Section 5.1. Polynomial functions and polynomial inequalities

1. Zeros of polynomial functions and solutions of polynomial equations.
2. Graphing factored polynomial functions.
3. Solving polynomial inequalities.

$\boxed{1.}$

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

$n \geq 0$

$a_n, a_{n-1}, \dots, a_0$ are constants ($\mathbb{C}$ or $\mathbb{R}$)

$a_n \neq 0$

$p(x)$ is a generic polynomial

Def.

The number c ($\mathbb{C}$ or $\mathbb{R}$) is a zero of the polynomial function $p(x)$ if $p(c) = 0$.

In other words, c is a root of

the polynomial or a solution of the equation $p(x) = 0$.

Def. (Polynomial equations)

A polynomial equation in variable x is an equation that can be written in the form

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0,$$

$n \geq 0$

$a_n, \dots, a_0$ are constants

$a_n \neq 0$ (equation is of a degree n; a_n is a leading coefficient)

Example

- $6x^2 - x^3 = 12 + 5x$

  $x = 4$

  $6 \cdot 4^2 - 4^3 = 12 + 5 \cdot 4$

  $6 \cdot 16 - 64 = 12 + 20$

  $32 = 32$ ✓

- $\dfrac{x}{1-i} = 3x^2$

  $x = 0$

  $0 = 3 \cdot 0^2 = 0$ ✓

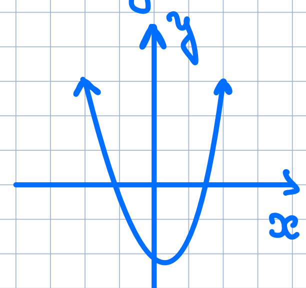
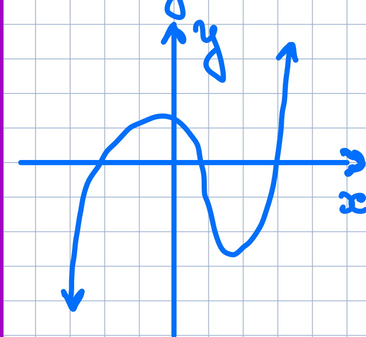
$\boxed{2.}$ Let us consider

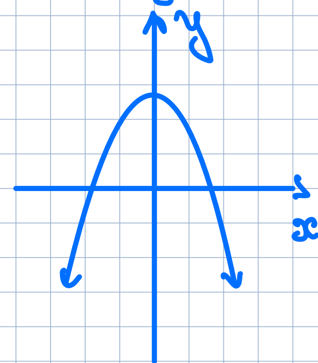
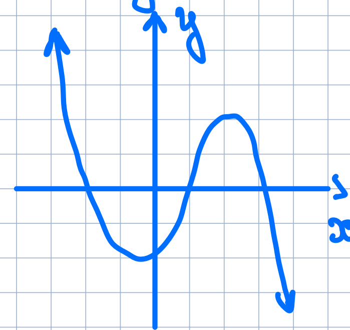
$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0,$$

where $a_n, \ldots, a_0 \in \mathbb{R}$

Properties (Behaviour of polynomials as $x \to \pm\infty$)

Given a polynomial function $p(x)$ with degree n, the behaviour of $p(x)$ as $x \to \pm\infty$ can be determined from the leading term $a_n x^n$.

	n is even	n is odd
a_n is positive	as $x \to -\infty$, $p(x) \to \infty$ as $x \to \infty$, $p(x) \to \infty$ The graph rises to the left and rises to the right  	as $x \to -\infty$, $p(x) \to -\infty$ as $x \to +\infty$, $p(x) \to \infty$ The graph falls to the left and rises to the right 

a_n is negative	as $x \to -\infty$, $p(x) \to -\infty$ as $x \to +\infty$, $p(x) \to -\infty$ The graph falls to the left and falls to the right.  	as $x \to -\infty$, $p(x) \to +\infty$ as $x \to +\infty$, $p(x) \to -\infty$ The graph rises to the left and falls to the right. 

Example

$f(x) = -x(2x+1)(x-2)$

$f(x) = 0$ at $x = 0$ or $x = -\frac{1}{2}$ or $x = 2$

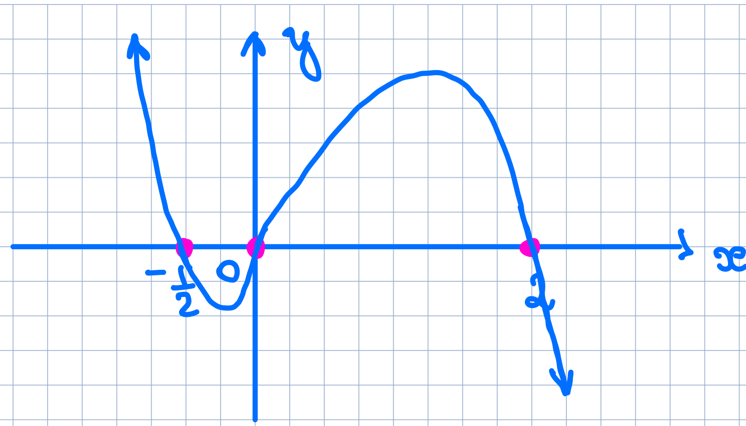
$f(x) = -x(2x^2 - 4x + x - 2) = -2x^3 + 4x^2 - x^2 + 2x$

$n = 3$ is an odd number

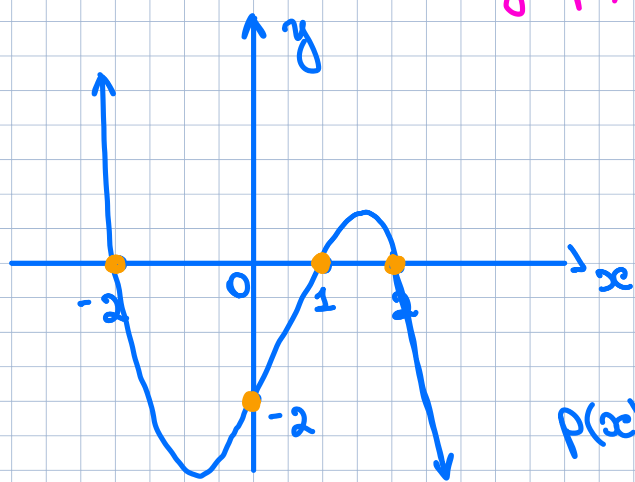
$a_n = -2 < 0$

Therefore, as $x \to +\infty$, $p(x) \to -\infty$

$x \to -\infty$, $p(x) \to +\infty$



Example (Constructing a polynomial function from a graph)



$$p(x) = a(x-x_1)(x-x_2)(x-x_3)$$

$$p(x) = a(x-1)(x-2)(x+2)$$

a - ?

But we know that $p(0) = -2$.

Therefore,

$$a(-1)(-2)\cdot 2 = -2$$

$$a = -\frac{1}{2}$$

Answer: $p(x) = -\frac{1}{2}(x-1)(x-2)(x+2)$.

$\boxed{3.}$

Def.

A <u>polynomial inequality</u> is any inequality that can be written in the form

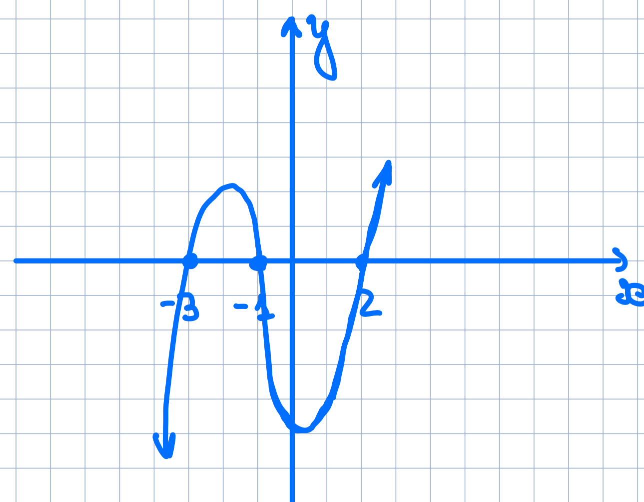
$$p(x)<0,\quad p(x)\leq 0,\quad p(x)>0,\quad p(x)\geq 0,$$

where $p(x)$ is a polynomial function.

Example

- $(x+3)(x+1)(x-2)<0$

$f(x)=(x+3)(x+1)(x-2)$



We have that $f(x)<0$ for

$x\in(-\infty;-3)\cup(-1,2)$.

Procedure (Solving Polynomial Inequalities using the Sign-test method)

To solve a polynomial inequality $p(x)<0$, $p(x)\leq 0$, $p(x)>0$, or $p(x)\geq 0$, we perform the following steps:

Step 1: Find the real zeros of $p(x)$. Equivalently, find the real solutions of $p(x) = 0$

Step 2: Place the zeros on a number line, splitting it into intervals.

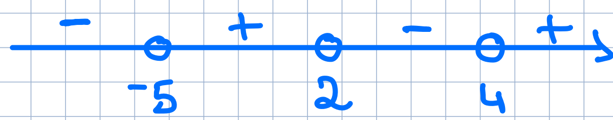
Step 3: Within each interval, select a test point and evaluate p at that number. If the result is positive, then $p(x) > 0$ for all x in the interval. If the result is negative, then $p(x) < 0$ for all x in the interval.

Step 4: Write the solution set, consisting of all of the intervals that satisfy the given inequality. If the inequality is not strict (uses $\leq$ or $\geq$), then the zeros are included in the solution set as well.

Example

$$(x-2)(x+5)(x-4) > 0$$

Step 1 + Step 2:



Step 3:

- $x = -6$ "−"
- $x = 0$ "+"
- $x = 3$ "−"
- $x = 5$ "+"

Step 4:

Therefore, we get

$$(-5, 2) \cup (4, +\infty)$$