

THEORETICAL PART:**Definition (Rational Functions):**

A **rational function** is a function that can be written in the form

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$. Even though q is not allowed to be identically zero, there will often be values of x for which $q(x)$ is zero, and at these values the function is undefined. Consequently, the **domain of f** consists of all real numbers except those for which $q(x) = 0$.

Definition (Vertical Asymptotes):

The vertical line $x = c$ is a **vertical asymptote** of a function f if $f(x)$ increases in magnitude without bound as x approaches c . The graph of a rational function cannot intersect a vertical asymptote.

Definition (Horizontal Asymptotes):

The horizontal line $y = c$ is a **horizontal asymptote** of a function f if $f(x)$ approaches the value c as $x \rightarrow -\infty$ or as $x \rightarrow \infty$. The graph of a rational function may intersect a horizontal asymptote near the origin, but will eventually approach the asymptote from one side only as x increases in magnitude.

Definition (Oblique Asymptotes):

A non-vertical, non-horizontal line may also be an asymptote of a function f . Again, the graph of a rational function may intersect an oblique asymptote near the origin, but will eventually approach the asymptote from one side only as $x \rightarrow \infty$ or $x \rightarrow -\infty$.

Definition (Asymptote Notation):

The notation $x \rightarrow c^-$ is used when describing the behaviour of a graph as x approaches the value c from the left (the negative side). The notation $x \rightarrow c^+$ is used when describing behaviour as x approaches c from the right (the positive side). The notation $x \rightarrow c$ is used when describing behaviour that is the same on both sides of c .

Theorem (Equations for Vertical Asymptotes):

If the rational function

$$f(x) = \frac{p(x)}{q(x)}$$

has been written in reduced form (so that p and q have no common factors), the vertical line $x = c$ is a **vertical asymptote** of f if and only if c is a zero of the polynomial q . In other words, f has vertical asymptotes at the x -intercepts of q .

Theorem (Equations for Horizontal and Oblique Asymptotes):

Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function, where p is an n -th degree polynomial with leading coefficient a_n , q is an m -th degree polynomial with leading coefficient b_m , and $p(x)$ and $q(x)$ have no

common factors other than constants. Then the asymptotes of f are found as follows:

1. If $n < m$, the horizontal line $y = 0$ (the x -axis) is the **horizontal asymptote** for f .
2. If $m = n$, the horizontal line $y = \frac{a_n}{b_m}$ is the **horizontal asymptote** for f .
3. If $n = m + 1$, the line $y = g(x)$ is an **oblique asymptote** for f , where g is the quotient polynomial obtained by dividing p by q .
4. If $n > m + 1$, there is **no** straight line **horizontal** or **oblique asymptote** for f .

PROCEDURE (Graphing Rational Functions):

Given a rational function f :

- Step 1. Factor the denominator in order to determine the domain of f . Any points excluded from the domain correspond to holes or vertical asymptotes in the eventual graph.
- Step 2. Factor the numerator as well and cancel any common factors. Zeros of the numerator and denominator arising from common linear factors are the x -coordinates of holes in the eventual graph.
- Step 3. Examine the remaining linear factors in the denominator to determine the equations for any vertical asymptotes.
- Step 4. Compare the degrees of the numerator and denominator to determine if there is a horizontal or oblique asymptote. If so, find its equations.
- Step 5. Determine the y -intercept if 0 is in the domain of f .
- Step 6. Determine the x -intercepts, if there are any, by setting the numerator of the reduced fraction equal to 0.
- Step 7. Plot enough points to determine the behaviour of f between x -intercepts and between vertical asymptotes.

Definition (Rational Inequalities):

A **rational inequality** is any inequality that can be written in the form:

$$f(x) < 0, \quad f(x) > 0, \quad f(x) \geq 0, \quad f(x) \leq 0,$$

where $f(x)$ is a rational function.

PROCEDURE (Solving Rational Inequalities Using the Sign-Test Method):

To solve a rational inequality $f(x) < 0$, $f(x) > 0$, $f(x) \geq 0$, $f(x) \leq 0$, where the rational function $f(x) = \frac{p(x)}{q(x)}$ is in reduced form:

- Step 1. Find the real zeros of the numerator $p(x)$. These values are the **zeros** of f .
- Step 2. Find the real zeros of the denominator $q(x)$. These values are the locations of the **vertical asymptotes** of f .
- Step 3. Place the values from Step 1 and Step 2 on a number line, splitting it into intervals.
- Step 4. Within each interval, select a **test point** and evaluate f at that number. If the result is positive, then $f(x) > 0$ for all x in the interval. If the result is negative, then $f(x) < 0$ for all x in the interval.
- Step 5. Write the **solution set**, consisting of all of the intervals that satisfy the given inequality. If the inequality is not strict (uses \geq or \leq), then the zeros of p are included in the solution set as well. The zeros of q are never included in the solution set.

PRACTICAL PART:

1. Find the domains and the equations for the vertical asymptotes of the following functions:

(a) $f(x) = \frac{32}{x+2}$

(b) $g(x) = \frac{x^2 + 1}{x^2 + 2x - 15}$

(c) $h(x) = \frac{x^2 - x}{x - 1}$

2. Find the equation for the horizontal or oblique asymptote of the following functions:

$$f(x) = \frac{x^2 + 1}{x^2 + 2x - 15}, \quad g(x) = \frac{x^3 + x^2 + 2x + 2}{x^2 + 9}$$

3. Sketch the graphs of the following rational functions:

$$f(x) = \frac{x^x - x}{x - 1}, \quad g(x) = \frac{x^2 + 1}{x^2 + 2x - 15}$$

4. Solve the rational inequality

$$\frac{x^2 + 1}{x^2 + 2x - 15} > 0.$$

5. Solve the rational inequalities $\frac{x}{x+2} < 3$ and $\frac{x}{x+2} \leq 3$.