Solutions

THEORETICAL PART:

Theorem (The Fundamental Theorem of Algebra):

If p is a polynomial of degree n, with $n \geq 1$, then p has **at least one zero**. That is, the equation $p(x) = 0$ has at least one solution. It is important to note that the zero of p, and consequently the solution of $p(x) = 0$, may be a non-real complex number.

Theorem (The Linear Factors Theorem):

Given the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $n \geq 1$ and $a_n \neq 0$, p can be factored as $p(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n)$, where $c_1, c_2, \cdots, c_n$ are constants (possibly non-real complex constants and not necessarily distinct). In other words, **an n-th degree polynomial can be factored as a product of n linear factors**.

CAUTION:

The Linear Factors Theorem does not tell us the following things:

1. The theorem does not tell us that a polynomial has all real zeros.
2. The theorem does not tell us that a polynomial has n distinct zeros.
3. The theorem does tell us that any polynomial can be written as a product of linear factors; it does not tell us how to determine the linear factors.

Theorem (Interpreting the Linear Factors Theorem):

The graph of an **n-th degree polynomial function has at most n x-intercepts and at most $n-1$ turning points**. This also means that an n-th degree polynomial function has at most n zeros.

Definition (Multiplicity of Zeros):

If the linear factor $(x-c)$ appears $k > 0$ times inn the factorization of a polynomial (or as $(x-c)^k$), we say the number c is a **zero of multiplicity** k.

PROPERTIES (Geometric Meaning of Multiplicity):

If c is a real zero of multiplicity k of a polynomial p (alternatively, if $(x-c)^k$ is a factor of p), the graph of p will touch the x-axis at $(c, 0)$ and

1. cross through the x-axis if k is odd, or
2. stay on the same side of the x-axis if k is even.

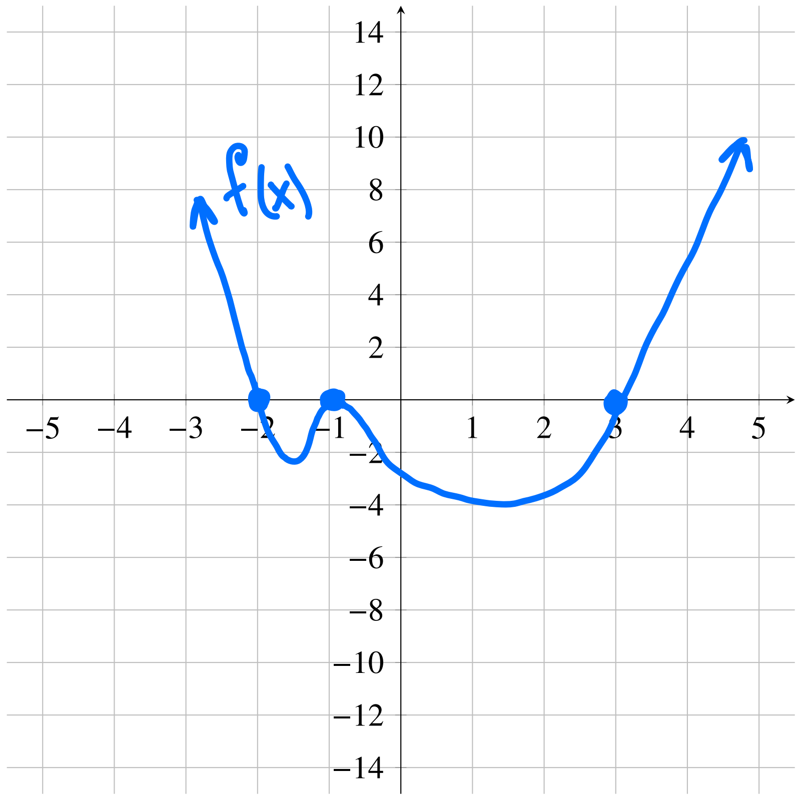
Further, if $k > 1$, the graph of p will "flatten out" near $(c, 0)$.

Theorem (The Conjugate Roots Theorem):

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with only real coefficients. If the complex number $a + bi$ is a zero of p, then so is the complex number $a - bi$. In terms of the linear factors of p, this means that if $x - (a + bi)$ is a factor of p, then so $x - (a - bi)$.

PRACTICAL PART:

1. Sketch the graph of the polynomial $f(x) = (x+2)(x+1)^2(x-3)^3$



• Since for $x=-1$ we have $k=2$, then $f(x)$ stays on the same side

• for $x=-2$ $x=3$ we have $k=1$ and $k=3$; therefore, $f(x)$ goes through x-axis at these points.

2. Given that $4-3i$ is a zero of the polynomial $f(x) = x^4 - 8x^3 + 200x - 625$ factor f completely.

$$\left.\begin{aligned} c_1 &= 4-3i \\ c_2 &= 4+3i \end{aligned}\right\} \text{zeros of } f(x)$$

$$(x-4+3i)(x-4-3i) = x^2 - 4x - \cancel{3ix} - 4x + 16 + \cancel{12i} + \cancel{3ix} - \cancel{12i} + 9 =$$
$$= x^2 - 8x + 25$$

Long division:

$$\begin{array}{rl} x^4 - 8x^3 + 200x - 625 & \big|\, \underline{x^2 - 8x + 25} \\ -\ \underline{x^4 - 8x^3 + 25x^2 \qquad\quad} & \ \ x^2 - 25 \\ -25x^2 + 200x - 625 & \\ -\ \underline{-25x^2 + 200x - 625} & \\ 0 & \end{array}$$

Hence, $x^4 - 8x^3 + 200x - 625 = (x^2 - 8x + 25)(x^2 - 25) =$

$= (x^2 - 8x + 25)(x-5)(x+5)$

$x^2 - 8x + 25 = (x - (4-3i))(x - (4+3i))$

Therefore, $f(x) = (x-(4-3i))(x-(4+3i))(x-5)(x+5)$

3. Construct a fourth-degree real-coefficient polynomial function f with zeros of 2, -5, and $1+i$ such that $f(1) = 12$.

We need to consider a complex conjugate root too.

$c_1 = 1+i \qquad c_3 = 2$

$c_2 = 1-i \qquad c_4 = -5$

$f(x) = a(x-2)(x+5)(x-(1+i))(x-(1-i))$

$f(1) = a\cdot(-1)\cdot 6\cdot(\cancel{1}-\cancel{1}-i)(\cancel{1}-\cancel{1}+i) = 12$

$-6a\cdot 1 = 12$

$\underline{a = -2}$

Therefore, a 4-th degree real-coeff. pol. function is

$f(x) = -2(x-2)(x+5)(x-(1+i))(x-(1-i))$

4. Use all available methods to factor the following polynomial function completely, and then sketch the graph of the polynomial function.

$$f(x) = x^5 + 4x^4 + x^3 - 10x^2 - 4x + 8$$

Let $\underbrace{x = 1}_{}$ (zero of f(x)). $f(1) = \cancel{1}+\cancel{4}+\cancel{1}-\cancel{10}-\cancel{4}+\cancel{8} = 0.$

Synthetic division:

	1	4	1	-10	-4	8
$\boxed{1}$	1	5	6	-4	-8	0

$f(x) = (x-1)\underbrace{(x^4 + 5x^3 + 6x^2 - 4x - 8)}$

$$g(x) = x^4 + 5x^3 + 6x^2 - 4x - 8$$

Let $x = 1$. $\quad g(1) = \cancel{1} + \cancel{5} + \cancel{6} - \cancel{4} - \cancel{8} = 0$

$\uparrow$ zero of $g(x)$

	1	5	6	-4	-8
$\boxed{1}$	1	6	12	8	0

Hence, $\quad f(x) = (x-1)^2(x^3 + 6x^2 + 12x + 8)$

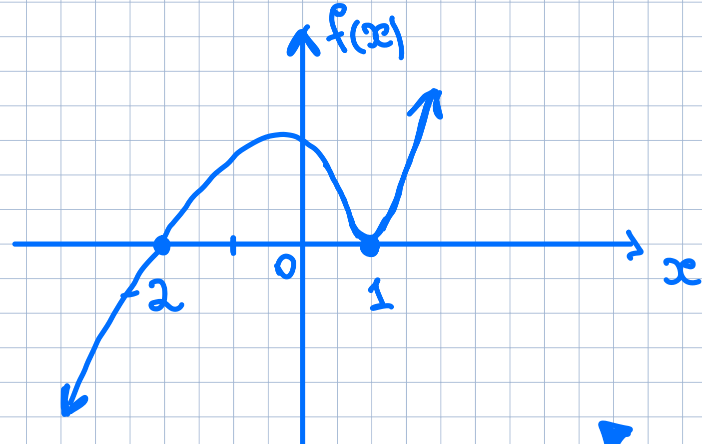
Now, let us factor $h(x) = x^3 + 6x^2 + 12x + 8$

Let $x = -2$. $\quad h(-2) = -8 + 24 - 24 + 8 = 0$

	1	6	12	8
-2	1	4	4	0

Hence, $\quad f(x) = (x-1)^2 (x+2)(x^2 + 4x + 4) =$

$$= (x-1)^2 (x+2)(x+2)^2 = (x-1)^2 (x+2)^3$$



5. Use all available methods to solve the following polynomial equation:

$$2x^4 - 5x^3 - 2x^2 + 15x = 0$$

$$x(2x^3 - 5x^2 - 2x + 15) = 0$$

$\boxed{x_1 = 0}$

$$2x^3 - 5x^2 - 2x + 15 = 0$$

$a_0 = 15$ factors: $\pm 1, \pm 3, \pm 5, \pm 15$

$a_3 = 2$ factors: $\pm 1, \pm 2$

Hence, a possible zero is $-\frac{3}{2}$.

$$2\cdot\left(-\frac{3}{2}\right)^3 - 5\left(-\frac{3}{2}\right)^2 + 2\cdot\frac{3}{2} + 15 = \frac{-27}{4} - \frac{45}{4} + 3 + 15 =$$

$$= -18 + 18 = 0,$$

Yes, $x = -\frac{3}{2}$ is a solution of $2x^3 - 5x^2 - 2x + 15$.

	2	-5	-2	15
$-\frac{3}{2}$	2	-8	10	0

$$2x^3 - 5x^2 - 2x + 15 = \left(x + \frac{3}{2}\right)(2x^2 - 8x + 10) =$$

$$= 2\left(x + \frac{3}{2}\right)(x^2 - 4x + 5)$$

$$x^2 - 4x + 5 = 0$$

$D = 16 - 20 = -4 < 0$ (two complex solutions)

$$x_3 = \frac{4+2i}{2} \qquad x_4 = \frac{4-2i}{2}$$

$$x_3 = 2+i \qquad x_4 = 2-i$$

Hence, $2x^4 - 5x^3 - 2x^2 + 15x = 0$

$$x\left(x+\frac{3}{2}\right)\left(x-(2+i)\right)\left(x-(2-i)\right) = 0$$

$x = 0$ or $x = \frac{-3}{2}$ or $x = 2+i$

or $x = 2-i$

Solutions: $\left\{0, -\frac{3}{2}, 2 \pm i\right\}$