THEORETICAL PART:

Solutions

Definition (Period of a function):

A function f is said to be **periodic** if there is a positive number p such that

$$f(x+p) = f(x)$$

for all x in the domain of f. The smallest such number p is called the **period** of f.

Identities (Even/Odd Identities):

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x), \quad \tan(-x) = -\tan(x),$$
$$\csc(-x) = -\csc(x), \quad \sec(-x) = \sec(x), \quad \cot(-x) = -\cot(x)$$

Definition (Amplitude of Sine and Cosine curves):

Given a fixed real number a, the **amplitude** of the function $f(x) = a\sin(x)$ or the function $g(x) = a\cos(x)$ is the value $|a|$. As we know, the multiplication of $\sin(x)$ or $\cos(x)$ by a stretches (or compresses, if $-1 < a < 1$) the graph vertically by a factor of $|a|$, so the amplitude represents the distance between the x-axis and the maximum value of the function.

Definition (Frequency of Sine and Cosine curves):

Given a fixed real number b, the **frequency** of the function $f(x) = \sin(bx)$ or the function $g(x) = \cos(bx)$ is the number $b/2\pi$. When the independent variable represents time, measured in seconds, the measurement of frequency is stated in terms of **cycles per second**, or **hertz (Hz)**.

Definition (Period Revisited):

Given a fixed real number b, the **period** of the function $f(x) = \sin(bx)$ or the function $g(x) = \cos(bx)$ is the number $2\pi/2$. The period and frequency of a sinusoidal function are reciprocals of one another.

Definition (Amplitude, Period, and Phase Shift Combined):

Given constants a, b (such that $b > 0$), and c, the functions

$$f(x) = a\sin(bx - c), \quad g(x) = a\cos(bx - c)$$

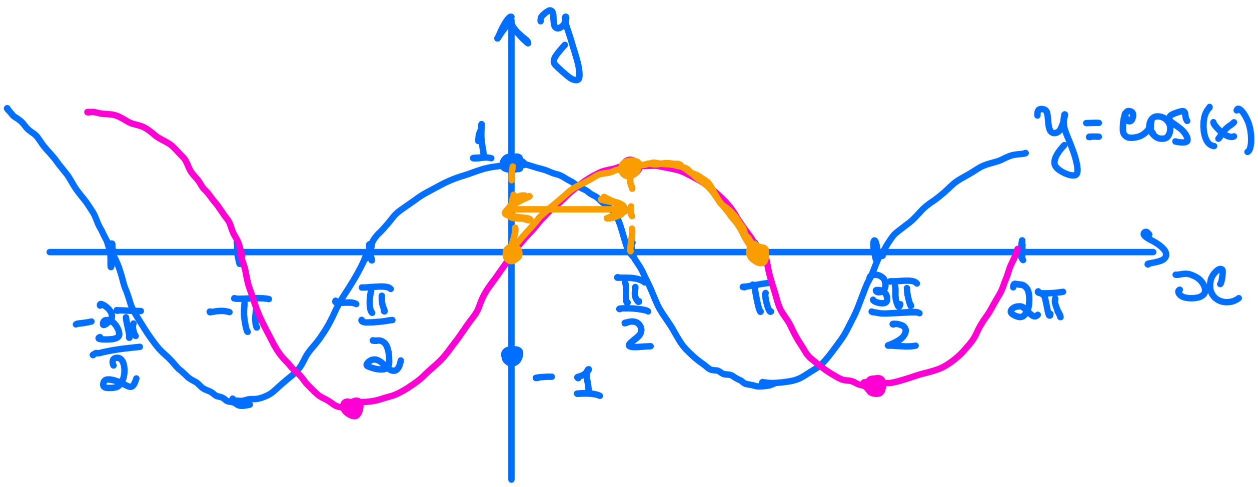
have **amplitude** $|a|$, **period** $2\pi/2$, and a **phase shift** of c/b. The left endpoint of one cycle of either function is c/b and the right endpoint is $c/b + 2\pi/b$.

Definition (Simple Harmonic Motion):

If an object is oscillating and its displacement from some midpoint at time t can be described by either $f(t) = a\sin(bt)$ or $g(t) = a\cos(bt)$, the object is said to be in **simple harmonic motion (SHM)**. In both cases, the maximum displacement of the object from its midpoint is the amplitude $|a|$ and its frequency of oscillation is $b/2\pi$.

PRACTICAL PART:

1. Based on graphs of sine and cosine functions and the values of sine and cosine, construct a transformation of cosine that is equal to sine.



$$\boxed{\sin(x) = \cos\left(x - \frac{\pi}{2}\right)}$$

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{2}\right) = \cos(0) = 1$$

$$\cos\left(-\frac{\pi}{2} - \frac{\pi}{2}\right) = \cos(-\pi) = -1$$

2. Determine the periods of the secant, cosecant, tangent, and cotangent functions.

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\sec(x+\underline{2\pi}) = \frac{1}{\cos(x+2\pi)} = \frac{1}{\cos(x)} = \sec(x)$$

$$\csc(x+\underline{2\pi}) = \csc(x)$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \tan(x+\underline{\pi}) = \tan(x)$$

$$\cot(x) = \cot(x+\underline{\pi})$$

3. Use a cofunction identity and an even/odd identity to prove the transformation statement $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$ for all x.

$$\cos\left(x - \frac{\pi}{2}\right) = \cos\left(-\left(\frac{\pi}{2} - x\right)\right) =$$

$$= \cos\left(\frac{\pi}{2} - x\right) = \sin(x) \text{ for all } x$$

4. Determine the amplitude and frequency of each of the following functions. Then use your results to sketch the graph of one complete cycle of each function starting at $x = 0$.

   a. $f(x) = 3\sin(x/2)$

   b. $g(x) = -\frac{1}{2}\cos(2\pi x)$

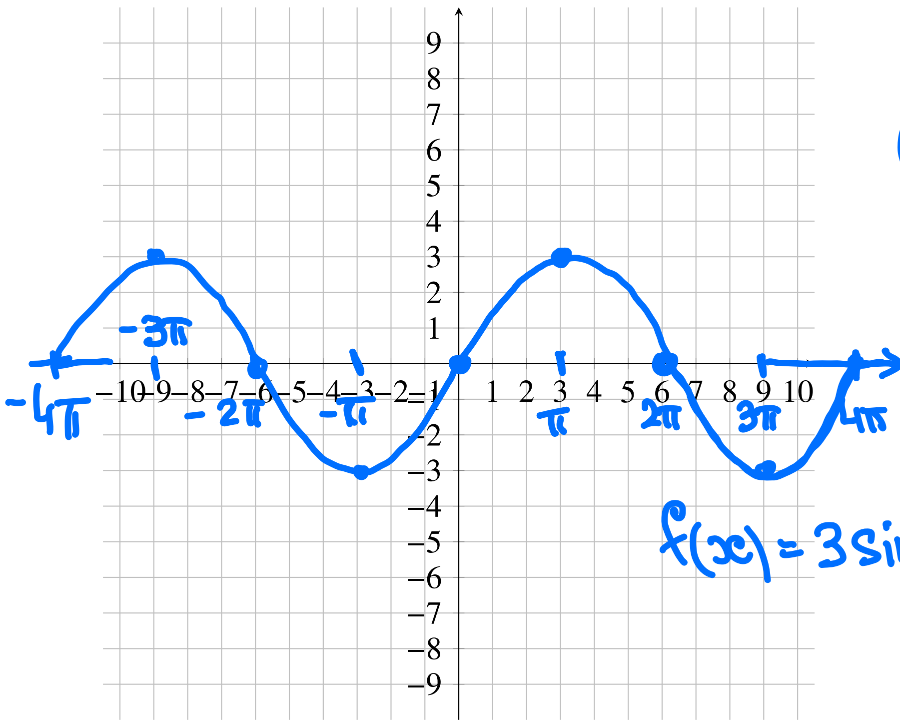
(a) $|a| = 3$

$\nu = \frac{1}{2} : 2\pi = \frac{1}{4\pi}$

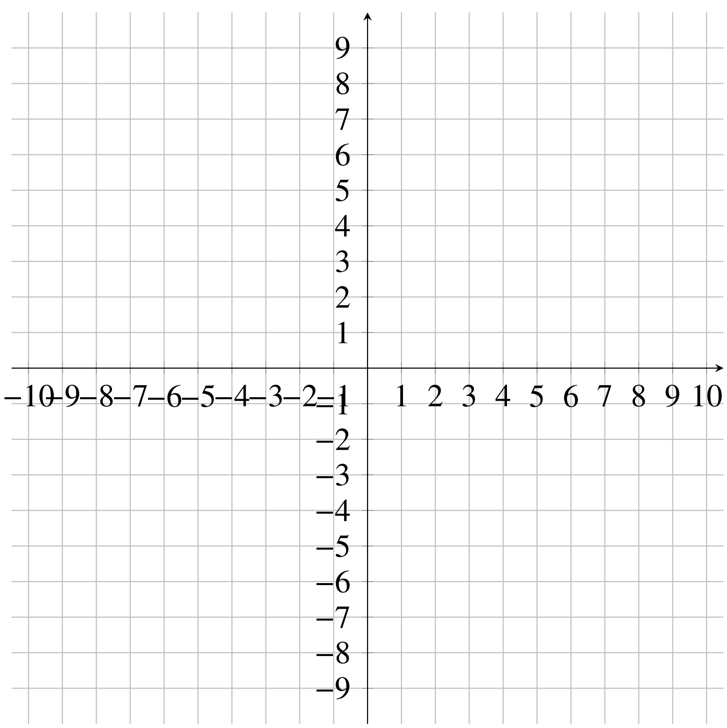
(b) $|a| = \frac{1}{2}$

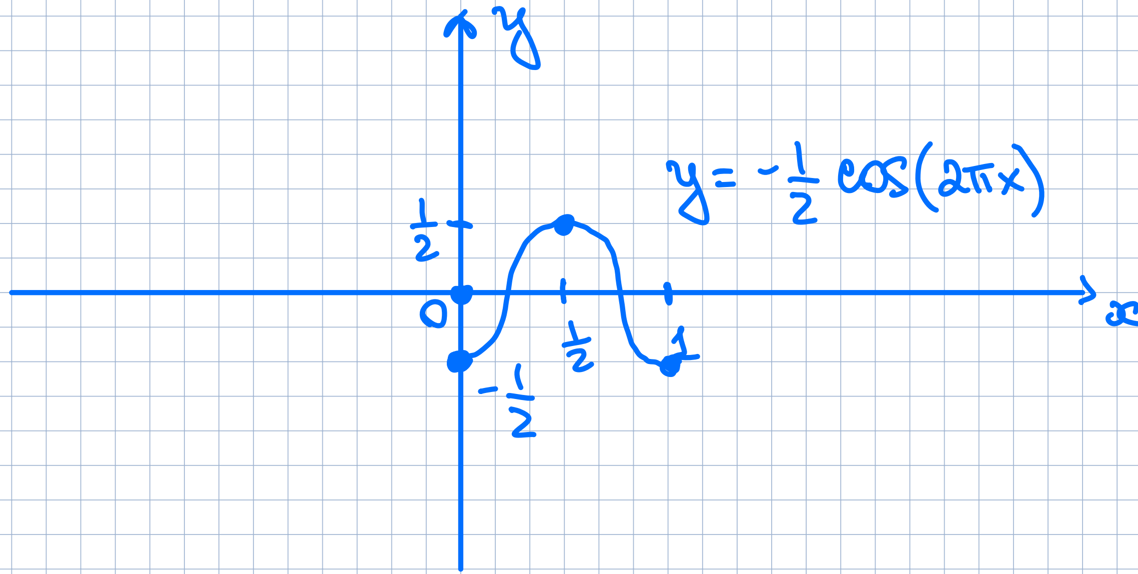
$\nu = \frac{2\pi}{2\pi} = 1$

$T = \frac{1}{\nu}$



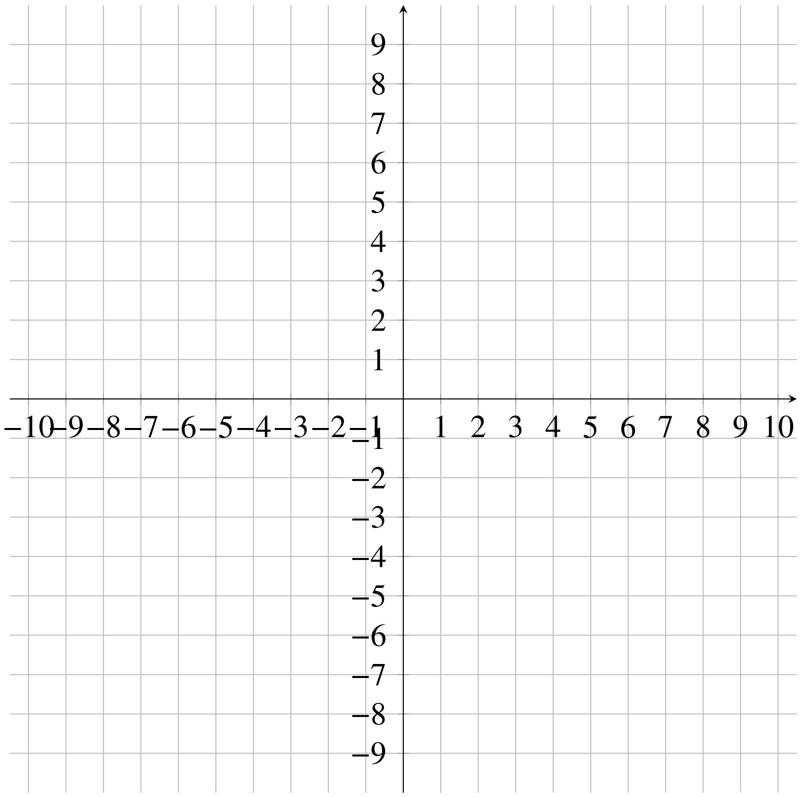
$f(x) = 3\sin\left(\frac{x}{2}\right)$





y
$y = -\frac{1}{2}\cos(2\pi x)$
$\frac{1}{2}$
0
$-\frac{1}{2}$
$\frac{1}{2}$
1
x

5. Sketch the graph of $f(x) = -2\cos(\pi x - \pi)$.



6. A heart rate of 1200 beats per minute (bpm) is typical for a hummingbird. What is the length of the period, in seconds, of such a heart rate?

$$1200 \text{ bpm} = 1200 \cdot \frac{1}{60} = \boxed{20} \text{ bps}$$

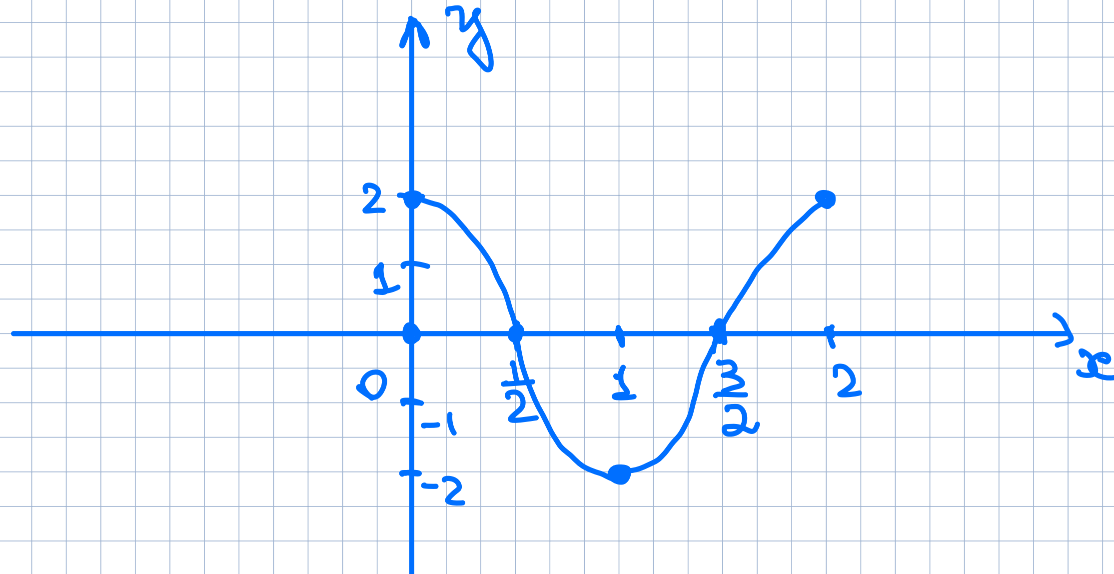
$$T = \frac{1}{\nu} = \frac{1}{20} = 0.05 \text{ (s)}$$

5. $y = -2\cos(\pi x - \pi)$

$|a| = 2$

$\lambda = \dfrac{\pi}{2\pi} = \dfrac{1}{2}$ $\qquad T = \dfrac{2\pi}{\pi} = 2$

phase shift : $\dfrac{C}{B} = \dfrac{\pi}{\pi} = 1$



7. Sketch the graph of $f(t) = -4e^{-t}\cos(6\pi t)$. (harmonic motion)