THEORETICAL PART: Solutions

Definition:

A **quadratic equation in one variable**, say the variable x, is an equation that can be transformed into the form

$$ax^2 + bx + c = 0,$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

We also call such equations as **second-degree** equations.

Completing the Square Procedure:

Step 1. Write the equation $ax^2 + bx + c = 0$ in the form $ax^2 + bx = -c$.

Step 2. Divide by $a \neq 1$, so that the coefficient of x^2 is 1: $x^2 + \frac{b}{a}x = -\frac{c}{a}$.

Step 3. Divide the coefficient of x by 2, square the result, and add this to both sides:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2.$$

Step 4. The trinomial on the left side will now be a perfect square trinomial. That is, it can be written as the square of a binomial.

The Quadratic Formula:

The solutions of the general quadratic equation $ax^2 + bx + x = 0$, with $a \neq 0$, are given by the **quadratic formula:**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We call $D = b^2 - 4ac$ the **discriminant**. Its value determines the number and type (real or complex) of solutions.

- $b^2 - 4ac > 0$: we have 2 real distinct solutions.
- $b^2 - 4ac = 0$: we have 1 repeated real solution.
- $b^2 - 4ac < 0$: we have 2 complex solutions (complex conjugate).

Definition: An equation is **quadratic-like**, or **quadratic in form**, if it can be written in the form

$$aA^2 + bA + c = 0,$$

where a, b, c are constants, $a \neq 0$, and A is an algebraic expression. Such equations can be solved by using a **substitution** method.

PRACTICAL PART:

1. Solve the quadratic equation by factoring:

   - $s^2 + 9 = 6s$

   $s^2 - 6s + 9 = 0$

   $(s-3)^2 = 0$

   $s - 3 = 0$

   $\boxed{s = 3}$ (double root)

2. Solve the quadratic equation by taking square roots:

   - $(2x + 3)^2 = 8$

   $2x + 3 = \pm\sqrt{8} = \pm 2\sqrt{2}$

   $2x = \pm 2\sqrt{2} - 3 \Rightarrow \boxed{x = \pm\sqrt{2} - \frac{3}{2}}$

3. Solve the quadratic equation by completing the square:

   - $x^2 - 2x - 6 = 0$

   $x^2 - 2x = 6$

   $x^2 - 2x + 1^2 = 6 + 1^2$

   $(x-1)^2 = 7$

   $x - 1 = \pm\sqrt{7}$

   $\boxed{x = \pm\sqrt{7} + 1}$

4. Solve the quadratic equation using the quadratic formula:

   - $8x^2 - 4x = 1$

   $8x^2 - 4x - 1 = 0$

   $D = 16 + 4\cdot 8 = 16 + 32 = 48 > 0$

   $x_{1,2} = \frac{4 \pm \sqrt{48}}{16}$

   $x_{1,2} = \frac{1 \pm 2\sqrt{6}}{4}$

5. For each of the following quadratic equations, calculate the discriminant and determine the number and type of solutions:

   - $-2x^2 + 12x - 18 = 0$ $D = 144 - 4\cdot 2\cdot 18 = 144 - 144 = 0$ (1 real repeating)
   - $5x^2 + 7x + 2 = 0$ $D = 49 - 4\cdot 2\cdot 5 = 9 > 0$ (2 real different roots)
   - $x^2 - 4x + 9 = 0$ $D = 16 - 4\cdot 9 = -20 < 0$ (2 complex conjugate roots)

6. Solve the quadratic-like equation:

   - $(x^2+2x)^2 - 7(x^2+2x) - 8 = 0$
   - $y^{\frac{2}{3}} + 4y^{\frac{1}{3}} - 5 = 0$

$(x^2+2x)^2 - 7(x^2+2x) - 8 = 0$

$t = x^2 + 2x$

$t^2 - 7t - 8 = 0$

$(t-8)(t+1) = 0$

$t = 8$ or $t = -1$

$x^2+2x = 8$ $\quad$ $x^2+2x = -1$

$x^2+2x-8 = 0$ $\quad$ $x^2+2x+1 = 0$

$(x+4)(x-2) = 0$ $\quad$ $(x+1)^2 = 0$

$x_1 = -4$ $\quad$ $x_{3,4} = -1$

$x_2 = 2$

$\boxed{\{-4, 2, -1\}}$

$y^{\frac{2}{3}} + 4y^{\frac{1}{3}} - 5 = 0$

$t = y^{\frac{1}{3}}$

$t^2 + 4t - 5 = 0$

$(t+5)(t-1) = 0$

$t = -5$ or $t = 1$

$y^{\frac{1}{3}} = -5$ or $y^{\frac{1}{3}} = 1$

$y = -125$ or $y = 1$

$\boxed{\{-125, 1\}}$

7. Solve the equation by factoring:

   - $8t^3 - 27 = 0$
   - $x^{\frac{7}{3}} + x^{\frac{4}{3}} - 2x^{\frac{1}{3}} = 0$

- $8t^3 - 27 = 0$

$(2t)^3 - 3^3 = 0$

$(2t-3)(4t^2+6t+9) = 0$

$\boxed{t = \frac{3}{2}}$ or $4t^2+6t+9 = 0$

$D = 36 - 16\cdot 9 = -108$

$$t_1 = \boxed{\frac{-6+i\sqrt{108}}{8}}$$

$$t_2 = \boxed{\frac{-6-i\sqrt{108}}{8}}$$

- $x^{\frac{7}{3}} + x^{\frac{4}{3}} - 2x^{\frac{1}{3}} = 0$

$$x^{\frac{1}{3}}\left(x^2 + x - 2\right) = 0$$

$\boxed{x = 0}$ or $x^2 + x - 2 = 0$

$$(x+2)(x-1) = 0$$

$\boxed{x = -2}$ or $\boxed{x = 1}$