Solutions

THEORETICAL PART:

Definition (y-Axis Symmetry):

The graph of a function f has **y-axis symmetry**, or is **symmetric with respect to the y-axis**, if $f(-x) = f(x)$ for all x in the domain of f. Such functions are called **even functions**.

Definition (Origin Symmetry):

The graph of a function f has **origin symmetry**, or is **symmetric with respect to the origin**, if $f(-x) = -f(x)$ for all x in the domain of f. Such functions are called **odd functions**.

Definition (Symmetry of Equations):

We say an equation in x and y is **symmetric with respect to**

1. the **y-axis** if replacing x with $-x$ results in an equivalent equation;
2. the **x-axis** if replacing y with $-y$ results in an equivalent equation;
3. the **origin** if replacing x with $-x$ and y with $-y$ results in an equivalent equation.

Definition (Increasing, decreasing, and Constant):

We say that a function f is

1. **increasing on an interval** if for any x_1 and x_2 in the interval with $x_1 < x_2$, it is the case that $f(x_1) < f(x_2)$;
2. **decreasing on an interval** if for any x_1 and x_2 in the interval with $x_1 < x_2$, it is the case that $f(x_1) > f(x_2)$;
3. **constant on an interval** if for any x_1 and x_2 in the interval, it is the case that $f(x_1) = f(x_2)$.

Definition (Local Extrema):

A function f has a **local maximum at c** if there is an open interval (a, b) containing c for which $f(x) \leq f(c)$ for all x in (a, b). In this case we say $f(c)$ is the **local maximum value** of f.

Similarly, f has a **local minimum at c** if there is an open interval (a, b) containing c for which $f(x) \geq f(c)$ for all x in (a, b), and in this case we say $f(c)$ is the **local minimum value** of f. The local maxima and minima of a function are collectively referred to as **local extrema**.

Definition (Average Rate of Change):

Given function f defined on an interval $[a, b]$, $a \neq b$, the **average rate of change** of f over $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}.$$

If $y = f(x)$, then any of the following expressions may be used to represent the average rate of change of f over $[a, b]$:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

The average rate of change of f over $[a, b]$ represents the slope of the **secant line** drawn between the points $(a, f(a))$ and $(b, f(b))$ on the graph of f.

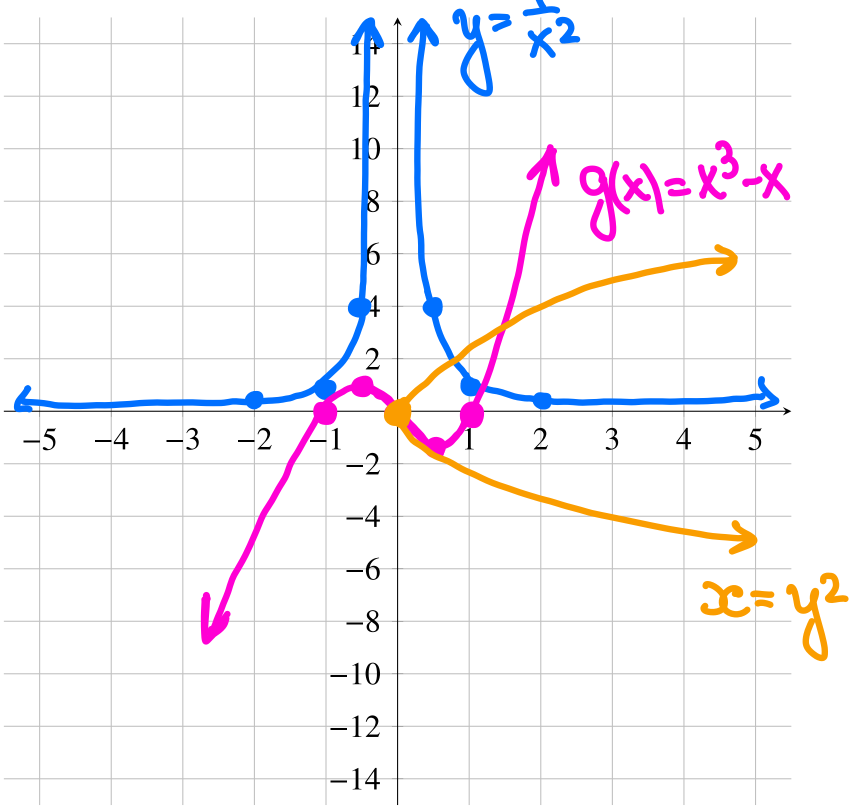
PRACTICAL PART:

1. Sketch the graphs of the following relations, making use of symmetry:

   (a) $f(x) = \dfrac{1}{x^2}$

   (b) $g(x) = x^3 - x$ $= x(x^2-1)$

   (b) $x = y^2$



$g(x) = x(x^2-1) =$

$= x(x-1)(x+1)$

$g(x) = 0$

$x(x-1)(x+1) = 0$

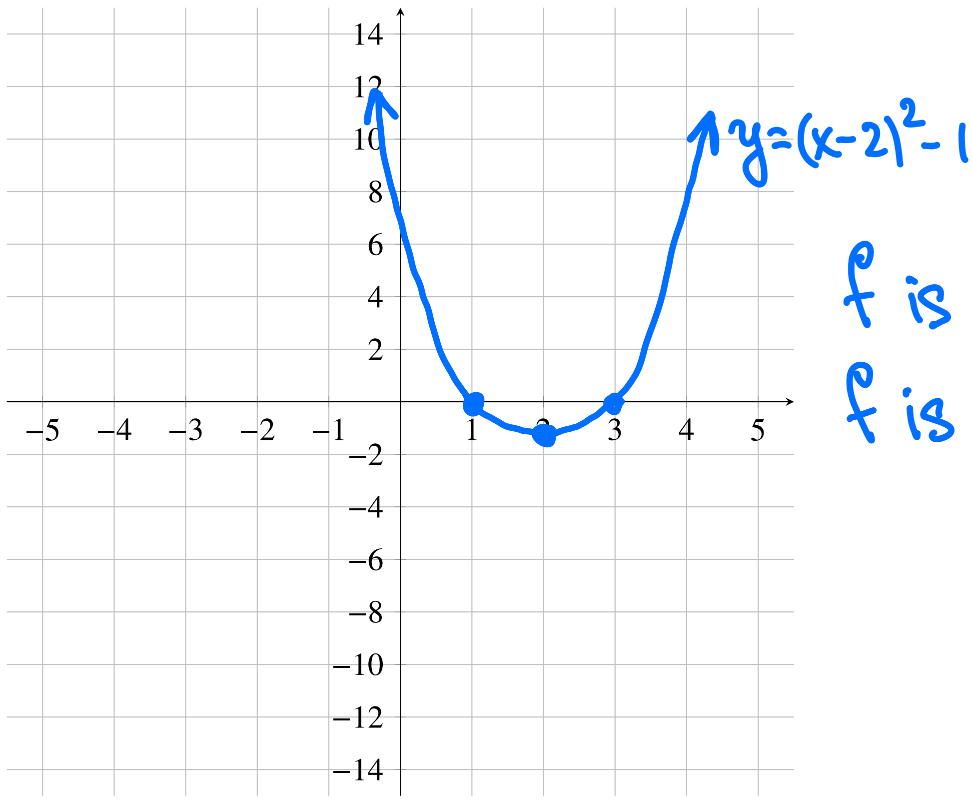
$x = 0$ or $x = 1$ or $x = -1$

(a) $f(x) = \frac{1}{x^2}$ is symmetric w.r. to y-axis

(b) $g(x) = x^3 - x$ is symmetric w.r. to origin

(c) $x = y^2$ is symmetric w.r. to x-axis

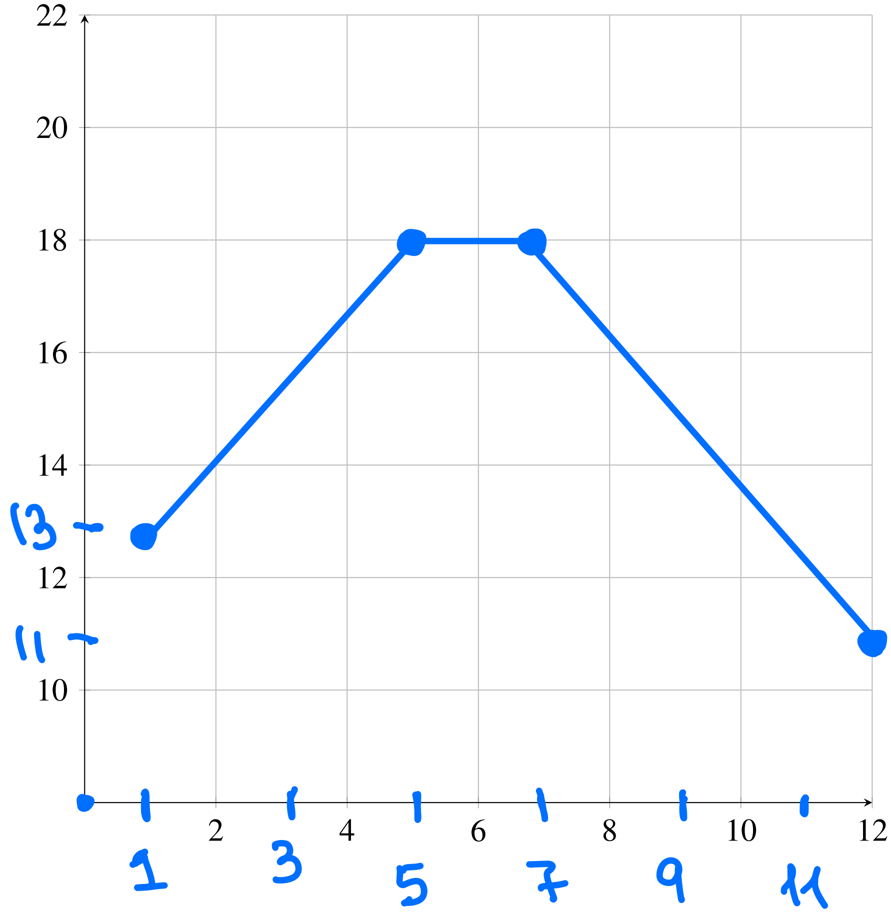
2. Determine the open intervals of monotonicity of the function $f(x) = (x-2)^2 - 1$.



f is ↑ on $(2, +\infty)$

f is ↓ on $(-\infty, 2)$

3. The water level of a certain river varied over the course of a year as follows. In January, the level was 13 feet. From that level, the water increased linearly to a level of 18 feet in May. The water remained constant at that level until July, at which point it began to decrease linearly to a final level of 11 feet in December. Graph the water level as a function of time and determine the open intervals of monotonicity.



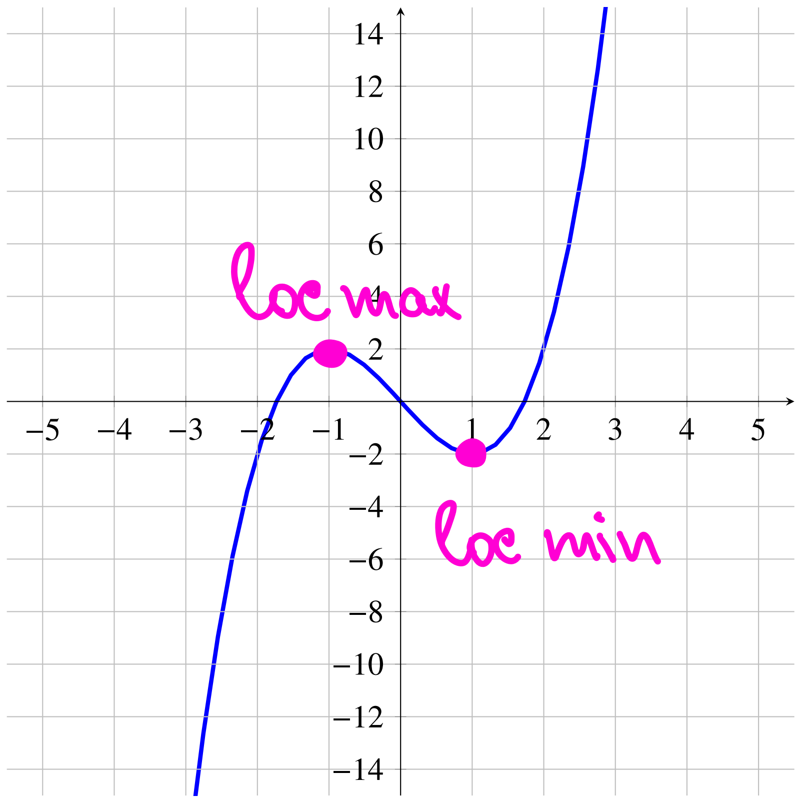
f is ↑ on $(1,5)$

f is ↓ on $(7,12)$

f is constant on $(5,7)$

4. For the given graph of the function f below determine:

   - locations and types of the local extrema of f;
   - the values of the local extrema of f.



f has loc.min. at $x = 1$

$f_{min}(1) = -2$

f has loc.max. at $x = -1$

$f_{max}(-1) = 2$

5. Given the function $f(x) = 3x^2 - 5x + 2$, determine the average rate of change over each of the following intervals:

   a. $[1, 3]$
   b. $[-2, 2]$
   c. $[c, c+h]$, $h \neq 0$

(a) $\dfrac{\Delta f}{\Delta x} = \dfrac{f(3)-f(1)}{3-1} = \dfrac{3\cdot 9 - 15 + 2 - 3 + 5 - 2}{2} =$

$= \boxed{7}$

(b) $\dfrac{\Delta f}{\Delta x} = \dfrac{f(2)-f(-2)}{2-(-2)} = \dfrac{\cancel{3\cdot 4} - \cancel{10} + \cancel{2} - \cancel{12} + \cancel{10} - \cancel{2}}{4} =$

$= \boxed{0}$

(c) $\frac{\Delta f}{\Delta x} = \frac{f(c+h) - f(c)}{c+h-h} =$

$= \frac{3(c+h)^2 - 5(c+h) + \cancel{2} - 3c^2 + 5c - \cancel{2}}{c} =$

$= \frac{\cancel{3c^2} + 6ch + 3h^2 - \cancel{5c} - 5h + \cancel{5c} - \cancel{3c^2}}{c} =$

$= \boxed{\frac{3h^2 - 5h + 6ch}{c}}$