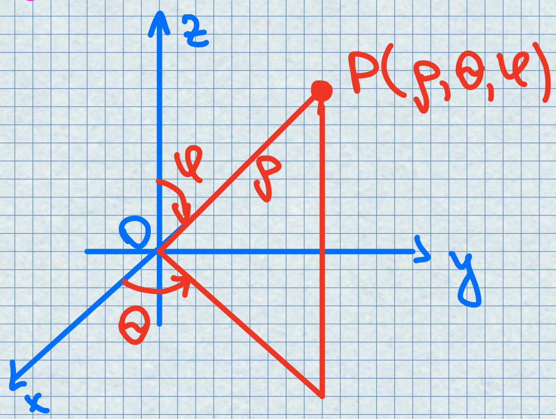
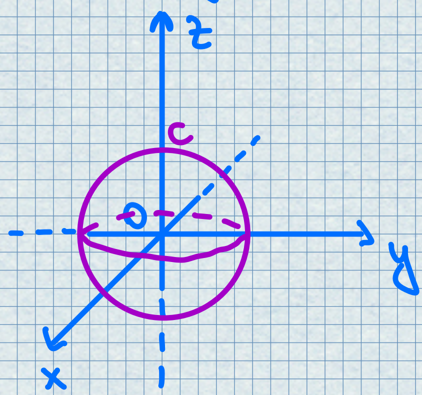
Lecture #27 - Week 9 - Triple integrals in spherical coordinates - 15.8

- Spherical coordinates

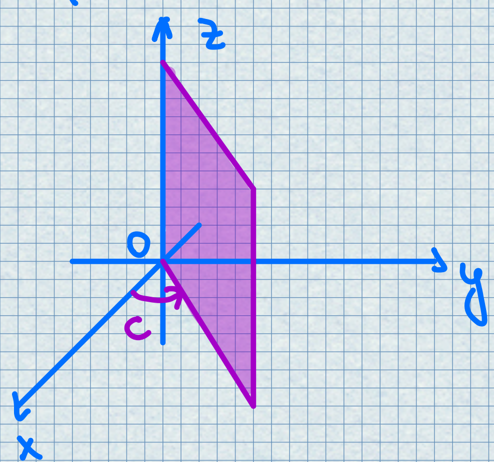


The <u>spherical coordinates</u> (ρ, θ, φ) of point P in space, where $\rho = |OP|$, θ is the same angle as in cylindrical coordinates, and φ is the angle between the positive z-axis and the line segment OP. Note that

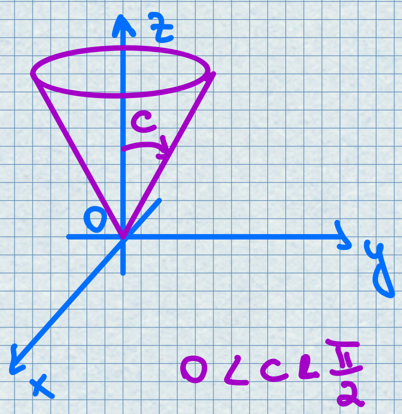
$$\rho \geq 0 \quad \text{and} \quad 0 \leq \varphi \leq \pi$$

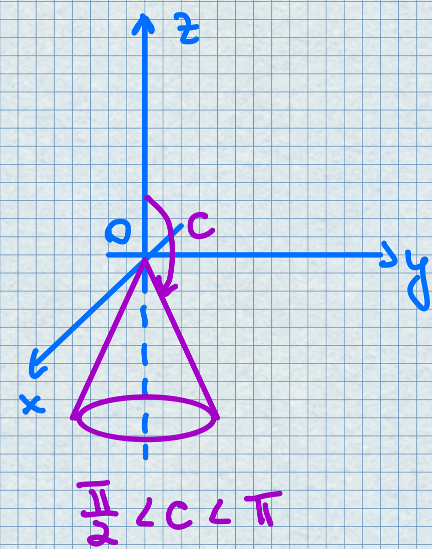


$\rho = c$, a sphere

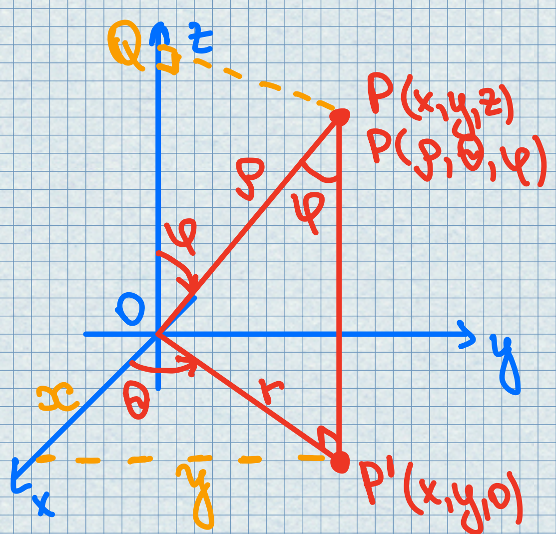


$\theta = c$, a half-plane





$\varphi = c$, a half-cone



The relationship between rectangular and spherical coordinates can be seen from picture above. From ΔOPQ and $\Delta OPP'$ we have

$$z = \rho\cos\varphi \quad \text{and} \quad r = \rho\sin\varphi$$

But

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

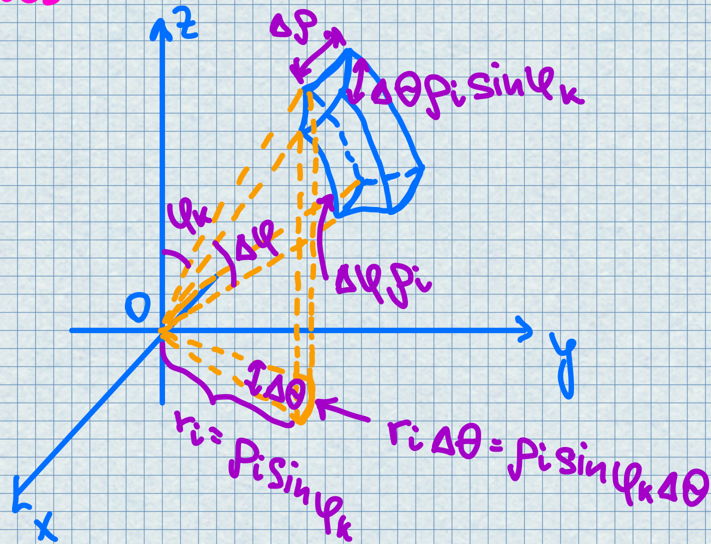
So

$$\boxed{x = \rho \sin\varphi \cos\theta \qquad y = \rho \sin\varphi \sin\theta \qquad z = \rho \cos\varphi}$$

Also,

$$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

- Evaluating triple integrals with spherical coordinates



In the spherical coordinate system the counterpart of a rectangular box is a <u>spherical wedge</u>

$E = \{(\rho, \theta, \varphi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \varphi \le d\}$,
where $\alpha \ge 0$ and $\beta - \alpha \le \pi$, and $d - c \le \pi$.

We divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\varphi = \varphi_k$. (see figure above).

So

$$\Delta V_{ijk} \approx (\Delta \rho)(\rho_i \Delta\varphi)(\rho_i \sin\varphi_k \Delta\theta) =$$

$$= \rho_i^2 \sin\varphi_k \Delta\rho \Delta\theta \Delta\varphi$$

It can be shown, with the aid of MVT, that

$$\Delta V_{ijk}(E_{ijk}) = \tilde{\rho}_i^2 \sin\tilde{\varphi}_k \Delta\rho \Delta\theta \Delta\varphi$$

where $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\varphi}_k)$ is some point in E_{ijk}.

Let $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ be the rectangular coordinates of this point. Then

$$\iiint_E f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V_{ijk}$$

$$= \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\tilde{\rho}_i \sin\tilde{\varphi}_k \cos\tilde{\theta}_j, \tilde{\rho}_i \sin\tilde{\varphi}_k \sin\tilde{\theta}_j, \tilde{\rho}_i \cos\tilde{\varphi}_k) \cdot$$

$$\cdot \tilde{\rho}_i^2 \sin\tilde{\varphi}_k \Delta\rho \Delta\theta \Delta\varphi$$

But this sum is a Riemann sum for

$$F(\rho,\theta,\varphi) = f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2 \sin\varphi$$

Formula for triple integration in spherical coordinates

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\cdot$$

$$\cdot\, \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi,$$

where E is a spherical wedge given by

$$E = \{(\rho,\theta,\varphi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \varphi \le d\}$$

This formula says that we convert a triple integral from rectangular coordinates to spherical coordinates by writing

$$x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta \qquad z = \rho\cos\varphi$$

$$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

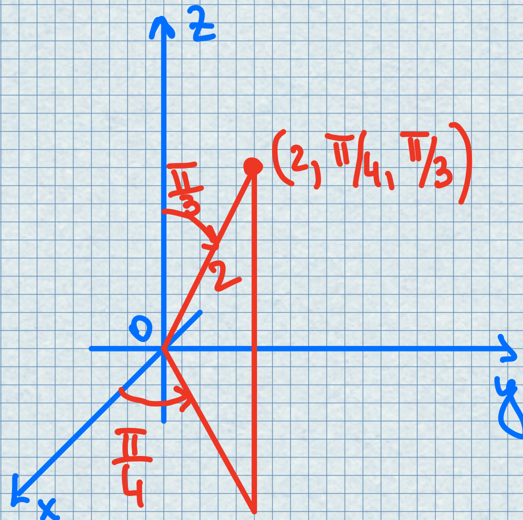
This formula can be extended to include more general spherical regions such as

$$E = \{(\rho,\theta,\varphi) \mid \alpha \le \theta \le \beta, c \le \varphi \le d, g_1(\theta,\varphi) \le \rho \le g_2(\theta,\varphi)\}$$

Examples

$\boxed{1.}$ The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

Solution



$$x = \rho \sin\varphi \cos\theta = 2 \sin\frac{\pi}{3} \cos\frac{\pi}{4} =$$
$$= 2\,\frac{\sqrt{3}}{2}\,\frac{1}{\sqrt{2}} = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin\varphi \sin\theta = 2 \sin\frac{\pi}{3} \sin\frac{\pi}{4} =$$
$$= 2\,\frac{\sqrt{3}}{2}\,\frac{1}{\sqrt{2}} = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos\varphi = 2 \cos\frac{\pi}{3} = 2\,\frac{1}{2} = 1$$

Thus the point $(2, \pi/4, \pi/3)$ is $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\right)$ in rectangular coordinates. ◀

$\boxed{2.}$ The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.

Solution

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{0+12+4} = 4$$

and

$$\cos\varphi = \frac{z}{\rho} = -\frac{2}{4} = -\frac{1}{2} \qquad \varphi = \frac{2\pi}{3}$$

$$\cos\theta = \frac{x}{\rho\sin\varphi} = 0 \qquad \theta = \frac{\pi}{2}$$

(Note that $\theta \neq \frac{3\pi}{2}$ because $y = 2\sqrt{3} > 0$).

Therefore, spherical coordinates of the given point are $(4, \pi/2, 2\pi/3)$. ▼

$\boxed{3.}$ Evaluate $\iiint\limits_B e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where B is:

$$B = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$$

Solution

$$B = \{(\rho,\theta,\varphi) \mid 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$$

$$x^2+y^2+z^2 = \rho^2$$

Thus

$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{(\rho^2)^{3/2}}\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi =$$

$$= \int_0^{\pi}\sin\varphi\,d\varphi\int_0^{2\pi}d\theta\int_0^{1}\rho^2 e^{\rho^3}\,d\rho = \left(-\cos\varphi\right)\Big|_0^{\pi}\,2\pi\left(\frac{1}{3}e^{\rho^3}\right)\Big|_0^{1} =$$

$$= \frac{4}{3}\pi(e-1).$$

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