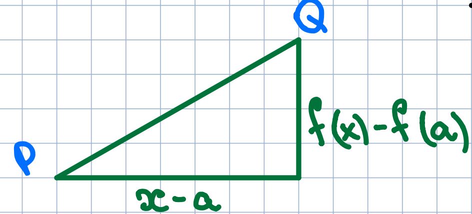
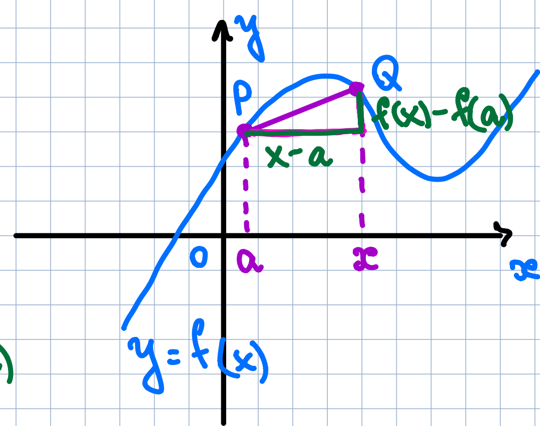
Section 2.7. Derivatives and Rates of Change

- **Tangents:**

  $P(a, f(a))$
  $Q(x, f(x))$





$$\boxed{m_{PQ}} = \frac{f(x)-f(a)}{x-a}$$

slope of the secant line PQ

Def. The tangent line to a curve $y=f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$\boxed{m = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}}$$

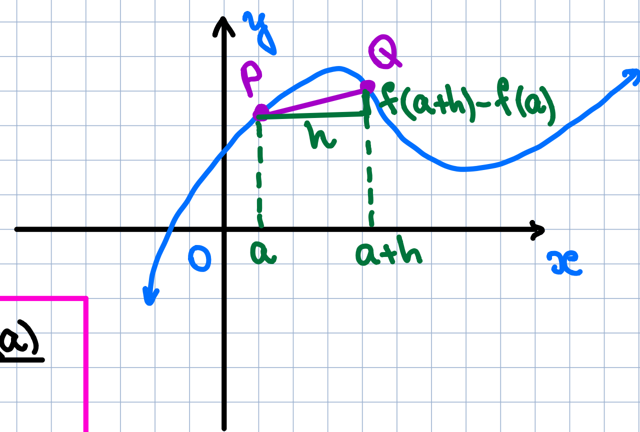
provided that this limit exists.

Def. Point-slope form for a line through the point (x_1, y_1) with slope m:

$$\boxed{y - y_1 = m(x - x_1)}$$

Let $x - a = h$. Then $x = a + h$.

$P(a, f(a))$
$Q(a+h, f(a+h))$



$$\boxed{m_{PQ} = \frac{f(a+h) - f(a)}{h}}$$

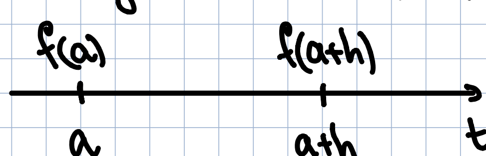
slope of a secant line

$$\boxed{m = \lim_{h \to 0} m_{PQ} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

slope of a tangent line

- **Velocities:**

Let us define a displacement function through $s = f(t)$ (position function)



$f(a+h) - f(a)$ is a change in position
$a + h - a = h$ is a change in time

$$\text{Average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h)-f(a)}{h}$$

Now let $h \to 0$.

Instantaneous velocity $v(a)$ at time $t=a$:

$$v(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

- **Derivatives:**

Def. The derivative of a function f at a number a, denoted by $f'(a)$, is

$$\boxed{f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}}$$

if this limit exists.

When $a+h=x$, $h=x-a$, then

$$\boxed{f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}}$$

Def. The tangent line to $y=f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a

$$\boxed{y-f(a) = f'(a)(x-a)}$$

- **Rates of change:**

Suppose quantity y depends on quantity x, that is $y=f(x)$.

$$\Delta x = x_2 - x_1 \quad (\text{change in } x)$$
$$\Delta y = f(x_2) - f(x_1) \quad (\text{change in } y)$$

Then

$\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2)-f(x_1)}{x_2-x_1}$ is an average rate of change of y with respect to x over $[x_1, x_2]$

Let $x_2 \to x_1$ $(\Delta x \to 0)$.

Def.
The limit of these average rates of change is called the instantaneous rate of change of y with respect to x at $x = x_1$ (slope of a tangent to the curve $y=f(x)$ at $P(x_1, f(x_1))$:

$$\text{instant. rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

The derivative $f'(a)$ is the instantaneous rate of change of $y=f(x)$ with respect to x when $x=a$.

Speed $= |f'(a)|$