Section 1.9. Rational and radical equations in one variable

1. Solving rational equations
2. Applications of rational equations
3. Solving radical equations
4. Solving equations with positive rational exponents
5. Solving equations for one variable.

1.

Def. A rational equation is an equation that contains at least one rational expression, while any nonrational expressions are polynomials.

Example

$$\frac{x^3+3x^2}{x^2-2x-15}=\frac{4x+5}{x-5}$$

$$\frac{x^3+3x^2}{x^2-2x-15} - \frac{4x+5}{x-5} = 0$$

$$\frac{(x^3+3x^2)(x-5)-(4x+5)(x^2-2x-15)}{(x^2-2x-15)(x-5)} = 0$$

$x \neq 5$ $\quad$ $x^2-2x-15 \neq 0$

$(x-5)(x+3) \neq 0$

$x \neq 5, \quad x \neq -3$

$$(x^3+3x^2)(x-5)-(4x+5)(x^2-2x-15) = 0$$

$$x^4 - 5x^3 + 3x^3 - 15x^2 - 4x^3 + 8x^2 + 60x - 5x^2 +$$
$$+ 10x + 75 = 0$$

$$x^4 - 6x^3 - 12x^2 + 70x + 75 = 0$$

$$x^2(x+3)(x-5) - (4x+5)(x-5)(x+3) = 0$$

$$(x-5)(x+3)(x^2-4x-5) = 0$$

$x \neq 5$

$x \neq -3$

$$x^2 - 4x - 5 = 0$$

$$(x-5)(x+1) = 0$$

$$\boxed{x = -1}$$

Example (Absolute value rational equation)

$$|x-5| = \frac{7}{x+1}$$

$$\begin{cases} 1)\quad x-5 = \dfrac{7}{x+1}, & x-5 \geq 0 \\ 2)\quad -(x-5) = \dfrac{7}{x+1}, & x-5 < 0 \end{cases}$$

$x \neq -1$

1) $(x-5)(x+1) = 7$

$x^2 - 4x - 12 = 0$

$(x-6)(x+2) = 0$

$\boxed{x_1 = 6}$ $\quad \cancel{x_2 = -2}$, $\underline{x \geq 5}$

2) $-x^2 + 4x + 5 - 7 = 0$

$-x^2 + 4x - 2 = 0$

$x^2 - 4x + 2 = 0$

$D = 16 - 8 = 8$

$x_1 = \dfrac{4+\sqrt{8}}{2} = \dfrac{4+2\sqrt{2}}{2} = \boxed{2+\sqrt{2}}$

$\boxed{x_2 = 2-\sqrt{2}}$ $\quad \underline{x < 5}$

$\boxed{2.}$

Goal: determine how fast the task at hand can be completed, either by the workers together or by one of the workers individually.

(1) The rate of work is the reciprocal of the time needed to complete the task.

If a given job can be done by a worker in x units of time, the worker works at a rate of $\frac{1}{x}$ jobs per unit of time.

(2) Rates of work are "additive".

$\boxed{3.}$ **Def.** A radical equation is an equation that has at least one radical expression containing a variable, while any nonradical expressions are polynomial terms.

Procedure:

1. Begin by isolating the radical expression on one side of the equation. If there is

more than one radical expression, choose one to isolate on one side.

2. Raise both sides of the equation by the power necessary to "undo" the isolated radical.

3. If any radical expressions remain, simplify the equation if possible and then repeat steps 1 and 2 until the result is a polynomial equation.
When a polynomial equation has been obtained, solve the equation using polynomial methods.

4. Check your solutions in the original equation! Any extraneous solutions must be discarded.

Example

(a) $\sqrt{1-x} - 1 = x$

$$\sqrt{1-x} = x + 1$$
$$1 - x = (x+1)^2$$
$$\cancel{1} - x = x^2 + 2x + \cancel{1}$$
$$x^2 + 3x = 0$$

$$x(x+3)=0$$

$\underline{x=0}$ or $\underline{x=-3}$

Check:

$$\sqrt{1-0}-1=0$$

$$\sqrt{1}=1 \quad \checkmark$$

$$\sqrt{1+3}-1=-3$$

$$\sqrt{4}=-3+1=-2$$

$$2\neq -2$$

Thus, -3 is an extraneous solution.

Answer: $\{0\}$ is the solution set.

$\boxed{4.}$

<u>Def.</u>

Meaning of $a^{\frac{m}{n}}$: if m and n are natural numbers with $n\neq 0$, if m and n have no common factors greater than 1, and if $\sqrt[n]{a}$ is a real number, then

$$a^{\frac{m}{n}}=\sqrt[n]{a^m}=\left(\sqrt[n]{a}\right)^m.$$

Example

- $x^{\frac{2}{3}} - 9 = 0$

$$x^{\frac{2}{3}} = 9$$

$$x^2 = 9^3$$

$$x = \pm\sqrt{9^3} = \pm 3^3 = \pm 27$$

Thus, if we plug (± 27) in $x^{\frac{2}{3}} - 9 = 0$ we see that the original equation holds.

$\boxed{5.}$ Solving equations for one variable.

Example

$v_e = \sqrt{\frac{2GM}{r}}$ – escape speed of a planet

Here G is a universal gravitational constant

M is a mass of a planet

We want to solve for r:

$$v_e^2 = \frac{2GM}{r}$$

$$r = \frac{2GM}{v_e^2}$$