THEORETICAL PART: Solutions

Definition (Inverse of a Relation):

Let R be a relation. The **inverse of R**, denoted R^{-1}, is the relation defined by switching the first and second coordinates of each ordered pair that is an element of R.

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$

CAUTION:

Note that f^{-1} does not stand for $\dfrac{1}{f}$ when f is a function!

Theorem (The Horizontal Line Test):

Let f be a function. We say that the graph of f passes the **horizontal line test** if every horizontal line in the plane intersects the graph no more than once. If f passes the horizontal line test, then f^{-1} is also a function.

Definition (One-to-One Function):

A function f is **one-to-one** if, for every pair of distinct elements x_1 and x_2 in the domain of f, we have $f(x_1) \neq f(x_2)$.

Procedure (Finding Formulas of Inverse Functions):

Let f be a one-to-one function, and assume that f is defined by a formula. To find a formula for f^{-1}, perform the following steps:

1. Replace $f(x)$ in the definition of f with the variable y. The result is an equation in x and y that is solved for y at this point.
2. Solve the equation for x.
3. Replace the x in the resulting equation with $f^{-1}(x)$ and replace each occurrence of y with x.

Theorem (Composition of Functions and Inverses):

Given a function f and its inverse f^{-1}, the following statements are true:

$$f(f^{-1}(x)) = x \quad \text{for all} \quad x \in Dom(f^{-1})$$

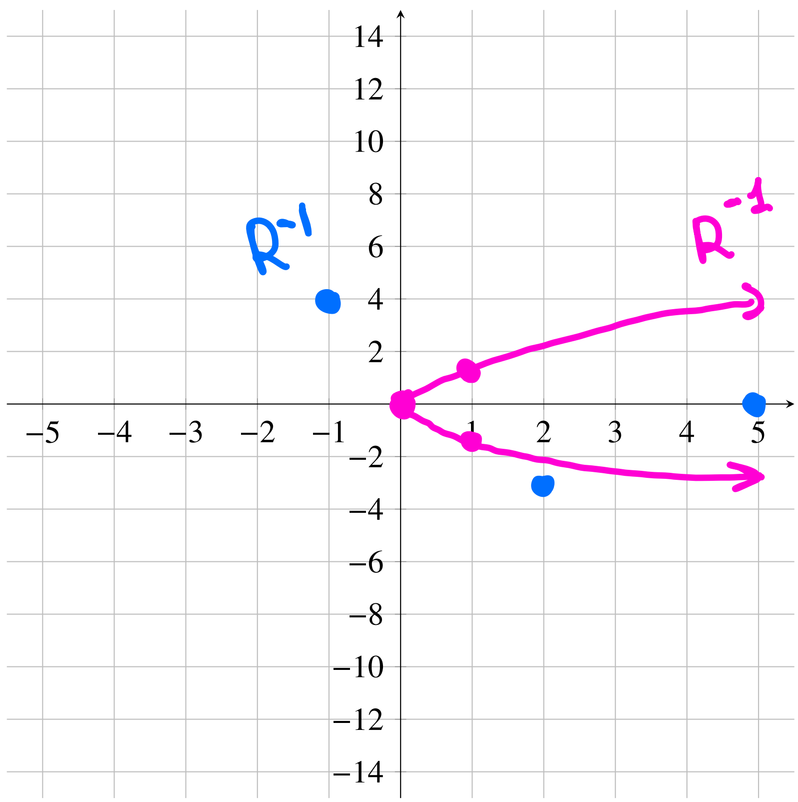
$$f^{-1}(f(x)) = x \quad \text{for all} \quad x \in Dom(f)$$

PRACTICAL PART:

1. Determine the inverse of each of the following relations. Then graph each relation and its inverse, and determine the domain and range of both:

   (a) $R = \{(4, -1), (-3, 2), (0, 5)\}$

   (b) $y = x^2$



(a) $R^{-1} = \{(-1,4),(2,-3),(5,0)\}$

(b) $R = \{(x,y) \mid y = x^2\}$

$R^{-1} = \{(x,y) \mid x = y^2\}$

(a)
$Dom(R^{-1}) = \{-1, 2, 5\}$
$Ran(R^{-1}) = \{4, -3, 0\}$

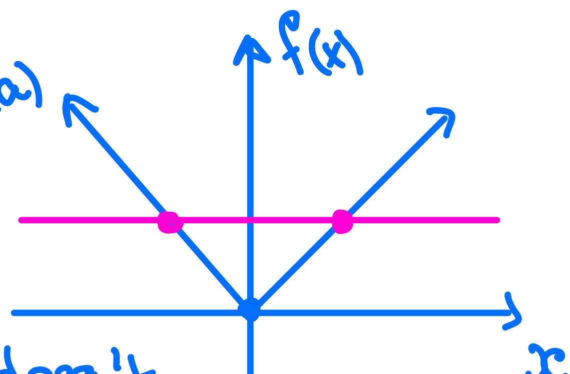
(b)
$Dom(R^{-1}) = [0, \infty)$
$Ran(R^{-1}) = (-\infty, \infty)$

2. Determine if the following functions have inverse functions:

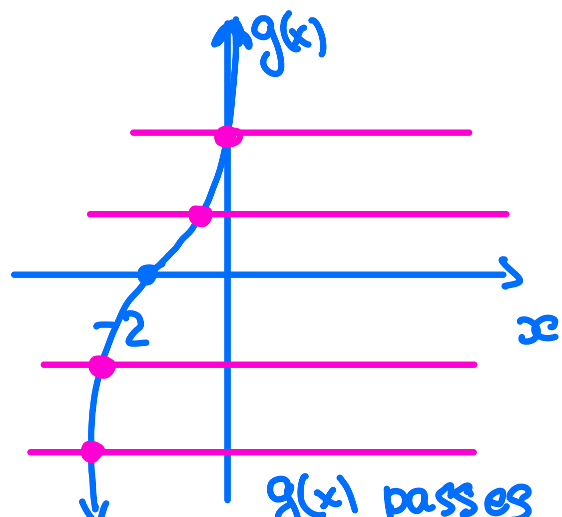
   (a) $f(x) = |x|$

   (b) $g(x) = (x + 2)^3$

(a)



(b)



$f(x)$ doesn't pass the h.l.t. Hence, $f^{-1}(x)$ does not exist

$g(x)$ passes h.l.t. g^{-1} exists

3. Find the inverse of each of the following functions:

   (a) $f(x) = (x-1)^3 + 2$

   (b) $g(x) = \dfrac{x-3}{2x+1}$

(a) $y = (x-1)^3 + 2$

$y - 2 = (x-1)^3$

$x - 1 = \sqrt[3]{y-2}$

$x = \sqrt[3]{y-2} + 1$

$\boxed{f^{-1}(x) = \sqrt[3]{x-2} + 1}$

(b) $y = \dfrac{x-3}{2x+1}$

$y(2x+1) = x - 3$

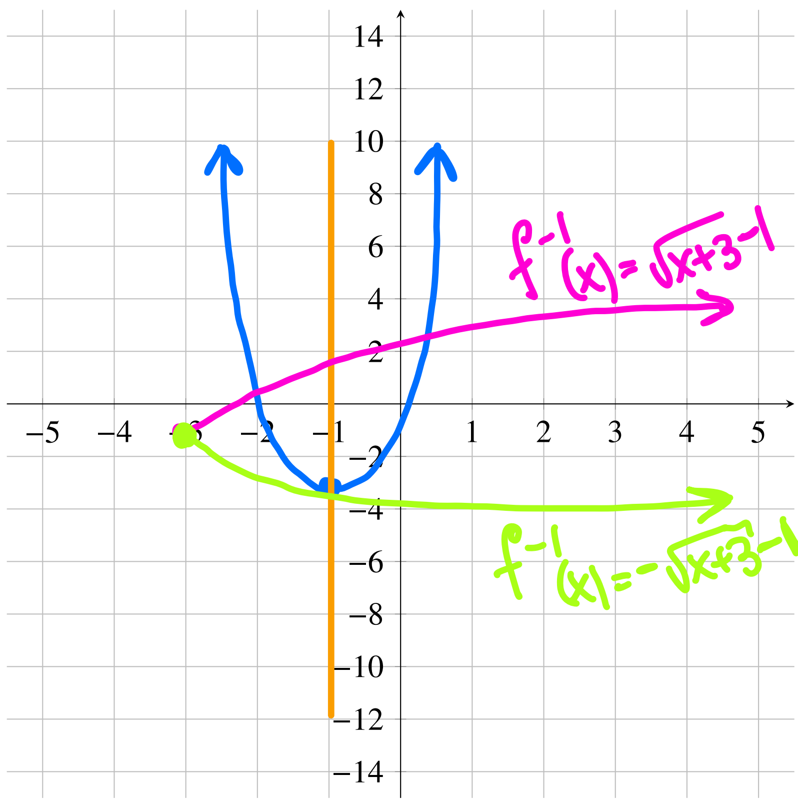
$2xy + y = x - 3$

$x(2y-1) = -y-3$

$x = \dfrac{-y-3}{2y-1}$

$\boxed{f^{-1}(x) = \dfrac{-x-3}{2x-1}}$

4. Find two suitable restrictions of the domain so that the function $f(x) = (x+1)^2 - 3$ has an inverse function, then find a formula for the inverse for each restricted function.



$f(x) = (x+1)^2 - 3$

$f(x)$ is not one-to-one on $\mathbb{R}$

- on $[-1, \infty)$ $f(x)$ is one-to-one

$y = (x+1)^2 - 3$

$y + 3 = (x+1)^2$

$x + 1 = +\sqrt{y+3}$

$x = +\sqrt{y+3} - 1$

$f^{-1}(x) = \sqrt{x+3} - 1$

- on $(-\infty, -1]$ $f(x)$ is also one-to-one

$$y = (x+1)^2 - 3$$

$$f^{-1}(x) = -\sqrt{x+3} - 1$$

5. Use functions $f(x) = (x-1)^3+2$ and $f^{-1}(x) = (x-2)^{\frac{1}{3}}+1$ to demonstrate that the composition of a function and its inverse leaves any input unchanged.

$$\underline{\underline{(f \circ f^{-1})}}(x) = f(f^{-1}(x)) = f((x-2)^{1/3}+1) =$$

$$= ((x-2)^{1/3}+\cancel{1}-\cancel{1})^3 + 2 = (x-2)+2 = x-\cancel{2}+\cancel{2} = \underline{\underline{x}}$$

$$\underline{\underline{(f^{-1} \circ f)}}(x) = f^{-1}(f(x)) = f^{-1}((x-1)^3+2) =$$

$$= ((x-1)^3+\cancel{2}-\cancel{2})^{1/3} + 1 = ((x-1)^3)^{1/3}+1 = x-\cancel{1}+\cancel{1} = \underline{\underline{x}}$$