Solutions

THEORETICAL PART:

Definition (Quadratic Function):

A **quadratic function** f in the variable x, also known as a **second-degree polynomial function**, is any function that can be written in the form

$$f(x) = ax^2 + bx + c,$$

where a, b, c are real numbers and $a \neq 0$.

The graph of any quadratic function is a **parabola**.

Definition (Vertex Form of a Quadratic Function):

The graph of the function $g(x) = a(x-h)^2 + k$, where a, h and k are real numbers and $a \neq 0$, is a parabola whose vertex is at (h, k). The parabola is narrower than $f(x) = x^2$ if $|a| > 1$ and is broader than $f(x) = x^2$ if $0 < |a| < 1$. The parabola opens upward if a is positive and downward if a is negative.

Formula (Vertex of a Quadratic Function):

Given a quadratic function $f(x) = ax^2 + bx + c$, the graph of f is a parabola with a vertex given by

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$$

Quadratic Regression:

Goal: for a given number of points $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$, we would like to find the **quadratic** equation whose graph comes closest to "fitting" the points. We use a **Least-Squares Method**.

PRACTICAL PART:

1. Sketch the graph of the following quadratic functions:

   (a) $f(x) = x^2$ and $f(x) = -x^2$

   (b) $f(x) = x^2 + 1$

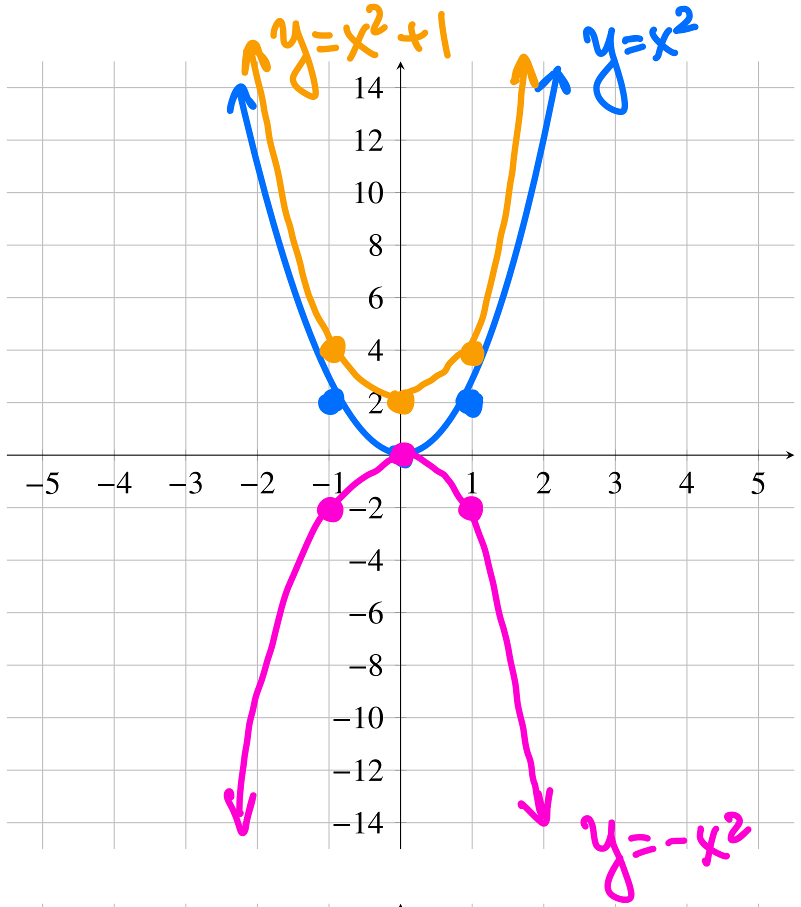
   (c) $f(x) = 2x^2 - 1$

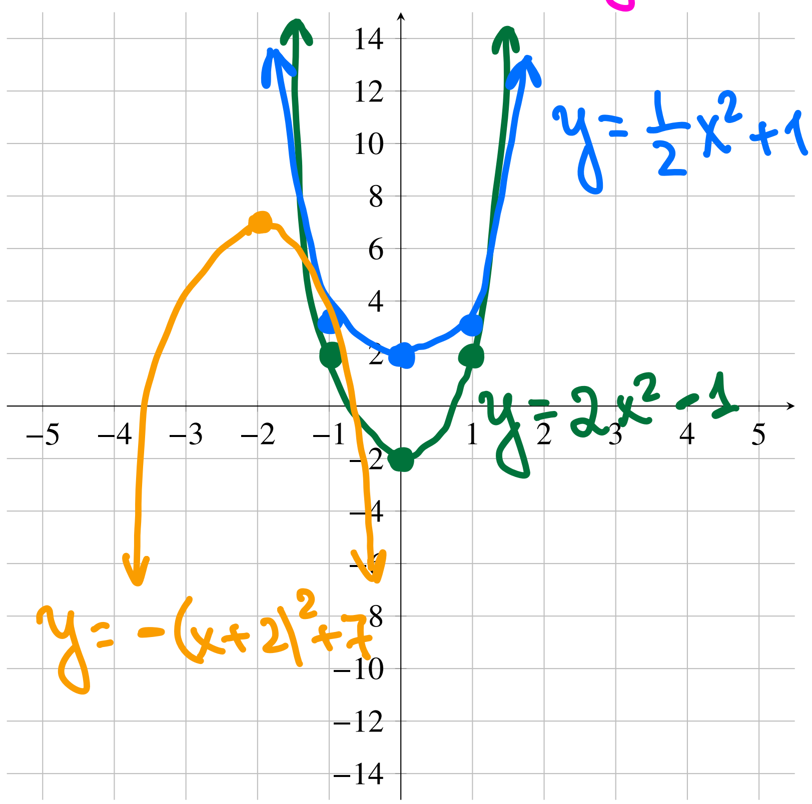
   (d) $f(x) = \frac{1}{2}x^2 + 1$

   (e) $f(x) = -x^2 - 2x + 3$ $= -(x^2+2x-3) = -(x^2+2x+4-7) =$

   $= -((x+2)^2-7) = -(x+2)^2 + 7$

   Vertex: $(-2, 7)$


y=x²+1
y=x²
y=-x²
14
12
10
8
6
4
2
−2
−4
−6
−8
−10
−12
−14
−5 −4 −3 −2 −1 1 2 3 4 5


y= ½x²+1
y=2x²-1
y=-(x+2)²+7
14
12
10
8
6
4
2
−2
−4
−8
−10
−12
−14
−5 −4 −3 −2 −1 1 2 3 4 5

2. Find the vertex of the following quadratic function using the vertex formula:

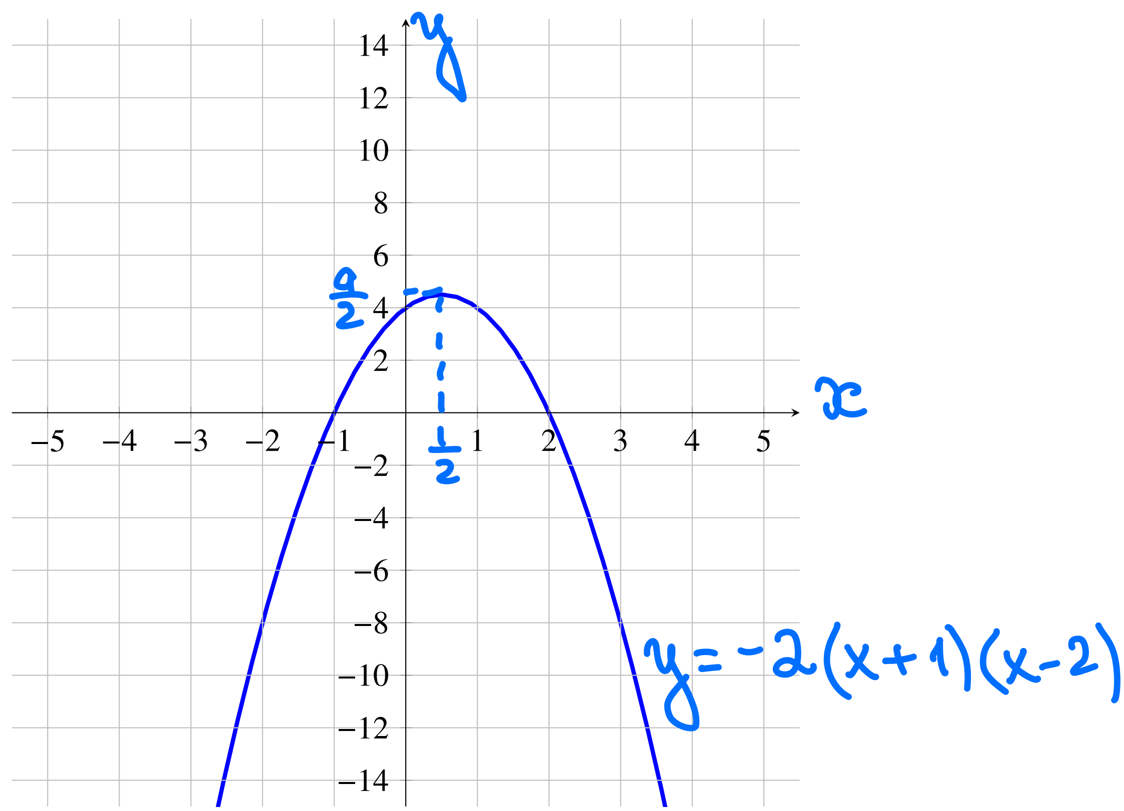
$$f(x) = x^2 - 4x + 8$$

$a = 1 \quad b = -4 \quad c = 8$

Vertex: $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

$\left(\frac{4}{2}, 2^2 - 8 + 8\right) = (2, 4)$

3. Find a formula for the quadratic function whose graph is given below.



$y = a(x-b)(x-c)$

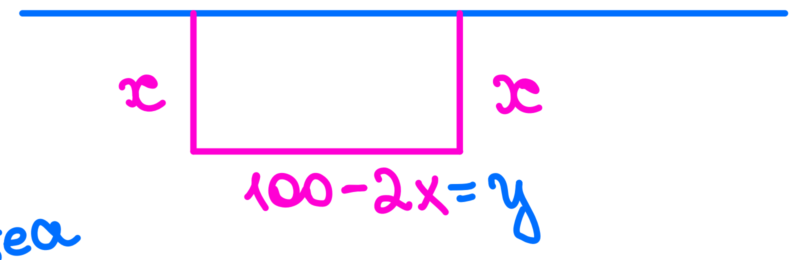
$y = a(x+1)(x-2)$

$\frac{9}{2} = a\left(\frac{1}{2}+1\right)\left(\frac{1}{2}-2\right)$

$4.5 = a \cdot 1.5 \cdot (-1.5) \Rightarrow a = -2$

4. (Maximization/Minimization Problems).

A farmer plans to use 100 feet of spare fencing material to form a rectangular garden plot against the side of a long barn, using the barn as one side of the plot. How should he split up the fencing among the other sides in order to maximize the area of the garden plot?



$A = h \cdot w$

area

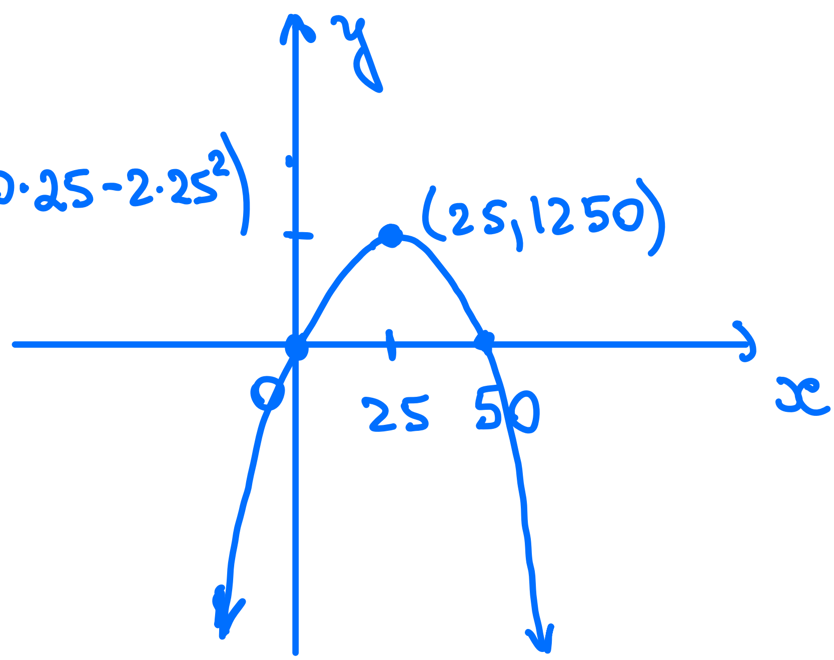
$A = \underbrace{x}_{h}\underbrace{(100-2x)}_{w} = 100x - 2x^2$

$A(x) = 100x - 2x^2$

Vertex: $\left(\frac{-100}{-4}, 100 \cdot 25 - 2 \cdot 25^2\right)$

Vertex: $(25, 2500 - 1250)$

$(25, 1250)$



Peak at $(25, 1250)$.

Hence $x_{max} = \boxed{25}$

$y_{max} = 100 - 2 \cdot 25 = \boxed{50}$

Height: 25 feet
Width: 50 feet