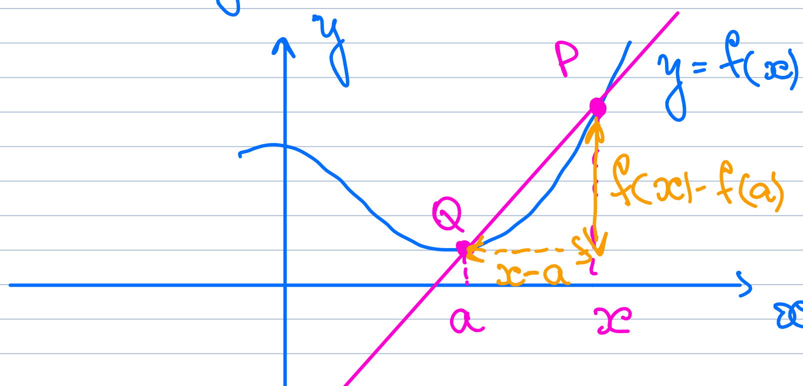
Section 2.7 Derivatives

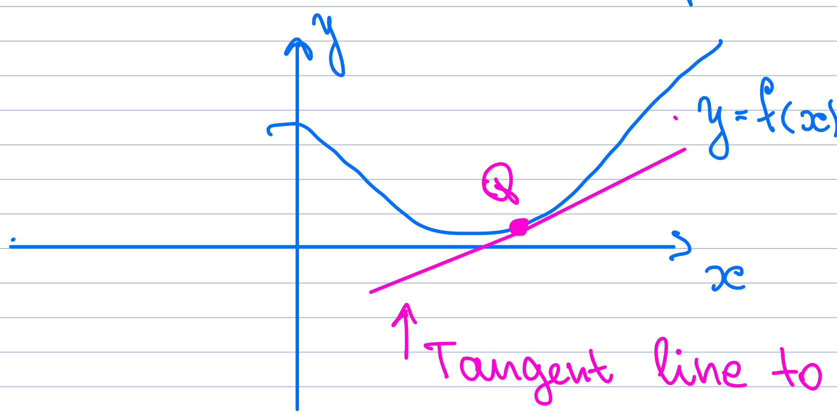
Let $y = f(x)$



$P(x, f(x))$
$Q(a, f(a))$

$$m_{PQ} = \frac{f(x) - f(a)}{x - a} = \frac{\Delta f}{\Delta x}$$

slope of a secant line



Tangent line to $y = f(x)$ at Q

$$\boxed{m_{TL} = \lim_{P \to Q} m_{PQ}}$$ slope of a tangent line

$$\boxed{m_{TL} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

Def. The derivative of a function $y=f(x)$ at point $x=a$ is

$$m_{TL} = \boxed{f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}} \quad (*)$$

slope of a TL to $y=f(x)$ at point $x=a$.

* Of course, in (*) the $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ has to exist.

Tangent line equation to $y=f(x)$ at $x=a$:

$$\boxed{y = f(x) = \underbrace{f'(a)}_{m}(x-a)+f(a)} \quad (**)$$

To find a Tangent line equation (**) to the curve $y=f(x)$ at $x=a$ you need to:

1. find $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$
2. find $f(a)$
3. Plug in (**) $f'(a)$, $f(a)$ and a.

WORKSHEET: SECTION 2.7

$\boxed{\text{Derivatives}}$

1. Complete the definition: The derivative of a function f at $x = a$ is

$$m = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$h = x - a \Rightarrow x = h + a$

$$f'(a) = \lim_{h \to 0} \frac{f(h+a) - f(a)}{h}$$

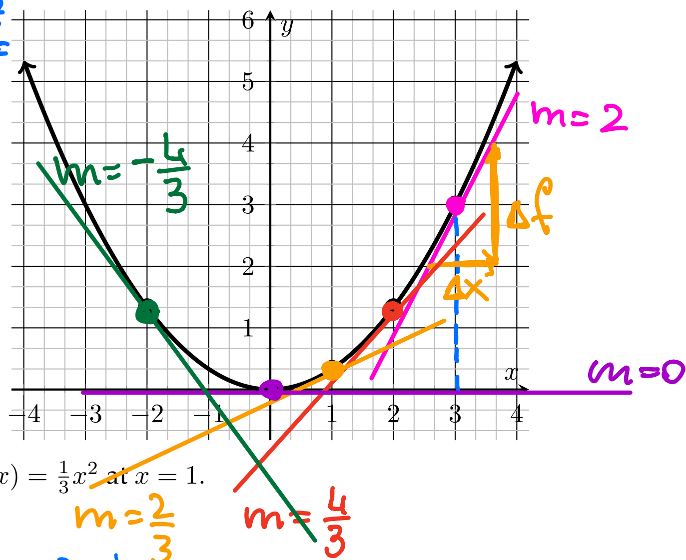
2. Consider the function $f(x) = \frac{1}{3}x^2$, shown in the graph below.

a

(a) Find the slope of the tangent line to $f(x)$ at $x = a$ by taking the limit of the slopes of secant lines. When you are done, check whether or not your solutions seems plausible, by sketching tangent lines at the marked points and determining the slopes of the tangent lines at those points using your calculation.

$$x = a: \; m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\frac{1}{3}x^2 - \frac{1}{3}a^2}{x - a} =$$

$$= \lim_{x \to a} \frac{\frac{1}{3}\cancel{(x-a)}(x+a)}{\cancel{x-a}} = \lim_{x \to a} \frac{1}{3}(x+a) = \frac{1}{3} \cdot 2a = \frac{2a}{3}$$



$x = 3: \; m = \frac{2 \cdot 3}{3} = 2$ $\qquad x = 0: m = 0$

$x = 2: \; m = \frac{4}{3}$ $\qquad x = -2: m = -\frac{4}{3}$

$x = 1: \; m = \frac{2}{3}$

(b) Write the equation of the line tangent to the graph of $f(x) = \frac{1}{3}x^2$ at $x = 1$.

$f(x) = \frac{1}{3}x^2 \qquad x = 1$

At $x = 1$: $f(1) = \frac{1}{3} \cdot 1^2 = \frac{1}{3}$

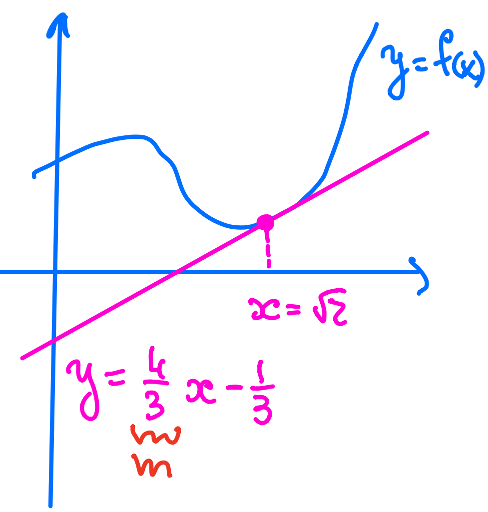
$y = mx + b \qquad m = \frac{2}{3}$

$\frac{1}{3} = \frac{2}{3} \cdot 1 + b \Rightarrow b = -\frac{1}{3}$

$y = \frac{2}{3}x - \frac{1}{3}$ – TL at $x = 1$

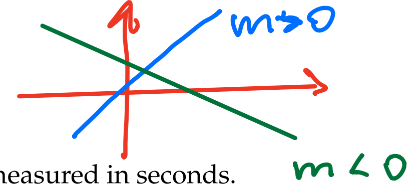
3. Assume the tangent line to the graph of $y = f(x)$ at $x = \sqrt{2}$ has equation $y = \frac{4}{3}x - \frac{1}{3}$. Determine:

(a) $f(\sqrt{2}) = \frac{4}{3} \cdot \sqrt{2} - \frac{1}{3} = \frac{1}{3}(\sqrt{2} - 1)$



(b) $f'(\sqrt{2})$ = slope of a tangent line at $x = \sqrt{2}$

$f'(\sqrt{2}) = \frac{4}{3}$



4. The height in meters of an object is given by the function $s(t) = \frac{2t}{t+1}$ where t is measured in seconds.
   (a) Find $s'(a)$ using the definition in # 1 on this sheet.
   (b) Determine the units of $s'(a)$.
   (c) Find and interpret in the context of the problem the meaning of $s'(1)$.

(a) $$s'(a) = \lim_{t\to a} \frac{s(t)-s(a)}{t-a} = \lim_{\Delta t\to 0} \frac{\Delta s}{\Delta t}$$

meters

seconds

$$s'(a) = \lim_{t\to a} \frac{\frac{2t}{t+1} - \frac{2a}{a+1}}{t-a} = \lim_{t\to a} \frac{\frac{2t(a+1)-2a(t+1)}{(t+1)(a+1)}}{t-a} =$$

$$= \lim_{t\to a} \frac{\cancel{2ta}+2t-\cancel{2at}-2a}{(t+1)(t-a)(a+1)} = \lim_{t\to a} \frac{2\cancel{(t-a)}}{(t+1)\cancel{(t-a)}(a+1)}$$

$$= \lim_{t\to a} \frac{2}{(t+1)(a+1)} = \boxed{\frac{2}{(a+1)^2}}$$

(b) $s'(a)$ has units meters/second

(c) $s'(1) = \frac{2}{2^2} = \frac{1}{2} > 0$: how fast the height of the object is increasing after $t = 1$ s.

5. Let $f(t) = \sqrt{90 - t}$
   (a) Find $f'(a)$ using the definition in # 1 on this sheet.
   (b) If f is measured in degrees Celsius and t is measured in minutes, determine the units of $f'(a)$.
   (c) Find and interpret $f'(0)$.

(a) $$f'(a) = \lim_{t\to a} \frac{f(t)-f(a)}{t-a} = \lim_{t\to a} \frac{\sqrt{90-t}-\sqrt{90-a}}{t-a} \begin{matrix}\times \text{ conjugate} \\ \times \text{ conjugate}\end{matrix} =$$

$$= \lim_{t\to a} \frac{(\sqrt{90-t}-\sqrt{90-a})(\sqrt{90-t}+\sqrt{90-a})}{(\sqrt{90-t}+\sqrt{90-a})(t-a)} = \lim_{t\to a} \frac{\cancel{90}-t-\cancel{90}+a}{(\sqrt{90-t}+\sqrt{90-a})(t-a)} =$$

$$= \lim_{t\to a} \frac{-\cancel{(t-a)}}{(\sqrt{90-t}+\sqrt{90-a})\cancel{(t-a)}} = \boxed{\frac{-1}{2\sqrt{90-a}}}$$

(b) $f'(a)$ is measured in °C/minutes

(c) $$f'(0) = \frac{-1}{2\sqrt{90-0}} = -\frac{1}{2\sqrt{90}} < 0$$

$f(t)$ is a temperature at time t.

$f'(0)$ means how fast is a temperature decreasing at $t = 0$ minutes.

The rate of decreasing is $\boxed{\dfrac{1}{2\sqrt{90}}}$ °C/min.