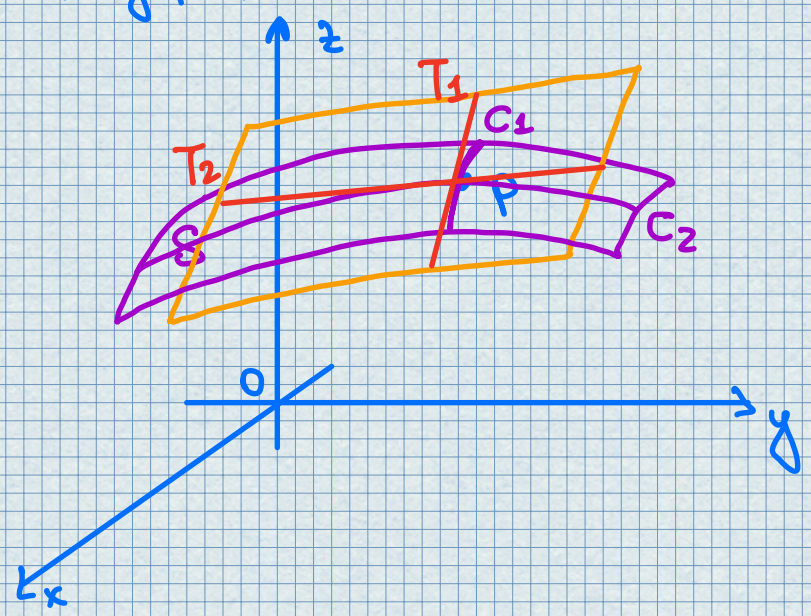
Lecture #15 - Week 5 - Tangent planes and linear approximations - 14.4

• Tangent Planes

$S: \quad z = f(x,y)$

f_x, f_y are continuous

$P(x_0, y_0, z_0) \in S$



Def. The tangent plane to the surface S at P is the plane that contains both tangent lines T_1 and T_2.

The tangent plane at P is the plane that most closely approximates the surface S near the point P.

Plane equation through the point $P(x_0, y_0, z_0)$:

$$\boxed{A(x-x_0) + B(y-y_0) + C(z-z_0) = 0} \qquad (1)$$

We can get from (1):

$$\boxed{z - z_0 = a(x - x_0) + b(y - y_0)} \quad (2)$$

Here, in (2), for $x = x_0$:

T_2: $z - z_0 = b(y - y_0)$

$b = f_y(x_0, y_0)$

for $y = y_0$:

T_1: $z - z_0 = a(x - x_0)$

$a = f_x(x_0, y_0)$

<u>Def.</u> Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x,y)$ at $P(x_0, y_0, z_0)$ is

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

- Linear Approximations

We have that the tangent plane to $z = f(x,y)$ at $(a, b, f(a,b))$ has equation:

$$z = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b)$$

<u>Def.</u> The linear function

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

is called the <u>linearization</u> of f at (a,b) and the approximation

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

is called the <u>linear approximation</u> or the <u>tangent plane approximation</u> of f at (a,b).

Recall,

$$\Delta y = f(a+\Delta x) - f(a)$$

If f is differ. at a, then

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x \quad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

Now, let us consider $z = f(x,y)$.

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b).$$

<u>Def.</u> If $z = f(x,y)$, then f is differentiable at (a,b) if Δz can be expressed in the form

$$\boxed{\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y}$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

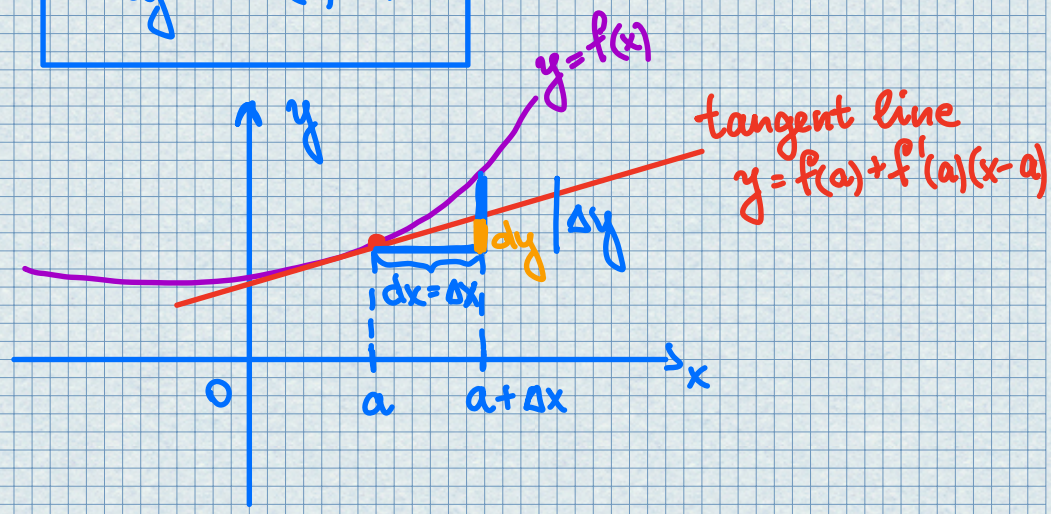
<u>Theorem</u> If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

- **Differentials**

Let $y = f(x)$

The differential of y is defined as

$$\boxed{dy = f'(x)\,dx}$$



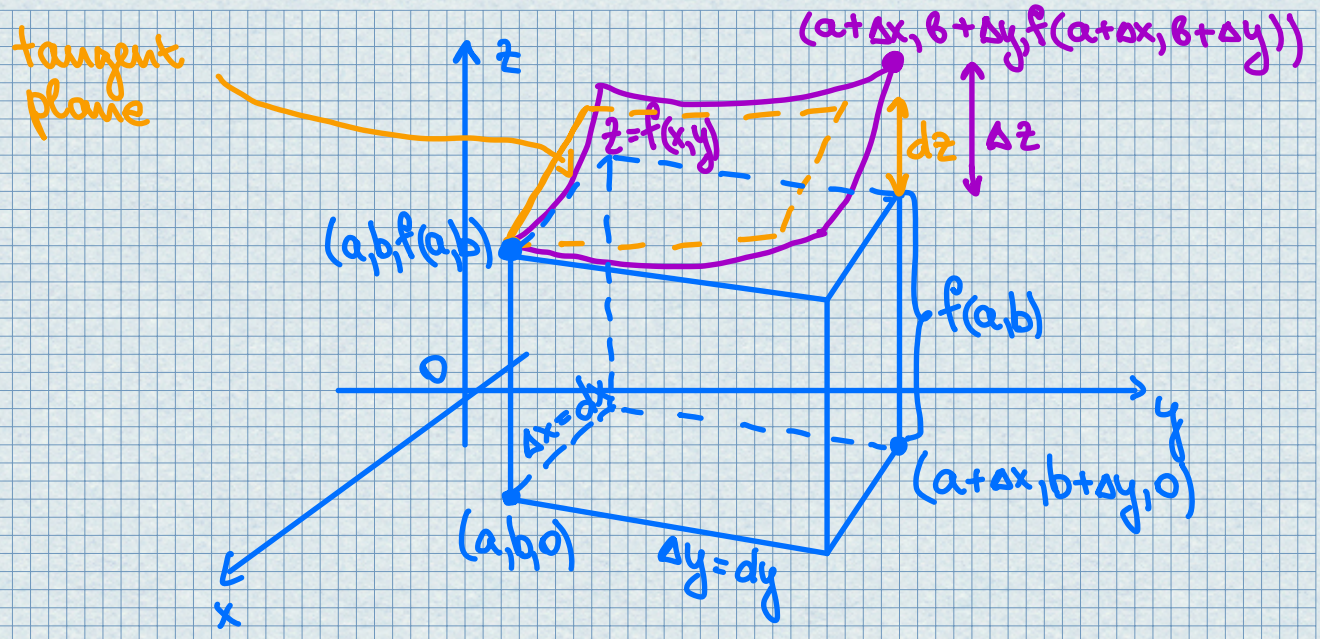
Δy – change in height of $y = f(x)$

dy – change in height of the tangent line when x changes by $dx = \Delta x$.

<u>Def.</u> For a differentiable function $z = f(x,y)$, differentials dx and dy are independent variables.

Then the <u>differential</u> dz (<u>total differential</u>), is defined by

$$\boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy}$$



tangent plane:

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

- Functions of three or more variables

For such functions the <u>linear approximation</u> is

$$f(x,y,z) \approx \underbrace{f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)}_{L(x,y,z)}$$

If $w = f(x,y,z)$, then the <u>increment</u> of w is

$$\boxed{\Delta w = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)}$$

The <u>differential</u> dw is defined by

$$\boxed{dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz}$$

Examples

1. Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at $(1,1,3)$.

Solution

$$f(x,y) = 2x^2 + y^2$$

$$f_x(x,y) = 4x \qquad f_x(1,1) = 4$$

$$f_y(x,y) = 2y \qquad f_y(1,1) = 2$$

Tangent plane at $(1,1,3)$:

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

$$z - 3 = 4(x-1) + 2(y-1)$$

or

$$\boxed{z = 4x + 2y - 3}$$

2. Show that $f(x,y) = xe^{xy}$ is differentiable at $(1,0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Solution

$$f_x(x,y) = e^{xy} + xye^{xy}$$

$$f_y(x,y) = xye^{xy}$$
$$f_x(1,0) = 1$$
$$f_y(1,0) = 1$$

Both f_x and f_y are continuous, so f is differentiable.

$$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) =$$
$$= 1 + 1(x-1) + 1 \cdot y = x + y$$

The corresponding linear approximation is

$$xe^{xy} \approx x+y$$

So
$$f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

Compare this with the actual value of $f(1.1, -0.1) = 1.1\, e^{-0.11} \approx 0.98542$.

$\blacktriangledown$

$\boxed{3.}$ (a) If $z = x^2 + 3xy - y^2$, find dz.

(b) If x changes from 2 to 2,05 and y changes from 3 to 2.96, compare the values of Δz and dz.

<u>Solution</u>

(a) $$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

$$\frac{\partial f}{\partial x} = 2x + 3y$$

$$\frac{\partial f}{\partial y} = 3x - 2y$$

$$dz = (2x+3y)\,dx + (3x-2y)\,dy$$

(b) $x = 2$

$dx = \Delta x = 0.05$

$y = 3$

$dy = \Delta y = -0.04$

$$dz = (2\cdot 2 + 3\cdot 3)\,0.05 + (3\cdot 2 - 2\cdot 3)(-0.04) = 0.65.$$

$$\Delta z = f(2.05, 2.96) - f(2,3) =$$

$$= (2.05^2 + 3\cdot 2.05\cdot 2.96 - 2.96^2) - (2^2 + 3\cdot 2\cdot 3 - 3^2) =$$

$$= 0.6449$$

Hence, $\Delta z \approx dz$ but dz is easier to compute.

▼

$\boxed{4.}$ The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively,

with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

Solution

$$V = \frac{\pi r^2 h}{3}$$

$$dV = \frac{dV}{\partial r} dr + \frac{\partial V}{\partial h} dh = \frac{2\pi r h}{3} dr + \frac{\pi r^2}{3} dh$$

We have

$$|\Delta r| \leq 0.1$$

$$|\Delta h| \leq 0.1$$

We take the largest error in the measurement of r and h. Therefore, we take

$$dr = 0.1$$
$$dh = 0.1$$
$$r = 10$$
$$h = 25$$

Hence,

$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1) = 20\pi$$

Thus the max error in the calculated volume is about $20\pi \text{ cm}^3 \approx 63 \text{ cm}^3$.