Solutions

THEORETICAL PART:

Identities (Sum and Difference Identities):

Sine Identities

$$\sin(u+v) = \sin u \cos v + \cos u \sin v, \quad \sin(u-v) = \sin u \cos v - \cos u \sin v$$

Cosine Identities

$$\cos(u+v) = \cos u \cos v - \sin u \sin v, \quad \cos(u-v) = \cos u \cos v + \sin u \sin v$$

Tangent Identities

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}, \quad \tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Theorem (Sum of Sines and Cosines)

$$A\sin(x) + B\cos(x) = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\sin(x) + \frac{B}{\sqrt{A^2+B^2}}\cos(x)\right) = \sqrt{A^2+B^2}\sin(x+\varphi),$$

where

$$\cos\varphi = \frac{A}{\sqrt{A^2+B^2}}, \quad \sin\varphi = \frac{B}{\sqrt{A^2+B^2}}$$

PRACTICAL PART:

1. Determine the exact value of $\sin 75°$.

$$\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° +$$

$$+ \cos 45° \sin 30° = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

2. Determine the exact value of $\cos 75°$.

$$\cos 75° = \cos(45° + 30°) = \cos 45° \cos 30° -$$
$$- \sin 45° \sin 30° = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

3. Determine the exact value of $\tan(\pi/12)$.

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{4}} =$$
$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3} \cdot 1} = \frac{\sqrt{3} - 1}{1 + \sqrt{3}}$$

4. Determine the exact value of $\sin 80° \cos 20° - \cos 80° \sin 20°$.

$$\sin 80° \cos 20° - \cos 80° \sin 20° = \sin(80° - 20°) =$$

$$= \sin 60° = \frac{\sqrt{3}}{2}$$

5. Use a difference identity to verify that $\sin\left(\frac{\pi}{2} - x\right) = \cos x$.

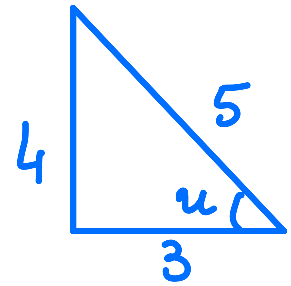
$$\sin\left(\frac{\pi}{2} - x\right) = \underbrace{\sin \frac{\pi}{2}}_{1} \cos x - \overbrace{\cos \frac{\pi}{2}}^{0} \sin x =$$

$$= 1 \cdot \cos x = \cos x$$

6. Evaluate the expression $\sin\left(\cos^{-1}(3/5) - \tan^{-1}(12/5)\right)$ =

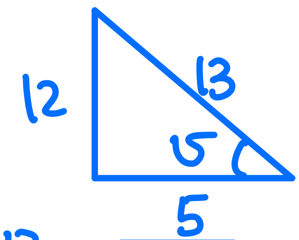
$u = \cos^{-1}(3/5)$, $v = \tan^{-1}(12/5)$

$= \sin u \cos v - \cos u \sin v =$



$\cos^{-1}\left(\frac{3}{5}\right) = u \Rightarrow \cos u = \frac{3}{5}$

$\tan^{-1}\left(\frac{12}{5}\right) = v \Rightarrow \tan v = \frac{12}{5}$



$\sin u = \frac{4}{5}$

$\sin v = \frac{12}{13}$

$\cos v = 5/13$

$= \frac{4}{5} \cdot \frac{5}{13} - \frac{3}{5} \cdot \frac{12}{13} = \boxed{-\frac{16}{65}}$

7. Express $\sin(\tan^{-1} x + \cos^{-1} x)$ as an algebraic function of x.

$\tan^{-1} x = u \Rightarrow x = \tan u$

$\cos^{-1} x = v \Rightarrow x = \cos v$

$\sin(u+v) = \sin u \underbrace{\cos v}_{= x} + \cos u \cdot \sin v$

$\sin v = \sqrt{1-\cos^2 v} = \sqrt{1-x^2}$

$\tan u = \frac{\sin u}{\cos u} = x \Rightarrow \sin u = x \cos u$

$\frac{1}{\cos^2 u} = 1 + \tan^2 u$

$$\cos^2 u = \frac{1}{1+\tan^2 u}$$

$$\cos u = \sqrt{\frac{1}{1+\tan^2 u}}$$

$$\cos u = \sqrt{\frac{1}{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$$

$$\sin u = \tan u \cdot \cos u = x \frac{1}{\sqrt{1+x^2}}$$

Hence, $\sin(u+v) = \frac{x}{\sqrt{1+x^2}} \cdot x + \frac{1}{\sqrt{1+x^2}} \cdot \sqrt{1-x^2} =$

$$= \boxed{\frac{x^2 + \sqrt{1-x^2}}{\sqrt{1+x^2}}}$$

8. Express the function $f(x) = \sin x - \sqrt{3}\cos x$ in terms of a single sine function, and graph the result.

$$f(x) = 1 \cdot \sin x - \sqrt{3} \cdot \cos x = 2\left(\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x\right) =$$

$$= 2\left(\cos\frac{\pi}{3}\sin x - \sin\frac{\pi}{3}\cos x\right) = 2\sin\left(x - \frac{\pi}{3}\right)$$

$$x - \frac{\pi}{3} = 0 \qquad\qquad x - \frac{\pi}{3} = 2\pi$$

$$x = \frac{\pi}{3} \qquad\qquad x = \frac{\pi + 6\pi}{3} = \frac{7\pi}{3}$$