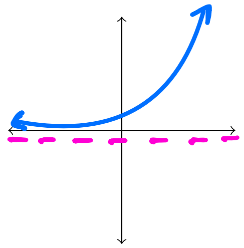
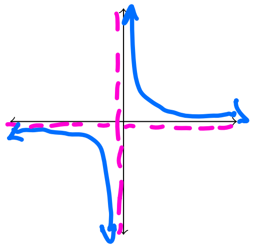
WORKSHEET: SECTION 2-6 (LIMITS AT INFINITY)

Solutions

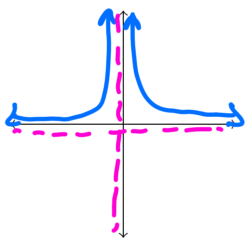
1. Sketch graphs of the following functions and then determine the limits at infinity below:



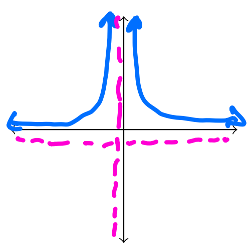
$\lim_{x\to -\infty} e^x = 0$ $\qquad$ $\lim_{x\to \infty} e^x = +\infty$



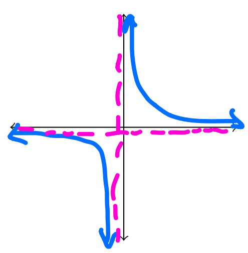
$\lim_{x\to -\infty} \frac{1}{x} = 0$ $\qquad$ $\lim_{x\to \infty} \frac{1}{x} = 0$



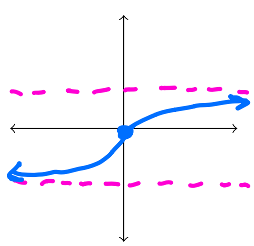
$\lim_{x\to -\infty} \frac{1}{x^2} = 0$ $\qquad$ $\lim_{x\to \infty} \frac{1}{x^2} = 0$



$\lim_{x\to -\infty} \frac{1}{x^{2k}} = 0$ $\qquad$ $\lim_{x\to \infty} \frac{1}{x^{2k}} = 0$



$\lim_{x\to -\infty} \frac{1}{x^{2k+1}} = 0$ $\qquad$ $\lim_{x\to \infty} \frac{1}{x^{2k+1}} = 0$

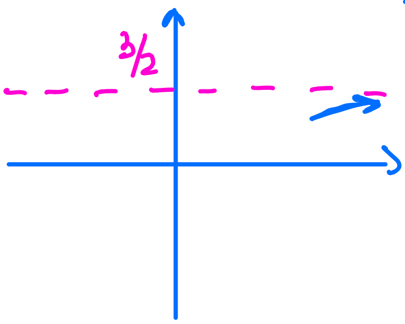


$\lim_{x\to -\infty} \arctan(x) = -\frac{\pi}{2}$ $\qquad$ $\lim_{x\to \infty} \arctan(x) = \frac{\pi}{2}$

2. **Algebraically** find the limits below and draw a picture demonstrating what this limit indicates about the graph of the function.

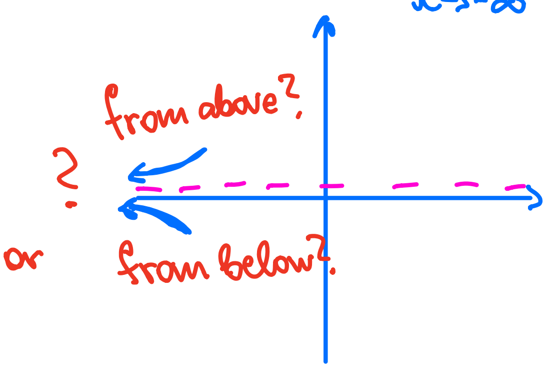
$$\lim_{x\to\infty}\frac{3x^2+4x}{2x^2+7} = \lim_{x\to\infty}\frac{\frac{1}{x^2}(3x^2+4x)}{\frac{1}{x^2}(2x^2+7)} = \lim_{x\to\infty}\frac{3+\frac{4}{x}}{2+\frac{7}{x^2}} = \frac{3}{2}$$

$\frac{4}{x} \to 0$, $\frac{7}{x^2} \to 0$



$$\lim_{x\to-\infty}\frac{3x^2+4x}{2x^4+7} = \lim_{x\to-\infty}\frac{\frac{1}{x^4}(3x^2+4x)}{\frac{1}{x^4}(2x^4+7)} = \lim_{x\to-\infty}\frac{\frac{3}{x^2}+\frac{4}{x^3}}{2+\frac{7}{x^4}} = 0$$

$\frac{3}{x^2} \to 0$, $\frac{4}{x^3} \to 0$, $\frac{7}{x^4} \to 0$



We can notice that if $x \to -\infty$ $y = \frac{2x^2+4x}{2x^4+7}$ is positive (since $\frac{2x^2}{2x^4}$ dominates)

3. Compute the following infinite limits:

(a) $\lim_{x\to\infty}\frac{1+5e^x}{7-e^x} = \lim_{x\to\infty}\frac{\frac{1}{e^x}(1+5e^x)}{\frac{1}{e^x}(7-e^x)} = \lim_{x\to\infty}\frac{\frac{1}{e^x}+5}{\frac{7}{e^x}-1} = -5$

$\frac{1}{e^x} \to 0$, $\frac{7}{e^x} \to 0$

(b) $\lim_{x\to\infty}[\ln(2+3x)-\ln(1+x)]$

$$\lim_{x\to\infty}(\ln(2+3x)-\ln(1+x)) = \lim_{x\to\infty}\ln\frac{2+3x}{1+x} = \ln\left(\lim_{x\to\infty}\frac{2+3x}{1+x}\right) = \ln(3).$$

Since $\ln(x)$ is continuous

(c) $\lim_{x\to\infty}(\sqrt{x^2+x}-x)$ = $\infty - \infty = 0$

wrong!!!

$$\lim_{x\to\infty}(\sqrt{x^2+x}-x) = \lim_{x\to\infty}\frac{\cancel{x^2}+x-\cancel{x^2}}{\sqrt{x^2+x}+x} = \lim_{x\to\infty}\frac{\frac{1}{x}\cdot x}{\frac{1}{x}(x+\sqrt{x^2+x})} =$$

$$= \lim_{x\to\infty}\frac{1}{1+\sqrt{1+\frac{1}{x}}} = \frac{1}{2}.$$

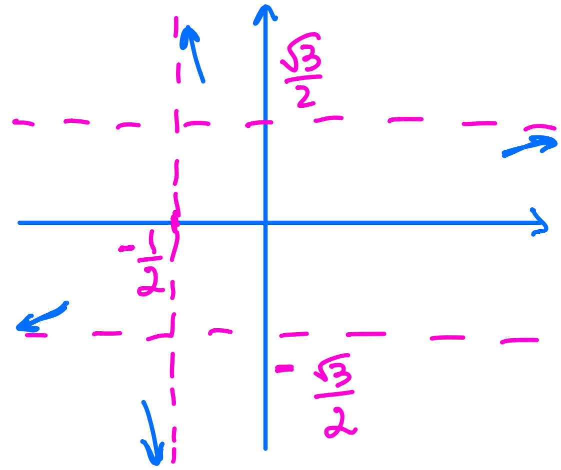
4. Find all vertical and horizontal asymptotes of the curves below. If none exists, state that explicitly. On quizzes and tests, you will be asked to show your work, so practice that now!

(a) $g(s) = \dfrac{\sqrt{3s^2+1}}{2s+1}$. (rational function)

$\sqrt{3s^2+1}$ is defined on $(-\infty, \infty)$

$2s+1=0 \Leftrightarrow s=-\frac{1}{2}$

$\boxed{\text{VA:}}$ $\lim_{x\to a} f(x) = \infty$



$$\lim_{s\to -\frac{1}{2}^+} \frac{\sqrt{3s^2+1}}{2s+1} = \frac{\sqrt{\frac{3}{4}+1}}{0^+} = +\infty$$

$$\lim_{s\to -\frac{1}{2}^-} \frac{\sqrt{3s^2+1}}{2s+1} = \frac{\sqrt{\frac{3}{4}+1}}{0^-} = -\infty$$

$\boxed{\text{HA:}}$ $\lim_{x\to\pm\infty} f(x) = L$

$$\lim_{s\to\infty} \frac{\sqrt{3s^2+1}}{2s+1} = \lim_{s\to\infty} \frac{\frac{1}{s}\sqrt{3s^2+1}}{\frac{1}{s}(2s+1)} = \lim_{s\to\infty} \frac{\sqrt{3+\frac{1}{s^2}}}{2+\frac{1}{s}} = \frac{\sqrt{3}}{2}$$

$$\lim_{s\to-\infty} \frac{\sqrt{3s^2+1}}{2s+1} = \boxed{\begin{array}{l} s=-t \\ s\to-\infty \\ t\to+\infty \end{array}} = \lim_{t\to\infty} \frac{\sqrt{3t^2+1}}{-2t+1} = -\frac{\sqrt{3}}{2}$$

(b) $y = \dfrac{2x^2 - x - 1}{3x^2 - 2x - 1}$

$\boxed{\text{VA:}}$ (usually appears at points for which the denominator is 0)

$3x^2 - 2x - 1 = 0$

$(3x+1)(x-1) = 0 \Rightarrow \boxed{x_1 = -\frac{1}{3}}, \boxed{x_2 = 1}$

$$\lim_{x\to -\frac{1}{3}^+} \frac{2x^2-x-1}{(3x+1)(x-1)} = \frac{\overbrace{\frac{2}{9}+\frac{1}{3}-1}^{<0}}{0^+\underbrace{\left(-\frac{1}{3}-1\right)}_{<0}} = \boxed{+\infty}$$

$$\lim_{x\to -\frac{1}{3}^-} \frac{2x^2-x-1}{(3x+1)(x-1)} = \frac{\overbrace{\frac{2}{9}+\frac{1}{3}-1}^{<0}}{0^-\underbrace{\left(-\frac{1}{3}-1\right)}_{<0}} = \boxed{-\infty}$$

$$\lim_{x\to 1} \frac{2x^2-x-1}{(3x+1)(x-1)} = \lim_{x\to 1^+} \frac{(2x+1)\cancel{(x-1)}}{(3x+1)\cancel{(x-1)}} = \frac{3}{4}$$

(There is a hole at $x=1$)

$\boxed{\text{HA:}}$ Def. $y = L$ is a HA if

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

$$\lim_{x\to\infty} \frac{2x^2-x-1}{3x^2-2x-1} = \lim_{x\to\infty} \frac{\frac{1}{x^2}(2x^2-x-1)}{\frac{1}{x^2}(3x^2-2x-1)} = \lim_{x\to\infty} \frac{2-\overset{\to 0}{\frac{1}{x}}-\overset{\to 0}{\frac{1}{x^2}}}{3-\underset{\downarrow 0}{\frac{2}{x}}-\underset{\downarrow 0}{\frac{1}{x^2}}} =$$

$$= \boxed{\frac{2}{3}}$$

$$\lim_{x\to-\infty} \frac{2x^2-x-1}{3x^2-2x-1} = \lim_{x\to\infty} \frac{2(-x)^2-(-x)-1}{3(-x)^2-2(-x)-1} = \lim_{x\to\infty} \frac{2x^2+x-1}{3x^2+2x-1} =$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x^2}(2x^2+x-1)}{\frac{1}{x^2}(3x^2+2x-1)} = \lim_{x\to\infty} \frac{2+\overset{\to 0}{\frac{1}{x}}-\overset{\to 0}{\frac{1}{x^2}}}{3+\underset{\downarrow 0}{\frac{2}{x}}-\underset{\downarrow 0}{\frac{1}{x^2}}} = \boxed{\frac{2}{3}}$$

Answer: VA: $x = -\frac{1}{3}$

HA: $y = \frac{2}{3}$