Section 4.8. Newton's Method

Ex.

$x^2 - 9 = 0$

$x^2 = 9$

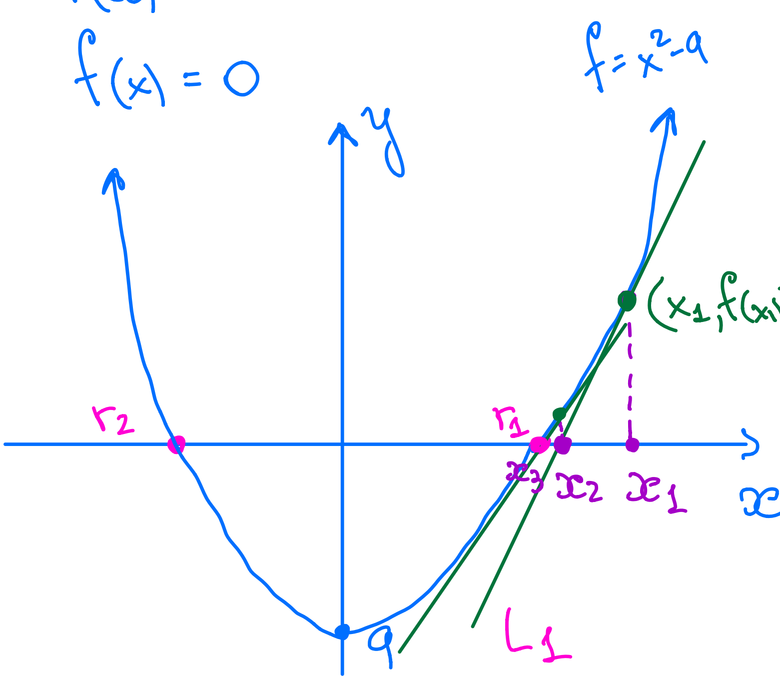
$x = \pm 3$

$\cos(x) + x^2 - \sqrt{x} = 0$

$\cos(x) = \sqrt{x} - x^2$

$f(x) = x^2 - 9$

$f(x) = 0$



$x_1 = 4$

$x_2 = 3.5$

$x_3 = 3.1$

$x_4 \simeq 3.0005$

Step 1:

Let x_1 be our initial guess

Step 2:

Build tangent line at your intial guess x_1.

Step 3:

pick x_2 as our new approximation and we build a new tangent line at x_2.

1. x_1 is given

2. TL equation at $x = x_1$

$L_1: f(x) = f'(x_1)(x - x_1) + f(x_1)$

3. $(x_2, 0)$ is a crossing point

$$0 = f'(x_1)(x_2 - x_1) + f(x_1)$$

$$f'(x_1)(x_2 - x_1) = -f(x_1)$$

$$x_2 - x_1 = -\frac{f(x_1)}{f'(x_1)}$$

$$\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

$f'(x_1) \neq 0$

$$\boxed{x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}}$$

$$\boxed{\underset{\text{new}}{x_{n+1}} = \underset{\text{old}}{x_n} - \frac{f(\underset{\text{old}}{x_n})}{f'(\underset{\text{old}}{x_n})}} \quad (*)$$

$n = 1:$ $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

is given

$n = 2:$ $$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

.....

Newton's Method is an iterative rule for finding roots.

Given: $F(x)$
Want: a so that $F(a) = 0$
Guess: x_0 close to a

Plug in and Repeat:
```
newX = oldX - F(oldX)/F'(oldX)
```
In math language:

$$x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)}$$

1. Let $F(x) = x^2 - 2$. $x^2 - 2 = 0$

   (a) Using elementary algebra, find a such that $F(a) = 0$. (Find a exactly and find a decimal approximation with at least 9 decimal places.)

   $$x_{1,2} = \pm\sqrt{2} \approx \pm 1.\underline{414213562}$$

   (b) Find a formula for x_{k+1}. Simplify it.

   $$x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)} \Rightarrow \boxed{x_{k+1} = x_k - \frac{x_k^2 - 2}{2x_k}}$$

   (c) Using an initial guess of $x_0 = 2$, complete 4 iterations of Newton's method to find x_4 and compare your answer to the one in part (a).

1. $x_0 = 2$

2. $k = 0$: $\quad x_1 = x_0 - \frac{x_0^2 - 2}{2x_0}$

   $$x_1 = 2 - \frac{4-2}{4} = 2 - \frac{1}{2} = \frac{3}{2} = \boxed{1.5}$$

3. $k = 1$: $\quad x_2 = x_1 - \frac{x_1^2 - 2}{2x_1}$

   $$x_2 = 1.5 - \frac{1.5^2 - 2}{2 \cdot 1.5} = 1.4166667$$

4. $k=2$: $$x_3 = x_2 - \frac{x_2^2 - 2}{2x_2}$$

$$x_3 = 1.4167 - \frac{1.4167^2 - 2}{2 \cdot 1.4167} = \boxed{1.414216}$$

2. This page is intended to illustrate *how* Newton's Method works.

   Again, consider the function $F(x) = x^2 - 2$.

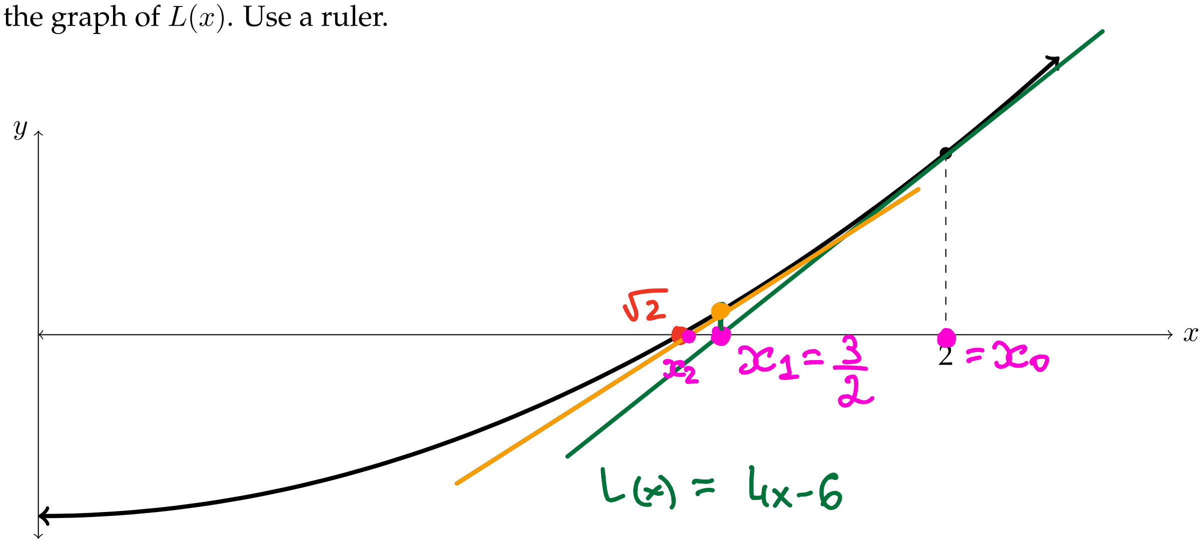
   $x^2 - 2 = 0$

   (a) Find the linearization $L(x)$ of $F(x)$ at $x = 2$. Leave your answer in point-slope form.

   $L(x) = F'(2)(x-2) + F(2) \qquad x = 2.$

   $L(x) = 2\cdot 2(x-2) + 2 \Rightarrow L(x) = 4(x-2) + 2 = \boxed{4x - 6}$

   (b) I've graphed $F(x)$ for you below. Mark where $\sqrt{2}$ is on this diagram and add to this diagram the graph of $L(x)$. Use a ruler.



   (c) Find the number x_1 such that $L(x_1) = 0$.

   $L(x) = 4x - 6 \quad : \quad L(x_1) = 0$

   $4x_1 - 6 = 0 \Rightarrow x_1 = \frac{6}{4} = \frac{3}{2} = 1.5$

   (d) In the diagram above, label the point x_1 on the x-axis.

   (e) Let's do it again! Find the linearization $L(x)$ of $F(x)$ at $x = x_1$.

   $L(x) = F'(x_1)(x - x_1) + F(x_1) \qquad F'(x) = 2x$

   $L(x) = 2x_1(x - x_1) + x_1^2 - 2 = \boxed{\cancel{2}\cdot\frac{3}{\cancel{2}}\left(x - \frac{3}{2}\right) + \frac{9}{4} - 2}$

   (f) Add the graph of this new linearization to your diagram above.

   (g) Find the number x_2 such that $L(x_2) = 0$. Then label the point $x = x_2$ in the diagram.

   $\boxed{L(x) = 0}$ when $L(x)$ crosses x-axis

   $3\left(x - \frac{3}{2}\right) + \frac{9}{4} - 2 = 0 \Rightarrow 3\left(x - \frac{3}{2}\right) = 2 - \frac{9}{4}$

   $x - \frac{3}{2} = \frac{2}{3} - \frac{3}{4}$

   (h) Compare your numbers for x_1 and x_2 to those on the previous page. They should be the same.

   $x_2 = x = \frac{2}{3} - \frac{3}{4} + \frac{3}{2}$

(i) Let's be a little more systematic. Suppose we have an estimate x_k for $\sqrt{2}$.

- Compute $F(x_k)$.
- Compute $F'(x_k)$.
- Compute the linearization of $F(x)$ at $x = x_k$.

$$x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)}$$

$L(x) =$ $F'(x_k)(x - x_k) + F(x_k)$

- Find the number x_{k+1} such that $L(x_{k+1}) = 0$. You should try to find as simple an expression as you can. Compare this to the formula we used on problem 1b from page 1.

$$L(x_{k+1}) = F'(x_k)(x_{k+1} - x_k) + F(x_k) = 0$$

$$\Downarrow$$

$$\boxed{x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)}} \qquad x_0 = a$$

3. Try to solve

$$e^{-x} - x = 0$$

by hand.

4. Explain why there is a solution between $x = 0$ and $x = 1$.

5. Starting with $x_0 = 1$, find an approximation of the solution of $e^{-x} - x = 0$ to 6 decimal places. During your computation, keep track of each x_k to at least 9 decimal places of accuracy.