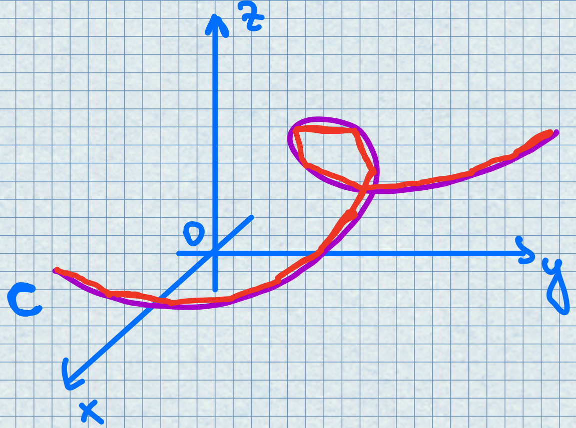
Lecture #9 - Week 3 - Arc Length and Curvature - 13.3

• Length of a Curve



$$C: \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad a \le t \le b$$

$$\boxed{L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Now, Suppose that

$$C: \quad r(t) = \langle f(t), g(t), h(t) \rangle, \quad a \le t \le b,$$

where $x = f(t)$, $y = g(t)$, $z = h(t)$ and f', g', h' are continuous.

Then

$$\boxed{L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt}$$

or

$$\boxed{L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt}$$

or

$$\boxed{L = \int_a^b |r'(t)|\, dt}$$

- The Arc Length Function

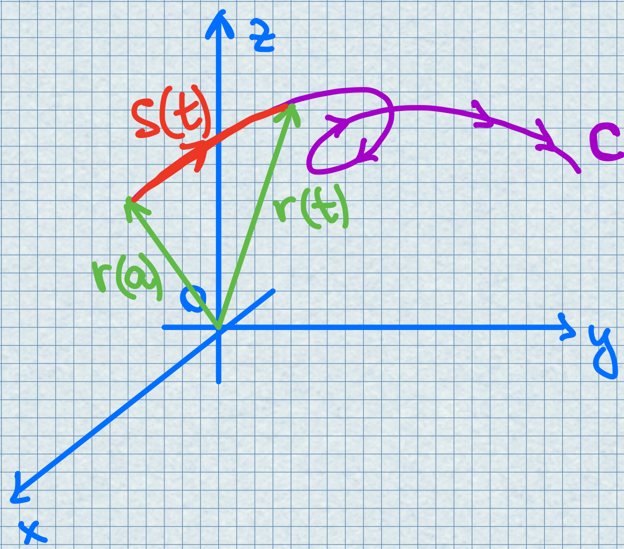
Let the curve C is given by a vector function

$$r(t) = f(t)i + g(t)j + h(t)k, \quad a \le t \le b$$

where r' is continuous and C is traversed exactly once as t increases from a to b.

Def. We define the arc length function by

$$\boxed{S(t) = \int_a^t |r'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du}$$



$$\boxed{\frac{ds}{dt} = |r'(t)|}$$

- Curvature

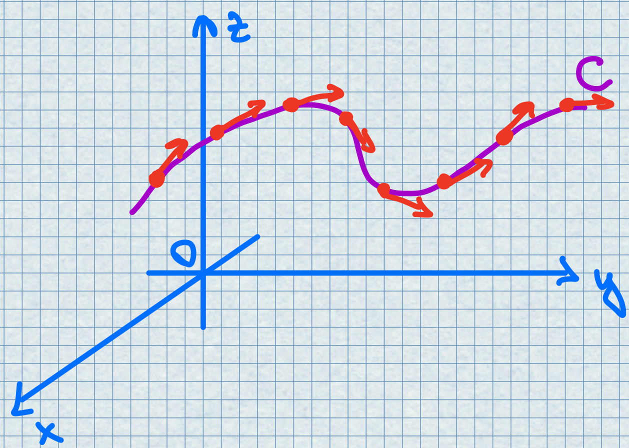
Def. A parametrization $r(t)$ is smooth on an interval I if $r'(t)$ is continuous and $r'(t) \neq 0$ on I.

A curve is called smooth if it has a smooth parametrization.

If C is a smooth curve defined by the vector function r, then

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

and $T(t)$ indicates the direction of the curve.



Def. The curvature of C at a given point is a measure of how quickly the curve changes direction at that point.

$$\boxed{\kappa = \left| \frac{dT}{ds} \right|}$$

where T is the unit tangent vector.

We have that

$$\frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} \quad \text{and} \quad \kappa = \left|\frac{dT}{ds}\right| = \left|\frac{dT/dt}{ds/dt}\right|$$

But $\frac{ds}{dt} = |r'(t)|$. So

$$\boxed{\kappa(t) = \frac{|T'(t)|}{|r'(t)|}}$$

Theorem The curvature of the curve given by the vector function r is

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and

$$r(x) = xi + f(x)j$$

Then

$$r'(x) = i + f'(x)j$$

$$r''(x) = f''(x)j$$

Since $i \times j = k$ and $j \times j = 0$, then

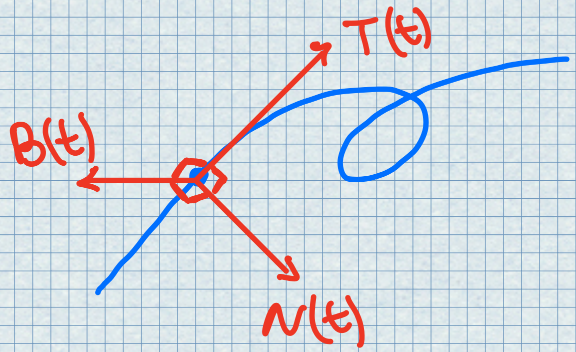
$$r'(x) \times r''(x) = f''(x)k,$$

$$|r'(x)| = \sqrt{1 + (f'(x))^2}.$$

So

$$\boxed{\kappa(x) = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}}}$$

- The Normal and Binormal Vectors



At a given point on a smooth space curve $r(t)$, there are many vectors that are orthogonal to the unit tangent vector $T(t)$.

$$|T(t)| = 1 \quad , \quad t \in \mathbb{R}$$

$$T(t) \cdot T'(t) = 0 \Rightarrow T'(t) \perp T(t)$$

Typically, $|T'(t)| \neq 1$.

But at any point where $\kappa \neq 0$, we define a unit normal vector $N(t)$

$$N(t) = \frac{T'(t)}{|T'(t)|}.$$

The vector

$$B(t) = T(t) \times N(t)$$

is called the Binormal vector.

$$B(t) \perp T(t) \text{ and } B(t) \perp N(t)$$

and $|B(t)| = 1$.

Def.

- The plane determined by $N(t)$ and $B(t)$ at a point P on a curve C is called the <u>normal plane</u> of C at P.
- The plane determined by $T(t)$ and $N(t)$ is called the <u>osculating plane</u> of C at P.
- The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C, and has radius $\rho = 1/\kappa$ is called the <u>osculating circle.</u> of C at P.

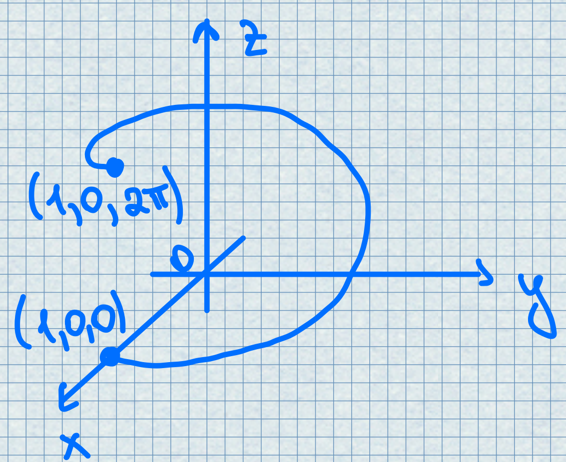
Examples

$\boxed{1.}$ Find the length of the arc of the circular helix with vector equation $r(t) = \cos t\, i + \sin t\, j + t\, k$ from the point $(1,0,0)$ to the point $(1,0,2\pi)$.

Solution

$$r'(t) = -\sin t\, i + \cos t\, j + k$$

$$|r'(t)| = \sqrt{1 + 1} = \sqrt{2}$$



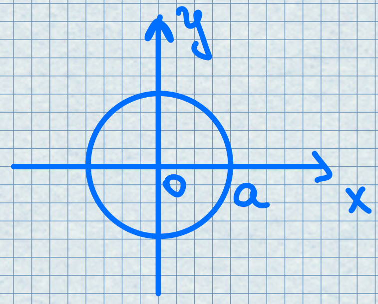
The arc from $(1,0,0)$ to $(1,0,2\pi)$ is described by $0 \le t \le 2\pi$ and

$$L = \int_0^{2\pi} |r'(t)|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi.$$

$\boxed{2.}$ Show that the curvature of a circle of radius a is $1/a$.

Solution

$$r(t) = a\cos t\, i + a\sin t\, j$$



$$r'(t) = -a\sin t\, i + a\cos t\, j$$

$$|r'(t)| = a$$

So

$$T(t) = \frac{r'(t)}{|r'(t)|} = -\sin t\, i + \cos t\, j$$

$$T'(t) = -\cos t\, i - \sin t\, j$$

$$|T'(t)| = 1$$

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{1}{a}.$$

$\boxed{3.}$ Find the curvature of the twisted cubic $r(t) = \langle t, t^2, t^3 \rangle$ at a general point and at $(0,0,0)$.

Solution

$$r'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$r''(t) = \langle 0, 2, 6t \rangle$$

$$|r'(t)| = \sqrt{1+4t^2+9t^4}$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2 i - 6t j + 2k$$

$$|r'(t) \times r''(t)| = 2\sqrt{9t^4 + 9t^2 + 1}.$$

Thus

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{2\sqrt{1+9t^2+9t^4}}{(1+4t^2+9t^4)^{3/2}}.$$

When $t=0$: $\kappa(0) = 2$. ▼

$\boxed{4.}$ Find the unit normal and binormal vectors for the circular helix

$$r(t) = \cos t\, i + \sin t\, j + k$$

Solution

$$r'(t) = -\sin t\, i + \cos t\, j + k$$

$$|r'(t)| = \sqrt{2}$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{2}}(-\sin t\, i + \cos t\, j + k)$$

$$T'(t) = \frac{1}{\sqrt{2}}(-\cos t\, i - \sin t\, j)$$

$$|T'(t)| = \frac{1}{\sqrt{2}}$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = -\cos t\, i - \sin t\, j =$$

$$= \langle -\cos t, -\sin t, 0 \rangle.$$

$$B(t) = T(t) \times N(t) = \frac{1}{\sqrt{2}} \begin{vmatrix} i & j & k \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} =$$

$$= \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle.$$

5. Find equations of the normal plane and osculating plane of the helix in example 4 at $P(0, 1, \pi/2)$.

Solution

P corresponds to $t = \pi/2$ and the normal plane there has normal vector $r'(\pi/2) = \langle -1, 0, 1 \rangle$, so

$$-1(x-0) + 0(y-1) + 1\left(z - \frac{\pi}{2}\right) = 0$$

or

$$\boxed{z = x + \frac{\pi}{2}}$$ Normal plane

The osculating plane at P contains T and

N, so its normal vector is $T \times N = B$.

$$B(t) = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle$$

$$B\left(\frac{\pi}{2}\right) = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

A simpler normal vector is $\langle 1, 0, 1 \rangle$, so an equation of the osculating plane is

$$1(x-0) + 0(y-1) + 1\left(z - \frac{\pi}{2}\right) = 0$$

or

$$\boxed{z = -x + \frac{\pi}{2}}$$ Osculating plane.

▼