

**Assessment Instructions:****Solutions**

- The Assessment 4 is 10 problems and is worth 40 points. Each numbered problem will earn you a score of 1-4 based on your set up of the function, your use of course methods to solve and prove your solution and your statement of the solution.
- You will have 1 hour to complete AS-4.
- The AS-4 is closed book and closed notes.
- **Calculators are not allowed** on the Assessment.

**5.1**

1. Solve the following polynomial equations by factoring and/or using the quadratic formula, making sure to identify all the solutions.
  - (a)  $x^3 - x^2 = 72x$
  - (b)  $x^4 - 8x^3 + 25x^2 = 0$
2. For each of the following polynomial functions, determine the behaviour of its graph as  $x \pm \infty$  and identify the  $x$ - and  $y$ -intercepts. Use this information to sketch the graph of each polynomial.
  - (a)  $r(x) = x^2 - 2x - 3$
  - (b)  $f(x) = (3 - x)(x + 2)(x + 4)$
3. Solve the following polynomial inequalities.
  - (a)  $-x^3 - x^2 + 30x > 0$
  - (b)  $(x^2 - 1)(x - 4)(x + 5) \leq 0$

**5.2**

4. Use polynomial long division to rewrite each of the following fractions in the form  $q(x) + \frac{r(x)}{d(x)}$ .
  - (a)  $\frac{x^3 + 2x^2 - 4x - 8}{x - 3}$
  - (b)  $\frac{2x^3 - 3ix^2 + 11x + (1 - 5i)}{2x - i}$

5. Use synthetic division to determine if the given value for  $c$  is a zero of the corresponding polynomial. If not, determine  $p(c)$ .

(a)  $p(x) = 12x^4 - 7x^3 - 32x^2 - 7x + 6$ ,  $c = 1$

(b)  $p(x) = 2x^2 - (3 - 5i)x + (3 - 9i)$ ,  $c = -3i$

6. Construct a polynomial function with the stated properties.

(a) Third-degree, zeros of -2, 1, and 3, and  $y$ -intercept of -12.

(b) Second-degree, zeros of -4 and 3, and goes to  $-\infty$  as  $x \rightarrow -\infty$ .

### 5.3

7. List all of the potential rational zeros of the following polynomials.

(a)  $q(x) = x^3 - 10x^2 + 23x - 14$

(b)  $k(x) = x^4 - 10x^2 + 24$

8. Use Descarte's Rule of Signs to determine the possible numbers of positive and negative real zeros of the following polynomials.

(a)  $f(x) = x^4 - 25$

(b)  $g(x) = x^3 + 6x^2 + 11x + 6$

9. Use the Intermediate Value Theorem to show that the following polynomial has a real zero between the indicated values.

(a)  $f(x) = 5x^3 - 4x^2 - 31x - 6$ ; -3 and -1

### 5.4

10. Sketch the graph of each factored polynomial.

(a)  $f(x) = (x + 1)^2(x - 2)^3$

(b)  $k(x) = -x^3(x - 2)(x + 1)^2$

11. Use all available methods to solve each polynomial equation.

(a)  $x^4 + 15 = 2x^3 + 8x^2 - 10x$

(b)  $x^3 - 5 = 5x^2 - 9x$

12. Use all available methods to solve each polynomial equation.

(a)  $x^4 + 15 = 2x^3 + 8x^2 - 10x$

(b)  $x^3 - 5 = 5x^2 - 9x$

## 5.5

13. Find equations for the vertical asymptotes, if any, for each of the following rational functions.

(a)  $f(x) = \frac{x^2 + 5}{x^3 - 27}$

(b)  $g(x) = \frac{x^2 - 1}{x^2 - 8x + 7}$

14. Find equations for the horizontal or oblique asymptotes, if any, for each of the following rational functions.

(a)  $f(x) = \frac{5}{x - 1}$

(b)  $g(x) = \frac{2x^2 - 5x + 6}{x - 3}$

(c)  $k(x) = \frac{-3x + 5}{x - 2}$

15. Sketch the graphs of the following rational functions.

(a)  $f(x) = \frac{x^2 - 4}{2x - x^2}$

(b)  $g(x) = \frac{5}{x - 1}$

16. Solve the following rational inequalities.

(a)  $\frac{5}{x - 2} > \frac{3}{x + 2}$

(b)  $\frac{x}{x^2 - x - 6} \leq \frac{-1}{x^2 - x - 6}$

**6.1**

17. Sketch the graphs of the following functions. State their domain and range.

(a)  $f(x) = (0.5)^x$

(b)  $g(x) = 10^x$

(c)  $h(x) = 2^{-x}$

(d)  $k(x) = 2 - 4^{2-x}$

18. Solve the following exponential equations.

(a)  $3^{2x-1} = 27$

(b)  $3^{x^2+4x} = 81^{-1}$

(c)  $10^x = 0.01$

**6.2**

19. Assuming a current world population of 7.75 billion people, an annual growth rate of 1.9% per year, and a worst-case scenario of exponential growth, what will the world population be in 10 years (in 50 years)?

20. Problem 19, page 455 (textbook).

## 5.1

1. (a)  $x^3 - x^2 = 72x$

$$x^3 - x^2 - 72x = 0$$

$$x(x^2 - x - 72) = 0$$

$$x(x-9)(x+8) = 0$$

$$x=0 \text{ or } x=9 \text{ or } x=-8$$

Answer:  $\{0, 9, -8\}$

(b)  $x^4 - 8x^3 + 25x^2 = 0$

$$x^2(x^2 - 8x + 25) = 0$$

$$x=0 \text{ or } x^2 - 8x + 25 = 0$$

$$\Delta = 64 - 100 = -36$$

$$x_1 = \frac{8 + i\sqrt{36}}{2} = \frac{8+6i}{2} = 4+3i$$

$$x_2 = \frac{8 - i\sqrt{36}}{2} = \frac{8-6i}{2} = 4-3i$$

Answer:  $\{0, 4 \pm 3i\}$

2. (a)  $r(x) = x^2 - 2x - 3$

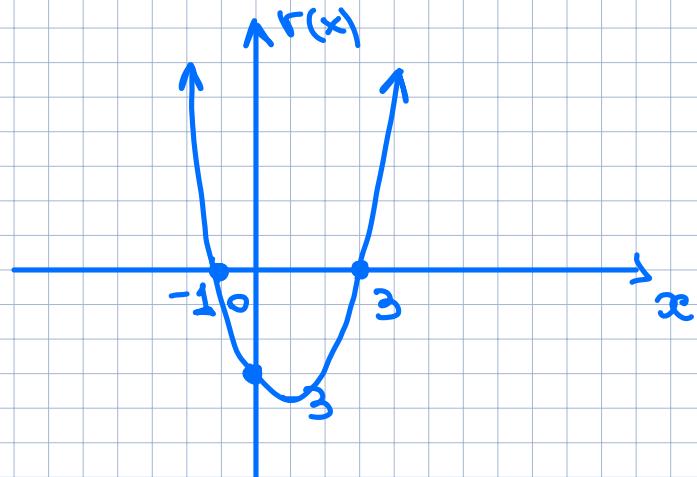
$$r(x) = (x-3)(x+1)$$

When  $x \rightarrow +\infty$ ,  $r(x) \rightarrow +\infty$

When  $x \rightarrow -\infty$ ,  $r(x) \rightarrow +\infty$

When  $x=0$ :  $r(0) = -3$

When  $r(x)=0$ :  $x=3$  or  $x=-1$



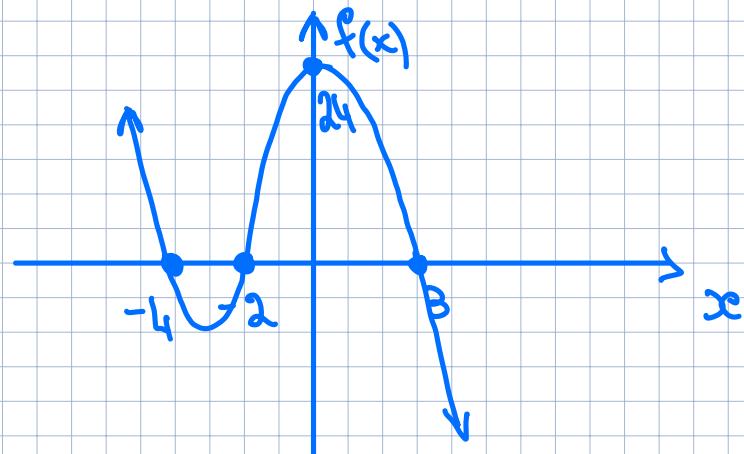
(b)  $f(x) = (3-x)(x+2)(x+4)$

When  $x \rightarrow \infty$ ,  $f(x) \rightarrow -\infty$

When  $x \rightarrow -\infty$ ,  $f(x) \rightarrow +\infty$

When  $x=0$ :  $f(0) = 3 \cdot 2 \cdot 4 = 24$

When  $f(x)=0$ :  $x=3$  or  $x=-2$  or  $x=-4$



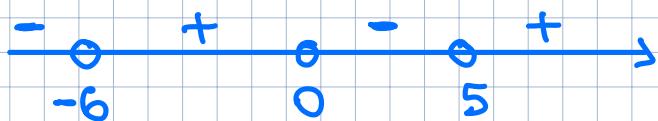
$$3. (a) -x^3 - x^2 + 30x > 0$$

$$x(-x^2 - x + 30) > 0 \quad | \cdot (-1)$$

$$x(x^2 + x - 30) < 0$$

$$x(x+6)(x-5) = 0$$

$$x(x+6)(x-5) < 0$$



Answer :  $(-\infty, -6) \cup (0, 5)$

$$(b) (x^2 - 1)(x-4)(x+5) \leq 0$$

$$(x-1)(x+1)(x-4)(x+5) \leq 0$$



Answer :  $[-5, -1] \cup [1, 4]$

5.2

$$4. (a) \begin{array}{r} x^3 + 2x^2 - 4x - 8 \\ -x^3 - 3x^2 \\ \hline -5x^2 - 4x - 8 \\ 5x^2 - 15x \\ \hline -11x - 8 \\ -11x - 33 \\ \hline 25 \end{array} \quad | \frac{x-3}{x^2 + 5x + 11}$$

Answer:  $(x^2 + 5x + 11) + \frac{25}{x-3}$

(b)

$$\begin{array}{r}
 -2x^3 - 3ix^2 + 11x + (1-5i) \\
 \underline{-2x^3 - ix^2} \\
 -2ix^2 + 11x + (1-5i) \\
 \underline{-2ix^2 + x} \\
 -10x + (1-5i) \\
 \underline{10x - 5i} \\
 1
 \end{array}$$

Answer:  $(x^2 + ix + 5) + \frac{1}{2x-i}$

5. (a)

$$\begin{array}{c|ccccc}
 & 12 & -7 & -32 & -7 & 6 \\
 \hline
 1 & 12 & 5 & -27 & -34 & -28
 \end{array}$$

$$p(c) = p(1) = -28$$

(b)

$$\begin{array}{c|cc|cc}
 & 2 & -(3-5i) & (3-9i) \\
 \hline
 -3i & 2 & -3 - i & 0
 \end{array}$$

$$p(-3i) = 0$$

6. (a)

$$p(x) = a(x+2)(x-1)(x-3)$$

$$p(0) = -12$$

$$a \cdot 2 \cdot (-1) \cdot (-3) = -12$$

$$6a = -12$$

$$a = -2$$

$$p(x) = -2(x+2)(x-1)(x-3)$$

(6)  $p(x) = a(x+4)(x-3)$

$$a = 1$$

Then  $p(x) \rightarrow -\infty$  as  $x \rightarrow -\infty$

$$P(x) = (x+4)(x-3)$$

5.3

7. (a)  $q(x) = x^3 - 10x^2 + 23x - 14$

$$a_0 = -14$$

factors:  $\{\pm 1, \pm 2, \pm 7, \pm 14\}$

$$a_n = 1$$

factors:  $\{\pm 1\}$

Potential rational zeros:  $\{\pm 1, \pm 2, \pm 7, \pm 14\}$

(b)  $k(x) = x^4 - 10x^2 + 24$

$$a_0 = +24$$

factors:  $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6,$

$$a_n = 1$$

$\pm 8, \pm 12, \pm 24\}$

↑ factors:  $\{\pm 1\}$

Potential rational zeros:  $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24\}$

8. (a)  $f(x) = \underbrace{x^4 - 25}_{\text{one sign change}}$

We have one positive real zero

$$f(-x) = (-x)^4 - 25 = \underbrace{x^4 - 25}_{\text{one sign change}}$$

We have one negative real zero

(b)  $g(x) = \underbrace{x^3 + 6x^2 + 11x + 6}_{\text{No change}}$

No positive real zero

$$g(-x) = \underbrace{-x^3 + 6x^2 - 11x + 6}_{\text{one change one one}}$$

We have 3 or 1 negative real zeros.

9. (a)  $f(x) = 5x^3 - 4x^2 - 31x - 6$ , -3 and -1

$$a = -3$$

$$b = -1$$

$$\begin{aligned} f(a) &= f(-3) = 5 \cdot (-27) - 4 \cdot 9 - 31(-3) - 6 = \\ &= -135 - 36 + 93 - 6 = -84 < 0 \end{aligned}$$

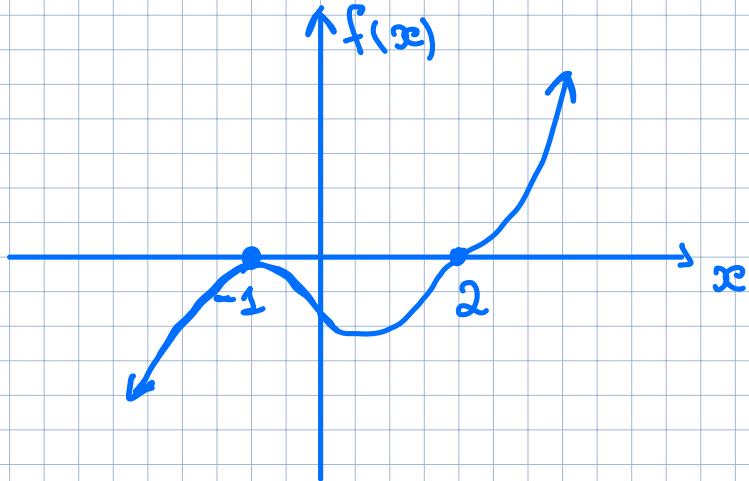
$$f(8) = f(-1) = -5 - 4 + 31 - 6 = 16 > 0$$

Hence, by IVT there is at least one  $-3 < c < -1$  such that

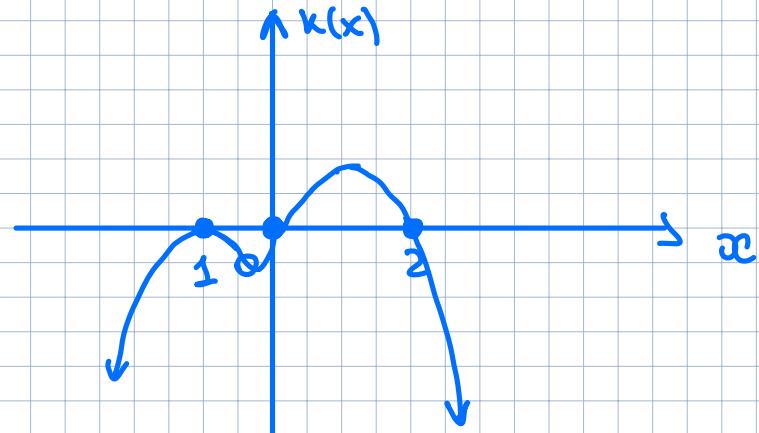
$$f(c) = 0$$

5.4

10. (a)  $f(x) = (x+1)^2(x-2)^3$



(b)  $k(x) = -x^3(x-2)(x+1)^2$



$$11. \text{ (a)} \quad x^4 + 15 = 2x^3 + 8x^2 - 10x$$

$$x^4 - 2x^3 - 8x^2 + 10x + 15 = 0$$

$$x = -1$$

$$1 + 2 - 8 - 10 + 15 = 0$$

	1	-2	-8	10	15
-1	1	-3	-5	15	0

$$(x+1)(x^3 - 3x^2 - 5x + 15) = 0$$

$$x = 3$$

$$27 - 27 - 15 + 15 = 0$$

	1	-3	-5	15
3	1	0	-5	0

$$(x+1)(x-3)(x^2 - 5) = 0$$

$$(x+1)(x-3)(x-\sqrt{5})(x+\sqrt{5}) = 0$$

Answer:  $\{-1, 3, \pm\sqrt{5}\}$

$$(B) \quad x^3 - 5 = 5x^2 - 9x$$

$$x^3 - 5x^2 + 9x - 5 = 0$$

$$x = 1$$

$$1 - 5 + 9 - 5 = 0$$

$$\begin{array}{c|ccccc} & 1 & -5 & 9 & -5 \\ \hline 1 & 1 & -4 & 5 & 0 \end{array}$$

$$(x-1)(x^2-4x+5) = 0$$

$$(x-1)(x-5)(x+1) = 0$$

Answer:  $\{1, 5, -1\}$

5.5

12. (a)  $f(x) = \frac{x^2+5}{x^3-27} = \frac{x^2+5}{(x-3)(x^2+3x+9)}$

$$x-3=0$$

V.A.:  $x = 3$

$$x^2 + 3x + 9 = 0$$

$$D = 9 - 36 < 0$$

$\emptyset$

Answer:  $x=3$  is a V.A.

(b)  $g(x) = \frac{x^2-1}{x^2-8x+7} = \frac{x^2-1}{(x-7)(x-1)}$

V.A.:  $x=7$  and  $x=1$

$$13. (a) f(x) = \frac{5}{x-1}$$

$$p(x) = 5, n=0$$

$$q(x) = x-1, m=1$$

$n < m$

Therefore,  $y=0$  is H.A.

$$(b) g(x) = \frac{2x^2 - 5x + 6}{x-3}$$

$$p(x) = 2x^2 - 5x + 6, n=2$$

$$q(x) = x-3, m=1$$

$n > m$

O.A. :

$$\begin{array}{r} -2x^2 - 5x + 6 \\ \hline 2x^2 - 6x \\ \hline -x + 6 \\ \hline x - 3 \\ \hline 9 \end{array} \quad | \begin{array}{l} x-3 \\ 2x+1 \end{array}$$

O.A. :

$$y = 2x + 1$$

$$(c) k(x) = \frac{-3x+5}{x-2}$$

$$p(x) = -3x + 5, m=1$$

$$q(x) = x-2, n=1$$

$m = h$

H.A. :  $y = \frac{-3}{1} = -3$

$y = -3$

14. (B)  $g(x) = \frac{5}{x-1}$

① Dom( $g$ ) =  $\mathbb{R} \setminus \{1\}$

② V.A. :  $x = 1$

③ H.A. :  $y = 0$

④ O.A. : None

⑤ x-intercept:  $g(x) = 0$

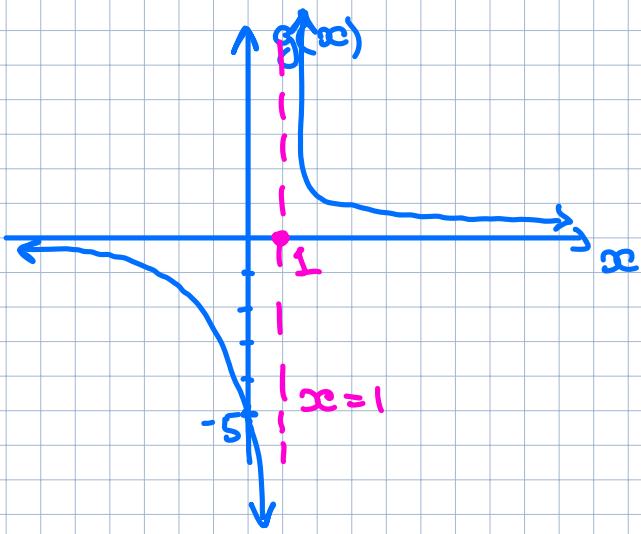
$$\frac{5}{x-1} = 0$$

None!

⑥ y-intercept:  $x = 0$

$$g(0) = -5$$

$$(0, -5)$$



15. (a)  $\frac{5}{x-2} > \frac{3}{x+2}$

$$\frac{5}{x-2} - \frac{3}{x+2} > 0$$

$$\frac{5(x+2) - 3(x-2)}{(x-2)(x+2)} > 0$$

$$\frac{2x + 16}{(x-2)(x+2)} > 0$$

①  $2x + 16 = 0$

$$x = -8$$

②  $(x-2)(x+2) = 0$

$$x = 2 \text{ or } x = -2$$

$$\begin{array}{ccccccc} - & 0 & + & 0 & - & 0 & + \\ \hline & -8 & & -2 & & 2 & \end{array}$$

Solution :  $(-8, -2) \cup (2, +\infty)$

$$(B) \frac{x}{x^2 - x - 6} \leq \frac{-1}{x^2 - x - 6}$$

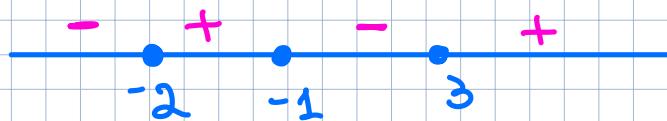
$$\frac{x+1}{x^2 - x - 6} \leq 0$$

①  $x = -1$

②  $x^2 - x - 6 = 0$

$$(x-3)(x+2) = 0$$

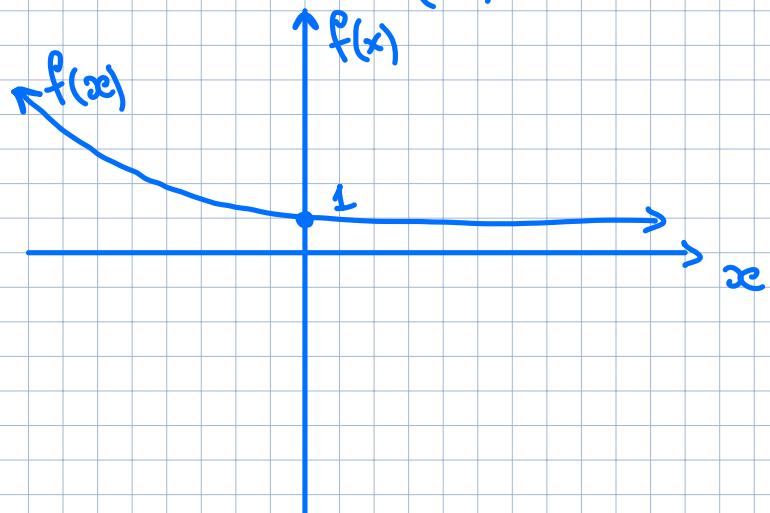
$$x = 3 \quad \text{or} \quad x = -2$$



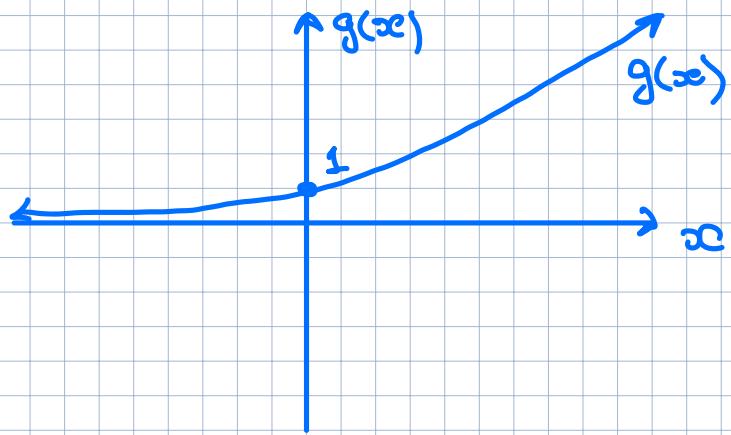
Solution :  $(-\infty, -2] \cup [-1, 3]$ .

6.1

17. (a)  $f(x) = \left(\frac{1}{2}\right)^x$



(B)  $g(x) = 10^x$



$$(c) \quad h(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$$

See (a)

$$(d) \quad k(x) = 2 - 4^{2-x} = 2 - 4^{-x+2}$$

$$① \quad b(x) = 4^x$$

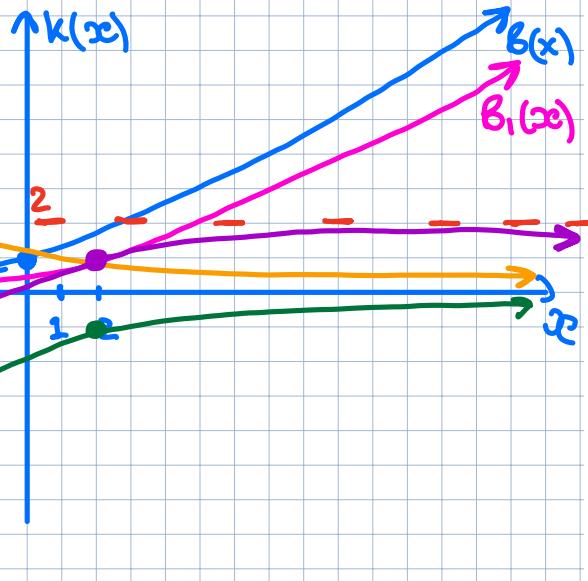
$$\nearrow b_2(x)$$

$$② \quad b_1(x) = 4^{x-2}$$

$$③ \quad b_2(x) = 4^{-(x-2)}$$

$$④ \quad b_3(x) = -4^{-(x-2)}$$

$$⑤ \quad b_4(x) = 2 - 4^{-(x-2)}$$



$$18. (a) \quad 3^{2x-1} = 27$$

$$3^{2x-1} = 3^3$$

$$2x-1 = 3$$

$$2x = 4$$

$$x = 2$$

$$(B) \quad 3^{x^2+4x} = 81^{-1}$$

$$3^{x^2+4x} = 3^{-4}$$

$$x^2 + 4x = -4$$

$$x^2 + 4x + 4 = 0$$

$$(x+2)^2 = 0$$

$$x = -2$$

$$(C) \quad 10^x = 0.01$$

$$10^x = 10^{-2}$$

$$x = -2$$

6.2

$$19. \quad P_0 = 7.75$$

$$r = 1.9\% = 0.019$$

$$t = 10 \text{ years}$$

$$P(t) = P_0 a^t$$

$$P(10) = 7.75 \cdot (0.019)^{10} \approx 9.35 \text{ Billion}$$

$$t = 50 \text{ years}$$

$$P(50) = 7.75 \cdot (0.019)^{50} \approx 19.66 \text{ billion}$$

20.

$$N(t) = \frac{10\ 000}{1 + 999 e^{-t}}$$

(a)  $N(0) = \frac{10\ 000}{1 + 999} = 10$

(b)  $N(t) \rightarrow 10\ 000$  as  $t \rightarrow \infty$ .