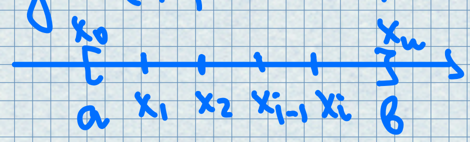
Lecture #20 - Week 7 – Double Integrals over Rectangles - 15.1

• Review of the Definite Integral

$y = f(x)$, $x \in [a, b]$



$\Delta x = (b-a)/n$

sample points: x_i^*

Then $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is a Riemann sum

and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

• Volumes and Double Integrals

$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$

and suppose $f(x,y) \geq 0$.

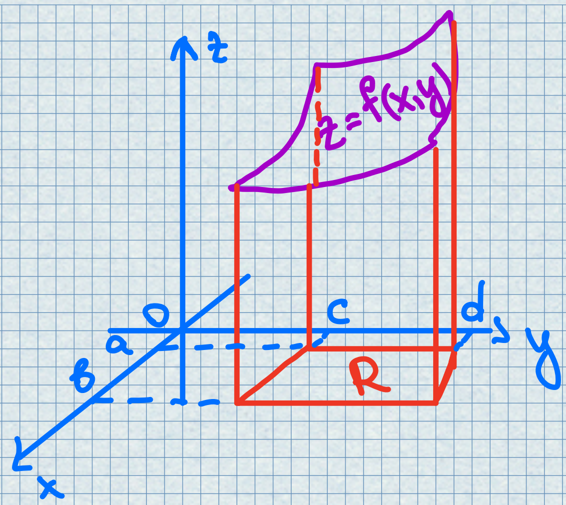
Let

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le z \le f(x,y),\ (x,y) \in R\}$$

Our goal is to find a volume of S.

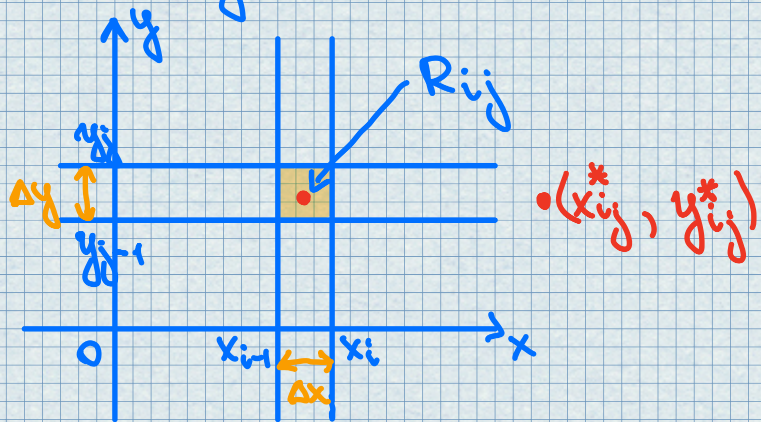
Let $\Delta x = (b-a)/m$

$\Delta y = (d-c)/n$



$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

$$\Delta A = \Delta x \Delta y$$



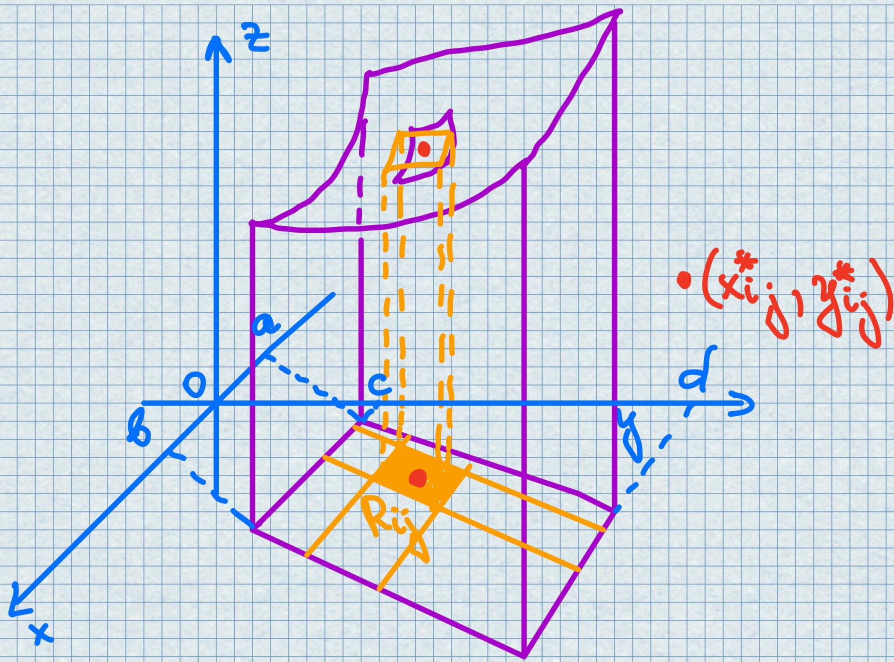
We choose a <u>sample point</u> $(x^*_{ij}, y^*_{ij}) \in R_{ij}$

Approximated volume:

$$\underbrace{f(x^*_{ij}, y^*_{ij})}_{\text{height}} \underbrace{\Delta A}_{\text{area}}$$

Thus

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta A$$



$$V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

Def. The double integral of f over the rectangle R is

double Riemann Sum

$$\iint_R f(x,y)\,dA = \lim_{m,n \to \infty} \boxed{\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A}$$

if this limit exists.

If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below $z = f(x,y)$ is

$$V = \iint_R f(x,y)\,dA$$

- The Midpoint Rule

We use a double Riemann sum to approximate the double integral, where (x^*_{ij}, y^*_{ij}) in R_{ij} is chosen to be the center $(\bar{x}_i, \bar{y}_j)$ in R_{ij}.

$\bar{x}_i$ is a midpoint of $[x_{i-1}, x_i]$

$\bar{y}_j$ is a midpoint of $[y_{j-1}, y_j]$

Midpoint Rule for Double Integrals

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\,\Delta A .$$

- Iterated Integrals

Suppose $f(x,y)$ is integrable on $R = [a,b] \times [c,d]$.

Then

$$A(x) = \int_c^d f(x,y)\,dy$$

$$\int_a^b A(x)\,dx = \boxed{\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx}$$

iterated integral

Usually we write

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Similarly,

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

Practical method for evaluating a double integral by expressing it as an iterated integral.

Fubini's Theorem

If f is continuous on $R = \{(x,y) \mid x \in [a,b], y \in [c,d]\}$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

If $f(x,y) = g(x)h(y)$ and $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b g(x)h(y)\,dx\,dy = \int_c^d \left(\int_a^b g(x)h(y)\,dx \right) dy$$

Based on that we can write

$$\boxed{\iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy\ , \quad R=[a,b]\times[c,d]}$$

- Average Value

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx \qquad \text{on } [a,b]$$

$$f_{ave} = \frac{1}{A(R)}\iint_R f(x,y)\,dA \qquad \text{on } R$$

If $f(x,y)\geq 0$, then

$$A(R)\times f_{ave} = \iint_R f(x,y)\,dA$$

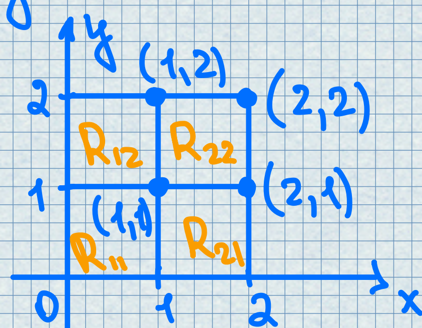
Examples

$\boxed{1.}$ Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose sample point to be the upper right corner of each square R_{ij}.

Solution

$$f(x,y) = 16 - x^2 - 2y^2$$

The area of each square is $\Delta A = 1$.



With $m = n = 2$, we have

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j) \Delta A$$

$$= f(1,1)\Delta A + f(1,2)\Delta A + f(2,1)\Delta A + f(2,2)\Delta A =$$

$$= 13 \cdot 1 + 7 \cdot 1 + 10 \cdot 1 + 4 \cdot 1 = 34$$

$\boxed{2.}$ Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral

$$\iint_R (x - 3y^2)\, dA, \quad \text{where } R = \{(x,y) \mid 0 \le x \le 2, 1 \le y \le 2\}.$$

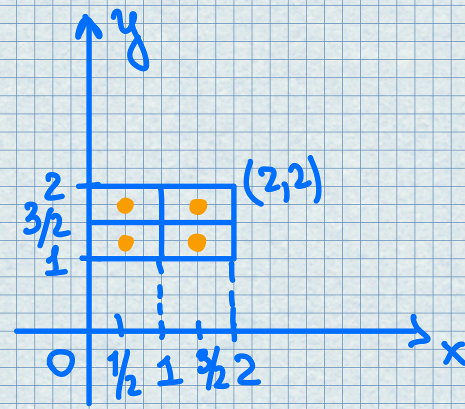
Solution

$m = n = 2$

$f(x,y) = x - 3y^2$

$\bar{x}_1 = \frac{1}{2} \qquad \bar{y}_1 = \frac{5}{4}$

$\bar{x}_2 = \frac{3}{2} \qquad \bar{y}_2 = \frac{7}{4}$



The area of each rectangle is $\Delta A = \frac{1}{2}$.

Thus

$$\iint_R (x - 3y^2)\, dA \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

$$= f(\bar{x}_1, \bar{y}_1)\Delta A + f(\bar{x}_1, \bar{y}_2)\Delta A + f(\bar{x}_2, \bar{y}_1)\Delta A + f(\bar{x}_2, \bar{y}_2)\Delta A$$

$$= f\left(\tfrac{1}{2}, \tfrac{5}{4}\right)\Delta A + f\left(\tfrac{1}{2}, \tfrac{7}{2}\right)\Delta A + f\left(\tfrac{3}{2}, \tfrac{5}{4}\right)\Delta A + f\left(\tfrac{3}{2}, \tfrac{7}{4}\right)\Delta A$$

$$= -\frac{67}{16} \cdot \frac{1}{2} + \left(-\frac{139}{16}\right) \cdot \frac{1}{2} + \left(-\frac{51}{16}\right) \frac{1}{2} + \left(-\frac{123}{16}\right) \frac{1}{2} =$$

$$= -\frac{95}{8} = -11.875$$

▼

$\boxed{3.}$ Evaluate the iterated integral

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx$$

Solution

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 x^2 \left(\frac{y^2}{2}\right)\Bigg|_1^2 dx =$$

$$= \int_0^3 x^2 \left(2 - \frac{1}{2}\right) dx = \frac{3}{2} \int_0^3 x^2 dx = \frac{3}{2} \cdot \frac{x^3}{3}\Bigg|_0^3 =$$

$$= \boxed{\frac{27}{2}}$$

▼

$\boxed{4.}$ Evaluate the double integral

$$\iint_R (x - 3y^2) \, dA, \text{ where}$$

$$R = \{(x,y) \mid 0 \le x \le 2, \; 1 \le y \le 2\}.$$

Solution Fubini's Theorem gives

$$\iint_R (x-3y^2)\,dA = \int_0^2 \int_1^2 (x-3y^2)\,dy\,dx =$$

$$= \int_0^2 (xy-y^3)\Big|_1^2 dx = \int_0^2 (x-7)\,dx = \frac{x^2}{2} - 7x\Big|_0^2 = -12.$$

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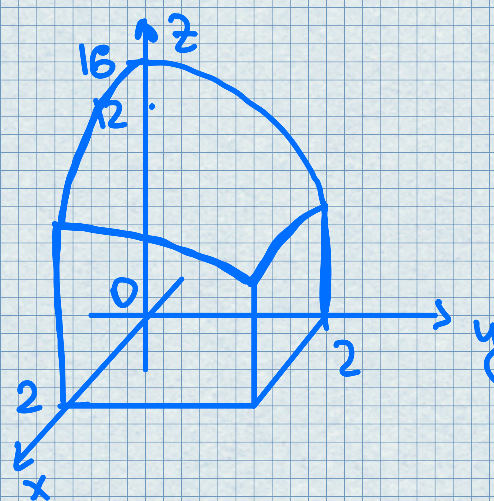
$\boxed{5.}$ Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2+2y^2+z=16$, the planes $x=2$ and $y=2$, and the three coordinate planes.

Solution



$z = 16 - x^2 - 2y^2$

$R = [0,2] \times [0,2]$

$$V = \iint_R (16-x^2-2y^2)\,dA =$$

$$= \int_0^2 \int_0^2 (16-x^2-2y^2)\,dx\,dy = \int_0^2 \left(16x - \frac{1}{3}x^3 - 2y^2x\right)\Big|_0^2 dy =$$

$$= \int_0^2 \left(\frac{88}{3} - 4y^2\right)dy = \left(\frac{88}{3}y - \frac{4}{3}y^3\right)\Big|_0^2 = 48$$

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$\boxed{6.}$ If $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$, then

$$\iint_R \sin x \cos y \, dA = \int_0^{\pi/2} \sin x \, dx \int_0^{\pi/2} \cos y \, dy =$$

$$= (-\cos x)\Big|_0^{\pi/2} (\sin y)\Big|_0^{\pi/2} = 1 \cdot 1 = 1.$$

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