THEORETICAL PART:

Solutions

Theorem (The division algorithm):

Let $p(x)$ and $d(x)$ be polynomials such that $d(x) \neq 0$ and with the degree of d less than or equal to the degree of p. Then there are unique polynomials $q(x)$ and $r(x)$, called the **quotient** and the **remainder**, respectively, such that

$$p(x) = q(x) \cdot d(x) + r(x).$$

Either the degree of the remainder r is less than the degree of the **divisor** d, or the remainder is 0, in which case we say d **divides evenly** into the polynomial p. If he remainder is 0, the two polynomials q and d are factors of p.

Theorem (Zeros and linear factors):

The number c is a **zero** of a polynomial p if and only if the linear polynomial $x - c$ is a factor of p. In this case, $p(x) = q(x)(x - c)$ for some quotient polynomial q. This also means that c is the solution of a polynomial equation $p(x) = 0$, and if p is a polynomial with real coefficients and if c is a real number, then c is an x-intercept of p.

Theorem (The remainder theorem):

If the polynomial $p(x)$ is divided by $x - c$, the remainder is $p(c)$.

$$p(x) = q(x)(x - c) + p(c)$$

Procedure (Polynomial long division):

1. Arrange the terms of each polynomial in descending order.
2. Divide the first term in the dividend by the first term in the divisor. This gives the first term of the quotient.
3. Multiply the entire divisor by the result (the first term of the quotient) and write the beneath the dividend so that like terms line up.
4. Subtract the product from the dividend.
5. Bring down the rest of the original dividend, forming a new dividend.
6. Repeat the process with the new dividend. Continue until the degree of the remainder is less than the degree of the divisor.

Procedure (Synthetic division):

1. If the divisor is $x - c$, write down c followed by the coefficients of the dividend.
2. Write the leading coefficient of the dividend on the bottom row.

3. Multiply c by the value placed on the bottom, and place the product in the next column, in the second row.
4. Add the values in this column, giving a new value in the bottom row.
5. Repeat this process until the table is complete.
6. The numbers in the bottom row are the coefficients of the quotient, plus the remainder (which is the final value in this row). Note that the first term of the quotient will have degree one less than the first term of the dividend.

PRACTICAL PART:

1. Divide the polynomial $x^2 + 2x - 24$ by the polynomial $x + 6$.

$$\begin{array}{r|l} x^2 + 2x - 24 & x+6 \\ -(x^2 + 6x) & x-4 \\ \hline -4x - 24 & \\ -(-4x - 24) & \\ \hline 0 & \end{array}$$

remainder

$p(x) = q(x) \cdot d(x) + r(x)$

$x^2 + 2x - 24 = (x+6)(x-4) + 0$

2. Divide the polynomial $6x^5 - 5x^4 + 10x^3 - 15x^2 - 19$ by the polynomial $2x^2 - x + 3$.

$$\begin{array}{r|l} 6x^5 - 5x^4 + 10x^3 - 15x^2 - 19 & 2x^2 - x + 3 \quad d(x) \\ -(6x^5 - 3x^4 + 9x^3) & 3x^3 - x^2 - 6 \\ \hline -2x^4 + x^3 - 15x^2 - 19 & \\ -(-2x^4 + x^3 - 3x^2) & \\ \hline -12x^2 - 19 & \\ -(-12x^2 + 6x - 18) & \\ \hline -6x - 1 \quad r(x) & \end{array}$$

$\deg(r) < \deg(d)$

3. Divide $p(x) = x^4 + 1$ by $d(x) = x^2 + i$.

$$\begin{array}{rl} x^4 + 1 & \big|\, x^2 + i \\ -\,(x^4 + ix^2) & \;\; x^2 - i \\ \hline -ix^2 + 1 & \\ -\,(-ix^2 + 1) & \\ \hline 0 & \end{array}$$

$r(x) = 0$

$x^4 + 1 = (x^2 - i)(x^2 + i)$

4. For the polynomial p below, divide p by $x - c$ using synthetic division. Use the result to determine if the given c is a zero. If not, determine $p(c)$.

$$p(x) = -2x^4 + 11x^3 - 5x^2 - 3x + 15; \quad c = 5$$

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	-2	11	-5	-3	15
5	-2	1	0	-3	0

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$p(x) = (x-5)(-2x^3 + x^2 - 3)$

$p(5) = 0$. Thus, $c = 5$ is a zero of $p(x)$.

5. Construct a polynomial that has the given properties:

   (a) Third degree, zeros of -3, 2, and 5, and goes to $-\infty$ as $x \to \infty$.

   (b) Fourth degree, zeros of -5, -2, 1, and 3, and y-intercept at (0, 15).

(a) $p(x) = a(x+3)(x-2)(x-5)$

$p(x) \to -\infty$ as $x \to \infty$

Thus, we need to put $a<0$.
For instance, let put $a=-1$.

Answer: $p(x) = -(x+3)(x-2)(x-5)$.

(b) $p(x) = a(x+5)(x+2)(x-1)(x-3)$

y-intercept: $(0, 15)$

$$p(0) = a \cdot 5 \cdot 2 \cdot (-1) \cdot (-3) = 15$$

$$a \cdot \cancel{15} \cdot 2 = \cancel{15}$$

$$a = \frac{1}{2}$$

Thus, $p(x) = \frac{1}{2}(x+5)(x+2)(x-1)(x-3)$