Section 5.4. The Fundamental Theorem of Algebra

1. The Fundamental Theorem of Algebra.
2. Multiple zeros and their geometric meaning.
3. Conjugate pairs of zeros.

$\boxed{1.}$

Theorem (The fundamental theorem of algebra)

If p is a polynomial of degree n, with $n \geq 1$, then p has <u>at least one zero.</u> That is, the equation $p(x) = 0$ has at least one solution. It is important to note that the zero of p, and consequently the solution of $p(x) = 0$, may be a nonreal complex number.

Theorem (The Linear Factors Theorem)

Given the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots +$

$+ a_1x + a_0$, where $n \geq 1$ and $a_n \neq 0$, p can be factored as

$$p(x) = a_n(x - c_1)(x - c_2) \cdot \ldots \cdot (x - c_n),$$

where $c_1, c_2, \ldots, c_n$ are constants (possibly nonreal complex constants and not necessarily distinct).

In other words, an <u>n^{th}-degree polynomial can be factored as a product of n linear factors.</u>

<u>Caution!</u>

- the theorem does not tell us that a polynomial has all real zeros.
- the theorem does not tell us that a polynomial has n distinct zeros.
- the theorem does tell us that any polynomial can be written as a product of linear factors.

<u>Def.</u> A <u>turning point</u> of a graph is a point where the graph changes behaviour from decreasing to increasing or vice versa.

Theorem (Interpreting the Linear Factors Theorem)

The graph of an n^{th}-degree polynomial function has at most n x-intercepts and at most $n-1$ turning points.

This also means that an n^{th}-degree polynomial function has at most n zeros.

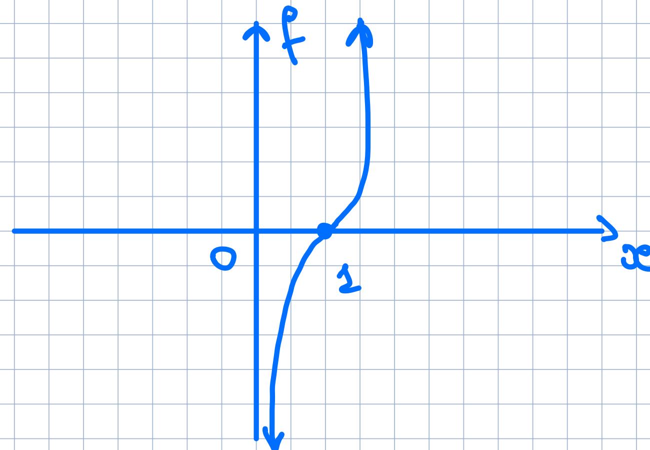
$\boxed{2.}$

Def. (Multiplicity of Zeros)

If the linear factor $(x-c)$ appears $k>0$ times in the factorization of a polynomial, we say the number c is a zero of multiplicity k.

Example

$f(x)=(x-1)^3$



Properties (Geometric Meaning of Multiplicity)

If c is a real zero of multiplicity k

of a polynomial p, the graph of p will touch the x-axis at $(c,0)$ and

- cross through the x-axis if k is odd
- Stay on the same side of the x-axis if k is even.

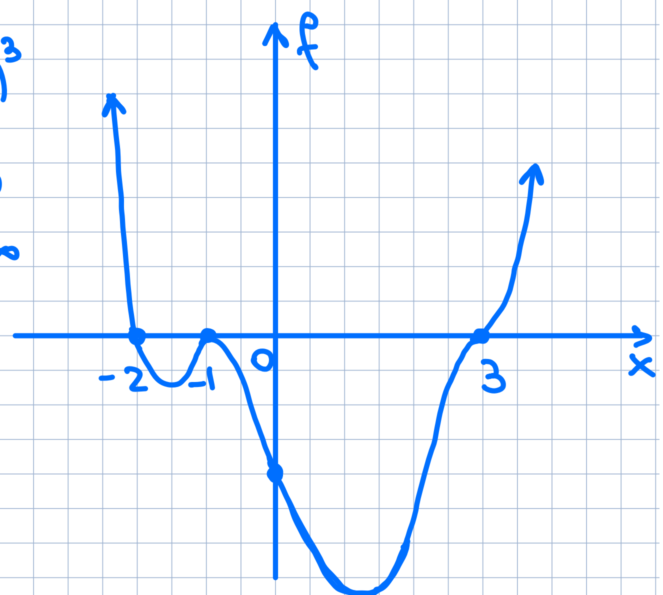
Further, if $k>1$, the graph of p will "flatten out" near $(c,0)$.

Example

$$f(x)=(x+2)(x+1)^2(x-3)^3$$

- as $x \to +\infty$: $f(x) \to +\infty$
- as $x \to -\infty$: $f(x) \to +\infty$

$x=0$: $f(0)=-54$



$\boxed{3.}$

Theorem (The conjugate roots theorem)

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ be a polynomial with only real coefficients. If the complex number $a+bi$ is a zero of p, then so is the complex number $a-bi$. In terms of the linear factors of p, this means that if $x-(a+bi)$ is a factor of p, then so is $x-(a-bi)$.

Example

$$f(x) = x^4 - 8x^3 + 200x - 625$$

$$x_0 = 4-3i$$

Factor $f(x)$ completely.

$$(x-(4-3i))(x-(4+3i)) =$$

$$= x^2 - x(4+3i) - x(4-3i) + (4-3i)(4+3i) =$$

$$= x^2 - 4x - \cancel{3i}x - 4x + \cancel{3i}x + 16 + 9 =$$

$$= x^2 - 8x + 25$$

$$\begin{array}{l|l}
x^4 - 8x^3 + 200x - 625 & x^2 - 8x + 25 \\ \hline
-(x^4 - 8x^3 + 25x^2) & x^2 - 25 \\ \hline
-25x^2 + 200x - 625 & \\
-(-25x^2 + 200x - 625) & \\ \hline
0 &
\end{array}$$

$$x^4 - 8x^3 + 200x - 625 = (x^2 - 8x + 25)(x^2 - 25) =$$

$$= (x-(4+3i))(x-(4-3i))(x-5)(x+5)$$

Example (Constructing polynomials)

Construct a 4th degree real-coefficient

polynomial function f with zeros of 2, -5, and $1+i$ s.t. $f(1)=12$.

$$f(x) = a(x-2)(x+5)(x-(1+i))(x-(1-i))$$

$$a(-1)(6)(i)(-i) = 12$$

$$-6a = 12$$

$$a = -2$$

Hence, $f(x) = -2(x-2)(x+5)(x-(1+i))(x-(1-i))$.