Section 3.1. Relations and functions

1. Relations, domain, and range.
2. Functions and the vertical line test
3. Function notation and function evaluation
4. Implied domain of a function.

$\boxed{1.}$

Def.

- A <u>relation</u> is a set of ordered pairs.
- The <u>domain</u> of a relation is the set of all the first coordinates.
- The <u>range</u> of a relation is the set of all second coordinates.

Example

(a) $R = \{(-4,2), (6,-1), (0,0), (-4,0), (\pi, \pi^2)\}$

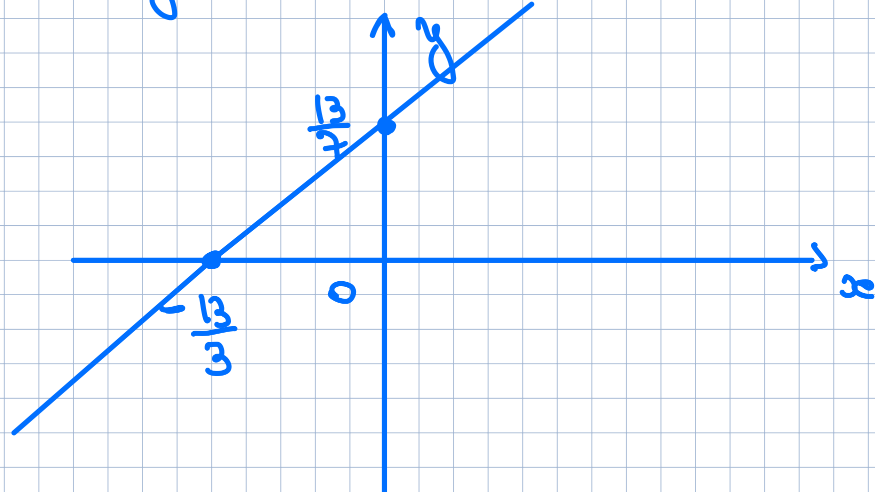
$\uparrow$

relation

$\mathrm{Dom}(R) = \{-4, 6, 0, -4, \pi\}$

$\mathrm{Ran}(R) = \{2, -1, 0, 0, \pi^2\}$

(b) $-3x+7y=13$ describes a relation.

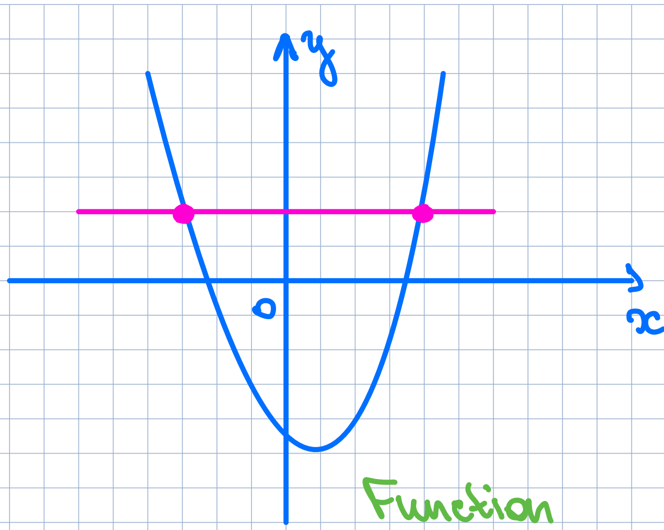


$\mathrm{Dom}(R) = \mathbb{R}$

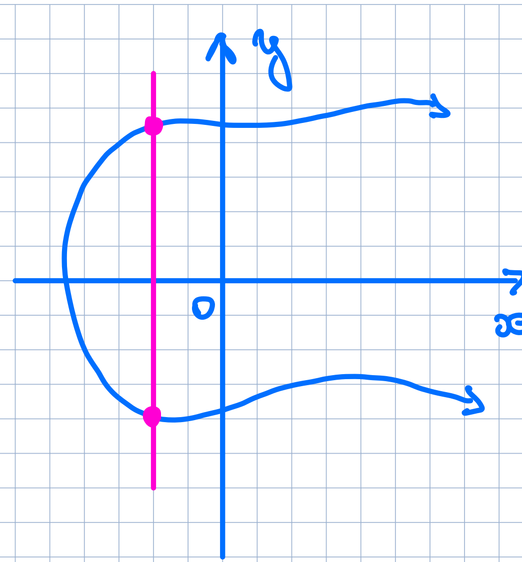
$\mathrm{Ran}(R) = \mathbb{R}$

$\boxed{2.}$ Def. (Function)

A function is a relation in which every element of the domain is paired with exactly one element of the range.
Equivalently, a function is a relation in which no two distinct ordered pairs have the same first coordinate.



Function



Not a function

Example

(a) $R = \{(-4,2), (6,-1), (0,0), (-4,0), (\pi, \pi^2)\}$

This relation is not a function.
We have $(-4,2)$ and $(-4,0)$ with the same first coordinate.

(b) Straight line is a function.

$$ax + by = c.$$

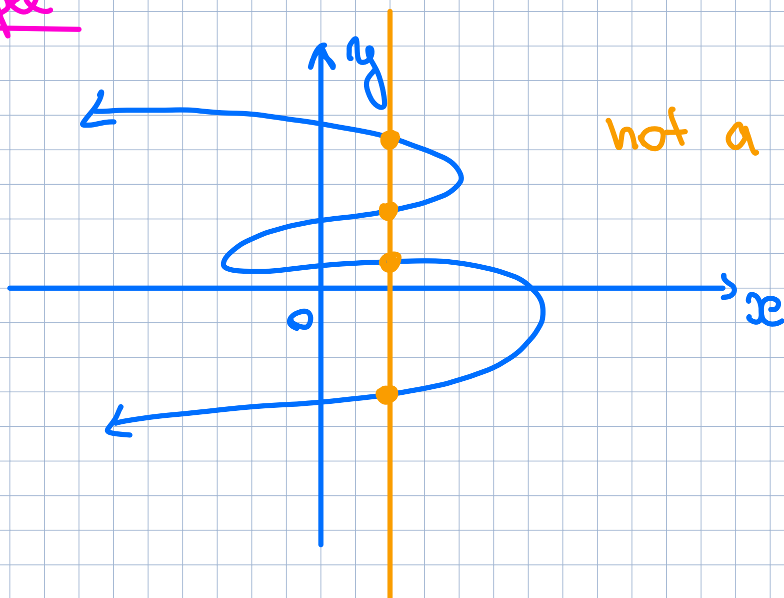
Theorem (The vertical line test)

If a relation can be graphed in the Cartesian plane, the relation is a function iff no vertical line passes through the graph more than once.
If even one vertical line intersects

the graph of the relation two or more times, the relation fails to be a function.

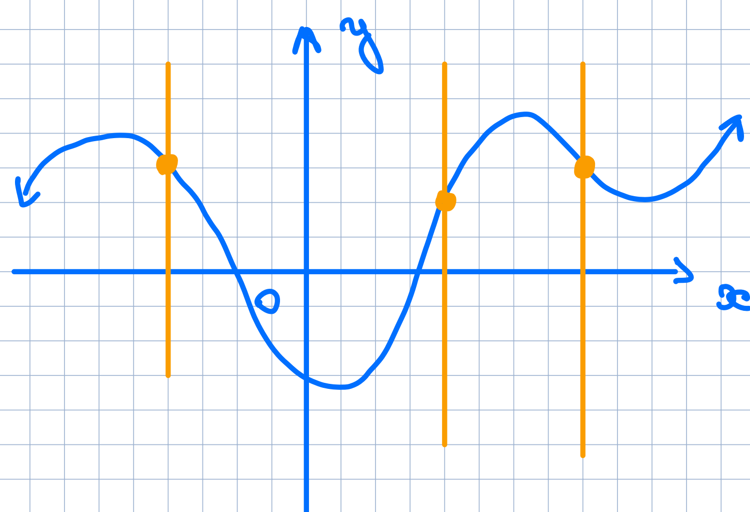
Example

(a)



not a function

(b)



$y = f(x)$ is a function

$\boxed{3.}$ Leonard Euler (1740s)

function notation

Def. $\boxed{y = f(x)}$ is a function notation.

x is an independent variable
y is a dependent variable

$x \in \mathrm{Dom}(f)$
$y \in \mathrm{Ran}(f)$

$$x \longrightarrow \underbrace{\boxed{\quad f \quad}}_{\text{machine}} \longrightarrow f(x) = y$$

Example

(a) $y = \dfrac{3}{x} + 2$ – function

$y(1) = \dfrac{3}{1} + 2 = 5$

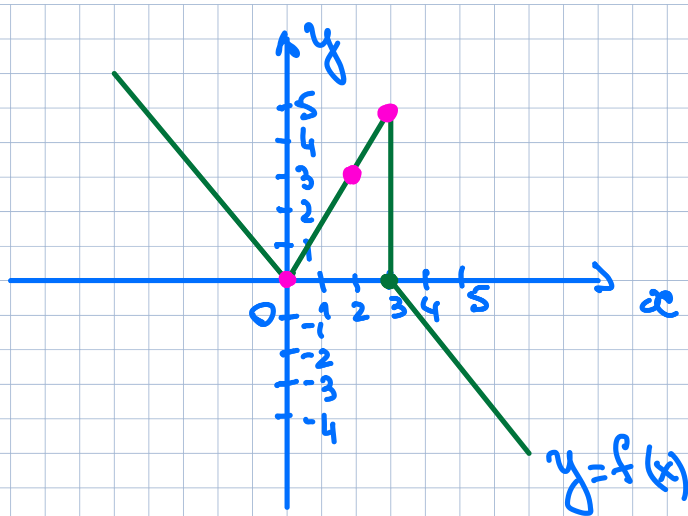
(b) $7x + 3 = 2y - 1$

$2y = 7x + 1 + 3 = 7x + 4$

$y = \dfrac{7}{2}x + 2$ – function

$y(0) = \dfrac{7}{2} \cdot 0 + 2 = 2$

Example (Interpreting function values from a graph)



$f(0)=0$
$f(2)=3$
$f(3)=5$

Example (Evaluating functions)

$f(x)=3x^2-2$

- $f(a)=3a^2-2$
- $f(x+h)=3(x+h)^2-2$
- $\dfrac{f(x+h)-f(x)}{h} = \dfrac{3(x+h)^2-2-3x^2+2}{h}$

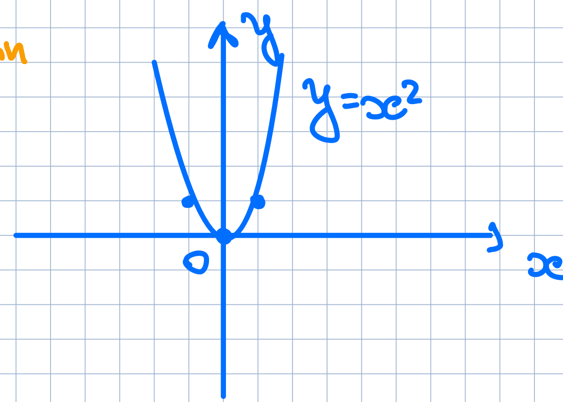
Def. (Domain and codomain notation)

The notation $f: A \to B$ implies that f is a function from the set A to the set B. The symbols indicate that the domain of f is the set A, and that the range of f is a subset of the set B. The set B is called the codomain of f.

Example

(a) $f: \mathbb{R} \to \mathbb{R}$ dom codomain

$f(x) = x^2$



$Dom(f) = \mathbb{R}$ dom

$Ran(f) = [0, \infty)$

(b) $g: \mathbb{R} \to [0, \infty)$ dom codomain

$g(x) = x^2$

$codomain(g) = Ran(g)$

$Dom(g) = \mathbb{R}$

$Ran(g) = [0, \infty)$

$\boxed{4.}$ Example (Implied domain of a function)

(a) $f(x) = 5x - \sqrt{3-x}$

$3 - x \geq 0$

$3 \geq x$ or $x \leq 3$

$Dom(f) = \{x \mid x \leq 3\} = (-\infty, 3]$

(b) $g(x) = \dfrac{x-3}{x^2-1} = \dfrac{x-3}{(x-1)(x+1)}$

$(x-1) \neq 0$ and $x+1 \neq 0$

Hence, $x \neq 1$ and $x \neq -1$.

$$\mathrm{Dom}(g) = \mathbb{R} \setminus \{1, -1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$