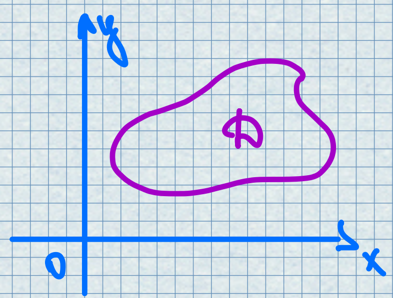
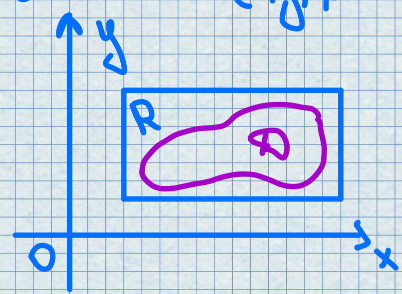
Lecture #21 - Week 7 - Double integrals over general regions - 15.2

Let

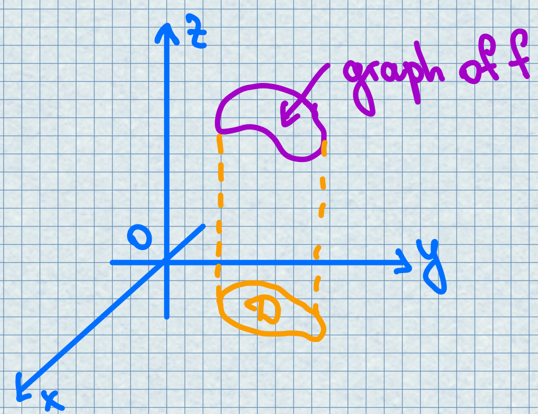
$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \in R \text{ and } (x,y) \notin D \end{cases}$$

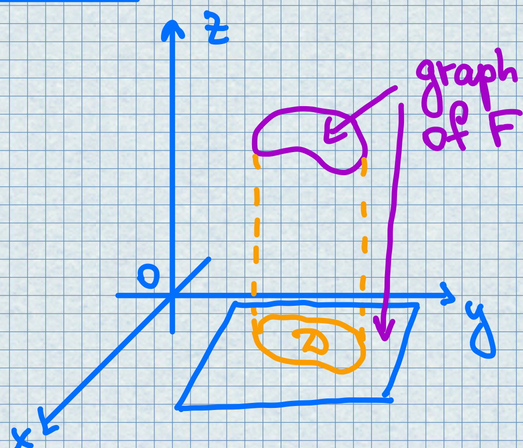




If F is integrable over R, then the <u>double integral</u> of f over D is

$$\boxed{\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA}$$

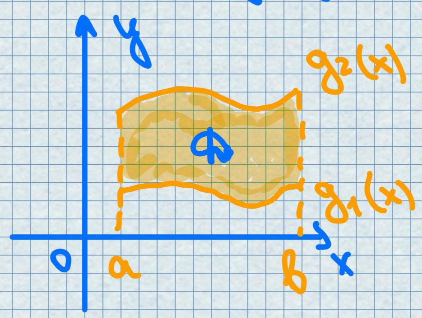


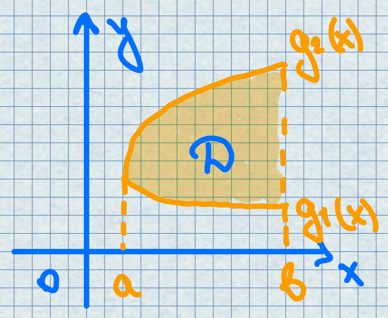


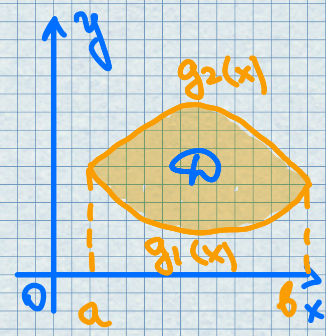
Plane region D of type I:

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

where $g_1, g_2 \in C[a,b]$.







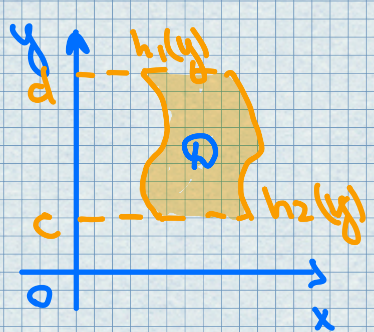
Statement If f is continuous on type I region D, then

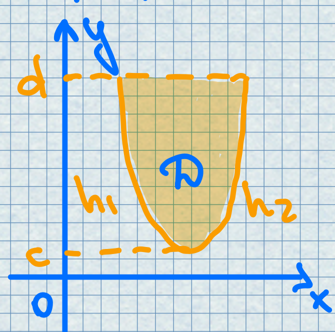
$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

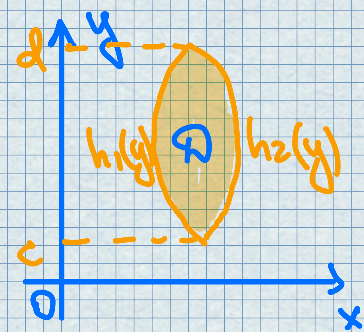
Plane region D of type II:

$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\},$$

where $h_1, h_2 \in C[c,d]$.







Statement

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

where D is a type II region.

- Properties of Double Integrals

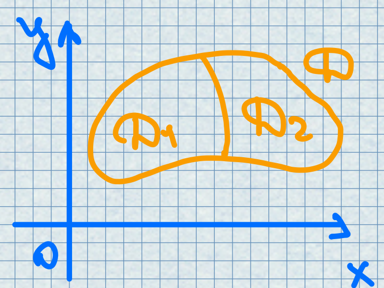
We assume that all of the following integrals exist.

1. $$\iint_D (f(x,y)+g(x,y))\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

2. $$\iint_D c f(x,y)\,dA = c\iint_D f(x,y)\,dA \,,$$ where c is a constant

3. If $f(x,y) \geq g(x,y)$ for all $(x,y) \in D$, then
$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$$

4. If $D = D_1 \cup D_2$, then



$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

5. If $f(x,y) = 1$ over D, then
$$\iint_D 1\,dA = A(D)$$

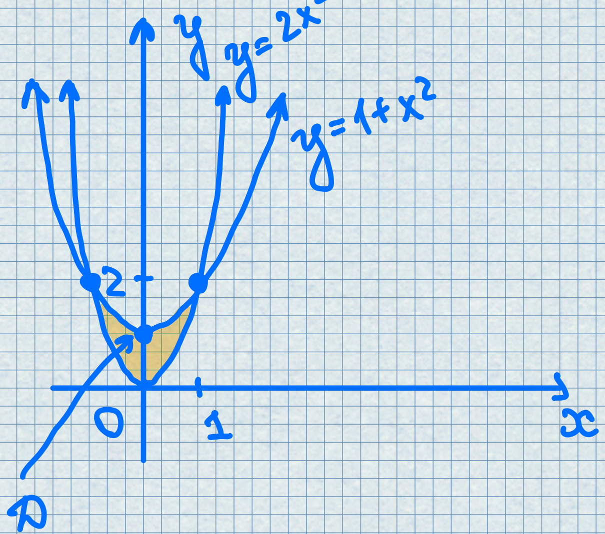
⑥ If $m \leq f(x,y) \leq M$ for all $(x,y) \in D$, then

$$m\,A(D) \leq \iint_D f(x,y)\,dA \leq M\,A(D)$$

Examples

$\boxed{1.}$ Evaluate $\iint_D (x+2y)\,dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1+x^2$.

Solution



$$\iint_D (x+2y)\,dA =$$

$$= \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx \;=$$

$$D = \{(x,y) \mid -1 \le x \le 1,\ 2x^2 \le y \le 1+x^2\}$$

$$= \int_{-1}^{1} \left(xy + y^2\right)\Big|_{y=2x^2}^{y=1+x^2} dx = \int_{-1}^{1} \Big(x(1+x^2) + (1+x^2)^2 -$$

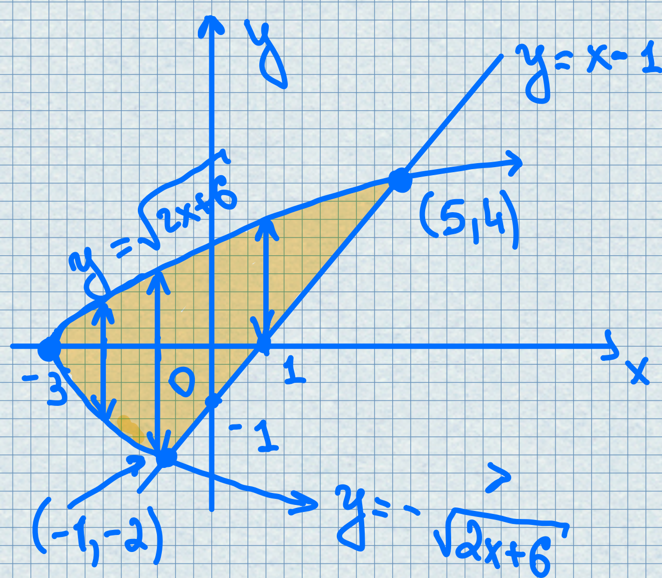
$$- 2x^3 - (2x^2)^2\Big)\,dx = \left(-3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\right)\Bigg|_{-1}^{1} =$$

$$= \frac{32}{15} \quad \blacktriangledown$$

2. Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y=x-1$ and the parabola $y^2=2x+6$.

Solution

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\,dx\,dy =$$



$$= \int_{-2}^{4}\left(\frac{x^2}{2}y\right)\Bigg|_{x=\frac{1}{2}y^2-3}^{x=y+1} dy = \frac{1}{2}\int_{-2}^{4} y\left((y+1)^2-\left(\frac{1}{2}y^2-3\right)^2\right)dy =$$

$$= \frac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4}+4y^3+2y^2-8y\right)dy =$$

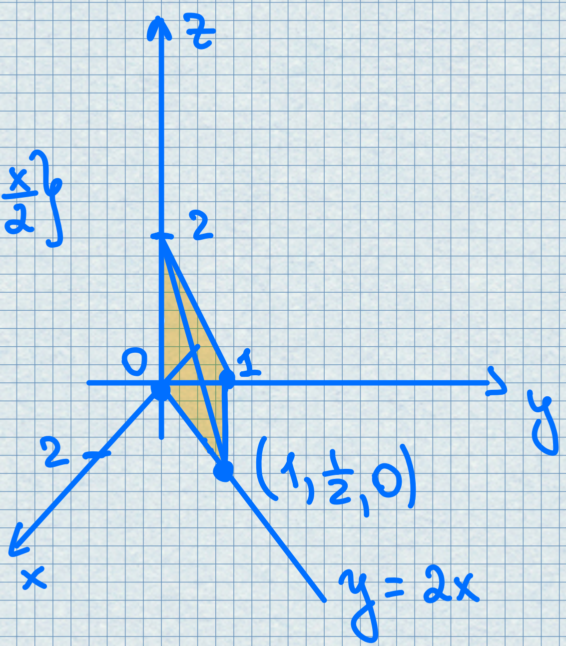
$$= \frac{1}{2}\left(-\frac{y^6}{24}+y^4+2\frac{y^3}{3}-4y^2\right)\Bigg|_{-2}^{4} = 36$$

$\boxed{3.}$ Find the volume of the tetrahedron bounded by the planes

$$x + 2y + z = 2$$
$$x = 2y$$
$$x = 0$$
$$z = 0.$$

Solution

$$D = \left\{ (x,y) \mid 0 \le x \le 1,\ \frac{x}{2} \le y \le 1 - \frac{x}{2} \right\}$$



Therefore,

$$V = \iint_D (2 - x - 2y)\, dA = \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx =$$

$$= \int_0^1 \left(2y - xy - y^2 \right) \Big|_{y=\frac{x}{2}}^{y=1-x/2} dx = \int_0^1 \Big(2 - x - x\left(1 - \frac{x}{2}\right) -$$

$$-\left(\left(1-\frac{x}{2}\right)^2 - x + \frac{x^2}{2} + \frac{x^2}{4}\right) dx =$$

$$= \int_0^1 (x^2 - 2x + 1)\, dx = \left(\frac{x^3}{3} - x^2 + x\right)\Big|_0^1 = \frac{1}{3}$$

$\boxed{4.}$ Use the property

$$\boxed{\begin{array}{l} \text{If } m \le f(x,y) \le M \text{ for all } (x,y) \in D, \text{ then} \\ \qquad mA(D) \le \iint_D f(x,y)\, dA \le MA(D) \end{array}}$$

to estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and radius 2.

Solution

Since
$$-1 \le \sin x \le 1$$
$$-1 \le \cos y \le 1$$
we have that
$$-1 \le \sin x \cos y \le 1$$
and
$$e^{-1} \le e^{\sin x \cos y} \le e^{1}$$

Thus, using $m = e^{-1} = \frac{1}{e}$, $M = e$ and $A(D) = \pi 2^2$,

we obtain

$$\frac{4\pi}{e} \le \iint_D e^{\sin x \cos y} \, dA \le 4\pi e.$$

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