Section 5.2. Polynomial division and the division algorithm

1. The division algorithm and the remainder theorem.
2. Polynomial long division.
3. Synthetic division.
4. Constructing polynomials with given zeros.

1.

Theorem

Let $p(x)$ and $q(x)$ be polynomials s.t. $d(x) \neq 0$ and with the degree of d less than or equal to the degree of p.
Then there are unique polynomials $q(x)$ and $r(x)$ (quotient and remainder) s.t.

$$\underbrace{p(x)}_{\text{dividend}} = \underbrace{q(x)}_{\text{quotient}} \cdot \underbrace{d(x)}_{\text{divisor}} + \underbrace{r(x)}_{\text{remainder}}$$

Either the degree of the remainder r is less than the degree of the <u>divisor</u> d, or the remainder is 0, in which case we say d <u>divides evenly</u> into the polynomial p. If the remainder is 0, the two polynomials q and d are <u>factors</u> of p.

Theorem (Zeros and Linear Factors)

The number c is a <u>zero</u> of a polynomial $p(x)$ iff the linear polynomial $x-c$ is a factor of p. In this case, $p(x) = q(x)(x-c)$ for some quotient polynomial q.
This also means that c is a solution of the polynomial equation $p(x)=0$, and if p is a polynomial with real coefficients and if c is a real number, then c is an x-intercept of p.

Theorem (The remainder theorem)

If the polynomial $p(x)$ is divided by $x-c$, the remainder is $p(c)$.

$$p(x) = q(x)(x-c) + p(c).$$

$\boxed{2.}$

Example 1

Divide the polynomial $x^2+2x-24$ by the polynomial $x+6$.

Solution

$$
\begin{array}{rl}
x^2+2x-24 & \big| \; \underline{x+6} \\
-\;\underline{x^2+6x} & \;\; x-4 \\
-4x-24 & \\
-\;\underline{-4x-24} & \\
0 &
\end{array}
$$

Thus, the quotient is $x-4$ with a remainder of 0.

Procedure (Polynomial Long Division)

Step 1: Arrange the terms of each polynomial in descending order

Step 2: Divide the first term in the dividend by the first term in the divisor.
This gives the first term of quotient

Step 3: Multiply the entire divisor by the result and write this beneath the dividend so that like terms line up.

Step 4: Subtract the product from the dividend.

Step 5: Bring down the rest of the original dividend, forming a new dividend.

Step 6: Repeat the process with the new dividend. Continue until the degree of the remainder is less than the degree of the divisor.

Example

$$p(x) = 6x^5 - 5x^4 + 10x^3 - 15x^2 - 19$$

$$q(x) = 2x^2 - x + 3$$

$$\begin{array}{rl|l}
 & 6x^5 - 5x^4 + 10x^3 - 15x^2 - 19 & 2x^2 - x + 3 \\
- & \underline{6x^5 - 3x^4 + 9x^3} & 3x^3 - x^2 - 6 \\
 & -2x^4 + x^3 - 15x^2 & \\
- & \underline{-2x^4 + x^3 - 3x^2} & \\
 & -12x^2 - 19 & \\
- & \underline{-12x^2 + 6x - 18} & \\
 & -6x - 1 &
\end{array}$$

Thus, the solution is

$$6x^5-5x^4+10x^3-15x^2-19=(2x^2-x+3)(3x^3-x^2-6)-(6x+1)$$

$\boxed{3.}$

Synthetic division is a shortened version of polynomial long division that can be used when the divisor is of the form $x-c$ for some constant c.

Procedure: (Synthetic division)

Step 1: If the divisor is $x-c$, write down c followed by the coefficients of the dividend.

Step 2: Write the leading coefficient of the dividend on the bottom row.

Step 3: Multiply c by the value placed on the bottom, and place the product in the next column, in the second row.

Step 4: Add the values in this column, giving a new value in the bottom row.

Step 5: Repeat this process until the table is complete.

Step 6: The numbers in the bottom row are the coefficients of the quotient, plus the remainder.
Note that the first term of the quotient will have degree one less than the first term of the dividend.

Example

$p(x) = -2x^4 + 11x^3 - 5x^2 - 3x + 15$

$c = 5$

	x^4	x^3	x^2	x^1	x^0
5	-2	11	-5	-3	15
	-2	1	0	-3	$\boxed{0}$

remainder

$c = 5$ is a zero of $p(x)$.

$p(x) = (-2x^3 + x^2 - 3)(x-5)$

Example

$$p(x) = -3x^3 + (5-2i)x^2 + (-4+i)x + (1-i)$$

$c = 1-i$

	x^3	x^2	x^1	x^0
$1-i$	-3	$5-2i$	$-4+i$	$1-i$
	-3	$2-i$	-1	$\boxed{0}$

remainder

- $-3+3i+5-2i = 2+i$
- $(2+i)(1-i) - 4 + i = 2 - \cancel{i} + 1 - 4 + \cancel{i} = -1$
- $-(1-i) + 1 - i = -1 + i + 1 - i = 0$

Thus, $p(x) = \left(-3x^2 + (2-i)x - 1\right)(x-1+i)$.

$\boxed{4.}$

Example (Constructing polynomials)

- Construct a third-degree polynomial with zeros of $-3, 2$ and 5, and s.t. it goes to $-\infty$ as $x \to \infty$.

$$p(x) = a(x+3)(x-2)(x-5)$$

Let $a = -1$.

Then, $p(x) = -(x+3)(x-2)(x-5)$.

Thus, $p(x) \to -\infty$ as $x \to \infty$.