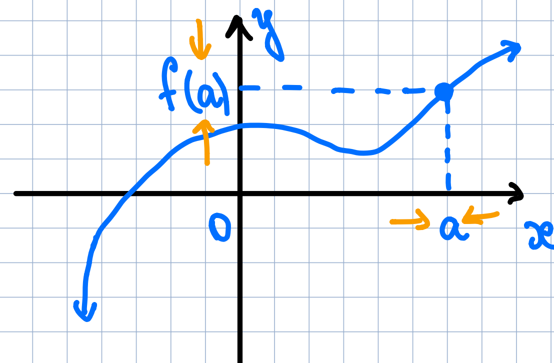
Section 2.5. Continuity

Def. A function f is continuous at a number a if

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

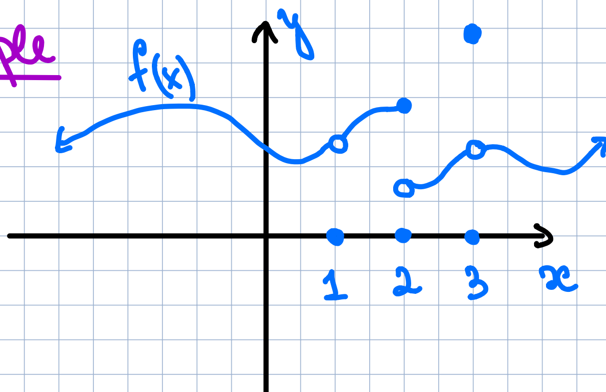
Requirements:

1. $f(a)$ is defined (a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$



f is discontinuous at a if f is not continuous at a.

Example



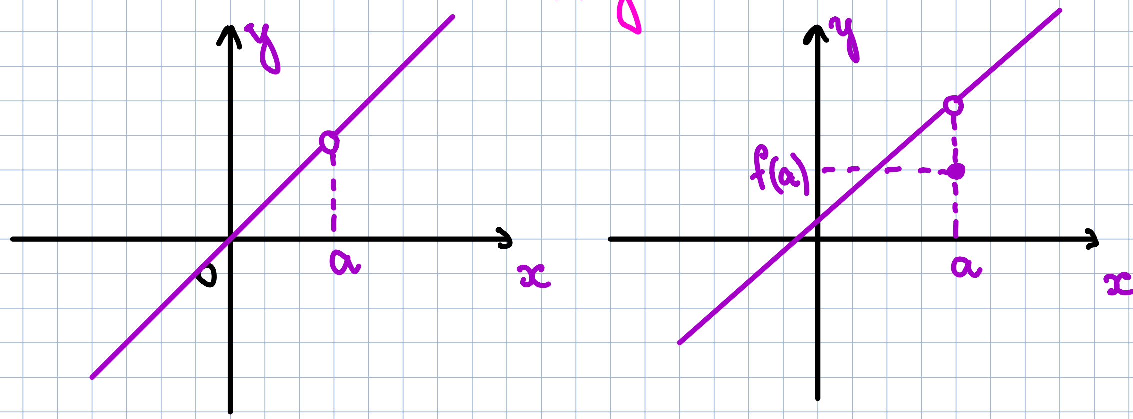
$f(1)$ is not defined

$\lim_{x \to 2} f(x)$ DNE

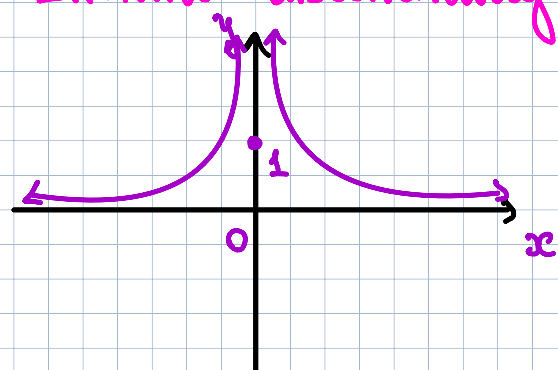
$\lim_{x \to 3} f(x) \neq f(3)$

Types of discontinuity:

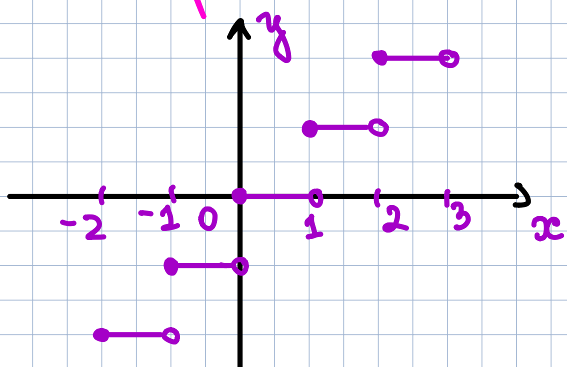
1. Removable discontinuity



2. Infinite discontinuity



3. Jump discontinuities



Def. A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Def. A function f is continuous on an interval if it is continuous at every number in the interval.

Remark* If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.

Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f+g$
2. $f-g$
3. $f \cdot g$
4. $\frac{f}{g}$, $g(a) \neq 0$
5. cf

Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$

(b) Any rational function is continuous whenever it is defined; that is, it is continuous on its domain.

Example

- $\lim_{\theta \to 0} \cos\theta = 1$
- $\lim_{\theta \to 0} \sin\theta = 0$

continuous at $\theta = 0$

- $f(x) = \tan x = \frac{\sin x}{\cos x}$ is continuous except the values of x for which $\cos x = 0$

$x = \pm\frac{\pi}{2} + 2\pi n$

Remark* The inverse function of any continuous one-to-one function is also continuous.

Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

Theorem If f is continuous at b and $\lim_{x \to a} g(x) = b$ then $\lim_{x \to a} f(g(x)) = f(b)$

In other words,

$$\boxed{\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)}$$

Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a,b) such that $f(c)=N$.