THEORETICAL PART: Solutions

Models of population growth

The population growth is described by exponential law:

$$P(t) = P_0 a^t,$$

where $P(t)$ is the size of population at time t, P_0 is an initial population (population at time $t = 0$), a is the growth rate of the population.

Models of Radioactive Decay

The radioactive decay is modeled by

$$A(t) = A_0 a^t,$$

where $A(t)$ represents the amount of a given substance at time t, A_0 is the amount at time $t = 0$, and a is a number between 0 and 1.

Compound Interest and the Number e

Formula: Compound Interest Formula

An investment of P dollars, compounded n times per year at an annual interest rate of r, has a value after t years of

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Definition (The number e):

The number e is defined as the value of $\left(1 + \frac{1}{m}\right)^m$ as $m \to \infty$.

$$e \approx 2.718281$$

Formula: Continuous Compounding Formula

An investment of P dollars, compounded continuously at an annual interest rate of r, has a value after t years of

$$A(t) = Pe^{rt}.$$

Definition:

Exponential regression can be used to fit an exponential curve to points that we suspect exhibit exponential behaviour.

The graph of the function

$$f(x) = ab^x$$

models the given data well.

Definition:

Logistic curves are a family of curves based on exponential functions that are designed to model behaviour often seen in biology, ecology, population studies, etc.

A logistic function can be written in the form:

$$f(x) = \frac{c}{1 + ae^{-bx}},$$

where a, b, c are positive constants.

Logistic regression is the process of using an algorithm to fit a logistic curve to a given collection of data.

PRACTICAL PART:

1. A biologist is culturing bacteria in a Petri dish. She begins with 1000 bacteria, and supplies sufficient food so that for the first five hours the bacteria population grows exponentially, doubling every hour.

   (a) Find a function that models the population growth of this bacteria culture.

   (b) Determine when the population reaches 16 000 bacteria.

   (c) Calculate the population two and half hours after the scientists begins.

(a) $P(t) = P_0 a^t$

$P_0 = 1000$

$P(1) = 2000$

$2000 = 1000 \cdot a^1$

$\boxed{a = 2}$

$\boxed{P(t) = 1000 \cdot 2^t}$ t (hours)

(b) $16\cancel{000} = 1\cancel{000} \cdot 2^t$

$16 = 2^t$

$2^4 = 2^t$

$t = 4$ hours

At $t = 4$ hours we have 16 000 bacteria.

(c) $P(2.5) = 1000 \cdot 2^{2.5} \approx 5657$ bacteria

2. Determine the base a so that the function $A(t) = A_0 a^t$ accurately describes the decay of carbon-14 as a function of t years.

$A(t) = A_0 a^t$

For $t = 5730$ years we have that

$A(5730) = A_0 a^{5730} = \frac{A_0}{2}$

$a^{5730} = \frac{1}{2}$

$a = \left(\frac{1}{2}\right)^{1/5730} \approx 0{,}9999$

Thus, $A(t) = A_0 \cdot (0{,}9999)^t$

3. Sandy invests \$10 000 in a savings account earning 4.5% annual interest compounded quarterly. What is the value of her investment after three and a half years?

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

$P = \$10\,000$ $t = 3.5$ years

$r = 0.045$

$n = 4$

Thus, $A(t) = 10\,000\left(1 + \frac{0.045}{4}\right)^{4 \cdot 3.5} \approx$

$\approx \boxed{\$\,11\,695.52}$

4. If Sandy has the option of investing her \$10 000 in a continuously compounded account earning 4.5% annual interest, what will be the value of her account in three and a half years?

$$A(t) = P e^{rt}$$

$P = \$10\,000$

$r = 0.045$

$t = 3.5$ years

$$A(t) = 10000\, e^{0.045 \cdot 3.5} \approx$$

$$\approx \boxed{\$\,11\,705.81}$$