THEORETICAL PART: Solutions

Definition (Addition, Subtraction, Multiplication, and Division of Functions):

Let f and g be two functions. The **sum** $f+g$, **difference** $f-g$, **product** $f \cdot g$, and **quotient** $\frac{f}{g}$ are four new functions defined as follows:

1. $(f+g)(x) = f(x) + g(x)$
2. $(f-g)(x) = f(x) - g(x)$
3. $(f \cdot g)(x) = f(x) \cdot g(x)$
4. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$.

Definition (Composing Functions):

Let f and g be two functions. The **composition** of f and g, denoted $f \circ g$, is the function defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ consists of all x in the domain of g for which $g(x)$ is in turn in the domain of f. The function $f \circ g$ is read "f composed with g".

CAUTION:

Note that the order of f and g is **IMPORTANT**.

CAUTION:

When evaluating the composition $(f \circ g)(x)$ at a point x, there are two reasons the value might be undefined:

1. If x is not in the domain of g, then $g(x)$ is undefined and we can't evaluate $f(g(x))$.
2. If $g(x)$ is not in the domain of f, then $f(g(x))$ is undefined and we can't evaluate it.

Definition (Recursion):

Recursion refers to using the output of a function as its input, and repeating the process a certain number of times.

For instance,

$$f^3(x) = f(f(f(x))) = (f \circ f \circ f)(x).$$

PRACTICAL PART:

1. Given that $f(-2) = 5$, $g(-2) = -3$, find:

   (a) $(f - g)(-2) =$ $f(-2) - g(-2) = 5 + 3 = 8$

   (b) $\left(\frac{f}{g}\right)(-2) =$ $\frac{f(-2)}{g(-2)} = \frac{5}{-3}$

2. Given the two functions $f(x) = 4x^2 - 1$ and $g(x) = \sqrt{x}$, find:

   (a) $(f + g)(x)$

   (b) $(f \cdot g)(x)$

(a) $(f+g)(x) = f(x) + g(x) = 4x^2 - 1 + \sqrt{x}$

$\text{Dom}(f) = \mathbb{R}$

$\text{Dom}(g) = [0, \infty)$

$\text{Dom}(f+g) = \text{Dom}(f) \cup \text{Dom}(g) = \mathbb{R}$

3. Given $f(x) = x^2$ and $g(x) = x - 3$, find the following:

   (a) $(f \circ g)(6) = f(g(6)) = (6-3)^2 = 3^2 = 9$

   (b) $(f \circ g)(x) = f(g(x)) = (x-3)^2$

   (c) $(g \circ f)(6) = (x^2 - 3)(6) = 6^2 - 3 = 33$

   (d) $(g \circ f)(x) = x^2 - 3$

2. (b) $(f \cdot g)(x) = f(x) \cdot g(x) =$

$= (4x^2 - 1)\sqrt{x} = 4x^{5/2} - \sqrt{x}$

$\mathrm{Dom}(f \cdot g) = \mathrm{Dom}(f) \cap \mathrm{Dom}(g) =$

$= [0, \infty)$

4. Let $f(x) = x^2 - 4$ and $g(x) = \sqrt{x}$. Find formulas and state the domains for the following:

- $f \circ g$
- $g \circ f$

$f \circ g = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 - 4 = x - 4$

$g \circ f = g(f(x)) = g(x^2 - 4) = \sqrt{x^2 - 4}$

$\mathrm{Dom}(f \circ g) = \mathbb{R}$ (we have a straigh line)

$\mathrm{Dom}(g \circ f) = (-\infty, -2] \cup [2, \infty)$

$x^2 - 4 \geq 0$

5. Decompose the function $f(x) = |x^2 - 3| + 2$ into the following:

a. a composition of two functions

b. a composition of three functions

(a) $f(x) = |x^2 - 3| + 2$

$g(x) = |x^2 - 3|$ $h(x) = x + 2$

$\boxed{f(x) = g \circ h}$

(b) $f(x) = |x^2 - 3| + 2$

$g(x) = x^2 - 3$ $h(x) = |x|$ $j(x) = x + 2$

$f(x) = j(h(g(x))) = j \circ h \circ g$