Section 8.3. Product-Sum Identities

1. Double-angle identities.
2. Power-reducing identities.
3. Half-angle identities.
4. Product-to-sum and sum-to-product identities.

$\boxed{1.}$

Identities (Double-angle identities)

- $\sin(2u) = 2\sin u \cdot \cos u$
- $\cos(2u) = \cos^2 u - \sin^2 u =$
  $= 1 - 2\sin^2 u$
- $\tan(2u) = \dfrac{2\tan u}{1-\tan^2 u}$

Example

$\cos x = -\frac{2}{\sqrt{5}}$

$\sin x > 0$

$$\cos(2x) = 1 - 2\sin^2(x)$$

$$\cos^2 x + \sin^2 x = 1$$

$$\sin^2 x = 1 - \frac{4}{5} = \frac{1}{5}$$

Therefore, $\cos(2x) = 1 - \frac{2}{5} = \frac{3}{5}$.

$\blacktriangledown$

$\boxed{2.}$

Identities (Power-reducing identities)

- $\sin^2 x = \frac{1}{2}\left(1 - \cos(2x)\right)$
- $\cos^2 x = \frac{1}{2}\left(1 + \cos(2x)\right)$
- $\tan^2 x = \dfrac{1 - \cos(2x)}{1 + \cos(2x)}$

$\boxed{3.}$

Identities (Half-angle identities)

- $\sin\left(\frac{x}{2}\right) = \pm\sqrt{\dfrac{1 - \cos x}{2}}$
- $\cos\left(\frac{x}{2}\right) = \pm\sqrt{\dfrac{1 + \cos x}{2}}$

- $\tan\left(\frac{x}{2}\right) = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$

Proof.

$$\tan^2\left(\frac{x}{2}\right) = \frac{1-\cos\left(2\left(\frac{x}{2}\right)\right)}{1+\cos\left(2\left(\frac{x}{2}\right)\right)}$$

$$\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$\tan\left(\frac{x}{2}\right) = \pm\sqrt{\left(\frac{1-\cos x}{1+\cos x}\right)\left(\frac{1-\cos x}{1-\cos x}\right)}$$

$$\tan\left(\frac{x}{2}\right) = \pm\frac{|1-\cos x|^{>0}}{|\sin x|}$$

↑ we need to consider all sign cases here.

$$\tan\left(\frac{x}{2}\right) = \frac{1-\cos x}{\sin x}$$

▼

$\boxed{4.}$ Identities (Product-to-Sum identities)

- $\sin x \cdot \cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$
- $\cos x \cdot \sin y = \frac{1}{2}\left(\sin(x+y) - \sin(x-y)\right)$
- $\sin x \cdot \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$

- $\cos x \cdot \cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$

Example

Express $\sin^5 x$ in terms containing only first powers of sine.

Solution

$$\sin^5 x = \frac{1}{4}\left(\frac{3\sin x}{2} - 2\cos(2x)\sin x + \right.$$
$$\left. + \frac{\cos(4x)\sin x}{2}\right) =$$
$$= \frac{1}{4}\left(\frac{3\sin x}{2} - 2\left(\frac{1}{2}\left(\sin(3x) - \sin(x)\right)\right) + \frac{1}{2}\cdot\right.$$
$$\left.\cdot\left(\frac{1}{2}\left(\sin(5x) - \sin(3x)\right)\right)\right) = \frac{5}{8}\sin x - \frac{5}{16}\sin(3x) +$$
$$+ \frac{1}{16}\sin(5x)$$

▼

Identities (Sum-to-Product Identities)

- $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
- $\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$
- $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
- $\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$