Name:

Instructions. (100 points) You have two hours. The exam is closed book, closed notes, and only simple calculators are allowed. Show all your work in order to receive full credit.

1. **[8 points]** Consider the points $A = (1, 0, -1)$, $B = (-2, 1, 3)$ and $C = (-1, 1, 0)$.

   (a) (3 pts) Give a parameterization of the straight line segment from B to C. **Be sure to state what the parameter may range over**.

$\vec{BC} = \langle -1+2, 1-1, 0-3 \rangle = \langle 1, 0, -3 \rangle$

$r(t) = \langle -2, 1, 3 \rangle + t\langle 1, 0, -3 \rangle = \langle -2+t, 1, 3-3t \rangle,$

where $0 \le t \le 1$.

   (b) (5pts) Find an equation (not a parameterization) for the plane containing points A, B, C.

$\vec{AB} = \langle -2-1, 1-0, 3+1 \rangle = \langle -3, 1, 4 \rangle$

$\vec{AC} = \langle -1-1, 1, 1 \rangle = \langle -2, 1, 1 \rangle$

Plane equation: $\begin{vmatrix} x-1 & y-0 & z+1 \\ -3 & 1 & 4 \\ -2 & 1 & 1 \end{vmatrix} = (x-1)\cdot(1-4) - (y-0)\cdot(-3+8) + (z+1)(-3+2) =$

$= -3(x-1) - y\cdot 5 + (z+1)(-1) = \boxed{-3x - 5y - z + 2 = 0}$

2. **[8 points]** Find the region of integration

$$\int_0^4 \int_{\sqrt{y}}^2 e^{(x^3+1)} dx\, dy. =$$

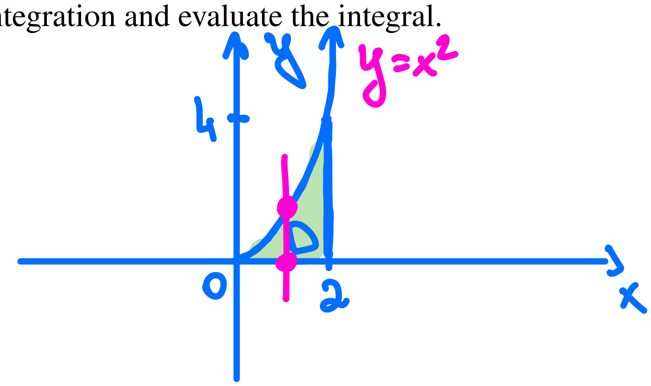
Then use your sketch to reverse the order of integration and evaluate the integral.

$\sqrt{y} \le x \le 2$

$0 \le y \le 4$

$0 \le x \le 2$

$0 \le y \le x^2$



$$\circled{=} \int_0^2\int_0^{x^2} e^{x^3+1}\,dy\,dx = \int_0^2 e^{x^3+1}\,y\Big|_0^{x^2}dx =$$

$$= \int_0^2 x^2 e^{x^3+1}\,dx = \frac{1}{3}\int_1^9 e^u\,du = \frac{1}{3}e^u\Big|_1^9 = \boxed{\frac{1}{3}(e^9-e)}$$

$u = x^3+1$

$du = 3x^2\,dx$

3. **[11 points]** Assume a particle has velocity $v(t) = (t^2+1)\,\mathbf{i} + 2e^t\,\mathbf{j} + (1-t)\,\mathbf{k}$, $t \geq 1$ with speed measured in m/s.

(a) (3pts) Find acceleration of the particle at $t = 2$.

$a(t) = v'(t) = \langle 2t, 2e^t, -1\rangle$

$a(2) = \langle 4, 2e^2, -1\rangle$

(b) (4pts) Set the formula for the distance traveled from $t = 1$ s to $t = 3$ s. (DO NOT EVALUATE)

$$d = \int_1^3 |v(t)|\,dt = \int_1^3 \sqrt{(t^2+1)^2 + 4e^{2t} + (1-t)^2}\,dt$$

$$|v(t)| = \sqrt{(t^2+1)^2 + 4e^{2t} + (1-t)^2}$$

(c) (4pts) Find the position vector $r(t)$ at all times if $r(1) = \mathbf{i} - 2\mathbf{k}$.

$$r(t) = \int v(t)\,dt = i\int (t^2+1)\,dt + j\int 2e^t\,dt + k\int (1-t)\,dt =$$

$$= i\left(\frac{t^3}{3}+t\right) + j\cdot 2e^t + k\left(t - \frac{t^2}{2}\right) + \langle c_1, c_2, c_3\rangle$$

$$r(1) = \langle \tfrac{4}{3}, 2e, \tfrac{1}{2}\rangle + \langle c_1, c_2, c_3\rangle = \langle 1, 0, -2\rangle$$

$c_1 = -\frac{1}{3}$, $c_2 = -2e$, $c_3 = -\frac{5}{2}$

4. **[8 points]** Use Lagrange multipliers to find the maximum product of two positive numbers satisfying $x^2 + y = 4$.

$f(x,y) = x \cdot y \to \max$

$\nabla f = \lambda \nabla g \qquad g = x^2 + y - 4$

$\begin{cases} y = \lambda \cdot 2x \\ x = \lambda \cdot 1 \end{cases} \Rightarrow y = 2\lambda^2$

$\lambda^2 + 2\lambda^2 = 4 \Rightarrow 3\lambda^2 = 4 \Rightarrow \lambda^2 = \frac{4}{3} \Rightarrow \lambda = \pm \frac{2}{\sqrt{3}}$

$\lambda = \frac{2}{\sqrt{3}}$

Hence, $x = \frac{2}{\sqrt{3}}, \; y = \frac{8}{3}$

Therefore $f_{max} = \frac{16}{3\sqrt{3}}$

5. **[12 points]** Compute the surface integral

$$\iint_S x^2 \, dS, \; =$$

where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

$\begin{cases} x = \sin\varphi \cos\theta \\ y = \sin\varphi \sin\theta \\ z = \cos\varphi \end{cases} \qquad \begin{array}{l} 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \\ dS = \sin\varphi \, d\varphi \, d\theta \end{array}$

$= \int_0^{2\pi} \int_0^{\pi} \sin^2\varphi \cos^2\theta \cdot \sin\varphi \, d\varphi \, d\theta = \int_0^{2\pi} \cos^2\theta \, d\theta \cdot$

$\cdot \int_0^{\pi} \sin^3\varphi \, d\varphi = \frac{1}{2} \int_0^{2\pi} (1 + \cos 2\theta) \, d\theta \cdot \int_0^{\pi} (-1 + \cos^2\varphi) \, d(\cos\varphi) =$

$= \frac{1}{2} \cdot (2\pi + 0) \cdot \left(-\cos\varphi + \frac{\cos^3\varphi}{3} \right) \Big|_0^{\pi} = \pi \cdot \left(1 - \frac{1}{3} + 1 - \frac{1}{3} \right) =$

$= \pi \cdot \left(2 - \frac{2}{3} \right) = \boxed{\frac{4\pi}{3}}$

6. **[8 points]** Sketch the two surfaces

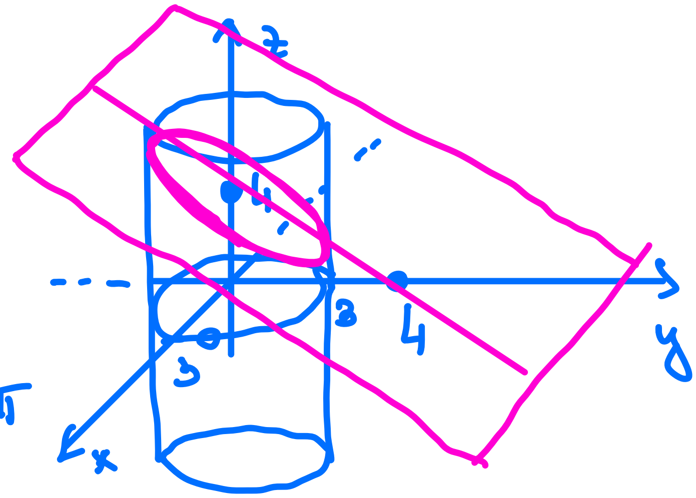
$$x^2 + y^2 = 9, \quad y + z = 4.$$

Highlight their curve of intersection. Give a parameterization of that curve.

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = 4 - r\sin\varphi \end{cases}$$

$r = 3$

$$\begin{cases} x = 3\cos\varphi \\ y = 3\sin\varphi \\ z = 4 - 3\sin\varphi \end{cases} \qquad 0 \le \varphi \le 2\pi$$



7. **[10 points]** Find all critical points of the function

$$f(x, y) = x^2 - 4xy + 6y^2$$

and, to the extent possible, determine whether they are local maxima, local minima, or saddle points.

$\nabla f = \langle f_x, f_y \rangle = \langle 2x - 4y, 12y - 4x \rangle = \vec{0}$

$$\begin{cases} 2x - 4y = 0 \\ 12y - 4x = 0 \end{cases} \Rightarrow \begin{cases} x = 2y \\ 3y = x \end{cases} \Rightarrow 3y - 2y = 0$$

$y = 0$
$x = 0$

CP: $(0,0)$

$f_{xx} = 2 \qquad f_{yy} = 12 \qquad f_{xy} = -4$

$D = 24 - 16 = 8 > 0$

$f_{xx} = 2 > 0$

Hence, $f(x,y)$ at $(0,0)$ has loc. min.

8. **[9 points]** Use cylindrical coordinates to set the formula for the volume of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$. (DO NOT EVALUATE)

$V = \iiint_E dV =$

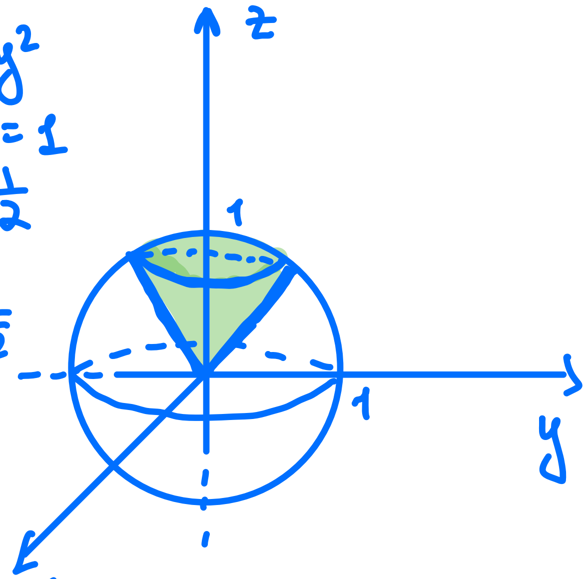
$z^2 = x^2 + y^2$

$2x^2 + 2y^2 = 1$

$x^2 + y^2 = \frac{1}{2}$

$r^2 = \frac{1}{2}$

$r = \frac{1}{\sqrt{2}}$



$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$dV = r\,dz\,dr\,d\theta$

$r \le z \le \sqrt{1-r^2}$

$0 \le \theta \le 2\pi$

$0 \le r \le \frac{1}{\sqrt{2}}$

$= \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \int_r^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta$

9. **[8 points]** Find a parametric representation for the cylinder

$$x^2 + y^2 = 16, \quad 0 \le z \le 1.$$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Rightarrow r = 4$

$\begin{cases} x = 4\cos\theta \\ y = 4\sin\theta \\ z = z \end{cases}$

$0 \le z \le 1$

$0 \le \theta \le 2\pi$

10. **[8 points]** Consider the force field

$$F(x, y) = \langle x^2, xy \rangle$$

(a) (5pts) Find a potential function for $F(x, y)$.

Since F is not conservative it doesn't have a potential function.

(b) (3pts) Find the work done by the force field $F(x, y)$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented in the counterclockwise direction.

$$\int_C F\cdot dr = \int_C F(r)\cdot r'(t)\,dt = \int_0^{2\pi} (-8\sin t\cos^2 t + 8\cos^2 t\sin t)\,dt = \boxed{0}$$

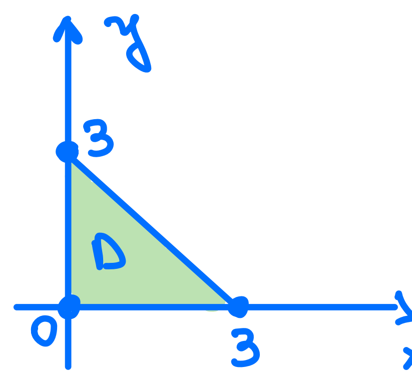
$$\begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases} \quad 0 \le t \le 2\pi$$

$$r'(t) = \langle -2\sin t, 2\cos t \rangle, \quad F(r) = \langle 4\cos^2 t, 4\cos t\cdot\sin t \rangle$$

11. **[10 points]** Use Green's Theorem to evaluate the line integral

$$\int_C \underbrace{ye^x}_{P}\,dx + \underbrace{2e^x}_{Q}\,dy =$$

along the positively oriented curve C, where C is the ~~rectangle~~ triangle with vertices $(0, 0)$, $(3, 0)$, ~~$(3, 4)$~~, and $(0, \cancel{4})$ 3.



$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA =$$

$$= \int_0^3 \int_0^{3-x} (2e^x - e^x)\,dy\,dx = \int_0^3\int_0^{3-x} e^x\,dy\,dx =$$

$$= \int_0^3 (3-x)e^x\,dx = 3\int_0^3 e^x\,dx - \int_0^3 xe^x\,dx = 3(e^3-1) - xe^x\Big|_0^3 + \int_0^3 e^x\,dx =$$

$$= 3(e^3-1) - 3e^3 + 0 + (e^3-1) = -3 + e^3 - 1 = \boxed{e^3-4}$$

12. **[Extra Credit, 8 points]** Let $f(x, y) = \dfrac{y}{x^2} + y^2 x$.

 (a) (4pts) Find the directional derivative of f at $(1, 2)$ when moving towards $(1, 0)$? What does this mean for function values?

$\nabla f = \langle -\frac{2y}{x^3} + y^2, \frac{1}{x^2} + 2xy \rangle$

$\nabla f(1,2) = \langle -4+4, 1+4 \rangle = \langle 0, 5 \rangle$

$v = \langle 1-1, 0-2 \rangle = \langle 0, -2 \rangle$

$u = \frac{v}{|v|} = \langle 0, -1 \rangle$

$D_u f(1,2) = \langle 0, 5 \rangle \cdot \langle 0, -1 \rangle = 0 - 5 = -5$

f values are decreasing

 (b) (4pts) Let $x(s, t) = ts^2$ and $y(s, t) = 4t - s$. Use the appropriate chain rule to find $\dfrac{\partial f}{\partial t}$. Your final answer should only contain s and t, but DO NOT simplify.

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = \boxed{\left(-\frac{8t-2s}{t^3 s^6} + (4t-s)^2\right) \cdot s^2 + \left(\frac{1}{t^2 s^4} + 2ts^2(4t-s)\right) \cdot 4}$$

$\frac{\partial f}{\partial x} = \frac{-2y}{x^3} + y^2$ $\quad \frac{\partial f}{\partial y} = \frac{1}{x^2} + 2xy$

$\frac{dx}{dt} = s^2$ $\quad \frac{dy}{dt} = 4$

Formulas:

- Surface integral formula

$$\iint_S f(x, y, z)\, dS = \iint_D f(r(u, v))\, |r_u \times r_v|\, dA$$

- The work done by a force field on a particle formula

$$W = \int_C F \cdot dr$$