Section 1.3. Polynomials and Factoring

1. The terminology of polynomial expressions.
2. Basic operations with polynomials
3. GCF
4. Factoring by grouping
5. Factoring special binomials
6. Factoring trinanials
7. Factoring expressions containing noninteger rational exponents.

$\boxed{1.}$ **Def.** A polynomial in the variable x of degree n can be written in the form

$$P = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

$a_n, a_{n-1}, \dots, a_0 \in \mathbb{R}$

$a_n \neq 0$, $n \in \mathbb{N} \cup \{0\}$

$a_n, \dots, a_0$ are coefficients

n is the degree of P

$$\underbrace{a_n x^n} + \underbrace{a_{n-1} x^{n-1}} + \dots + \underbrace{a_0}$$

monomials

Examples

1) $\sqrt{x}$ is not a polynomial
2) $x^3 + 2x + 7x^{-1}$ is not a polynomial
3) 5 is a polynomial

$\boxed{2.}$

Adding and Subtracting Polynomials

We combine similar terms

$$(2x^3y - 3y + z^2x) + (3y + z^2 - 3xz^2 + 4) =$$

$$= 2x^3y - \cancel{3y} + \underset{\sim\sim}{z^2x} + \cancel{3y} + z^2 \underset{\sim\sim\sim}{- 3xz^2} + 4 =$$

$$= 2x^3y - 2z^2x + z^2 + 4$$

Multiplying polynomials

We use the distributive property.

$$\underbrace{(\qquad)}_{n\text{ terms}} \cdot \underbrace{(\qquad)}_{m\text{ terms}} = \underbrace{n \cdot m}_{\text{terms}}$$

$$(3ab - a^2)(ab + a^2) = 3a^2b^2 + 3a^3b - a^3b - a^4 =$$

$$= 3a^2b^2 + 2a^3b - a^4$$

Special Product Formulas:

Let A and B be algebraic expressions.

1. $(A-B)(A+B) = A^2 - B^2$
2. $(A+B)^2 = A^2 + 2AB + B^2$
3. $(A-B)^2 = A^2 - 2AB + B^2$

$\boxed{3.}$ **Def.** A polynomial with integer coefficients is <u>factorable</u> if it can be written as a product of two or more polynomials, all of which also have integer coefficients.

If it cannot be done, the polynomial is <u>irreducible</u> (over $\mathbb{Z}$), or <u>prime</u>.

The GCF among all terms is the product of all the factors common to each.

Example

- $(a^2-b)-3(a^2-b)=(a^2-b)(1-3)=-2(a^2-b)$
- $12x^5-4x^2+8x^3z^3=4x^2(3x^3-1+2xz^3)$

4. Factoring by grouping.

Example

1) $6x^2-y+2x-3xy=2x(3x+1)-y(3x+1)=$
$=(3x+1)(2x-y)$

2) $ax-ay-bx+by=x(a-b)+y(b-a)=$
$=x(a-b)-y(a-b)=(a-b)(x-y)$

5. Factoring special binomials.

Difference of two squares:

$$A^2-B^2=(A-B)(A+B)$$

Difference of two cubes:

$$A^3-B^3=(A-B)(A^2+AB+B^2)$$

Sum of two cubes:

$$A^3 + B^3 = (A+B)(A^2 - AB + B^2)$$

$\boxed{6.}$ Factoring trinomials.

$$a x^2 + bx + c = (px+q)(rx+s)$$

$$(px+q)(rx+s) = \underbrace{pr}_{a}x^2 + \underbrace{(ps+qr)}_{b}x + \underbrace{qs}_{c}$$

a is a leading coefficient.

Case 1: $\boxed{a = 1.}$

$$x^2 + x - 12 = (x + e)(x + f)$$

e, f - ?

$e \cdot f = -12$

$e + f = 1$

$$\boxed{\begin{matrix} e = -3 \\ f = 4 \end{matrix}}$$

Hence, $x^2 + x - 12 = (x-3)(x+4)$

Case 2: $\boxed{a \neq 1}$

Procedure:

1. Multiply a and c.
2. Factor ac into two integers whose sum is b. If no such factors

exist, the trinomial is irreducible over integers.

3. Rewrite b in the trinomial with the sum found in step 2, and distribute. The resulting polynomial of four terms may now be factored by grouping.

Example

$$6x^2 - x - 12 = (6x + \ldots)(x + \ldots) \text{ or } (2x+\ldots)(3x+\ldots)$$

$$6\cdot(-12) = -72$$

$$-72 = -9\cdot 8$$

$$6x^2 + (-9+8)x - 12 = 6x^2 - 9x + 8x - 12 =$$

$$= 3x(2x-3) + 4(2x-3) = (3x+4)(2x-3)$$

Perfect Square Trinomials:

$$A^2 + 2AB + B^2 = (A+B)^2$$

$$A^2 - 2AB + B^2 = (A-B)^2$$

$\boxed{7.}$ **Factoring expressions containing noninteger rational exponents.**

$$3x^{-\frac{2}{3}} - 6x^{\frac{1}{3}} + 3x^{\frac{4}{3}} = 3x^{\frac{1}{3}}(x^{-2} - 2x + x^4) =$$

$$= 3x^{-\frac{2}{3}}(1 - 2x + x^2) = 3x^{-\frac{2}{3}}(x-1)^2 =$$

$$= 3x^{-\frac{2}{3}}(x-1)(x-1).$$