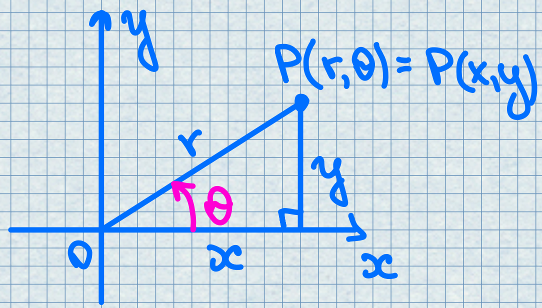
Lecture #27 - Week 9 - Triple integrals in cylindrical coordinates - 15.7



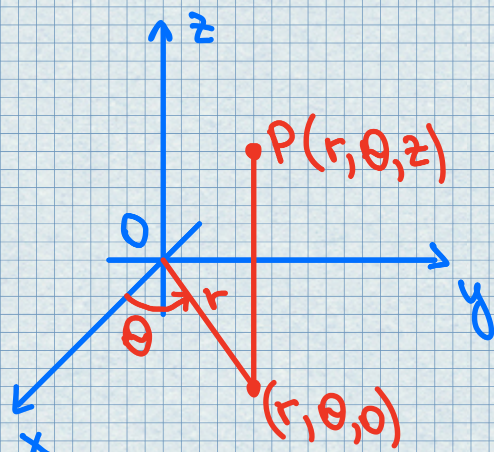
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

- Cylindrical Coordinates

In the cylindrical coordinate system, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P.

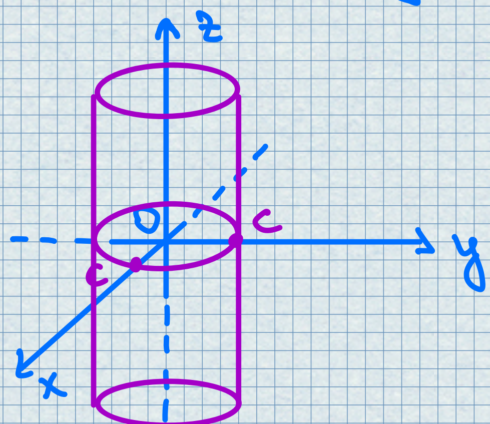


$$\boxed{x = r\cos\theta \qquad y = r\sin\theta \qquad z = z}$$

or

$$\boxed{r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z}$$

- Cylindrical coordinates are useful in problems that involve symmetry about an axis, and the z-axis is chosen to coincide with this axis of symmetry.
  For instance, the axis of the circular cylinder with Cartesian equation $x^2 + y^2 = c^2$ is the z-axis. In cylindrical coordinates this cylinder has the very simple equation $r = c$.



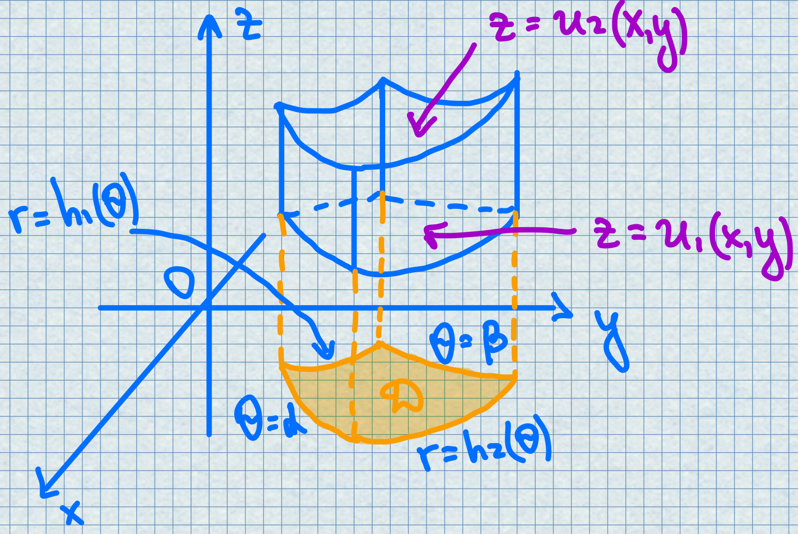
- Evaluating Triple integrals with cylindrical coordinates

  Suppose that E is a type 1 region whose projection D onto the xy-plane is conveniently described in polar coordinates.
  In particular, suppose that f is continuous and

$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$$

where

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$



$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA}$$

The formula for triple integration in cylindrical coordinates

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Examples

$\boxed{1.}$ (a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.

(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

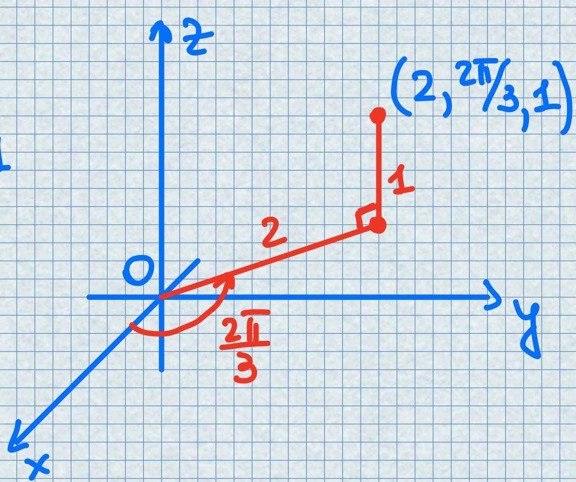
Solution

(a)

$$x = 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

$$y = 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$z = 1$$



So the point is $(-1, \sqrt{3}, 1)$ in rectangular coordinates.

(b)

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = \frac{-3}{3} = -1 \implies \theta = \frac{7\pi}{4} + 2\pi n$$

$$z = -7$$

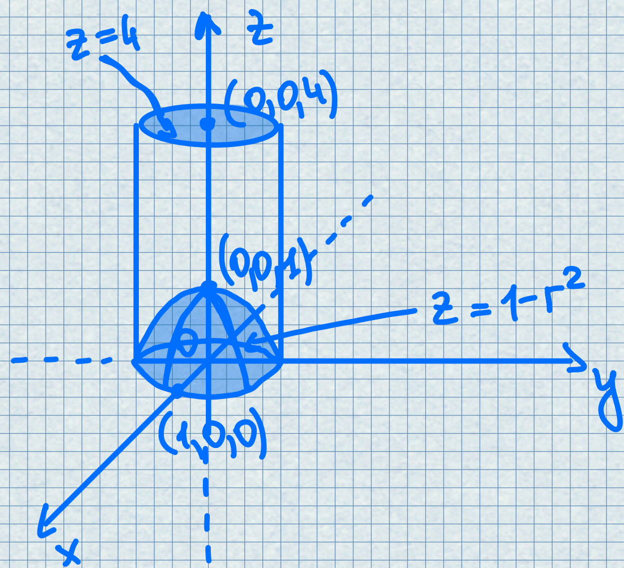
Therefore, one set of cylindrical coordinates

is $\left(3\sqrt{2}, 7\pi/4, -7\right)$. Another is $\left(3\sqrt{2}, -\pi/4, -7\right)$.
There are infinitely many choices. ◤

$\boxed{2.}$ A solid E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above paraboloid $z=1-x^2-y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

Solution

$r=1$ and $z=1-r^2$



$E=\{(r,\theta,z) \mid 0\le\theta\le 2\pi, 0\le r\le 1,$
$1-r^2\le z\le 4\}$

$f(x,y,z)=K\sqrt{x^2+y^2}=Kr$

where K is the proportionally constant.
Thefore,

$$m=\iiint_E K\sqrt{x^2+y^2}\,dV=\int_0^{2\pi}\int_0^1\int_{1-r^2}^{4}(Kr)r\,dz\,dr\,d\theta =$$

$$= \int_0^{2\pi}\int_0^{1} Kr^2\left(4-(1-r^2)\right)\, dr\, d\theta = K\int_0^{2\pi} d\theta \int_0^{1} (3r^2 + r^4)\, dr =$$

$$= 2\pi K \left(r^3 + \frac{r^5}{5} \right)\Big|_0^1 = \frac{12\pi K}{5}$$

▼

$\boxed{3.}$ Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dz\, dy\, dx.$

Solution

$E = \{(x,y,z) \mid -2 \le x \le 2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2},\ \sqrt{x^2+y^2} \le z \le 2\}$

and the projection of E onto xy-plane is $x^2+y^2 \le 4$.

The lower surface of E is $z = \sqrt{x^2+y^2}$ and its upper surface is $z = 2$.

Thus E in cylindrical coordinates is:

$$E = \{(r,\theta,z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 2,\ r \le z \le 2\}$$

Therefore,

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx = \iiint_E (x^2+y^2)\,dV =$$

$$= \int_0^{2\pi}\int_0^{2}\int_r^{2} r^2\, r\, dz\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 r^3(2-r)\,dr =$$

$$= 2\pi\left(\frac{1}{2}r^4 - \frac{1}{5}r^5\right)\Big|_0^2 = \frac{16}{5}\pi.$$

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