Section 1.5. Complex numbers

1. The imaginary unit "i" and its properties.
2. Basic operations with complex numbers
3. Roots and complex numbers.

1. $a \neq 0$

Find x such that

$$x^2 = a$$

$$\Downarrow$$

$$x = \pm \sqrt{a}$$

If $a \geq 0$, then $\pm\sqrt{a}$ are two real solutions

If $a < 0$, then there are no real solutions.

Def. The imaginary unit "i" is defined as $i = \sqrt{-1}$, or, in other words,

$$i^2 = -1.$$

Def. If a is a positive real number,

$$\sqrt{-a} = i\sqrt{a}.$$

Note, that $(i\sqrt{a})^2 = i^2 \cdot a = -a.$

Example

- $\sqrt{-16} = \sqrt{i^2 4^2} = 4i$
- $(-i)^2 = i^2 = -1$
- $i^3 = i^2 \cdot i = -i$
- $i^4 = i^2 \cdot i^2 = 1$
- $i^{28} = (i^2)^{14} = (-1)^{14} = 1$

$28 : \underline{4} = 7$ and the remainder $= 0$,

Hence, $i^{28} = 1$

Def. For $a, b \in \mathbb{R}$:

$$z = a + ib \in \mathbb{C}$$

$\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R} \}$ – the set of complex numbers

a is a real part $\underline{\operatorname{Re} z}$

b is an imaginary part $\underline{\operatorname{Im} z}$.

2. Operations with complex numbers.

① Addition:

$z_1 = a + bi$
$z_2 = c + di$

$$z_1 + z_2 = (a+c) + (b+d)i$$

② Subtraction:

$z_1 = a + bi$
$z_2 = c + di$

$$z_1 - z_2 = (a-c) + (b-d)i$$

③ Multiplication:

$$z_1 \cdot z_2 = (a+bi)(c+di) = ac + adi + bci + bd \cdot i^2$$

$$= (ac - bd) + (ad + bc)i$$

Def. Given any complex number $a+bi$, the complex number $a-bi$ is called its complex conjugate.

Very useful property:

$$(a+bi)(a-bi) = a^2 + b^2$$

④ Division:

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} =$$

$$= \frac{ac - adi + bci - bd\,i^2}{c^2 + d^2} =$$

$$= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

Example

$$\frac{2+3i}{3-i} = \frac{(2+3i)(3+i)}{(3-i)(3+i)} = \frac{6+2i+9i-3}{9+1} =$$

$$= \frac{11i + 3}{10} = \frac{3}{10} + \frac{11}{10}i$$

$\boxed{3.}$

$$\left.\begin{array}{l} a \geq 0: \quad \sqrt{a} \in \mathbb{R}_+ \\ \sqrt{-a} = i\sqrt{a} \end{array}\right\} \text{principal square roots}$$

We distinguish them from $-\sqrt{a}$ and $-i\sqrt{a}$.

Caution:

- if $\sqrt{a}, \sqrt{b} \in \mathbb{R}$: $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$

  $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

- if $\sqrt{a}, \sqrt{b} \in \mathbb{C}$, then the properties do not necessarily hold.

$$\sqrt{(-9)(-4)} = \sqrt{36} = 6$$

$$\sqrt{-9}\,\sqrt{-4} = (3i)(4i) = -6$$

Example $(2-\sqrt{-3})^2 = 4 - 4\sqrt{-3} + (-3) =$

$$= 1 - 4\sqrt{-3} = 1 - 4i\sqrt{3}.$$