WRH-9-Solutions

15.6:   4, 13, 27
15.7:   3, 17
15.8:   4, 9(a), 23
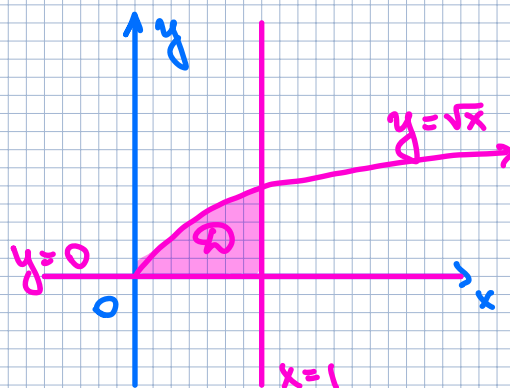
15.6

④

$$\int_0^1 \int_y^{2y} \int_0^{x+y} 6xy \, dz \, dx \, dy = \int_0^1 \int_y^{2y} 6xyz \Big|_0^{x+y} dx \, dy =$$

$$= \int_0^1 \int_y^{2y} 6xy(x+y) \, dx \, dy = \int_0^1 \left(\frac{6x^3 y}{3} + \frac{6x^2 y^2}{2} \right) \Big|_y^{2y} dy$$

$$= \int_0^1 \left(2x^3 y + 3x^2 y^2 \right) \Big|_y^{2y} dy = \int_0^1 \left(16y^4 + 12y^4 - 2y^4 - \right.$$

$$\left. - 3y^4 \right) dy = \int_0^1 23y^4 \, dy = \frac{23}{5} y^5 \Big|_0^1 = \boxed{\frac{23}{5}}$$

▼

⑬

$$\iiint_E 6xy \, dV$$

E: z = 1+x+y
    y = √x
    z = 0
    x = 1

$$\iiint_E 6xy \, dz \, dy \, dx = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy \, dz \, dy \, dx =$$

$$= \int_0^1 \int_0^{\sqrt{x}} 6xyz \Big|_0^{1+x+y} dy \, dx =$$

$$= \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y) \, dy \, dx = \int_0^1 \int_0^{\sqrt{x}} (6xy + 6x^2y + 6xy^2) \, dy \, dx =$$

$$= 6 \int_0^1 \left(\frac{xy^2}{2} + \frac{x^2y^2}{2} + \frac{xy^3}{3} \right) \Big|_0^{\sqrt{x}} dx =$$

$$= 6 \int_0^1 \left(\frac{x^2}{2} + \frac{x^3}{2} + \frac{x^{5/2}}{3} \right) dx =$$

$$= 6 \left(\frac{x^3}{6} + \frac{x^4}{8} + \frac{x^{7/2}}{3 \cdot 7/2} \right) \Big|_0^1 =$$

$$= 6 \left(\frac{1}{6} + \frac{1}{8} + \frac{2}{14} \right) = \boxed{\frac{65}{28}}$$

▼

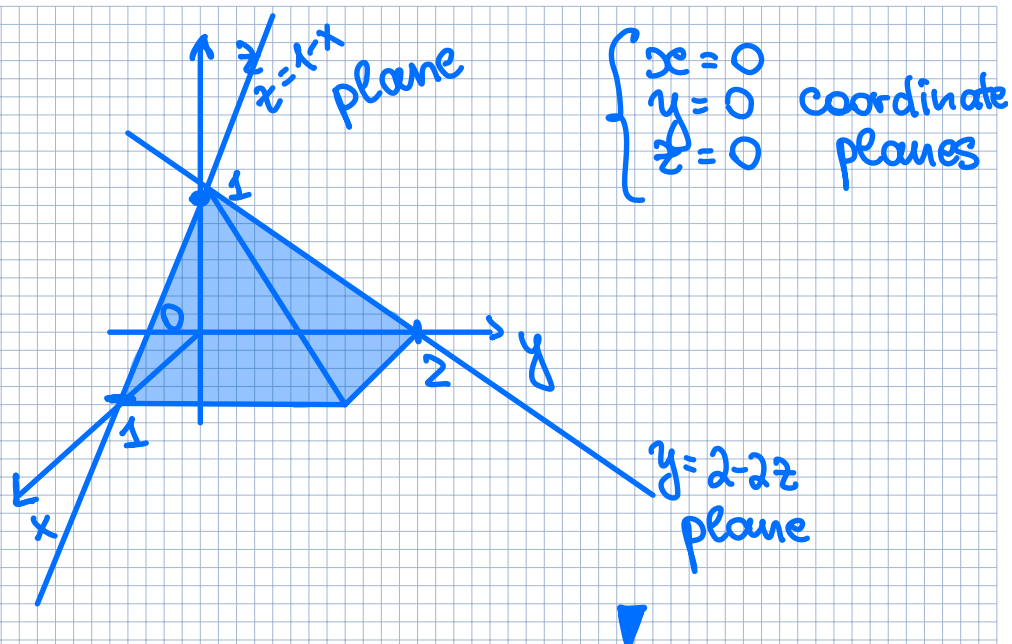㉗ $\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy \, dz \, dx$

$0 \leq x \leq 1$
$0 \leq y \leq 2 - 2z$
$0 \leq z \leq 1-x$
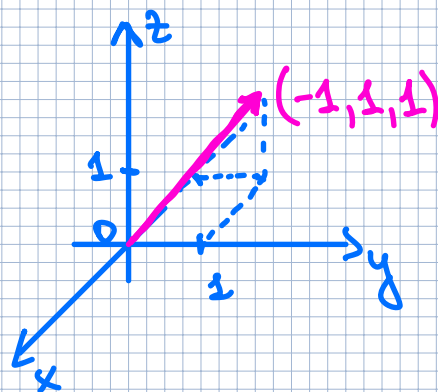
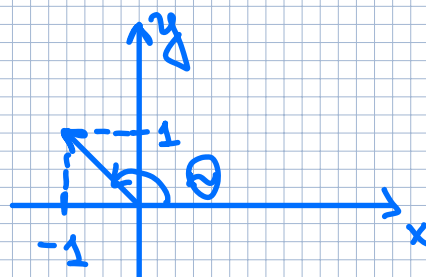$\begin{cases} x=0 \\ y=0 \\ z=0 \end{cases}$ coordinate planes

$z = 1-x$ plane

1

O

2 $\quad y$

1

x

$y = 2-2z$ plane

▼

③ (a) $(-1,1,1)$

$\begin{cases} x(t) = -1 \\ y(t) = 1 \\ z(t) = 1 \end{cases}$

z

$(-1,1,1)$

1

O

1 $\quad y$

x

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

y

1

θ

-1

x

$\begin{cases} 0 \leq r \leq \sqrt{2} \\ 0 \leq \theta \leq \dfrac{5\pi}{4} \\ 0 \leq z \leq 1 \end{cases}$

$$\boxed{(r, \theta, z) = \left(\sqrt{2}, \frac{5\pi}{4}, 1\right)}$$

(B) $(-2, 2\sqrt{3}, 3)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$
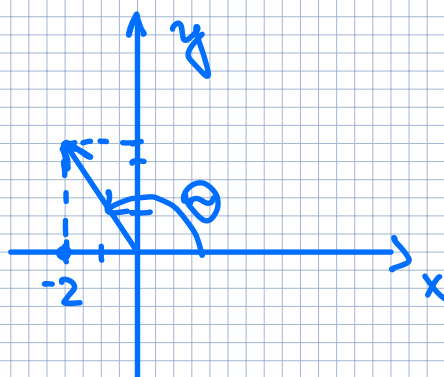
$$r = \sqrt{4 + 4 \cdot 3} = 4$$

$$\tan\theta = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$$

$$\theta = \frac{2\pi}{3} + 2\pi n$$
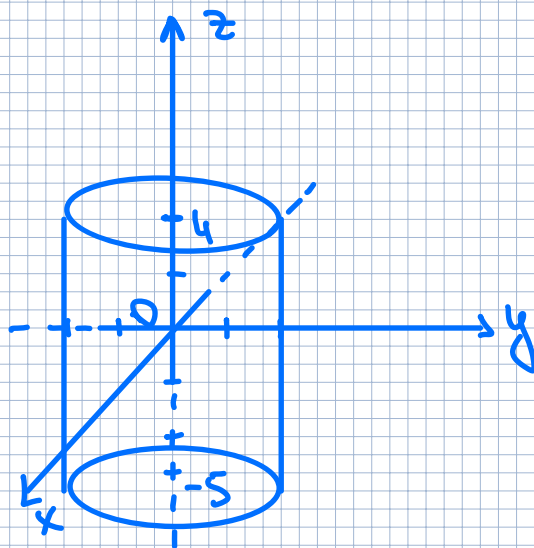
$$z = 3$$

Thus, one set of cylindrical coordinates is

$$\boxed{\left(4, \frac{2\pi}{3}, 3\right)}$$

▼

(17)

$$\iiint_E \sqrt{x^2+y^2}\, dV =$$

E: $x^2 + y^2 = 16$
     $z = -5$
     $z = 4$

$$\stackrel{=}{\boxed{=}} \begin{bmatrix} x = r\cos\theta & x^2+y^2=r^2=16 \\ y = r\sin\theta & r=4 \\ z = z & \theta \in [0, 2\pi] \\ & -5 \leq z \leq 4 \end{bmatrix} \stackrel{=}{\boxed{=}}$$

$$\stackrel{=}{\boxed{=}} \int_0^4 \int_0^{2\pi} \int_{-5}^4 \sqrt{r^2}\, r\, dz\, d\theta\, dr =$$

$$= \int_0^4 \int_0^{2\pi} r^2 z \Big|_{-5}^4 d\theta\, dr = \int_0^4 \int_0^{2\pi} 9r^2\, d\theta\, dr =$$

$$= 9 \cdot 2\pi \cdot \frac{r^3}{3} \Big|_0^4 = \frac{\overset{3}{\cancel{9}} \cdot 2\pi \cdot 64}{\cancel{3}} = 6\pi \cdot 64 =$$

$$= \boxed{384\pi}$$

▼

④ (a) $(1, 0, \sqrt{3})$

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 0 + 3} = 2$$

$$\boxed{r = 2}$$

$(1, 0, \sqrt{3})$

$$\cos \varphi = \frac{z}{r} = \frac{\sqrt{3}}{2} \implies \varphi = \frac{\pi}{6}$$

$$\cos \theta = \frac{x}{r \sin \varphi} = \frac{1}{2 \cdot \frac{1}{2}} = 1$$

$$\theta = 0$$

Thus spherical coordinates are
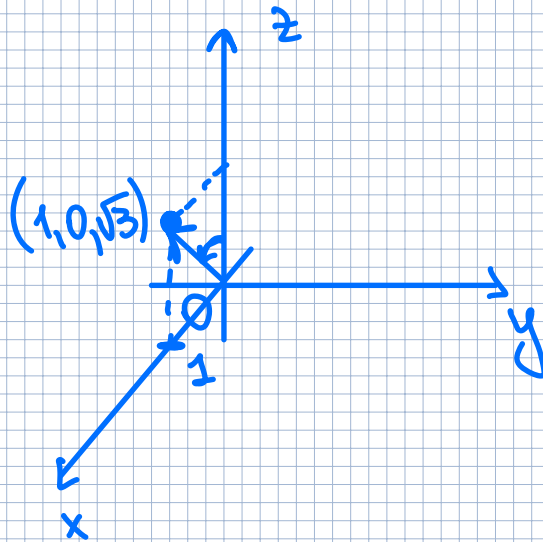
$$\boxed{\left(2, 0, \frac{\pi}{6}\right)}$$

(b) $(\sqrt{3}, -1, 2\sqrt{3})$

$$\begin{cases} x = r \sin \varphi \cos \theta \\ y = r \sin \varphi \sin \theta \\ z = r \cos \varphi \end{cases}$$

$$r = \sqrt{3+1+4\cdot3} = 4$$

$$\cos \varphi = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

$$\varphi = \frac{\pi}{6}$$

$$\cos \theta = \frac{\sqrt{3}}{4\cdot\sin\frac{\pi}{6}} = \frac{\sqrt{3}}{4\cdot\frac{1}{2}} = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{11\pi}{6} \quad (\text{since } y<0)$$

Thus   spherical   coordinates are $\boxed{\left(4, \frac{11\pi}{6}, \frac{\pi}{6}\right).}$

▼

9(a)
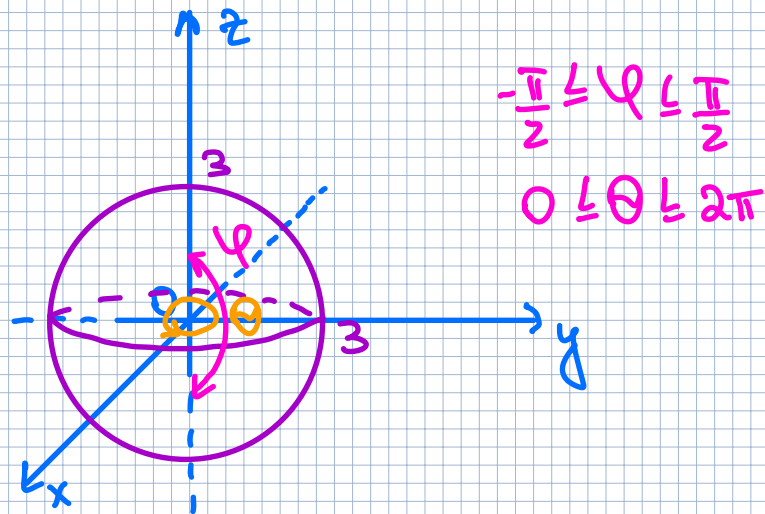
$$x^2 + y^2 + z^2 = 9$$

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$$

$$(r\sin\varphi\cos\theta)^2 + (r\sin\varphi\sin\theta)^2 + (r\cos\varphi)^2 = r^2 = 9$$

$$\boxed{r = 3}$$

$x^2 + y^2 + z^2 = 9$ is a sphere of a radius
$r = 3$ and   center $(0,0,0)$.

$$-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$$

$$0 \leq \theta \leq 2\pi$$

The equation is $\boxed{r=3}$

▼

(23) $\displaystyle\iiint\limits_{E} (x^2 + y^2)\, dV$

E: $x^2 + y^2 + z^2 = 4$

$x^2 + y^2 + z^2 = 9$

$E = \{ (r, \theta, \varphi) \mid 2 \leq r \leq 3, 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi \}$

$x^2 + y^2 = (r\sin\varphi\cos\theta)^2 + (r\sin\varphi\sin\theta)^2 =$

$= r^2 \sin^2\varphi.$

Thus $\displaystyle\iiint\limits_{E} (x^2 + y^2)\, dV = \int_{0}^{\pi}\int_{0}^{2\pi}\int_{2}^{3} r^2 \sin^2\varphi \, r^2 \sin\varphi \, dr\, d\theta\, d\varphi =$

$$= \int_0^\pi \sin^3 \varphi \, d\varphi \int_0^{2\pi} d\theta \int_2^3 r^4 \, dr = \int_0^\pi (1 - \cos^2 \varphi) \sin \varphi \, d\varphi \cdot$$

$$\cdot \left. \theta \right|_0^{2\pi} \left. \frac{1}{5} r^5 \right|_2^3 = \boxed{\frac{1688\pi}{15}}$$