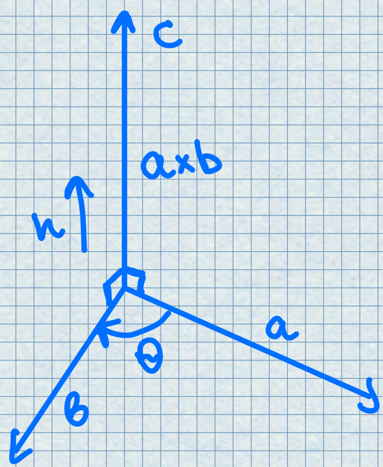
Lecture #4 - Week 1 — The cross product - 12.4



$c \perp a$

$c \perp b$

$c = \langle c_1, c_2, c_3 \rangle$

$a = \langle a_1, a_2, a_3 \rangle$

$b = \langle b_1, b_2, b_3 \rangle$

$$\begin{cases} a \cdot c = 0 \\ b \cdot c = 0 \end{cases} \qquad (1)$$

Solving (1) we obtain

$$\langle c_1, c_2, c_3 \rangle = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Def. If $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$, then the cross product of a and b is the vector

$$a \times b = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

↑
vector

Determinant of order 2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Determinant of order 3

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

If $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$, then

$$\boxed{a \times b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k}$$

$$\boxed{a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

Theorem The vector $a \times b$ is orthogonal to both a and b.

Theorem If θ is the angle between a and b $(0 \leq \theta \leq \pi)$, then

$$\boxed{|a \times b| = |a|\,|b| \sin\theta}$$

Corollary Two nonzero vectors a and b are parallel if and only if

$$\boxed{a \times b = 0}$$

The length of the $a \times b$ is equal to the area of the parallelogram.

Let i, j, k be standard basis vectors

and $\theta = \frac{\pi}{2}$. Then

$$\boxed{\begin{array}{lll} i \times j = k & j \times k = i & k \times i = j \\ j \times i = -k & k \times j = -i & i \times k = -j \end{array}}$$

Observe that

$$i \times j \neq j \times i \quad (\text{not commutative})$$

Also,

$$i \times (i \times j) = i \times k = -j$$

$$(i \times i) \times j = 0 \times j = 0$$

$$(a \times b) \times c \neq a \times (b \times c) \quad (\text{not associative})$$

Properties of the Cross Product

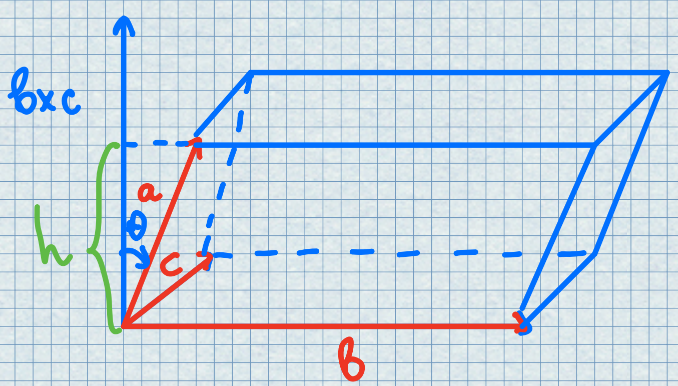
If a, b, and c are vectors and α is a scalar, then

1. $a \times b = -b \times a$
2. $(\alpha a) \times b = \alpha (a \times b) = a \times (\alpha b)$
3. $a \times (b + c) = a \times b + a \times c$
4. $(a + b) \times c = a \times c + b \times c$
5. $a \cdot (b \times c) = (a \times b) \cdot c$
6. $a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$

• Triple Products

Def. The product $a\cdot(b\times c)$ is the <u>scalar triple product</u> of vectors a, b, and c.

$$a\cdot(b\times c)=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



$$\boxed{V_{\text{parallelepiped}} = A\cdot h = |b\times c|\,|a|\,|\cos\theta| = |a\cdot(b\times c)|}$$

The volume of the parallelepiped determined by the vectors a, b, and c is the magnitude of their triple product:

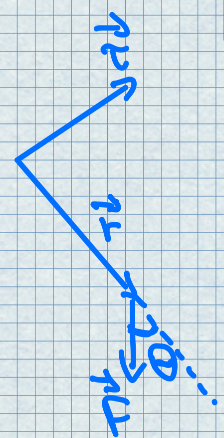
$$\boxed{V=|a\cdot(b\times c)|}$$

Def. If $|a\cdot(b\times c)|=0$, then vectors a, b, and c <u>lie in the same plane</u>, they are coplanar.

Torque

The torque τ is defined to be the cross product of the position and force vectors

$$\boxed{\tau = r \times F}$$



The torque τ measures the tendency of the body to rotate about the origin.
The direction of the torque vector indicates the axis of rotation.

$$|\tau| = |r \times F| = |r||F| \sin\theta ,$$

where θ is the angle between the position and force vectors.

Examples

1. If $a = \langle 1,3,4 \rangle$ and $b = \langle 2,7,-5 \rangle$, then

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} i - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} j +$$

$$+ \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} k = (-15-28)i - (-5-8)j + (7-6)k =$$

$$= -43i + 13j + k.$$

2. Find a vector perpendicular to the plane that passes through the points $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.

Solution

$\overrightarrow{PQ} \times \overrightarrow{PR}$ is $\perp$ to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and is therefore $\perp$ to the plane through P, Q, R.

$$\overrightarrow{PQ} = (-2-1)i + (5-4)j + (-1-6)k = -3i + j - 7k$$

$$\overrightarrow{PR} = (1-1)i + (-1-4)j + (1-6)k = -5j - 5k$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = (-5-35)i - (15-0)j +$$

$$+ (15-0)k = -40i - 15j + 15k$$

So $\langle -40, -15, 15 \rangle$ is $\perp$ to the given plane. ▼

$\boxed{3.}$ Use the scalar triple product to show that the vectors $a = \langle 1, 4, -7 \rangle$, $b = \langle 2, -1, 4 \rangle$, and $c = \langle 0, -9, 18 \rangle$ are coplanar.

Solution

$$a \cdot (b \times c) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1 \begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} -$$

$$-7 \begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix} = 1(18) - 4(36) - 7(-18) = 0$$

Therefore, the volume of the parallelepiped determined by a, b, and c is 0.

This means that a, b, and c are coplanar. ▼