Name:

Instructions. (100 points) You have two hours. The exam is closed book, closed notes, and only simple calculators are allowed. Show all your work in order to receive full credit.

1. **[17 points]** Consider points $A(2,3,1)$ and $B(3,4,c)$ and vectors $u = \langle 1,2,3\rangle$ and $v = \langle 1,1,2\rangle$.

   (a) (4pts) Find the vector projection of u along v.

   $$\text{proj}_v u = \frac{u\cdot v}{|v|^2} v = \frac{9}{\sqrt{6}}\langle 1,1,2\rangle = \boxed{\left\langle \frac{9}{\sqrt{6}}, \frac{9}{\sqrt{6}}, \frac{18}{\sqrt{6}} \right\rangle}$$

   $$u\cdot v = 1+2+6 = 9$$

   $$|v| = \sqrt{1+1+4} = \sqrt{6}$$

   (b) (3 pts) Find all values of c such that the length of $\overrightarrow{AB}$ equals 5.

   $$\overrightarrow{AB} = \langle 1, 1, c-1\rangle$$

   $$|\overrightarrow{AB}| = \sqrt{1+1+(c-1)^2} = 5$$

   $$2+(c-1)^2 = 25$$

   $$(c-1)^2 = 23$$

   $$c-1 = \pm\sqrt{23}$$

   $$\boxed{c = 1\pm\sqrt{23}}$$

   (c) (3 pts) Find all values of c such that $\overrightarrow{AB}$ is parallel to v.

   $$\overrightarrow{AB} \parallel v \iff \begin{cases} 1 = k \\ 1 = k \\ c-1 = 2k \end{cases} \qquad k = 1$$

   $$c-1 = 2$$

   $$\boxed{c = 3}$$

   (d) (3 pts) Find all values of c such that $\overrightarrow{AB}$ is orthogonal to u.

   $$\overrightarrow{AB} \perp u \iff \overrightarrow{AB}\cdot u = 0 \iff 1\cdot 1 + 1\cdot 2 + (c-1)3 = 0$$

   $$\cancel{1} + \cancel{2} + 3c - \cancel{3} = 0 \Rightarrow \boxed{c = 0}$$

   (e) (4 pts) Find the cross product of vectors u and v.

   $$u\times v = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} = (4-3)i - (2-3)j + (2-3)k =$$

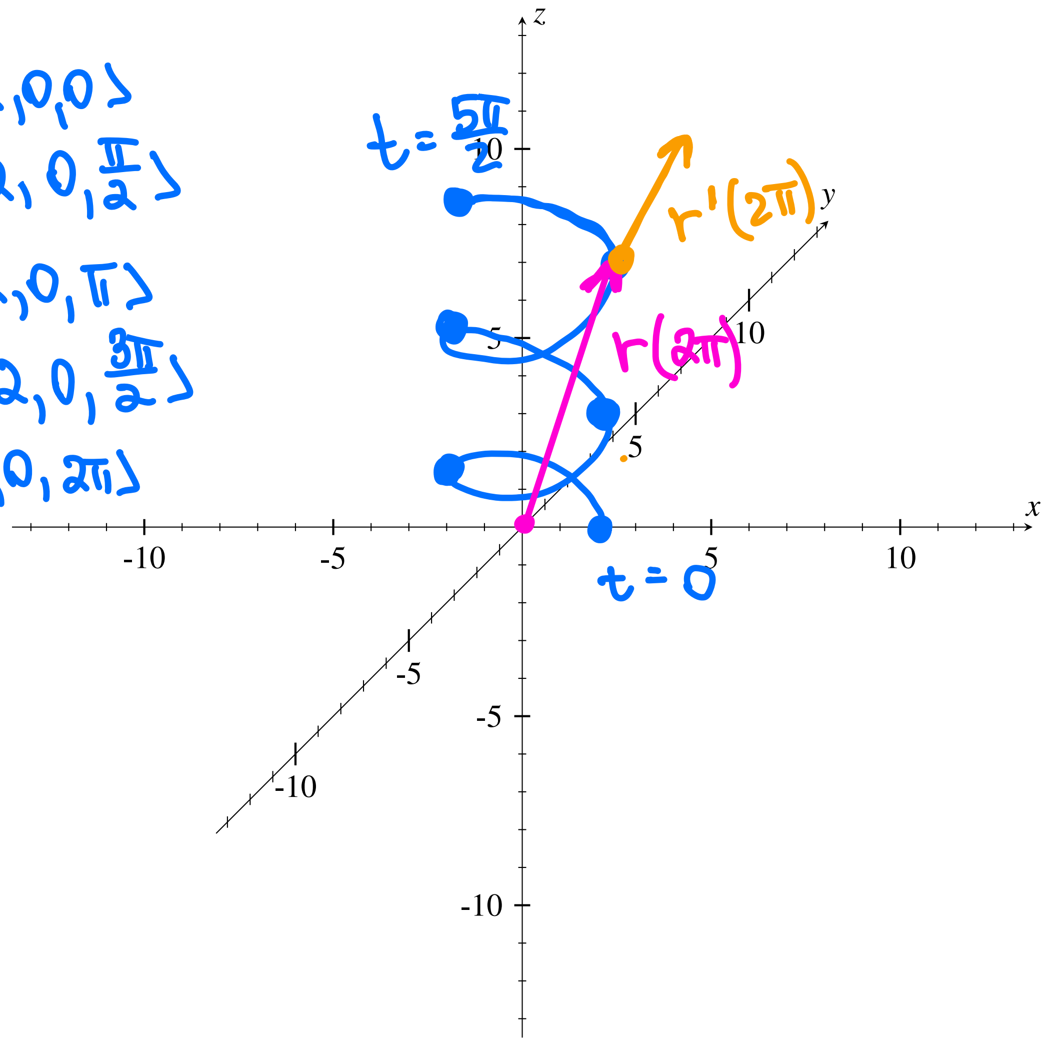
   $$= i + j - k = \langle 1, 1, -1\rangle$$

2. **[15 points]** You are given the following space curve:

$$r(t) = \langle 2\cos(2t),\ 2\sin(2t),\ t\rangle, \quad 0 \le t \le \frac{5\pi}{2}$$

(a) (8 pts) Draw the trajectory of the vector function $r(t)$ for the given value t.

$r(0) = \langle 2, 0, 0\rangle$
$r(\frac{\pi}{2}) = \langle -2, 0, \frac{\pi}{2}\rangle$
$r(\pi) = \langle 2, 0, \pi\rangle$
$r(\frac{3\pi}{2}) = \langle -2, 0, \frac{3\pi}{2}\rangle$
$r(2\pi) = \langle 2, 0, 2\pi\rangle$



(b) (4 pts) Draw on the above trajectory the position and velocity vectors for $t = 2\pi$.

$r(2\pi) = \langle 2, 0, 2\pi\rangle$
$r'(t) = \langle -4\sin 2t, 4\cos 2t, 1\rangle$
$r'(2\pi) = \langle 0, 4, 1\rangle$

(c) (3 pts) Find the speed at time t and simplify your result.

$v = |r'(t)| = \sqrt{16\sin^2 2t + 16\cos^2 2t + 1} = \boxed{\sqrt{17}}$

3. **[15 points]** Sketch the following surfaces.

(a) (10 pts) For $x = y^2 + 4z^2$, sketch the given traces, then the surface in 3-D.

1) traces: $x = 0, 4$

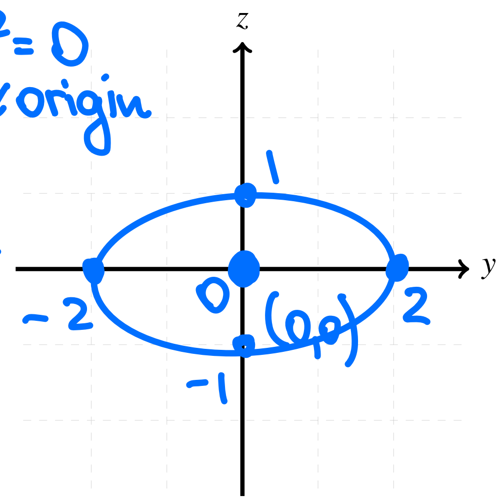
$x=0: y^2+4z^2=0$

$y=z=0$ ← origin

$x=4: 4=y^2+4z^2$

$1=\frac{y^2}{4}+\frac{z^2}{1}$

↑ ellipse



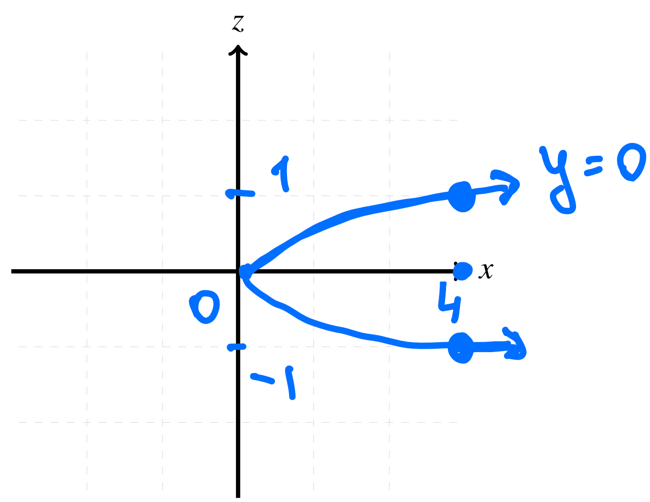
2) traces: $y = 0, \pm 2$

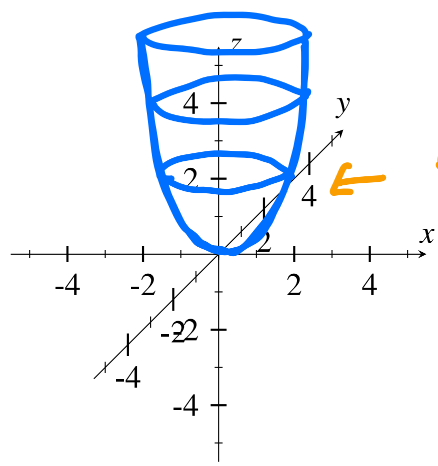
$y=0: x=4z^2$

↑ parabola

$y=\pm 2: x=4+4z^2$

↑ parabola





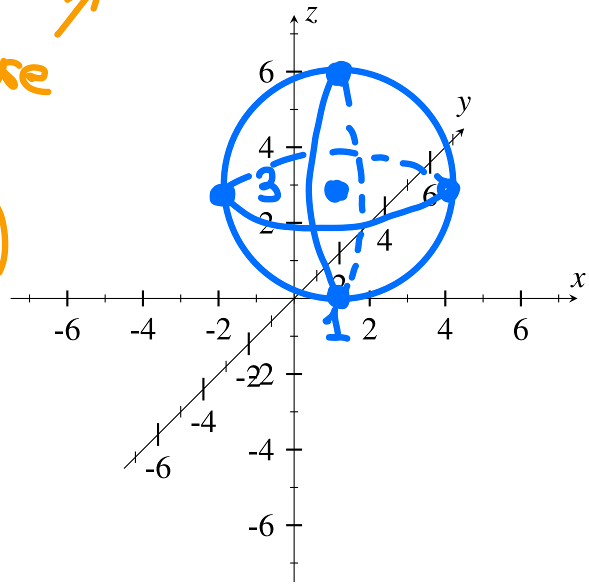
← elliptic paraboloid

(b) (5 pts) Sketch the surface $(x-1)^2 + y^2 + (z-3)^2 = 9$.

sphere

$r = 3$

center: $(1,0,3)$



4. **[21 points]** Consider the following point, line, and plane:

$$B(3, -2, 1)$$

$$\vec{l}(t) = \langle t,\ 1-t,\ 2t+3\rangle$$

$$Plane\ \ P: 2x + 5y - z = 10$$

(a) (5 pts) Give the equation of a plane parallel to the plane P that passes through B.

$n = \langle 2, 5, -1\rangle$

$B(3,-2,1)$

Hence,

$2(x-3) + 5(y+2) - 1(z-1) = 0$

$2x - 6 + 5y + 10 - z + 1 = 0$

$\boxed{2x + 5y - z + 5 = 0}$

(b) (4 pts) Find the point of intersection of the line $\vec{l}(t)$ and the plane P.

From $\vec{l}(t)$: $\begin{cases} x = t \\ y = 1-t \\ z = 2t+3 \end{cases}$. Plug in P:

$2t + 5(1-t) - (2t+3) = 10$

$\cancel{2t} + 5 - 5t - \cancel{2t} - 3 = 10$

$-5t = 8$

$\boxed{t = -8/5}$

Hence,

$\begin{cases} x = -8/5 \\ y = 13/5 \\ z = -1/5 \end{cases}$

(c) (5 pts) Find the angle the line $\vec{l}(t)$ makes with the normal to the plane P. (Your answer may involve an inverse trigonometric function.)

$$\begin{cases} x = t \\ y = 1-t \\ z = 2t+3 \end{cases} \Rightarrow \begin{matrix} t = x \\ t = 1-y \\ t = \frac{z-3}{2} \end{matrix} \qquad x = 1-y = \frac{z-3}{2}$$

$$\frac{x-0}{1} = \frac{y-1}{-1} = \frac{z-3}{2}$$

$$\vec{u} = \langle 1, -1, 2 \rangle, \quad \vec{n} = \langle 2, 5, -1 \rangle \Rightarrow \cos\theta = \frac{u \cdot n}{|u| \cdot |n|} \Rightarrow \cos\theta = \frac{-5}{\sqrt{6}\sqrt{30}} = \frac{-5}{\sqrt{180}}$$

$$\boxed{\theta = \arccos(5/\sqrt{180})}$$

(d) (7 pts) Find an equation for the plane containing the point B and the line $\vec{l}(t)$.

$B(3,-2,1)$

$$\vec{u} \times \vec{n} = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & 5 & -1 \end{vmatrix} = (1-10)i - (-1-4)j + (5+2)k =$$

$$= -9i + 5j + 7k = \langle -9, 5, 7 \rangle$$

Hence, the plane equation is

$$-9(x-3) + 5(y+2) + 7(z-1) = 0$$

$$-9x + 27 + 5y + 10 + 7z - 7 = 0$$

$$\boxed{-9x + 5y + 7z + 30 = 0}$$

5. **[8 points]** An object moves in 3-D with acceleration

$$a(t) = \langle \sin(t),\ 2\cos(t),\ 6t \rangle.$$

At time $t = 0$ it has velocity $\langle 0, 0, -1 \rangle$. Find a function $v(t) = r'(t)$ giving its velocity at all times $t > 0$.

$v(0) = \langle 0, 0, -1 \rangle$

$$v(t) = \int a(t)\,dt = i\int \sin t\,dt + j\int 2\cos t\,dt +$$

$$+ k\int 6t\,dt + C = -\cos t \cdot i + 2\sin t \cdot j + 3t^2 \cdot k + C$$

$$v(0) = \langle -1 + c_1, 0 + c_2, 0 + c_3 \rangle = \langle 0, 0, -1 \rangle$$

$$C = \langle 1, 0, -1 \rangle$$

Hence $\boxed{v(t) = \langle -\cos t + 1,\ 2\sin t,\ 3t^2 - 1 \rangle}$

+7 points to each student

6. **[20 points]** A particle moves with *velocity* $v(t) = \langle t^3, t^2, 2t\rangle$.

(a) (7 pts) Find the distance the particle travels between times $t = 0$ and $t = 2$.

$$d(2)-d(0) = \int_0^2 |v(t)|\,dt = \int_0^2 \sqrt{t^6+t^4+4t^2}\,dt =$$

$$= \int_0^2 t\sqrt{t^4+t^2+4}\,dt = \int_0^2 t\sqrt{\left(t^2+\tfrac{1}{2}\right)^2+\tfrac{15}{4}}\,dt = \frac{1}{2}\int_0 \sqrt{u^2+\tfrac{15}{4}}\,du =$$

$t^2+\frac{1}{2}=u$

$2t\,dt = du$

$$= \frac{1}{2}\left(\frac{u}{2}\sqrt{u^2+\tfrac{15}{4}}+\frac{15}{8}\ln\left(u+\sqrt{u^2+\tfrac{15}{4}}\right)\right)\Bigg|_0^2 =$$

$$= \frac{1}{2}\left(\frac{t^2+\frac{1}{2}}{2}\sqrt{\left(t^2+\tfrac{1}{2}\right)^2+\tfrac{15}{4}}+\frac{15}{8}\ln\left(t^2+\tfrac{1}{2}\sqrt{(\ldots)^2+15/4}\right)\right)\Bigg|_0^2$$

(b) (8 pts) Calculate the curvature

$$\kappa = \frac{|r'(t)\times r''(t)|}{|r'(t)|^3}$$

of the trajectory at time $t = 1$.

$r'(t) = \langle t^3, t^2, 2t\rangle$

$r''(t) = \langle 3t^2, 2t, 2\rangle$

$r'(1) = \langle 1, 1, 2\rangle$

$r''(1) = \langle 3, 2, 2\rangle$

$|r'(t)| = \sqrt{t^6+t^4+4t^2}$

$$|r'\times r''| = \begin{vmatrix} i & j & k \\ t^3 & t^2 & 2t \\ 3t^2 & 2t & 2 \end{vmatrix} = (2t^2-4t^2)i-(2t^3-6t^3)j+(2t^3-6t^3)k$$

$|r'(1)| = \sqrt{6}$

$|r'(1)\times r''(1)| = \sqrt{4+16+16} = 6$

$$\kappa = \frac{6}{(\sqrt{6})^3} = \frac{6}{6\sqrt{6}} = \boxed{\frac{1}{\sqrt{6}}}$$

(c) (5 pts) Find the unit tangent vector **T(t)** and the tangential component of acceleration

$$a_T = \frac{r'(t) \cdot r''(t)}{|r'(t)|}$$

at $t = 1$.

$$T(t) = \frac{r'(t)}{|r'(t)|}\ ; \ T(1) = \frac{r'(1)}{|r'(1)|} = \frac{\langle 1,1,2\rangle}{\sqrt{6}} =$$

$$= \boxed{\left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle}$$

$$r'(1)\cdot r''(1) = 3+2+4 = 9$$

$$a_T = \boxed{\frac{9}{\sqrt{6}}}$$

7. **[4 points]** Evaluate the following integral

$$\int_1^4 \left(2t^{3/2}\,\mathbf{i} + (t+1)\sqrt{t}\,\mathbf{k}\right) dt. =$$

$$= i\int_1^4 2t^{3/2}\,dt + k\int_1^4 \left(t\sqrt{t}+\sqrt{t}\right)dt =$$

$$= 2i\,\frac{t^{5/2}}{5/2}\Big|_1^4 + k\left(\frac{t^{5/2}}{5/2} + \frac{t^{3/2}}{3/2}\right)\Big|_1^4 =$$

$$= 2\left(\frac{2}{5}\cdot 32 - \frac{2}{5}\right)i + \left(\frac{2}{5}\cdot 32 + \frac{2}{3}\cdot 8 - \frac{2}{5} - \frac{2}{3}\right)k =$$

$$= \frac{124}{5}i + 0\cdot j + \left(\frac{62}{5} + \frac{14}{3}\right)k = \boxed{\left\langle \frac{124}{5}, 0, \frac{256}{15} \right\rangle.}$$

$$\| \quad \frac{186+70}{15} = \frac{256}{15}$$