Section 5-3: The Fundamental Theorem of Calculus, Part 1

1. Suppose f is the function whose graph is shown and that $g(x) = \int_0^x f(t)dt$.

   (a) Find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, $g(5)$, and $g(6)$. Then, sketch a rough graph of g.

   (a) $g(0) =$ $\int_0^0 f(t)\,dt = 0$

   (b) $g(1) =$ $\int_0^1 f(t)\,dt = 1$

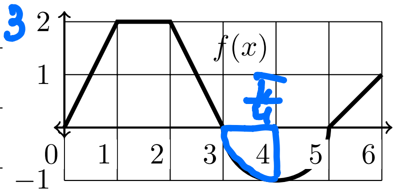
   (c) $g(2) =$ $3\int_0^2 f(t)\,dt = 3$

   (d) $g(3) =$ $\int_0^3 f(t)\,dt = 4$

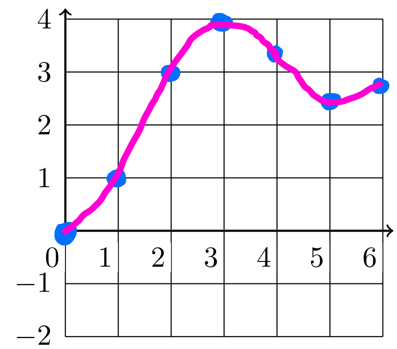
   (e) $g(4) =$ $4-\frac{\pi}{4}$

   (f) $g(5) =$ $4-\frac{\pi}{2}$

   (g) $g(6) =$ $4\frac{1}{2}-\frac{\pi}{2}$



Sketch of $g(x)$



   (i) Where is $g(x)$ increasing? $(0,3)\cup(5,6)$

   (ii) Describe f when $g(x)$ is increasing. positive

   (iii) Where is $g(x)$ decreasing? $(3,5)$

   (iv) Describe f when $g(x)$ is decreasing. negative

   (v) Where does $g(x)$ have a local maximum? $x=3$ $\quad g_{max}=4$

   (vi) Describe f when $g(x)$ has a local max. f is changing its sign from + to −

   (vii) Where does $g(x)$ have a local minimum? $x=5$

   (viii) Describe f when $g(x)$ has a local min. f is changing its sign from − to +

   (b) Make a guess: what is the relationship between $g(x)$ and $f(x)$?

$$\boxed{f(x) = g'(x)} \leftarrow g(x) = \int_0^x f(t)dt$$

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a,b]$, the function g defined by

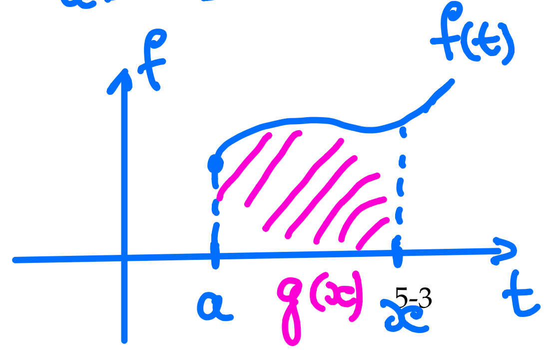
$$g(x) = \int_a^x f(t)\,dt \quad a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) = f(x)$.

a b

2. Find the derivative of $g(x) = \int_2^x t^2\,dt$. $\quad f(t) = t^2 \qquad a \le t \le x$

$g'(x) = x^2$



$g(x) = \int_0^x f(t)\,dt \Rightarrow g'(x) = f(x)$

3. The Fresnel function $S(x) = \int_0^x \sin(\pi t^2/2)\,dt$ first appeared in Fresnel's theory of the diffraction of light waves. Recently it was be applied to the design of highways. Find the derivative of the Fresnel function.

$S'(x) = \sin\left(\frac{\pi x^2}{2}\right)$

$g(x) = \int_0^x f(t)\,dt$

$g(x) = \int_a^x f(t)\,dt$

4. Consider $g(x) = \displaystyle\int_1^{x^4} \sec t\,dt$.

   Let $u = x^4$ and $h(x) = \int_1^x \sec t\,dt$.

   (a) Write $g(x)$ as a composition.

   $x^4 = u$

   $\boxed{g(x) = h(u(x))}$

   $g(x) = h(u) = \int_1^u \sec t\,dt$

   (b) Use FTC1 and the chain rule to differentiate $g(x)$.

5. Consider $g(x) = \displaystyle\int_{2x+1}^{2} \sqrt{t}\,dt$.

   (a) Write $g(x)$ as a composition.

   $g(x) = -\int_2^{2x+1} \sqrt{t}\,dt$

   $g'(x) = -\sqrt{2x+1}\cdot 2$

   (b) Use FTC1 and the chain rule to differentiate $g(x)$.

$g'(x) = \underbrace{h'(u)}\cdot\underbrace{\frac{du}{dx}} = \sec(u)\cdot 4x^3 = \boxed{\sec(x^4)\cdot 4x^3}$

By FTC1 : $h'(u) = \sec(u)$

$\frac{du}{dx} = 4x^3$

6. Consider the function $g(x) = \displaystyle\int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt$. Observe that

$$\int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt = \int_{\tan x}^{0} \frac{1}{\sqrt{2+t^4}}\,dt + \int_{0}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt.$$

Use properties of definite integrals, FTC1, and the chain rule to determine $g'(x)$.

$g(x) = \int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt = \int_{\tan x}^{1} \frac{1}{\sqrt{2+t^4}}\,dt + \int_{1}^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt =$

$= -\int_1^{\tan x} \frac{1}{\sqrt{2+t^4}}\,dt + \int_1^{x^2} \frac{1}{\sqrt{2+t^4}}\,dt$

$$g'(x) = \boxed{\frac{-1}{\sqrt{2+(\tan x)^4}} \cdot \sec^2(x) + \frac{1}{\sqrt{2+x^8}} \cdot 2x}$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Fundamental Theorem of calculus Part I.

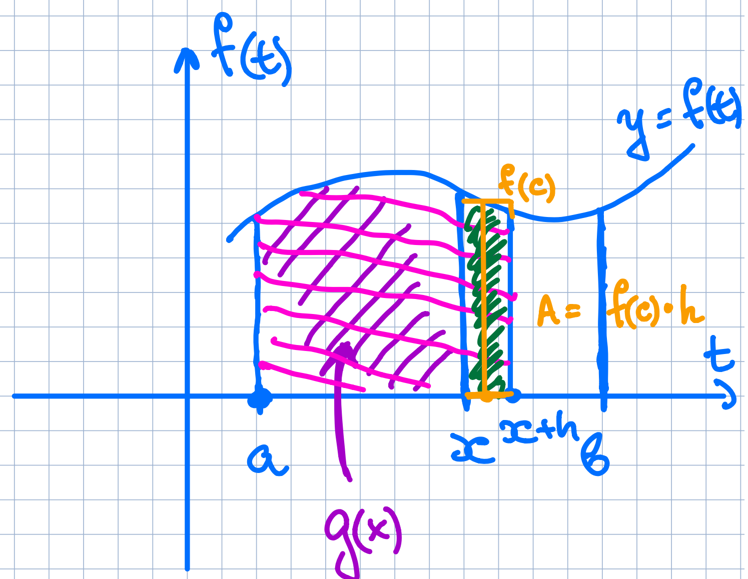
$$g(x) = \int_a^x f(t)\,dt$$

Why $g'(x) = f(x)$

Explanation

$$g(x) = \int_a^x f(t)\,dt$$

$a \le x \le b$



$$\boxed{g'(x)} = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} =$$

$$= \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} =$$

$$= \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} =$$

The mean value theorem for the definite integral:

there exists the value c such that $x \le c \le x+h$

$$A = \int_x^{x+h} f(t)\,dt = f(c)\cdot h$$

$$= \lim_{h\to 0} \frac{f(c)\cdot \cancel{h}}{\cancel{h}} = f(c) = \boxed{f(x)}$$

$$x \le c \le x+h$$

$h \to 0$

x

Squeeze Theorem

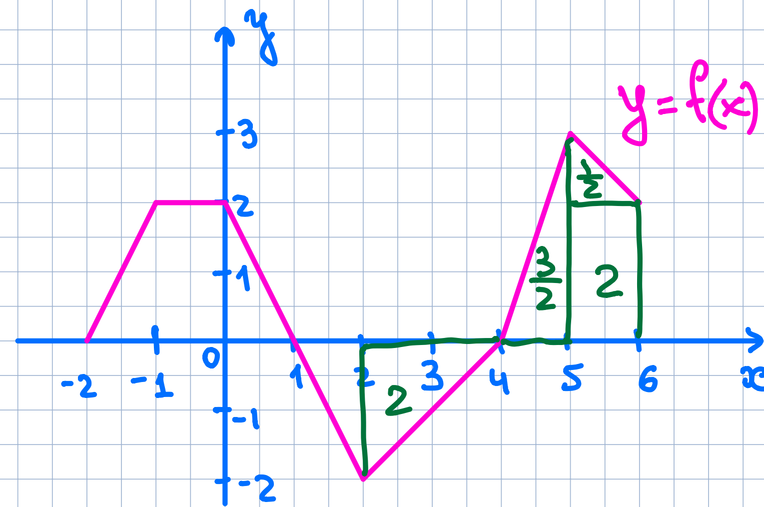
Definite Integral. Review

Problem # 1

$$\int_{-2}^{1} f(x)\,dx = 4$$

$$\int_{0}^{2} f(x)\,dx = 0$$

$$\int_{2}^{6} f(x)\,dx = \frac{1}{2} + \cancel{2} + \frac{3}{2} - \cancel{2} = 2$$



Problem # 2

If $\int_{1}^{5} f(x)\,dx = 3$ $\int_{1}^{8} f(x)\,dx = 10$ $\int_{1}^{5} g(x)\,dx = -4$

compute

$$\int_{1}^{5} 5 f(x)\,dx = 5\int_{1}^{5} f(x)\,dx = 5 \cdot 3 = 15$$

$$\int_1^5 (2f-6)\,dx = 2\int_1^5 f(x)\,dx - \underbrace{\int_1^5 6\,dx}_{\text{area of a rectangle}} = 2\cdot 3 - 6\cdot\underbrace{(5-1)}$$

$$= 6 - 24 = -18$$

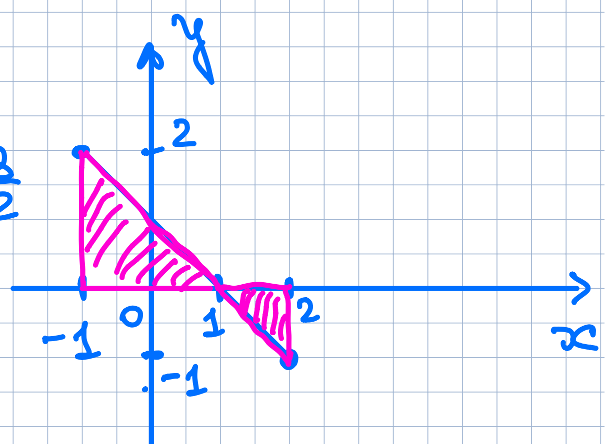
$$\int_3^3 g(x)\,dx = 0$$

Problem #3

Evaluate the integral by interpreting it in terms of areas.

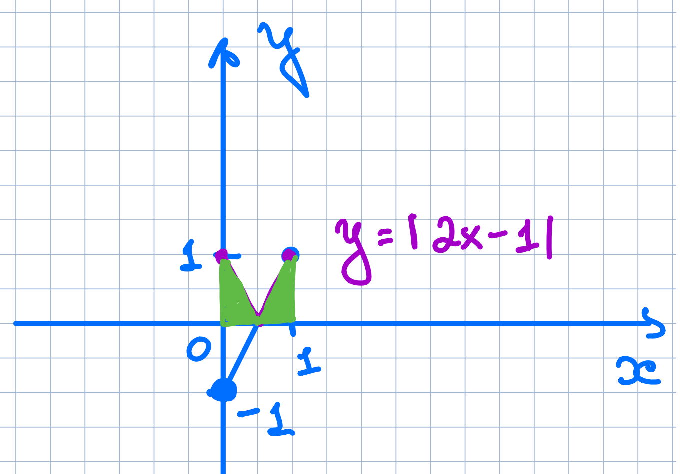
$$\int_{-1}^{2} (1-x)\,dx = 2 - \frac{1}{2} = \frac{3}{2}$$

$f(x) = 1-x$



$$\int_0^1 |2x-1|\,dx = \frac{1}{2}$$

$f(x) = |2x-1|$



Section 5-3: The Fundamental Theorem of Calculus, Part 2

The Fundamental Theorem of Calculus (Part 2) If f is continuous on $[a, b]$, then

antiderivative

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is **any antiderivative** of f, that is, is a function such that $F' = f$. To evaluate, we write

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

evaluating $F(x)$ at boundaries

1. Evaluate the following integrals.

   (a) $\int_0^1 x^2\,dx =$

   $= \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$

   (b) $\int_1^4 (1 + 3y - y^2)\,dy =$

   $= \left(y + \frac{3}{2}y^2 - \frac{y^3}{3}\right)\Big|_1^4 =$

   $= \left(4 + \frac{3}{2}16 - \frac{64}{3}\right) - \left(1 + \frac{3}{2} - \frac{1}{3}\right)$

2. Review from §4.9: To compute integrals effectively you **must** have your basic antidifferentiation formulas down. You should know that antiderivatives to the following functions. We are using the $\int$ symbol to mean "find the antiderivative" of the function right after the symbol.

Antiderivatives of common functions:

indefinite

- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
- $\int \sin x\,dx = -\cos(x) + C$
- $\int \cos x\,dx = \sin(x) + C$
- $\int \sec^2 x\,dx = \tan(x) + C$
- $\int \sec x \tan x\,dx = \sec(x) + C$
- $\int \csc^2 x\,dx = -\cot(x) + C$
- $\int \csc x \cot x\,dx = -\csc(x) + C$
- $\int e^x\,dx = e^x + C$
- $\int a^x\,dx = \frac{a^x}{\ln a} + C$
- $\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$
- $\int \frac{1}{x}\,dx = \ln|x| + C$

3. Evaluate the following integrals.

   (a) $\int_2^5 \frac{3}{x}\,dx =$

   $= 3\ln|x|\Big|_2^5 =$

   $= 3\ln 5 - 3\ln 2 = 3\ln\frac{5}{2}$

   (b) $\int_0^{\pi/2} \cos x\,dx = \sin(x)\Big|_0^{\frac{\pi}{2}} =$

   $= \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1$

4. Evaluate the following integrals.

(a) $\int_1^8 \sqrt[3]{x}\, dx$

(b) $\int_{\pi/6}^{\pi/2} \csc x \cot x \, dx =$

$= -\csc(x) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} =$

$= -\csc\left(\frac{\pi}{2}\right) + \csc\left(\frac{\pi}{6}\right)$

(c) $\int_0^1 \frac{9}{1+x^2}\, dx =$

$= 9 \arctan(x) \Big|_0^1 =$

$= 9\frac{\pi}{4} - 9 \cdot 0 = \frac{9\pi}{4}$

5. We do not have any product or quotient rules for antidifferentiation. To evaluate an integral that is expressed as a product or quotient you must try to manipulate the integrand (the stuff inside the $\int$ sign) to look like something you know how to antidifferentiate. The following integrals are examples of this. Evaluate the following integrals.

(a) $\int_1^3 \frac{x^3 + 3x^6}{x^4}\, dx =$

$= \int_1^3 \left(\frac{x^3}{x^4} + \frac{3x^6}{x^4} \right) dx =$

$= \int_1^3 \left(\frac{1}{x} + 3x^2 \right) dx =$

$= \left(\ln|x| + \cancel{3} \frac{x^3}{\cancel{3}} \right) \Big|_1^3 = (\ln 3 + 27) - (\overset{0}{\ln 1} + 1) = \ln 3 + 26$

(b) $\int_0^1 x(3+\sqrt{x})\, dx =$

$= \int_0^1 (3x + x^{3/2})\, dx =$

$= \left(\frac{3}{2}x^2 + \frac{2}{5}x^{5/2} \right) \Big|_0^1 = \left(\frac{3}{2} + \frac{2}{5} \right) - (0) =$

$= \frac{19}{10} = 1.9$

6. Evaluate the following integrals.

(a) $\int_0^2 (5^x + x^5)\, dx =$

$= \left(\frac{5^x}{\ln 5} + \frac{x^6}{6} \right) \Big|_0^2 =$

$= \left(\frac{25}{\ln 5} + \frac{64}{6} \right) - \frac{1}{\ln 5}$

(b) $\int_{1/2}^{\sqrt{2}/2} \frac{1}{\sqrt{1-x^2}}\, dx =$

$= \arcsin(x) \Big|_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} =$

$= \arcsin\left(\frac{\sqrt{2}}{2}\right) - \arcsin\left(\frac{1}{2}\right) =$

$= \frac{\pi}{4} - \frac{\pi}{6} = \frac{2\pi}{24} = \frac{\pi}{12}$

7. What is wrong with the following calculation? (Hint: draw a picture!)

$$\int_{-1}^{3} \frac{1}{x^2}\, dx = \frac{x^{-1}}{-1} \Bigg|_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$