THEORETICAL PART:

Solutions

Properties of logarithms:

Let a (the logarithmic base) be a positive real number not equal to 1, let x and y be positive real numbers, and let r be any real number.

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^r) = r\log_a x$

Formula (Change of Base Formula):

Let a and b be positive real numbers, neither of them equal to 1, and let x be a positive real number. Then

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Definition (The pH Scale):

The **pH** of a solution is defined to be $-\log[H_3O^+]$, where $[H_3O^+]$ is the concentration of hydronium ions in units of moles/liter. Solutions with a pH less than 7 are said to be acidic, while those with a pH greater than 7 are basic.

Definition (The Richter Scale):

Earthquake intensity is measured on the **Richter Scale**:

$$R = \log\left(\frac{I}{I_0}\right),$$

where I_0 is the intensity of a just-discernible earthquake, I is the intensity of an earthquake being analyzed, and R is its ranking on the Richter scale.

Scale: $R < 4$ – minor, $4 \le R < 5$ – light, $5 \le R < 6$ – moderate, $6 \le R < 7$ – strong, $7 \le R < 8$ – major, $8 \le R$ – great.

Definition (The Decibel Scale):

$$D = 10\log\left(\frac{I}{I_0}\right),$$

where I_0 is the intensity of a just-discernible sound, I is the intensity of the sound being analyzed, and D is its decibel level.

Scale: $0 < D < 60$ – normal conversation, $60 < D < 80$ – heavy traffic, $80 < D < 120$ – loud rock concert, $120 < D < 160$ – eardrum is likely to rupture.

PRACTICAL PART:

1. Use properties of logarithms to expand the following expressions as much as possible.

   (a) $\log_4 (64x^3 \sqrt{y})$

   (b) $\log_a \left(\sqrt[3]{\frac{xy^2}{z^4}} \right)$

   (c) $\log \left(\frac{2.7 \times 10^4}{x^{-2}} \right)$

(a) $\log_4(64x^3\sqrt{y}) = \log_4 64 + \log_4 x^3 + \log_4 \sqrt{y} =$

$= 4\underbrace{\log_4 4}_{=1} + 3\log_4 x + \frac{1}{2}\log_4 y =$

$= 4 + 3\log_4 x + \frac{1}{2}\log_4 y$

(b) $\log_a\left(\sqrt[3]{\frac{xy^2}{z^4}}\right) = \frac{1}{3}\log_a \frac{xy^2}{z^4} =$

$= \frac{1}{3}\log_a x + \frac{1}{3}\log_a y^2 - \frac{1}{3}\log_a z^4 =$

$= \frac{1}{3}\log_a x + \frac{2}{3}\log_a y - \frac{4}{3}\log_a z$

(c) $\log\left(\frac{2.7\cdot 10^4}{x^{-2}}\right) = \log 2.7 + \log 10^4 -$

$- \log x^{-2} = \log 2.7 + 4 + 2\log x$

2. Use the properties of logarithms to condense the following expressions as much as possible.

(a) $2\log_3\left(\frac{x}{3}\right) - \log_3\left(\frac{1}{y}\right)$

(b) $\ln(x^2) - \frac{1}{2}\ln y + \ln 2$

(c) $\log_b 5 + 2\log_b(x^{-1})$

(a) $\log_3\left(\frac{x}{3}\right)^2 - \log_3\left(\frac{1}{y}\right) = \log_3\left(\frac{x}{3}\right)^2 -$

$- \log_3 y^{-1} = \log_3\left(\frac{x^2}{9y^{-1}}\right) = \log_3\left(\frac{x^2 y}{9}\right)$

(b) $\ln(x^2) - \ln\sqrt{y} + \ln 2 =$

$= \ln\left(\frac{2x^2}{\sqrt{y}}\right)$

(c) $\log_b 5 + \log_b (x^{-1})^2 = \log_b 5 + \log_b x^{-2} =$

$= \log_b\left(\frac{5}{x^2}\right)$

3. Evaluate the following logarithmic expressions, using the base of your choice.

   (a) $\log_7 15$

   (b) $\log_{\frac{1}{2}} 3$

   (c) $\log_\pi 5$

(a) $\log_7 15 = \frac{\log_{15} 15}{\log_{15} 7} = \frac{1}{\log_{15} 7} \approx 1.392$

(b) $\log_{\frac{1}{2}} 3 = \frac{\log_3 3}{\log_3 \frac{1}{2}} = \frac{1}{\log_3 \frac{1}{2}} \approx -1.585$

(c) $\log_\pi 5 = \frac{\log_5 5}{\log_5 \pi} = \frac{1}{\log_5 \pi} \approx 1.406$

4. If a sample of orange juice is determined to have a $[H_3O^+]$ concentration of 1.58×10^{-4} moles/liter, what is its pH?

$pH = -\log(H_3O^+) = -\log(1.58 \cdot 10^{-4}) =$

$= -\log 1.58 - \log 10^{-4} = -\log 1.58 + 4 \approx$

≈ 3.8

5. Given that $I_0 = 10^{-12}$ $watts/meter^2$, what is the decibel level of jet airliner's engines at a distance of 45 meters, for which the sound intensity is 50 $watts/meter^2$?

$$D = \log\left(\frac{I}{I_0}\right) \quad \frac{\text{watts}/m^2}{\text{watts}/m^2}$$

$$D = \log\left(\frac{50}{10^{-12}}\right) = \log 50 + 12 =$$

$$= 13 + \log 5 \approx \boxed{137}$$

The sound will be quite painful.