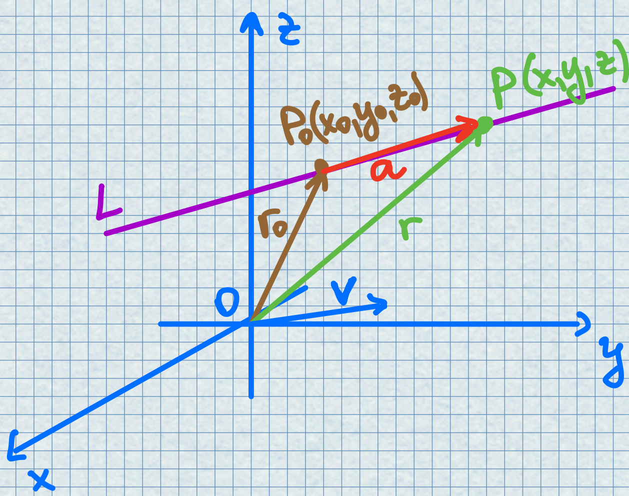
Lecture #5 - Week 2 - Equations of Lines and Planes - 12.5

• **Lines**



$a = \overrightarrow{P_0P}$

$$r = r_0 + a$$

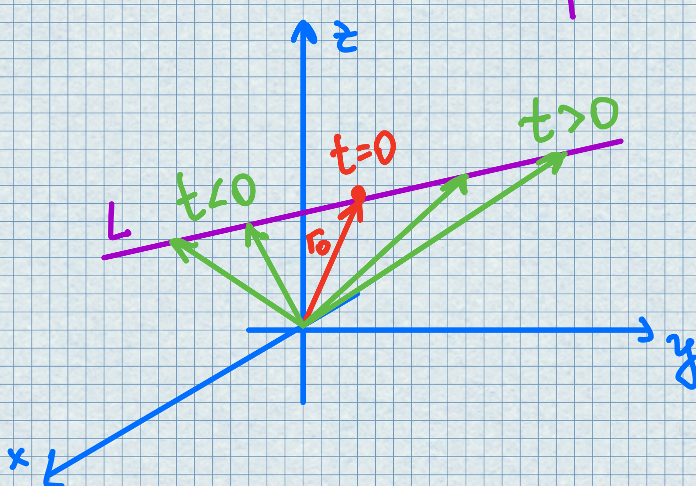
Since $\vec{a} \parallel \vec{v}$, then $a = tv$, where t is a scalar.

Thus

$$\boxed{r = r_0 + tv}$$

↑

vector equation of a line L



L in 3D is determined by P_0 on L and

the direction of L.

If $v = \langle a, b, c\rangle$, then $tv = \langle ta, tb, tc\rangle$

$r = \langle x, y, z\rangle$

$r_0 = \langle x_0, y_0, z_0\rangle$

$$\langle x, y, z\rangle = \langle x_0 + ta,\ y_0 + tb,\ z_0 + tc\rangle$$

$$\Updownarrow$$

$$(1) \quad \begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}, \quad t \in \mathbb{R}$$

parametric equations of L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $v = \langle a, b, c\rangle$.

Def. If $v = \langle a, b, c\rangle$ is used to describe the direction of L, then a, b, and c are called direction numbers of L.

From (1) we get

$$t = \frac{x - x_0}{a}, \quad t = \frac{y - y_0}{b}, \quad t = \frac{z - z_0}{c}$$

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$ Symmetric equations of L

In general if $(a,b,c) = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$, then

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

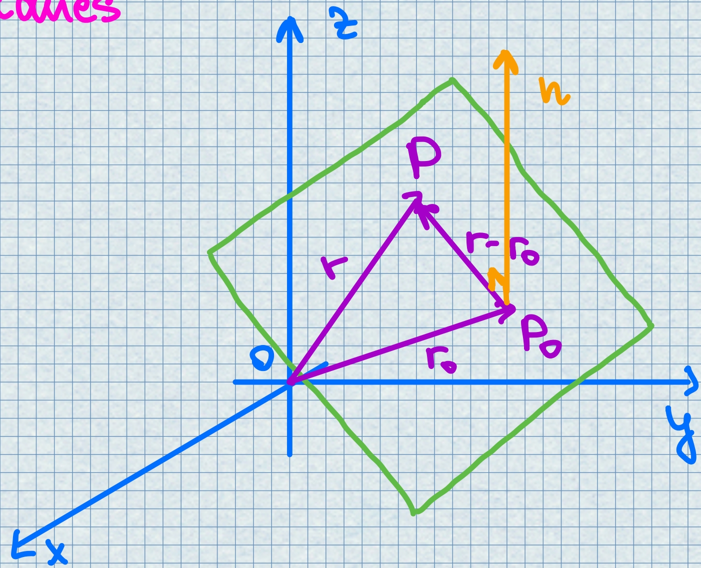
and

$$r = r_0 + tv = r_0 + t(r_1 - r_0) =$$
$$= (1-t)r_0 + t r_1$$

The line segment from r_0 to r_1 is given by the vector equation

$$r(t) = (1-t)r_0 + t r_1, \quad 0 \le t \le 1$$

- **Planes**



$P(x, y, z)$

$P_0(x_0, y_0, z_0)$

n is a normal vector

$$n \perp r - r_0 \iff n \cdot (r - r_0) = 0$$

(2) $\boxed{n\cdot(r-r_0)=0}$ vector equation of the plane

If $n=\langle a,b,c\rangle$ and $r=\langle x,y,z\rangle$, $r_0=\langle x_0,y_0,z_0\rangle$,
then (2) becomes

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

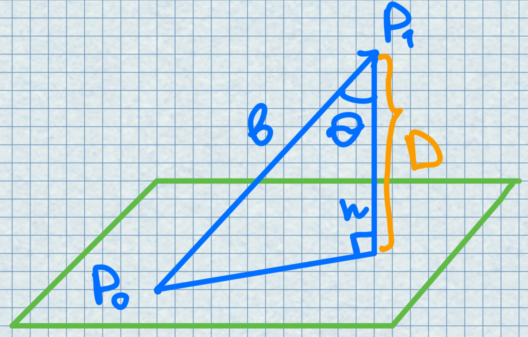
A scalar equation of the plane through point $P_0(x_0,y_0,z_0)$ with normal vector $n=\langle a,b,c\rangle$ is

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

$$\Updownarrow$$

$$\boxed{Ax+By+Cz+D=0}$$

- Distances



$$D=\frac{|n\cdot b|}{|n|}$$

$P_0=(x_0,y_0,z_0)$
$P_1=(x_1,y_1,z_1)$
$b=\overrightarrow{P_0P_1}$

Thus,
$$D = \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

or
$$\boxed{D = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{a^2 + b^2 + c^2}}}$$

Examples

$\boxed{1.}$ (a) Find a vector equation and parametric equations for the line that passes through the point $(5,1,3)$ and is parallel to the vector $i+4j-2k$.

(b) Find two other points on the line.

Solution

(a) $r_0 = \langle 5,1,3 \rangle = 5i + j + 3k$

$v = i + 4j - 2k$

$$r = (5i + j + 3k) + t(i + 4j - 2k)$$

or

$$r = (5+t)i + (1+4t)j + (3-2t)k$$

Parametric equations are

$$x = 5 + t \qquad y = 1 + 4t \qquad z = 3 - 2t$$

(b) Choosing $t = 1$: $x = 6, y = 5, z = 1$

$(6,5,1) \in \ell$

$t = -1$: $x = 4, y = -3, z = 5$

$(4,-3,5) \in \ell$.

$\boxed{2.}$ Find parametric equations and symmetric equations of the line that passes through the points $A(2,4,-3)$ and $B(3,-1,1)$.

Solution

We are not explicitly given a vector parallel to the line, but observe that the vector v with representation $\overrightarrow{AB}$ is parallel to the line and

$$v = \langle 3-2, -1-4, 1-(-3) \rangle = \langle 1, -5, 4 \rangle$$

Thus direction numbers are $a=1$, $b=-5$, and $c=4$.

Taking the point $(2,4,-3)$ as P_0, we get

parametric $\boxed{x = 2+t \qquad y = 4-5t \qquad z = -3+4t}$

symmetric $\boxed{\dfrac{x-2}{1} = \dfrac{y-4}{-5} = \dfrac{z+3}{4}}$

▼

3. Find an equation of the plane through the point $(2,4,-1)$ with normal vector $n = \langle 2,3,4 \rangle$. Find the intercepts and sketch the plane.

Solution

Putting $a=2$, $b=3$, $c=4$, $x_0=2$, $y_0=4$, and

$z_0 = -1$, we get

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

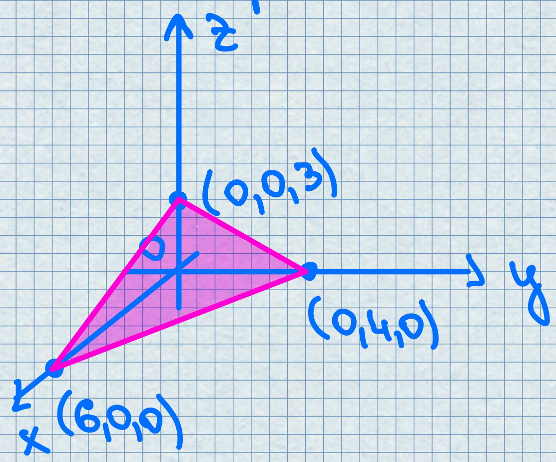
$$2(x-2) + 3(y-4) + 4(z+1) = 0$$

$$\boxed{2x + 3y + 4z = 12}$$

To find the x-intercept we set $y = z = 0$: $x = 6$.

y-intercept: $y = 4$

z-intercept: $z = 3$.



4. Find an equation of the plane that passes through the points $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$.

Solution

$$a = \overrightarrow{PQ} = \langle 2, -4, 4 \rangle$$

$$b = \overrightarrow{PR} = \langle 4, -1, -2 \rangle$$

Since a, b lie in the plane,
$a \times b \perp$ to the plane
and $a \times b$ can be taken as a normal vector.

$$n = a \times b = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12i + 20j + 14k$$

With $P(1,3,2)$ and n, an equation of the plane is

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

or

$$6x + 10y + 7z = 50.$$

$\boxed{5.}$ Find the point at which the line with parametric equations

$$x = 2 + 3t \quad y = -4t \quad z = 5 + t$$

intersects the plane $4x + 5y - 2z = 18$.

Solution

We substitute the expressions for x, y, and z from the parametric equations into the plane equation:

$$4(2+3t) + 5(-4t) - 2(5+t) = 18$$

$$\Downarrow$$

$$-10t = 20 \quad , \quad t = -2$$

Therefore, the point of intersection occurs when $t=-2$. Then

$$x = 2+3(-2) = -4$$
$$y = -4(-2) = 8$$
$$z = 5-2 = 3$$

And so the point of intersection is $(-4, 8, 3)$.

$\boxed{6.}$ Find the distance between the parallel planes $10x+2y-2z=5$ and $5x+y-z=1$.

<u>Solution</u>

The planes are parallel because their normal vectors are parallel.

$$\langle 10, 2, -2 \rangle \parallel \langle 5, 1, -1 \rangle$$

For $10x+2y-2z=5$, we set $y=z=0$. So $10x=5$, $x=\frac{1}{2}$. And $(\frac{1}{2}, 0, 0)$ lie on the plane $10x+2y-2z=5$.

Thus the distance between $(\frac{1}{2}, 0, 0)$ and the plane $5x+y-z-1=0$ is

$$D = \frac{\left|5\cdot\frac{1}{2}+1\cdot 0-1\cdot 0-1\right|}{\sqrt{5^2+1^2+(-1)^2}} = \frac{\frac{3}{2}}{3\sqrt{3}} = \boxed{\frac{\sqrt{3}}{6}}$$