Section 3.6. Derivatives of Logarithmic Functions

$$y = \log_b(x) \quad \text{or} \quad y = \ln(x)$$

Let $y = \log_b(x)$. Then $x = b^y$. Use implicit differentiation:

$$1 = b^y \ln b \cdot \frac{dy}{dx}$$

$$\frac{1}{\ln b \cdot b^y} = \frac{dy}{dx}$$

$$\boxed{\frac{d}{dx}\left(\log_b(x)\right) = \frac{dy}{dx} = \frac{1}{x \ln b}}$$

If $b = e$, then $y = \ln(x)$

$$\boxed{\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}}$$

$$\underline{\frac{d}{dx}\left(\ln(u)\right) = \frac{1}{u} \cdot \frac{du}{dx}} \quad \text{or} \quad \underline{\frac{d}{dx}\left(\ln(g(x))\right) = \frac{g'(x)}{g(x)}}$$

Example $\frac{d}{dx}\left(\ln(\sin(x))\right) = \frac{1}{\sin x} \cdot \cos(x) = \cot(x)$

Example $f(x) = \ln|x|$

$$f(x) = \begin{cases} \ln(x), & x > 0 \\ \ln(-x), & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{-1}{-x} = \frac{1}{x}, & x < 0 \end{cases}$$

$$f'(x) = \frac{1}{x}, \quad x \neq 0$$

$$\boxed{\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}}$$

Logarithmic Differentiation:

Example

$$y = \frac{x^{3/4}\left(\sqrt{x^2+1}\right)}{(3x+2)^5}$$

$$\ln y = \ln x^{3/4} + \ln \sqrt{x^2+1} - \ln(3x+2)^5$$

$$\frac{dy}{dx} \cdot \frac{1}{y} = \frac{3}{4}\,\frac{1}{x} + \frac{1}{2}\,\frac{1}{x^2+1} \cdot 2x - 5 \cdot \frac{1}{3x+2} \cdot 3$$

Express $\frac{dy}{dx}$, Substitute $y \longrightarrow \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

Steps in Logarithmic Differentiation:

1. Take natural logarithms of both sides

of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.

2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

The Power Rule: If n is any real number and $f(x) = x^n$, then

$$f'(x) = n \cdot x^{n-1}$$

Proof. Let $y = x^n$.

$$\ln|y| = \ln|x|^n = n \ln|x|$$

$$\frac{1}{y}\frac{dy}{dx} = n\frac{1}{x}$$

$$\frac{dy}{dx} = n\frac{y}{x} = n x^{n-1}$$

Four cases for exponents and bases:

1. $\frac{d}{dx}(b^n) = 0 \quad (b, n = \text{const})$
2. $\frac{d}{dx}(f(x))^n = n(f(x))^{n-1} \cdot f'(x)$
3. $\frac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)} \ln(b) \cdot (g(x))'$

④ $\frac{d}{dx}\left(f(x)^{g(x)}\right) = \frac{d}{dx}\left(e^{g(x)\ln f(x)}\right) = f^g \cdot \left(g' \ln f + \right.$

$\left. + g \frac{1}{f} \cdot f'\right).$

__Example__

$$y = x^{\sqrt{x}}$$

$$\ln y = \sqrt{x} \ln(x)$$

$$\frac{1}{y} \frac{dy}{dx} = \sqrt{x} \cdot \frac{1}{x} + \frac{1}{2\sqrt{x}} \ln(x)$$

$$y = x^{\sqrt{x}} = e^{\sqrt{x} \ln x}$$

$$y' = x^{\sqrt{x}} \left(\frac{1}{2\sqrt{x}} \ln(x) + \sqrt{x} \cdot \frac{1}{x} \right).$$

The number e as a limit:

$$f(x) = \ln(x), \quad f'(x) = \frac{1}{x}, \quad f'(1) = 1.$$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x} =$$

$$= \lim_{x \to 0} \frac{\ln(1+x) - \ln(1)}{x} = \lim_{x \to 0} \ln(1+x)^{1/x} = 1$$

$$\boxed{e = \lim_{x \to 0} (1 + x)^{1/x}}$$

$$e \approx 2.7182818\ldots$$

If we put $\frac{1}{x} = n$, then

$$\boxed{e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n}$$