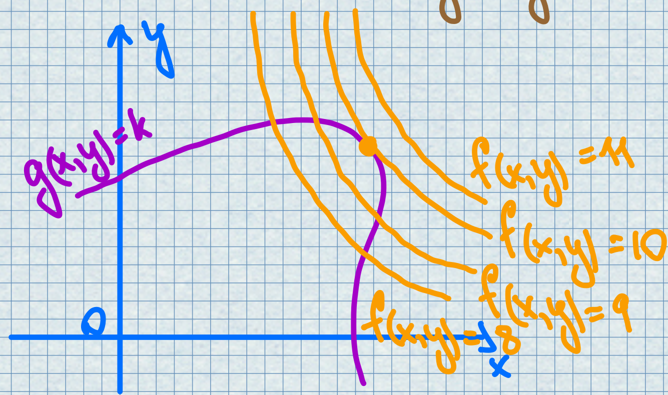
Lecture #19 - Week 6 - Lagrange Multipliers ~ 14.8



$\mathbb{R}^2$ case: we start by trying to find the extreme values of $f(x,y)$ subject to a constraint of the form $g(x,y) = k$.

In other words, we seek the extreme values of $f(x,y)$ when point (x,y) is restricted to lie on the level curve $g(x,y) = k$.

To maximize $f(x,y)$ subj. to $g(x,y)=k$ is to find the largest value of c such that the level curve $f(x,y) = c$ intersects $g(x,y)=k$. This means that the normal lines at (x_0,y_0), where they touch are identical.

So

$$\boxed{\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)}$$ for some scalar λ

$\mathbb{R}^3$ case: Suppose that f has an extreme value at $P(x_0,y_0,z_0)$ on the surface S and let C be a curve with vector equation

$r(t) = \langle x(t), y(t), z(t) \rangle$ that lies on S and passes through P.

If t_0 is the parameter value corresponding to P, then $r(t_0) = \langle x_0, y_0, z_0 \rangle$.

Function $h(t) = f(x(t), y(t), z(t))$ represents the values that f takes on C.

We have that h has an extreme value at t_0, so $h'(t_0) = 0$.

But

$$0 = h'(t_0) = f_x(x_0,y_0,z_0)x'(t_0) + f_y(x_0,y_0,z_0)y'(t_0) + f_z(x_0,y_0,z_0)z'(t_0) = \nabla f(x_0,y_0,z_0) \cdot r'(t_0)$$

$$\Downarrow$$

$$\boxed{\nabla f \perp r'(t_0)}$$

And we also have that

$$\boxed{\nabla g \perp r'(t_0)}$$

Hence,

$$\boxed{\nabla f \parallel \nabla g}$$

Therefore, if $\nabla g(x_0,y_0,z_0) \neq 0$, there is a number λ such that

$$\boxed{\nabla f(x_0,y_0,z_0) = \lambda \nabla g(x_0,y_0,z_0)} \qquad (1)$$

The number λ in (1) is called a **Lagrange multiplier**.

Method of Lagrange Multipliers

To find the max and min values

of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$ (assuming that these extreme values exist and $\nabla g \neq 0$ on $g(x,y,z)=k$):

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

and

$$g(x,y,z) = k$$

(b) Evaluate f at all points (x,y,z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

$$\nabla f = \lambda \nabla g$$

$$\Updownarrow$$

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z \qquad g(x,y,z) = k$$

$$\Updownarrow$$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x,y,z) = k \end{cases} \qquad (2)$$

(2) is a system of 4 equations in the four unknowns x, y, z, and λ.

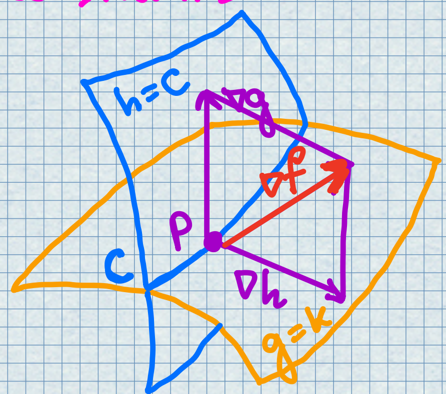
For $f = f(x,y)$:

$$\nabla f(x,y) = \lambda \nabla g(x,y) \quad \text{and} \quad g(x,y) = k$$

$$\Updownarrow$$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = k \end{cases} \qquad (3)$$

- Two Constraints



Suppose now we want to find the max and min values of $f(x,y,z)$ subject to

$$g(x,y,z) = k \quad \text{and} \quad h(x,y,z) = c.$$

Geometrically, this means that we are looking for the extreme values of f when (x,y,z) is restricted to lie on the curve of intersection C of the level surfaces $g=k$ and $h=c$.

Suppose f has such an extreme value at $P(x_0,y_0,z_0)$. We know that

$$\nabla f \perp C \text{ at } P.$$

But also

$$\nabla g \perp g(x,y,z)=k$$
$$\nabla h \perp h(x,y,z)=c$$

So

$$\nabla g \perp C \quad \text{and} \quad \nabla h \perp C.$$

Hence, $\nabla f(x_0,y_0,z_0)$ is in the plane determined by $\nabla g(x_0,y_0,z_0)$ and $\nabla h(x_0,y_0,z_0)$.

So there numbers λ and μ such that

$$\boxed{\nabla f(x_0,y_0,z_0)=\lambda \nabla g(x_0,y_0,z_0)+\mu \nabla h(x_0,y_0,z_0)}$$

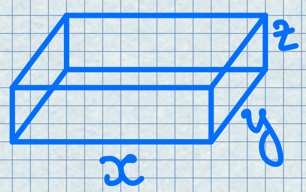
$$\Updownarrow$$

$$\begin{cases} f_x = \lambda g_x + \mu h_x \\ f_y = \lambda g_y + \mu h_y \\ f_z = \lambda g_z + \mu h_z \\ g(x,y,z)=k \\ h(x,y,z)=c \end{cases}$$

Examples

$\boxed{1.}$ A rectangular box without a lid is to be made from $12\,m^2$ of cardboard. Find the maximum volume of such a box.

Solution



$$V = xyz \to \max$$

Subject to

$$g(x,y,z) = 2xz + 2yz + xy = 12$$

By the method of Lagrange multipliers:

$$\nabla V = \lambda \nabla g \quad \text{and} \quad g(x,y,z) = 12$$

$$\Downarrow$$

$$\begin{cases} V_x = \lambda g_x \\ V_y = \lambda g_y \\ V_z = \lambda g_z \\ g = 12 \end{cases}$$

From this one gets

$$\begin{cases} yz = \lambda(2z+y) \;\Big|\cdot x & (1) \\ xz = \lambda(2z+x) \;\Big|\cdot y & (2) \\ xy = \lambda(2x+2y) \;\Big|\cdot z & (3) \\ 2xz + 2yz + xy = 12 & (4) \end{cases}$$

Then
$$\begin{cases} xyz = \lambda(2xz+xy) & (5) \\ xyz = \lambda(2yz+xy) & (6) \\ xyz = \lambda(2xz+2yz) & (7) \end{cases}$$

$\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from (1),(2),(3) and this would contradict (4). Therefore,

(5)+(6): $\quad 2xz + xy = 2yz + xy$

$$xz = yz$$
$$z \neq 0$$
So $\boxed{x = y}$

(6)+(7): $\quad 2yz + xy = 2xz + 2yz$

$$xy = 2xz$$
$$y = 2z \quad (x \neq 0)$$

So $x = y = 2z$.

Now we put it in (4). Hence,

$$4z^2 + 4z^2 + 4z^2 = 12.$$

Since, $x>0, y>0, z>0$, we get $z = 1$, $x = 2$, $y = 2$.

Thus, $(x,y,z) = (2,2,1)$ ▼

$\boxed{2.}$ Find the extreme values of the function $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Solution

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = x^2 + y^2 = 1 \end{cases}$$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \begin{cases} 2x = \lambda 2x & (1) \\ 4y = \lambda 2y & (2) \\ x^2 + y^2 = 1 & (3) \end{cases}$$

From (1): $x = 0$ or $\lambda = 1$

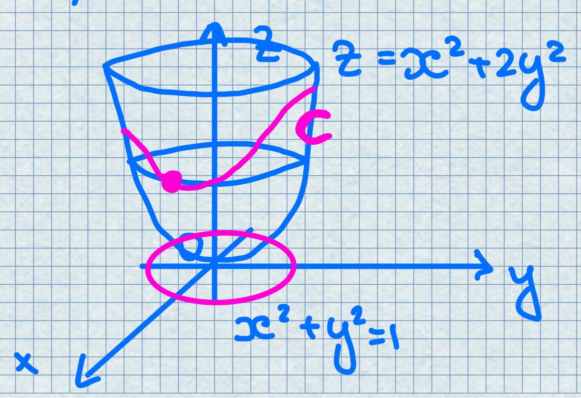
If $x = 0$, then from (3): $y = \pm 1$

If $\lambda = 1$, then from (2): $y = 0$ and from (3): $x = \pm 1$.

Therefore, CP: $(0,1), (0,-1), (1,0), (-1,0)$.

$f(0,1) = 2$, $f(0,-1) = 2$, $f(1,0) = 1$, $f(-1,0) = 1$

Hence, f_{max} is on the circle $x^2 + y^2 = 1$ and $f_{max} = f(0, \pm 1) = 2$ and $f_{min} = f(\pm 1, 0) = 1$.



$\boxed{3.}$ Find the maximum value of the function $f(x,y,z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Solution

$$f \to \max$$

subject to $x - y + z = 1$

$x^2 + y^2 = 1$.

$$\nabla f = \lambda \nabla g + \mu \nabla h,$$

where

$$g(x,y,z) = x - y + z = 1$$
$$h(x,y,z) = x^2 + y^2 = 1$$

So, we get

$$1 = \lambda + 2x\mu$$
$$2 = -\lambda + 2y\mu$$
$$3 = \lambda$$
$$x - y + z = 1$$
$$x^2 + y^2 = 1$$

Since $\lambda = 3$, we get $2\mu x = -2$, $x = -\frac{1}{\mu}$.

Similarly, $y = \frac{5}{2\mu}$.

Then

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

So

$$\mu^2 = \frac{29}{4} \Rightarrow \mu = \pm \frac{\sqrt{29}}{2}.$$

Then $x = \pm \frac{2}{\sqrt{29}}$, $y = \pm \frac{5}{\sqrt{29}}$, and

$$z = 1 - x + y = 1 \pm \frac{7}{\sqrt{29}}.$$

The corresponding values of f are

$$\pm \frac{2}{\sqrt{29}} + 2\left(\pm \frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}$$

Therefore, the f_{max} on the given curve is $3 + \sqrt{29}$.