Solutions

THEORETICAL PART:

Definitions:

- Let $a \in \mathbb{R}$, $n \in \mathbb{N}$. Then a^n is the product of n factors of a. Here a is called the base and n is the exponent.
- For any $a \in \mathbb{R}$, $a \neq 0$:
$$a^0 = 1.$$
- For any $a \in \mathbb{R}$, $a \neq 0$, and $n \in \mathbb{N}$:
$$a^{-n} = \frac{1}{a^n}.$$

Properties of Exponents:

In the following properties, a and b may be taken to represent variables, constants, or more complicated algebraic expressions. Letters n and m represent integers.

- $a^n \cdot a^m = a^{n+m}$
- $\frac{a^n}{a^m} = a^{n-m}$
- $a^{-n} = \frac{1}{a^n}$
- $(a^n)^m = a^{n \cdot m}$
- $(ab)^n = a^n \cdot b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Definitions:

- The number is in scientific notation when it is written in the form:
$$a \times 10^n,$$
where $1 \leq |a| < 10$ and $n \in \mathbb{Z}$. If n is a positive integer, the number the number is large in magnitude; if n is a negative integer, the number is small in magnitude (close to 0).

Definition (Radical Notation):

- n is an even natural number, $a \in \mathbb{R}$ and $a \geq 0$: $\sqrt[n]{a} = b$ if and only if $a = b^n$.
- n is an odd natural number, $a \in \mathbb{R}$: $\sqrt[n]{a} = b$ if and only if $a = b^n$.
- A **perfect square** is an integer equal to the square of another integer. The square root of a perfect square is always an integer.

Properties of radicals:

Let a and b be constants, variables, or more complicated algebraic expressions, and $n \in \mathbb{N}$. Then

- $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}$
- $\sqrt[n]{a^n} = \begin{cases} |a|, & n \quad \text{is even} \\ a, & n \quad \text{is odd} \end{cases}$

Rational Number Exponents:

- **meaning of $a^{\frac{1}{n}}$:** If $n \in \mathbb{N}$ and $\sqrt[n]{a} \in \mathbb{R}$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$.
- **meaning of $a^{\frac{m}{n}}$:** If $m, n \in \mathbb{N}$, $n \neq 0$, if m and n have no common factors greater than 1, and if $\sqrt[n]{a} \in \mathbb{R}$, then $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

PRACTICAL PART:

1. Simplify each of the following expressions. Write your answer with only positive exponents.

   (a) $\frac{x^5}{x^2} =$ x^3

   (b) $n^2 \cdot n^5 =$ n^7

   (c) $(-2)^4 =$ 16

   (d) $5^0 5^{-3} =$ $\frac{1}{125}$

(a) $\frac{x^5}{x^2} = x^3$

(b) $n^2 \cdot n^5 = n^7$

(c) $(-2)^4 = (-2)^2 \cdot (-2)^2 = 16$

(d) $5^0 \cdot 5^{-3} = 1 \cdot 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

2. Simplify the following expressions (use properties of exponents). Write your result with only positive exponents.

   (a) $$\frac{s^5 y^{-5} z^{-11}}{s^8 y^{-7}} = s^{5-8} \cdot y^{-5+7} \cdot z^{-11} = s^{-3} y^2 z^{-11}$$

   (b) $$\left[\frac{y^6 (xy^2)^{-3}}{3x^{-3}z}\right]^{-2} = \frac{y^{-12}(xy^2)^6}{3^{-2} \cdot x^6 z^{-2}} = \frac{\cancel{y^{-12}} \cdot \cancel{x^6} \cdot \cancel{y^{12}} \cdot 3^2}{\cancel{x^6} \cdot z^{-2}} = 9 \cdot z^2$$

3. Convert each number from scientific notation to standard notation, or vice versa.

   (a) 0.00000021; convert to scientific.

   (b) A white blood cell is approximately 3.937×10^{-4} inches in diameter. Express this diameter in standard notation.

(a) $0.00000021 = 2.1 \cdot 10^{-7}$

(b) $3.937 \times 10^{-4} = 0.0003937$

4. Evaluate the following expression using the properties of exponents:

$$(2 \times 10^{-13})(5.5 \times 10^{10})(-1 \times 10^{3}) =$$

$$= 2 \cdot 5.5 \cdot (-1) \cdot 10^{-13+10+3} = -11 \cdot 10^{0} = -11$$

$$\frac{(3.6 \times 10^{-12})(-6 \times 10^{4})}{1.8 \times 10^{-6}} =$$

$$= \frac{3.6 \cdot (-6) \cdot 10^{-12+4}}{1.8 \cdot 10^{-6}} = -12 \cdot 10^{-12+4+6} = -12 \cdot 10^{-2} = -0.12$$

5. Evaluate the following radical expression:

$$\sqrt[3]{\frac{-27}{125}} = \frac{\sqrt[3]{-27}}{\sqrt[3]{125}} = \frac{-3}{5}$$

$$-\sqrt[4]{16} = -\sqrt[4]{2^4} = -2$$

$$\sqrt{0} = 0$$

6. Simplify the following radical expressions:

$$\sqrt[7]{x^{14}y^{49}z^{21}} = x^{\frac{14}{7}} \cdot y^{\frac{49}{7}} \cdot z^{\frac{21}{7}} =$$

$$= x^2 \cdot y^7 \cdot z^3$$

$$\sqrt{8z^6} = \sqrt{8} \cdot \sqrt{z^6} = 2\sqrt{2} \cdot z^{\frac{6}{2}} =$$

$$= 2\sqrt{2} \cdot z^3$$

$$\sqrt[3]{\frac{72x^2}{y^3}} = \frac{\sqrt[3]{72x^2}}{\sqrt[3]{y^3}} = \frac{\sqrt[3]{8 \cdot 9} \cdot x^{\frac{2}{3}}}{y^1} =$$

$$= \frac{2\sqrt[3]{9} \cdot x^{\frac{2}{3}}}{y}$$

7. Simplify the following radicals by rationalizing the denominators:

$$\frac{-\sqrt{3a^3}}{\sqrt{6a}} = \frac{-\sqrt{3a^3} \cdot \sqrt{6a}}{\sqrt{6a} \cdot \sqrt{6a}} =$$

$$= \frac{-\sqrt{3^2 \cdot 2 \cdot a^4}}{6a} = \frac{-\cancel{3}\sqrt{2}\,a^2}{2\,\cancel{6}a} = \frac{-\sqrt{2}a}{2} = -\frac{a}{\sqrt{2}}$$

$$\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}} =$$

$$= \frac{(\sqrt{x}+\sqrt{y})(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} = \frac{(\sqrt{x}+\sqrt{y})^2}{x-y} =$$

$$= \frac{x + 2\sqrt{x}\sqrt{y} + y}{x-y}$$

8. Rationalize the numerator of the fraction

$$\frac{\sqrt{4x} - \sqrt{6y}}{2x - 3y} =$$

$$= \frac{(\sqrt{4x} - \sqrt{6y})(\sqrt{4x} + \sqrt{6y})}{(2x-3y)(\sqrt{4x} + \sqrt{6y})} = \frac{4x - 6y}{(2x-3y)(\sqrt{4x} + \sqrt{6y})}$$

9. Combine the radical expressions, if possible.

$$\sqrt[3]{-16x^4} + 5x\sqrt[3]{2x} =$$

$$= -2x\sqrt[3]{2x} + 5x\sqrt[3]{2x} = 3x\sqrt[3]{2x}$$

10. Simplify each of the following expressions, writing your answer using the same notation as the original expression.

$$27^{-\frac{2}{3}} = \frac{1}{\sqrt[3]{27^2}} = \frac{1}{(\sqrt[3]{27})^2} =$$

$$= \frac{1}{3^2} = \frac{1}{9}$$

$$\sqrt[5]{\sqrt[3]{x^2}} = \left(\left(x^2\right)^{\frac{1}{3}}\right)^{\frac{1}{5}} = x^{\frac{2}{15}}$$

11. Convert the following expressions from radical notation to exponential notation, or vice versa.

$$(36n^4)^{\frac{5}{6}} = \left(\sqrt[6]{36n^4}\right)^5 =$$

$$= \sqrt[6]{(36n^4)^5} = \sqrt[6]{36^5 \cdot n^{20}}$$

$$\sqrt[12]{x^3} = x^{\frac{3}{12}} = x^{\frac{1}{4}}$$