Name:

Instructions. (100 points) You have two hours. The exam is closed book, closed notes, and only simple calculators are allowed. Show all your work in order to receive full credit.

1. **[17 points]** Consider points $A(2, 3, 1)$ and $B(3, 4, c)$ and vectors $u = \langle 1, 2, 3\rangle$ and $v = \langle 1, 1, 2\rangle$.

   (a) (4pts) Find the vector projection of u along v.

   $$\text{proj}_v u = \frac{u\cdot v}{|v|^2} v = \frac{9}{6}\langle 1,1,2\rangle = \langle \frac{9}{6}, \frac{9}{6}, \frac{18}{6} \rangle =$$

   $u\cdot v = 1+2+6=9$

   $|v| = \sqrt{1+1+4} = \sqrt{6}$

   $$= \boxed{\langle \frac{3}{2}, \frac{3}{2}, 3 \rangle}$$

   (b) (3 pts) Find all values of c such that the length of $\overrightarrow{AB}$ equals 5.

   $\overrightarrow{AB} = \langle 1, 1, c-1\rangle$

   $|\overrightarrow{AB}| = \sqrt{1+1+(c-1)^2} = 5$

   $2+(c-1)^2 = 25$

   $(c-1)^2 = 23$

   $c-1 = \pm\sqrt{23}$

   $\boxed{c = 1\pm\sqrt{23}}$

   (c) (3 pts) Find all values of c such that $\overrightarrow{AB}$ is parallel to v.

   $$\overrightarrow{AB} \parallel v \Leftrightarrow \begin{cases} 1 = k \\ 1 = k \\ c-1 = 2k \end{cases}$$

   $k = 1$

   $c-1 = 2$

   $\boxed{c = 3}$

   (d) (3 pts) Find all values of c such that $\overrightarrow{AB}$ is orthogonal to u.

   $\overrightarrow{AB} \perp u \Leftrightarrow \overrightarrow{AB}\cdot u = 0 \Leftrightarrow 1\cdot 1 + 1\cdot 2 + (c-1)3 = 0$

   $\cancel{1} + \cancel{2} + 3c - \cancel{3} = 0 \Rightarrow \boxed{c = 0}$

   (e) (4 pts) Find the cross product of vectors u and v.

   $$u\times v = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = (4-3)i - (2-3)j + (2-3)k =$$

   $$= i + j - k = \langle 1, 1, -1\rangle$$

2. **[15 points]** You are given the following space curve:

$$r(t) = \langle 2\cos(2t),\ 2\sin(2t),\ t\rangle, \quad 0 \le t \le \frac{5\pi}{2}$$

(a) (8 pts) Draw the trajectory of the vector function $r(t)$ for the given value t.

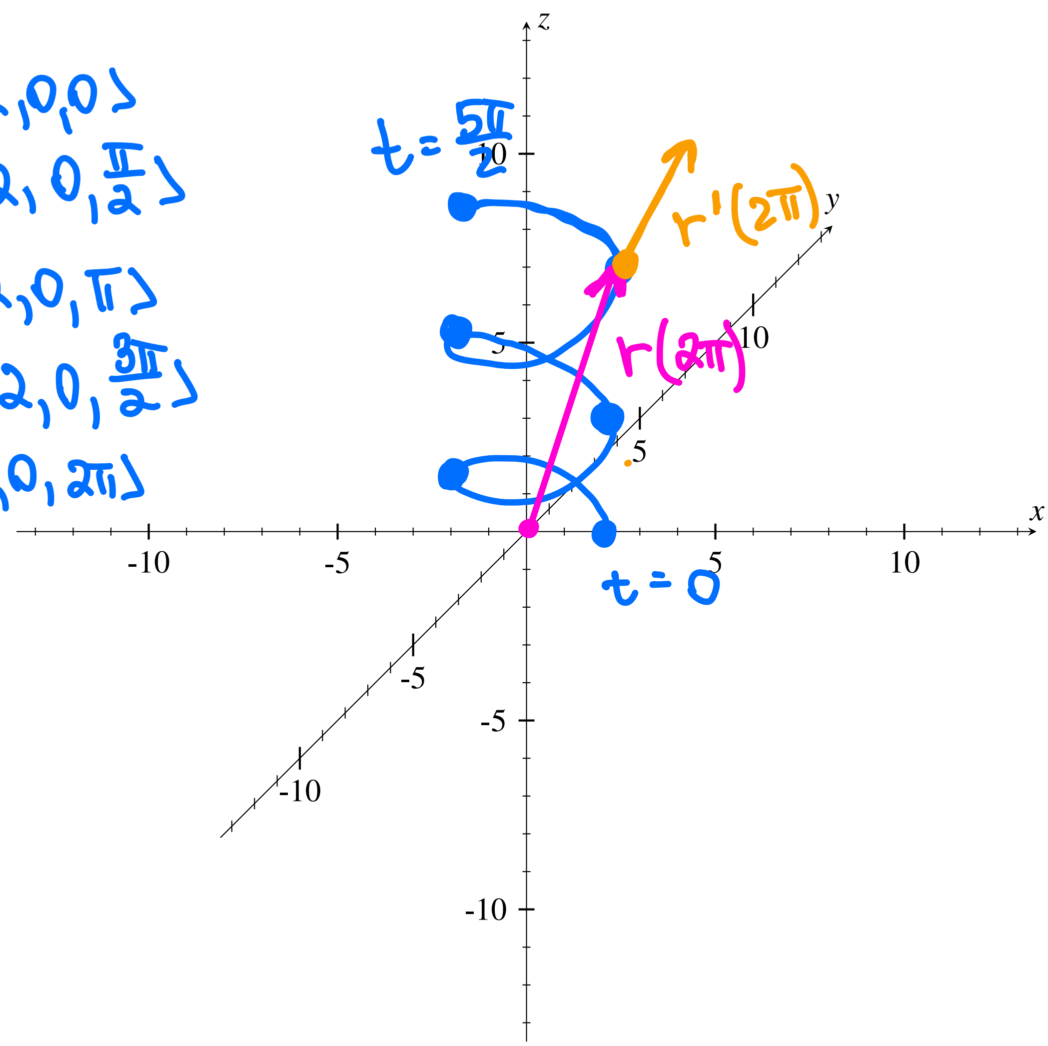
$r(0) = \langle 2, 0, 0\rangle$

$r(\frac{\pi}{2}) = \langle -2, 0, \frac{\pi}{2}\rangle$

$r(\pi) = \langle 2, 0, \pi\rangle$

$r(\frac{3\pi}{2}) = \langle -2, 0, \frac{3\pi}{2}\rangle$

$r(2\pi) = \langle 2, 0, 2\pi\rangle$



(b) (4 pts) Draw on the above trajectory the position and velocity vectors for $t = 2\pi$.

$r(2\pi) = \langle 2, 0, 2\pi\rangle$

$r'(t) = \langle -4\sin 2t, 4\cos 2t, 1\rangle$

$r'(2\pi) = \langle 0, 4, 1\rangle$

(c) (3 pts) Find the speed at time t and simplify your result.

$v = |r'(t)| = \sqrt{16\sin^2 2t + 16\cos^2 2t + 1} = \boxed{\sqrt{17}}$

3. **[15 points]** Sketch the following surfaces.

(a) (10 pts) For $x = y^2 + 4z^2$, sketch the given traces, then the surface in 3-D.

1) traces: $x = 0, 4$

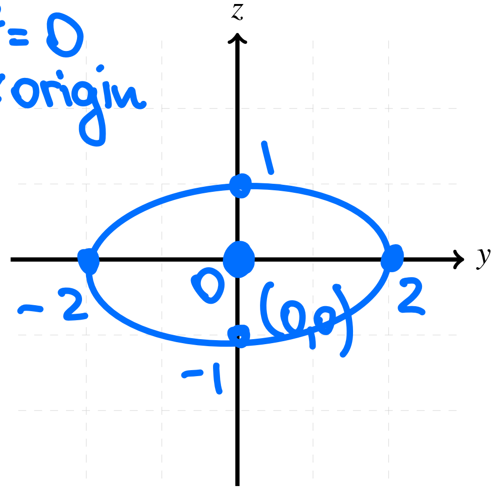
$x=0: y^2+4z^2=0$

$y=z=0$ ← origin

$x=4: 4=y^2+4z^2$

$1=\frac{y^2}{4}+\frac{z^2}{1}$

↑ ellipse



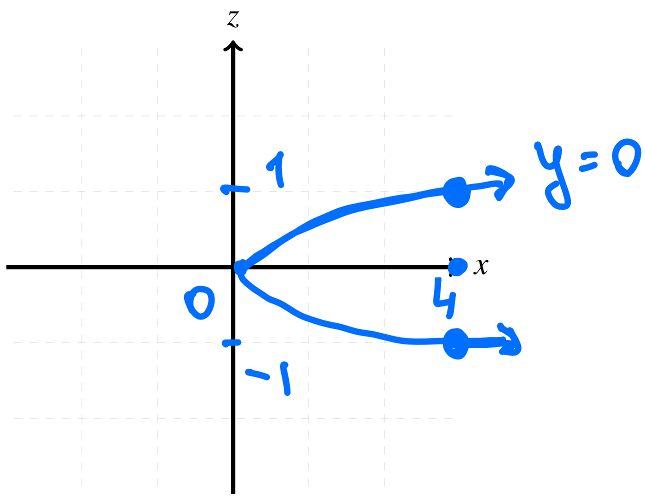
2) traces: $y = 0, \pm 2$

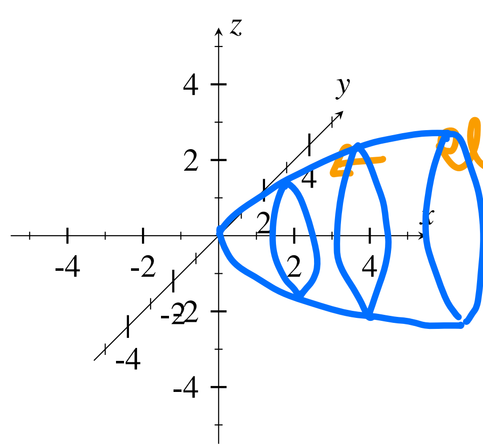
$y=0: x=4z^2$

↑ parabola

$y=\pm 2: x=4+4z^2$

↑ parabola





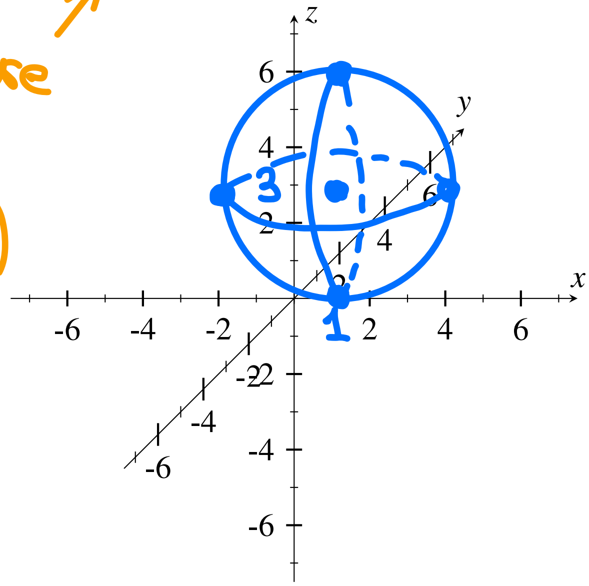
← elliptic paraboloid

(b) (5 pts) Sketch the surface $(x-1)^2 + y^2 + (z-3)^2 = 9$.

sphere

$r = 3$

center: $(1, 0, 3)$



4. **[21 points]** Consider the following point, line, and plane:

$$B(3, -2, 1)$$
$$\vec{l}(t) = \langle t,\ 1-t,\ 2t+3 \rangle$$
$$Plane\ \ P : 2x + 5y - z = 10$$

(a) (5 pts) Give the equation of a plane parallel to the plane P that passes through B.

$n = \langle 2, 5, -1 \rangle$

$B(3, -2, 1)$

Hence,

$2(x-3) + 5(y+2) - 1(z-1) = 0$

$2x - 6 + 5y + 10 - z + 1 = 0$

$\boxed{2x + 5y - z + 5 = 0}$

(b) (4 pts) Find the point of intersection of the line $\vec{l}(t)$ and the plane P.

From $\vec{l}(t)$: $\begin{cases} x = t \\ y = 1 - t \\ z = 2t + 3 \end{cases}$. Plug in P:

$2t + 5(1-t) - (2t+3) = 10$

$\cancel{2t} + 5 - 5t - \cancel{2t} - 3 = 10$

$-5t = 8$

$\boxed{t = -8/5}$

Hence,

$\begin{cases} x = -8/5 \\ y = 13/5 \\ z = -1/5 \end{cases}$

(c) (5 pts) Find the angle the line $\vec{l}(t)$ makes with the normal to the plane P. (Your answer may involve an inverse trigonometric function.)

$$\begin{cases} x=t \\ y=1-t \\ z=2t+3 \end{cases} \Rightarrow \begin{matrix} t=x \\ t=1-y \\ t=\frac{z-3}{2} \end{matrix} \qquad x=1-y=\frac{z-3}{2}$$

$$\frac{x-0}{1}=\frac{y-1}{0-1}=\frac{z-3}{2}$$

$\vec{u}=\langle 1,-1,2\rangle$
$\vec{n}=\langle 2,5,-1\rangle$ $\Rightarrow \cos\theta=\frac{u\cdot n}{|u|\cdot|n|} \Rightarrow \cos\theta=\frac{-5}{\sqrt{6}\sqrt{30}}=\frac{-5}{\sqrt{180}}$

$$\boxed{\theta=\arccos(5/\sqrt{180})}$$

(d) (7 pts) Find an equation for the plane containing the point B and the line $\vec{l}(t)$.

$B(3,-2,1)$

$$\vec{u}\times\vec{n}=\begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & 5 & -1 \end{vmatrix}=(1-10)i-(-1-4)j+(5+2)k=$$

$=-9i+5j+7k=\langle -9,5,7\rangle$

Hence, the plane equation is

$$-9(x-3)+5(y+2)+7(z-1)=0$$
$$-9x+27+5y+10+7z-7=0$$
$$\boxed{-9x+5y+7z+30=0}$$

5. **[8 points]** An object moves in 3-D with acceleration

$$a(t)=\langle \sin(t),\, 2\cos(t),\, 6t\rangle.$$

At time $t=0$ it has velocity $\langle 0,0,-1\rangle$. Find a function $v(t)=r'(t)$ giving its velocity at all times $t>0$.

$v(0)=\langle 0,0,-1\rangle$

$v(t)=\int a(t)\,dt = i\int \sin t\,dt + j\int 2\cos t\,dt +$
$+k\int 6t\,dt + C = -\cos t\cdot i + 2\sin t\cdot j + 3t^2\cdot k + C$

$v(0)=\langle -1+c_1, 0+c_2, 0+c_3\rangle=\langle 0,0,-1\rangle$

$C=\langle 1,0,-1\rangle$

Hence $\boxed{v(t)=\langle -\cos t+1,\ 2\sin t,\ 3t^2-1\rangle}$

+7 points to each student

6. **[20 points]** A particle moves with *velocity* $v(t) = \langle t^3, t^2, 2t \rangle$.

(a) (7 pts) Find the distance the particle travels between times $t = 0$ and $t = 2$.

$$d(2)-d(0) = \int_0^2 |v(t)|\,dt = \int_0^2 \sqrt{t^6+t^4+4t^2}\,dt =$$

$$= \int_0^2 t\sqrt{t^4+t^2+4}\,dt = \int_0^2 t\sqrt{\left(t^2+\tfrac{1}{2}\right)^2+\tfrac{15}{4}}\,dt = \frac{1}{2}\int_0 \sqrt{u^2+\tfrac{15}{4}}\,du =$$

$t^2+\frac{1}{2}=u$

$2t\,dt = du$

$$= \frac{1}{2}\left(\frac{u}{2}\sqrt{u^2+\tfrac{15}{4}} + \frac{15}{8}\ln\left(u+\sqrt{u^2+\tfrac{15}{4}}\right)\right)\Big|^2 =$$

$$= \frac{1}{2}\left(\frac{t^2+\frac{1}{2}}{2}\sqrt{\left(t^2+\tfrac{1}{2}\right)^2+\tfrac{15}{4}} + \frac{15}{8}\ln\left(t^2+\tfrac{1}{2}\sqrt{(\ldots)^2+15/4}\right)\right)\Big|_0^2$$

(b) (8 pts) Calculate the curvature

$$\kappa = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

of the trajectory at time $t = 1$.

$r'(t) = \langle t^3, t^2, 2t \rangle \qquad r'(1) = \langle 1, 1, 2 \rangle$

$r''(t) = \langle 3t^2, 2t, 2 \rangle \qquad r''(1) = \langle 3, 2, 2 \rangle$

$|r'(t)| = \sqrt{t^6+t^4+4t^2}$

$$|r' \times r''| = \begin{vmatrix} i & j & k \\ t^3 & t^2 & 2t \\ 3t^2 & 2t & 2 \end{vmatrix} = (2t^2-4t^2)i - (2t^3-6t^3)j + (2t^4-3t^4)k$$

$|r'(1)| = \sqrt{6}$

$|r'(1) \times r''(1)| = \sqrt{4+16+1} = \sqrt{21}$

$$\kappa = \boxed{\frac{\sqrt{21}}{(\sqrt{6})^3}}$$

(c) (5 pts) Find the unit tangent vector **T(t)** and the tangential component of acceleration

$$a_T = \frac{r'(t) \cdot r''(t)}{|r'(t)|}$$

at $t = 1$.

$$T(t) = \frac{r'(t)}{|r'(t)|}\ ; \quad T(1) = \frac{r'(1)}{|r'(1)|} = \frac{\langle 1,1,2\rangle}{\sqrt{6}} =$$

$$= \boxed{\left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle}$$

$$r'(1)\cdot r''(1) = 3+2+4 = 9$$

$$a_T = \boxed{\frac{9}{\sqrt{6}}}$$

7. **[4 points]** Evaluate the following integral

$$\int_1^4 \left(2t^{3/2}\,\mathbf{i} + (t+1)\sqrt{t}\,\mathbf{k}\right) dt. =$$

$$= i\int_1^4 2t^{3/2}\,dt + k\int_1^4 \left(t\sqrt{t} + \sqrt{t}\right) dt =$$

$$= 2i\, \frac{t^{5/2}}{5/2}\Bigg|_1^4 + k\left(\frac{t^{5/2}}{5/2} + \frac{t^{3/2}}{3/2}\right)\Bigg|_1^4 =$$

$$= 2\left(\frac{2}{5}\cdot 32 - \frac{2}{5}\right)i + \left(\frac{2}{5}\cdot 32 + \frac{2}{3}\cdot 8 - \frac{2}{5} - \frac{2}{3}\right)k =$$

$$= \frac{124}{5}i + 0\cdot j + \left(\frac{62}{5} + \frac{14}{3}\right)k = \boxed{\left\langle \frac{124}{5}, 0, \frac{256}{15} \right\rangle}.$$

$$\frac{186+70}{15} = \frac{256}{15}$$