THEORETICAL PART:

Definition 1.

The **Imaginary unit i** is defined as $i = \sqrt{-1}$. In other words, i has the property that its square is -1:

$$i^2 = -1.$$

Definition 2.

If a is a positive real number, $\sqrt{-a} = i\sqrt{a}$.

Definition 3.

For any two real numbers a and b, the sum $a + bi$ is a **complex number**. The collection $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ is called the set of complex numbers. The number a is called the **real part** of $a + bi$, and the number b is called the **imaginary part**. If $a = 0$, then we obtain simply a real number. If $b = 0$, then we obtain a pure imaginary number.

Simplifying Complex Expressions:

- Add, subtract, or multiply the complex numbers, as required, by treating every complex number $a + bi$ as a polynomial expression.
- Complete the simplification by using the fact that $i^2 = -1$.

Definition 4. Given any complex number $a + bi$, the complex number $a - bi$ is called its **complex conjugate**.

A very useful property:

$$(a + bi)(a - bi) = a^2 + b^2$$

Definition 5 (Principal square roots). Given $a \in \mathbb{R}$, $a > 0$, we have:

$$\sqrt{a} \in \mathbb{R}, \ \sqrt{a} > 0$$

$$\sqrt{-a} = i\sqrt{a}.$$

Caution: If a and b are both real numbers, then:

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

As for complex numbers, first simplify any square roots of negative numbers by rewriting them as pure imaginary numbers.

PRACTICAL PART:

1. Simplify the following expressions:

   (a) $\sqrt{-16} =$

   (b) $\sqrt{-8} =$

   (c) $i^3 =$

   (d) $i^8 =$

   (e) $i^{102} =$

2. Simplify the following complex expressions:

   (a)
$$(4 + 3i) + (-5 + 7i) =$$

   (b)
$$(3 + 2i)(-2 + 3i) =$$

   (c)
$$(2 - 3i)^2 =$$

3. Simplify the following expressions:

   (a)
   $$\frac{2+3i}{3-i} =$$

   (b)
   $$(4-3i)^{-1} =$$

4. Simplify the following expressions:

   (a)
   $$(2-\sqrt{-3})^2 =$$

   (b)
   $$\frac{\sqrt{4}}{\sqrt{-4}} =$$