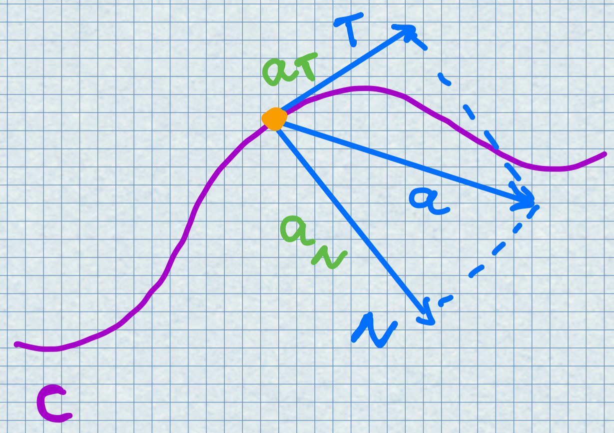
Lecture # 11 - Week 4 - Motion in Space: Acceleration - 13.4

- Tangential and Normal Components of Acceleration



Let $v = |\mathbf{v}|$. Then

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{v(t)}{|v(t)|} = \frac{v(t)}{v}$$

So

$$v(t) = v\,T(t)$$

$$a = v'(t) = v'\,T(t) + T'(t)\,v$$

We have that

$$\kappa = \frac{|T'|}{|r'|} = \frac{|T'|}{v}$$

So

$$|T'| = v\kappa$$

The unit normal vector is defined as follows

$$N = \frac{T'(t)}{|T'(t)|}$$

and

$$T'(t) = N|T'(t)| = \kappa v N$$

Hence,

$$\boxed{a = v'T + \kappa v^2 N}$$

or

$$\boxed{a = a_T T + a_N N}$$

where

$$a_T = v' \quad \text{and} \quad a_N = \kappa v^2$$

Since

$$v \cdot a = vT \cdot (v'T + \kappa v^2 N) =$$

$$= vv' \underbrace{T \cdot T}_{=1} + \kappa v^3 \underbrace{T \cdot N}_{=0} = vv'$$

Therefore,

$$\boxed{a_T = v' = \frac{r'(t) \cdot r''(t)}{|r'(t)|}}$$

$$\boxed{\begin{aligned} a_N &= \kappa v^2 = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} |r'(t)|^2 = \\ &= \frac{|r'(t) \times r''(t)|}{|r'(t)|} \end{aligned}}$$

- Kepler's Laws of Planetary Motion

① A planet revolves around the sun in an elliptical orbit with the sun at one focus.

② The line joining the sun to a planet sweeps out equal areas in equal times.

③ The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Examples

$\boxed{1.}$ A particle moves with position function $r(t) = \langle t^2, t^2, t^3 \rangle$. Find the tangential and normal components of acceleration.

Solution

$$r(t) = t^2 i + t^2 j + t^3 k$$

$$r'(t) = 2t i + 2t j + 3t^2 k$$

$$r''(t) = 2i + 2j + 6t k$$

$$|r'(t)| = \sqrt{8t^2 + 9t^4}$$

$$a_T = \frac{r'(t) \cdot r''(t)}{|r'(t)|} = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} = 6t^2 i - 6t^2 j$$

$$a_N = \frac{|r'(t) \times r''(t)|}{|r'(t)|} = \frac{6\sqrt{2}\, t^2}{\sqrt{8t^2 + 9t^4}}.$$

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