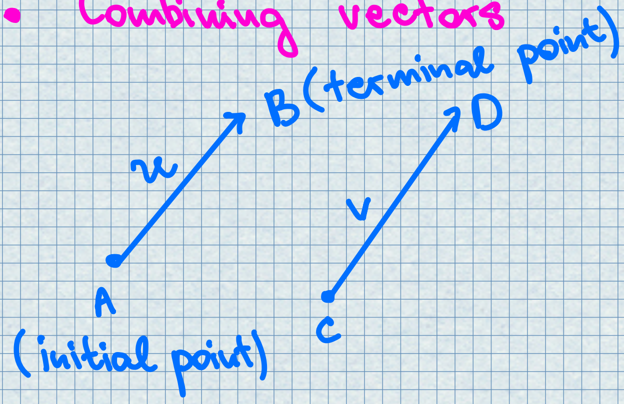
Lecture #2 - Week 1 - Vectors - 12.2

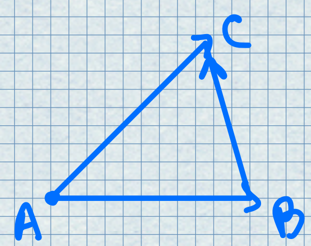
- Combining vectors



$\overrightarrow{AB} = u$

$\overrightarrow{CD} = v$

$\vec{0} = (0,0,0)$

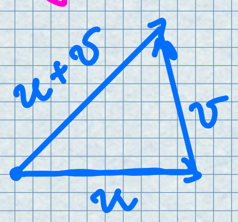


$$\boxed{\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}}$$

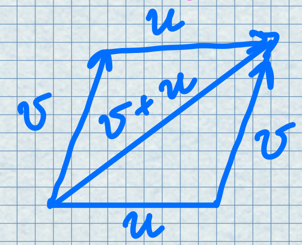
Def. (Vector addition)

If u and v are vectors positioned so the initial point of v is a terminal point of u, then the sum $u+v$ is the vector from the initial point of u to the terminal point of v.

Triangle law

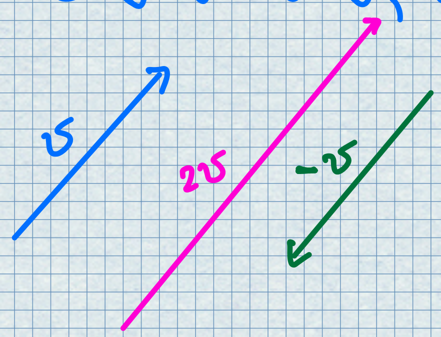


Parallelogram Law

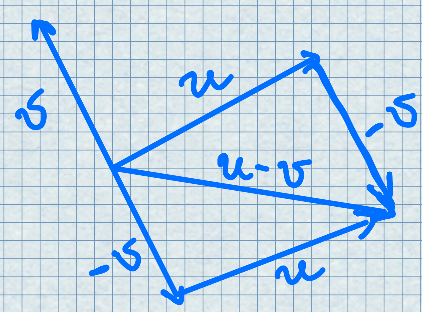


Def. (Scalar Multiplication)

If c is a scalar and v is a vector, then the scalar multiple cv is a vector whose length is $|c|$ times the length of v and whose direction is the same as v if $c>0$ and is opposite to v if $c<0$.
If $c=0$ or $v=0$, then $cv=0$.



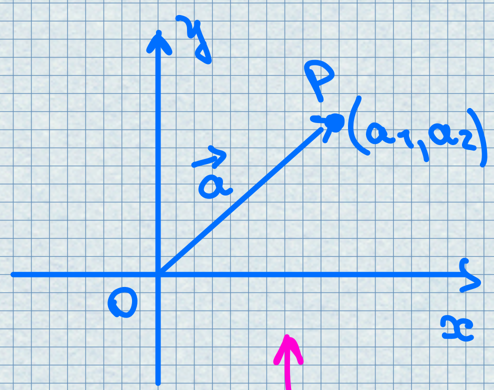
$$u-v = u+(-v)$$

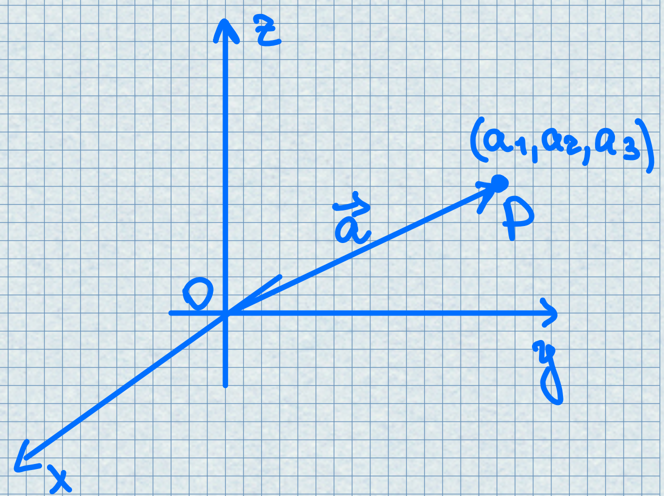


- **Components**

$\mathbb{R}^2$: $\vec{a} = \langle \underbrace{a_1, a_2}_{\text{components}} \rangle$

$\mathbb{R}^3$: $\quad \vec{a} = \langle \underbrace{a_1, a_2, a_3}_{\text{components}} \rangle$





$\vec{OP} = \vec{a} = \langle a_1, a_2 \rangle$

$\vec{OP}$ is a position vector of the point $P(a_1, a_2)$.

Given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector

$$a = \vec{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

The length of $a = \langle a_1, a_2 \rangle$ is

$$|a| = \sqrt{a_1^2 + a_2^2}$$

The length of $a = \langle a_1, a_2, a_3 \rangle$ is

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

If $a = \langle a_1, a_2 \rangle$ and $b = \langle b_1, b_2 \rangle$, then

$$a + b = \langle a_1 + b_1, a_2 + b_2 \rangle$$
$$a - b = \langle a_1 - b_1, a_2 - b_2 \rangle$$
$$ca = \langle ca_1, ca_2 \rangle$$

Similarly,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$
$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$
$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

In $\mathbb{R}^n$ we have

$$a = \langle a_1, a_2, \ldots, a_n \rangle$$

Properties of vectors

If a, b, c are vectors in V_n (set of all n-dimensional vectors) and α, β are scalars, then

1. $a + b = b + a$
2. $a + 0 = a$
3. $\alpha(a + b) = \alpha a + \alpha b$
4. $a + (b + c) = (a + b) + c$
5. $a + (-a) = 0$
6. $(\alpha + \beta)a = \alpha a + \beta a$
7. $1 \cdot a = a$
8. $(\alpha\beta)a = \alpha(\beta a)$

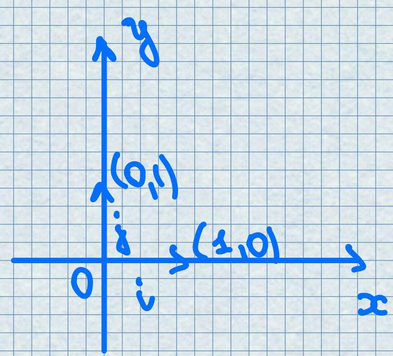
Let us consider V_3 (set of all three-dimensional vectors).

Let

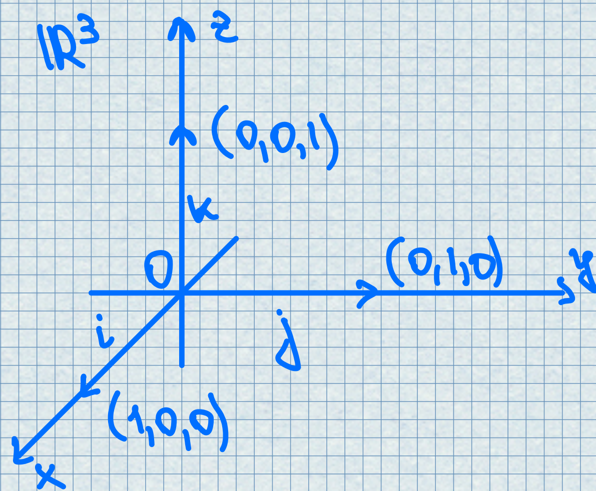
$$\mathbf{i} = \langle 1,0,0 \rangle \qquad \mathbf{j} = \langle 0,1,0 \rangle \qquad \mathbf{k} = \langle 0,0,1 \rangle$$

$\mathbf{i}, \mathbf{j}, \mathbf{k}$ are standard basis vectors.

$\mathbb{R}^2$



$\mathbb{R}^3$



Then

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle =$$

$$= a_1 \langle 1,0,0 \rangle + a_2 \langle 0,1,0 \rangle + a_3 \langle 0,0,1 \rangle =$$

$$= \boxed{a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}}$$

$$\boxed{\mathbf{a} = \langle a_1, a_2 \rangle = a_1 \mathbf{i} + a_2 \mathbf{j}}$$

Def. A unit vector is a vector whose length is 1.

i, j, k are unit vectors.

$$u = \frac{1}{|a|} a = \frac{a}{|a|} \quad \text{if} \quad a \neq 0$$

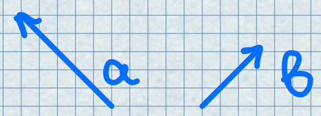
u has the same direction as a.

$$u = \underbrace{\frac{1}{|a|}}_{c>0} \cdot a = c \cdot a \quad (\text{Same direction})$$

$$|u| = \frac{1}{|a|} \cdot |a| = 1$$

Examples

$\boxed{1.}$ If a and b are vectors shown below, draw $a-2b$.



Solution

$\boxed{2.}$ Find the vector represented by the directed line segment with initial point $A(2,-3,4)$ and terminal point $B(-2,1,1)$.

Solution

$$\overrightarrow{AB} = \langle -2-2,\ 1-(-3),\ 1-4 \rangle = \langle -4, 4, -3 \rangle.$$

$\boxed{3.}$ If $a = \langle 4,0,3 \rangle$, $b = \langle -2,1,5 \rangle$ find $|a|$ and $a+b$.

Solution

$$|a| = \sqrt{4^2+0^2+3^2} = \sqrt{25} = 5$$

$$a+b = \langle 4,0,3 \rangle + \langle -2,1,5 \rangle = \langle 2,1,8 \rangle.$$

$\boxed{4.}$ If $a = i+2j-3k$, $b = 4i+7k$, express the vector

$2a+3b$ in terms of i, j, k.

Solution

$$2a+3b = 2(i+2j-3k)+3(4i+7k) =$$

$$= 2i+4j-6k+12i+21k = 14i+4j+15k.$$

$\boxed{5.}$ Find the unit vector in the direction of the vector $2i-j-2k$.

Solution

$$|2i-j-2k| = \sqrt{2^2+1+(-2)^2} = \sqrt{9} = 3$$

Thus $$\frac{1}{3}(2i-j-2k) = \frac{2}{3}i - \frac{1}{3}j - \frac{2}{3}k.$$