Section 1.8. Polynomial and polynomial-like equations in one variable

1. Solving quadratic equations by factoring.
2. Solving "perfect square" quadratic equations.
3. Solving quadratic equations by completing the square.
4. The quadratic formula.
5. Applications of quadratic equations
6. Solving quadratic-like equations.
7. Solving general polynomial equations by factoring.
8. Solving polynomial-like equations by factoring.

1.

Def. A quadratic equation in one variable x is the equation that can be transformed into the form

$$ax^2 + bx + c = 0,$$

where $a, b, c \in \mathbb{R}$, $a \neq 0$.

By factoring we get:

$$ax^2 + bx + c = \underbrace{A}_{\text{factor}} \cdot \underbrace{B}_{\text{factor}} = 0$$

$$A \cdot B = 0 \iff A = 0 \text{ or } B = 0$$

Example

- $x^2 + \frac{5x}{2} = \frac{3}{2} \quad | \cdot 2$

  $2x^2 + 5x - 3 = 0$

  $\underbrace{(2x - 1)}_{\text{factor}}\underbrace{(x + 3)}_{\text{factor}} = 0$

  $\boxed{x = \frac{1}{2} \quad \text{or} \quad x = -3}$

- $5x^2 + 10x = 0$

  $5x(x + 2) = 0$

  $x = 0 \quad \text{or} \quad x + 2 = 0$

  $\boxed{x = 0 \quad \text{or} \quad x = -2}$

$\boxed{2.}$ Let A be an algebraic expression and c be a constant.
Then $A^2 = c$ means $A = \sqrt{c}$
or $A = -\sqrt{c}$

Example

- $(2x+3)^2 = 8$

$$2x+3 = \pm\sqrt{8} = \pm 2\sqrt{2}$$
$$2x = \pm 2\sqrt{2} - 3$$
$$x = \pm\sqrt{2} - \frac{3}{2}$$

$\boxed{3.}$ What if the equation doesn't have the form $A^2 = c$?

We need to use the method of completing the square.

Procedure:

Step 1: $ax^2 + bx + c = 0 \Rightarrow ax^2 + bx = -c$

Step 2: if $a \neq 1$: $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Step 3: divide the coef. of x by 2, Square the result, and add this to both

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Step 4: the trinomial on the left side is now a perfect square trinomial.

Example

- $x^2 - 2x - 6 = 0$

1. $x^2 - 2x = 6$
2. Skip since $a = 1$
3. $x^2 - 2x + 1 = 6 + 1$
4. $x^2 - 2x + 1 = 7$

$$(x-1)^2 = 7$$

$$x - 1 = \pm\sqrt{7}$$

$$\boxed{x = 1 \pm \sqrt{7}}$$

4.

$$ax^2 + bx + c = 0, \quad a \neq 0$$

Then, by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$D = b^2 - 4ac$

↑ discriminant

Discriminant	# of distinct solutions	Type of solutions	Notes
$b^2 - 4ac > 0$	2	$\mathbb{R}$	two different solutions
$b^2 - 4ac = 0$	1	$\mathbb{R}$	double root sol.
$b^2 - 4ac < 0$	2	$\mathbb{C}$	complex conjugates solutions

Example

1) $8x^2 - 4x = 1$

$8x^2 - 4x - 1 = 0$

$a = 8, \ b = -4, \ c = -1$

$$x_{1,2} = \frac{4 \pm \sqrt{16 - 4 \cdot 8 \cdot (-1)}}{2 \cdot 8}$$

$$x_{1,2} = \frac{4 \pm \sqrt{16 + 32}}{16} = \frac{4 \pm \sqrt{48}}{16}$$

$$\boxed{x_{1,2} = \frac{1 \pm \sqrt{3}}{4}}$$

2) $-2x^2 + 12x - 18 = 0$

$D = 12^2 - 4 \cdot 2 \cdot 18 = 144 - 144 = 0$

Therefore, we have one double root

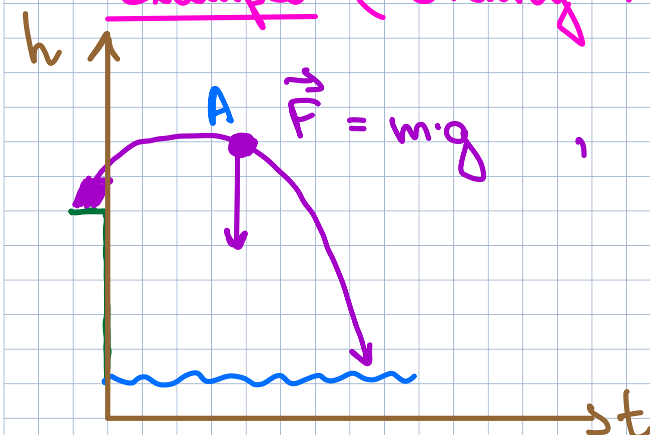
3) $x^2 - 4x + 9 = 0$

$$D = 16 - 4 \cdot 9 = -20 < 0$$

Therefore, we have two complex conjugate roots.

$\boxed{5.}$

Example (Gravity Problems)



$\vec{F} = m \cdot g$, where
- m is a mass
- g is the force due to gravity

We want to find the trajectory of an object A.

$$h = h(t) = -\frac{1}{2} g t^2 + v_0 t + h_0,$$

where

- g, v_0 and h_0 are constants

  g – force due to gravity

v_0 – initial velocity (v at $t=0$)
h_0 – initial position (h at $t=0$)

$g = 32\ ft/s^2 = 9.8\ m/s^2.$

$\boxed{6.}$ **Def.** An equation is quadratic-like, or quadratic in form, if it can be written in the form

$$\boxed{aA^2 + bA + c = 0}$$

where $a \neq 0, b, c$ are constants
A is an algebraic expression

We use here a substitution method.

Example

- $(x^2+2x)^2 - 7(x^2+2x) - 8 = 0$

$$x^2 + 2x = A$$

$$A^2 - 7A - 8 = 0$$

$$A_{1,2} = \frac{7 \pm \sqrt{49+32}}{2} = \frac{7 \pm 9}{2} = 8, -1$$

$A_1 = 8$ $\qquad$ $A_2 = -1$

$x^2 + 2x = 8$ $\qquad$ $x^2 + 2x = -1$

$$x^2 + 2x - 8 = 0$$
$$(x+1)^2 - 9 = 0$$
$$(x+1)^2 = 9$$
$$x+1 = \pm 3$$
$$x_1 = 2$$
$$x_2 = -4$$

$$x^2 + 2x + 1 = 0$$
$$(x+1)^2 = 0$$
$$x = -1$$
double root

Answer: $x_1 = 2$, $x_2 = -4$, $x_{3,4} = -1$.

$\boxed{7.}$

Example

- $x^4 = 9$

$$(x^2)^2 = 9 \Rightarrow x^2 = \pm 3$$

$$\boxed{x_1 = \pm\sqrt{3} \qquad x_2 = \pm i\sqrt{3}}$$

- $8t^3 - 27 = 0$

$$(2t)^3 - 3^3 = 0$$
$$(2t-3)((2t)^2 + 6t + 9) = 0$$
$$2t - 3 = 0 \quad \text{or} \quad 4t^2 + 6t + 9 = 0$$
$$\boxed{t = \frac{3}{2}} \qquad D = 36 - 4 \cdot 4 \cdot 9$$

$$D = 36 - 144 = -108$$

$$t_{2,3} = \frac{-6 \pm \sqrt{-108}}{8}$$

$$t_{2,3} = -\frac{3}{4} \pm \frac{6i\sqrt{3}}{8}$$

$$\boxed{t_{2,3} = -\frac{3}{4} \pm \frac{3i\sqrt{3}}{4}}$$

$\boxed{8.}$

Example

- $x^{\frac{7}{3}} + x^{\frac{4}{3}} - 2x^{\frac{1}{3}} = 0$

$$x^{\frac{1}{3}}\left(x^{\frac{6}{3}} + x^{\frac{3}{3}} - 2\right) = 0$$

$$x^{\frac{1}{3}}\left(x^2 + x - 2\right) = 0$$

✓ $\boxed{x = 0}$ or $x^2 + x - 2 = 0$

$$D = 1 + 8 = 9$$

$$x_{1,2} = \frac{-1 \pm 3}{2}$$

$$\boxed{x_{1,2} = 1, -2}$$ ✓

- $(x-1)^{\frac{1}{2}} - (x-1)^{-\frac{1}{2}} = 0$

$$(x-1)^{-\frac{1}{2}}\left((x-1)^1 - 1\right) = 0$$

$(x-1)^{-\frac{1}{2}}(x-1-1)=0$

$(x-1)^{-\frac{1}{2}}(x-2)=0$

✓ $\boxed{x=2}$

$\frac{1}{(x-1)^{\frac{1}{2}}}=0$

$\boxed{x \neq 1}$