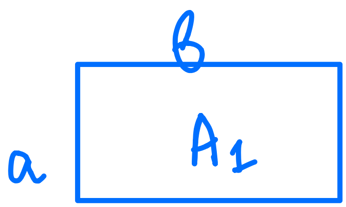
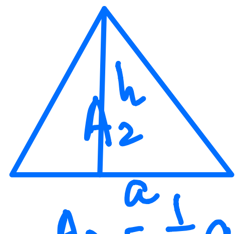
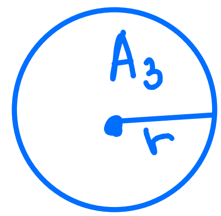
Section 5.1 Areas and Distances



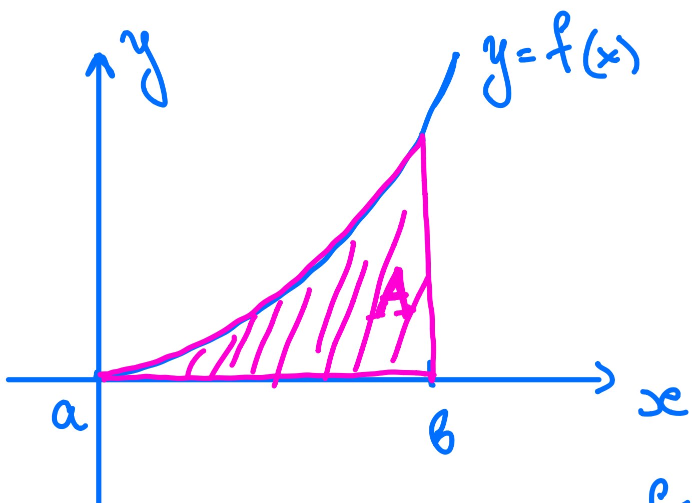
$A_1 = a \cdot b$

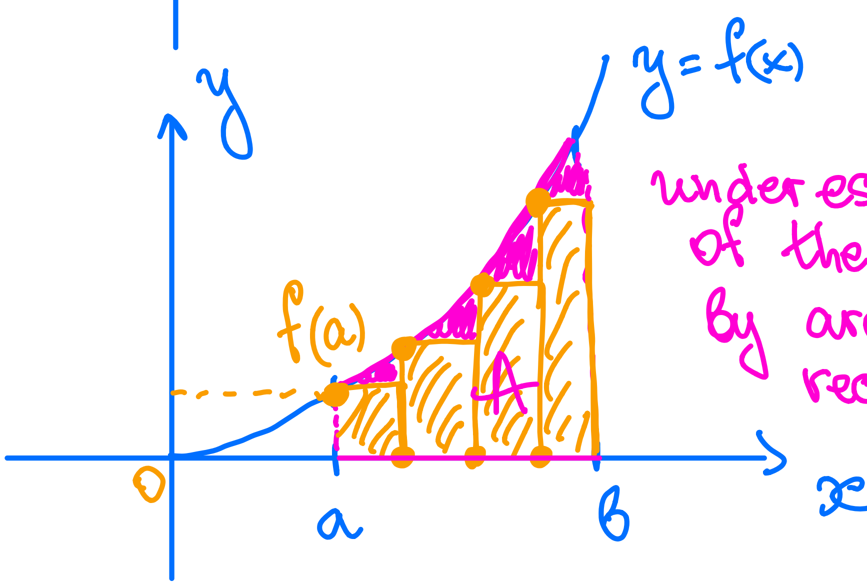


$A_2 = \frac{1}{2} a \cdot h$

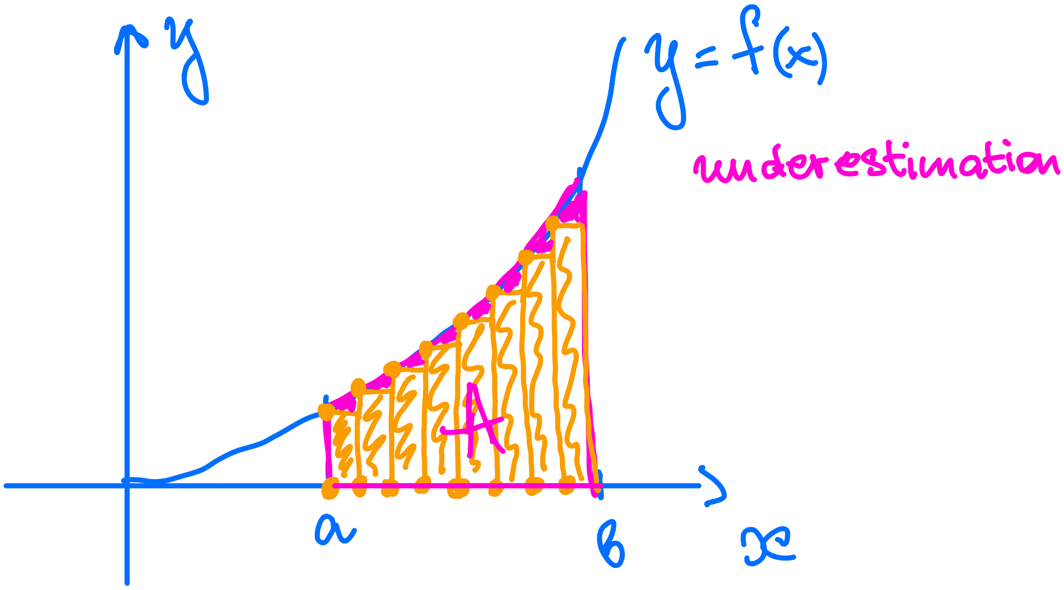


$A_3 = \pi r^2$

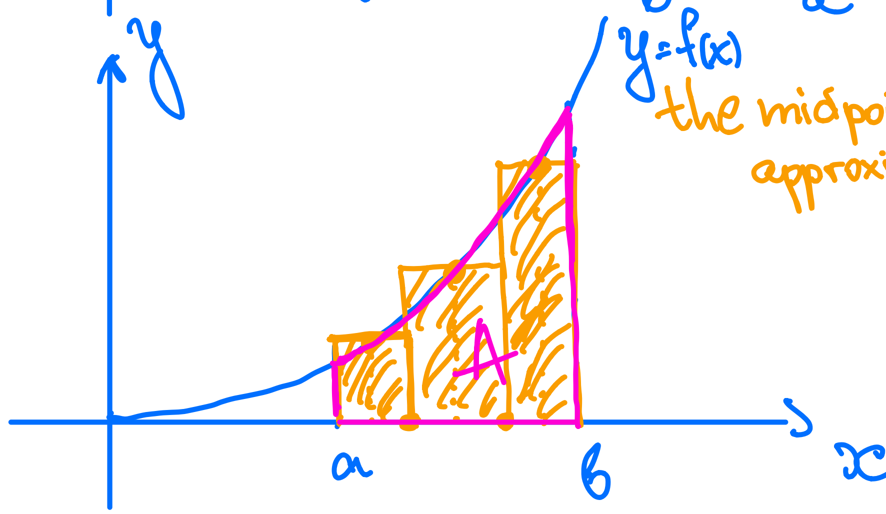




underestimation of the area A by areas of rectangles

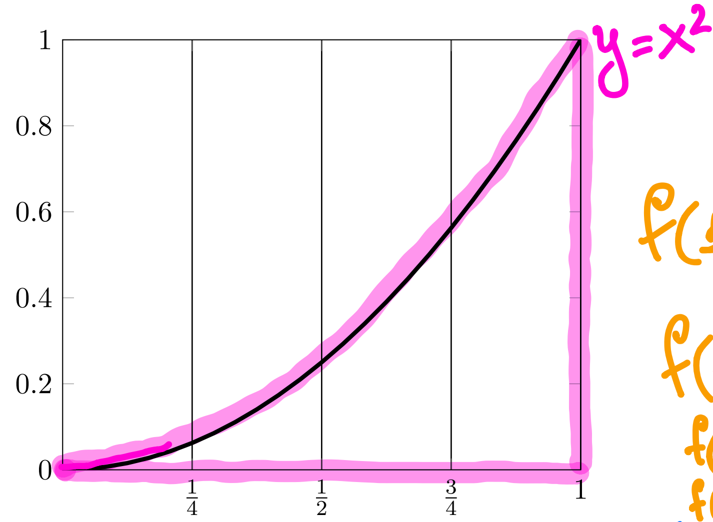

y
y = f(x)
underestimation
A
a
b
x

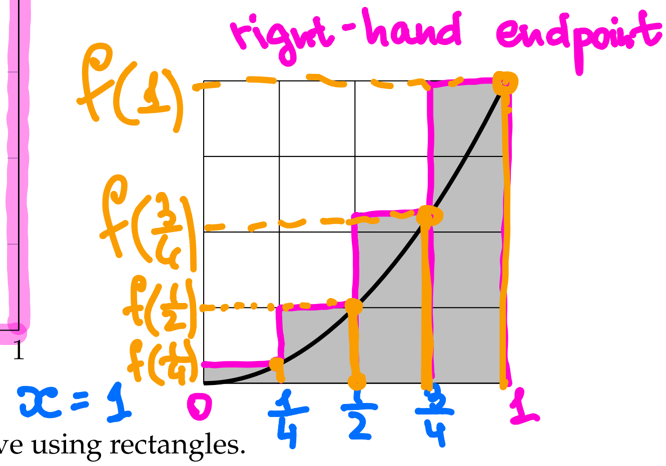

y
y = f(x)
overestimation of A by areas of rectangles
A
a
b
x


y
y = f(x)
the midpoint approximation
A
a
b
x

Approximating the area under a curve

Below is a piece of the graph of $y = x^2$. We need to estimate the area under the curve bounded between the x-axis, $x = 0$, $x = 1$, and the curve.





1. We want to approximate the area under the curve using rectangles.

   (a) On the left-hand graph, $\boxed{\text{draw}}$ in **four** equal rectangles (divide the x-axis between $x = 0$ and $x = 1$ into four equal pieces, and use the **right-hand endpoint** and the function to determine the height of the rectangle). Your picture should look something like the right hand smaller figure.

   (b) What is the width of each rectangle? $\frac{1}{4}$

   $[a, b]$

   $\frac{b-a}{n}$

   (c) Compute the total area of all the rectangles.

   To get started: Right-hand endpoint of the first rectangle is $x = \frac{1}{4}$, and the height is $f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^2$, so the area of the first rectangle is ...

   total area

   $+A_1: \frac{1}{4} \cdot f\left(\frac{1}{4}\right) = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2$

   $+A_2: \frac{1}{4} \cdot f\left(\frac{1}{2}\right) = \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2$

   $+A_3: \frac{1}{4} \cdot f\left(\frac{3}{4}\right) = \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2$

   $A_4: \frac{1}{4} \cdot f(1) = \frac{1}{4} \cdot 1^2$

   $\boxed{\Delta x = \frac{b-a}{n}}$

   a Δx Δx Δx .... b

   x_0 x_1 x_2 x_3 x_n

   $x_1 = x_0 + \Delta x$

   $x_2 = x_0 + 2 \cdot \Delta x$

   $x_3 = x_0 + 3 \cdot \Delta x$

   $x_n = x_0 + n \cdot \Delta x$

   Total area of the rectangles: ____________

   (d) Is your estimated area bigger or smaller than the total area under the curve? Why?

   Larger. Overestimation case.

2. Now we want to estimate the area using left-hand endpoints.

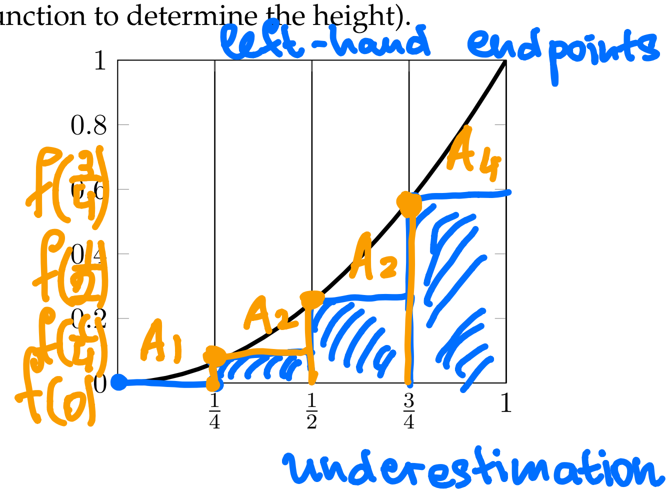
   (a) Again, slice up the area under the curve into four equally-spaced slices. $\boxed{\text{Draw}}$ in 4 rectangles and use the LEFT-HAND ENDPOINT of each slice to determine the height of the rectangle.

   (b) What is the width of each rectangle? $\frac{1}{4}$

   (c) Determine the total area of the rectangles (use the function to determine the height).

total area

$$\begin{aligned} &A_1: f(0)\cdot\tfrac{1}{4} = 0\\ +\ &A_2: f\left(\tfrac{1}{4}\right)\cdot\tfrac{1}{4} = \left(\tfrac{1}{4}\right)^2\cdot\tfrac{1}{4}\\ +\ &A_3: f\left(\tfrac{1}{2}\right)\cdot\tfrac{1}{4} = \left(\tfrac{1}{2}\right)^2\cdot\tfrac{1}{4}\\ +\ &A_4: f\left(\tfrac{3}{4}\right)\cdot\tfrac{1}{4} = \left(\tfrac{3}{4}\right)^2\cdot\tfrac{1}{4} \end{aligned}$$



   (d) Total area of the rectangles: __________

   (e) Is this an overestimation or an underestimation? Why?

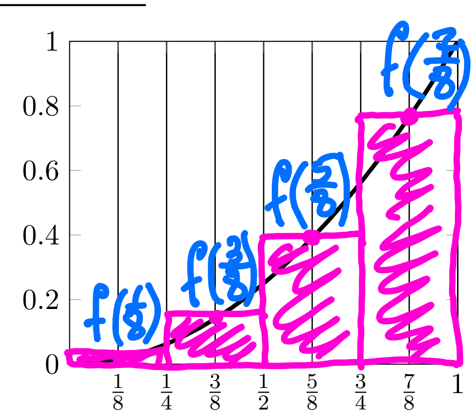
The area under the curve is greater than the total area of the rectangles.

3. Can we get a better estimation using 4 rectangles?

   (a) One more time, slice up the segment between $x = 0$ and $x = 1$ into four equal pieces. Using the MIDPOINT of each piece to determine the height of each rectangle, $\boxed{\text{draw}}$ in four rectangles, and determine the area of each rectangle (use the function to determine the heights).

   (b) What is the width of each rectangle? $\frac{1}{4}$

$$\begin{aligned} A_1 &: \tfrac{1}{4}\cdot f\left(\tfrac{1}{8}\right) = \tfrac{1}{4}\left(\tfrac{1}{8}\right)^2\\ A_2 &: \tfrac{1}{4}\cdot f\left(\tfrac{3}{8}\right) = \tfrac{1}{4}\left(\tfrac{3}{8}\right)^2\\ A_3 &: \tfrac{1}{4}\cdot f\left(\tfrac{5}{8}\right) = \tfrac{1}{4}\left(\tfrac{5}{8}\right)^2\\ A_4 &: \tfrac{1}{4}\cdot f\left(\tfrac{7}{8}\right) = \tfrac{1}{4}\left(\tfrac{7}{8}\right)^2 \end{aligned}$$



   (c) Determine the total area of the rectangles (use the function to determine the heights, not the picture!):

Total area $= A_1 + A_2 + A_3 + A_4$

(d) What do you think: based on your rectangles, is this an overestimation or an underestimation? (The other two rectangle types should have been obvious; this one is considerably more subtle. Make some sort of argument in favor of one choice or the other.)

Since $y = x^2$ is increasing, the underestimated area under the curve will be slightlty bigger than the area above the curve

4. Suppose you want a really good approximation of the area under the curve $y = x^2$ on the interval $[0, 1]$. What could you compute to get it? Explain in a sentence or two.

5. Suppose the odometer on our car is broken and we want to estimate the distance driven over a 1.5 hour time period. We take speedometer readings every 15 minutes and then record them in the table below. Estimate the distance traveled by the car. What method are you using?

Time (minutes)	0	15	30	45	60	75	90
Velocity (mi/h)	17	21	24	29	32	31	28

6. Oil leaked out of a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at 2 hour time intervals are shown in the table. Estimate how much oil leaked out. What method are you using? Is it an overestimate or an underestimate.

t (h)	0	2	4	6	8	10
r(t) (L/h)	8.7	7.6	6.8	6.2	5.7	5.3