Section 4.4. Indeterminate Forms and L'Hospital's Rule

Let us consider $F(x) = \dfrac{\ln x}{x-1}$

How F behaves near 1 ?

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \frac{0}{0}$$

may or may not exist

indeterminate form of $\frac{0}{0}$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ (we use a geometric method)

There exists another method for finding a limit l'Hospital's Rule

$$\lim_{x\to \infty} \frac{\ln x}{x-1} = \frac{\infty}{\infty}$$

indeterminate form of $\frac{\infty}{\infty}$

L' Hospital's Rule

Suppose f and g are differentiable

and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a).
Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

$$\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty} \right)$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Note: L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity:

'$x \to a$' $\Rightarrow$ '$x \to a^+$', '$x \to a^-$', $x \to \infty$
$x \to -\infty$

Example

1) $\displaystyle \lim_{x \to 1} \frac{\ln x}{x-1} = \boxed{\frac{0}{0}} \stackrel{L'H}{=} \lim_{x \to 1} \frac{\frac{1}{x}}{1} = \lim_{x \to 1} \frac{1}{x} = 1$

2) $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2} = \boxed{\dfrac{\infty}{\infty}} \overset{L'H}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{2x} = \boxed{\dfrac{\infty}{\infty}} \overset{L'H}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{2} = \infty$

3) $\lim\limits_{x\to\pi^-} \dfrac{\sin x}{1-\cos x} = \dfrac{0}{2} = 0$

- ## Indeterminate Products

If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = \infty \, (-\infty)$, then

$$\lim_{x\to a} f(x)\cdot g(x) = ?$$

$0\cdot\infty$ indeterminate form

How to deal with that:

$$f\cdot g = \frac{f}{1/g} \quad \text{or} \quad f\cdot g = \frac{g}{1/f}$$

$$\downarrow$$

$\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ $\rightarrow$ use L'Hospital's Rule

Example

$$\lim_{\substack{x\to 0^+}} \underset{0\cdot\infty}{x\ln x} = \lim_{x\to 0^+} \frac{x}{1/\ln x} = \lim_{x\to 0^+} \frac{\ln x}{1/x} =$$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = 0.$$

- **Indeterminate Differences**

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a} f(x) = \infty$$

$$\lim_{x\to a}\left(f(x) - g(x)\right) = \infty - \infty$$

indeterminate form of type $\infty - \infty$

How to deal with that:
convert difference into a quotient, so that we have $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1^+} \frac{x-1-\ln x}{(x-1)\ln x} =$$

$$= \lim_{x\to 1^+} \frac{1-\frac{1}{x}}{\ln x + (x-1)\frac{1}{x}} = \lim_{x\to 1^+} \frac{\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}.$$

- **Indeterminate Powers**

$$\lim_{x\to a} f(x)^{g(x)}$$

1. 0^0 2. ∞^0 3. 1^∞

How to deal with that:

- take the natural logarithm:

$$y = f^g$$

$$\ln y = \overset{0}{g} \cdot \overset{\infty}{\ln f}$$

- writing the function as an exponential:

$$f^g = e^{g \ln f} \qquad 0 \cdot \infty$$

Example

$$\lim_{x \to 0^+} (1 + \sin(4x))^{\cot x} = \lim_{x \to 0^+} e^{\cot x \cdot \ln(1+\sin 4x)} =$$

$$= \lim_{x \to 0^+} e^{\frac{\ln(1+\sin 4x)}{1/\cot x}} = \lim_{x \to 0^+} e^{\frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x}} = e^4$$