Section 1.1. Real Numbers and Algebraic Expressions

1. Common subsets of real numbers
2. The real number line
3. Order on the real number line
4. Set-builder notation and interval notation
5. Basic set operations and Venn diagrams
6. Absolute value and distance on the real number line

1. Def.

- The set of natural numbers:

  $$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$$

  (this set is countable)

- The set of whole numbers:

  $$\{0, 1, 2, 3, 4, \dots\}$$

- The set of integer numbers:

  $$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

- The set of <u>rational</u> numbers:

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{Z},\ q \neq 0 \right\}$$

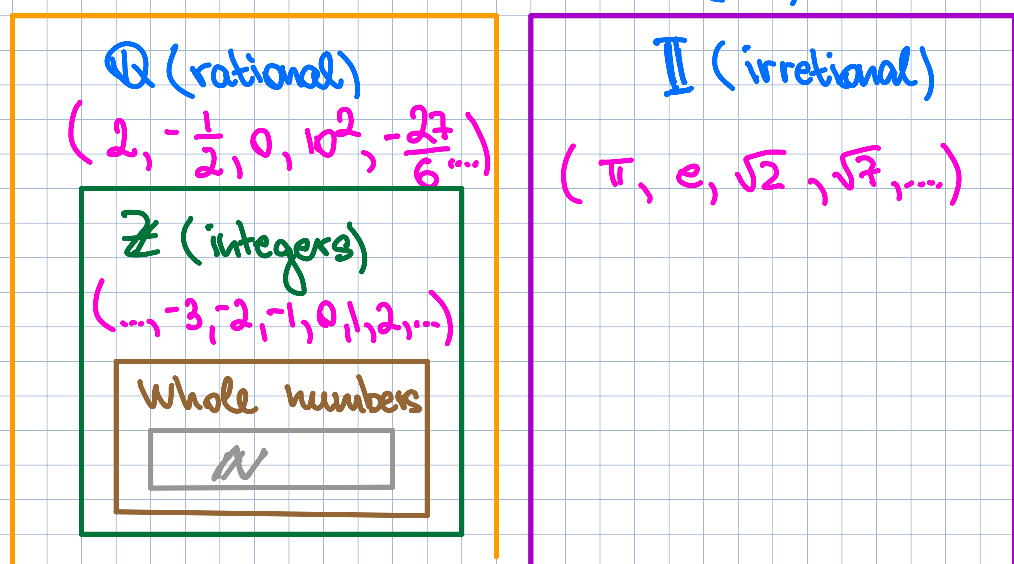
- The set of <u>irrational</u> numbers:

$\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ (all real numbers except of rational numbers)

- The set of <u>real</u> numbers:

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$$

Real numbers ($\mathbb{R}$)



<u>Example</u>

$A = \{ 0, \sqrt{2}, \pi, 10^5, 1.\bar{6}, -7.5 \}$

$\mathbb{Q}$: $0,\ 10^5,\ 1.\bar{6},\ -7.5$

$\mathbb{R}$: A

$\mathbb{I}$: $\sqrt{2},\ \pi$

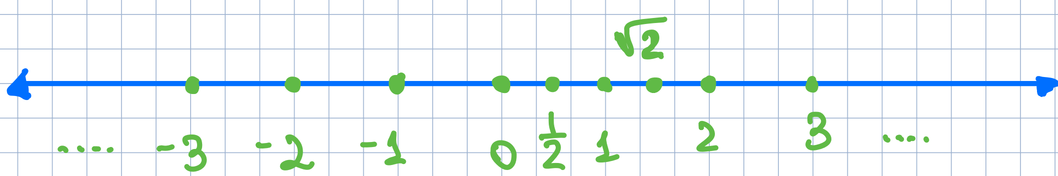
$\mathbb{N}$: 10^5

Whole numbers: $0,\ 10^5$

$\mathbb{Z}$: $0,\ 10^5$.

$\boxed{2.}$

The real number line



$\boxed{3.}$ **Def.** (inequality symbols (order))

$a < b$ "a is less than b"

$a \leq b$ "a is less than or equal to b"

$b > a$ "b is greater than a"

$b \geq a$ "b is greater than or equat to a"

$\boxed{4.}$ Set - builder notation and interval notation.

Def.

$$A = \{x \mid x \text{ has property } P\}.$$

Such that

Examples

- $\{x \mid x \text{ is an even integer}\}$
- $\{2n \mid n \text{ is an integer}\}$
- $\{x \in \mathbb{Z} \mid -2 \leq x \leq 1\} = \{-2, 0\}$.

Def. $\{\}$, $\emptyset$ – empty set/null set

Def. (Interval notations)

$(a,b) = \{x \mid a < x < b\}$ – an open interval

$[a,b] = \{x \mid a \leq x \leq b\}$ – a closed interval

$(a,b] = \{x \mid a < x \leq b\}$ – a half-open

$(-\infty, b) = \{x \mid x < b\}$

$[a,\infty) = \{x \mid x \geq a\}$

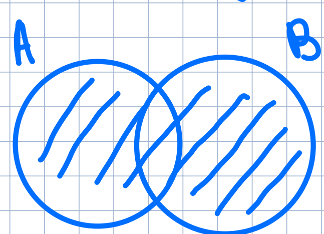
$-\infty, \infty, +\infty$ are just symbols!!!

$\boxed{5.}$ Basic set operations and Venn Diagrams

Notation: "$\in$" is an element of

Venn Diagrams (1880s)

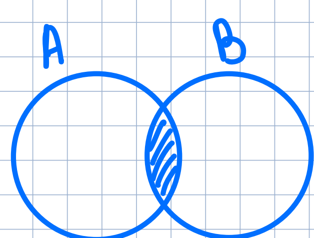
1.



Union

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

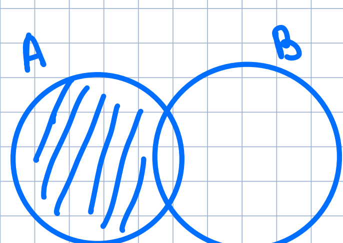
2.



Intersection

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

3.



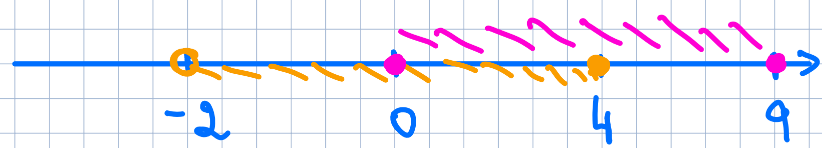
Difference

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$$

Examples

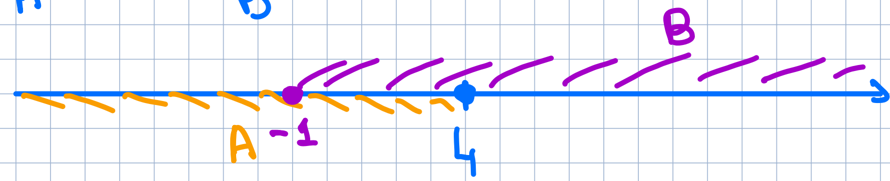
(a) $(-2,4] \cup [0,9] = (-2,9]$

$\quad\;\; A \qquad\qquad B$



(b) $(-\infty,4] \cap (-1,\infty) = [-1,4]$

$\quad\;\; A \qquad\qquad B$

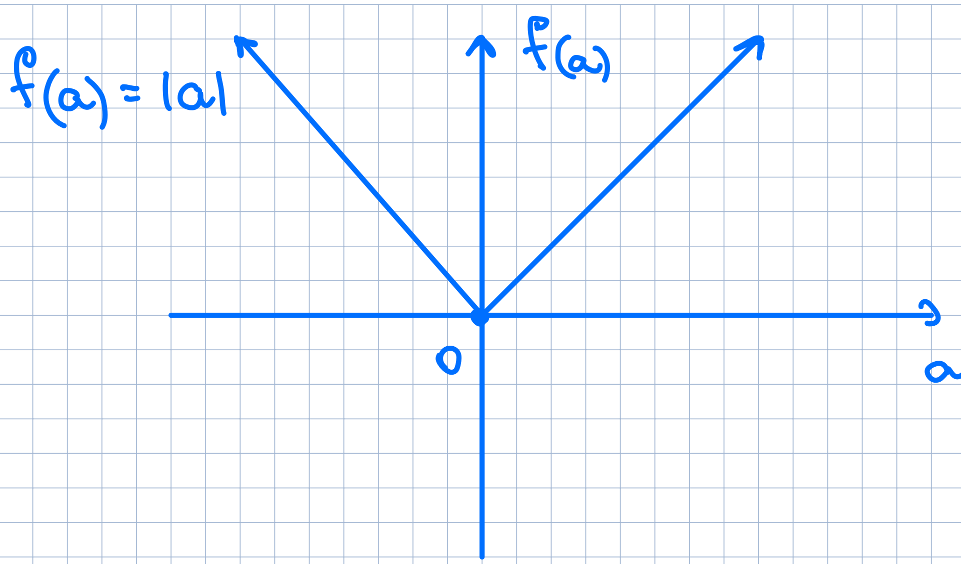


$\boxed{6.}$ **Absolute value and distance on the real number line.**

Def. The <u>absolute value</u> of a real number a, denoted as $|a|$, is defined as follows.

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$$

absolute value of a $\equiv$ magnitude of a



Def. Given two real numbers a and b, the distance between them is defined to be $|a-b|$. In particular,

$$|a-0| = |a|.$$

The distance is symmetric:

$$|a-b| = |b-a|$$

Examples

(a) $|17-3| = |3-17| = 14.$

(b) $|-\pi| = \pi$

(c) $\dfrac{|7|}{7} = \dfrac{7}{7} = 1$

Properties of Absolute Value

$a, b \in \mathbb{R}$

1. $|a| \geq 0$

2. $|-a| = |a|$

$|-a| = |-1 \cdot a| = |-1| \cdot |a| = |a|$

3. $a \leq |a|$

$a < 0$:

$a \leq |-a| = |a|$

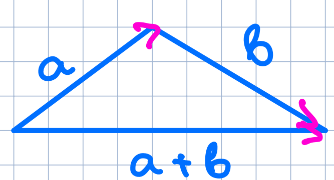
$a > 0$:

$a \leq |a|$

4. $|ab| = |a||b|$

5. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \quad b \neq 0$

6. $|a+b| \leq |a| + |b|$



$\boxed{7.}$ Components and terminology of algebraic expressions.

Example

$$-17x(x^2+4y) + 5\sqrt{x} - 13$$

$$\underbrace{\underbrace{-17}_{\text{coefficient}}\,\underbrace{x(x^2+4y)}_{\text{variable factor}}}_{\text{term 1}} \qquad \underbrace{5\sqrt{x}}_{\text{term 2}} \qquad \underbrace{-13}_{\text{term 3}} \ \text{constant}$$

$\boxed{8.}$ The field properties and their use in algebra.

The set of real numbers forms what is known mathematically as a field.

Field Properties:

Let $a, b, c \in \mathbb{R}$

Closure: $a+b \in \mathbb{R}$ $\quad ab \in \mathbb{R}$

Commutative: $a+b = b+a$ $\quad ab = b \cdot a$

Associative: $a+(b+c) = (a+b)+c$ $\quad a(bc) = (ab)c$

Identity: $a+0 = 0+a = a$ $\quad a \cdot 1 = 1 \cdot a = a$

Inverse: $a+(-a) = 0$ $\quad a \cdot \frac{1}{a} = 1 \ (a \neq 0)$

Distributive: $a(b+c) = ab + ac$

Cancellation Properties:

$A = B \iff A + C = B + C$ Additive cancellation

For $C \neq 0$, $A = B \iff A \cdot C = B \cdot C$ Multiplicative cancellation

Zero-factor property:

$AB = 0 \Rightarrow A = 0$ or $B = 0$