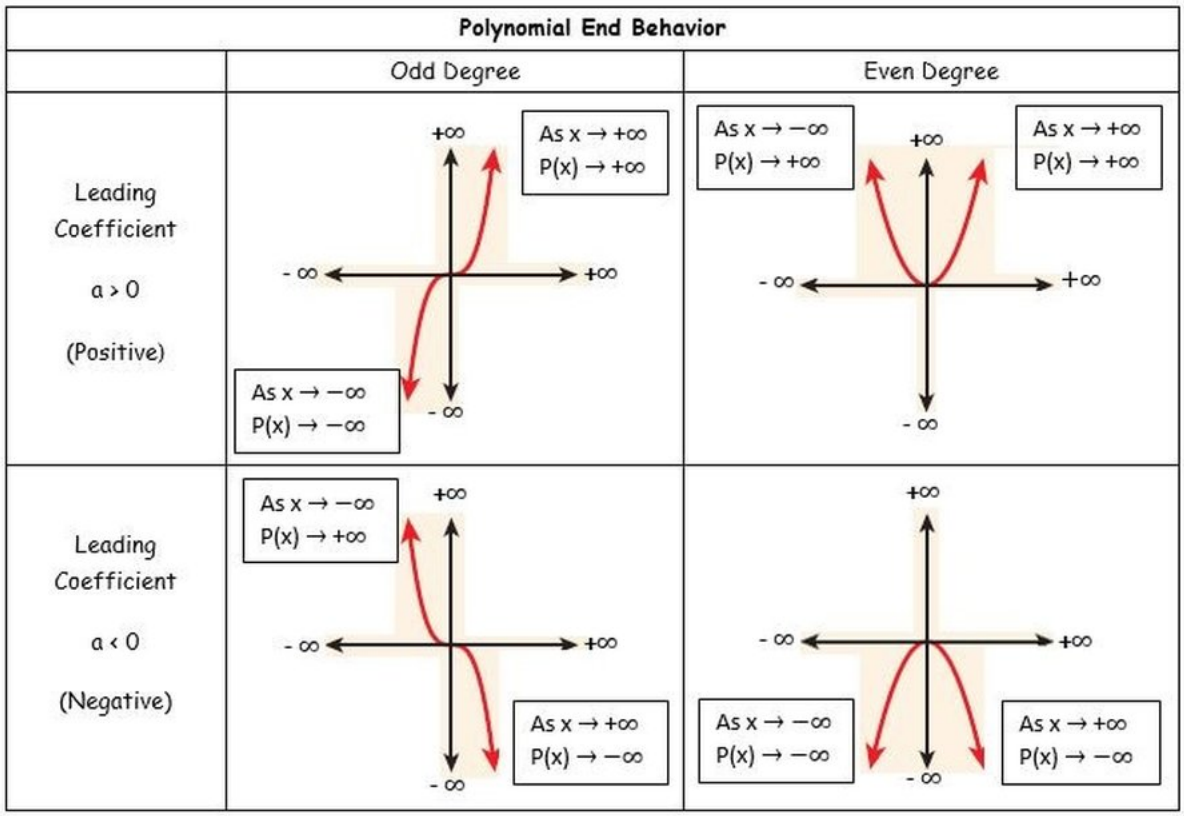
THEORETICAL PART: Solutions

Definition (Zeros of a polynomial):

The number c (which may be a complex number) is a **zero** of a polynomial function $p(x)$ if $p(c) = 0$. This is also expressed by saying that c is a **root** of the polynomial or a **solution** of the equation $p(x) = 0$.

Definition (Polynomial equations):

A polynomial equation in one variable, say x, is an equation that can be written in the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$, where n is a nonnegative integer and $a_0, a_1, \cdots, a_n$ are constants. Assuming $a_n \neq 0$, we say such an equation is of **degree** n and call a_n as a **leading coefficient**.



Definition (Polynomial Inequalities):

A **polynomial inequality** is any inequality that can be written in the form

$$p(x) > 0, \quad p(x) < 0, \quad p(x) \geq 0, \quad p(x) \leq 0,$$

where $p(x)$ is a polynomial function.

Procedure (Solving polynomial inequalities using the sign-test method):

To solve the polynomial inequality $p(x) < 0$, $p(x) > 0$, $p(x) \leq 0$, or $p(x) \geq 0$, perform the following steps:

1. Find the real zeros of $p(x)$. Equivalently, find the real solutions of the equation $p(x) = 0$.
2. Place the zeros on a number line, splitting it into intervals.
3. Within each interval, select a **test point** and evaluate p at that number. If the result is positive, then $p(x) > 0$ for all x in the interval. If the result is negative, then $p(x) < 0$ for all x in the interval.
4. Write the solution set, consisting of all the intervals that satisfy the given inequality. If the inequality is not strict ($\geq$ or $\leq$), then the zeros are included in the solution set as well.

PRACTICAL PART:

1. Verify that the given values of x solve the corresponding polynomial equations:
   (a) $6x^2 - x^3 = 12 + 5x$, $x = 4$
   (b) $x^2 = 2x - 5$, $x = 1 + 2i$

(a) $6x^2 - x^3 - 12 - 5x = 0$

$-x^3 + 6x^2 - 5x - 12 = 0 \quad | \cdot (-1)$

$x^3 - 6x^2 + 5x + 12 = 0$

at $x = 4$: $64 - 96 + 20 + 12 = 0$

(b) $x^2 - 2x + 5 = 0$

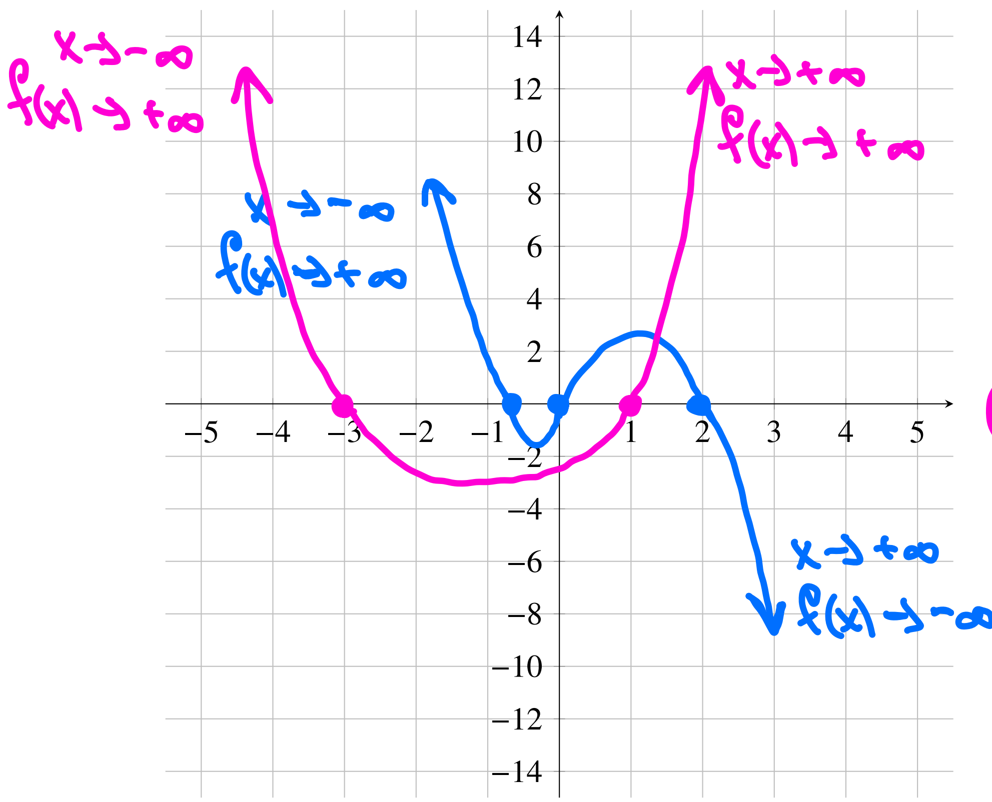
at $x = 1 + 2i$: $(1+2i)^2 - 2(1+2i) + 5 = \cancel{1} + \cancel{4i} - \cancel{4} - \cancel{2} - \cancel{4i} + \cancel{5} =$

$= 0$

2. Sketch the graphs of the following polynomial functions, paying attention to the x-intercept(s), y-intercept, and the behaviour as $x \to \pm\infty$:

   (a) $f(x) = -x(2x+1)(x-2)$

   (b) $g(x) = x^2 + 2x - 3 = (x+3)(x-1)$



(a) x- intercepts:

$(0,0)$

$(-\frac{1}{2},0)$

$(2,0)$

$x \to +\infty : f(x) \to -\infty$

$x \to -\infty : f(x) \to +\infty$

(b) x- intercepts

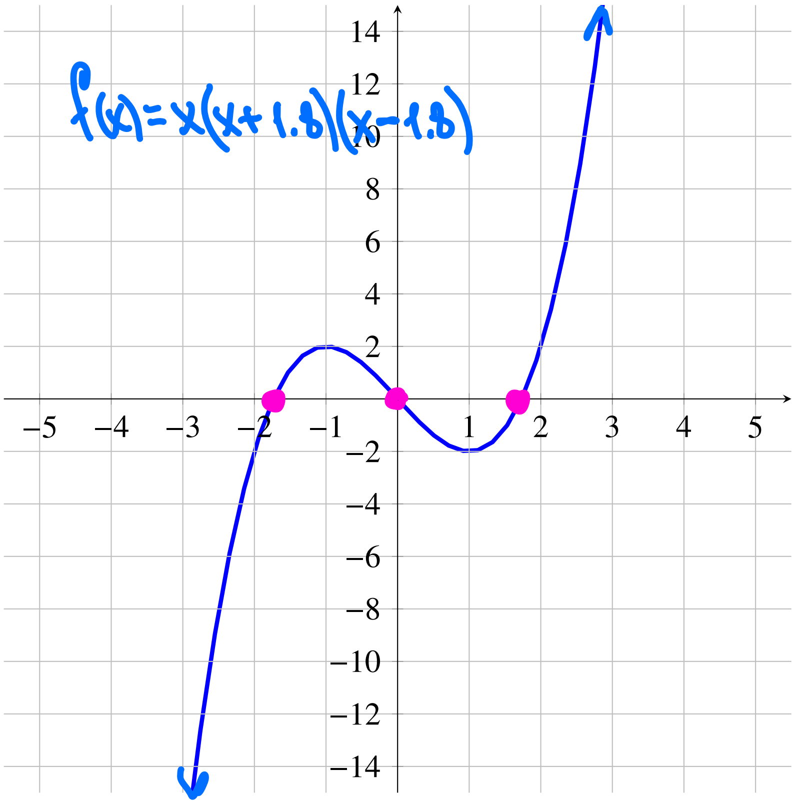
$(-3,0)$

$(1,0)$

$x \to +\infty : f(x) \to +\infty$

$x \to -\infty : f(x) \to +\infty$

3. Find the polynomial of lowest possible degree that corresponds to the graph below:



x- intercepts:
$(-1.8,0)$
$(0,0)$
$(1.8,0)$

$f(x)=a(x+1.8)(x-1.8)\cdot x$

$a>0$ since $f(x)\to+\infty$ as $x\to\infty$ and $f(x)\to-\infty$ as $x\to-\infty$

So, we can choose $\boxed{a=1}$

4. Solve the following polynomial inequalities:

(a) $\underbrace{(x+3)(x+1)(x-2)}_{=f(x)} \leq 0$

(b) $(x+3)(x+1)(x-2) \geq 0$

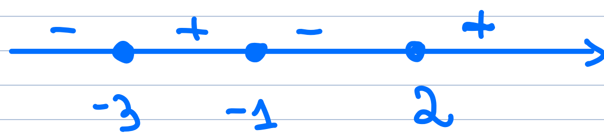
(a) Zeros: $x=-3, x=-1, x=2$

$-$ (−3) $+$ (−1) $-$ (2) $+$

$x=-4$: $f(-4) = -1\cdot(-3)\cdot(-6)<0$
$x=-2$: $f(-2) = 1\cdot(-1)\cdot(-4)>0$
$x=0$: $f(0) = 3\cdot1\cdot(-2)<0$
$x=3$: $f(3) = 6\cdot4\cdot1>0$

Answer: $(-\infty,-3)\cup(-1,2)$

(b) Zeros: $x=-3$, $x=-1$, $x=2$



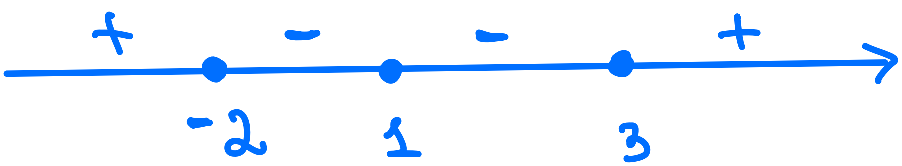
Answer: $[-3,-1] \cup [2,+\infty)$

5. Solve the polynomial inequality:

$$(x+2)(x-1)^2(x-3) \leq 0$$

$$(x+2)(x-1)(x-1)(x-3) \leq 0$$

Zeros: $x=-2$, $x=1$, $x=3$



$x=-3$: $f(-3) = -1\cdot(-4)^2(-6) = -1\cdot 16\cdot(-6) > 0$

$x=0$: $f(0) = 2\cdot(-1)^2\cdot(-3) < 0$

$x=2$: $f(2) = 4\cdot 1\cdot(-1) < 0$

$x=4$: $f(4) = 6\cdot 3^2\cdot 1 > 0$

Answer: $[-2,3]$.