THEORETICAL PART: Solutions

Theorem (The rational zero theorem):

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a polynomial with integer coefficients with $a_n \neq 0$, then any rational zero of f must be of the form $\frac{p}{q}$, where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

Theorem (Descarte's Rule of Signs):

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a polynomial with real coefficients, and assume $a_n \neq 0$. A **variation in sign** of f is a change in the sign of one coefficient of f to the next, either from positive to negative or vice versa.

1. The number of **positive real zeros** of f is either the number of variations in sign of $f(x)$ or is less than this number by a positive even integer.
2. The number of **negative real zeros** of f is either the number of variations in sign of $f(-x)$ or is less than this number by a positive even integer.

Theorem (Upper and Lower bounds of zeros):

Let $f(x)$ be a polynomial with real coefficients, a positive leading coefficient, and degree ≥ 1. Let a be a negative number and b be a positive number. Then:

1. No real zero of f is larger than b if the last row in the synthetic division of $f(x)$ by $x - b$ contains no negative numbers. That is, b is an upper bound of the zeros if the quotient and remainder have no negative coefficients when $f(x)$ is divided by $x - b$.
2. No real zero of f is smaller than a if the last row in the synthetic division of $f(x)$ by $x - a$ has entries that alternate in sign (0 can count as either positive or negative).

Theorem (The Intermediate Value Theorem):

Assume that $f(x)$ is a polynomial with real coefficients, and that a and b are real numbers with $a < b$. If $f(a)$ and $f(b)$ differ in sign, then there is at least one point c such that $a < c < b$ and $f(c) = 0$. That is at least one zero of f lies between a and b.

PRACTICAL PART:

1. For the polynomial function $f(x) = 2x^3 + 5x^2 - 4x - 3$ list all of the potential rational zeros. Then write the polynomial in factored form and identify the actual zeros.

$a_0 = -3$ Factors of -3: $\pm 1, \pm 3 = p$

$a_3 = 2$ Factors of 2: $\pm 1, \pm 2 = q$

Therefore, all potential zeros are:

$$\frac{p}{q} = \left\{ \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2} \right\}$$

Let $c = 1$. Then $p(c) = 2 + 5 - 4 - 3 = 0$

	2	5	-4	-3
1	2	7	3	0

$f(x) = (x-1)(2x^2 + 7x + 3) = (x-1)(2x+1)(x+3)$

Actual zeros are: $\left\{ 1, -\frac{1}{2}, -3 \right\}$

2. Divide the polynomial $6x^5 - 5x^4 + 10x^3 - 15x^2 - 19$ by the polynomial $2x^2 - x + 3$.

See the solution in WS-5-2.

3. Use Descarte's Rule of Signs to determine the possible numbers of positive and negative real zeros of each of the following polynomials:

   (a) $f(x) = 2x^3 + 3x^2 - 14x - 21$

   (b) $g(x) = 3x^3 - 10x^2 + \frac{51}{4}x - \frac{13}{4}$

(a) $f(x) = 2x^3 + \underbrace{3x^2 - 14x}_{\text{one change in sign}} - 21$

$f(-x) = \underbrace{-2x^3 + 3x^2}_{\text{one change in sign}} + \underbrace{14x - 21}_{\text{one change in sign}}$

Therefore, f(x) has one positive real zero and 0 or 2 negative real zeros.

(b) $g(x) = \underbrace{3x^3 - 10x^2}_{\text{one}} + \underbrace{\frac{51}{4}x - \frac{13}{4}}_{\text{one}}$

No negative real zeros.

$g(-x) = -3x^3 - 10x^2 - \frac{51}{4}x - \frac{13}{4}$

Therefore, g(x) has 0 or 2 positive real zeros.

4. Use synthetic division to identify lower and upper bounds of the real zeros of the polynomial $f(x) = 2x^3 + 3x^2 - 14x - 21$.

$a_3 = 2$ $\quad \pm 1, \pm 2$

$a_0 = 21$ $\quad \pm 1, \pm 3, \pm 7$

	2	3	-14	-21
2	2	7	0	-19

	2	3	-14	-21
3	2	9	13	18
	>0	>0	>0	>0

Hence, x=3 is an upper bound.

Let $c = -2$

	2	3	-14	-21
-2	2	-1	-11	1

	2	3	-14	-21
-4	2	-5	6	-45
	>0	<0	>0	<0

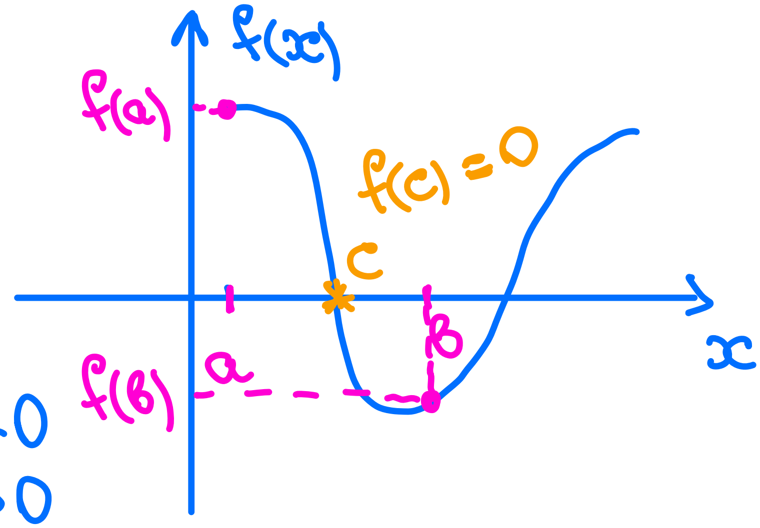
Hence, $x = -4$ is a lower bound.

All zeros of f lie in the interval $[-4, 3]$.

5. (a) Show that $f(x) = x^3 + 3x - 7$ has zeros between 1 and 2.
   (b) Find an approximation of the zero to the nearest tenth.

Use <u>Intermidiate Value Theorem</u>

(a) Let $a = 1$
$b = 2$



Compute;

$f(a) = f(1) = 1 + 3 - 7 = -3 < 0$

$f(b) = f(2) = 8 + 6 - 7 = 7 > 0$

Since $f(a) < 0$ and $f(b) > 0$, then there exists at least one $x = c$ s.t.

$1 < c < 2$ and $f(c) = 0$.

That is, there exists at least one zero of f that lies between 1 and 2.

(b) $f(1.5) = 0.875$

$f(1.4) = -0.056$

Hence, c lies between 1.5 and 1.4.

$f(1.45) = 0.398625$.

Hence, c lies between 1.45 and 1.4.

The value of our zero, rounded to the nearest tenth, is $\boxed{1.4}$.