Section 3.3. Derivatives of Trigonometric Functions

- $f(x) = \sin(x)$

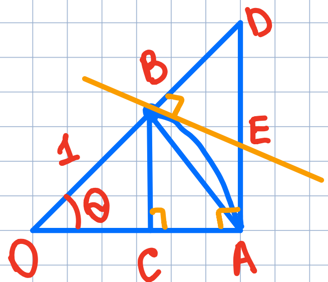
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} =$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} =$$

$$= \underbrace{\lim_{h\to 0} \sin(x)}_{=\sin(x)} \lim_{h\to 0} \frac{\cos(h)-1}{h} + \underbrace{\lim_{h\to 0} \cos(x)}_{\cos(x)} \cdot \lim_{h\to 0} \frac{\sin(h)}{h}$$

Let us prove that

$$\boxed{\lim_{h\to 0} \frac{\sin h}{h} = 1}$$



$0 < \theta < \frac{\pi}{2}$

$BC \perp OA$

$AB = \theta$

$|BC| = \sin\theta \cdot |OB| = \sin\theta$

$|BC| < |AB| < \text{arc } AB$

Therefore,

$$\sin\theta < \theta \quad \text{or} \quad \frac{\sin\theta}{\theta} < 1.$$

From the picture we see that

$$\text{arc } AB < |AE| + |BE|$$

$$\theta = \text{arc } AB < |AE| + |EB| < |AE| + |ED| =$$

$$= |AD| = |OA| \tan\theta = \tan\theta.$$

Therefore,

$$\theta < \frac{\sin\theta}{\cos\theta}.$$

So

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

By the Squeeze theorem, we have

$$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1.$$

But, $\frac{\sin\theta}{\theta}$ is an even function, so

$$\lim_{\theta \to 0^+} f(\theta) = \lim_{\theta \to 0^-} f(\theta).$$

$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$

Now, we calculate another limit

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = \lim_{\theta \to 0} \left(\frac{\cos\theta - 1}{\theta} \cdot \frac{\cos\theta + 1}{\cos\theta + 1} \right) =$$

$$= \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = - \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \lim_{\theta \to 0} \frac{\sin\theta}{\cos\theta + 1} = 0.$$

$$\boxed{\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0}$$

$\frac{d}{dx}(\sin x) = \cos(x)$	$\frac{d}{dx}(\csc x) = -\csc(x)\cot(x)$
$\frac{d}{dx}(\cos x) = -\sin(x)$	$\frac{d}{dx}(\sec x) = \sec(x)\tan(x)$
$\frac{d}{dx}(\tan x) = \sec^2(x)$	$\frac{d}{dx}(\cot x) = -\csc^2(x)$