THEORETICAL PART: Solutions

Identities (Double-angle Identities):

Sine Identity

$$\sin(2u) = 2\sin u \cos u$$

Cosine Identities

$$\cos(2u) = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$$

Tangent Identity

$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$

Identities (Power-Reducing Identities)

Sine Identity

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

Cosine Identity

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

Tangent Identity

$$\tan^2 x = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

Identities (Half-Angle Identities)

Sine Identity

$$\sin(x/2) = \pm\sqrt{\frac{1 - \cos x}{2}}$$

Cosine Identity

$$\cos(x/2) = \pm\sqrt{\frac{1 + \cos x}{2}}$$

Tangent Identities

$$\tan(x/2) = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$$

Identities (Product-to-Sum Identities)

$$\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$$
$$\cos x \sin y = \frac{1}{2}(\sin(x+y) - \sin(x-y))$$
$$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$$
$$\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$$

Identities (Sum-to-Product Identities)

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

<u>PRACTICAL PART:</u>

1. Given that $\cos x = -\frac{2}{\sqrt{5}}$ and that $\sin x$ is positive, determine $\cos(2x)$, $\sin(2x)$, and $\tan(2x)$.

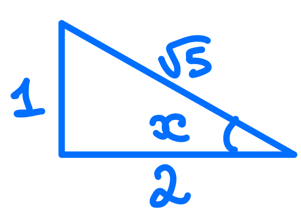
$\cos(2x) = 1 - 2\sin^2 x = 2\cos^2 x - 1$

$\cos(2x) = 2\cdot\left(\frac{4}{5}\right) - 1 = \frac{8}{5} - \frac{5}{5} = \frac{3}{5}$

$\sin(2x) = 2\sin x \cdot \cos x$

$\sin(2x) = 2\cdot\frac{1}{\sqrt{5}}\cdot\left(-\frac{2}{\sqrt{5}}\right) = -\frac{4}{5}$

$\tan(2x) = -\frac{4}{3}$



2. Prove that $\sin(3x) = 3\sin x - 4\sin^3 x$.

$\sin(3x) = \sin(2x + x) = \sin(2x)\cos x + \cos(2x)\sin x =$

$= 2\sin x\cos^2 x + (1 - 2\sin^2 x)\sin x =$

$= 2\sin x\cos^2 x + \sin x - 2\sin^3 x =$

$= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x =$

$= 3\sin x - 4\sin^3 x$

3. Express $\sin^5 x$ in terms containing only first powers of sine and cosine.

$\sin^5 x = \sin^2 x \cdot \sin^2 x \cdot \sin x =$

$= \frac{1}{4}(1 - \cos(2x))(1 - \cos(2x))\sin x =$

$= \frac{1}{4}\left(\sin x - 2\cos(2x)\sin x + \cos^2(2x)\sin x\right) =$

$= \frac{1}{4}\left(\sin x - 2\cos(2x)\sin x + \frac{\sin x}{2} + \frac{\cos(4x)\sin x}{2}\right) =$

$= \frac{1}{4}\left(\frac{3\sin x}{2} - 2\cos(2x)\sin x + \frac{\cos(4x)\sin x}{2}\right)$

4. Determine the exact values of $\sin(\pi/8)$, $\cos(\pi/8)$ and $\tan(\pi/8)$.

$$\sin\left(\frac{\pi}{8}\right) = \sin\left(\frac{\frac{\pi}{4}}{2}\right) = \pm\sqrt{\frac{1-\cos\frac{\pi}{4}}{2}} =$$

$$= \pm\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{2-\sqrt{2}}{4}}$$

$$\cos\left(\frac{\pi}{8}\right) = \pm\sqrt{\frac{2+\sqrt{2}}{4}}$$

$$\tan\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}}{2\left(1+\frac{\sqrt{2}}{2}\right)} = \frac{\sqrt{2}}{2+\sqrt{2}}$$

5. Express $\sin^5 x$ in terms containing only first powers of sine.

$$\sin^5 x = \frac{1}{4}\left(\frac{3\sin x}{2} - 2\cos(2x)\sin x + \frac{\cos(4x)\sin x}{2}\right) =$$

$$= \frac{1}{4}\left(\frac{3}{2}\sin x - 2\left(\frac{1}{2}(\sin(3x)-\sin x)\right) + \frac{1}{2}\left(\frac{1}{2}(\sin(5x) - \sin(3x))\right)\right) = \frac{5}{8}\sin x - \frac{5}{16}\sin(3x) + \frac{1}{16}\sin(5x)$$

6. Verify the identity
$$\frac{\sin(2x) + \sin(4x)}{\cos(2x) + \cos(4x)} = \tan(3x).$$

$$\frac{\sin(2x) + \sin(4x)}{\cos(2x) + \cos(4x)} = \frac{2\sin(3x)\cancel{\cos x}}{2\cos(3x)\cancel{\cos x}} = \tan(3x)$$