Section 5.1 (Day two)

(d) What do you think: based on your rectangles, is this an overestimation or an underestimation? (The other two rectangle types should have been obvious; this one is considerably more subtle. Make some sort of argument in favor of one choice or the other.)

Since $y = x^2$ is increasing, the underestimated area under the curve will be slighley bigger than the area above the curve

4. Suppose you want a really good approximation of the area under the curve $y = x^2$ on the interval $[0, 1]$. What could you compute to get it? Explain in a sentence or two.

5. Suppose the odometer on our car is broken and we want to estimate the distance driven over a 1.5 hour time period. We take speedometer readings every 15 minutes and then record them in the table below. Estimate the distance traveled by the car. What method are you using?

	0	$1/4$	$1/2$	$3/4$	1	$1\frac{1}{4}$	$1\frac{1}{2}$
Time (minutes)	0	15	30	45	60	75	90
Velocity (mi/h)	17	21	24	29	32	31	28

1.5h

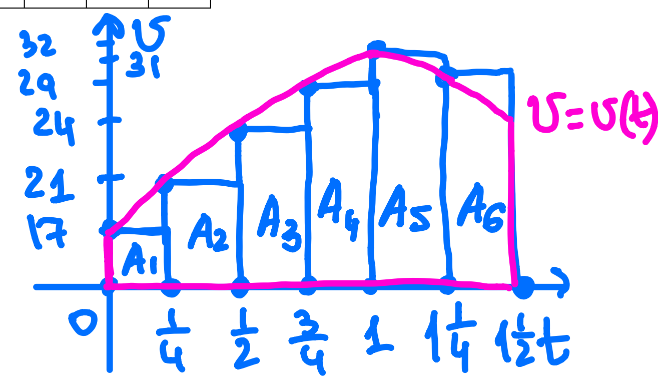
• left-hand endpoints

Total area $A = A_1 + A_2 + A_3 + A_4 + A_5 + A_6$

$A_1 = \frac{17}{4}$ (mi) $\quad A_4 = \frac{29}{4}$ (mi)

$A_2 = \frac{21}{4}$ (mi) $\quad A_5 = \frac{32}{4}$ (mi)

$A_3 = \frac{24}{4}$ (mi) $\quad A_6 = \frac{31}{4}$ (mi)



6. Oil leaked out of a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at 2 hour time intervals are shown in the table. Estimate how much oil leaked out. What method are you using? Is it an overestimate or an underestimate.

t (h)	0	2	4	6	8	10
r(t) (L/h)	8.7	7.6	6.8	6.2	5.7	5.3

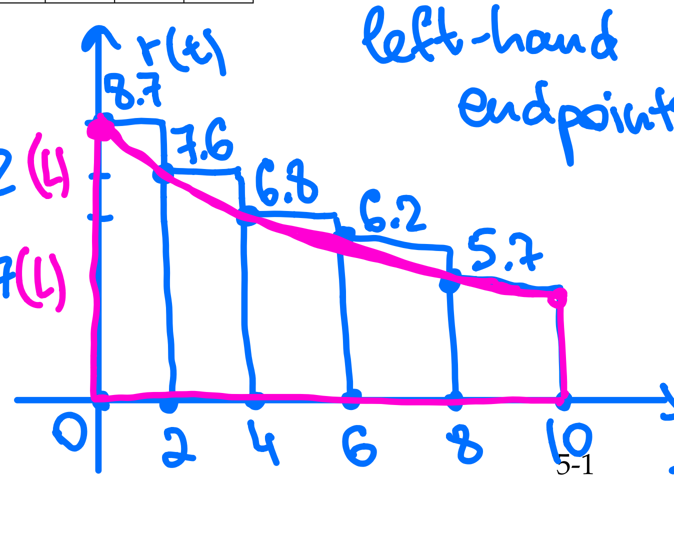
$A = A_1 + A_2 + A_3 + A_4 + A_5$

left-hand endpoints

$A_1 = 2 \cdot 8.7$ (L) $\quad A_4 = 2 \cdot 6.2$ (L)

$A_2 = 2 \cdot 7.6$ (L) $\quad A_5 = 2 \cdot 5.7$ (L)

$A_3 = 2 \cdot 6.8$ (L)



Units: $v = \frac{S}{t}$

(pr. #5)

$v = S'(t)$

$S = \int v(t)\,dt$

$A = a \cdot b$

$a \sim h$

$b \sim mi/h$

$A = \cancel{h} \cdot mi/\cancel{h} =$

$= mi$

 (Day one)

Definite Integrals and Areas "under" Curves

1. Estimate the area (actually draw rectangles) under $f(x) = x^2 - 2x$ on $[1,3]$ with $n = 4$ using the

   (a) Right-hand endpoints

   $A = A_1 + A_2 + A_3 + A_4$

   $A_1 = \frac{1}{2} \cdot f(\frac{3}{2}) = \frac{1}{2}((\frac{3}{2})^2 - 2 \cdot \frac{3}{2})$

   $A_2 = \frac{1}{2} \cdot 0 = 0$

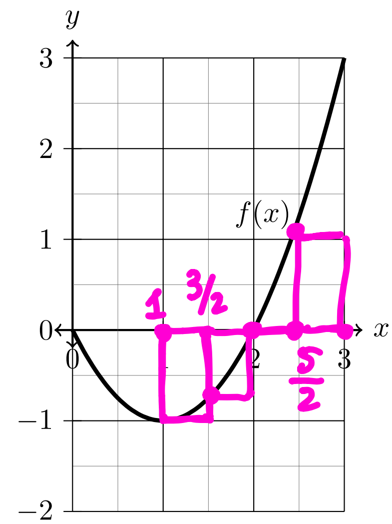
   $A_3 = f(\frac{5}{2}) \cdot \frac{1}{2}$ $\quad A_4 = f(3) \cdot \frac{1}{2}$

   (b) Left-hand endpoints

   $A = A_1 + A_2 + A_3 + A_4$

   $A_1 = \frac{1}{2} \cdot f(1)$ $\quad A_3 = \frac{1}{2} \cdot 0 = 0$

   $A_2 = \frac{1}{2} \cdot f(\frac{3}{2})$ $\quad A_4 = \frac{1}{2} \cdot f(\frac{5}{2})$



Definition of a Definite Integral If f is a function defined for $a \leq x \leq b$, we divide the interval $[a,b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. We let $x_0 = (a), x_1, x_2, \cdots, x_n = (b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \cdots, x_n^*$ be **sample points**[1] in these subintervals, so x_i^* lies in the i-th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

sigma Σ

$$A = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a,b]$.

[1] For example, we could choose our sample points to be right-hand endpoints, left-hand endpoints, midpoints, a combination of these, or any other sample points in the interval that we choose!

2. Consider again $f(x) = x^2 - 2x$ on the interval $[1,3]$. Suppose that we are dividing the interval $[1,3]$ into n subintervals. (Think about your answers to #1.)

   (a) What is the length of each subinterval? $\frac{2}{n}$

   (b) What is the right-hand endpoint of the first subinterval? $1 + \frac{2}{n}$
   What is the height of the first right-hand rectangle? $f(1 + \frac{2}{n})$

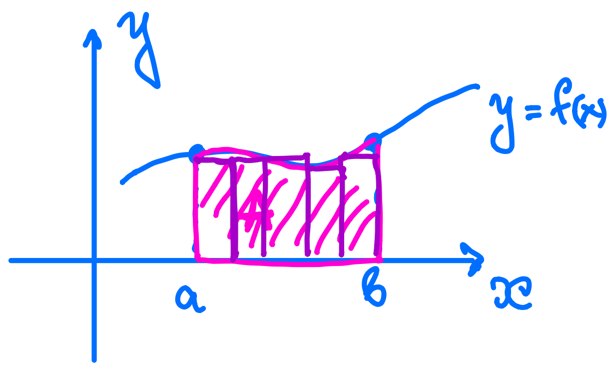
   (c) What is the right-hand endpoint of the second subinterval? $1 + 2 \cdot \frac{2}{n}$
   What is the height of the second right-hand rectangle? $f(1 + 2 \cdot \frac{2}{n})$

   (d) What is the right-hand endpoint of the third subinterval? $1 + 3 \cdot \frac{2}{n}$
   What is the height of the third right-hand rectangle? $f(1 + 3 \cdot \frac{2}{n})$

   (e) What is the right-hand endpoint of the i^{th} rectangle? $x_i = 1 + i \cdot \frac{2}{n}$
   What is the height of the i-th right-hand rectangle? $f(1 + i \cdot \frac{2}{n})$
   What is the **area** of the i-th right-hand rectangle? $f(1 + i \cdot \frac{2}{n}) \cdot \frac{2}{n} = A_i$

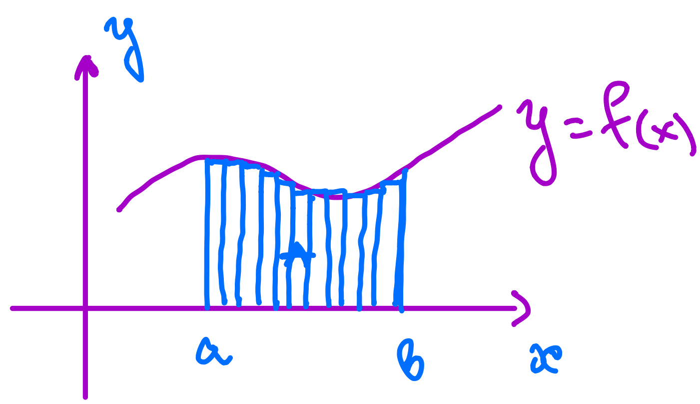
Definite Integrals

$\int f(x)\,dx \rightarrow$ indefinite integral



$A = \int_a^b f(x)\,dx$

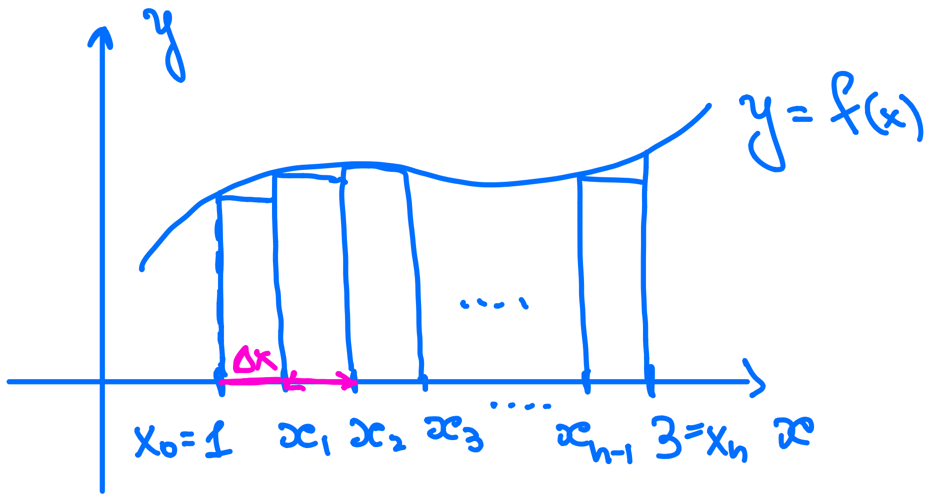
$\int_a^b f(x)\,dx \rightarrow$ definite integral.



$$\int_a^b f(x)dx = A = \lim_{n\to\infty} \sum_{i=1}^{n} \overbrace{\underbrace{f(x_i)}_{\text{height}} \cdot \underbrace{\Delta x}_{\text{width}}}^{A_i}$$

A_i (area of the ith rectangle)

sample points (left, right, middle endpoints)



$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

$$x_1 = x_0 + \Delta x = 1 + \frac{2}{n}$$

$$x_2 = x_0 + 2\Delta x = 1 + 2\cdot\frac{2}{n}$$

$$\cdots$$

$$x_n = x_0 + n\cdot\Delta x = 1 + n\cdot\frac{2}{n} =$$

$$= \boxed{3}$$

3. Using your answers to the previous problem, write down a limit that equals $\int_1^3 x^2 - 2x\,dx$.

$$\int_1^3 (x^2-2x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} A_i = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f\left(1+i\cdot\frac{2}{n}\right)}_{\text{height}} \cdot \underbrace{\frac{2}{n}}_{\text{width}}$$

4. Write down a limit that equals $\int_2^8 e^x\,dx$, using right-hand endpoints as your sample points.

$$A = \int_2^8 e^x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} A_i = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(2+i\frac{6}{n}\right)\cdot\frac{6}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} e^{2+i\frac{6}{n}}\cdot\frac{6}{n}$$

$A_i = f(x_i)\cdot\Delta x$

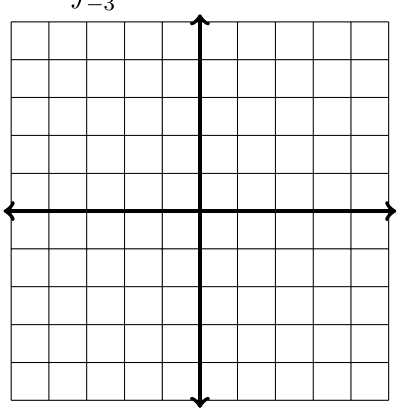
$\Delta x = \frac{8-2}{n} = \frac{6}{n}$

$x_i = x_0 + i\cdot\Delta x = \underline{2+i\cdot\frac{6}{n}}$

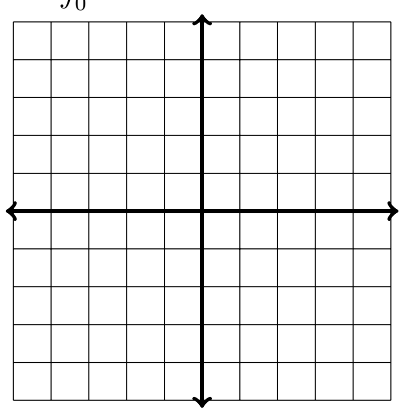
> *A definite integral represents the* **signed** *area under a curve (that is, the signed area between the curve and the x-axis). If a curve is above the x-axis that area is _____________; if the curve is below the x-axis the area is _____________.*

5. Evaluate the following definite integrals by drawing the function and interpreting the integral in terms of areas. Shade in the area you are computing with the integral and then actually compute the integral by adding up areas.

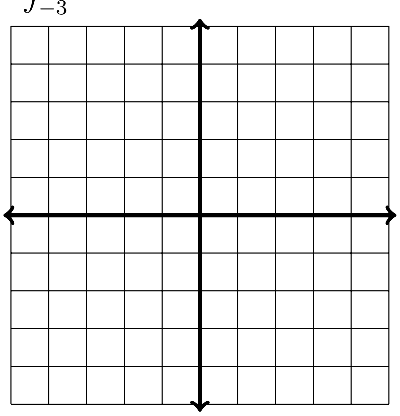
(a) $\int_{-3}^{3} (x-1)\,dx =$________



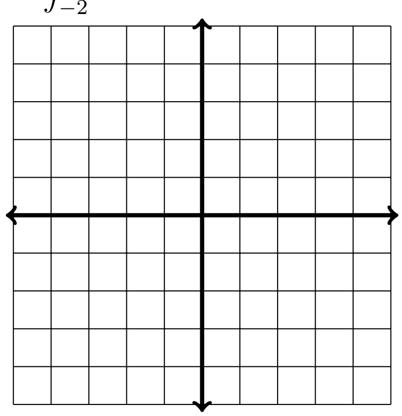
(b) $\int_0^4 \sqrt{16-x^2}\,dx =$ ________



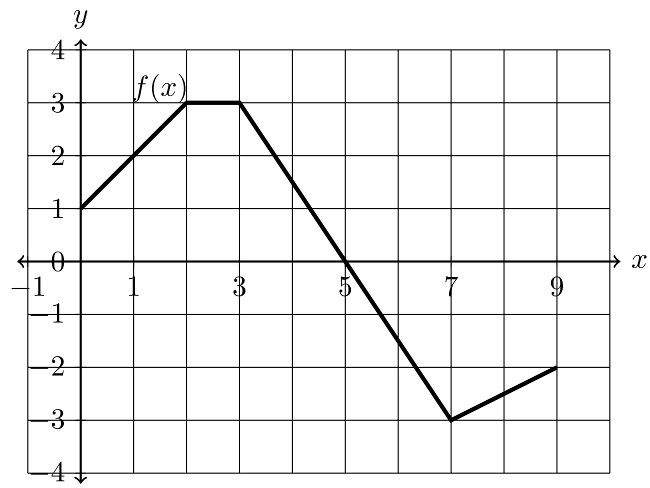
(c) $\int_{-3}^{3} (2+\sqrt{9-x^2})\,dx =$ ________



$\int_{-2}^{3} (|x|-3)\,dx =$ ________



6. The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.



(a) $\int_2^5 f(x)dx =$

(b) $\int_5^9 f(x)dx =$

(c) $\int_3^7 f(x)dx =$

Properties of the Definite Integral:

- $\int_a^a f(x)\,dx =$ ____________________
- $\int_a^b c\,dx =$ ____________________
- $\int_a^b cf(x)\,dx =$ ____________________
- $\int_a^b [f(x) \pm g(x)]dx =$ ____________________
- $\int_a^b f(x) + \int_b^c f(x)\,dx =$ ____________________
- $\int_b^a f(x)\,dx =$ ____________________

7. Using the fact that $\int_0^1 x^2dx = \frac{1}{3}$ and $\int_1^2 x^2\,dx = \frac{7}{3}$, evaluate the following using the properties of integrals.

(a) $\int_1^0 x^2\,dx$ (b) $\int_0^1 5x^2\,dx$ (c) $\int_0^1 (4 + 3x^2)\,dx$ (d) $\int_0^2 x^2\,dx$.