# Chapters 3 & 4 Review

## MT-2

## Problem # 1

$$f(x) = x - 2\sin(x)$$

(a) Find CP of $f(x)$ on $[0, \pi]$

(b) Find abs. max and abs. min values of $f(x)$ on $[0, \pi]$.

## Solution

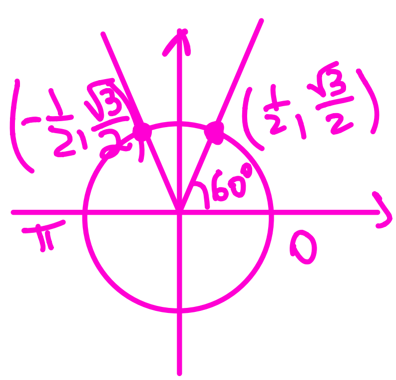
(a) $f'(x) = 0$ or $f'(x)$ DNE

$f'(x) = 1 - 2\cos(x)$

$f'(x)$ is defined for all $x$ in $\mathbb{R}$

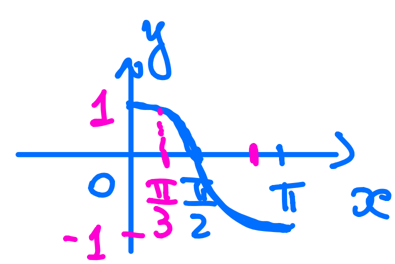
$f'(x) = 0 \Rightarrow 1 - 2\cos(x) = 0$

$\cos(x) = \frac{1}{2}$




$$\boxed{x = \frac{\pi}{3}}$$

CP



(b) $f(0)$ , $f(\pi)$ , $f\left(\frac{\pi}{3}\right)$

$f(x) = x - 2\sin(x)$

$f(0) = 0 - 2\cdot\sin(0) = \underline{0}$

$f(\pi) = \pi - 2\sin(\pi) = \underline{\pi}$

$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2\sin\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}} = \underline{\frac{\pi}{3} - \sqrt{3}}$

$$\boxed{\begin{array}{l} f_{max} = \pi \quad \text{at } x = \pi \\ f_{min} = \frac{\pi}{3} - \sqrt{3} \quad \text{at } x = \frac{\pi}{3} \end{array}}$$

## Problem # 2

(a) Find the linearization of $f(x) = \sqrt{x}$ at $a=4$.

(b) Use the result from (a) to approximate $\sqrt{3.9}$

## Solution

(a) $L(x) = f'(a)(x-a) + f(a)$ (tangent line equation to $f(x)=\sqrt{x}$ at $x=a$)

$f(x) = \sqrt{x}$ , $a=4$

$f(4) = \sqrt{4} = 2$

$f'(x) = \dfrac{1}{2\sqrt{x}}$ $\qquad f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$

$$L(x) = \frac{1}{4}(x-4) + 2 = \frac{1}{4}x + 1$$

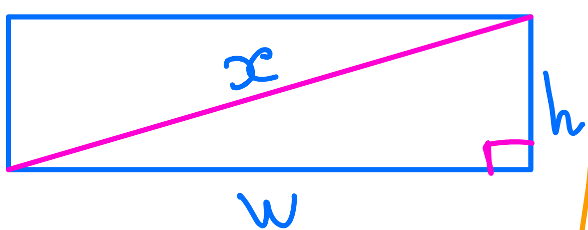
$$\boxed{L(x) = \frac{1}{4}x + 1}$$

(b) $\sqrt{3.9} \approx L(3.9) = \frac{1}{4} \cdot 3.9 + 1$

## Problem # 3 (Related Rates)

A rectangular photograph whose width is 3 times the height is being enlarged, and the height is increasing at a rate of 2 in/min. How fast is the diagonal of the rectangle increasing when the height is 4 in? Indicate units.

## Solution




$$w = 3h$$

$$\frac{dh}{dt} = h'(t) = 2 \text{ in/min}$$

$$\frac{dx}{dt} \text{ when } h = 4 \text{ in?}$$

$$x^2 = w^2 + h^2$$

$$x^2(t) = w^2(t) + h^2(t)$$

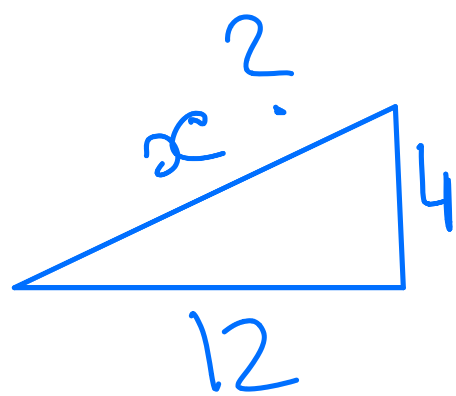
$$x^2(t) = 9h^2(t) + h^2(t) = 10\, h^2(t)$$

$$x^2(t) = 10h^2(t)$$

$$2x \cdot x' = 20h \cdot h'$$

$$x' = \frac{20h \cdot h'}{2x} = \frac{10h \cdot h'}{x}$$

$$x' = \frac{10 \cdot 4 \cdot 2}{x}$$




$$x^2 = 4^2 + 12^2$$

$$x^2 = 144 + 16$$

$$x^2 = 160$$

$$x = \sqrt{160} = 4\sqrt{10} \text{ (in)}$$

$$\boxed{x'} = \frac{10 \cdot \cancel{4} \cdot 2}{\cancel{4}\sqrt{10}} = 2\sqrt{10} \text{ (in/min)}$$

## Problem #4 (Optimization)

Suppose an open cup in the shape of a cylinder is to be made with surface area 48 in². What dimensions will maximize the volume of the cup?

$S_A = \pi r^2 + 2\pi r h$

$V = \pi r^2 h$

## Solution

Find $h, r$ such that $V \to \max$.


$$S_A = 48 \text{ in}^2$$

$$\pi r^2 + 2\pi r h = 48 \rightsquigarrow 2\pi r h = 48 - \pi r^2$$

$$V = \pi r^2 h \to \max \qquad \underbrace{h}_{} = \frac{48 - \pi r^2}{2\pi r}$$

$$V = \cancel{\pi} r^{\cancel{2}} \cdot \left( \frac{48 - \pi r^2}{2\cancel{\pi}\cancel{r}} \right) = \frac{r}{2}\left(48 - \pi r^2\right)$$

$$V(r) = \frac{r}{2}\left(48 - \pi r^2\right) \to \max$$

$r > 0,\ h > 0$

- $V(r) = 24r - \frac{\pi}{2} r^3$

  $V'(r) = 24 - \frac{3\pi}{2} r^2 = 0$

  $-\frac{3\pi}{2} r^2 = -24$

  $r^2 = \cancel{24}^{\,8} \cdot \frac{2}{\cancel{3}\pi} = \frac{16}{\pi}$

  $\boxed{r = \frac{4}{\sqrt{\pi}}}$ CP

- Justification:

  2nd Der. Test:

  $V''(r) = -3\pi r$

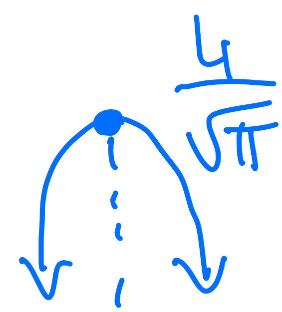
  $V''\left(\frac{4}{\sqrt{\pi}}\right) = -3\pi \cdot \frac{4}{\sqrt{\pi}} = -12\sqrt{\pi} < 0$

Therefore, at $r = \frac{4}{\sqrt{\pi}}$

$V(r)$ is concave down,

that is, at $r = \frac{4}{\sqrt{\pi}}$

$V(r)$ attains its abs. max value.




Answer:

$$\boxed{r = \frac{4}{\sqrt{\pi}}} \text{ (in.)}$$

$$h = \frac{48 - \pi r^2}{2\pi r}$$

$$\boxed{h = \frac{48 - \pi \frac{16}{\pi}}{2\pi \frac{4}{\sqrt{\pi}}}} \text{ (in.)}$$

**Remark**: $x = -1$ is a loc. max

**Correct**: At $x = -1$ $f(x)$ has a loc. max.

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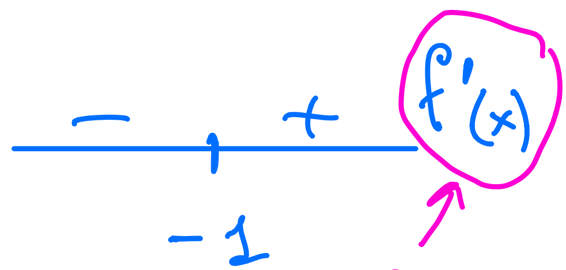
**Remark**:




use notations

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No Section 4.9 on MT-2

Sections on MT-2:

4.3, 4.4, 4.5, 4.7, 3.10, 3.9, 4.1

## Problem #5  Evaluate the following limits.

(a) $\lim_{t \to 0} \frac{\sin(t^2)}{t^2} =$

(b) $\lim_{x \to 0^+} \sqrt{x}\,\ln(x) =$

## Solution

(a) $\lim_{t \to 0} \frac{\sin(t^2)}{t^2} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{t \to 0} \frac{\cos(t^2) \cdot \cancel{2t}}{\cancel{2t}} = \boxed{1}$

(b) $\lim_{x \to 0^+} \sqrt{x}\,\ln(x) \overset{0 \cdot (-\infty)}{=} \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{\sqrt{x}}} =$

$\overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}\frac{1}{x^{3/2}}} = \lim_{x \to 0^+} \frac{-1}{2\,x \cdot x^{-3/2}} =$

$= \lim_{x \to 0^+} -\frac{1}{2} \cdot \sqrt{x} = \boxed{0}$

**Problem #6**

1. (15 points) Consider the function $f(x) = \dfrac{(x-1)^3}{x^2}$. We have computed for you

$$f'(x) = \frac{(x-1)^2(x+2)}{x^3} \quad \text{and} \quad f''(x) = \frac{6x-6}{x^4}.$$

**For full credit, show your work.**

(a) Find the intervals where $f(x)$ is increasing and decreasing.

$f$ is $\uparrow$ on $I$ if $f'(x) > 0$ on $I$
$f$ is $\downarrow$ on $I$ if $f'(x) < 0$ on $I$

$f'(x) = \dfrac{(x-1)^2(x+2)}{x^3}$

CP: $f'(x) = 0$
$(x-1)^2(x+2) = 0$
$x = 1$ or $x = -2$

$f'(x)$ DNE at $x = 0$, but $x = 0$ is not in the domain of $f$

$f'$: $+$ $-2$ $-$ $0$ $+$ $1$ $+$
$f$: $\nearrow$ $-2$ $\searrow$ $0$ $\nearrow$ $1$ $\nearrow$

Ans: $f$ is $\uparrow$ on $(-\infty, -2) \cup (0, \infty)$ and is $\downarrow$ on $(-2, 0)$

(b) Find the intervals where $f(x)$ is concave up and concave down.

$f''(x) = \dfrac{6x-6}{x^4} = 0$ when $6x - 6 = 0$
$x = 1$

$f''$: $-$ $0$ $-$ $1$ $+$
$f$: $\frown$ $0$ $\frown$ $1$ $\smile$

$x = 0$ is not in the dom of $f(x)$ $\Rightarrow$ is not a CP

Ans: $f$ is cc up on $(1, \infty)$ and cc down on $(-\infty, 0) \cup (0, 1)$

(c) Classify all critical points of $f(x)$.

- At $x = 1$ $f(x)$ attains neither loc. max nor loc. min value.

  $(1, 0)$ is an inflection point

- At $x = -2$ $f(x)$ attains its loc. max value; $f''(-2) < 0$.

Problem #7    Spring 2019 (#7).

# REVIEW OF CHAPTERS 3 & 4

Summary of Topics

Chapter 3

- Sections 1-6 primarily involve derivative rules. You will *not* be explicitly tested on "can you take a derivative", but you will need to be able to accurately compute derivatives to answer all the problems.
- Section 5 involves implicilty defined functions. You are expected to be able to use implicit differentiation to, for example, find the equation of a tangent line to an implicitly defined curve at a given point.
- Sections 7 and 8 focus on applications of the derivative in science and particularly to exponential growth and decay. Position, velocity and acceleration were again discussed. The overall emphasis is on *interpretation* of the derivative in the context of an applied problem.
- Section 9 Related Rate Problems. In these problems you are always taking the derivative implicitly with respect to time and almost always seeking of find a rate of change at a particular instant.
- Section 10 Linear Approximations and Differentials. The crucial idea here is that the derivative can be used to estimate function-values or changes in function-values.
- Section 11 we did not cover.

Chapter 4

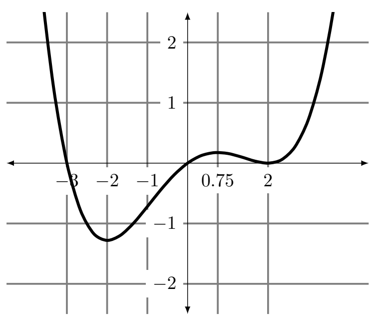
- Section 1 makes a careful study of the ideas of local/absolute maximum/minimum and the difference between an extreme value ($y$-value) and where it occurs ($x$-value).
- Section 4.2 The Mean Value Theorem. Know, roughly, what it says and be able to draw a picture.
- Section 4.3 discussed how the sign of $f'$ and $f''$ can tell us things about $f$ such as intervals on which $f$ is increasing, decreasing, concave up, concave down, local/absolute extreme values.
- Section 4.4 involved L'Hôpital's Rule. Recall that before using this rule one should make sure it applies.
- Section 4.5 put a whole bunch of Calculus together to sketch a graph. In addition to topics from Section 1 and 2, we also included things like $x$- and $y$-intercepts, vertical and horizontal asymptotes, and the function's domain.
- Section 4.6 was not discussed.
- Section 4.7 involved Optimization. Recall that by this time we have a clear understanding of how the domain of the function may determine the techniques we use to determine the answer.
- Section 4.8 will be discussed at the end of the semester and will not appear on this midterm.
- Section 4.9 involves antiderivatives.

Note that the problems provided below are not necessarily comprehensive; they are intended to remind you of the sorts of problems we have discussed, but there may be other problems on the Midterm that don't look just like these!

1. Find the linearization of $f(x) = \sqrt{x}$ at $a = 4$ and use it to estimate $\sqrt{4.1}$ and $\sqrt{3.8}$.

2. Find the differential of $y = \sqrt{x}$ and use it to estimate how much $y$ will change as $x$ changes from $x = 4$ to $x = 4.1$.

3. If the $\boxed{\textbf{derivative}}$ of a function is shown below, identify all local maxima, minima, intervals of increase and decrease, intervals of concave up and concave down, and inflection points.




4. Evaluate the following limits. Show your work.

(a) $\lim_{x \to 0} \dfrac{1 + x - e^x}{\sin x}$

(b) $\lim_{x \to \infty} x \ln(1 + \dfrac{2}{x})$

5. (a) What are critical numbers of a function $f$?

(b) How do you find the absolute maximum and minimum of a function $f$ on a closed interval? (Assume $f$ is continuous on the interval.)

(c) Find the critical numbers of $f(x) = \sin(x) + \cancel{\sec(x)}(\cos(x))^2$ in $[-2\pi, \pi]$.

6. (a) State the Mean Value Theorem and draw a picture to illustrate it.

(b) Determine whether the Mean Value Theorem applies to $f(x) = x(x^2 - x - 2)$ on $[-1, 1]$. If it can be applied find all numbers that satisfy the conclusion of the Mean Value Theorem.

7. Consider $f(x) = 2x - 2\cos x$ on the interval $[-\pi, 2\pi]$

   (a) Find the open intervals on which the function is increasing or decreasing.

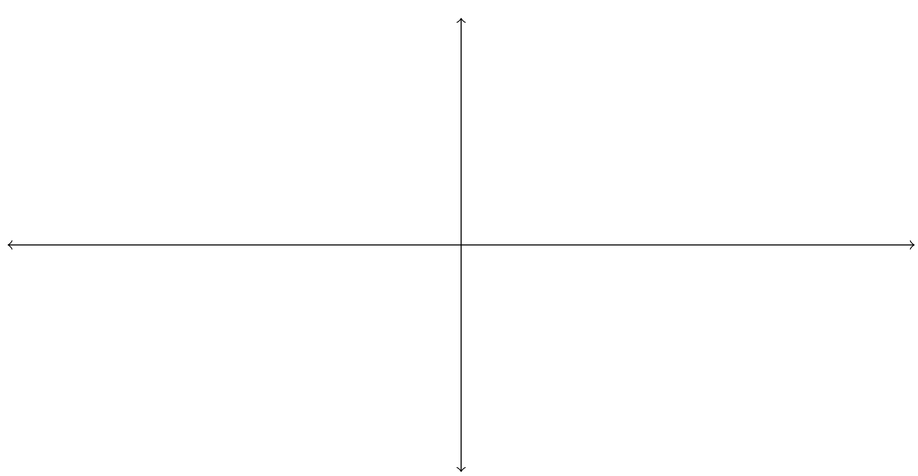
   (b) Apply the first derivative test to identify all relative extrema. Classify each as a local maximum or local minimum.

   (c) Find the open intervals on which the function is concave up or concave down.

   (d) Find the inflection points.

   (e) What are the absolute maximum and minimum values of the function on the interval?

   (f) Sketch the graph.



8. Find the rectangle of maximum area that can be inscribed inside the region bounded above by $y = 20 - x^2$ and bounded below by the $x$-axis. (Assume the base of the rectangle lies on the $x$-axis.) Begin by sketching a picture and labelling useful information.

9. Find the equation of the tangent line to the function $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

10. The angle of elevation of the sun is decreasing at a rate of 0.25 radians/hour. How fast is the shadow cast by a 400 foot tall building increasing when the angle of elevation of the sun is $\frac{\pi}{6}$ radians?