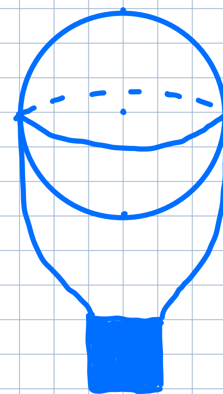
Section 3.9. Related Rates

Idea: compute the rate of change of one quantity in terms of the rate of change of another

Problem Solving Strategy:

1. Read the problem carefully
2. Draw a diagram if possible
3. Introduce notations. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem.
6. Use the <u>Chain Rule</u> to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

Example 1



V - volume

t - time (s)

r - radius (cm)

$\frac{dV}{dt} = 100\ cm^3/s$

$\frac{dr}{dt}$ - ? when $d = 50$ cm

Given information:

$\frac{dV}{dt} = 100$ $\quad d = 2r = 50,\quad r = 25$ cm

Unknown information:

$$\left.\frac{dr}{dt}\right|_{r=25} - ?$$

Equation that relates two variables:

$$V = \frac{4}{3}\pi r^3$$

Differentiate it + Chain Rule with respect to time:

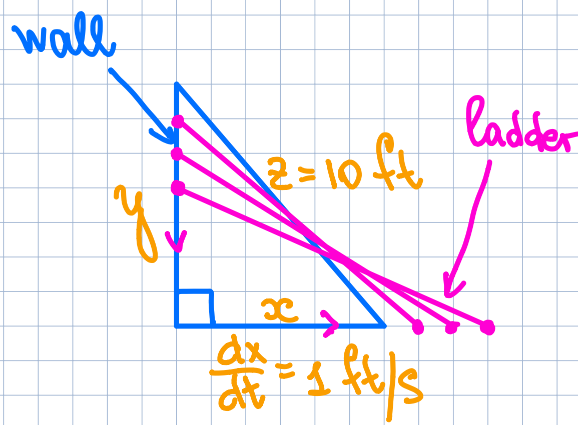
$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3 \cdot r^2 \frac{dr}{dt}$$

$$100 = \frac{4}{3}\pi \cdot 3 \cdot 25^2 \frac{dr}{dt}$$

$$\boxed{\frac{dr}{dt} = \frac{1}{25\pi}}$$

Example 2

$\frac{dy}{dt}$ - ? when $x = 6$ ft



Given Info:

$\frac{dx}{dt} = 1$ ft/s

$z = 10$ ft

Unknown info:

$\frac{dy}{dt}$ - ?

Write an equation:

$$x^2 + y^2 = 100$$

$$y = \sqrt{100 - x^2}$$

$$y = 8 \text{ ft}$$

Differentiate + Chain Rule:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

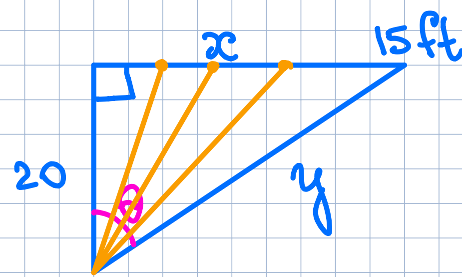
$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{6}{8}\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{3}{4} \text{ ft/s}$$

The top of the ladder is sliding down the wall at a rate $\frac{3}{4}$ ft/s.

Example 3



$$\frac{dx}{dt} = 4 \text{ ft/s}$$

$$\left.\frac{d\theta}{dt}\right|_{x=15\text{ ft}} = ?$$

$$\tan\theta = \frac{x}{20}$$

$$\sec^2\theta \, \frac{d\theta}{dt} = \frac{1}{20} \, \frac{dx}{dt}$$

$$\sec\theta = \frac{1}{\cos\theta}, \qquad \cos\theta = \frac{20}{y}$$

$$y^2 = 20^2 + 15^2 = 400 + 225 = 625$$

$$y = 25 \text{ ft}$$

$$\cos\theta = \frac{20}{25} = \frac{4}{5}$$

$$\sec\theta = \frac{5}{4}$$

$$\frac{d\theta}{dt} = \frac{1}{20} \, \frac{dx}{dt} \cdot \frac{16}{25} = \frac{16}{125}.$$