Assessment Instructions: Solutions

- The Assessment 4 is 10 problems and is worth 40 points. Each numbered problem will earn you a score of 1-4 based on your set up of the function, your use of course methods to solve and prove your solution and your statement of the solution.
- You will have 1 hour to complete AS-4.
- The AS-4 is closed book and closed notes.
- **Calculators are not allowed** on the Assessment.

5.1

1. Solve the following polynomial equations by factoring and/or using the quadratic formula, making sure to identify all the solutions.

   (a) $x^3 - x^2 = 72x$

   (b) $x^4 - 8x^3 + 25x^2 = 0$

2. For each of the following polynomial functions, determine the behaviour of its graph as $x \pm \infty$ and identify the x- and y-intercepts. Use this information to sketch the graph of each polynomial.

   (a) $r(x) = x^2 - 2x - 3$

   (b) $f(x) = (3 - x)(x + 2)(x + 4)$

3. Solve the following polynomial inequalities.

   (a) $-x^3 - x^2 + 30x > 0$

   (b) $(x^2 - 1)(x - 4)(x + 5) \leq 0$

5.2

4. Use polynomial lonf division to rewrite each of the following fractions in the form $q(x) + \frac{r(x)}{d(x)}$.

   (a) $\dfrac{x^3 + 2x^2 - 4x - 8}{x - 3}$

   (b) $\dfrac{2x^3 - 3ix^2 + 11x + (1 - 5i)}{2x - i}$

5. Use synthetic division to determine if the given value for c is a zero of the corresponding polynomial. If not, determine $p(c)$.

   (a) $p(x) = 12x^4 - 7x^3 - 32x^2 - 7x + 6$, $c = 1$

   (b) $p(x) = 2x^2 - (3 - 5i)x + (3 - 9i)$, $c = -3i$

6. Construct a polynomial function with the stated properties.

   (a) Third-degree, zeros of -2, 1, and 3, and y-intercept of -12.

   (b) Second-degree, zeros of -4 and 3, and goes to $-\infty$ as $x \to -\infty$.

5.3

7. List all of the potential rational zeros of the following polynomials.

   (a) $q(x) = x^3 - 10x^2 + 23x - 14$

   (b) $k(x) = x^4 - 10x^2 + 24$

8. Use Descarte's Rule of Signs to determine the possible numbers of positive and negative real zeros of the following polynomials.

   (a) $f(x) = x^4 - 25$

   (b) $g(x) = x^3 + 6x^2 + 11x + 6$

9. Use the Intermediate Value Theorem to show that the following polynomial has a real zero between the indicated values.

   (a) $f(x) = 5x^3 - 4x^2 - 31x - 6$; -3 and -1

5.4

10. Sketch the graph of each factored polynomial.

    (a) $f(x) = (x + 1)^2(x - 2)^3$

    (b) $k(x) = -x^3(x - 2)(x + 1)2$

11. Use all available methods to solve each polynomial equation.

    (a) $x^4 + 15 = 2x^3 + 8x^2 - 10x$

    (b) $x^3 - 5 = 5x^2 - 9x$

12. Use all available methods to solve each polynomial equation.

    (a) $x^4 + 15 = 2x^3 + 8x^2 - 10x$

    (b) $x^3 - 5 = 5x^2 - 9x$

5.5

13. Find equations for the vertical asymptotes, if any, for each of the following rational functions.

    (a) $f(x) = \dfrac{x^2 + 5}{x^3 - 27}$

    (b) $g(x) = \dfrac{x^2 - 1}{x^2 - 8x + 7}$

14. Find equations for the horizontal or oblique asymptotes, if any, for each of the following rational functions.

    (a) $f(x) = \dfrac{5}{x - 1}$

    (b) $g(x) = \dfrac{2x^2 - 5x + 6}{x - 3}$

    (c) $k(x) = \dfrac{-3x + 5}{x - 2}$

15. Sketch the graphs of the following rational functions.

    (a) $f(x) = \dfrac{x^2 - 4}{2x - x^2}$

    (b) $g(x) = \dfrac{5}{x - 1}$

16. Solve the following rational inequalities.

    (a) $\dfrac{5}{x - 2} > \dfrac{3}{x + 2}$

    (b) $\dfrac{x}{x^2 - x - 6} \leq \dfrac{-1}{x^2 - x - 6}$

6.1

17. Sketch the graphs of the following functions. State their domain and range.

    (a) $f(x) = (0.5)^x$

    (b) $g(x) = 10^x$

    (c) $h(x) = 2^{-x}$

    (d) $k(x) = 2 - 4^{2-x}$

18. Solve the following exponential equations.

    (a) $3^{2x-1} = 27$

    (b) $3^{x^2+4x} = 81^{-1}$

    (c) $10^x = 0.01$

6.2

19. Assuming a current world population of 7.75 billion people, an annual growth rate of 1.9% per year, and a worst-case scenario of exponential growth, what will the world population be in 10 years (in 50 years)?

20. Problem 19, page 455 (textbook).

5.1

1. (a) $x^3 - x^2 = 72x$

$x^3 - x^2 - 72x = 0$

$x(x^2 - x - 72) = 0$

$x(x-9)(x+8) = 0$

$x = 0$ or $x = 9$ or $x = -8$

Answer: $\{0, 9, -8\}$

(b) $x^4 - 8x^3 + 25x^2 = 0$

$x^2(x^2 - 8x + 25) = 0$

$x = 0$ or $x^2 - 8x + 25 = 0$

$D = 64 - 100 = -36$

$x_1 = \frac{8 + i\sqrt{36}}{2} = \frac{8+6i}{2} = 4 + 3i$

$x_2 = \frac{8 - i\sqrt{36}}{2} = \frac{8-6i}{2} = 4 - 3i$

Answer: $\{0, 4 \pm 3i\}$

2. (a) $r(x) = x^2 - 2x - 3$

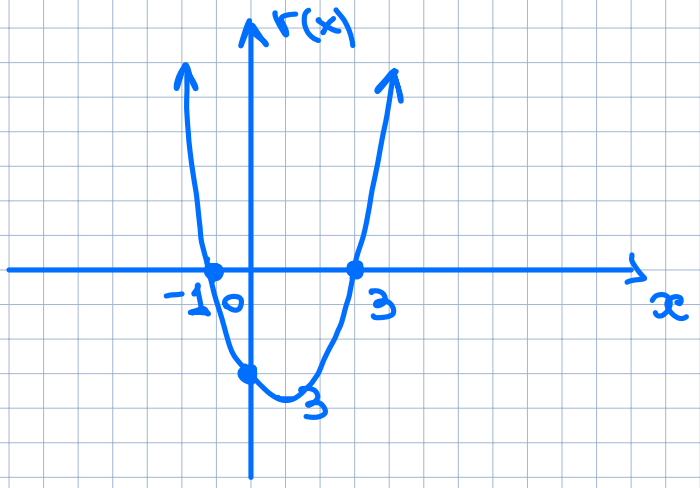
$r(x) = (x-3)(x+1)$

When $x \to +\infty$, $r(x) \to +\infty$

When $x \to -\infty$, $r(x) \to +\infty$

When $x = 0$: $r(0) = -3$

When $r(x) = 0$: $x = 3$ or $x = -1$



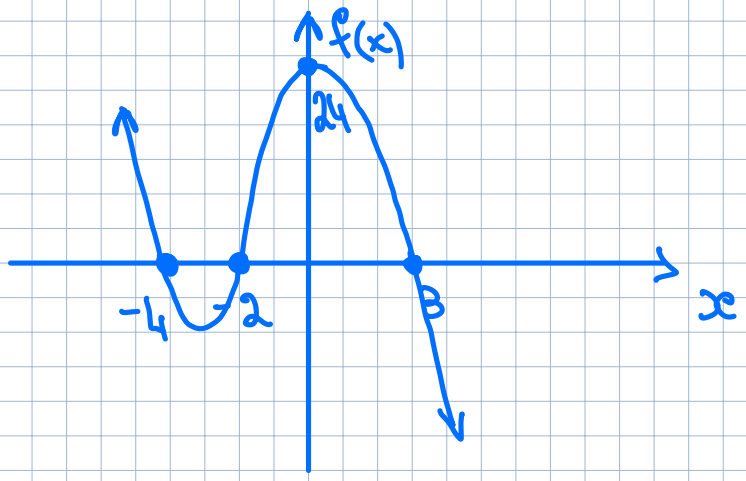
(b) $f(x) = (3-x)(x+2)(x+4)$

When $x \to \infty$, $f(x) \to -\infty$

When $x \to -\infty$, $f(x) \to +\infty$

When $x = 0$: $f(0) = 3 \cdot 2 \cdot 4 = 24$

When $f(x) = 0$: $x = 3$ or $x = -2$ or $x = -4$



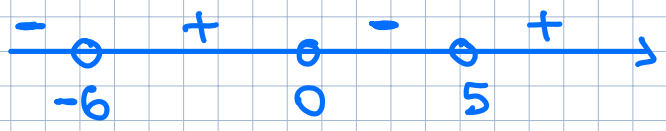
3. (a) $-x^3 - x^2 + 30x \geq 0$

$x(-x^2 - x + 30) \geq 0 \quad | \cdot (-1)$

$x(x^2 + x - 30) \leq 0$

$x(x^2 + x - 30) = 0$

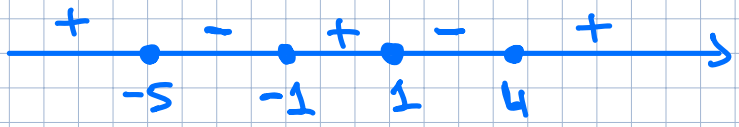
$x(x+6)(x-5) = 0$



Answer: $(-\infty, -6) \cup (0, 5)$

(b) $(x^2 - 1)(x-4)(x+5) \leq 0$

$(x-1)(x+1)(x-4)(x+5) \leq 0$



Answer: $[-5, -1] \cup [1, 4]$

5.2

4. (a)

$$\begin{array}{r|l} x^3 + 2x^2 - 4x - 8 & x - 3 \\ \underline{-(x^3 - 3x^2)} & \\ 5x^2 - 4x - 8 & x^2 + 5x + 11 \\ \underline{-(5x^2 - 15x)} & \\ 11x - 8 & \\ \underline{-(11x - 33)} & \\ 25 & \end{array}$$

Answer: $(x^2 + 5x + 11) + \frac{25}{x-3}$

(b)

$$\begin{array}{r|l} 2x^3 - 3ix^2 + 11x + (1-5i) & 2x - i \\ \cline{2-2} 2x^3 - ix^2 & x^2 + ix + 5 \\ \cline{1-1} -2ix^2 + 11x + (1-5i) & \\ -2ix^2 + x & \\ \cline{1-1} -10x + (1-5i) & \\ -10x - 5i & \\ \cline{1-1} 1 & \end{array}$$

Answer: $(x^2 + ix + 5) + \frac{1}{2x - i}$

5. (a)

	12	-7	-32	-7	6
1	12	5	-27	-34	-28

$p(c) = p(1) = -28$

(b)

	2	$-(3-5i)$	$(3-9i)$
$-3i$	2	$-3-i$	0

$p(-3i) = 0$

6. (a)

$p(x) = a(x+2)(x-1)(x-3)$

$p(0) = -12$

$$a \cdot 2 \cdot (-1) \cdot (-3) = -12$$

$$6a = -12$$

$$a = -2$$

$$\boxed{p(x) = -2(x+2)(x-1)(x-3)}$$

(b) $p(x) = a(x+4)(x-3)$

$a = 1$

Then $p(x) \to -\infty$ as $x \to -\infty$

$$\boxed{p(x) = (x+4)(x-3)}$$

5.3

7. (a) $q(x) = x^3 - 10x^2 + 23x - 14$

$a_0 = -14$ factors: $\{\pm 1, \pm 2, \pm 7, \pm 14\}$

$a_n = 1$ factors: $\{\pm 1\}$

Potential rational zeros: $\{\pm 1, \pm 2, \pm 7, \pm 14\}$

(b) $k(x) = x^4 - 10x^2 + 24$

$a_0 = +24$ factors: $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24\}$

$a_n = 1$

↑ factors: $\{\pm 1\}$

Potential rational zeros: $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24\}$

8. (a) $f(x) = \underbrace{x^4 - 25}_{\text{one sign change}}$

We have one positive real zero

$f(-x) = (-x)^4 - 25 = \underbrace{x^4 - 25}_{\text{one sign change}}$

We have one negative real zero

(b) $g(x) = \underbrace{x^3 + 6x^2 + 11x + 6}_{\text{No change}}$

No positive real zero

$g(-x) = \underbrace{-x^3}_{\text{one change}} + \underbrace{6x^2}_{\text{one}} - \underbrace{11x}_{\text{one}} + 6$

We have 3 or 1 negative real zeros.

9. (a) $f(x) = 5x^3 - 4x^2 - 31x - 6$, -3 and -1

$a = -3$

$b = -1$

$f(a) = f(-3) = 5 \cdot (-27) - 4 \cdot 9 - 31(-3) - 6 =$

$= -135 - 36 + 93 - 6 = -84 < 0$

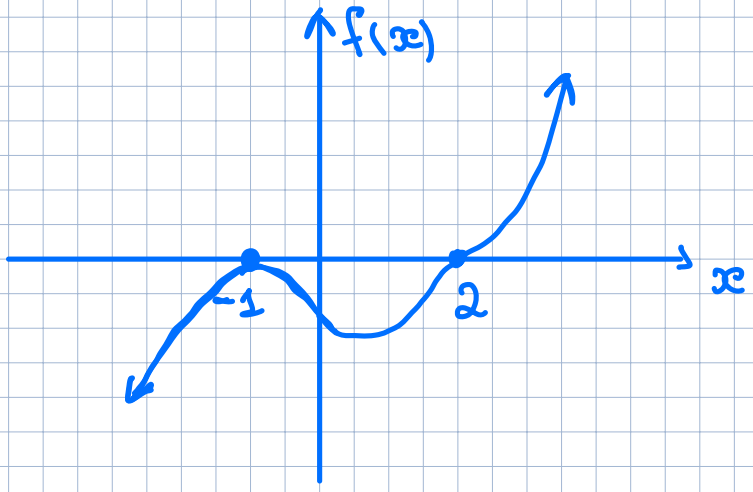
$f(b) = f(-1) = -5 - 4 + 31 - 6 = 16 > 0$

Hence, by IVT there is at least one $-3 < c < -1$ such that

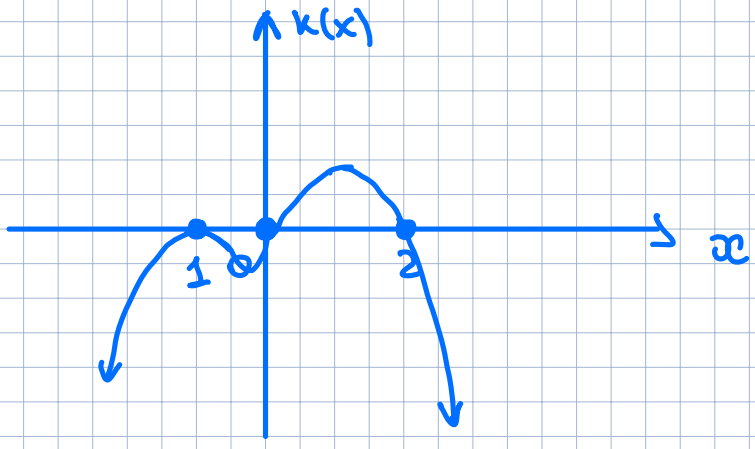
$$\boxed{f(c) = 0}$$

5.4

10. (a) $f(x) = (x+1)^2(x-2)^3$



(b) $k(x) = -x^3(x-2)(x+1)^2$



11. (a) $x^4 + 15 = 2x^3 + 8x^2 - 10x$

$x^4 - 2x^3 - 8x^2 + 10x + 15 = 0$

$\boxed{x = -1}$

$1 + 2 - 8 - 10 + 15 = 0$

	1	-2	-8	10	15
-1	1	-3	-5	15	0

$(x+1)(x^3 - 3x^2 - 5x + 15) = 0$

$\boxed{x = 3}$

$27 - 27 - 15 + 15 = 0$

	1	-3	-5	15
3	1	0	-5	0

$(x+1)(x-3)(x^2 - 5) = 0$

$(x+1)(x-3)(x-\sqrt{5})(x+\sqrt{5}) = 0$

Answer: $\{-1, 3, \pm\sqrt{5}\}$

(b) $x^3 - 5 = 5x^2 - 9x$

$x^3 - 5x^2 + 9x - 5 = 0$

$x = 1$

$1 - 5 + 9 - 5 = 0$

	1	-5	9	-5
1	1	-4	5	0

$(x-1)(x^2 - 4x + 5) = 0$

$(x-1)(x-5)(x+1) = 0$

Answer : $\{1, 5, -1\}$

5.5 12. (a) $f(x) = \frac{x^2+5}{x^3-27} = \frac{x^2+5}{(x-3)(x^2+3x+9)} =$

$x - 3 = 0$

V.A.: $\boxed{x = 3}$

$x^2 + 3x + 9 = 0$

$D = 9 - 36 < 0$

$\emptyset$

Answer : $x = 3$ is a V.A.

(b) $g(x) = \frac{x^2-1}{x^2-8x+7} = \frac{x^2-1}{(x-7)(x-1)}$

V.A. : $x = 7$ and $x = 1$

13. (a) $f(x) = \frac{5}{x-1}$

$p(x) = 5$, $n = 0$

$q(x) = x - 1$, $m = 1$

$n < m$

Therefore, $y = 0$ is H.A.

(b) $g(x) = \frac{2x^2 - 5x + 6}{x - 3}$

$p(x) = 2x^2 - 5x + 6$, $n = 2$

$q(x) = x - 3$, $m = 1$

$n > m$

O.A. :

$$\begin{array}{r|l} 2x^2 - 5x + 6 & x - 3 \\ \cline{2-2} 2x^2 - 6x & 2x + 1 \\ \cline{1-1} -x + 6 & \\ x - 3 & \\ \cline{1-1} 9 & \end{array}$$

O.A. : $\boxed{y = 2x + 1}$

(c) $k(x) = \frac{-3x + 5}{x - 2}$

$p(x) = -3x + 5$, $m = 1$

$q(x) = x - 2$, $n = 1$

$m = n$

H.A. : $y = \frac{-3}{1} = -3$

$\boxed{y = -3}$

14. (B) $g(x) = \frac{5}{x-1}$

① $Dom(g) = \mathbb{R} \setminus \{1\}$

② V.A. : $x = 1$

③ H.A. : $y = 0$

④ O.A. : none

⑤ x- intercept: $g(x) = 0$

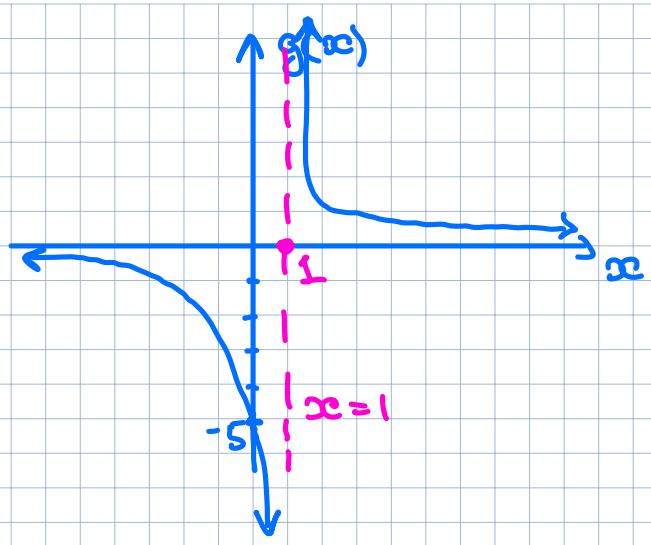
$\frac{5}{x-1} = 0$

none!

⑥ y- intercept: $x = 0$

$g(0) = -5$

$(0, -5)$



15. (a) $\frac{5}{x-2} > \frac{3}{x+2}$

$\frac{5}{x-2} - \frac{3}{x+2} > 0$

$\frac{5(x+2)-3(x-2)}{(x-2)(x+2)} > 0$

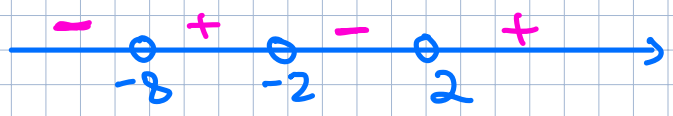
$\frac{2x+16}{(x-2)(x+2)} > 0$

① $2x+16=0$

$x=-8$

② $(x-2)(x+2)=0$

$x=2$ or $x=-2$



Solution : $(-8,-2)\cup(2,+\infty)$

(b) $\dfrac{x}{x^2-x-6} \leq \dfrac{-1}{x^2-x-6}$

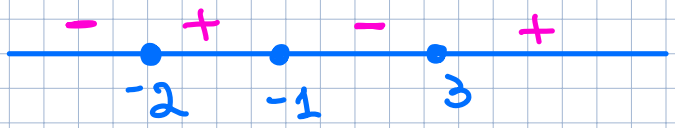
$$\frac{x+1}{x^2-x-6} \leq 0$$

① $x = -1$

② $x^2 - x - 6 = 0$

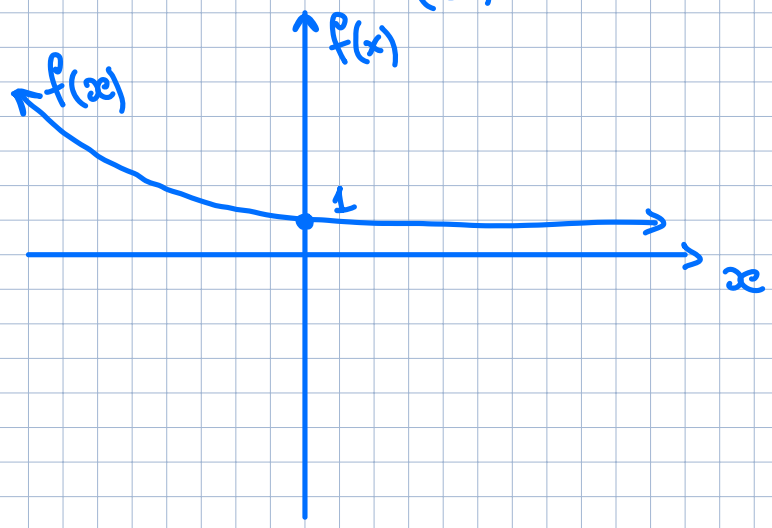
$(x-3)(x+2) = 0$

$x = 3$ or $x = -2$

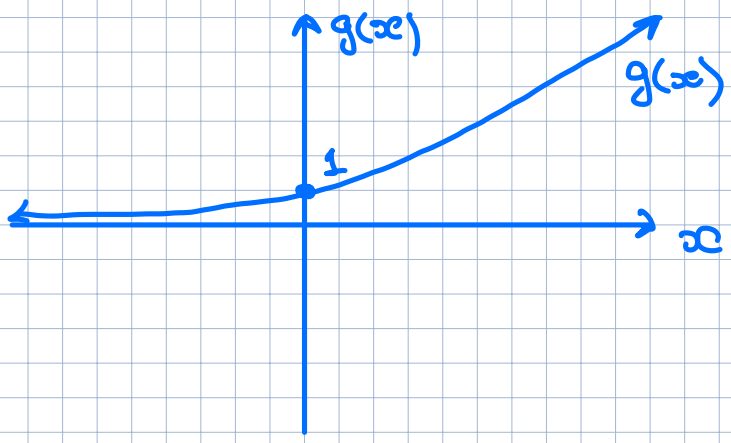


Solution : $(-\infty, -2] \cup [-1, 3]$.

6.1 17. (a) $f(x) = \left(\frac{1}{2}\right)^x$



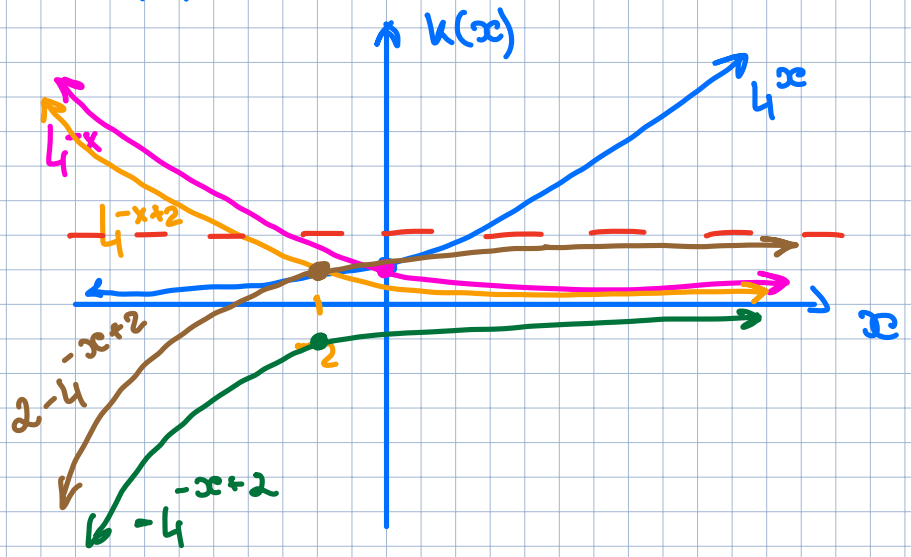
(b) $g(x) = 10^x$



(c) $h(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$

See (a)

(d) $k(x) = 2 - 4^{2-x} = 2 - 4^{-x+2}$



18. (a) $3^{2x-1} = 27$

$3^{2x-1} = 3^3$

$2x - 1 = 3$

$2x = 4$

$x = 2$

(b) $3^{x^2+4x} = 81^{-1}$

$3^{x^2+4x} = 3^{-4}$

$x^2 + 4x = -4$

$x^2 + 4x + 4 = 0$

$(x+2)^2 = 0$

$x = -2$

(c) $10^x = 0.01$

$10^x = 10^{-2}$

$x = -2$

6.2

19. $P_0 = 7.75$

$r = 1.9\% = 0.019$

$t = 10$ years

$P(t) = P_0 a^t$

$P(10) = 7.75 \cdot (0.019)^{10} \approx 9.35$ Billion

$t = 50$ years

$$P(50) = 7.75 \cdot (0.019)^{50} \approx 19.86 \text{ Billion}$$

20. $$N(t) = \frac{10\,000}{1 + 999\,e^{-t}}$$

(a) $$N(0) = \frac{10\,000}{1 + 999} = 10$$

(b) $N(t) \to 10\,000$ as $t \to \infty$.