Lecture #34 - Week 11 - Parametric surfaces and their areas - 16.6

• Parametric Surfaces

We can describe a surface by a vector function $r(u,v)$ of two parameters u and v.

We suppose that

$$\boxed{r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k} \quad (1)$$

is a vector-valued function defined on a region D in the uv-plane.

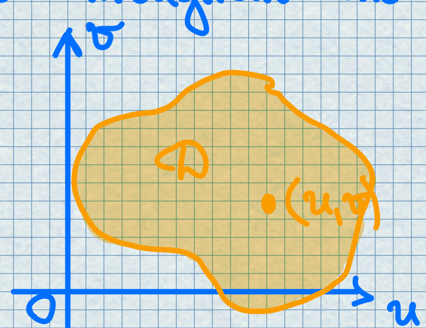
So the set of all points $(x,y,z) \in \mathbb{R}^3$ such that

$$\boxed{x = x(u,v) \qquad y = y(u,v) \qquad z = z(u,v)} \quad (2)$$

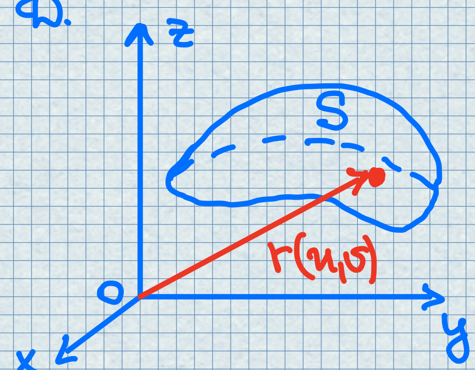
and (u,v) varies throughout D, is called a <u>parametric surface</u> S and (2) are called <u>parametric equations</u> of S.

Each choice of u and v gives a point on S.

The surface S is traced out by the tip of the position vector $r(u,v)$ as (u,v) moves throughout the region D.

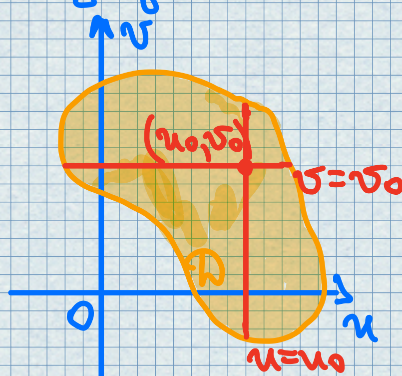


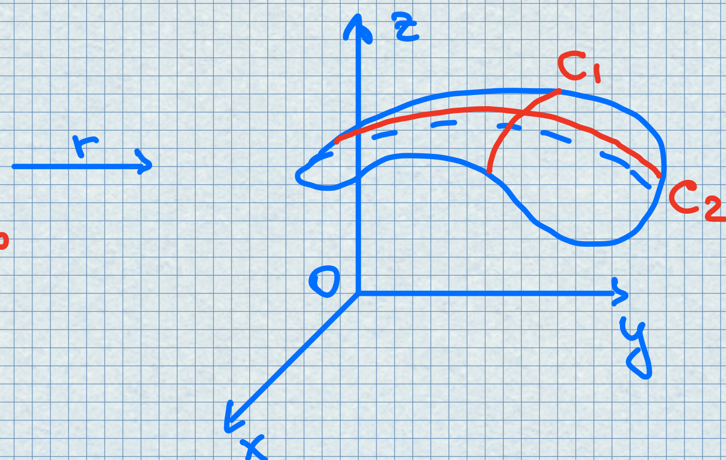
$\xrightarrow{r}$



If a parametric surface S is given by a vector function $r(u,v)$, then there are two useful families of curves that lie on S, one family with u constant and the other with v constant.

These families correspond to vertical and horizontal lines in the uv-plane.
If we keep u constant by putting $u = u_0$, then $r(u_0, v)$ is a vector function of the single parameter v and defines a curve C_1 lying on S.





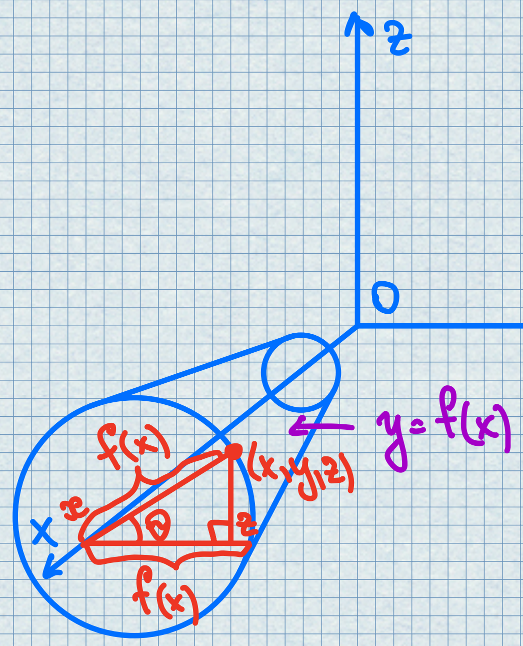
Similarly, for $v = v_0$, we get a curve C_2 given by $r(u, v_0)$ that lies on S.
We call C_1 and C_2 <u>grid curves.</u>

- **Surfaces of Revolution**

Let us consider the surface S obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis, where $f(x) \ge 0$.
Let θ be the angle of rotation. If (x,y,z) is a point on S, then

(3) $\boxed{x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta}$

Therefore, we take x and θ as parameters and regard (3) as parametric equations of S. The parameter domain is given by $a \le x \le b$, $0 \le \theta \le 2\pi$.



- Tangent Planes

We now find the tangent plane to a parametric surface S traced out by a vector function

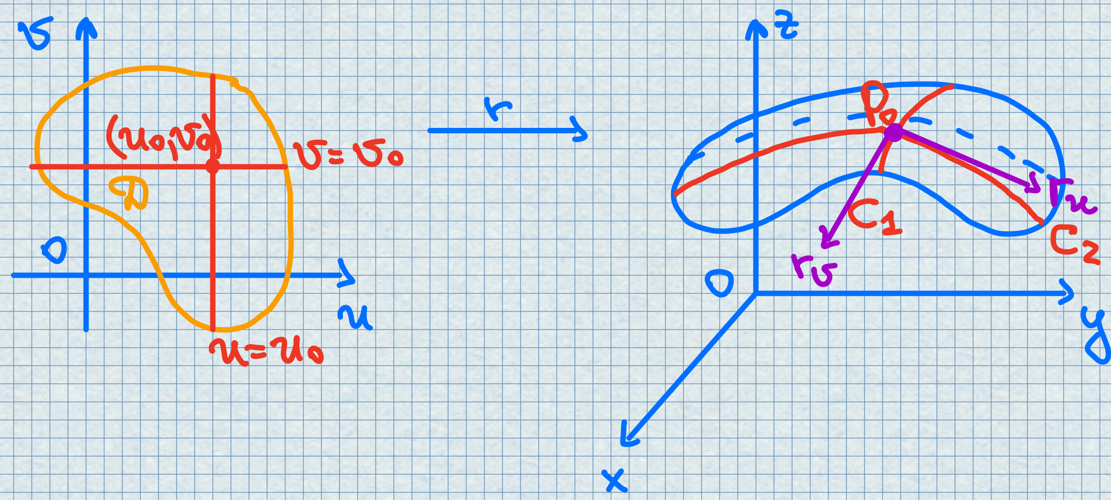
$$r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$$

at a point P_0 with position vector $r(u_0, v_0)$.

If $u = u_0$, then C_1: $r(u_0, v) \in S$

The tangent vector to C_1 at P_0 is obtained by taking the partial derivative of r with respect to v:

(4) $$\boxed{r_v = \frac{\partial x}{\partial v}(u_0,v_0)i + \frac{\partial y}{\partial v}(u_0,v_0)j + \frac{\partial z}{\partial v}(u_0,v_0)k}$$



Similarly, if $v=v_0$, we get $C_2: r(u,v_0) \in S$, and its tangent vector at P_0 is

$$\boxed{r_u = \frac{\partial x}{\partial u}(u_0,v_0)i + \frac{\partial y}{\partial u}(u_0,v_0)j + \frac{\partial z}{\partial u}(u_0,v_0)k} \quad (5)$$

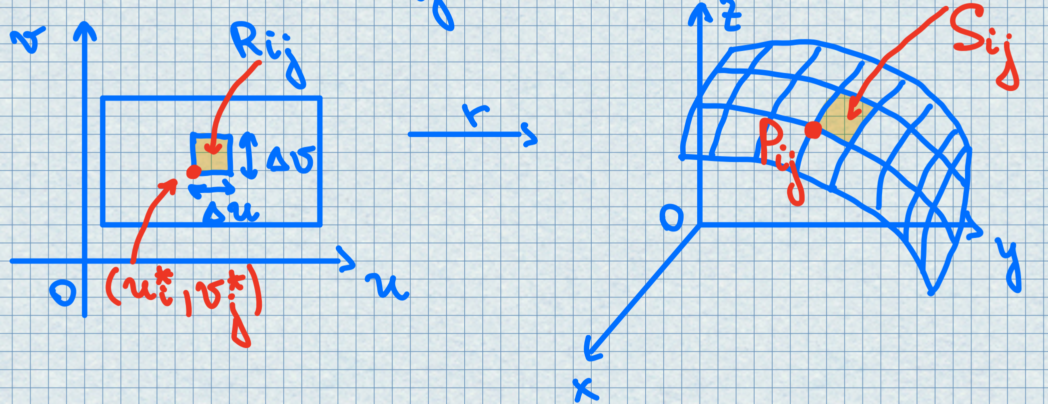
Def. If $r_u \times r_v \neq 0$, then S is called smooth. For a smooth surface, the <u>tangent plane</u> is the plane that contains the tangent vectors r_u and r_v, and the vector $r_u \times r_v$ is a normal vector to the tangent plane.

- **Surface Area**

For simplicity we start by considering a surface whose parameter domain D is a rectangle, and we divide it into

Subrectangles R_{ij}.

Let's choose (u_i^*, v_j^*) to be the lower left corner of R_{ij}.

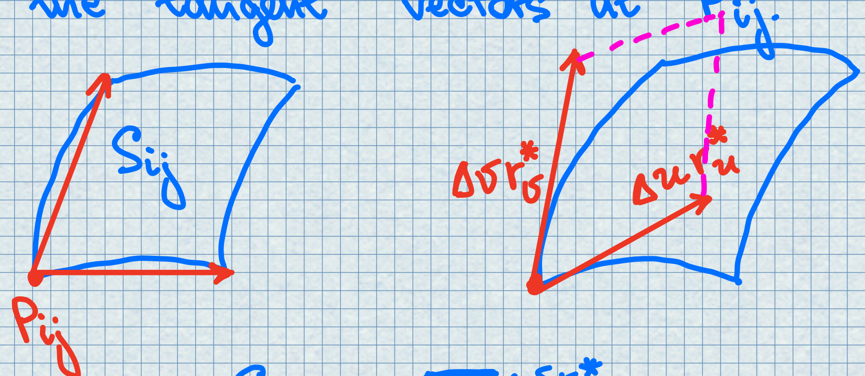


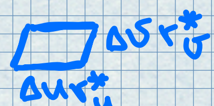
The part S_{ij} of S that corresponds to R_{ij} is called a <u>patch</u> and has the point P_{ij} with position vector $r(u_i^*, v_j^*)$.

Let

$$r_u^* = r_u(u_i^*, v_j^*) \quad \text{and} \quad r_v^* = r_v(u_i^*, v_j^*)$$

be the tangent vectors at P_{ij}.



$S_{ij} \approx$ 

$$A(\text{▱}) = |(\Delta u\, r_u^*) \times (\Delta v\, r_v^*)| = |r_u^* \times r_v^*|\, \Delta u\, \Delta v$$

$$A(S) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} |r_u^* \times r_v^*|\, \Delta u\, \Delta v$$

Def. If a smooth parametric surface S is given by the equation

$$r(u,v) = x(u,v)\,i + y(u,v)\,j + z(u,v)\,k$$
$$(u,v) \in D$$

and S is covered just once as (u,v) ranges throughout the parameter domain D, then the surface area of S is

$$\boxed{A(S) = \iint_D |r_u \times r_v| \, dA}$$

where

$$r_u = \frac{\partial x}{\partial u} i + \frac{\partial y}{\partial u} j + \frac{\partial z}{\partial u} k$$

$$r_v = \frac{\partial x}{\partial v} i + \frac{\partial y}{\partial v} j + \frac{\partial z}{\partial v} k$$

- **Surface Area of the Graph of a Function**

For the special case of a surface S with equation $z = f(x,y)$, where $(x,y) \in D$ and $f_x, f_y \in C(D)$, we take x and y as parameters.

The parametric equations are

$$x = x \qquad y = y \qquad z = f(x,y)$$

so

$$r_x = i + \frac{\partial f}{\partial x} k \qquad r_y = j + \frac{\partial f}{\partial y} k$$

and

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x} i - \frac{\partial f}{\partial y} j + k$$

Thus we have

$$\boxed{|r_x \times r_y| = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}$$

and the surface area formula becomes

$$\boxed{A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA}$$

Single-variable calculus case

S is obtained by rotating $y = f(x)$, $a \le x \le b$, about the x-axis, where $f(x) \ge 0$ and $f' \in C(a,b)$.

Then

$$x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta$$

$$a \le x \le b \qquad 0 \le \theta \le 2\pi$$

$$r_x = i + f'(x)\cos\theta\, j + f'(x)\sin\theta\, k$$

$$r_\theta = -f(x)\sin\theta\, j + f(x)\cos\theta\, k$$

Therefore,

$$\boxed{A(S) = \iint_D |r_x \times r_\theta| \, dA = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx}$$

Examples

$\boxed{1.}$ Identify and sketch the surface with vector equation

$$r(u,v) = 2\cos u \, i + v j + 2\sin u \, k$$

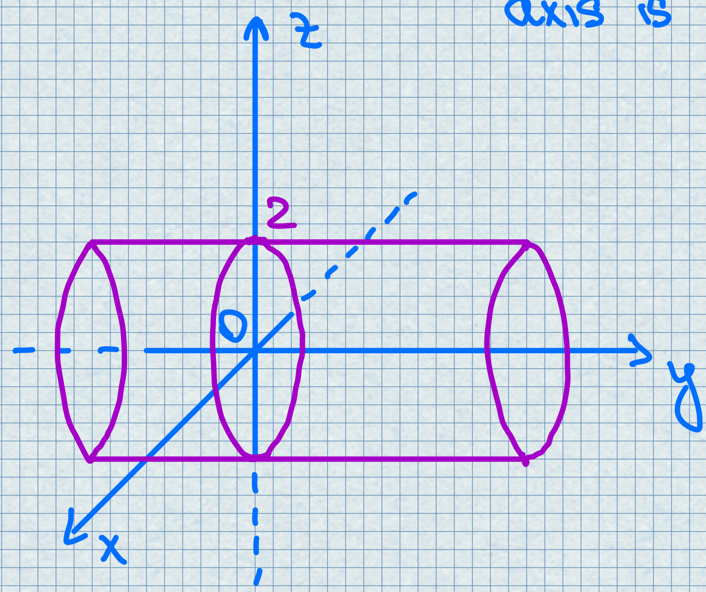
Solution

$$x = 2\cos u \qquad y = v \qquad z = 2\sin u$$

So for any point on the surface, we have

$$x^2 + z^2 = 4\cos^2 u + 4\sin^2 u = 4$$

$x^2 + z^2 = 4$ (cylinder with radius 2 whose axis is y-axis)



$\boxed{2.}$ Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2$$

Solution

in spherical coordinates: $\rho = a$.
We choose φ and θ as parameters.
Then, putting $\rho = a$ in the equations for conversion from spherical to rectangular coordinates, we obtain:

$$x = a \sin\varphi \cos\theta \qquad y = a\sin\varphi \sin\theta$$
$$z = a\cos\varphi$$

as the parametric equations of the sphere.
Then

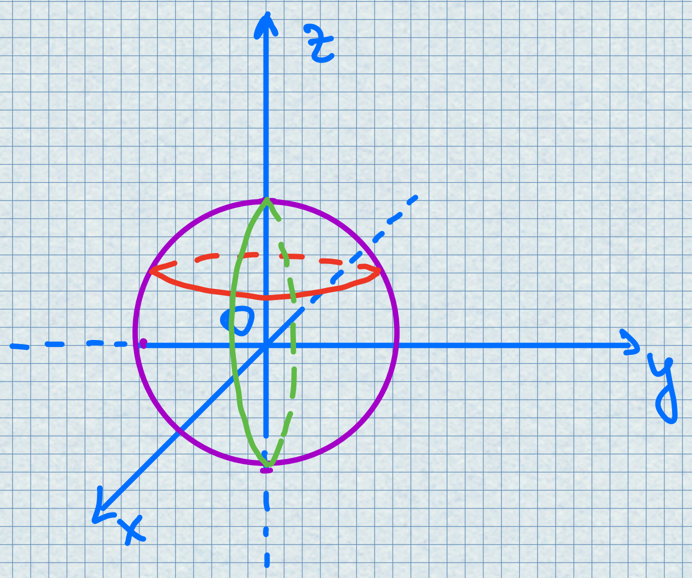
$$r(\varphi, \theta) = a\sin\varphi\cos\theta\, i + a\sin\varphi\sin\theta\, j + a\cos\varphi\, k.$$

$$0 \le \varphi \le \pi \qquad 0 \le \theta \le 2\pi$$

and

$$D = [0, \pi] \times [0, 2\pi].$$

The grid curves with $\varphi = \text{const}$ are latitude. The grid curves with $\theta = \text{const}$ are the meridians.



$\boxed{3.}$ Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$.

Solution

If we set x and y as parameters, then

$$x = x \qquad y = y \qquad z = x^2 + 2y^2$$

and the vector equation is

$$r(x,y) = x\,i + y\,j + (x^2 + 2y^2)\,k$$

$\boxed{4.}$ Find parametric equations for the surface generated by rotating the curve $y = \sin x$, $0 \le x \le 2\pi$, about x-axis.

Solution

From equations

$$x = x \quad y = f(x)\cos\theta \quad z = f(x)\sin\theta$$
$$a \le x \le b, \quad 0 \le \theta \le 2\pi,$$

the parametric equations are

$$x = x \quad y = \sin x \cos\theta \quad z = \sin x \sin\theta$$

and $0 \le x \le 2\pi, \quad 0 \le \theta \le 2\pi.$

$\boxed{5.}$ Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point $(1,1,3)$.

Solution

$$r_u = \frac{\partial x}{\partial u} i + \frac{\partial y}{\partial u} j + \frac{\partial z}{\partial u} k = 2ui + k$$

$$r_v = \frac{\partial x}{\partial v} i + \frac{\partial y}{\partial v} j + \frac{\partial z}{\partial v} k = 2vj + 2k$$

Thus a normal vector to the tangent plane is

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2vi - 4uj + 4uvk$$

$(1,1,3)$ corresponds to $u=1, v=1.$
So the normal vector there is

$$-2i - 4j + 4k$$

Therefore, an equation of the tangent plane at $(1,1,3)$ is

$$-2(x-1)-4(y-1)+4(z-3)=0$$

or

$$x+2y-2z+3=0$$

▼

$\boxed{6.}$ Find the surface area of a sphere of radius a.

Solution

$$x = a\sin\varphi\cos\theta \qquad y = a\sin\varphi\sin\theta$$
$$z = a\cos\varphi$$

where the parameter domain is

$$D = \{(\varphi,\theta) \mid 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi\}$$

$$r_\varphi \times r_\theta = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} =$$

$$= \begin{vmatrix} i & j & k \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix} =$$

$$= a^2\sin^2\varphi\cos\varphi\, i + a^2\sin^2\varphi\sin\theta\, j +$$

$+ a^2 \sin\varphi \cos\varphi \, k$

Thus

$$|r_\varphi \times r_\theta| = \sqrt{a^4 \sin^4\varphi \cos^2\theta + a^4 \sin^4\varphi \sin^2\theta + a^4 \sin^2\varphi \cos^2\varphi} = \sqrt{a^4 \sin^4\varphi + a^4 \sin^2\varphi \cos^2\varphi} =$$

$$= a^2 \sqrt{\sin^2\varphi} = a^2 \sin\varphi$$

Since $\sin\varphi \geq 0$ for $0 \leq \varphi \leq \pi$. Therefore, the area of the sphere is

$$A = \iint_D |r_\varphi \times r_\theta| \, dA = \int_0^{2\pi} \int_0^{\pi} a^2 \sin\varphi \, d\varphi \, d\theta =$$

$$= a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi = a^2 (2\pi) 2 = 4\pi a^2.$$

$\blacktriangledown$

$\boxed{7.}$ Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Solution

The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$.

Therefore, the given surface lies above the disk D with center the origin and radius 3.

Then

$$A = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA =$$

$$= \iint_D \sqrt{1+(2x)^2+(2y)^2}\, dA = \iint_D \sqrt{1+4(x^2+y^2)}\, dA$$

Converting to polar coordinates, we obtain

$$A = \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1+4r^2}\, dr =$$

$$= 2\pi\left(\frac{1}{8}\right)\frac{2}{3}\left(1+4r^2\right)^{3/2}\Big|_0^3 = \frac{\pi}{6}\left(37\sqrt{37}-1\right)$$

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