Review Part :

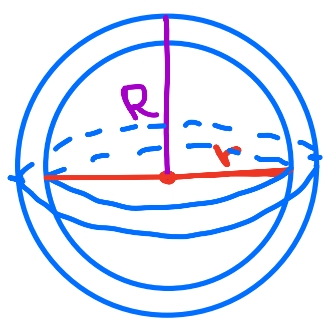
Quiz #6

- Related Rates
- Approx. By Liniarization
- Abs max and min values of the given function f(x)

Pr. #1.

The radius of a sphere is ↗ at a rate of 4 mm/s. How fast is the volume increasing when d = 80 mm?

1.



2. What we know

$$r'(t) = \frac{dr}{dt} = 4 \text{ mm/s}$$

$d = 80$ mm

$r = 40$ mm

3. What we want

$V'(t) = \frac{dV}{dt}$ when $d = 80$ mm ?
$r = 40$ mm.

4. $V = \frac{4}{3}\pi r^3$ $\qquad r = r(t)$

5. Implicit differentiation:

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3} r^2 \cdot \frac{dr}{dt}$$

$$(r^3)' = 3r^2 \cdot \frac{dr}{dt}$$

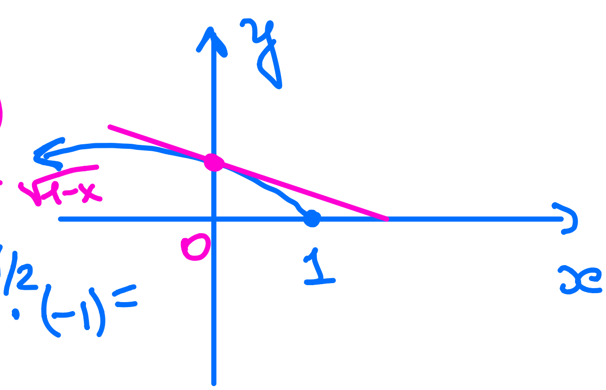
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$\frac{dV}{dt}$ when $r = 40$ mm $= 4\pi \cdot 40^2 \cdot 4 =$

$$= 25600\pi \left(\mathrm{mm}^3/s\right)$$

Pr. #2 Find the linear approx. of $f(x) = \sqrt{1-x}$ at $a = 0$.

$$L(x) = f'(a)(x-a) + f(a)$$



$$f'(x) = \left(\sqrt{1-x}\right)' = \frac{1}{2}(1-x)^{-1/2} \cdot (-1) =$$

$$= \frac{-1}{2\sqrt{1-x}}$$

$$f'(0) = -\frac{1}{2\sqrt{1-0}} = -\frac{1}{2}$$

$$f(0) = \sqrt{1-0} = 1$$

$$L(x) = -\frac{1}{2}(x-0) + 1 = -\frac{1}{2}x + 1$$

$$L(x) = -\frac{1}{2}x + 1$$

$$\sqrt{0.9} \approx L(0.9) = -\frac{1}{2} \cdot 0.9 + 1 =$$

$$= -\frac{1}{2} \cdot \frac{9}{10} + 1 = \frac{-9}{20} + \frac{20}{20} =$$

$$= \frac{11}{20}$$

Pr. #3. Find the abs max and min values of $f(x)$ on $[a, b]$.

1. (a): Find cp for $f(x)$:

$f'(c) = 0$ or $f'(c)$ DNE

(b): $f(c) =$

2. $f(a) =$

$f(b) =$

3. $\max\{f(c), f(a), f(b)\}$ = abs max of $f(x)$

$\min\{\ldots\ldots\}$ = abs min of $f(x)$

Section 4.3. How Derivatives affect the shape of the graph

Let us consider $y = f(x)$ on its domain D.

Def. (a) if $f'(x) > 0$ on an interval I, then $f(x)$ is increasing on that interval

(B) if $f'(x) < 0$ on an interval I, then $f(x)$ is $\searrow$ on that interval.

Example $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

1) where $f(x) \nearrow$
2) where $f(x) \searrow$

$\text{Dom}(f) = \mathbb{R}$

$$f'(x) = 12x^3 - 12x^2 - 24x =$$
$$= 12x(x^2 - x - 2) =$$
$$= 12x(x-2)(x+1)$$

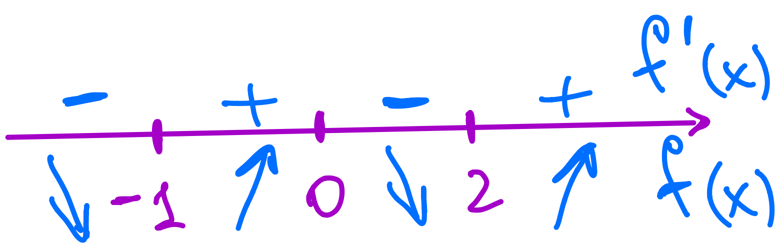
$f'(x) > 0$ $\qquad\qquad\qquad\qquad$ $f'(x) < 0$

$f'(x) = 0$ when

$$12x(x-2)(x+1) = 0$$

$x = 0$ or $x - 2 = 0$ or $x + 1 = 0$

$\qquad\qquad x = 2 \qquad\qquad x = -1$



$f(x)$ is increasing on $(-1, 0) \cup (2, \infty)$

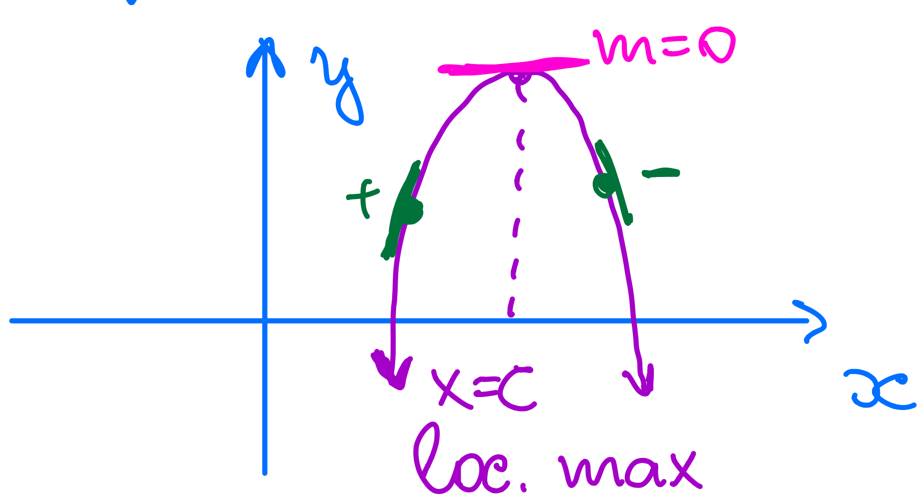
$f(x)$ is decreasing on $(-\infty, -1) \cup (0, 2)$

The first Derivative Test:

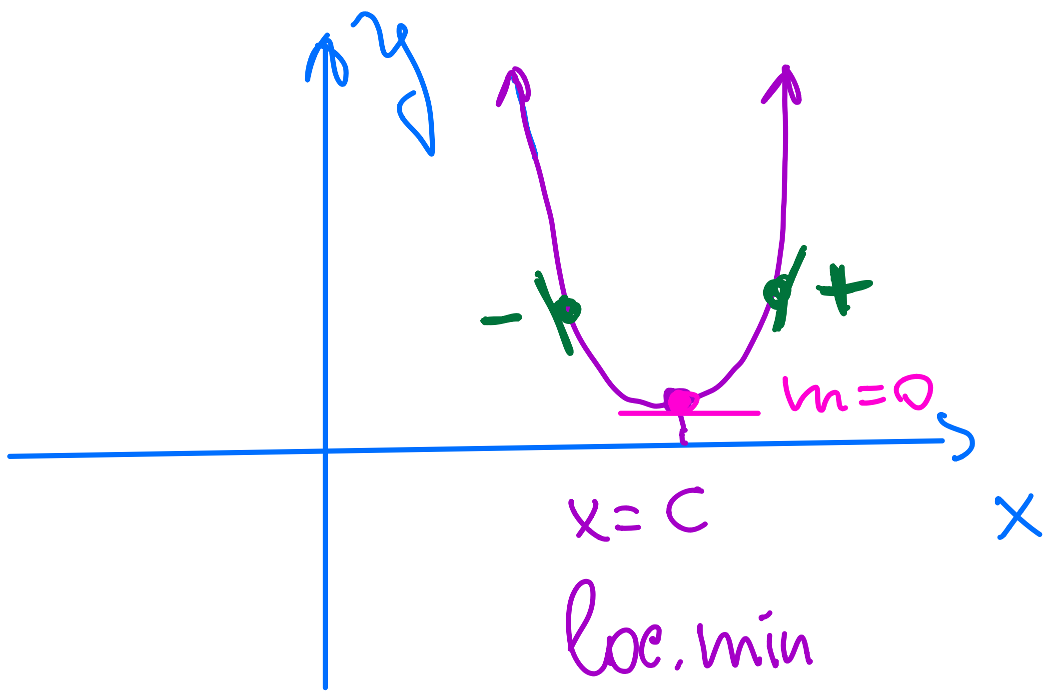
Suppose that c is a critical number of a continuous function f(x).

Then

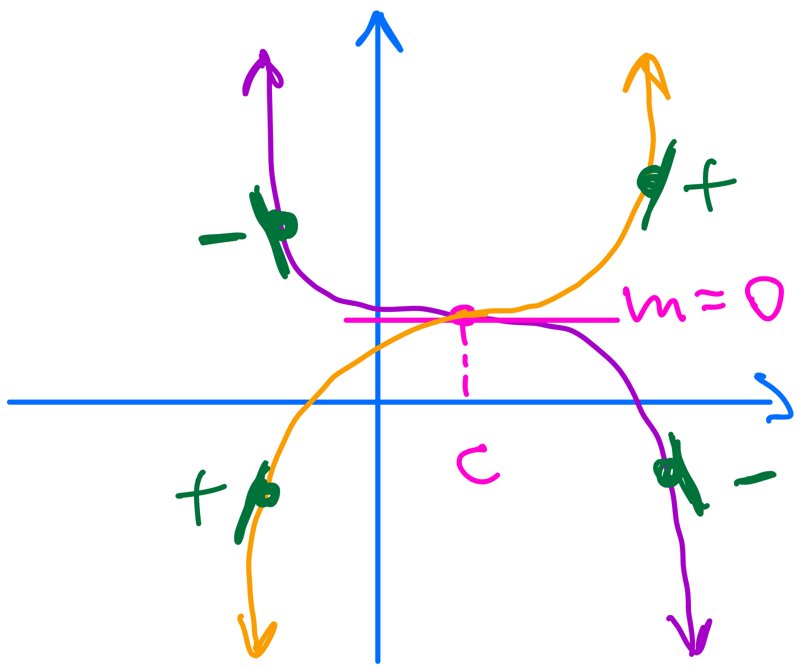
(a) if $f'(x)$ changes from + to − near that CP c, then f has a loc. max at $x=c$



(b) if $f'(x)$ changes from $-$ to $+$ near that CP c, then f has a loc. min at $x=c$



(c) If (a) or (b) does not hold, then $x=c$ is neither loc. max or loc. min.

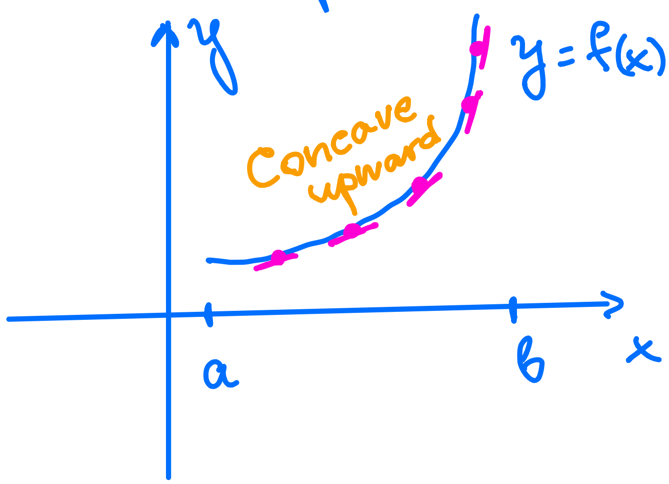


Section 4.3. How derivatives affect the shape of a graph (Day two)

Second Derivative Test

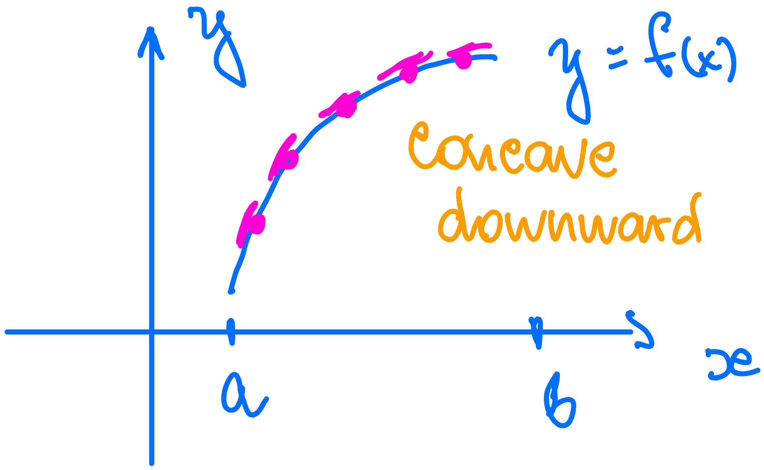
Def.

- If the graph of the function $f(x)$ lies above all its tangents on the interval I, then it is called concave upward on I.



CU ∪

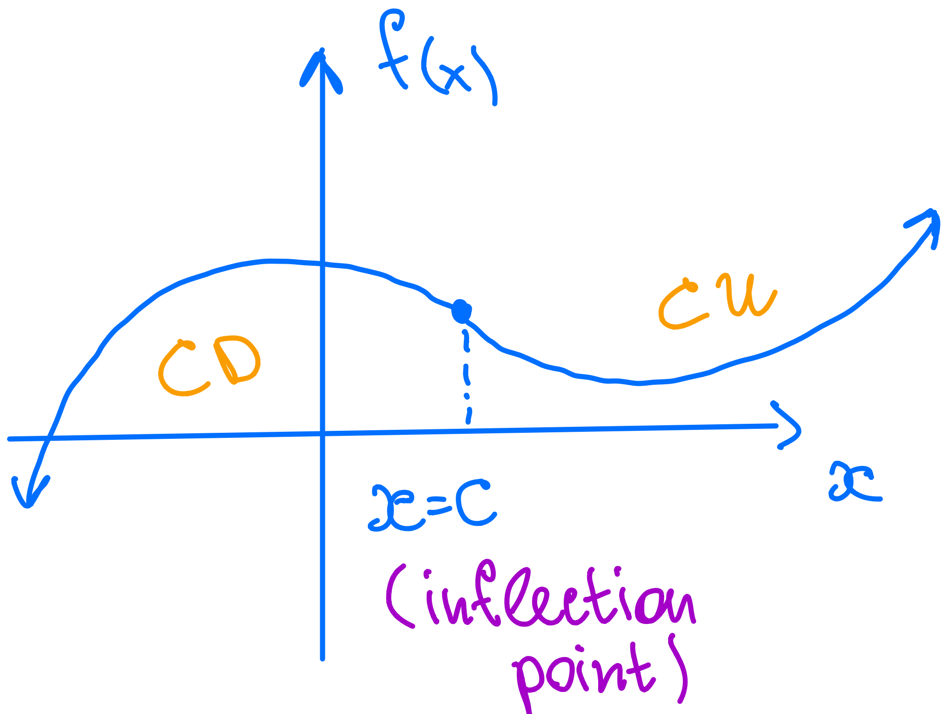
- If the graph of $f(x)$ lies below all its tangents on I, then it is called concave downward on I.



CD ∩

Concavity Test

- If $f''(x) > 0$ for all x in I, then the graph of the $f(x)$ is concave upward on I ∪
- If $f''(x) < 0$ for all x in I, then the graph of the $f(x)$ is concave downward ∩



Def. A point $x=c$ on the curve $y=f(x)$ is called an INFLECTION POINT if f is continuous there and the curve changes from

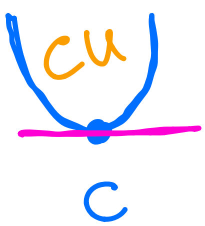
CU to CD
CD to CU

at $x=c$.

The Second Derivative Test

Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a loc. min at c



$f'(c) = 0$
$f''(c) > 0$

- If $f'(c) = 0$ and $f''(c) < 0$, then f has a loc. max at c

$f'(c) = 0$
$f''(c) < 0$

1. Consider $f(x) = \frac{2}{3}x^3 + x^2 - 12x + 7$, and observe $f'(x) = 2x^2 - 2x - 12 = 2(x-2)(x+3)$.

   (a) What are the critical points of $f(x)$? (Where does $f'(x) = 0$?) $x = 2, -3$

   $f'(x) = 0$ or $f'(x)$ DNE $\quad 2(x-2)(x+3) = 0$

   $x = 2$ or $x = -3$

   (b) Fill in the following table, by evaluating $f'(x)$ at "sample points" in the intervals:

x	$x < -3$	-3	$-3 < x < 2$	2	$x > 2$
sample point	-4	-3	0	2	5
sign or value of f'	+	0	−	0	+
Increasing/decreasing: f is ↗ or ↘	↗	loc max	↘	loc min	↗

   (c) On what interval(s) is $f(x)$ increasing? $(-\infty, -3) \cup (2, \infty)$ decreasing? $(-3, 2)$

   (d) Use the First Derivative Test to determine where f has a local max and local min (if any):

   i. Local max at $x =$ -3 because f' goes from + to −.

   ii. Local min at $x =$ 2 because f' goes from − to +

   (e) It is a fact that $f''(x) = 4x - 2$, so $f''(x) = 0$ when $x =$ $\frac{1}{2}$.

   Fill in the expanded chart:

x	$x < -3$	-3	$-3 < x < 1/2$	$1/2$	$1/2 < x < 2$	2	$x > 2$
sample point	-4	-3	0	$1/2$	1	2	5
sign or value of f'	+	0	−	−	−	0	+
sign or value of f''	−	−	−	0	+	+	+
concavity: f is ↗↘⤴⤵	↘	loc max	↘	IP	↗	loc min	↗

   Inflection point: $f''(x) = 0 \Rightarrow 4x - 2 = 0$

   $x = \frac{1}{2}$

   (f) Use the Second Derivative Test to determine where f has local maxima or minima:

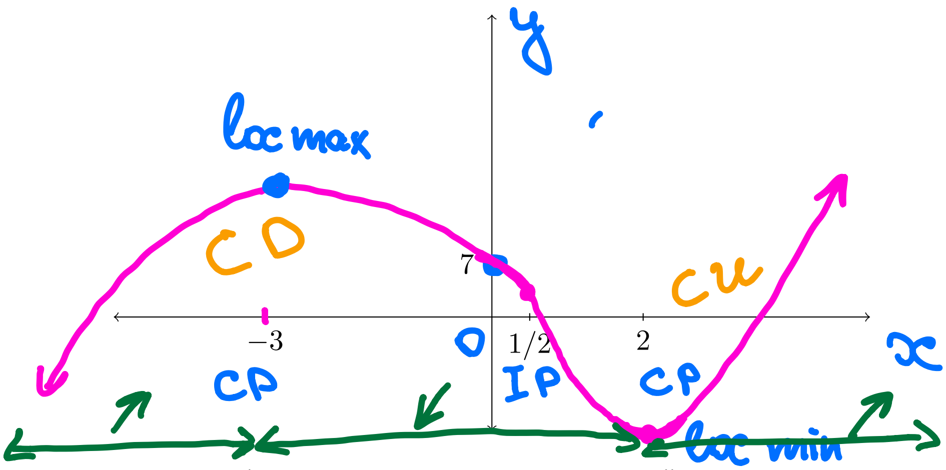
   i. Local max at $x =$ -3 because $f'(-3) =$ 0 and $f''(-3)$ < 0.

   ii. Local min at $x =$ 2 because $f'(2) =$ 0 and $f''(2)$ > 0.

   (g) Where does f have an inflection point? $x =$ $\frac{1}{2}$

   How do you know? $f''(\frac{1}{2}) = 0$ and $f(x)$ is changing its concavity

(h) Use the information you collected to sketch the graph of $f(x)$. You don't have to be accurate with the y-values, but they should be correct relative to each other. Because $f(0) = 7$, you can use that to "nail down" the position of your curve on the graph. Note that



2. Consider $g(x) = xe^x$, and note $g'(x) = xe^x + x = e^x(x+1)$ and $g''(x) = e^x(x+2)$.

(a) What are the critical point(s) of $g(x)$?

$g'(x) = 0$ or ~~$g'(x)$ DNE~~ $e^x(x+1) = 0$

$x = -1$ CP

(b) Where is g increasing? $(-1, +\infty)$

− + $g'(x)$

−1

(c) Use the First Derivative Test to determine whether g has a local max or min at its critical point.

x	$x<-1$	-1	$x>-1$
$g'(x)$	−	0	+
$g(x)$	↘	loc min	↗

x is a CP

$g'(x)$ + −

$g'(x)$ − +

(d) Use the Second Derivative Test to determine whether g has a local max or min at its critical point.

IP: $g''(x) = 0$

$e^x(x+2) = 0$

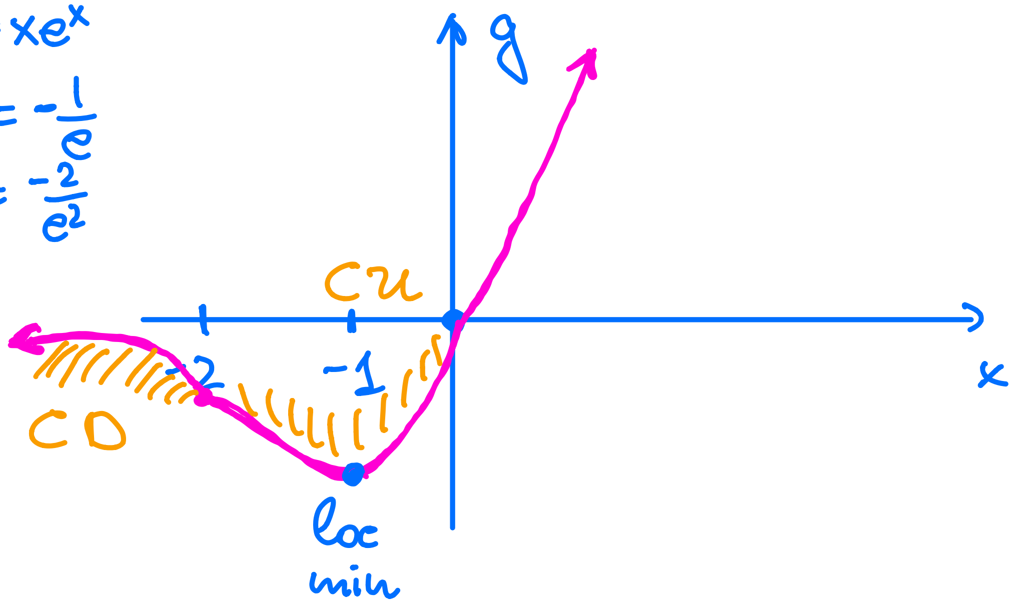
$x = -2$

x	$x<-2$	-2	$-2<x<-1$	-1	$x>-1$
$g'(x)$	−	−	−	0	+
$g''(x)$	−	0	+	+	+
	↷	IP	↖	loc min	↗

$g(x) = xe^x$

$g(-1) = -\frac{1}{e}$

$g(-2) = -\frac{2}{e^2}$



II Der. Test:

- if $f'(c)=0$ and $f''(c)>0$, then at $x=c$ $f(x)$ has a loc. min 
- if $f'(c)=0$ and $f''(c)<0$, then at $x=c$ $f(x)$ has a loc. max 

3. Consider the function $h(x) = x^3$ and observe $h'(x) = 3x^2$ and $h''(x) = 6x$.

   (a) What are the critical point(s) of $h(x)$?

   (b) What happens when you try to use the Second Derivative Test to determine whether h has a local max or min at its critical point?

   (c) Make a table of first and second derivatives to determine where h is increasing, decreasing, concave up, and/or concave down. Then sketch h.

4. Consider the function $j(x) = x^4$ and observe $j'(x) = 4x^3$ and $j''(x) = 12x^2$.

   (a) What are the critical point(s) of $j(x)$?

   (b) What happens when you try to use the Second Derivative Test to determine whether j has a local max or min at its critical point?

   (c) Make a table of first and second derivatives to determine where j is increasing, decreasing, concave up, and/or concave down. Then sketch j.