Section 4.4. Indeterminate Forms and L'Hospital's Rule

$F(x) = \dfrac{\ln x}{x-1}$

$$\lim_{x \to 1} \frac{\ln x}{x-1} = \boxed{\frac{0}{0}}$$

Indeterminate forms:

$\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞

$\dfrac{0}{0}$: L'H

$\dfrac{\infty}{\infty}$: L'H

$0 \cdot \infty$: $\dfrac{1}{\infty} \cdot \infty$ L'H; $0 \cdot \dfrac{1}{0}$ L'H

$\infty - \infty$: Algebra, $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ L'H

0^0, ∞^0, 1^∞: take ln of our function, $0 \cdot \infty$, $\dfrac{\infty}{\infty}$ or $\dfrac{0}{0}$ L'H

L'Hospital's Rule

Suppose that f and g are differentiable functions and $g'(x) \neq 0$ on some open interval I that contains a.

Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \underbrace{\lim_{x \to a} \frac{f'(x)}{g'(x)}}_{\text{exists or is equal } \pm\infty.}$$

$\frac{0}{0}$ $\quad$ $\frac{\pm\infty}{\pm\infty}$

Evaluate:

1. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2-5x+6}$ (type $\dfrac{4-4}{10-10}$) $= \dfrac{0}{0}$

L'H

$$\lim_{x\to 2} \frac{x^2-4}{x^2-5x+6} \overset{L'H}{=} \lim_{x\to 2} \frac{2x}{2x-5} = \frac{2\cdot 2}{2\cdot 2-5} = \frac{4}{-1} = -4$$

2. $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$ (type $\dfrac{0}{0}$)

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{\cos x}{1} = \frac{1}{1} = 1$$

3. $\lim\limits_{x\to 0} \dfrac{\tan(5x)}{\sin(3x)}$ (type $\dfrac{0}{0}$)

$$\lim_{x\to 0} \frac{\tan(5x)}{\sin(3x)} \overset{L'H}{=} \lim_{x\to 0} \frac{\sec^2(5x)\cdot 5}{\cos(3x)\cdot 3} = \frac{1\cdot 5}{1\cdot 3} = \frac{5}{3}$$

4. $\lim\limits_{u\to\infty} \dfrac{e^{u/10}}{u^2}$ (type $\dfrac{\infty}{\infty}$)

$$\lim_{u\to\infty} \frac{e^{u/10}}{u^2} \overset{L'H}{=} \lim_{u\to\infty} \frac{e^{u/10}\cdot\frac{1}{10}}{2u} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{u\to\infty} \frac{e^{\frac{u}{10}}\cdot\frac{1}{100}}{2} =$$

$$= \infty$$

Indeterminate forms.
Indeterminate products and differences

$\frac{0}{0}$ $\frac{\infty}{\infty}$

L'H $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$

$0 \cdot \infty$ $\lim\limits_{x\to a} f(x) = 0$

$\lim\limits_{x\to a} g(x) = \pm\infty$

$$\lim_{x\to a} \left(f(x) \cdot g(x) \right) = 0 \cdot (\pm\infty)$$

$$f \cdot g = \frac{1}{1/f} \cdot g = f \cdot \frac{1}{1/g}$$

$$0 \cdot \infty = \frac{1}{1/0} \cdot \infty = \frac{1}{\infty} \cdot \infty = \frac{\infty}{\infty}$$

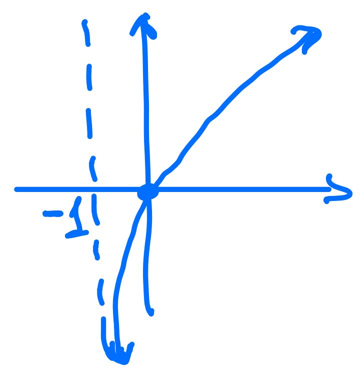
$$0 \cdot \infty = 0 \cdot \frac{1}{1/\infty} = 0 \cdot \frac{1}{0} = \frac{0}{0}$$

Example (Indeterminate Product Form)

$$\lim_{x \to \infty} e^{-2x} \cdot \ln(x+1) = 0 \cdot \infty$$

$$\lim_{x \to \infty} e^{-2x} = \lim_{x \to \infty} \frac{1}{e^{2x}} = 0$$

$$\lim_{x \to \infty} \ln(x+1) = \infty$$



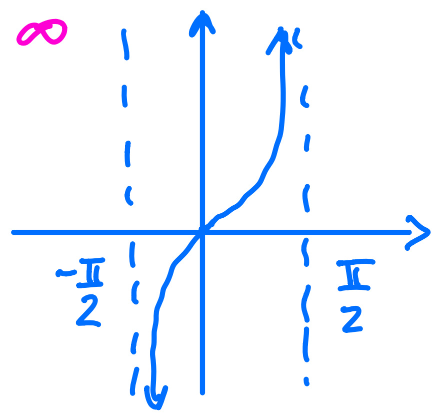
$$\lim_{x \to \infty} \underbrace{e^{-2x}}_{f} \cdot \underbrace{\ln(x+1)}_{g} = \lim_{x \to \infty} \frac{1}{e^{2x}} \cdot \ln(x+1) =$$

$$= \lim_{x \to \infty} \frac{\ln(x+1)}{e^{2x}} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x \to \infty} \frac{\frac{1}{x+1} \cdot 1}{e^{2x} \cdot 2} =$$

$$= \lim_{x \to \infty} \frac{1}{2(x+1) \cdot e^{2x}} = 0$$

Example (Indeterminate Product Form)

$$\lim_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2}\right) \cdot \tan(x) = 0 \cdot \infty$$



Approach #1

$$\lim_{x \to \frac{\pi}{2}} \left(x - \frac{\pi}{2}\right) \cdot \tan(x) =$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1}{\frac{1}{x - \frac{\pi}{2}}} \cdot \tan(x) \overset{\frac{1}{\infty} \cdot \infty}{\underset{L'H}{=}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\tan(x)}{\frac{1}{(x - \frac{\pi}{2})}} = \lim_{x \to \frac{\pi}{2}} \frac{\sec^2(x)}{\frac{-1}{(x - \frac{\pi}{2})^2} \cdot 1} =$$

$$= -\lim_{x \to \frac{\pi}{2}} \sec^2(x) \cdot \left(x - \frac{\pi}{2}\right)^2 =$$

$$= -\lim_{x \to \frac{\pi}{2}} \underbrace{\frac{1}{\cos^2(x)}}_{\to \infty} \cdot \underbrace{\left(x - \frac{\pi}{2}\right)^2}_{\to 0} = \infty \cdot 0$$

Approach # 2

$$\lim_{x \to \frac{\pi}{2}} \frac{\left(x - \frac{\pi}{2}\right)}{1/\tan(x)} \quad 0 \cdot \frac{1}{0} = \text{L'H}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1}{-\frac{1}{\tan^2(x)} \cdot \sec^2(x)} =$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1}{-\frac{\cancel{\cos^2(x)}}{\sin^2(x)} \cdot \frac{1}{\cancel{\cos^2(x)}}} =$$

$$= \lim_{x \to \frac{\pi}{2}} -\sin^2(x) = -1.$$

5. $\lim\limits_{x\to 0}\dfrac{\cos(4x)}{e^{2x}}$ (type $\frac{1}{1}$) $= 1$

$$\lim_{x\to 0}\frac{\cos(4x)}{e^{2x}} = \frac{1}{1} = 1.$$

6. $\lim\limits_{x\to 0}\dfrac{xe^{x}}{2^{x}-1}$ (type $\frac{0}{0}$)

$$\lim_{x\to 0}\frac{xe^{x}}{2^{x}-1} \overset{L'H}{=} \lim_{x\to 0}\frac{e^{x}(x+1)}{2^{x}\ln 2} = \frac{1}{1\cdot \ln 2} = \frac{1}{\ln 2}$$

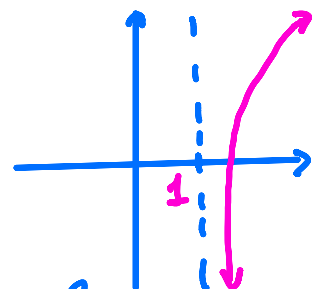
$\sqrt{\dots} - \sqrt{\dots}$ Indeterminate differences

$-\infty + \infty = \infty - \infty$

7. $\lim\limits_{x\to 1^{+}}\left(\ln(x^{4}-1) - \ln(x^{9}-1)\right)$ (type ______)

$$\lim_{x\to 1^{+}}\ln\left(\frac{x^{4}-1}{x^{9}-1}\right) =$$

$$= \ln\left(\lim_{x\to 1^{+}}\frac{x^{4}-1}{x^{9}-1}\right) \overset{L'H}{\underset{\frac{0}{0}}{=}} \ln\left(\lim_{x\to 1^{+}}\frac{4x^{3}}{9x^{8}}\right) = \ln\left(\frac{4}{9}\right)$$



8. $\lim\limits_{x\to\infty}\sqrt{x}e^{-x/2}$ (type ______)

9. $\lim\limits_{x\to 0^{+}}\underbrace{(1+\sin(2x))}^{1/x}$ (type 1^{∞})

$$y = (1+\sin(2x))^{1/x}$$

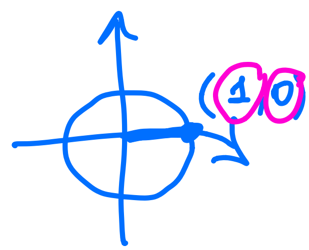
$$\ln y = \ln(1+\sin(2x))^{1/x} = \frac{1}{x}\ln(1+\sin(2x))$$

$$\lim_{x\to 0^{+}}\ln y = \lim_{x\to 0^{+}}\frac{1}{x}\cdot\ln(1+\sin(2x)) =$$

$$\frac{1}{0}\cdot 0 = \frac{0}{0} \quad \underset{L'H}{=} \quad \lim_{x\to 0^+} \frac{\frac{1}{1+\sin(2x)}\cdot \cos(2x)\cdot 2}{1} =$$

$$= \lim_{x\to 0^+} \frac{2\cdot \cos(2x)}{1+\sin(2x)} = \frac{2\cdot 1}{1} = 2$$

$$\lim_{x\to 0^+} \ln y = 2$$



$$e^{\ln\left(\lim_{x\to 0^+} y\right)} = e^{2}$$

$$e^{\ln(x)} = x$$

$$\lim_{x\to 0^+} y = \boxed{e^{2}}$$

8. $\lim\limits_{x\to\infty} \sqrt{x}\, e^{-\frac{x}{2}} = \infty \cdot 0$

$$= \lim_{x\to\infty} \sqrt{x} \cdot \frac{1}{e^{\frac{x}{2}}} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^{\frac{x}{2}}} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{\cancel{2}\sqrt{x}}}{e^{\frac{x}{2}} \cdot \frac{1}{\cancel{2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{x} \cdot e^{\frac{x}{2}}} =$$

$= 0.$