Section 4.1. Transformations of Functions

1. Shifting graphs vertically and horizontally.
2. Reflecting graphs.
3. Stretching graphs vertically and horizontally.
4. Order of transformations.

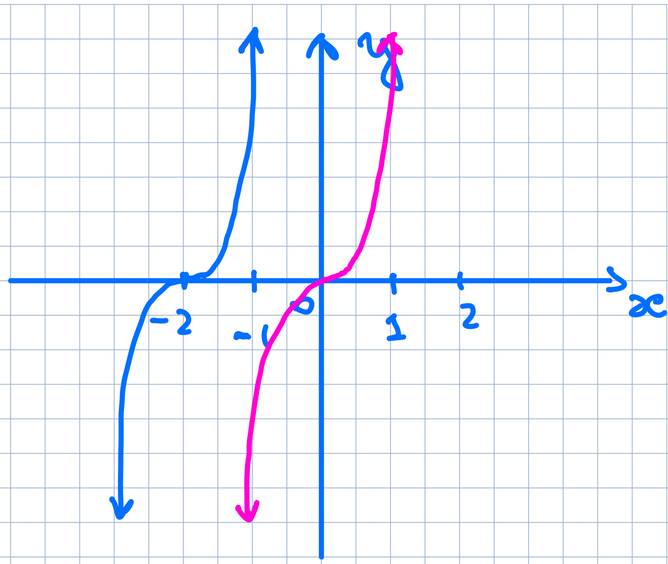
$\boxed{1.}$

Theorem (Horizontal Shifting / Translation)

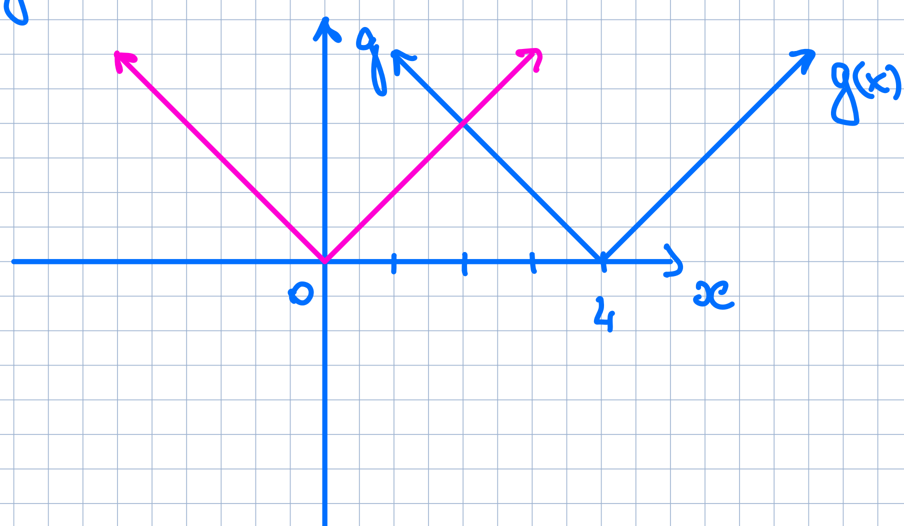
Let $f(x)$ be the function, and let h be a fixed real number.
Now, let us set $g(x) = f(x-h)$.
The graph of g has the same shape as the graph of f, but shifted h units to the right if $h>0$ and shifted h units to the left if $h<0$.

Example:

- $f(x) = (x+2)^3$



- $g(x) = |x-4|$



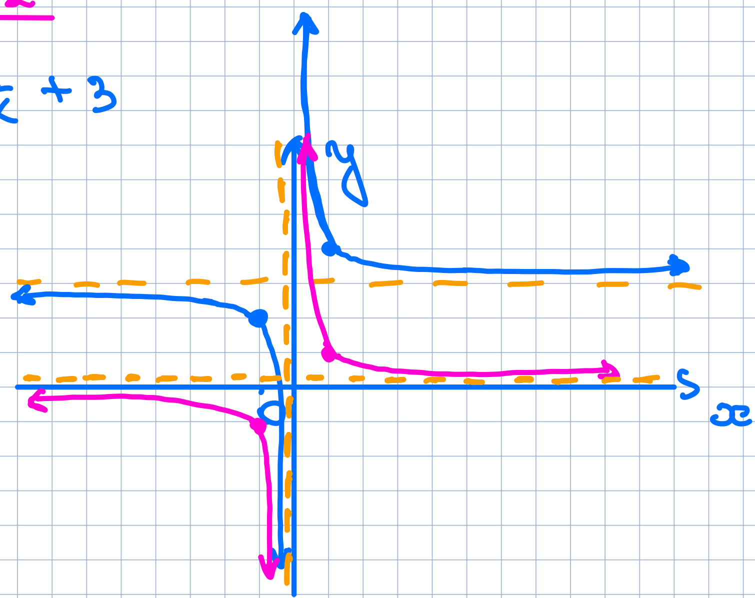
Theorem (Vertical shifting / translation)

Let $f(x)$ be a function whose graph is known, and let k be a fixed real number. The graph of the function $g(x) = f(x) + k$ is the same shape as the graph of f, but shifted k

units up if $k>0$ and k units down if $k<0$.

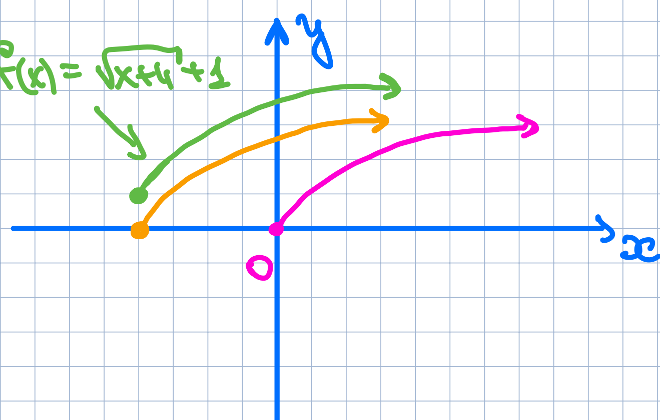
Example 2

- $f(x) = \frac{1}{x} + 3$



Example 3

- $f(x) = \sqrt{x+4} + 1$



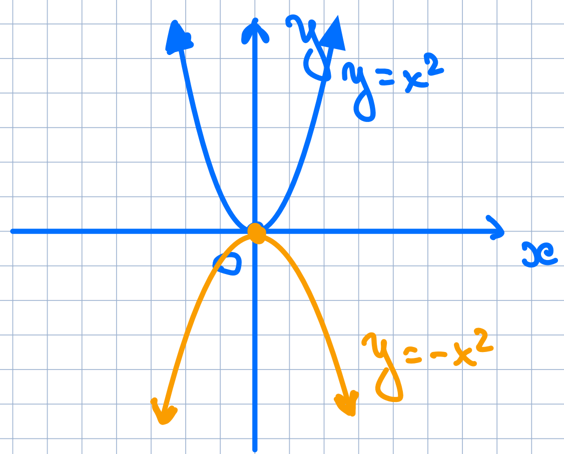
$\boxed{2.}$ Theorem (Reflecting with respect to the Axes)

Given a function $f(x)$:

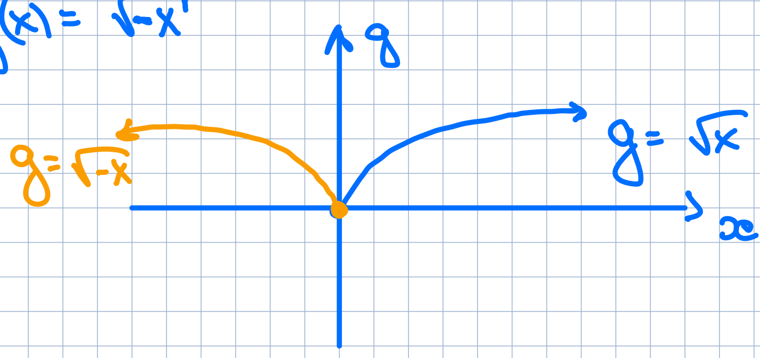
1. the graph of the function $g(x) = -f(x)$ is the reflection of the graph of f with respect to the x-axis
2. the graph of the function $g(x) = f(-x)$ is the reflection of the graph of f with respect to the y-axis.

Example

- $f(x) = -x^2$



- $g(x) = \sqrt{-x}$

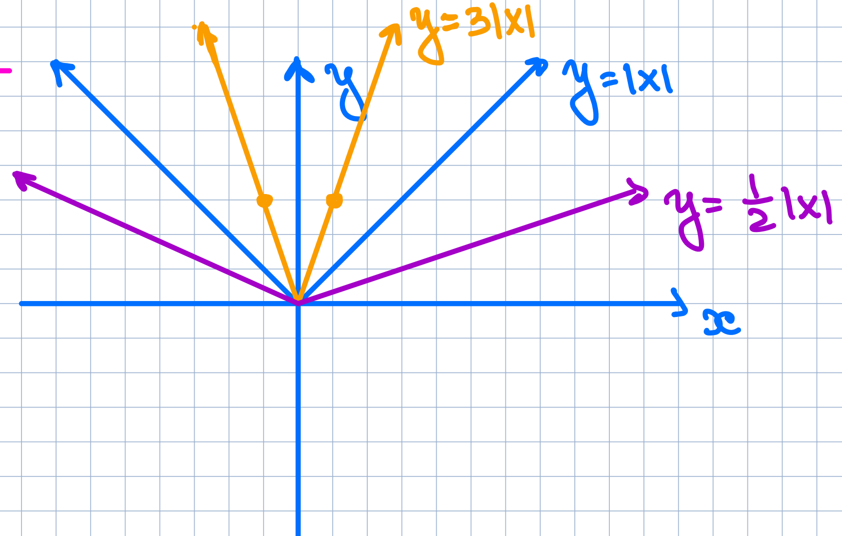


3. **Theorem** (Vertical stretching and compressing)

Let $f(x)$ be a function and let a be a positive real number.

1. The graph of the function $g(x) = af(x)$ is stretched vertically by a factor a if $a > 1$.
2. The graph of the function $g(x) = af(x)$ is compressed vertically by a factor a if $0 < a < 1$.

Example



Theorem (Horizontal stretching and compressing)

Let $f(x)$ be a function and a be

a positive real number.

1. The graph of the function $g(x) = f(ax)$ is stretched horizontally by a factor of $\boxed{\frac{1}{a}}$ if $0 < a < 1$.

2. The graph of the function $g(x) = f(ax)$ is compressed horizontally by a factor of $\boxed{\frac{1}{a}}$ if $a > 1$.

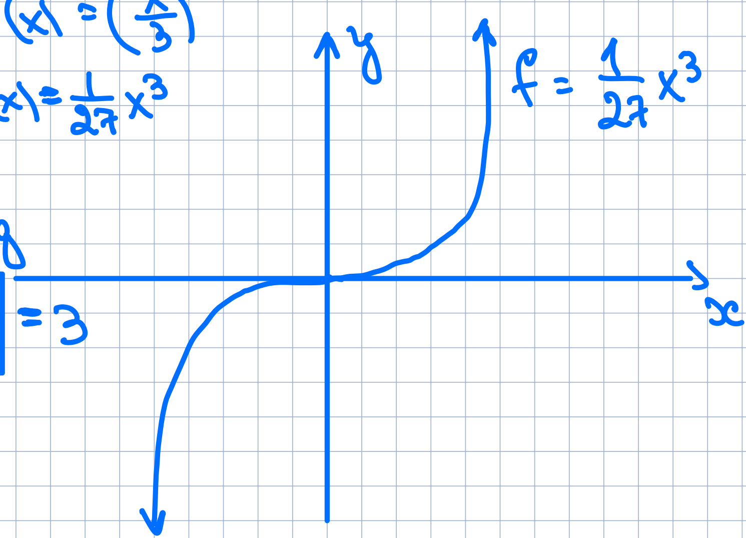
Example

$\boxed{a = \frac{1}{3}}$

$f(x) = \left(\frac{x}{3}\right)^3$

$f(x) = \frac{1}{27}x^3$

stretching by a factor $\boxed{\frac{1}{a}} = 3$



$\boxed{4.}$ Procedure (Order of Transformations):

1. Horizontal shifts.
2. Horizontal and vertical stretching and compressing.

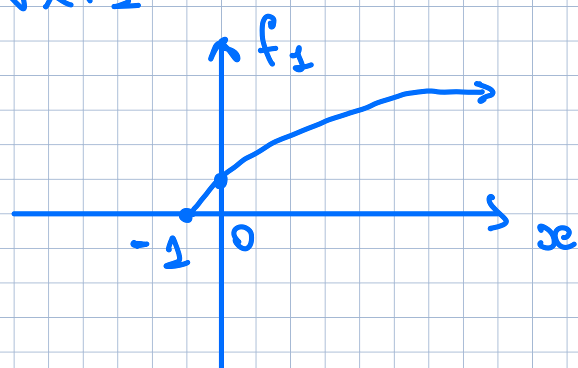
3. Reflections.
4. Vertical shifts.

Example

$$g(x) = -2\sqrt{2x+2} + 3$$

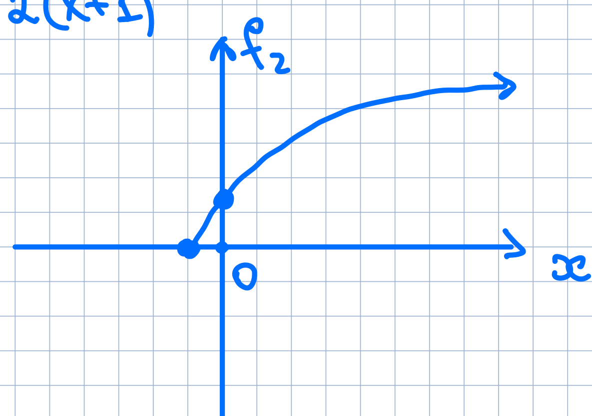
1. $f_1(x) = \sqrt{x+1}$

horizontal shift



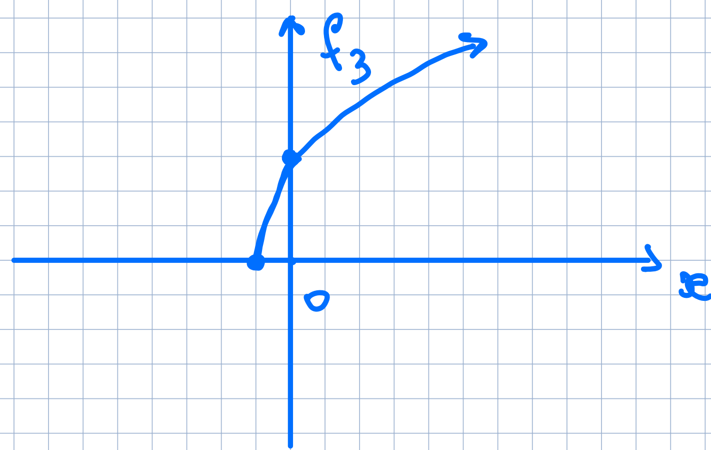
2. $f_2(x) = \sqrt{2(x+1)}$

compressing horizontally by $\frac{1}{2}$



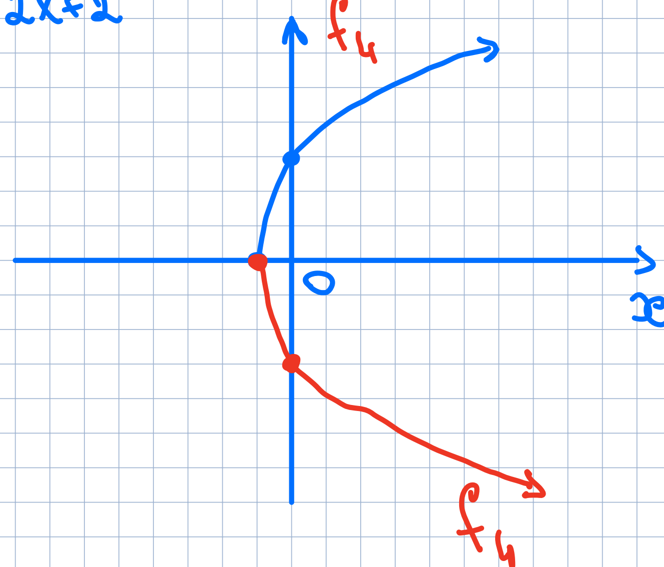
3. $f_3(x) = 2\sqrt{2(x+1)}$

vertical stretching by 2



4. $f_4(x) = -2\sqrt{2x+2}$

reflection with respect to x-axis



5. $f_5(x) = -2\sqrt{2x+2} + 3$

vertical shift per 3 units up