THEORETICAL PART:

Theorem (The Fundamental Theorem of Algebra):

If p is a polynomial of degree n, with $n \geq 1$, then p has **at least one zero**. That is, the equation $p(x) = 0$ has at least one solution. It is important to note that the zero of p, and consequently the solution of $p(x) = 0$, may be a non-real complex number.

Theorem (The Linear Factors Theorem):

Given the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $n \geq 1$ and $a_n \neq 0$, p can be factored as $p(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n)$, where $c_1, c_2, \cdots, c_n$ are constants (possibly non-real complex constants and not necessarily distinct). In other words, **an n-th degree polynomial can be factored as a product of n linear factors**.

CAUTION:

The Linear Factors Theorem does not tell us the following things:

1. The theorem does not tell us that a polynomial has all real zeros.
2. The theorem does not tell us that a polynomial has n distinct zeros.
3. The theorem does tell us that any polynomial can be written as a product of linear factors; it does not tell us how to determine the linear factors.

Theorem (Interpreting the Linear Factors Theorem):

The graph of an **n-th degree polynomial function has at most n x-intercepts and at most $n - 1$ turning points**. This also means that an n-th degree polynomial function has at most n zeros.

Definition (Multiplicity of Zeros):

If the linear factor $(x - c)$ appears $k > 0$ times inn the factorization of a polynomial (or as $(x - c)^k$), we say the number c is a **zero of multiplicity k**.

PROPERTIES (Geometric Meaning of Multiplicity):

If c is a real zero of multiplicity k of a polynomial p (alternatively, if $(x - c)^k$ is a factor of p), the graph of p will touch the x-axis at $(c, 0)$ and

1. cross through the x-axis if k is odd, or
2. stay on the same side of the x-axis if k is even.

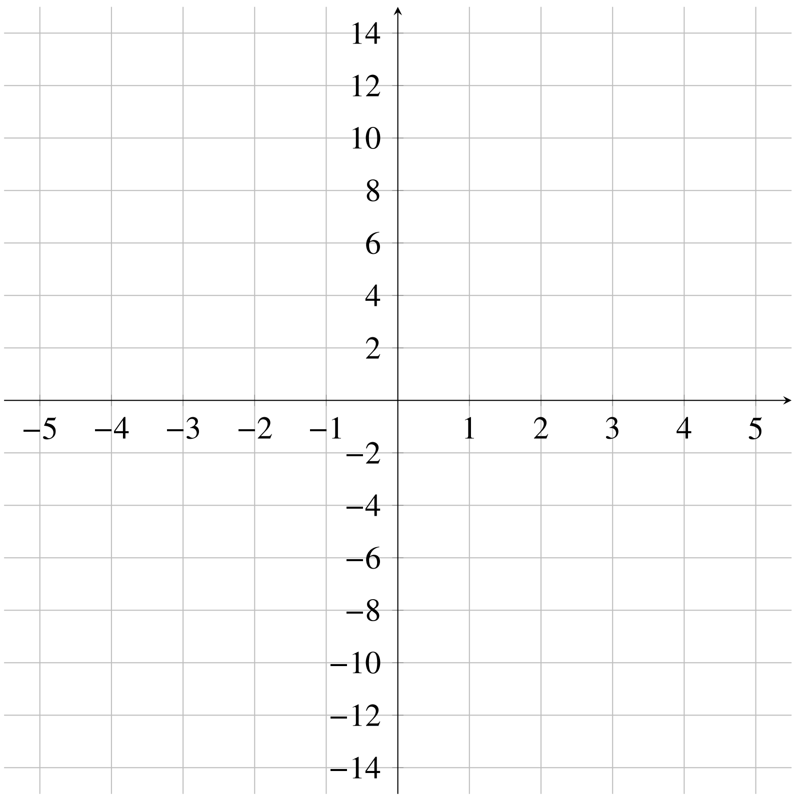
Further, if $k > 1$, the graph of p will ”flatten out” near $(c, 0)$.

Theorem (The Conjugate Roots Theorem):

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with only real coefficients. If the complex number $a + bi$ is a zero of p, then so is the complex number $a - bi$. In terms of the linear factors of p, this means that if $x - (a + bi)$ is a factor of p, then so $x - (a - bi)$.

PRACTICAL PART:

1. Sketch the graph of the polynomial $f(x) = (x+2)(x+1)^2(x-3)^3$



2. Given that $4-3i$ is a zero of the polynomial $f(x) = x^4 - 8x^3 + 200x - 625$ factor f completely.

3. Construct a fourth-degree real-coefficient polynomial function f with zeros of 2, -5, and $1 + i$ such that $f(1) = 12$.

4. Use all available methods to factor the following polynomial function completely, and then sketch the graph of the polynomial function.

$$f(x) = x^5 + 4x^4 + x^3 - 10x^2 - 4x + 8$$

5. Use all available methods to solve the following polynomial equation:

$$2x^4 - 5x^3 - 2x^2 + 15x = 0$$