Section 2.3. Calculating limits using the limits laws

Limit Laws:

Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

exists. Then

1. (Sum Law) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. (Difference Law) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

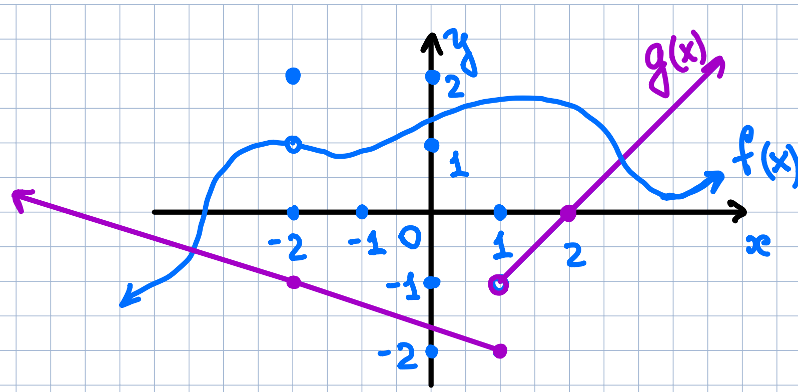
3. (Constant multiple law) $\lim_{x \to a} [c f(x)] = c \lim_{x \to a} f(x)$

4. (Product Law) $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. (Quotient Law) $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} f(x) \Big/ \lim_{x \to a} g(x)$ if

$\lim_{x \to a} g(x) \neq 0$

Example



- $\lim_{x\to -2}[f(x)+5g(x)] = \lim_{x\to -2} f(x) + 5\lim_{x\to -2} g(x) = 1+5\cdot(-1) = -4$
- $\lim_{x\to 1}[f(x)\cdot g(x)] = \text{DNE}$ since $\lim_{x\to 1} g(x)$ DNE
- $\lim_{x\to 2}\dfrac{f(x)}{g(x)} = \text{DNE}$ since $g(x)\to 0$ when $x\to 2$

6. (Power Law) $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$, $n\geq 0$

7. $\lim_{x\to a} c = c$

   constant

8. $\lim_{x\to a} x = a$

9. $\lim_{x\to a} x^n = a^n$, $n\geq 0$

10. $\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}$, $n\geq 0$ ($a>0$ when $n=2k$)

11. (Root Law) $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, $n\geq 0$

    (if $n=2k$ we assume that $\lim_{x\to a} f(x)>0$)

Direct Substitution property:

If f is a polynomial or a rational function and a is in the domain of f, then

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

Example

1. $\displaystyle \lim_{x \to 1} \frac{x^2 - 2}{x + 1} = \frac{1^2 - 2}{1 + 1} = -1$

2. $\displaystyle \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = 2$

Remark

$f(x) = \frac{x^2 - 1}{x - 1}$ and $g(x) = x + 1$

We have that $f(x) = g(x)$ except when $x = 1$.

In computing a limit we don't look what happens when $x = 1$.

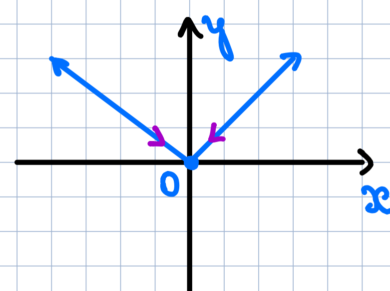
$$\boxed{\begin{array}{l} \text{If } f(x) = g(x) \text{ when } x \neq a, \text{ then} \\ \displaystyle \lim_{x \to a} f(x) = \lim_{x \to a} g(x), \text{ provided the limits exist} \end{array}}$$

Theorem

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

Example

$$\lim_{x\to 0} |x| = 0$$



$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\lim_{x\to 0^-} |x| = 0 = \lim_{x\to 0^+} |x| = 0$$

Example

$$\lim_{x\to 0} \frac{|x|}{x} = DNE$$

Explanation: $\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} 1 = 1 \neq \lim_{x\to 0^-} \frac{|x|}{x} = -1$

Theorem

If $f(x) \leq g(x)$ when x is near a (except possibly a) and the limits of f and g both exist as x approaches a, then

$$\boxed{\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)}$$

Theorem (The squeeze theorem)

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\boxed{\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,}$$

then

$$\boxed{\lim_{x \to a} g(x) = L}$$

Example Show that

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

$$x^2 \leq x^2 \sin \frac{1}{x^2} \leq x^2$$

since $|\sin(x)| \leq 1$

$x \to 0$ $\qquad$ $x \to 0$

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