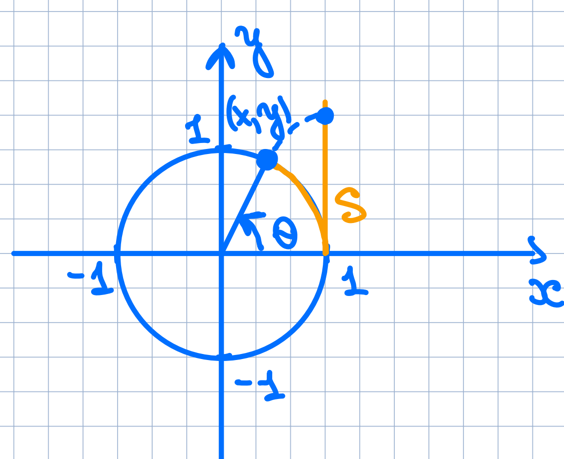
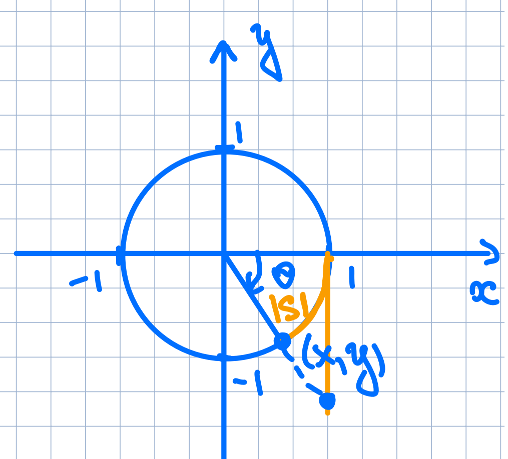
Section 7.3. Trigonometric functions and the unit circle

1. Extending the domains of the trigonometric functions.
2. Evaluating trigonometric functions using reference angles.
3. Relationships between trigonometric functions.

1.



$S > 0$

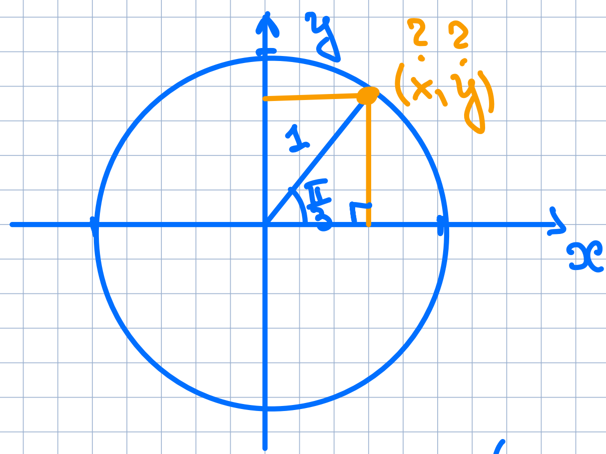


$S < 0$

Example

Determine (x, y) when $S = \frac{\pi}{3}$.

Solution



$x = \cos\left(\frac{\pi}{3}\right) \cdot 1$

$x = \frac{1}{2}$

$y = \sin\left(\frac{\pi}{3}\right) \cdot 1$

$y = \frac{\sqrt{3}}{2}$

$$(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

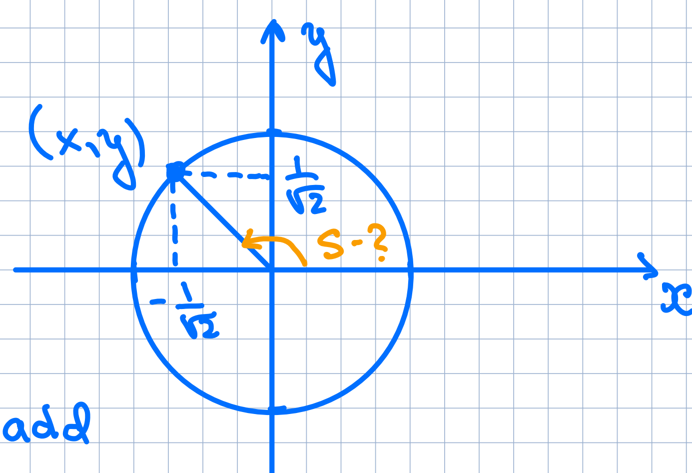
Example

Determine all real numbers s associated with the point

$$(x, y) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

on the unit circle.

Solution



$s = \frac{\pi}{2} + \frac{\pi}{4}$

$s = \frac{3\pi}{4}$

We should add a period:

$$s = \frac{3\pi}{4} + 2\pi n, \quad n \in \mathbb{Z}$$

Def. Let s be a real number and let (x,y) be the point on the unit circle associated with s. We define the six trigonometric functions with argument s as follows:

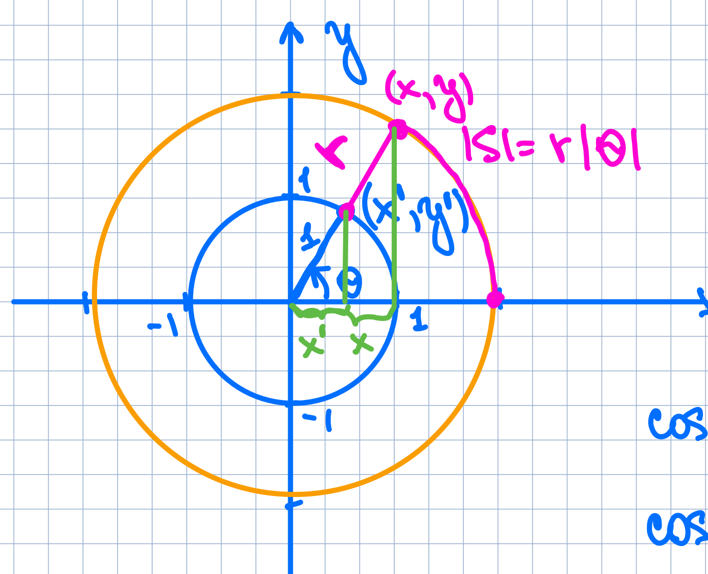
$$\sin s = y \qquad \cos s = x$$

$$\tan s = \frac{y}{x}, \quad x \neq 0$$

$$\csc s = \frac{1}{y}, \quad y \neq 0$$

$$\sec s = \frac{1}{x}, \quad x \neq 0$$

$$\cot s = \frac{x}{y}, \quad y \neq 0.$$



$$r = \sqrt{x^2 + y^2}$$

$$x' = \frac{x}{r}$$

$$y' = \frac{y}{r}$$

$$\cos\theta = \frac{x'}{1}$$

$$\cos\theta = \frac{x}{r}$$

$$\frac{x'}{1} = \frac{x}{r} \Rightarrow x' = \frac{x}{r}$$

Def.

Let θ be an angle in standard position, let (x,y) be any point (other than origin) on the terminal side of the angle θ, and let $r=\sqrt{x^2+y^2}$. We define the six trigonometric functions with argument θ as follows:

$$\sin\theta=\frac{y}{r} \qquad \cos\theta=\frac{x}{r} \qquad \tan\theta=\frac{y}{x},\ x\neq 0$$

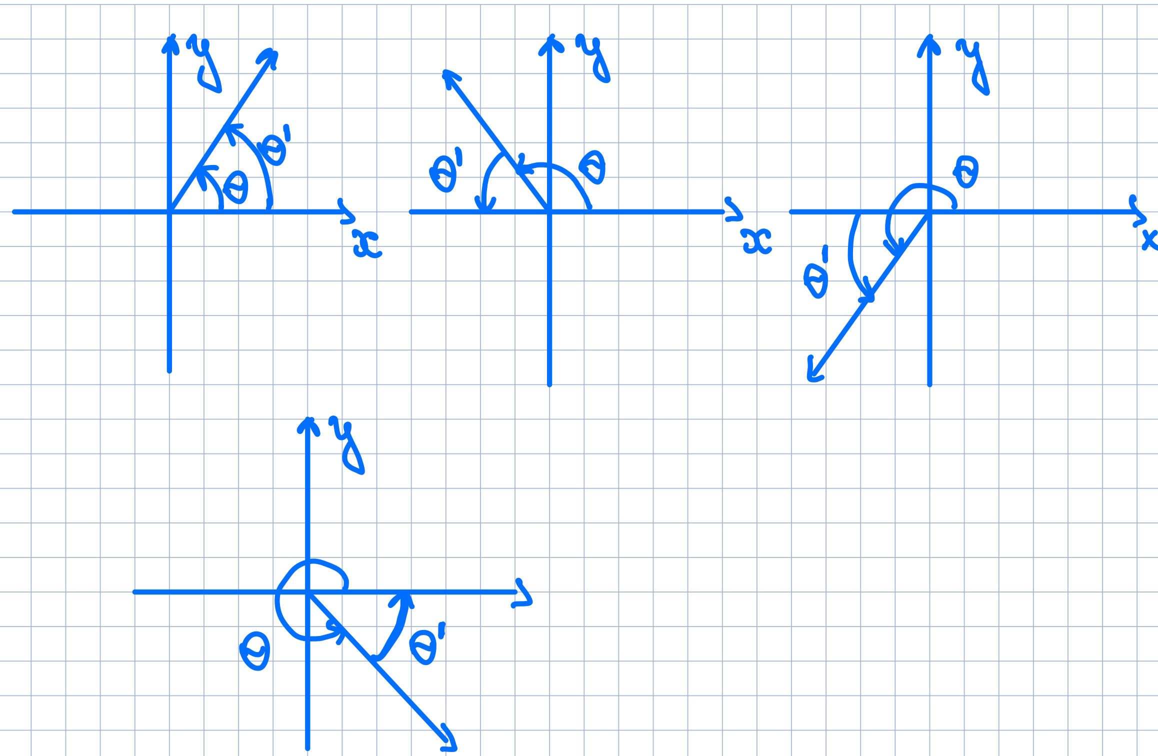
$$\csc\theta=\frac{r}{y},\ y\neq 0 \qquad \sec\theta=\frac{r}{x},\ x\neq 0$$

$$\cot\theta=\frac{x}{y},\ y\neq 0$$

$\boxed{2.}$

Def. (Reference angles)

Given an angle θ in standard position, the <u>reference angle θ'</u> associated with it is the angle formed by the x-axis and the terminal side of θ. Reference angles are always greater than or equal 0 and less than or equal to $\frac{\pi}{2}$ radians; that is, $0\leq\theta'\leq\frac{\pi}{2}$.

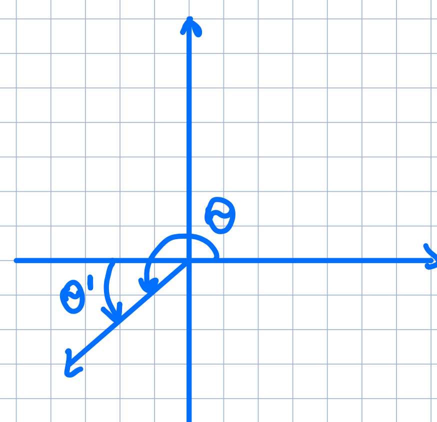


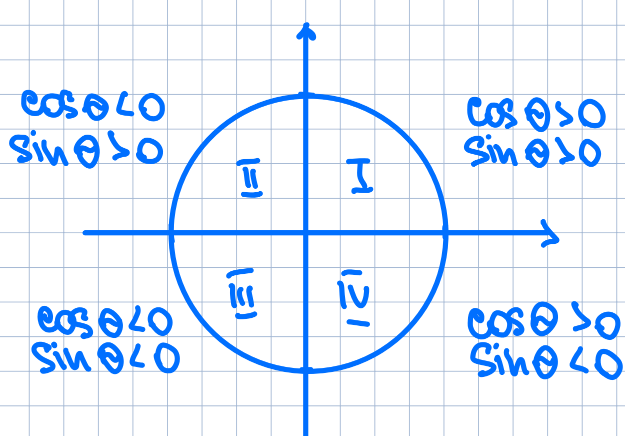
Example

$$\theta = \frac{9\pi}{8} = \pi + \frac{\pi}{8}$$

Reference angle: $\theta' = \frac{9\pi}{8} - \pi$

$$\theta' = \frac{\pi}{8}.$$





$\boxed{3.}$

Def. Complementary angles are two angles that, when combined, form a right angle. That is, the sum of their measures is $\frac{\pi}{2}$.

Identities: (Cofunction identities)

Given angle θ (measured in radians), $\frac{\pi}{2} - \theta$ is the measure of its complement, so

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$
$$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$$
$$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

Identities: (Reciprocal identities)

For a given angle θ for which both sides of the equation are defined,

$$\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}.$$

Identities: (Quotient identities)

For a given angle θ for which both sides of the equation are defined,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

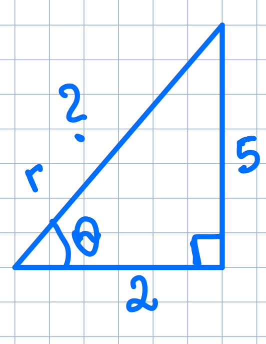
Example (using cofunction identities)

- $\cos\left(-\frac{5\pi}{11}\right) = \sin\left(\frac{\pi}{2} + \frac{5\pi}{11}\right) = \sin\left(\frac{21\pi}{22}\right).$

Example

Given that $\cot\theta = \frac{2}{5}$ and θ lies in the first quadrant, determine $\sin\theta$.

Solution



$$r = \sqrt{4+25} = \sqrt{29}$$

$$\sin\theta = \frac{5}{\sqrt{29}}.$$