Section 3.10. Linear approximation and differentials

$y = \sqrt{x}$

$\sqrt{4.05} \approx ?$

$\sqrt{4}$

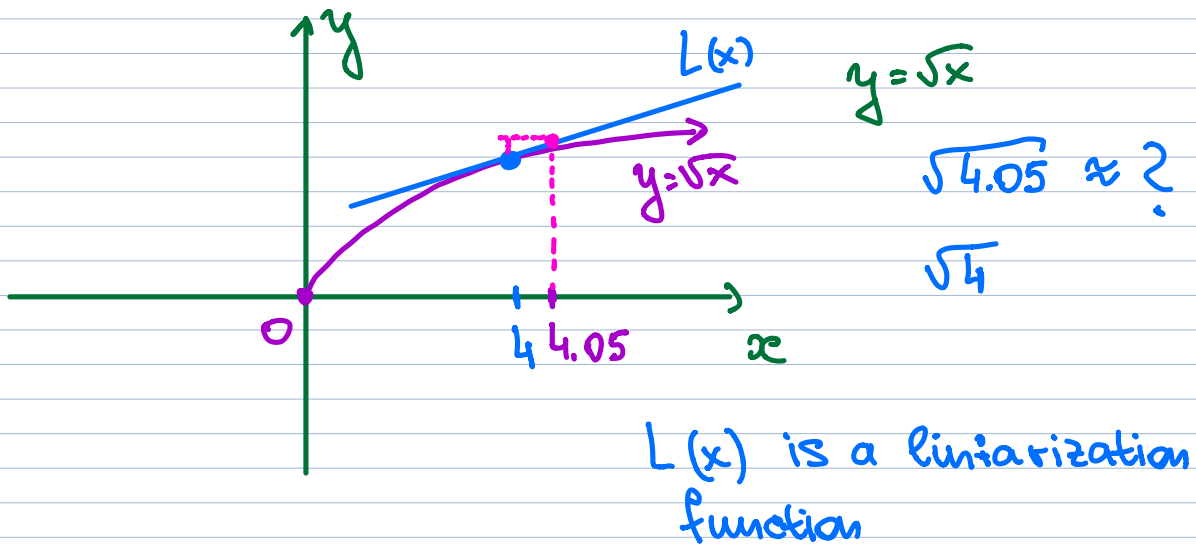$L(x)$ is a liniarization function

$$\sqrt{4.05} \approx L(4.05)$$

$L(x)$ is a tangent line to $y = \sqrt{x}$ at $x = 4$.

$$L(x) = f'(a)(x-a) + f(a)$$

$a = 4 \qquad f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f'(4) = \frac{1}{4}$$

$$f(4) = 2$$

$$L(x) = \frac{1}{4}(x-4) + 2$$

$$L(x) = \frac{1}{4}x + 1$$

$$\sqrt{4.05} \approx L(4.05) = \frac{1}{4}(4+0.05)+1 =$$

$$= 1 + \frac{\cancel{5}}{\underset{20}{4\cdot\cancel{100}}} + 1 = 2 + \frac{1}{80}$$

$$\sqrt{4} = 2$$

Example

$$x^2 y + \cos(y) = e^{xy}$$

$$\frac{dy}{dx} = y'(x) = y' \; ?$$

$$\frac{d}{dx}\left(x^2 y + \cos(y)\right) = 2x\cdot y + y'\cdot x^2 +$$

$$+ (-\sin(y))\cdot y'$$

$$\frac{d}{dx}\left(e^{xy}\right) = e^{xy}\cdot(y + y'\cdot x)$$

$$2xy + \underline{\underline{y'}} \cdot x^2 - \sin(y) \cdot \underline{\underline{y'}} = e^{xy}(y + \underline{\underline{y'}}x)$$

$$y'(x^2 - \sin y - e^{xy} \cdot x) = e^{xy} \cdot y - 2xy$$

$$y' = \frac{e^{xy} \cdot y - 2xy}{x^2 - \sin(y) - e^{xy} \cdot x}$$

1. Use the linear approximation of $f(x) = \sqrt{x}$ at $x = 4$ to approxmiate $\sqrt{4.1}$ and compare your result to its approximation computed by your calculator.

$$L(x) = f'(a)(x-a) + f(a) \qquad a = 4$$
$$f'(x) = \frac{1}{2\sqrt{x}}$$
$$f'(a) = f'(4) = \frac{1}{4}$$
$$f(a) = \sqrt{4} = 2$$
$$L(x) = \frac{1}{4}(x-4) + 2$$

$$\sqrt{4.1} \approx L(4.1) = \frac{1}{4}(4.1-4) + 2$$
$$= 2.025$$

By calculator:
$$\sqrt{4.1} \approx 2.02484$$

error is $\underline{0.00016}$

2. Use the linear approximation to approximate the cosine of $29° = \frac{29}{30}\frac{\pi}{6}$ radians.

$$L(x)$$
$$f(x) = \cos(x)$$
$$\cos(29°) \approx ?$$

$$L(x) = -\frac{1}{2}\left(x - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}$$

$$L(x) = f'(a)(x-a) + f(a)$$
$$a = 30° = \frac{\pi}{6}$$
$$f(a) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$
$$f'(x) = (\cos(x))' = -\sin(x)$$
$$f'(a) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$\cos(29°) \approx L(29°) = -\frac{1}{2}\left(\frac{29}{30}\frac{\pi}{6} - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2} = -\frac{1}{2}\cdot\left(-\frac{1}{30}\right)\frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

3. Find the linear approximation of $f(x) = \ln(x)$ at $a = 1$ and use it to approximate $\ln(0.5)$ and $\ln(0.9)$. Compare your approximation with your calculator's. Sketch both the curve $y = \ln(x)$ and $y = L(x)$ and label the points $A = (0.5, \ln(0.5))$ and $B = (0.5, L(0.5))$

$$L(x) = f'(1)(x-1) + f(1)$$
$$f'(x) = \frac{1}{x}$$
$$f'(1) = 1$$
$$f(1) = 0$$
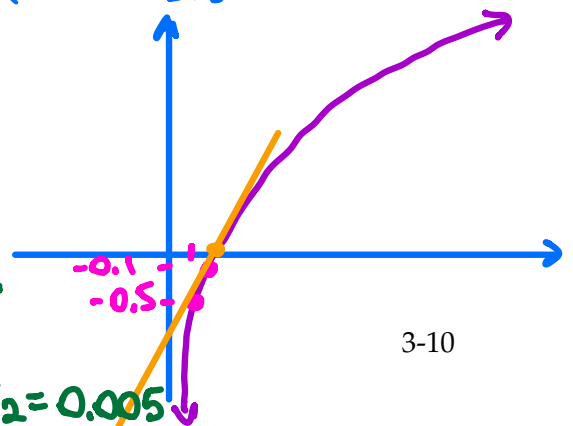
$$L(x) = 1\cdot(x-1) = \underline{x-1}$$

$$\ln(0.5) \approx L(0.5) = -0.5$$
$$\ln(0.9) \approx L(0.9) = -0.1$$
By calculator:
$$\ln(0.5) \approx -0.6931$$
$$\ln(0.9) \approx -0.105$$

We see that at $x = 0.5$ we have not such a good appr. as at $x = 0.9$.

$$error_1 = 0.1931 \qquad error_2 = 0.005$$

−0.1
−0.5

4. A tree is growing and the radius of its trunk in centimeters is $r(t) = 2\sqrt{t}$ where t is measured in years. Use the differential to estimate the change in radius of the tree from 4 years to 4 years and one month.

$r(t) = 2\sqrt{t}$

$t_1 = 4 \text{ years}$
$t_2 = 4 \text{ years} + 1 \text{ mon.}$

Δr is a change in radius ?

$\Delta r = r(t_2) - r(t_1)$

Δt is a change in time

$\Delta t = \frac{1}{12} \text{ year} = 1 \text{ month}$

$\frac{\Delta r}{\Delta t} \approx \frac{dr}{dt} = r'(t)$

$\Delta r = \boxed{\frac{dr}{dt}} \cdot \Delta t$

$\frac{dr}{dt} = \frac{1}{\sqrt{t}}$

$\frac{dr}{dt}\Big|_{t=4 \text{ years}} = \frac{1}{2}$

5. A coat of paint of thickness 0.05cm is being added to a hemispherical dome of radius 25m. Estimate the volume of paint needed to accomplish this task. [Challenge: will this be an underestimate or an overestimate? Thinking geometrically or thinking algebraically will both give you the same answer.]

$\Delta r = \frac{1}{2} \cdot \frac{1}{12} = \frac{1}{24} \text{ (centimet)}$

$r(4) = 2 \cdot \sqrt{4} = 4 \text{ cen.}$

6. The radius of a disc is 24cm with an error of ± 0.5cm. Estimate the error in the area of the disc as an absolute and as a relative error.