Section 2.2. The Limit of a Function

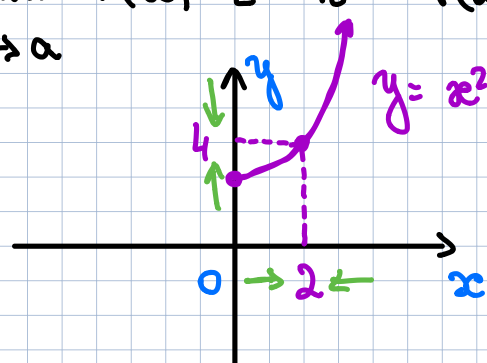
Def. (Intuitive definition of a limit)

Suppose $f(x)$ is defined when x is near the number a (f is defined on some open interval that contains a, except possibly at a itself). Then we write

$$\lim_{x \to a} f(x) = L$$

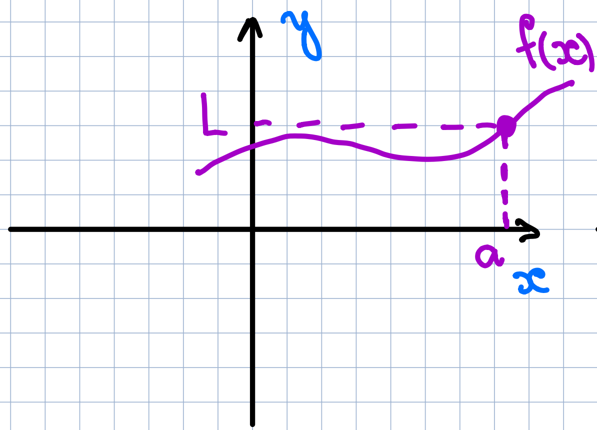
and say "the limit of $f(x)$, as $x \to a$, is equal to L" if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

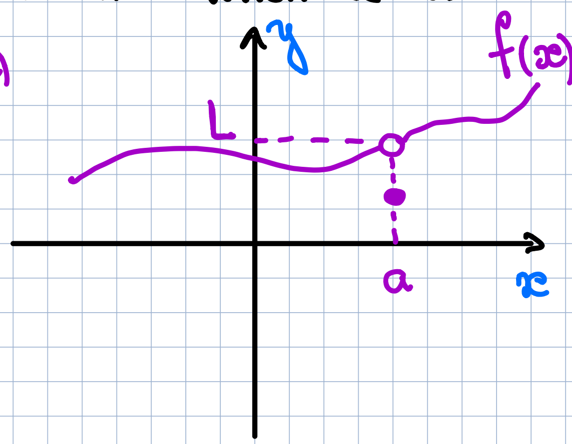
$\lim_{x \to a} f(x) = L$ is $f(x) \to L$ as $x \to a$

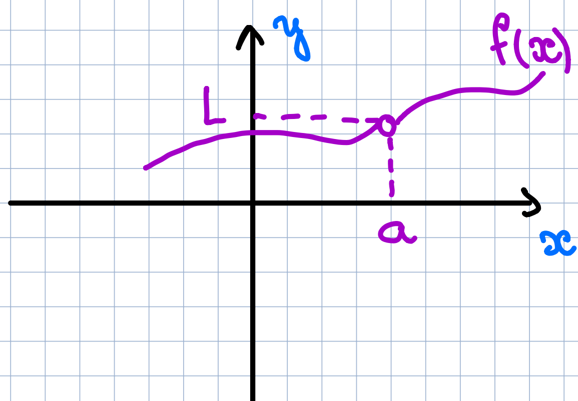


Remark*

- we never consider $x = a$
- f need not to be defined when $x = a$





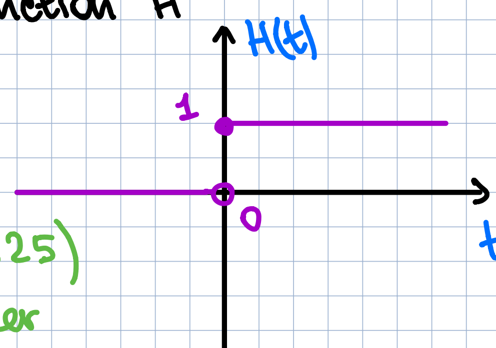


One-sided Limits:

Example

The Heaviside function H

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



Oliver Heaviside (1850-1925)

electrical engineer

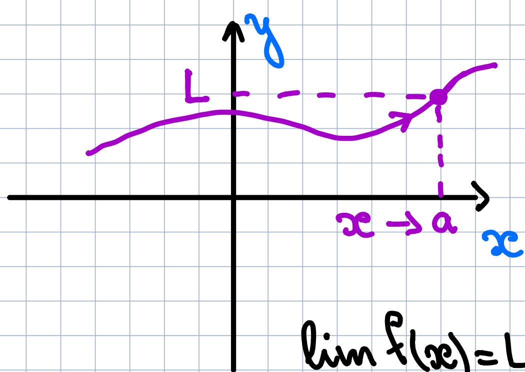
$$\lim_{t \to 0^-} H(t) = 0 \qquad \lim_{t \to 0^+} H(t) = 1$$

Def. (One-sided limits)

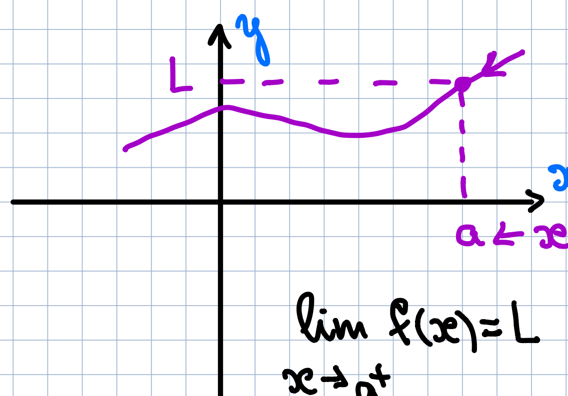
We write

$$\lim_{x \to a^-} f(x) = L \quad \left(\lim_{x \to a^+} f(x) = L \right)$$

and say the left-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x less than a.



$$\lim_{x \to a^-} f(x) = L$$



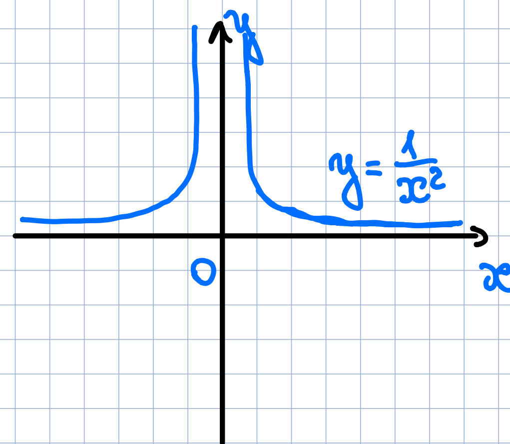
$$\lim_{x \to a^+} f(x) = L$$

The following is true

$$\boxed{\lim_{x \to a} f(x) = L} \text{ if and only if } \boxed{\begin{array}{l} \lim_{x \to a^-} f(x) = L \\ \lim_{x \to a^+} f(x) = L \end{array}}$$

Infinite Limits:

Example $\lim_{x \to 0} \frac{1}{x^2} = ?$



$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

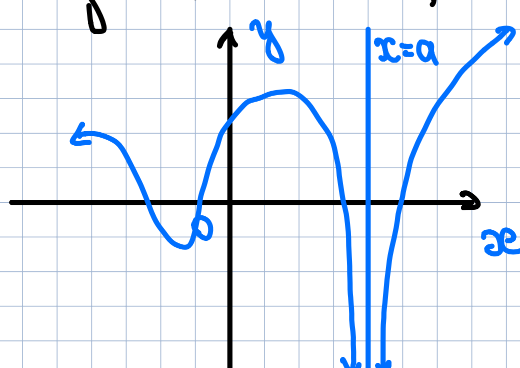
- We are not regarding ∞ as a number
- it does not mean that the limit exists

Def. (Intuitive definition of an infinite limit)

Let f be a function defined on both sides of a, except possibly at a itself. Then

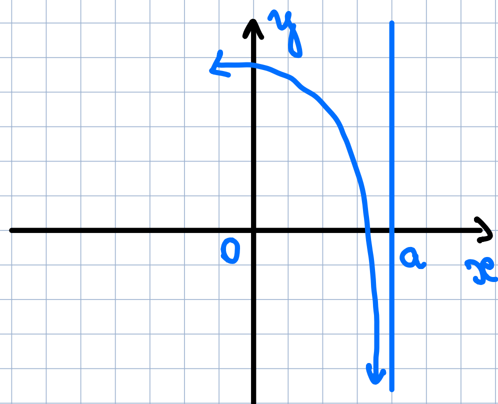
$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.

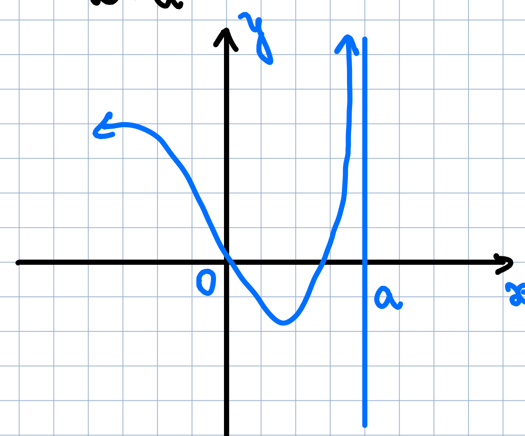


Remark* (similar for one-sided limits)

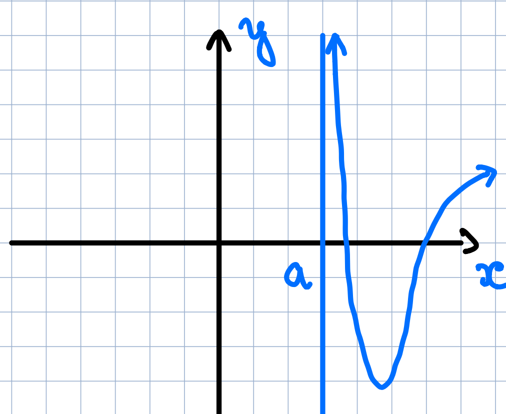
$$\lim_{x \to a^+} f(x) = \infty$$



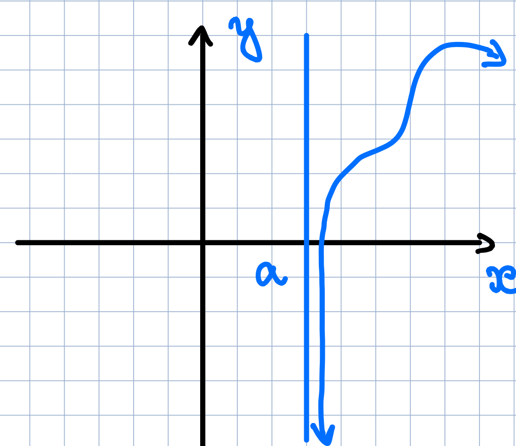
$$\lim_{x \to a^+} f(x) = -\infty$$



$$\lim_{x \to a^+} f(x) = +\infty$$



$$\lim_{x \to a^+} f(x) = -\infty$$



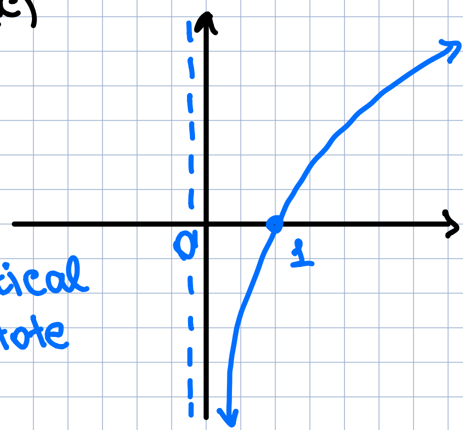
Def. The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$\lim_{x \to a} f(x) = \infty$ $\lim_{x \to a^+} f(x) = \infty$ $\lim_{x \to a^-} f(x) = \infty$

$\lim_{x \to a} f(x) = -\infty$ $\lim_{x \to a^+} f(x) = -\infty$ $\lim_{x \to a^-} f(x) = -\infty$

Example $y = \ln(x)$

$\lim_{x \to 0^+} f(x) = -\infty$



$x = 0$ is a vertical asymptote