# Section 5.3. Locating real zeros of polynomial functions

1. The Rational Zero Theorem.
2. Descart's Rule of Signs.
3. Bounds of real zeros.
4. The Intermediate Value Theorem.

$\boxed{1.}$

<u>Theorem</u> (The Rational Zero Theorem)

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ is a polynomial with integer coefficients with $a_n \neq 0$, then any rational zero of $f$ must be of the form $\frac{p}{q}$, where $p$ is a factor of the constant term $a_0$ and $q$ is a factor of the leading coefficient $a_n$.

Caution!

1. The theorem doesn't necessarily

find even a single zero of a polynomial.

2. The theorem says nothing about irrational or complex zeros.

## Example

$$f(x) = 2x^3 + 5x^2 - 4x - 3$$

$a_0 = -3$ $\quad\quad \pm 1, \pm 3$

$a_n = a_3 = 2$ $\quad\quad \pm 1, \pm 2$

factors

$$\frac{p}{q} = \left\{ \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2} \right\}$$

Let us take $c = 1$. Then

| 1 | 2 | 5 | -4 | -3 |
|---|---|---|---|---|
| | 2 | 7 | 3 | $\boxed{0}$ |

$c = 1$ is a root of $f(x)$.

$$f(x) = (x-1)(2x^2 + 7x + 3) = (x-1)(2x+1)(x+3)$$

Actual zeros: $\left\{ 1, -3, -\frac{1}{2} \right\}$.

$\boxed{2.}$ **Theorem** (Descarte's Rule of Signs)

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

be a polynomial with real coefficients, and assume $a_n \neq 0$.

A <u>variation of sign</u> of f is a change in the sign of one coefficient of f to the next, either from positive to negative or vice versa.

1. The number of <u>positive real zeros</u> of f is either the number of variations in sign of f(x) or is less than this number by a positive even integer.

2. The number of <u>negative real zeros</u> of f is either the number of variations in sign of f(-x) or is less than this number by a positive even integer.

<u>Example</u>

$$f(x) = 2x^3 + 3x^2 - 14x - 21$$

Sign change

$$f(-x) = 2(-x)^3 + 3(-x)^2 - 14(-x) - 21 =$$

$$= -2x^3 + 3x^2 + 14x - 21$$

Sign change     Sign change

Thus, there is exactly 1 positive real zero and either 2 or 0 negative real zeros of $f(x)$.

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$\boxed{3.}$ Theorem (Upper and Lower Bounds of Zeros)

Let $f(x)$ be a polynomial with real coefficients, a positive leading coefficient, and degree $\geq 1$. Let $a$ be a negative number and $b$ be a positive number. Then:

1. No real zero of $f$ is larger than $b$ (upper bound) if the last row in the synthetic division of $f(x)$ by $x-b$ contains no negative numbers.
   That is, $b$ is an upper bound of the zeros if the quotient and remainder have no negative coefficients when $f(x)$ is divided by $x-b$.
2. No real zero of $f$ is smaller than $a$ (lower bound) if the last row in the synthetic division of $f(x)$ by $x-a$ has entries that alternate in sign (0 can count as either

positive or negative).

## Example

$f(x) = 2x^3 + 3x^2 - 14x - 21.$

p: $\pm 1, \pm 2$
q: $\pm 1, \pm 3, \pm 7, \pm 21$

| 2 | 2 | 3 | -14 | -21 |
|---|---|---|---|---|
| | 2 | 7 | 0 | -21 |

| 3 | 2 | 3 | -14 | -21 |
|---|---|---|---|---|
| | 2 | 9 | 13 | 18 |
| | $\geq 0$ | $\geq 0$ | $\geq 0$ | $\geq 0$ |

$x = 3$ is an upper bound

| -3 | 2 | 3 | -14 | -21 |
|---|---|---|---|---|
| | 2 | -3 | -5 | - |

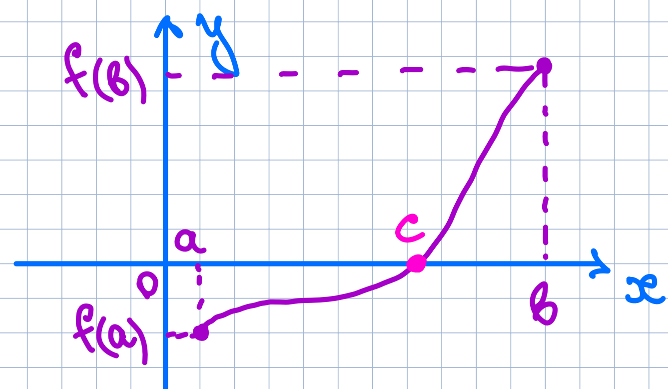
| -4 | 2 | 3 | -14 | -21 |
|---|---|---|---|---|
| | 2 | -5 | 6 | -45 |

$x = -4$ is a lower bound.

All real zeros of f lie in the interval $[-4, 3]$.

$\boxed{4.}$ **Theorem** (Intermediate Value Theorem)

Assume that $f(x)$ is a polynomial with real coefficients, and that $a$ and $b$ are real numbers with $a < b$. If $f(a)$ and $f(b)$ differ in sign, then there is at least one point $c$ such that $a < c < b$ and $f(c) = 0$.
That is, at least one zero of $f$ lies between $a$ and $b$.




**Caution!**
The IVT can only tell us that there is a zero between two x-values, it can not prove that a zero does not exist between two values.

**Example**
Show that $f(x) = x^3 + 3x - 7$ has a zero

between 1 and 2.

- $f$ is a polynomial
- $a = 1 < b = 2$
- $f(1) = 1 + 3 - 7 = -3 < 0$

  $f(2) = 8 + 6 - 7 = 7 > 0$

Therefore, by IVT $f$ has a zero between 1 and 2.

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