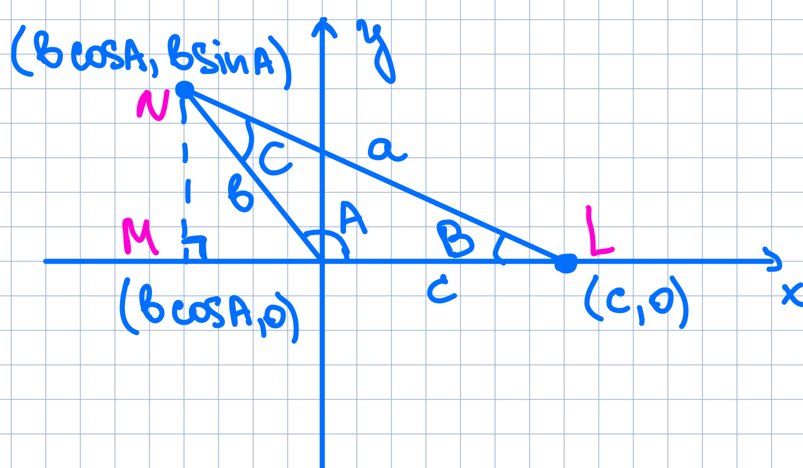
Section 9.2. The Law of Cosines

1. The Law of Cosines.
2. Applications of the Law of Cosines.
3. Heron's Formula and the area of a triangle.

$\boxed{1.}$



Theorem (The Law of Cosines)

Given a triangle ABC, with sides labeled conventionally, the following equations are all true. These equations represent the Law of Cosines.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Proof

Let us consider the right triangle MNL.

By Pythagorean theorem we have

$$a^2 = b^2\sin^2 A + (c - b\cos A)^2 =$$

$$= b^2\sin^2 A + c^2 - 2bc\cos A +$$

$$+ b^2\cos^2 A$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

▼

Law of Cosines
Two sides and the included angle (SAS)
Three sides (SSS)

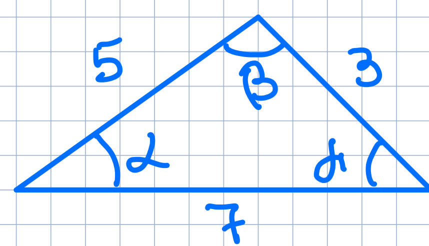
$\boxed{2.}$

Example (using the Law of Cosines in an SSS situation)

Determine the three angles for a triangle in which $a = 3$ (in), $b = 5$ (in), and $c = 7$ (in).

Solution

$\alpha, \beta, \gamma - ?$



$$7^2 = 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cdot \cos\beta$$

$$49 = 34 - 30\cos\beta$$

$$\cos\beta = -\frac{1}{2}$$

$$\beta = 120^\circ$$

Now, by the Law of Sines:

$$\frac{\sin\alpha}{3} = \frac{\sin 120^\circ}{7}$$

So $\sin\alpha \approx 0.37 \Rightarrow \alpha \approx 21.79^\circ$.

We can then determine an angle

$$\gamma = 180^\circ - \alpha - \beta = 38.21^\circ$$

▼

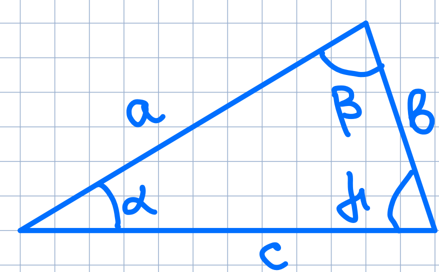
$\boxed{3.}$

Theorem (Area of a triangle (Heron's formula))

Given a triangle with sides a, b, and c, let
$$S = \frac{a+b+c}{2}.$$
Then the following is true:

$$\boxed{\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}}$$

Proof.



$$\text{Area}^2 = \frac{1}{4} a^2 b^2 \sin^2 \beta =$$

$$= \frac{1}{4} a^2 b^2 (1 - \cos^2 \beta) =$$

$$= \frac{1}{4} a^2 b^2 (1 - \cos \beta)(1 + \cos \beta)$$

Now, let us rewrite

- $1 - \cos \beta = 1 - \dfrac{a^2 + b^2 - c^2}{2ab} =$

$$= \frac{2ab - a^2 - b^2 + c^2}{2ab} = \frac{c^2 - (a^2 - 2ab + b^2)}{2ab} =$$

$$= \frac{c^2 - (a-b)^2}{2ab} = \frac{(c+a-b)(c-a+b)}{2ab}$$

Similarly,

- $1 + \cos\beta = \frac{(a+b+c)(a+b-c)}{2ab}$

Therefore,

$$\text{Area}^2 = \frac{1}{4} a^2 b^2 \left(\frac{(c+a-b)(c-a+b)}{2ab} \right) \cdot$$

$$\cdot \left(\frac{(a+b+c)(a+b-c)}{2ab} \right) = \frac{1}{16} (a+b+c)(c-a+b)(c+a-b) \cdot$$

$$\cdot (a+b-c) = S(S-a)(S-b)(S-c)$$

$\blacktriangledown$