

Name:

**Instructions.** (100 points) You have two hours. The exam is closed book, closed notes, and only simple calculators are allowed. Show all your work in order to receive full credit.

1. [17 points] Consider points  $A(2, 3, 1)$  and  $B(3, 4, c)$  and vectors  $u = \langle 1, 2, 3 \rangle$  and  $v = \langle 1, 1, 2 \rangle$ .

- (a) (4pts) Find the vector projection of  $u$  along  $v$ .

$$\text{proj}_v u = \frac{u \cdot v}{|v|^2} v = \frac{9}{\sqrt{6}} \langle 1, 1, 2 \rangle = \left\langle \frac{9}{\sqrt{6}}, \frac{9}{\sqrt{6}}, \frac{18}{\sqrt{6}} \right\rangle$$

$$u \cdot v = 1 + 2 + 6 = 9$$

$$|v| = \sqrt{1+1+4} = \sqrt{6}$$

- (b) (3 pts) Find all values of  $c$  such that the length of  $\vec{AB}$  equals 5.

$$\vec{AB} = \langle 1, 1, c-1 \rangle$$

$$|\vec{AB}| = \sqrt{1+1+(c-1)^2} = 5$$

$$2+(c-1)^2 = 25$$

$$(c-1)^2 = 23$$

$$c-1 = \pm\sqrt{23}$$

$$c = 1 \pm \sqrt{23}$$

- (c) (3 pts) Find all values of  $c$  such that  $\vec{AB}$  is parallel to  $v$ .

$$\vec{AB} \parallel v \Leftrightarrow \begin{cases} 1 = k \\ 1 = k \\ c-1 = 2k \end{cases} \quad k=1$$

$$c-1 = 2$$

$$c = 3$$

- (d) (3 pts) Find all values of  $c$  such that  $\vec{AB}$  is orthogonal to  $u$ .

$$\vec{AB} \perp u \Leftrightarrow \vec{AB} \cdot u = 0 \Leftrightarrow 1 \cdot 1 + 1 \cdot 2 + (c-1)3 = 0$$

$$1+2+3c-3=0 \Rightarrow c=0$$

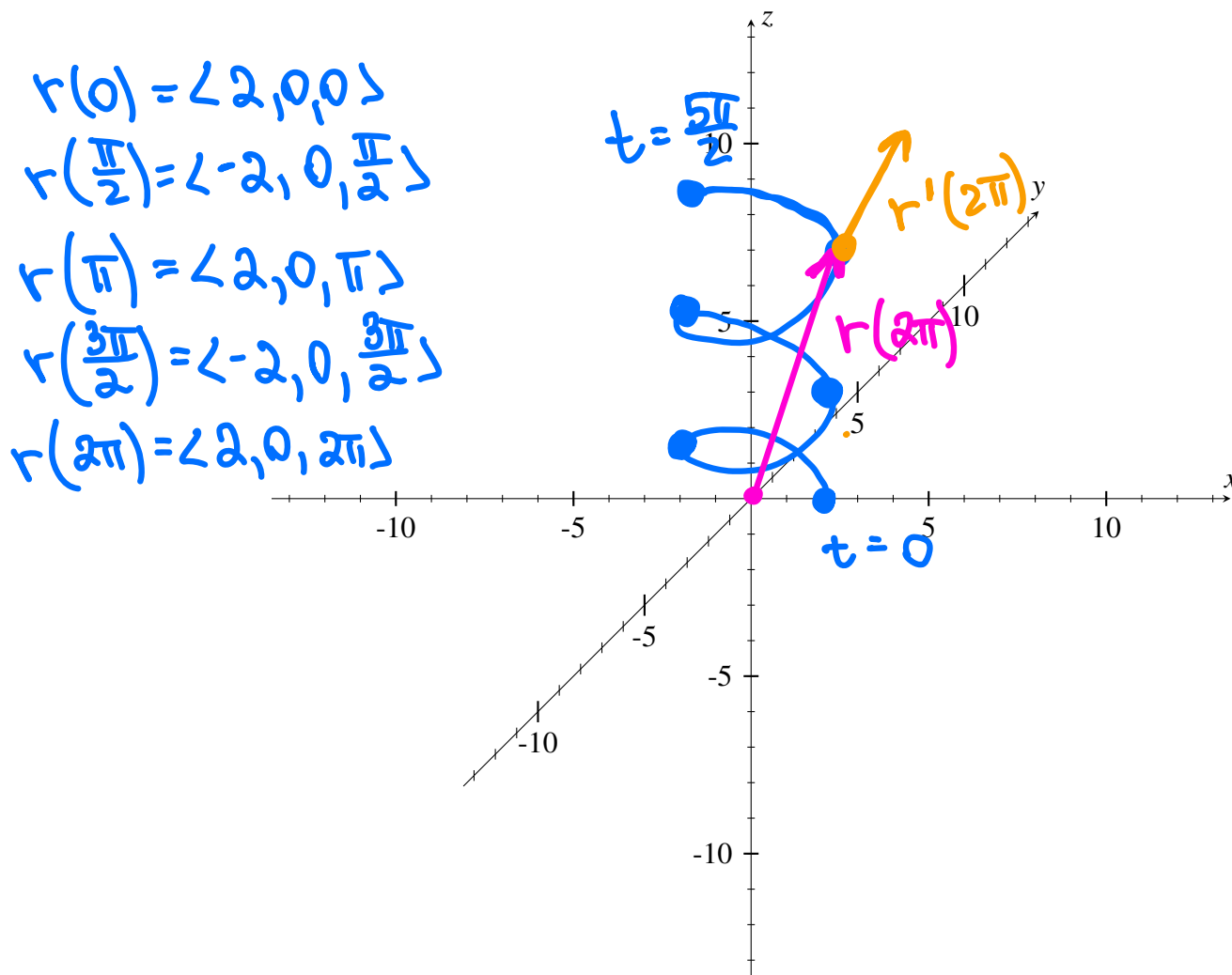
- (e) (4 pts) Find the cross product of vectors  $u$  and  $v$ .

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = (4-3)i - (2-3)j + (2-3)k = i + j - k = \langle 1, 1, -1 \rangle$$

2. [15 points] You are given the following space curve:

$$r(t) = \langle 2 \cos(2t), 2 \sin(2t), t \rangle, \quad 0 \leq t \leq \frac{5\pi}{2}$$

(a) (8 pts) Draw the trajectory of the vector function  $r(t)$  for the given value  $t$ .



(b) (4 pts) Draw on the above trajectory the position and velocity vectors for  $t = 2\pi$ .

Handwritten calculations for part (b):

$$\begin{aligned} r(2\pi) &= \langle 2, 0, 2\pi \rangle \\ r'(t) &= \langle -4 \sin 2t, 4 \cos 2t, 1 \rangle \\ r'(2\pi) &= \langle 0, 4, 1 \rangle \end{aligned}$$

(c) (3 pts) Find the speed at time  $t$  and simplify your result.

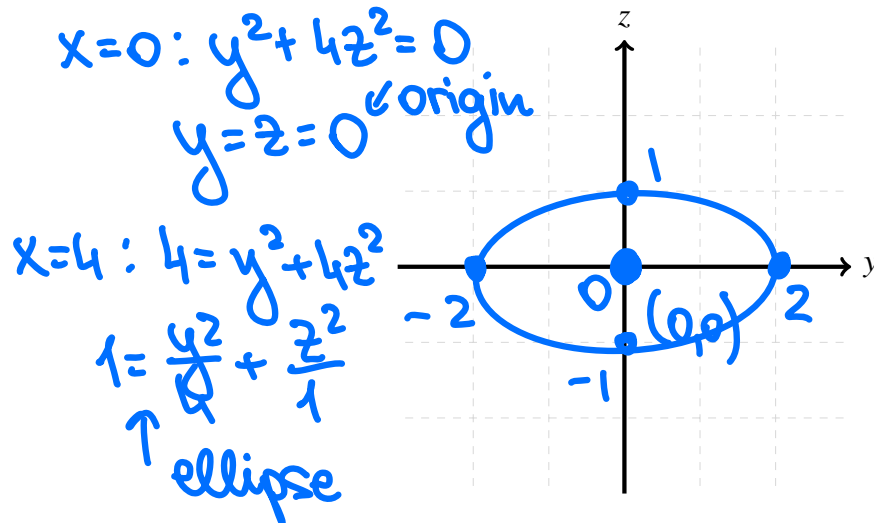
Handwritten calculation for part (c):

$$v = |r'(t)| = \sqrt{16 \sin^2 2t + 16 \cos^2 2t + 1} = \boxed{\sqrt{17}}$$

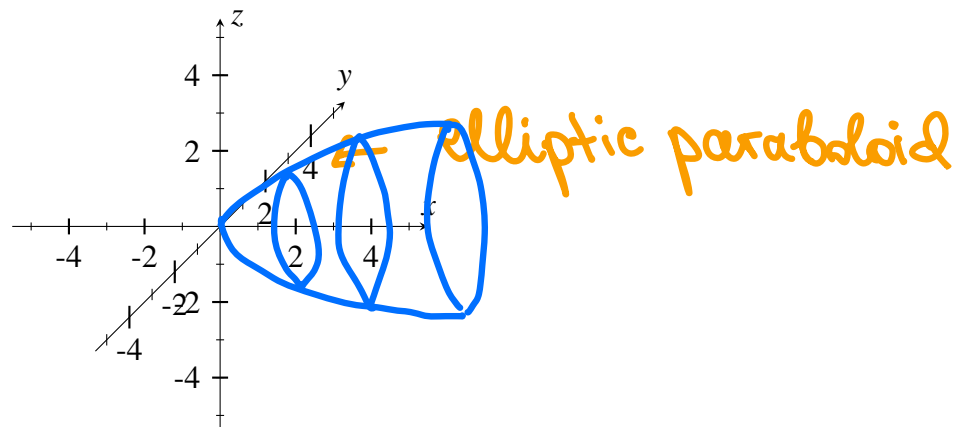
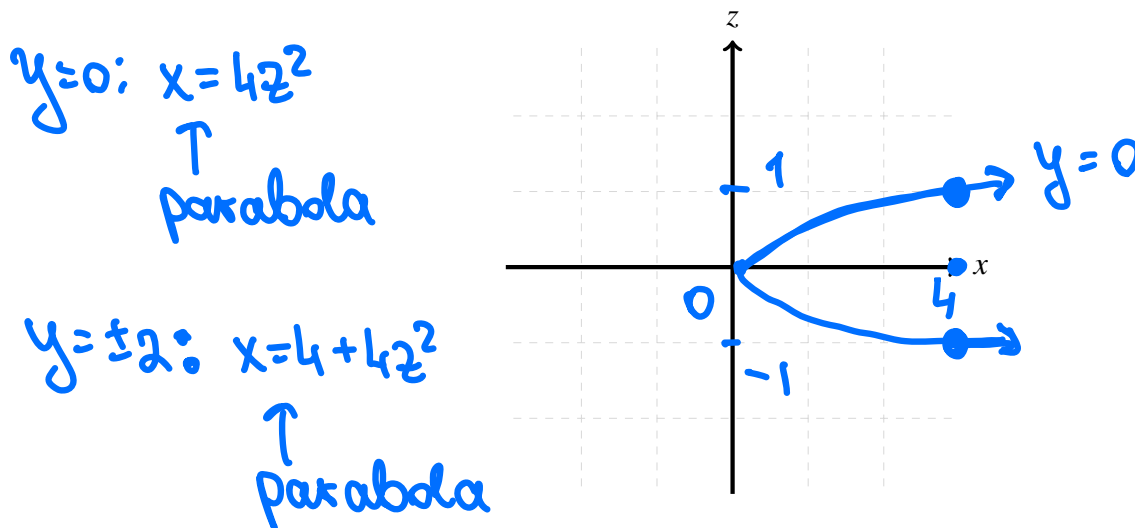
3. [15 points] Sketch the following surfaces.

(a) (10 pts) For  $x = y^2 + 4z^2$ , sketch the given traces, then the surface in 3-D.

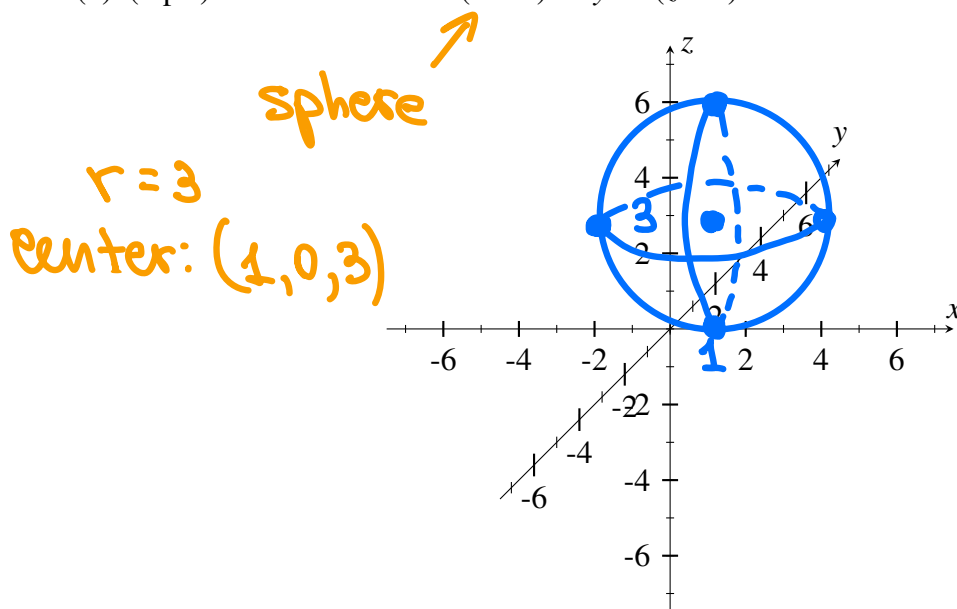
1) traces:  $x = 0, 4$



2) traces:  $y = 0, \pm 2$



(b) (5 pts) Sketch the surface  $(x-1)^2 + y^2 + (z-3)^2 = 9$ .



4. [21 points] Consider the following point, line, and plane:

$$B(3, -2, 1)$$

$$\vec{l}(t) = \langle t, 1-t, 2t+3 \rangle$$

$$\text{Plane } P: 2x + 5y - z = 10$$

(a) (5 pts) Give the equation of a plane parallel to the plane  $P$  that passes through  $B$ .

$n = \langle 2, 5, -1 \rangle$   
 $B(3, -2, 1)$   
Hence,

$$2(x-3) + 5(y+2) - 1(z-1) = 0$$

$$2x - 6 + 5y + 10 - z + 1 = 0$$

$$2x + 5y - z + 5 = 0$$

(b) (4 pts) Find the point of intersection of the line  $\vec{l}(t)$  and the plane  $P$ .

From  $\vec{l}(t)$ :  $\begin{cases} x = t \\ y = 1-t \\ z = 2t+3 \end{cases}$  . Plug in  $P$ :

$$2t + 5(1-t) - (2t+3) = 10$$

$$2t + 5 - 5t - 2t - 3 = 10$$

$$-5t = 8$$

$$t = -8/5$$

Hence,

$$\begin{cases} x = -8/5 \\ y = 13/5 \\ z = -1/5 \end{cases}$$

- (c) (5 pts) Find the angle the line  $\vec{l}(t)$  makes with the normal to the plane  $P$ . (Your answer may involve an inverse trigonometric function.)

$$\begin{aligned} \begin{cases} x=t \\ y=1-t \\ z=2t+3 \end{cases} &\Rightarrow \begin{cases} t=x \\ t=1-y \\ t=\frac{z-3}{2} \end{cases} & \begin{aligned} x &= 1-y = \frac{z-3}{2} \\ \frac{x-0}{1} &= \frac{y-1}{0-1} = \frac{z-3}{2} \end{aligned} \\ \vec{u} &= \langle 1, -1, 2 \rangle & \Rightarrow \cos \theta = \frac{\vec{u} \cdot \vec{w}}{|\vec{u}| |\vec{w}|} & \Rightarrow \cos \theta = \frac{-5}{\sqrt{6} \sqrt{30}} = \frac{-5}{\sqrt{180}} \\ \vec{w} &= \langle 2, 5, -1 \rangle & & \end{aligned}$$

$$\theta = \arccos\left(\frac{5}{\sqrt{180}}\right)$$

- (d) (7 pts) Find an equation for the plane containing the point  $B$  and the line  $\vec{l}(t)$ .

$$B(3, -2, 1)$$

$$\begin{aligned} \vec{u} \times \vec{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 2 & 5 & -1 \end{vmatrix} = (1-10)\mathbf{i} - (-1-4)\mathbf{j} + (5+2)\mathbf{k} = \\ &= -9\mathbf{i} + 5\mathbf{j} + 7\mathbf{k} = \langle -9, 5, 7 \rangle \end{aligned}$$

Hence, the plane equation is

$$\begin{aligned} -9(x-3) + 5(y+2) + 7(z-1) &= 0 \\ -9x + 27 + 5y + 10 + 7z - 7 &= 0 \\ -9x + 5y + 7z + 30 &= 0 \end{aligned}$$

5. [8 points] An object moves in 3-D with acceleration

$$a(t) = \langle \sin(t), 2 \cos(t), 6t \rangle.$$

At time  $t = 0$  it has velocity  $\langle 0, 0, -1 \rangle$ . Find a function  $v(t) = r'(t)$  giving its velocity at all times  $t > 0$ .

$$\begin{aligned} v(0) &= \langle 0, 0, -1 \rangle \\ v(t) &= \int a(t) dt = \mathbf{i} \int \sin t dt + \mathbf{j} \int 2 \cos t dt + \\ &+ \mathbf{k} \int 6t dt + C = -\cos t \cdot \mathbf{i} + 2 \sin t \cdot \mathbf{j} + 3t^2 \cdot \mathbf{k} + C \\ v(0) &= \langle -1 + C_1, 0 + C_2, 0 + C_3 \rangle = \langle 0, 0, -1 \rangle \\ C &= \langle 1, 0, -1 \rangle \end{aligned}$$

Hence

$$v(t) = \langle -\cos t + 1, 2 \sin t, 3t^2 - 1 \rangle$$

6. [20 points] A particle moves with velocity  $v(t) = \langle t^3, t^2, 2t \rangle$ .

(a) (7 pts) Find the distance the particle travels between times  $t = 0$  and  $t = 2$ .

+ 7 points to each student

$$d(2) - d(0) = \int_0^2 |v(t)| dt = \int_0^2 \sqrt{t^6 + t^4 + 4t^2} dt =$$

$$= \int_0^2 t \sqrt{t^4 + t^2 + 4} dt = \int_0^2 t \sqrt{(t^2 + \frac{1}{2})^2 + \frac{15}{4}} dt = \frac{1}{2} \int_0^2 \sqrt{u^2 + \frac{15}{4}} du$$

$t^2 + \frac{1}{2} = u$   
 $2t dt = du$

$$= \frac{1}{2} \left( \frac{u}{2} \sqrt{u^2 + \frac{15}{4}} + \frac{15}{8} \ln(u + \sqrt{u^2 + \frac{15}{4}}) \right) \Big|_0^2 =$$

$$= \frac{1}{2} \left( \frac{t^2 + \frac{1}{2}}{2} \sqrt{(t^2 + \frac{1}{2})^2 + \frac{15}{4}} + \frac{15}{8} \ln(t^2 + \frac{1}{2} \sqrt{(\dots)^2 + \frac{15}{4}}) \right) \Big|_0^2$$

(b) (8 pts) Calculate the curvature

$$\kappa = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

of the trajectory at time  $t = 1$ .

$$r'(t) = \langle t^3, t^2, 2t \rangle$$

$$r''(t) = \langle 3t^2, 2t, 2 \rangle$$

$$r'(1) = \langle 1, 1, 2 \rangle$$

$$r''(1) = \langle 3, 2, 2 \rangle$$

$$|r'(t)| = \sqrt{t^6 + t^4 + 4t^2}$$

$$|r' \times r''| = \begin{vmatrix} i & j & k \\ t^3 & t^2 & 2t \\ 3t^2 & 2t & 2 \end{vmatrix} = (2t^2 - 4t^2)i - (2t^3 - 6t^3)j + (2t^4 - 3t^4)k$$

$$|r'(1)| = \sqrt{6}$$

$$|r'(1) \times r''(1)| = \sqrt{4 + 16 + 1} = \sqrt{21}$$

$$\kappa = \frac{\sqrt{21}}{(\sqrt{6})^3}$$

(c) (5 pts) Find the unit tangent vector  $\mathbf{T}(t)$  and the tangential component of acceleration

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

at  $t = 1$ .

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} ; \quad \mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{\langle 1, 1, 2 \rangle}{\sqrt{6}} =$$

$$= \boxed{\left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle}$$

$$\mathbf{r}'(1) \cdot \mathbf{r}''(1) = 3 + 2 + 4 = 9$$

$$a_T = \boxed{\frac{9}{\sqrt{6}}}$$

7. [4 points] Evaluate the following integral

$$\int_1^4 (2t^{3/2} \mathbf{i} + (t+1) \sqrt{t} \mathbf{k}) dt.$$

$$= \mathbf{i} \int_1^4 2t^{3/2} dt + \mathbf{k} \int_1^4 (t\sqrt{t} + \sqrt{t}) dt =$$

$$= 2\mathbf{i} \left. \frac{t^{5/2}}{5/2} \right|_1^4 + \mathbf{k} \left( \left. \frac{t^{5/2}}{5/2} + \frac{t^{3/2}}{3/2} \right|_1^4 \right) =$$

$$= 2 \left( \frac{2}{5} \cdot 32 - \frac{2}{5} \right) \mathbf{i} + \left( \frac{2}{5} \cdot 32 + \frac{2}{3} \cdot 8 - \frac{2}{5} - \frac{2}{3} \right) \mathbf{k} =$$

$$= \frac{124}{5} \mathbf{i} + 0 \cdot \mathbf{j} + \left( \frac{62}{5} + \frac{14}{3} \right) \mathbf{k} = \boxed{\left\langle \frac{124}{5}, 0, \frac{256}{15} \right\rangle}.$$

$$\frac{186 + 70}{15} = \frac{256}{15}$$