Solutions

THEORETICAL PART:

Definition 1.

The **Imaginary unit i** is defined as $i = \sqrt{-1}$. In other words, i has the property that its square is -1:

$$i^2 = -1.$$

Definition 2.

If a is a positive real number, $\sqrt{-a} = i\sqrt{a}$.

Definition 3.

For any two real numbers a and b, the sum $a + bi$ is a **complex number**. The collection $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ is called the set of complex numbers. The number a is called the **real part** of $a + bi$, and the number b is called the **imaginary part**. If $a = 0$, then we obtain simply a real number. If $b = 0$, then we obtain a pure imaginary number.

Simplifying Complex Expressions:

- Add, subtract, or multiply the complex numbers, as required, by treating every complex number $a + bi$ as a polynomial expression.
- Complete the simplification by using the fact that $i^2 = -1$.

Definition 4. Given any complex number $a + bi$, the complex number $a - bi$ is called its **complex conjugate**.

A very useful property:

$$(a + bi)(a - bi) = a^2 + b^2$$

Definition 5 (Principal square roots). Given $a \in \mathbb{R}$, $a > 0$, we have:

$$\sqrt{a} \in \mathbb{R},\ \sqrt{a} > 0$$

$$\sqrt{-a} = i\sqrt{a}.$$

Caution: If a and b are both real numbers, then:

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

As for complex numbers, first simplify any square roots of negative numbers by rewriting them as pure imaginary numbers.

PRACTICAL PART:

1. Simplify the following expressions:

   (a) $\sqrt{-16} =$ $\sqrt{-1} \cdot \sqrt{16} = i \cdot 4 = 4i$

   (b) $\sqrt{-8} =$ $\sqrt{-1} \cdot \sqrt{8} = i \cdot 2\sqrt{2} = 2i\sqrt{2}$

   (c) $i^3 =$ $-i$

   (d) $i^8 =$ 1

   (e) $i^{102} =$ -1

   $102 = 25 \cdot 4 + 2$

   reminder

2. Simplify the following complex expressions:

   (a)

   $$(4 + 3i) + (-5 + 7i) =$$

   $$= (4-5) + (3+7)i = \boxed{-1 + 10i}$$

   (b)

   $$(3 + 2i)(-2 + 3i) =$$

   $$= -6 + 9i - 4i + 6 \cdot i^2 = -6 + 5i - 6 = \boxed{-12 + 5i}$$

   (c)

   $$(2 - 3i)^2 =$$

   $$= 4 - 12i + 9i^2 = 4 - 12i - 9 = \boxed{-5 - 12i}$$

3. Simplify the following expressions:

(a)

$$\frac{2+3i}{3-i} = \frac{(2+3i)(3+i)}{(3-i)(3+i)}$$

$$= \frac{6+2i+9i-3}{3^2-i^2} = \frac{3+11i}{9+1} =$$

$$= \frac{3+11i}{10} = \boxed{\frac{3}{10}+\frac{11}{10}i}$$

(b)

$$(4-3i)^{-1} =$$

$$= \frac{1}{(4-3i)} = \frac{4+3i}{(4-3i)(4+3i)} = \frac{4+3i}{16-(9i)^2} =$$

$$= \frac{4+3i}{16+9} = \frac{4+3i}{25} = \boxed{\frac{4}{25}+\frac{3}{25}i}$$

4. Simplify the following expressions:

(a)

$$(2-\sqrt{-3})^2 = (2-i\sqrt{3})^2 =$$

$$= 4-4i\sqrt{3}+i^2\cdot 3 = 4-4i\sqrt{3}-3 =$$

$$= \boxed{1-4i\sqrt{3}}$$

(b)

$$\frac{\sqrt{4}}{\sqrt{-4}} = \frac{\sqrt{4}}{i\sqrt{4}} = \frac{2}{2i} = \frac{1}{i} =$$

$$= \frac{i}{i^2} = \frac{i}{-1} = \boxed{-i}$$