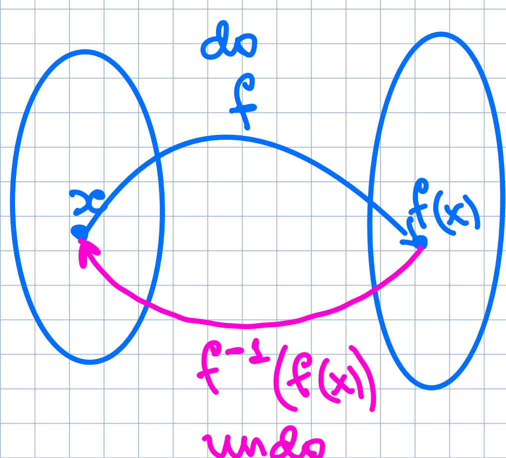
Section 4.4. Inverses of Functions

1. Inverses of relations.
2. Inverse functions and the horizontal line test.
3. Finding the inverse of a function.

$\boxed{1.}$



Def. (Inverse of a Relation)

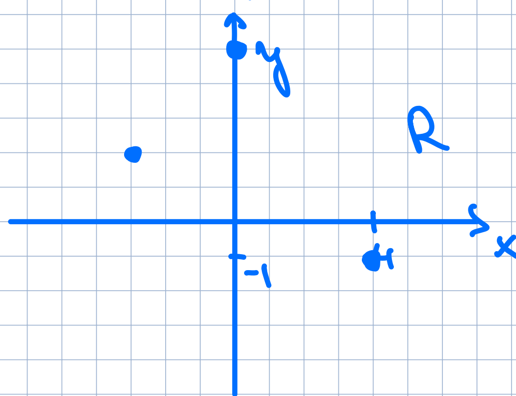
Let R be a relation. The <u>inverse of R</u>, denoted R^{-1}, is the relation defined by switching the first and second coordinates of each ordered pair that is an element of R.

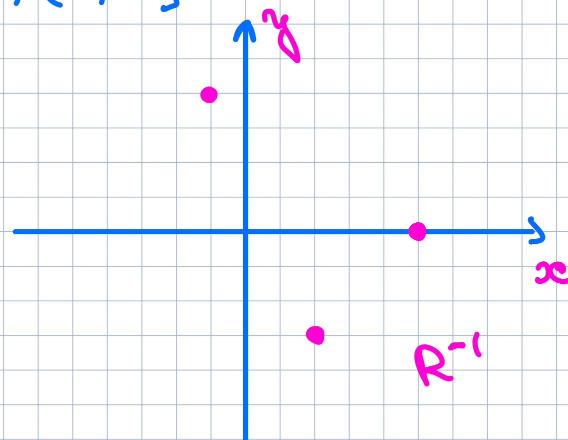
$$R^{-1} = \{(b,a) \mid (a,b) \in R\}$$

Example

- $R = \{(4,-1), (-3,2), (0,5)\}$

  $R^{-1} = \{(-1,4), (2,-3), (5,0)\}$





$Dom(R) = \{4, -3, 0\}$

$Dom(R^{-1}) = \{-1, 2, 5\}$

$Ran(R) = \{-1, 2, 5\}$

$Ran(R^{-1}) = \{4, -3, 0\}$

$\boxed{2.}$

Caution!

$f^{-1} \neq \frac{1}{f}$, f is a function

Theorem (The horizontal line test)

Let f be a function. We say that the graph of f passes the <u>horizontal line test</u> if every horizontal line in the plane intersects the graph no more than once. If f passes the horizontal line test, then f^{-1} is also a function.

Def. (One-to-One function)

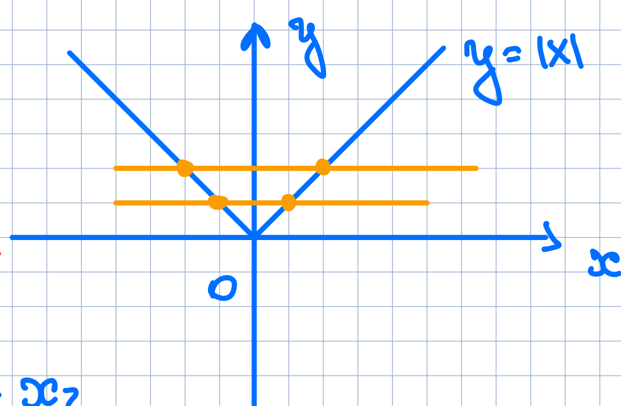
A function f is <u>one-to-one</u> if, for every pair of distinct elements x_1 and x_2 in the domain of f, we have $f(x_1) \neq f(x_2)$.

This means that every element of the range of f is paired with exactly one element of the domain of f.

$$\boxed{f^{-1} \text{ is a function iff } f \text{ is one-to-one}}$$

Example

- $f(x) = |x|$



f is not 1-to-1

$x_1 = 1$

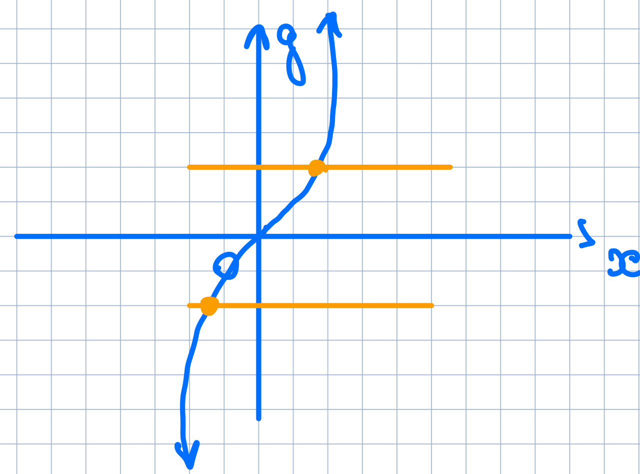
$x_2 = -1$

$x_1 \neq x_2$

But $f(x_1) = |1| = 1 = |-1| = f(x_2)$.

Therefore, $f^{-1}(x)$ does not exist.

- $g(x) = x^3$



g is 1-to-1

Therefore,
$g^{-1}(x)$ exists.

$\boxed{3.}$

Procedure (Finding formulas of inverse functions)

Let f be a 1-to-1 function, and assume that f is defined by a formula. To find a formula for f^{-1}, perform the following steps:

<u>Step 1:</u> Replace $f(x)$ in the definition of f with the variable y. The result is an equation in x and y that is solved for y at this point.

<u>Step 2:</u> Solve the equation for x.

Step 3: Replace x in the resulting equation with $f^{-1}(x)$ and replace each occurrence of y with x.

Example

$$f(x) = (x-1)^3 + 2$$

Step 1 $\quad y = (x-1)^3 + 2$

Step 2

$$y - 2 = (x-1)^3$$

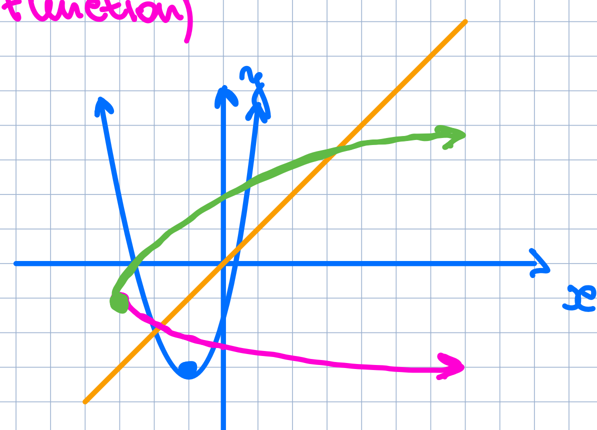
$$(x-1) = \sqrt[3]{y-2}$$

$$x = \sqrt[3]{y-2} + 1$$

Step 3 $\quad f^{-1}(x) = \sqrt[3]{x-2} + 1$

Example (Finding the inverse of a Restricted-domain function)

$$f(x) = (x+1)^2 - 3$$



f is not 1-to-1

Let us consider cases:

1) $x \in (-\infty ; -1]$

f is 1-to-1 on $(-\infty, -1]$

$y = (x+1)^2 - 3$

$y+3 = (x+1)^2$

$-\sqrt{y+3} = x+1$

$x = -\sqrt{y+3} - 1$

$f^{-1}(x) = -\sqrt{x+3} - 1$

2) $x \in [-1, \infty)$

f is 1-to-1 on $[-1, \infty)$

$y = (x+1)^2 - 3$

$y+3 = (x+1)^2$

$(x+1) = \sqrt{y+3}$

$x = \sqrt{y+3} - 1$

$f^{-1}(x) = \sqrt{x+3} - 1$

Theorem (Composition of functions and inverses)

Given a function f and its inverse f^{-1}, the following statements are true:

$$f(f^{-1}(x)) = x \text{ for all } x \in \mathrm{Dom}(f^{-1})$$

$$f^{-1}(f(x)) = x \text{ for all } x \in \mathrm{Dom}(f).$$

Example

- $f(x) = (x-1)^3 + 2$

  $f^{-1}(x) = (x-2)^{\frac{1}{3}} + 1$

$$f(f^{-1}(x)) = f\left((x-2)^{1/3} + 1\right) =$$

$$= \left((x-2)^{1/3} + 1 - 1\right)^3 + 2 = \left((x-2)^{1/3}\right)^3 + 2 =$$

$$= (x-2) + 2 = x$$

$$f(f^{-1}(x)) = x$$