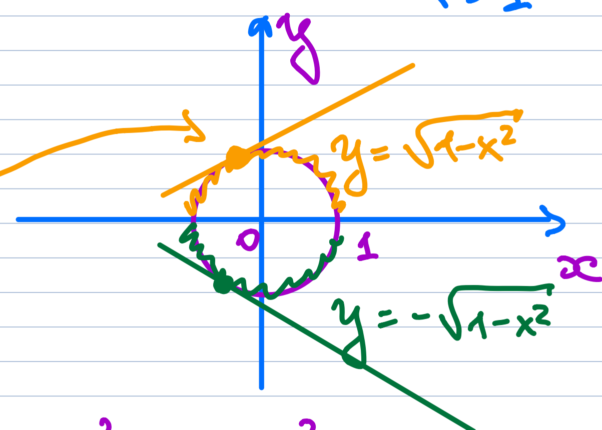
Implicit Differentiation

Explicitey defined func. have the representation
$y = f(x)$

- $y = x^2$
- $y = \cos(x)$

Curve:
$x^2 + y^2 = 1$

circle $(0,0)$
$r = 1$



$y^2 = 1 - x^2$

$y = \pm\sqrt{1-x^2}$

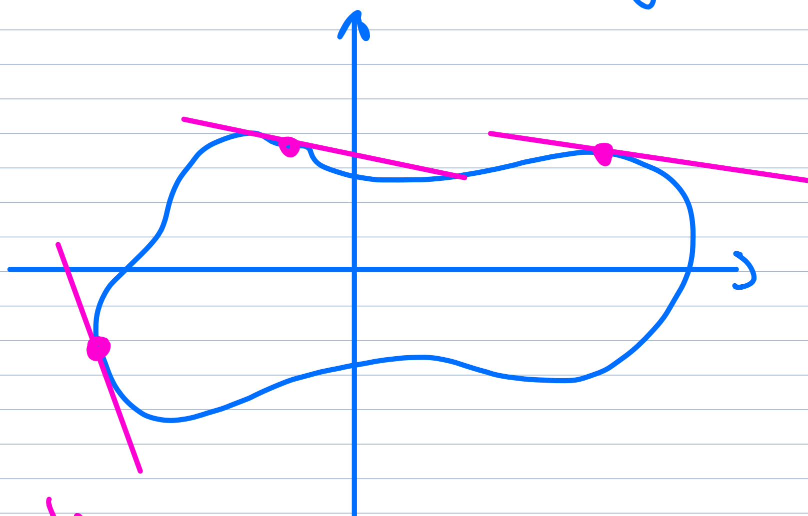
- $y = \sqrt{1-x^2}$

  $y' = \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x)$

- $y = -\sqrt{1-x^2}$

  $y' = -\frac{1}{2}(1-x^2)^{-1/2}(-2x)$

Given curve: $x^4 + 3y^3 + \cos(y) = \sin(x)$



Find:

$$\frac{dy}{dx} = y'(x) = ?$$

1. $$\frac{d}{dx}\left(x^4 + 3y^3 + \cos(y)\right) = \frac{d}{dx}\left(\sin(x)\right)$$

independent variable

function of x

$y = y(x)$

2. $$4 \cdot x^3 + 3 \cdot 3 \cdot \left(y(x)\right)^2 \cdot y'(x) +$$

$$+ \left(-\sin(y(x))\right) \cdot y'(x) = \cos(x)$$

3. Solve for y':

$$y'\left(9y^2 - \sin(y)\right) + 4x^3 = \cos(x)$$

$$y'\left(9y^2 - \sin(y)\right) = \cos(x) - 4x^3$$

$$\boxed{y' = \frac{\cos(x) - 4x^3}{9y^2 - \sin(y)}}$$

Implicit Differentiation

$$\tan(y) + xy = e^{x} \qquad y = y(x)$$

$$\frac{dy}{dx} = y'(x) = y'$$

1. $$\frac{d}{dx}\left(\overbrace{\tan(y)}^{\text{chain rule}} + \underbrace{xy}_{\text{Product rule}}\right) = y' \cdot \sec^2(y) +$$

$$+ 1 \cdot y + xy'$$

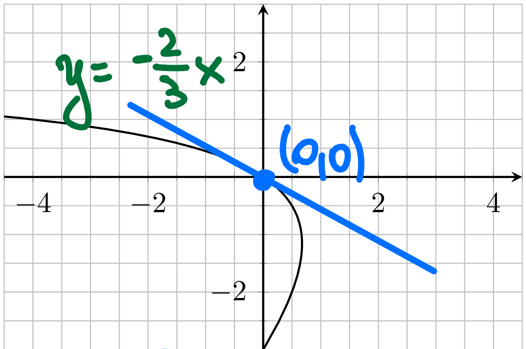
$$\frac{d}{dx}\left(e^x\right) = e^x$$

$$y' \cdot \sec^2(y) + y + xy' = e^x$$

$$y'\left(\sec^2(y) + x\right) = e^x - y$$

$$y' = \frac{e^x - y}{\sec^2(y) + x}$$

1. Find $\frac{dy}{dx}$ for $2x + 3y = xy - y^2$ and find the equations of tangents to the graph when $x = 0$. Use the portion of the curve shown below as an aid and to determine the plausibility of your answers.



$y'(x) - ?$

1. $\frac{d}{dx}(2x+3y) = \frac{d}{dx}(xy - y^2)$

chain rule

Product rule

2.

$y' = -\frac{2}{3}$

$y = -\frac{2}{3}(x-0)+0$

$2 + 3\cdot y'(x) = 1\cdot y + y'(x)\cdot x - 2y\cdot y'(x)$

$3y' - x\cdot y' + 2y\cdot y' = y - 2$

$y'(3 - x + 2y) = y - 2 \Rightarrow y' = \frac{y-2}{3-x+2y}$

2. Find $\dfrac{da}{db}$ for $a^3 \sin(3b) = a^2 - b^2$. (Pay attention here: b is the independent variable (like x) and a is the dependent variable (like y).

$b \sim x$

$a = a(b) \sim y(x)$

$a^3 \sin(3b) = \underset{\text{function}}{a^2} - \underset{\text{indepen. variable}}{b^2}$

$a' = \frac{da}{db} = a'(b)$

$\frac{d}{db}(a^3 \sin(3b)) = a^3 \cos(3b)\cdot 3 +$
$+ 3a^2\cdot a'\cdot \sin(3b)$

$\frac{d}{db}(a^2 - b^2) = 2a\cdot a' - 2b$

$3a^3 \cos(3b) + 3a^2 a' \sin(3b) = 2aa' - 2b$

$a'(3a^2 \sin(3b) - 2a) = -2b - 3a^3 \cos(3b)$

$a' = \frac{-2b - 3a^3 \cos(3b)}{3a^2 \sin(3b) - 2a}$

3. Find $\frac{dy}{dx}$ for $e^{xy} = x + y + 1$

3. $e^{xy} = x + y + 1$

$y' = \frac{dy}{dx} - ?$

1. $\frac{d}{dx}\left(e^{xy}\right) = e^{xy} \cdot (y + x \cdot y')$

product rule

$\frac{d}{dx}(x + y + 1) = 1 + y'$

$e^{xy}(y + x y') = 1 + y'$

$y e^{xy} + x e^{xy} \cdot y' = 1 + y'$

$x e^{xy} \cdot y' - y' = 1 - y e^{xy}$

$y'(x e^{xy} - 1) = 1 - y \cdot e^{xy}$

$$y' = \frac{1 - y e^{xy}}{x e^{xy} - 1}$$

4. You are going to derive the formula for the derivative of inverse tangent the way we found the derivative of inverse sine in the video.

   (a) Find dy/dx for the expression $x = \tan(y)$.

$y = \arctan(x)$

$y' - ?$

$x = \tan(y)$

$\frac{d}{dx}(x) = \frac{d}{dx}(\tan(y))$

$1 = \sec^2(y) \cdot y'$

$\tan(y) = x$

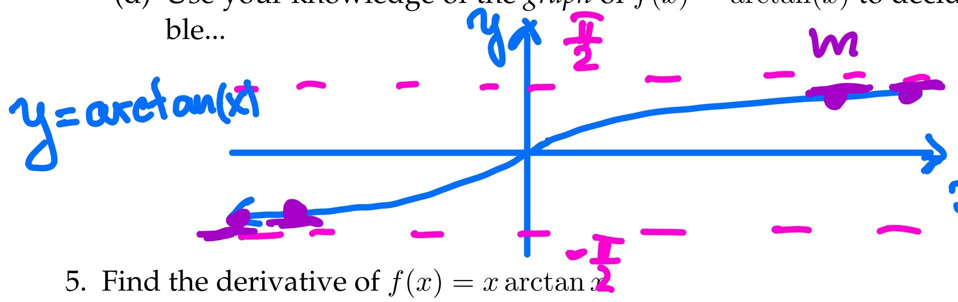
$x = \tan(y)$

   (b) Use the identity $1 + (\tan(\theta))^2 = (\sec(\theta))^2$ to rewrite you answer in part (a) and *write your* dy/dx *in terms of* x *only*.

$$y' = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}$$

   (c) Now fill in the blank $\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$

   (d) Use your knowledge of the *graph* of $f(x) = \arctan(x)$ to decide if your answer seems plausible...



$m \to 0$ when $x \to \infty$

$m = \frac{1}{1+x^2} \to 0$

$$\lim_{x\to\infty} \frac{1}{1+x^2} = 0$$

5. Find the derivative of $f(x) = x \arctan x$

$f'(x) = (x \cdot \arctan x)' =$

$= 1 \cdot \arctan(x) + (\arctan(x))' \cdot x = \arctan(x) + \frac{1}{1+x^2} \cdot x$

6. Find the derivative of $f(x) = \arctan(4 - x^2)$. Chain rule

$f'(x) = \frac{1}{1+(4-x^2)^2} \cdot (-2x)$.

Derivatives of inverse functions

$$(\arctan(x))' = \frac{1}{1+x^2}$$

$$(\operatorname{arccot}(x))' = -\frac{1}{1+x^2}$$

$$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arccos(x))' = -\frac{1}{\sqrt{1-x^2}}$$

$y = \arcsin(x)$

$y' - ?$

$$\sin y = x$$

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

$$\cos y \cdot y' = 1$$

$$y' = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}}$$

$$\sin^2(y) + \cos^2(y) = 1$$

$$\cos(y) = \sqrt{1-\sin^2 y}$$

$$y' = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}$$

$$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$$

$$\left(\operatorname{arcsec}(x)\right)' = \frac{1}{x\sqrt{x^2-1}}$$

$$\left(\operatorname{arccsc}(x)\right)' = -\frac{1}{x\sqrt{x^2-1}}$$