THEORETICAL PART:

Theorem (The rational zero theorem):

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a polynomial with integer coefficients with $a_n \neq 0$, then any rational zero of f must be of the form $\dfrac{p}{q}$, where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

Theorem (Descarte's Rule of Signs):

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a polynomial with real coefficients, and assume $a_n \neq 0$. A **variation in sign** of f is a change in the sign of one coefficient of f to the next, either from positive to negative or vice versa.

1. The number of **positive real zeros** of f is either the number of variations in sign of $f(x)$ or is less than this number by a positive even integer.
2. The number of **negative real zeros** of f is either the number of variations in sign of $f(-x)$ or is less than this number by a positive even integer.

Theorem (Upper and Lower bounds of zeros):

Let $f(x)$ be a polynomial with real coefficients, a positive leading coefficient, and degree ≥ 1. Let a be a negative number and b be a positive number. Then:

1. No real zero of f is larger than b if the last row in the synthetic division of $f(x)$ by $x - b$ contains no negative numbers. That is, b is an upper bound of the zeros if the quotient and remainder have no negative coefficients when $f(x)$ is divided by $x - b$.
2. No real zero of f is smaller than a if the last row in the synthetic division of $f(x)$ by $x - a$ has entries that alternate in sign (0 can count as either positive or negative).

Theorem (The Intermediate Value Theorem):

Assume that $f(x)$ is a polynomial with real coefficients, and that a and b are real numbers with $a < b$. If $f(a)$ and $f(b)$ differ in sign, then there is at least one point c such that $a < c < b$ and $f(c) = 0$. That is at least one zero of f lies between a and b.

PRACTICAL PART:

1. For the polynomial function $f(x) = 2x^3 + 5x^2 - 4x - 3$ list all of the potential rational zeros. Then write the polynomial in factored form and identify the actual zeros.

2. Divide the polynomial $6x^5 - 5x^4 + 10x^3 - 15x^2 - 19$ by the polynomial $2x^2 - x + 3$.

3. Use Descarte's Rule of Signs to determine the possible numbers of positive and negative real zeros of each of the following polynomials:

   (a) $f(x) = 2x^3 + 3x^2 - 14x - 21$

   (b) $g(x) = 3x^3 - 10x^2 + \frac{51}{4}x - \frac{13}{4}$

4. Use synthetic division to identify lower and upper bounds of the real zeros of the polynomial $f(x) = 2x^3 + 3x^2 - 14x - 21$.

5. (a) Show that $f(x) = x^3 + 3x - 7$ has zeros between 1 and 2.
   (b) Find an approximation of the zero to the nearest tenth.