \quad x = 3\pi/2, \quad x = -\pi/2$

(e) What are the absolute maximum and minimum values of the function on the interval?

Since f is always increasing, the absolute min on $[-\pi, 2\pi]$ must occur at $x = -\pi$, min is $f(-\pi) = -2\pi + 2$; abs max at $x = 2\pi$ equal to $f(2\pi) = 4\pi + 2$

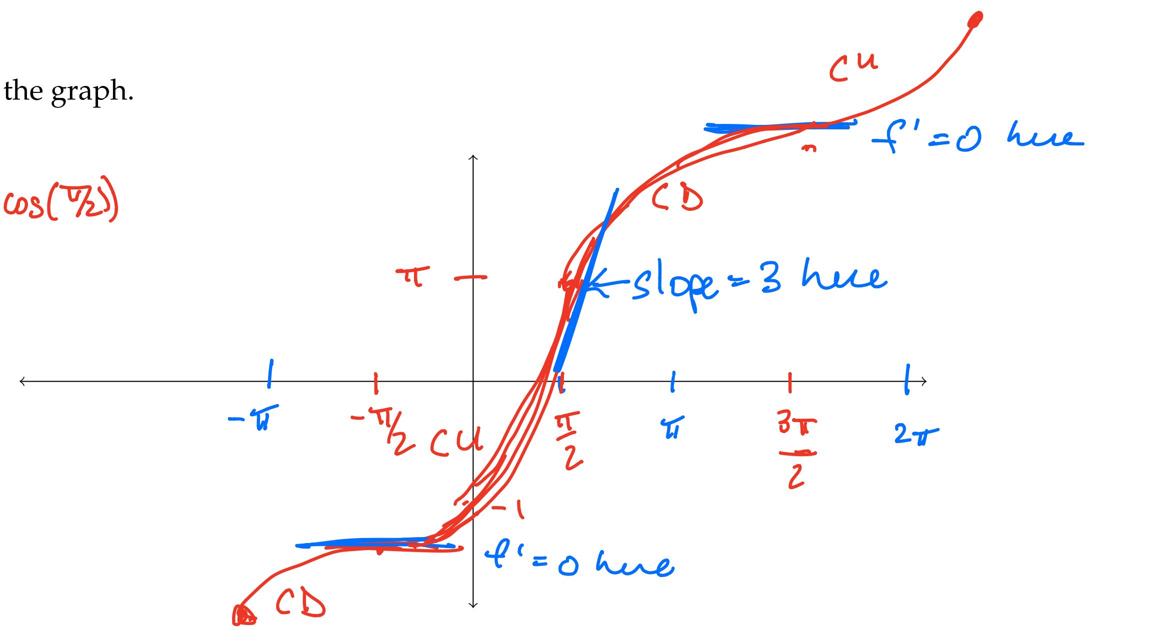
(f) Sketch the graph.

Note $f(0) = -1$

$f(\pi/2) = \pi - 2(\cos(\pi/2))$
$= \pi$

$f'(\frac{\pi}{2}) =$

$2 + \sin(\pi/2) = 3$

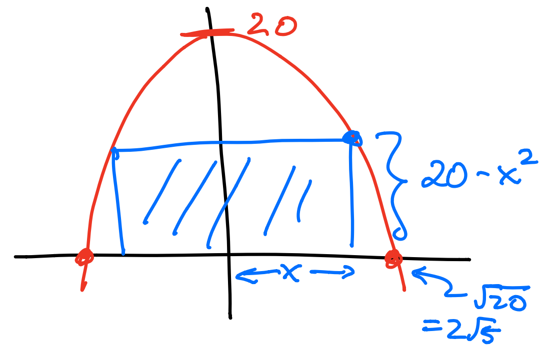


8. Find the rectangle of maximum area that can be inscribed inside the region bounded above by $y = 20 - x^2$ and bounded below by the x-axis. (Assume the base of the rectangle lies on the x-axis.) Begin by sketching a picture and labelling useful information.

Maximize $A = 2xy$ when $y = 20 - x^2$.

$A(x) = 2x(20 - x^2)$ Domain $= [0, 2\sqrt{5}]$

$= 40x - 2x^3$



$A'(x) = 40 - 6x^2$

$A'(x) = 0 \Rightarrow \frac{40}{6} = x^2 \Rightarrow x = \sqrt{\frac{40}{6}} = \sqrt{\frac{20}{3}}$

$20 - x^2 = 0 \Rightarrow x = \pm\sqrt{20}$

$= 2\sqrt{\frac{5}{3}}$. Is it a max? $A''(x) = -12x$ so $A''\left(\sqrt{\frac{20}{3}}\right) < 0 \Rightarrow \cap$.

dimensions are $4\sqrt{\frac{5}{3}} \leftrightarrow$ and $\frac{40}{3} \updownarrow$ $\quad 20 - \frac{20}{3} = \frac{40}{3}$

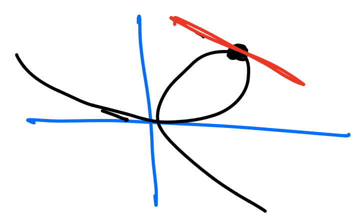
9. Find the equation of the tangent line to the function $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(6xy) \Rightarrow 3x^2 + 3y^2 y' = 6xy' + 6y$

at $x = 3, y = 3$, $3(3)^2 + 3(3)^2 y' = 6(3)y' + 6(3) \Rightarrow 27(1 + y') = 18(y' + 1)$

$\Rightarrow 9(1 + y') = 0 \Rightarrow y' = -1$.

So TL is $y = -1(x - 3) + 3$.

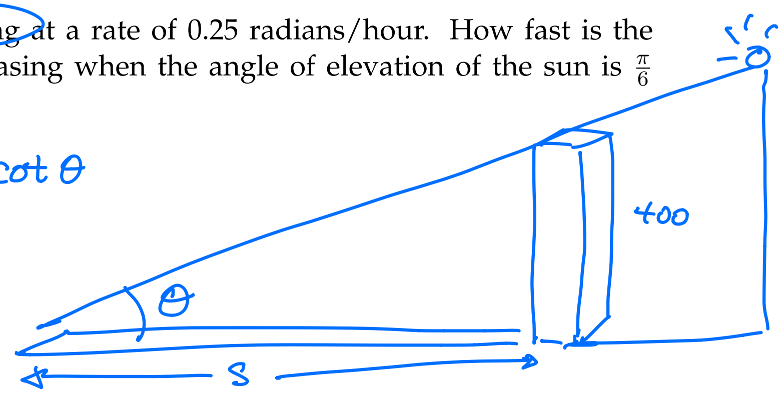


10. The angle of elevation of the sun is decreasing at a rate of 0.25 radians/hour. How fast is the shadow cast by a 400 foot tall building increasing when the angle of elevation of the sun is $\frac{\pi}{6}$ radians?

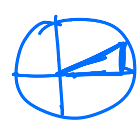
Know $\tan\theta = \frac{400}{s} \Rightarrow s = 400\cot\theta$

$\frac{ds}{dt} = 400\left(-(\csc(\theta))^2\right)\frac{d\theta}{dt}$

Know when $\theta = \pi/6$, $\frac{d\theta}{dt} = -0.25$



so $\frac{ds}{dt} = -400(4)\left(\frac{-1}{4}\right) = +400$ ft/min



$\leftarrow \frac{1}{2} = \sin\theta$