WORKSHEET: SECTIONS 2.7-2.8

Solutions

The function

$$f'(x) = \lim_{h \to 0} \frac{f(h+x) - f(x)}{h}$$

is called the **derivative of** f. The value of f' at x can be interpreted geometrically as the slope of the tangent line to f at the point $(x, f(x))$. *Note: f' is called the derivative because it has been derived from f using the limit operation defined above. The domain of f' is the set of all x such that this limit exists and may be smaller than the domain of f.*

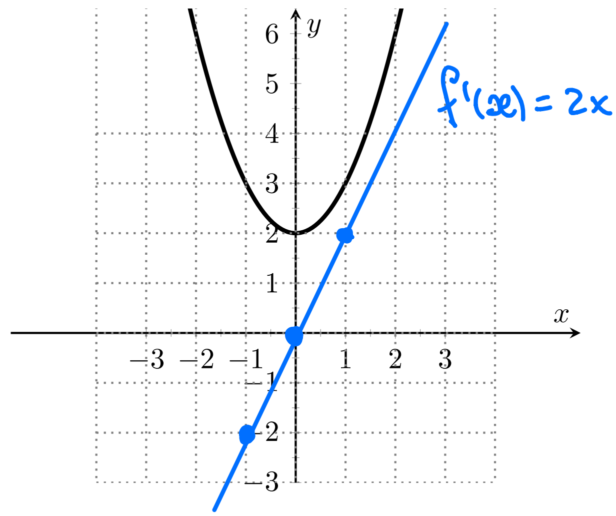
1. Let $f(x) = x^2 + 2$, shown below. Use the definition of the derivative as a function to compute $f'(x)$. Then graph $f'(x)$ on the same axes.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 2 - x^2 - 2}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{2} - \cancel{x^2} - \cancel{2}}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \boxed{2x}$$



2. Let $f(x) = \frac{1}{3}x^3 - 4x$.

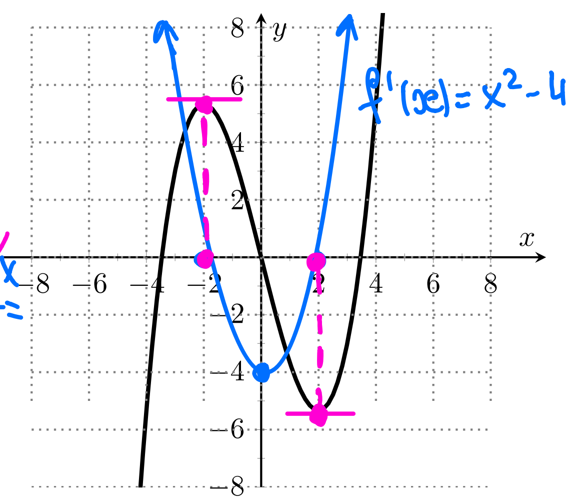
   (a) Use the definition of the derivative (as a function) to find a formula for $f'(x)$. You may find it helpful to use the fact that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{\frac{1}{3}(x+h)^3 - 4(x+h) - \frac{1}{3}x^3 + 4x}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{\tfrac{1}{3}x^3} + x^2h + xh^2 + \frac{1}{3}h^3 \cancel{- 4x} - 4h \cancel{- \tfrac{1}{3}x^3} \cancel{+ 4x}}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{h}(x^2 + xh + \frac{1}{3}h^2 - 4)}{\cancel{h}} = \boxed{x^2 - 4}$$



   (b) Factor the formula and use the factorization to plot the graph of $f'(x)$ on the same axes that show $f(x)$.

$$f'(x) = x^2 - 4 = (x-2)(x+2)$$

   (c) What do you notice about the relationship between the zeroes of $f'(x)$ and the tangent lines to $f(x)$?

$f'(x) = 0$ at $x = \pm 2$ where the slope of a tangent line to $y = f(x)$ is 0

or where the tangent line is horizontal (at $x=\pm 2$)

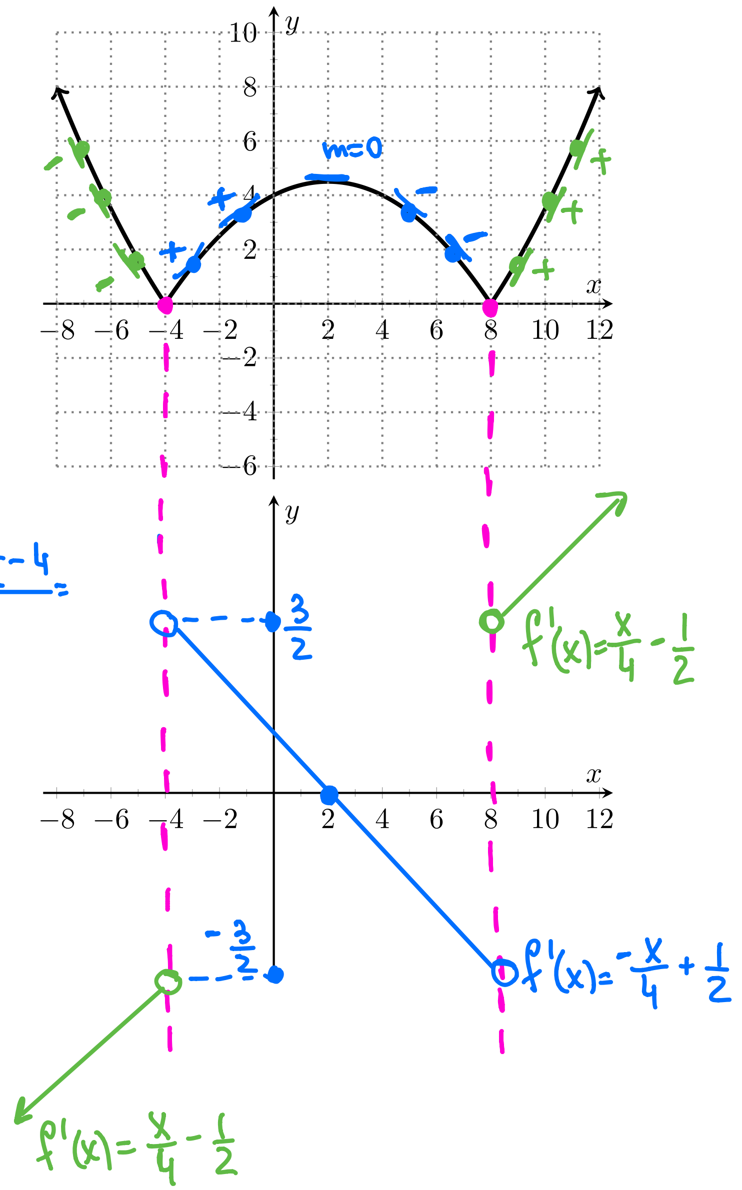
3. Consider the function

$$f(x)=\left|\frac{x^2}{8}-\frac{x}{2}-4\right|=\begin{cases}\frac{x^2}{8}-\frac{x}{2}-4 & \text{if} \quad x\le -4 \text{ or } x\ge 8\\ -(\frac{x^2}{8}-\frac{x}{2}-4) & \text{if} \quad -4<x<8\end{cases}.$$

(a) The graph of $f(x)$ is given on the top set of axes shown below. By thinking about slopes of tangent lines, sketch a graph of the derivative on the second set of axes.

When I ask you to sketch, I am interested in the qualitative behavior of the derivative: Where does it cross the x-axis? Is it positive or negative? Is it a lot positive or a little positive? Are the slopes growing steeper or getting less steep? (This is why the y-axis is unmarked on the answer graph.)

(b) Use the definition of the derivative to determine $f'(x)$ algebraically, for two cases: (i) $x<-4$ or $x>8$; (ii) $-4<x<8$. Explain why your algebraic calculations match your sketch.



(i) $f'(x)$ for $x<-4$ or $x>8$:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{(x+h)^2}{8}-\frac{(x+h)}{2}-4-\frac{x^2}{8}+\frac{x}{2}+4}{h}=$$

$$=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2-\cancel{4x}-4h-\cancel{16}-\cancel{x^2}+\cancel{4x}+\cancel{16}}{8h}=$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2x+h-4)}{8\cancel{h}}=\frac{2x-4}{8}=\boxed{\frac{x}{4}-\frac{1}{2}}$$

(ii) $f'(x)$ for $-4<x<8$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{-\frac{(x+h)^2}{8}+\frac{(x+h)}{2}+4+\frac{x^2}{8}-\frac{x}{2}-4}{h}=$$

$$=\lim_{h\to 0}\frac{-\cancel{x^2}-2xh-h^2+\cancel{4x}+4h+\cancel{x^2}-\cancel{4x}-\cancel{16}+\cancel{16}}{8h}=$$

$$=\lim_{h\to 0}\frac{\cancel{h}(-2x-h+4)}{8\cancel{h}}=\frac{-2x+4}{8}=$$

$$=\boxed{-\frac{x}{4}+\frac{1}{2}}$$

(c) Using your formula from (a), compute

- $\lim_{x\to -4^-} f'(x)=\lim_{x\to -4^-}\left(\frac{x}{4}-\frac{1}{2}\right)=-\frac{3}{2}$
- $\lim_{x\to -4^+} f'(x)=\lim_{x\to -4^+}\left(-\frac{x}{4}+\frac{1}{2}\right)=\frac{3}{2}$
- $\lim_{x\to 8^-} f'(x)=\lim_{x\to 8^-}\left(-\frac{x}{4}+\frac{1}{2}\right)=-\frac{3}{2}$
- $\lim_{x\to 8^+} f'(x)=\lim_{x\to 8^+}\left(\frac{x}{4}-\frac{1}{2}\right)=\frac{3}{2}$

Using the language of calculus, what can you say about $f'(x)$ at $x=-4$ and $x=8$? Why does this make sense geometrically? (Does it match your picture?)

$y=f'(x)$ is discontinuous at $x=-4$ and $x=8$.
$y=f(x)$ has corners at $x=-4$ and $x=8$.