Section 3.3. Quadratic Functions

1. Quadratic functions and their graphs
2. Quadratic regression.
3. Maximization / minimization problems.

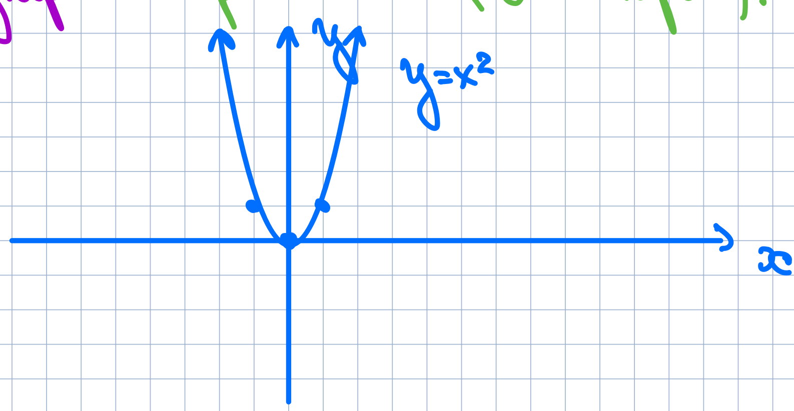
$\boxed{1.}$

<u>Def.</u> A quadratic function $f(x)$, also known as a second-degree polynomial function, is any function that can be written in the form

$$f(x) = ax^2 + bx + c,$$

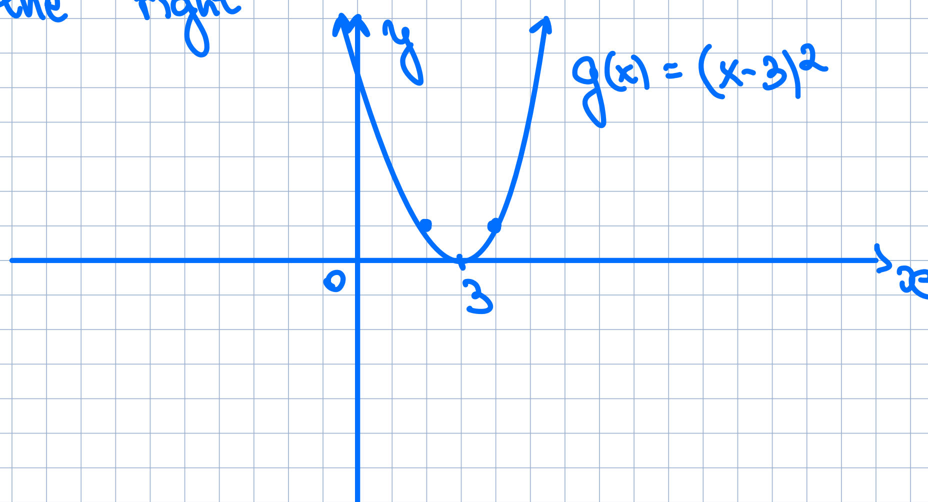
where $a, b, c \in \mathbb{R}$, $a \neq 0$.

The graph: parabola (U - shaped).



$g(x) = (x-3)^2$

parabola, shifted per 3 units to the right



If $a > 0$: parabola opens upward
$a < 0$: parabola opens downward

Def. (vertex form of a quadratic function)
The graph of the function

$$g(x) = a(x-h)^2 + k,$$

$$a, h, k \in \mathbb{R}, \quad a \neq 0$$

is a parabola whose vertex is at (h,k).
The parabola is narrower than $f(x) = x^2$ if $|a| > 1$ and is broader than $f(x) = x^2$ if $0 < |a| < 1$.

Example

- $f(x)=x^2$ – basic parabola with vertex $(0,0)$
- $g(x)=(x-h)^2$ – horizontal shift of h
- $g(x)=(x-h)^2+k$ – add a vertical shift per k units, have a new vertex (h,k).
- $g(x)=a(x-h)^2+k$ – stretch/compress factor of a.

Now, let us consider the quadratic function

$$f(x)=ax^2+bx+c=$$

$$=a\left(x^2+\frac{b}{a}x\right)+c=a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)+$$

$$+c-\frac{ab^2}{4a^2}=a\left(x+\frac{b}{2a}\right)^2+c-\frac{ab^2}{4a^2}=$$

$$=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}$$

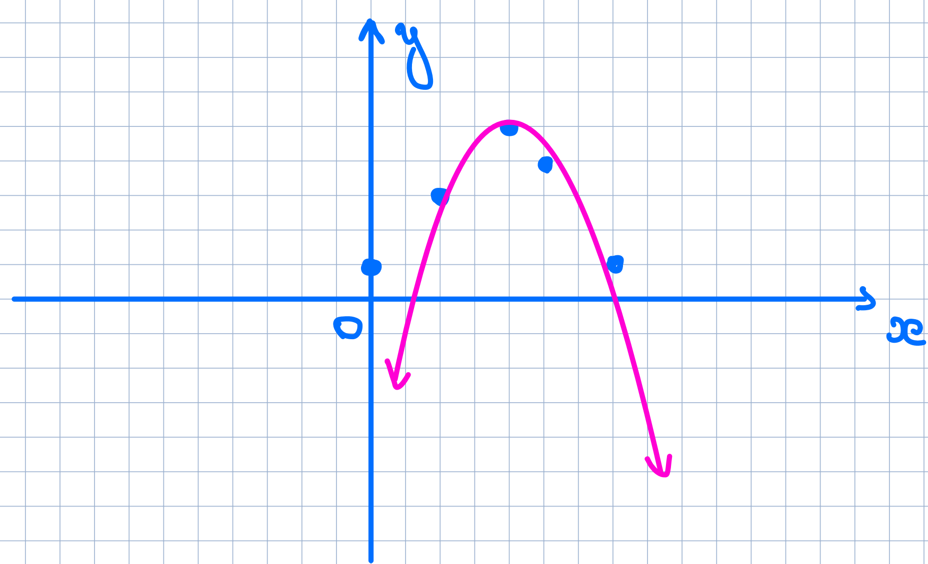
Formula: vertex of a quadratic function

Given a quadratic function $f(x)=ax^2+bx+c$, the graph of f is a parabola with a vertex given by

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$$

$\boxed{2.}$

Goal: find a parabola that better reflects the behaviour of the points (quadratic regression)



Let us consider the quadratic function written in general form:

(1) $$y = ax^2 + bx + c, \quad a \neq 0.$$

We want to find a, b, c such that (1) fits the points (x_i, y_i) in the best way.

Finding a, b, c:

$$\begin{cases} \left(\sum_i x_i^4\right) \cdot a + \left(\sum_i x_i^3\right) b + \left(\sum_i x_i^2\right) c = \sum_i x_i^2 y_i \\ \left(\sum_i x_i^3\right) a + \left(\sum_i x_i^2\right) b + \left(\sum_i x_i\right) c = \sum_i x_i y_i \\ \left(\sum_i x_i^2\right) a + \left(\sum_i x_i\right) b + n \cdot c = \sum_i y_i \end{cases}$$

Solving the above system gives us values of $\boxed{a, b, c.}$

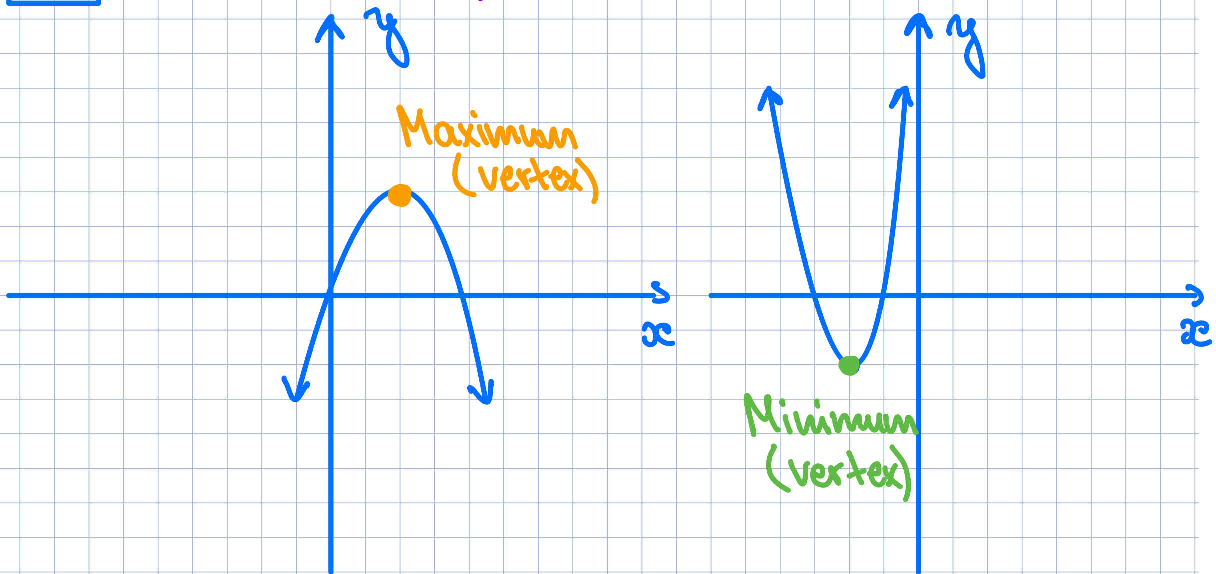
To calculate the Pearson correlation coefficient we use the following formula:

$$r = 1 - \frac{SSE}{SST}, \quad \text{where}$$

$$SSE = \sum_i \left(y_i - a x_i^2 - b x_i + i - c\right)^2$$

$$SST = \sum_i \left(y_i - \bar{y}\right)^2$$

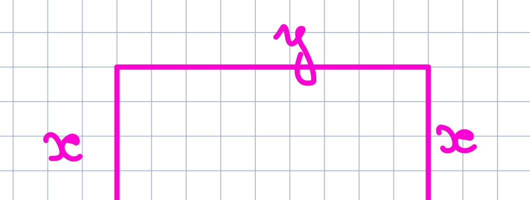
3. Maximization / minimization Problems



Example

A farmer plans to use 100 feet of square fencing material to form a rectangular garden plot against the side of a long barn, using the barn as one side of the plot. How should he split up the fencing among the other three sides in order to maximize the area of the garden plot?

Solution:



$$A = x \cdot y$$

Total fencing is $2x + y = 100$.

Therefore,

$$y = 100 - 2x.$$

Thus,

$$A = x(100 - 2x) = -2x^2 + 100x$$

$$A \to \max$$

$$y(x) = -2x^2 + 100x \to \max$$

parabola (open downwards since $a = -2 < 0$)

The peak of parabola:

$$\left(-\frac{b}{2a}, y\left(-\frac{b}{2a}\right)\right)$$

$b = 100$

$a = -2$

$$\left(-\frac{100}{-4}, y\left(\frac{100}{4}\right)\right) = (25, y(25))$$

$$(25, -2 \cdot 625 + 2500) =$$

$$= (25, 1250)$$

Hence, $x = 25$ (feet) and $y = 100 - 50 = 50$ (feet)

Therefore, the maximum area is:

$$A = 25 \cdot 50 = 1250 \, (ft)^2.$$