Section 5.5. Rational Functions and Rational Inequalities

1. Characteristics of rational functions.
2. Vertical asymptotes.
3. Horizontal and oblique asymptotes.
4. Graphing rational functions.
5. Solving rational inequalities.

$\boxed{1.}$ **Def.** (Rational Functions)

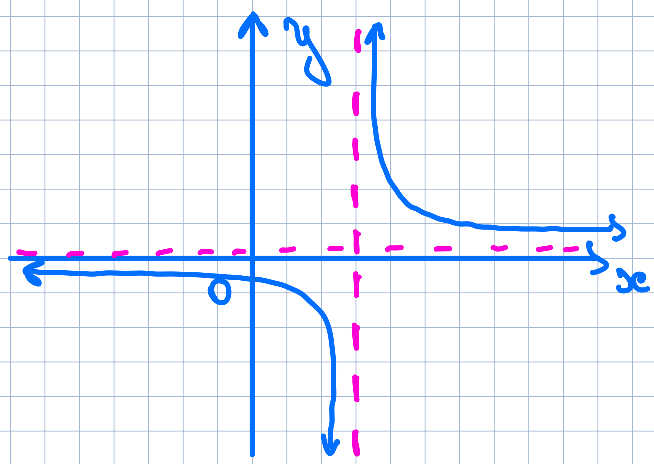
A <u>rational function</u> is a function that can be written in the form

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

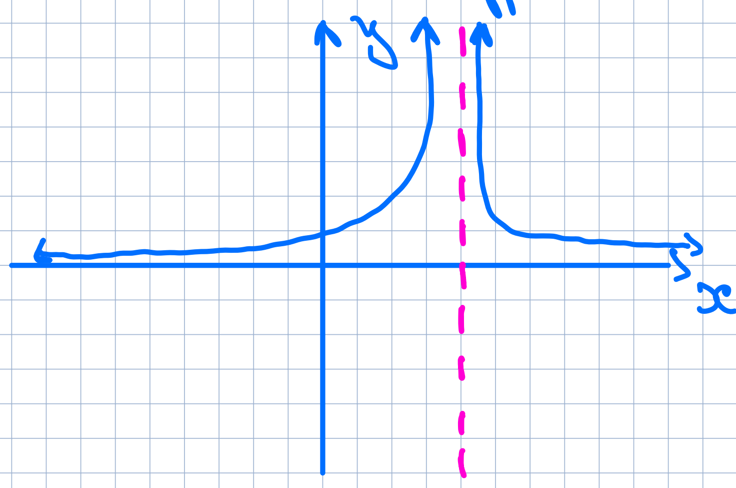
At x values s.t. $q(x) = 0$ the function $f(x)$ is undefined.

$$\text{Dom}(f) = \{x \in \mathbb{R} \mid q(x) \neq 0\}.$$



Def. (Vertical asymptotes)

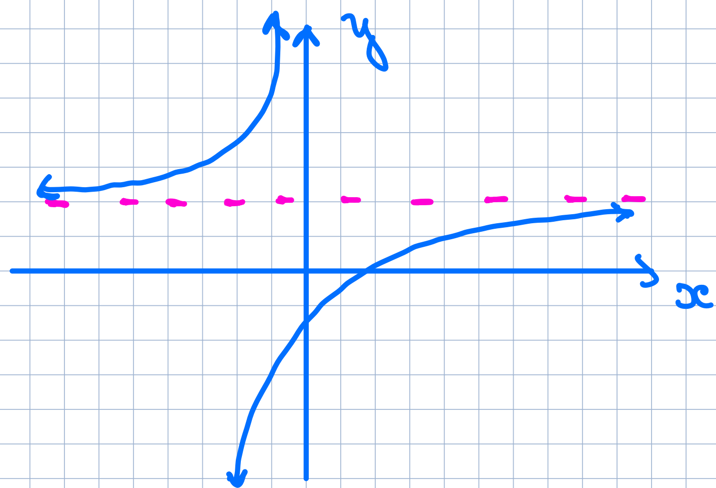
The vertical line $x=c$ is a vertical asymptote of a function f if $f(x)$ increases in magnitude without bound as x approaches c.



Def. (Horizontal asymptotes)

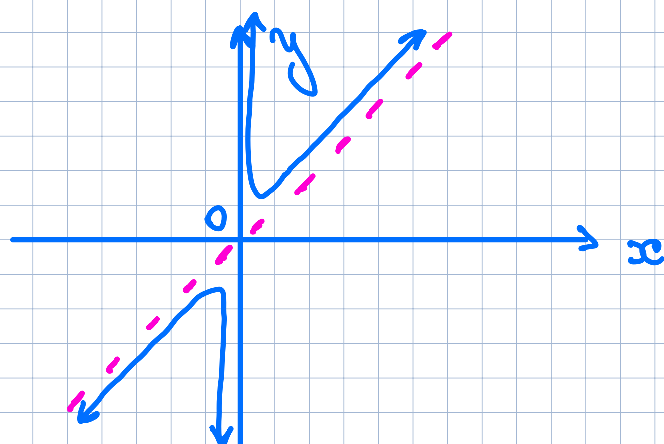
The horizontal line $y=c$ is a horizontal asymptote of a function

f if $f(x)$ approaches the value c as $x \to -\infty$ or $x \to \infty$.



Def. (Oblique asymptotes)

A nonvertical, nonhorizontal line may also be an asymptote of a function f.



Def. (Asymptote Notation)

- The notation $x \to c^-$ is used when describing the behaviour of a graph as x approaches the value c from the left.

- The notation $x \to c^+$ is used when describing behaviour as x approaches c from the right.
- The notation $x \to c$ is used when describing behaviour that is the same on both sides of c.

$\boxed{2.}$

Theorem (Equations for Vertical Asymptotes)

If the rational function

$$f(x) = \frac{p(x)}{q(x)}$$

has been written in reduced form, the vertical line $x = c$ is a vertical asymptote of f iff c is a zero of the polynomial q. In other words, f has vertical asymptotes at the x-intercepts of q.

Example

- $f(x) = \dfrac{32}{x+2}$, $\quad \mathrm{Dom}(f) = (-\infty, -2) \cup (-2, \infty)$

$$q(x) = x+2 = 0$$
$$x = -2$$

Thus, $x=-2$ is a vertical asymptote.

$\boxed{3.}$

Theorem (Equations for Horizontal and Oblique Asymptotes)

Let $f(x) = \dfrac{p(x)}{q(x)}$ be a rational function, where p is an n^{th}-degree polynomial with leading coefficient a_n, q is an m^{th}-degree polynomial with leading coefficient b_m, and $p(x)$ and $q(x)$ have no common factors other than constants. Then the asymptotes of f are found as follows.

1. If $n<m$, the horizontal line $y=0$ is the horizontal asymptote for f.

2. If $n=m$, the horizontal line $y = \dfrac{a_n}{b_m}$ is the horizontal asymptote for f.

3. If $n = m+1$, the line $y = g(x)$ is an <u>oblique asymptote</u> for f, where g is the quotient polynomial obtained by dividing p by q.

4. If $n > m+1$, there is <u>no</u> straight line <u>horizontal</u> or <u>oblique asymptote</u> for f.

Example

- $f(x) = \dfrac{x^2+1}{x^2+2x-15}$ $\qquad a_2 = 1$, $b_2 = 1$

$y = \frac{1}{1} = 1$

$y = 1$ is a horizontal asymptote

- $f(x) = \dfrac{x^3+x^2+2x+2}{x^2+9}$

$$
\begin{array}{l|l}
x^3 + x^2 + 2x + 2 & x^2+9 \\ \cline{2-2}
-\underline{(x^3 + 9x)} & x+1 \\
\phantom{-x^3+{}} x^2 - 7x + 2 & \\
 -\underline{(x^2 + 9)} & \\
 -7x - 7 &
\end{array}
$$

$y = x+1$ is an oblique asymptote.

Procedure: (Graphing Rational Functions)

Given a rational function f,

Step 1: Factor the denominator in order to determine the domain of f.
Any points excluded from the domain correspond to holes or vertical asymptotes in the eventual graph.

Step 2: Factor the numerator as well and cancel any common factors.
Zeros of the numerator and denominator arising from common linear factors are the x-coordinates of holes in the eventual graph.

Step 3: Examine the remaining linear factors in the denominator to determine the equations for any vertical asymptotes.

Step 4: Compare the degrees of the numerator and denominator to determine if there is a horizontal or oblique asymptote. If so, find its equation.

Step 5: Determine the y-intercept if 0 is in the domain of f.

Step 6: Determine the x-intercepts, if there are any, by setting the numerator of the reduced fraction equal to 0.

Step 7: Plot enough points to determine the behaviour of f between x-intercepts and between vertical asymptotes.

$\boxed{5.}$

Def. (Rational Inequalities)

A rational inequality is any inequality that can be written in the form:

$$f(x) < 0, \; f(x) \leq 0, \; f(x) > 0, \text{ or } f(x) \geq 0,$$

where $f(x)$ is a rational function.

Procedure: (Sign-Test Method)

To solve a rational inequality $f(x) < 0$, $f(x) > 0$, $f(x) \leq 0$, or $f(x) \geq 0$, where the rational function

$f(x) = \frac{p(x)}{q(x)}$ is in reduced form:

Step 1: Find the real zeros of the numerator $p(x)$. These values are the zeros of f.

Step 2: Find the real zeros of the denominator $q(x)$. These values are the locations of the vertical asymptotes of f.

Step 3: Place the values from Step 1 and Step 2 on a number line, splitting it into intervals.

Step 4: Within each interval, select a test point and evaluate f at that number. If the result is positive, then $f(x) > 0$ for all x in the interval. If the result is negative, then $f(x) < 0$ for all x in the interval.

Step 5: Write the solution set, consisting of all of the intervals that satisfy the given inequality.

If the inequality is not strict ($\leq, \geq$), then the zeros of p are included in the solution set as well.
The zeros of q are never included in the solution set.

Example

$$\frac{x^2+1}{x^2+2x-15} > 0$$

$x^2+1>0$ for all $x \in \mathbb{R}$

$x^2+2x-15=(x+5)(x-3)=0$

$x=-5$ or $x=3$

Solution: $(-\infty,-5)\cup(3,+\infty)$.