Section 4.3. Combining functions

1. Combining functions arithmetically.
2. Composing functions.
3. Decomposing functions.
4. Recursive graphics.

1.

Def.

Let f and g be two functions. The <u>sum</u> $f+g$, <u>difference</u> $f-g$, <u>product</u> $f \cdot g$, and <u>quotient</u> $\frac{f}{g}$ are four new functions defined as follows:

1. $(f+g)(x) = f(x) + g(x)$
2. $(f-g)(x) = f(x) - g(x)$
3. $(f \cdot g)(x) = f(x) \cdot g(x)$
4. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$.

$\text{Dom}(f+g) = \text{Dom}(f) \cap \text{Dom}(g)$

$\text{Dom}(f-g) = \text{Dom}(f) \cap \text{Dom}(g)$

$\text{Dom}(f \cdot g) = \text{Dom}(f) \cap \text{Dom}(g)$

$\text{Dom}\left(\frac{f}{g}\right) = \text{Dom}(f) \cap \text{Dom}(g), \quad g(x) \neq 0.$

Example

- $f(-2) = 5$

  $g(-2) = -3$

Find $(f-g)(-2)$ - ?

$(f-g)(-2) = f(-2) - g(-2) = 5 + 3 = 8$

Find $\left(\frac{f}{g}\right)(-2)$ - ?

$\left(\frac{f}{g}\right)(-2) = \frac{f(-2)}{g(-2)} = \frac{5}{-3} = -\frac{5}{3}.$

Example

$f(x) = 4x^2 - 1$

$g(x) = \sqrt{x}$

$(f+g)(x) = f(x) + g(x) = 4x^2 - 1 + \sqrt{x}$

$(f \cdot g)(x) = f(x) \cdot g(x) = (4x^2 - 1)\sqrt{x} = 4x^{5/2} - x^{1/2}.$

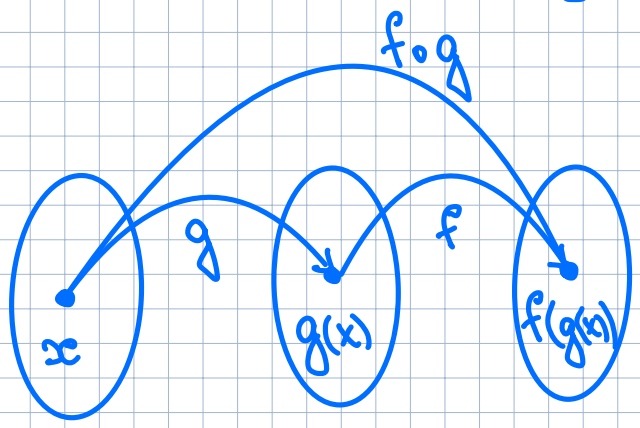
2. **Def.** (Composing functions)

Let f and g be two functions.

The composition of f and g $(f \circ g)$ is the function defined by

$$(f \circ g)(x) = f(g(x)).$$

$$\text{Dom}(f \circ g) = \{x \in \text{Dom}(g) \mid g(x) \in \text{Dom}(f)\}.$$



Caution! The order of f and g is important.

Example

- $f(x) = x^2$, $g(x) = x - 3$

$(f \circ g)(6) = ?$

$(f \circ g)(x) = f(g(x)) = (x-3)^2$

$(f \circ g)(6) = (6-3)^2 = 9$

Caution!

When evaluating the composition $(f \circ g)(x)$ at a point x, there are two reasons the value might be undefined:

1. If x is not in the domain of g, then $g(x)$ is undefined and we can't evaluate $f(g(x))$.
2. If $g(x)$ is not in the domain of f, then $f(g(x))$ is undefined and we can't evaluate it.

Example

Let $f(x) = \sqrt{x-5}$

$g(x) = \dfrac{2}{x+1}$

$(f \circ g)(-1) = f(g(-1))$

We have that $g(-1)$ is undefined since

$-1 \notin \text{dom}(g)$ $\left[\dfrac{1}{-1+1} = \dfrac{1}{0} \right]$.

Therefore,

$(f \circ g)(-1)$ is undefined.

▼

3. Decomposing Functions

- Let us consider a function

$$h(x) = (x-2)^2.$$

$h(x) = (f \circ g)(x) = f(g(x))$, where

$f(x) = x^2$ and $g(x) = x-2$.

- Let us consider a function

$$h(x) = \sqrt[3]{5x^2 - 1}$$

$h(x) = (f \circ g)(x) = f(g(k(x)))$

$f(x) = \sqrt[3]{x}$, $g(x) = 5x - 1$,

$k(x) = x^2$.

4.

Recursion refers to using the output of a function as its input, and repeating the process a certain number of times.

$$f(f(f \dots (x)))$$

Notation: $f^2(x) = f(f(x)) = (f \circ f)(x)$

$$f^3(x) = f(f(f(x))) = f \circ (f \circ f) =$$

$$= (f \circ f \circ f)(x)$$

The functions $f^2, f^3, \ldots$ are called <u>iterates</u> of f, with f^n being the n-th iterate of f.