THEORETICAL PART: Solutions

Definition (Rational Functions):

A **rational function** is a function that can be written in the form

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$. Even though q is not allowed to be identically zero, there will often be values of x for which $q(x)$ is zero, and at these values the function is undefined. Consequently, the **domain of** f consists of all real numbers except those for which $q(x) = 0$.

Definition (Vertical Asymptotes):

The vertical line $x = c$ is a **vertical asymptote** of a function f if $f(x)$ increases in magnitude without bound as x approaches c. The graph of a rational function cannot intersect a vertical asymptote.

Definition (Horizontal Asymptotes):

The horizontal line $y = c$ is a **horizontal asymptote** of a function f if $f(x)$ approaches the value c as $x \to -\infty$ or as $x \to \infty$. The graph of a rational function may intersect a horizontal asymptote near the origin, but will eventually approach the asymptote from one side only as x increases in magnitude.

Definition (Oblique Asymptotes):

A non-vertical, non-horizontal line may also be an asymptote of a function f. Again, the graph of a rational function may intersect an oblique asympotote near the origin, but will eventually approach the asymptote from one side only as $x \to \infty$ or $x \to -\infty$.

Definition (Asymptote Notation):

The notation $x \to c^-$ is used when describing the behaviour of a graph as x approaches the value c from the left (the negative side). The notation $x \to c^+$ is used when describing behaviour as x approaches c from the right (the positive side). The notation $x \to c$ is used when describing behaviour that is the same on both sides of c.

Theorem (Equations for Vertical Asymptotes):

If the rational function

$$f(x) = \frac{p(x)}{q(x)}$$

has been written in reduced form (so that p and q have no common factors), the vertical line $x = c$ is a **vertical asymptote** of f if and only if c is a zero of the polynomial q. In other words, f has vertical asymptotes at the x-intercepts of q.

Theorem (Equations for Horizontal and Oblique Asymptotes):

Let $f(x) = \dfrac{p(x)}{q(x)}$ be a rational function, where p is an n-th degree polynomial with leading coefficient a_n, q is an m-th degree polynomial with leading coefficient b_m, and $p(x)$ and $q(x)$ have no common factors other than constants. Then the asymptotes of f are found as follows:

1. If $n < m$, the horizontal line $y = 0$ (the x-axis) is the **horizontal asymptote** for f.
2. If $m = n$, the horizontal line $y = \dfrac{a_n}{b_m}$ is the **horizontal asymptote** for f.
3. If $n = m + 1$, the line $y = g(x)$ is an **oblique asymptote** for f, where g is the quotient polynomial obtained by dividing p by q.
4. If $n > m + 1$, there is **no** straight line **horizontal** or **oblique asymptote** for f.

PROCEDURE (Graphing Rational Functions):

Given a rational function f:

Step 1. Factor the denominator in order to determine the domain of f. Any points excluded from the domain correspond to holes or vertical asymptotes in the eventual graph.

Step 2. Factor the numerator as well and cancel any common factors. Zeros of the numerator and denominator arising from common linear factors are the x-coordinates of holes in the eventual graph.

Step 3. Examine the remaining linear factors in the denominator to determine the equations for any vertical asymptotes.

Step 4. Compare the degrees of the numerator and denominator to determine if there is a horizontal or oblique asymptote. If so, find its equations.

Step 5. Determine the y-intercept if 0 is in the domain of f.

Step 6. Determine the x-intercepts, if there are any, by setting the numerator of the reduced fraction equal to 0.

Step 7. Plot enough points to determine the behaviour of f between x-intercepts and between vertical asymptotes.

Definition (Rational Inequalities):

A **rational inequality** is any inequality that can be written in the form:

$$f(x) < 0, \quad f(x) > 0, \quad f(x) \geq 0, \quad f(x) \leq 0,$$

where $f(x)$ is a rational function.

PROCEDURE (Solving Rational Inequalities Using the Sigh-Test Method):

To solve a rational inequality $f(x) < 0$, $f(x) > 0$, $f(x) \geq 0$, $f(x) \leq 0$, where the rational function $f(x) = \dfrac{p(x)}{q(x)}$ is in reduced form:

Step 1. Find the real zeros of the numerator $p(x)$. These values are the **zeros** of f.

Step 2. Find the real zeros of the denominator $q(x)$. These values are the locations of the **vertical asymptotes** of f.

Step 3. Place the values from Step 1 and Step 2 on a number line, splitting it into intervals.

Step 4. Within each interval, select a **test point** and evaluate f at that number. If the result is positive, then $f(x) > 0$ for all x in the interval. If the result is negative, then $f(x) < 0$ for all x in the interval.

Step 5. Write the **solution set**, consisting of all of the intervals that satisfy the given inequality. If the inequality is not strict (uses $\geq$ or $\leq$), then the zeros of p are included in the solution set as well. The zeros of q are never included in the solution set.

PRACTICAL PART:

1. Find the domains and the equations for the vertical asymptotes of the following functions:

   (a) $f(x) = \dfrac{32}{x+2}$

   (b) $g(x) = \dfrac{x^2+1}{x^2+2x-15}$

   (c) $h(x) = \dfrac{x^2-x}{x-1}$

(a) $x+2 \neq 0$ V.A.: $x=-2$

$x \neq -2$

$\text{Dom}(f) = \mathbb{R} \setminus \{-2\}$

(b) $x^2+2x-15 \neq 0$ V.A.: $x=-5$ and $x=3$

$(x+5)(x-3) \neq 0$

$x \neq -5,\ x \neq 3$

$\text{Dom}(g) = \mathbb{R} \setminus \{-5, 3\}$

(c) $x - 1 \neq 0$

$x \neq 1$

$\mathrm{Dom}(h) = \mathbb{R} \setminus \{1\}$

$h(x) = \dfrac{x\cancel{(x-1)}}{\cancel{(x-1)}} = x$

We have a hole at $x=1$, but $\boxed{\text{no}}$ V.A.

2. Find the equation for the horizontal or oblique asymptote of the following functions:

$$f(x) = \frac{1 \cdot x^2 + 1}{1 \cdot x^2 + 2x - 15}, \quad g(x) = \frac{x^3 + x^2 + 2x + 2}{x^2 + 9}$$

- For $f(x)$:

  $p(x) = x^2 + 1$, $n = 2$

  $q(x) = x^2 + 2x - 15$, $m = 2$

  Hence, we have one H.A.: $y = \frac{1}{1} = 1$

  $\boxed{y = 1}$

  O.A.: None

- For $g(x)$:

  $\deg(p) = 3$

  $\deg(q) = 2$

  H.A.: None

  O.A.:

$$\begin{array}{r|l} x^3 + x^2 + 2x + 2 & x^2 + 9 \\ -(x^3 + 9x) & x + 1 \\ \hline x^2 - 7x + 2 & \\ -(x^2 + 9) & \\ \hline -7x - 7 & \end{array}$$

"$q(x)$"

O.A.: $y = x + 1$

3. Sketch the graphs of the following rational functions:

$$f(x) = \frac{x^2 - x}{x - 1}, \quad g(x) = \frac{x^2 + 1}{x^2 + 2x - 15}$$

- $f(x) = \dfrac{x^2 - x}{x-1}$

<u>Analysis</u>:

① $Dom(f) = \mathbb{R} \setminus \{1\}$

② V.A.: $f(x) = \dfrac{x\cancel{(x-1)}}{\cancel{x-1}} = x$

None

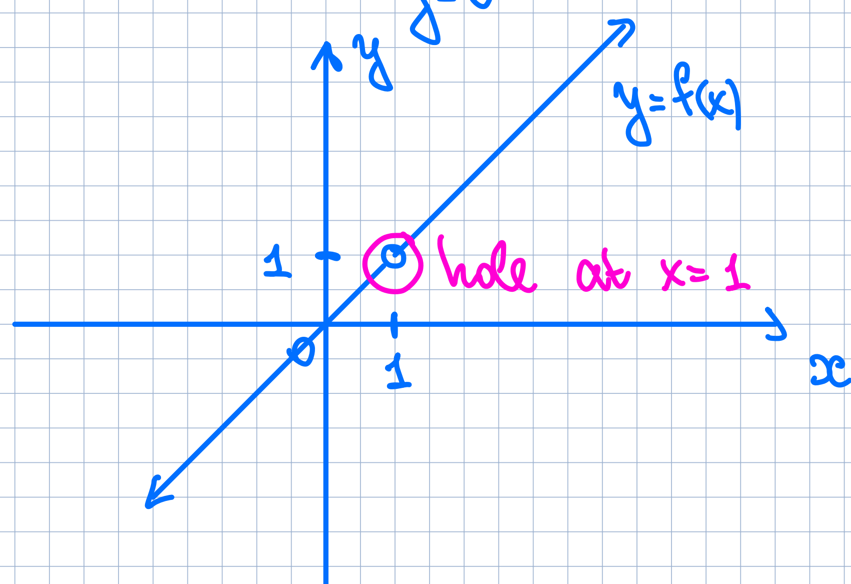
We have a hole at $x = 1$.

③ H.A.: None

O.A.: None

④ x- intercept: $y = 0$ $x = 0$ $(0,0)$

⑤ y- intercept: $x = 0$ $y = 0$ $(0,0)$



- $g(x) = \dfrac{x^2+1}{x^2+2x-15} = \dfrac{x^2+1}{(x+5)(x-3)}$ >0

① $\text{Dom}(g) = \mathbb{R} \setminus \{-5, 3\}$

② V.A. : $x=-5$ and $x=3$

③ H.A. : Since $n=2$ and $m=2$, we have that $y=1$ is a H.A.

④ O.A. : None

⑤ x-intercept: $y=0 \Rightarrow \dfrac{x^2+1}{(x+5)(x-3)} = 0$

None

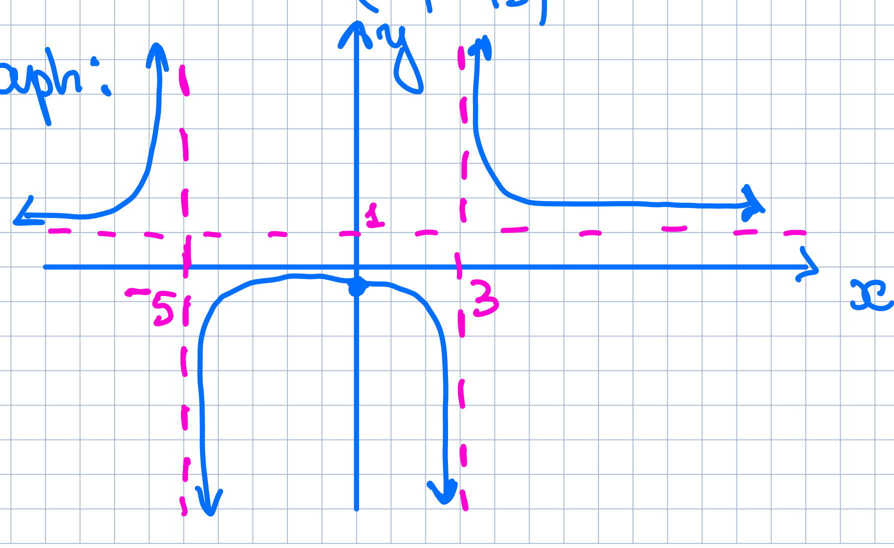
$x^2+1 \neq 0$

Never

⑥ y-intercept: $x=0$, $y=-\dfrac{1}{15}$

$\left(0, -\dfrac{1}{15}\right)$

⑦ Graph:



4. Solve the rational inequality

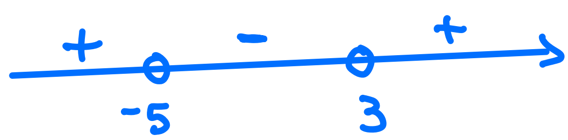
$$\frac{x^2+1}{x^2+2x-15} > 0.$$

① $x^2+1>0$

$x^2+2x-15=0$

$(x+5)(x-3)=0 \Rightarrow x=-5,\ x=3$

②



Solution: $(-\infty,-5)\cup(3,+\infty)$

5. Solve the rational inequalities $\frac{x}{x+2} < 3$ and $\frac{x}{x+2} \leq 3$.

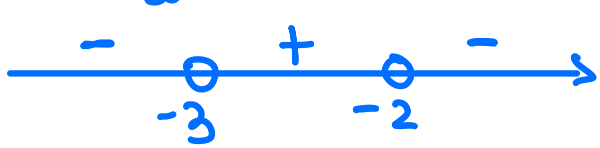
① $\frac{x}{x+2} < 3 \Leftrightarrow \frac{x}{x+2} - 3 < 0$

$\frac{x-3x-6}{x+2} < 0$

$\frac{-2x-6}{x+2} < 0$

$x+2=0$ and $-2x-6=0$

$x=-2$ $\quad$ $x=-3$



Solution: $(-\infty,-3)\cup(-2,\infty)$

(2) $\frac{x}{x+2} \leq 3$

$\frac{x}{x+2} - 3 \leq 0$

$\frac{x - 3x - 6}{x+2} \leq 0$

$\frac{-2x-6}{x+2} \leq 0$

$-2x-6 = 0$
$x = -3$

$x+2 = 0$
$x = -2$

we exclude -2 since it is not in the domain of f(x)

Solution: $(-\infty, -3] \cup (-2, +\infty)$