Section 8.2. Sum and Difference Identities

1. Sum and difference identities.
2. Using sum and difference identities for exact evaluation.
3. Using sum and difference identities for verification and simplification.

$\boxed{1.}$

Identities (Sum and difference)

- $\sin(u+v) = \sin u \cos v + \cos u \sin v$
- $\sin(u-v) = \sin u \cos v - \cos u \sin v$

- $\cos(u+v) = \cos u \cos v - \sin u \sin v$
- $\cos(u-v) = \cos u \cos v + \sin u \sin v$

- $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$
- $\tan(u-v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

$\boxed{2.}$

Example

$$\sin(75^\circ) = \sin(45^\circ + 30^\circ) =$$

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ =$$

$$= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right).$$

Example

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) =$$

$$= \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3} \cdot \tan\frac{\pi}{4}} =$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3} \cdot 1} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$\boxed{3.}$

Example

Verify: $\sin\left(\frac{\pi}{2} - x\right) = \cos x.$

Solution

$$\sin\left(\frac{\pi}{2} - x\right) = \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x =$$

$$= 1 \cdot \cos x - 0 = \cos x.$$

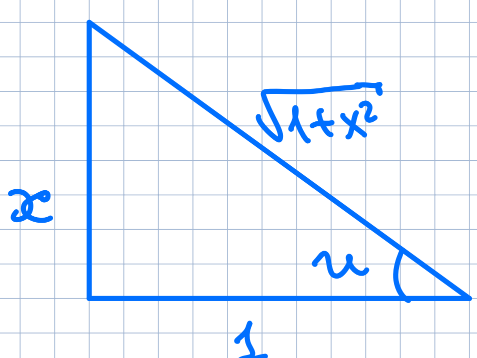
Example

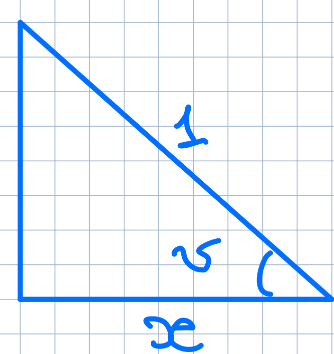
Express $\sin(\tan^{-1}x + \cos^{-1}x)$ as function of x.

Solution

$$u = \tan^{-1}x \Rightarrow x = \tan u$$

$$v = \cos^{-1}x \Rightarrow x = \cos v$$





$$\sin(\tan^{-1}x + \cos^{-1}x) = \sin(u+v) =$$

$= \sin u \cdot \cos v + \cos u \sin v =$

$$= \frac{x^2}{\sqrt{1+x^2}} + \frac{\sqrt{1-x^2}}{\sqrt{1+x^2}} = \frac{x^2+\sqrt{1-x^2}}{\sqrt{1+x^2}}$$

$\blacktriangledown$

Theorem (Sum of Sines and Cosines)

$$A \sin x + B \cos x = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}} \sin x + \right.$$
$$\left. + \frac{B}{\sqrt{A^2+B^2}} \cos x\right) = \sqrt{A^2+B^2} \sin(x+\varphi),$$

where $\cos\varphi = \frac{A}{\sqrt{A^2+B^2}}$ and $\sin\varphi = \frac{B}{\sqrt{A^2+B^2}}$

Example

$f(x) = \sin x - \sqrt{3}\cos x$

$A = 1$
$B = -\sqrt{3}$

$$f(x) = \sqrt{1+3}\left(\frac{1}{2} \sin x - \frac{\sqrt{3}}{2}\cos x\right) =$$

$$= 2 \cdot \sin(x+\varphi)$$

$$\sin\varphi = -\frac{\sqrt{3}}{2}, \quad \cos\varphi = \frac{1}{2}$$

$$\varphi = -\frac{\pi}{3}$$

Thus, $f(x) = 2 \cdot \sin\left(x - \frac{\pi}{3}\right).$ ▼