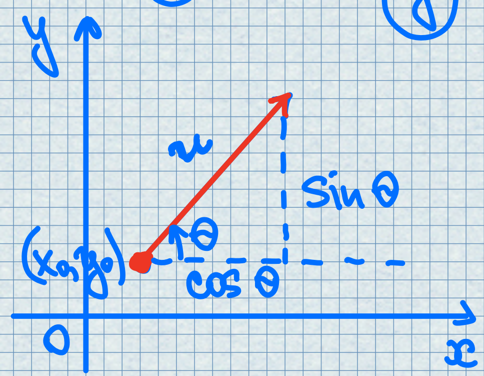
Lecture #17 - Week 6 - Directional Derivatives and the Gradient Vector - 14.6

• Directional Derivatives

If $z = f(x,y)$, then

$$\boxed{\begin{aligned} f_x(x_0,y_0) &= \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h} \\ f_y(x_0,y_0) &= \lim_{h\to 0} \frac{f(x_0,y_0+h) - f(x_0,y_0)}{h} \end{aligned}}$$

These are the rates of change of z in x- and y-directions, that is, in the directions of the unit vectors i and j.

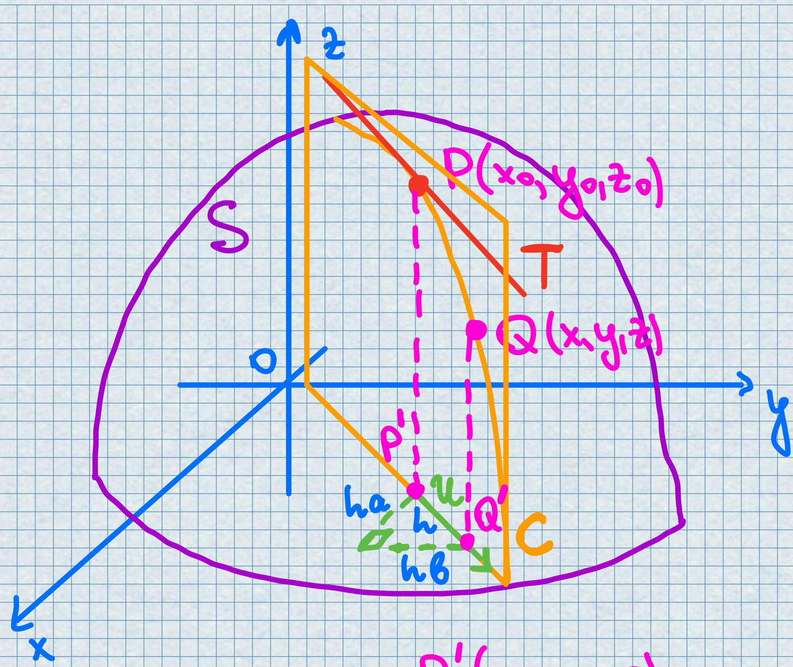


Let $u = \langle a, b \rangle$ be an arbitrary unit vector.

We consider S: $z = f(x,y)$

$z_0 = f(x_0,y_0)$

$P(x_0,y_0,z_0) \in S$



$P'(x_0, y_0, 0)$
$Q'(x, y, 0)$

If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q onto xy-plane, then

$$\overrightarrow{P'Q'} \parallel \vec{u} \quad \text{and}$$

$$\overrightarrow{P'Q'} = hu = \langle ha, hb \rangle, \quad h\text{ - scalar}$$

Therefore,

$$x - x_0 = ha$$
$$y - y_0 = hb$$

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Def. The <u>directional derivative</u> of f at (x_0, y_0) in the direction of a unit vector $u = \langle a, b \rangle$ is

$$\boxed{D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}}$$

if this limit exists.

Theorem If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector

$u = \langle a, b \rangle$ and

$$\boxed{D_u f(x,y) = f_x(x,y)a + f_y(x,y)b}$$

If the unit vector u makes an angle θ with the positive x-axis, then

$$u = \langle \cos\theta, \sin\theta \rangle$$

and

$$\boxed{D_u f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta}$$

- The Gradient Vector

We have that

$$D_u f(x,y) = f_x(x,y)a + f_y(x,y)b =$$

$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle =$$

$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot u$$

Def. If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by

$$\boxed{\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j}$$

With this notation for ∇f, we can rewrite

$$\boxed{D_u f(x,y) = \nabla f(x,y) \cdot u}$$

This expresses the directional derivative in the direction of a unit vector u as the scalar projection of the gradient vector onto u.

- **Functions of Three Variables**

Def. The <u>directional derivative</u> of f at (x_0, y_0, z_0) in the direction of a unit vector $u = \langle a, b, c\rangle$ is

$$D_u f(x_0,y_0,z_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0,y_0,z_0)}{h}$$

if this limits exists.

Equivalently,

$$\boxed{D_u f(\vec{x}_0) = \lim_{h\to 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h}}$$

where $\vec{x}_0 = \langle x_0, y_0\rangle$ if $n=2$.

If $f(x,y,z)$ is differentiable and $u = \langle a, b, c\rangle$, then

$$\boxed{D_u f(x,y,z) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c}$$

$$\boxed{\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle}$$

or

$$\boxed{\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k}$$

$$\boxed{D_u f(x,y,z) = \nabla f(x,y,z) \cdot u}$$

- **Maximizing the Directional Derivative**

Suppose we have a function f of two or three variables and we consider all possible directional derivatives of f at a given point. These give the rates of change of f in all possible directions.

Question: in which of these directions does f change fastest and what is the maximum rate of change?

<u>**Theorem**</u> Suppose f is a differentiable function of two or three variables. The maximum value of the directional

derivative $D_u f(\vec{x})$ is $|\nabla f(\vec{x})|$ and it occurs when u has the same direction as the gradient vector $\nabla f(\vec{x})$.

- **Tangent Planes to Level Surfaces.**

Suppose S: $F(x,y,z) = k$

$P(x_0,y_0,z_0) \in S$

C is a curve on S that goes through P.

C: $r(t) = \langle x(t), y(t), z(t) \rangle$.

$t_0 \to P$: $r(t_0) = \langle x(t_0), y(t_0), z(t_0) \rangle$.

Since $C \in S$:

$$\boxed{F(x(t), y(t), z(t)) = k}$$

If x,y,z and F are differentiable, then

$$\boxed{\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0}$$

But since

$\nabla F = \langle F_x, F_y, F_z \rangle$ and $r'(t) = \langle x'(t), y'(t), z'(t) \rangle$,

we get

$$\boxed{\nabla F \cdot r'(t) = 0}$$

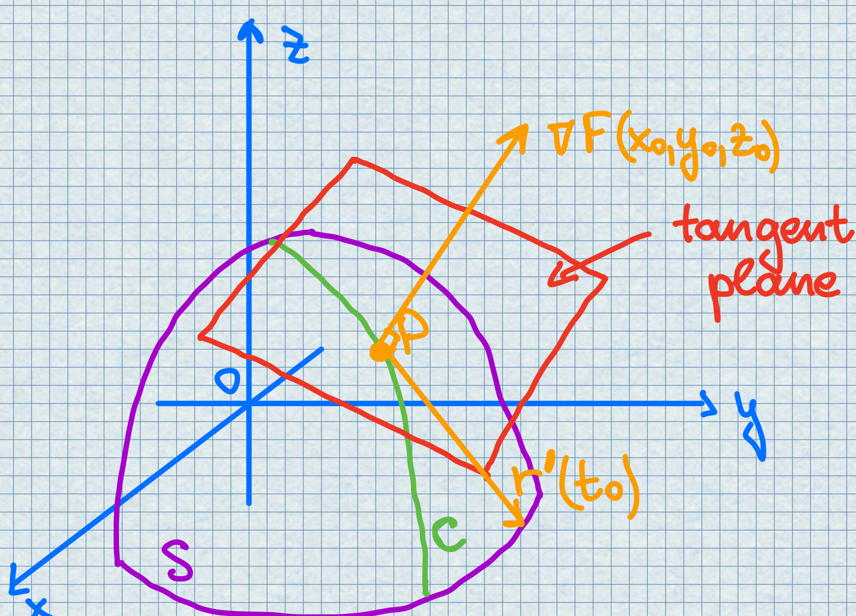
When $t = t_0$: $r(t_0) = \langle x_0, y_0, z_0 \rangle$, so

$$\boxed{\nabla F(x_0, y_0, z_0) \cdot r'(t_0) = 0} \qquad (1)$$

(1) says: $\nabla F(x_0, y_0, z_0)$ is $\perp$ to $r'(t_0)$ to any curve C on S that passes through P.

If $\nabla F(x_0, y_0, z_0) \neq 0$, then we can define the <u>tangent plane to the level surface</u> $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$.

$$\boxed{F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0}$$



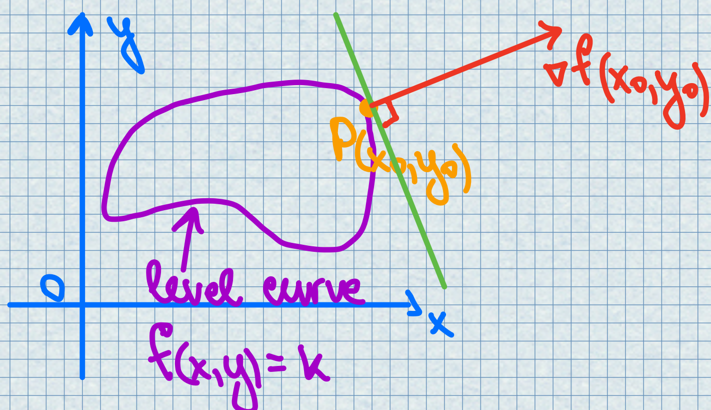
The <u>normal line</u> to S at P is the line passing through P and $\perp$ to the tangent plane.

The direction of the normal line is given by the gradient vector $\nabla F(x_0, y_0, z_0)$.

And

$$\boxed{\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}}$$

- Significance of the Gradient Vector



- $\nabla f(x_0,y_0)$ gives the direction of fastest increase of f.
- $\nabla f(x_0,y_0)$ is $\perp$ to $f(x,y)=k$ that passes through P.

Examples

$\boxed{1.}$ Find the directional derivative $D_u f(x,y)$ if

$$f(x,y) = x^3 - 3xy + 4y^2$$

and u is the unit vector given by angle $\theta = \pi/6$. What is $D_u f(1,2)$?

Solution

$$D_u f(x,y) = f_x(x,y)\cos\frac{\pi}{6} + f_y(x,y)\sin\frac{\pi}{6}$$

$$D_u f(x,y) = (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2}$$

Therefore,

$$D_u f(1,2) = \frac{\sqrt{3}}{2}(3-6) + (-3+16)\frac{1}{2} = \frac{13 - 3\sqrt{3}}{2}.$$

▼

$\boxed{2.}$ If $f(x,y) = \sin x + e^{xy}$, then

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

$$\nabla f(0,1) = \langle 2, 0 \rangle.$$

▼

$\boxed{3.}$ Find the directional derivative of the function $f(x,y) = x^2y^3 - 4y$ at $(2,-1)$ in the direction of the vector $v = 2i + 5j$.

Solution

$$\nabla f(x,y) = 2xy^3 i + (3x^2y^2 - 4) j$$
$$\nabla f(2,-1) = -4i + 8j$$

Note that v is not a unit vector.
We have that $|v| = \sqrt{29}$.

$$u = \frac{v}{|v|} = \frac{2}{\sqrt{29}} i + \frac{5}{\sqrt{29}} j.$$

Therefore,

$$D_u f(2,-1) = \nabla f(2,-1) \cdot u = (-4i + 8j)\left(\frac{2}{\sqrt{29}} i + \frac{5}{\sqrt{29}} j\right) =$$
$$= \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

▼

4. (a) If $f(x,y) = xe^y$, find the rate of change of f at the point $P(2,0)$ in the direction from P to $Q\left(\frac{1}{2}, 2\right)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Solution

(a)

$$\nabla f(x,y) = \langle e^y, xe^y \rangle$$
$$\nabla f(2,0) = \langle 1, 2 \rangle$$

The unit vector in the direction $\overrightarrow{PQ} = \langle -\frac{3}{2}, 2 \rangle$

is $u = \langle -\frac{3}{5}, \frac{4}{5} \rangle$.

So

$$D_u f(2,0) = \nabla f(2,0) \cdot u = \langle 1,2 \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle =$$

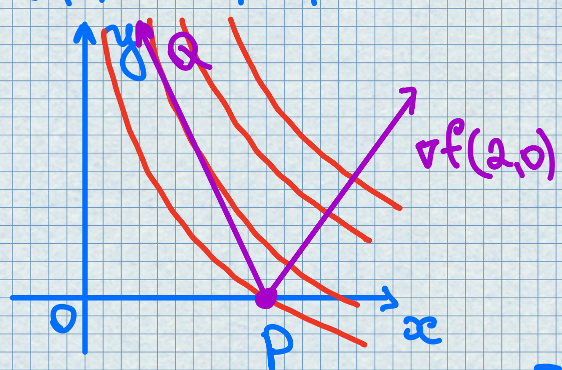
$$= 1.$$

(b) f increases fastest in the direction of the gradient vector

$$\nabla f(2,0) = \langle 1,2 \rangle.$$

The maximum rate of change is

$$|\nabla f(2,0)| = |\langle 1,2 \rangle| = \sqrt{5}.$$



$\boxed{5.}$ Find the equations of the tangent plane and normal line at the point $(-2,1,-3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$.

Solution

$$F(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

$$F_x = \frac{1}{2}x \qquad F_y = 2y \qquad F_z = \frac{2}{9}z$$

$$F_x(-2,1,-3) = -1 \qquad F_y(-2,1,-3) = 2 \qquad F_z(-2,1,-3) = -\frac{2}{3}$$

The tangent plane equation at $(-2,1,-3)$ is

$$-1(x+2)+2(y-1)-\frac{2}{3}(z+3)=0$$

or

$$3x-6y+2z+18=0.$$

The symmetric equations of the normal line are

$$\frac{x+2}{-1}=\frac{y-1}{2}=\frac{z+3}{-\frac{2}{3}}$$

▼