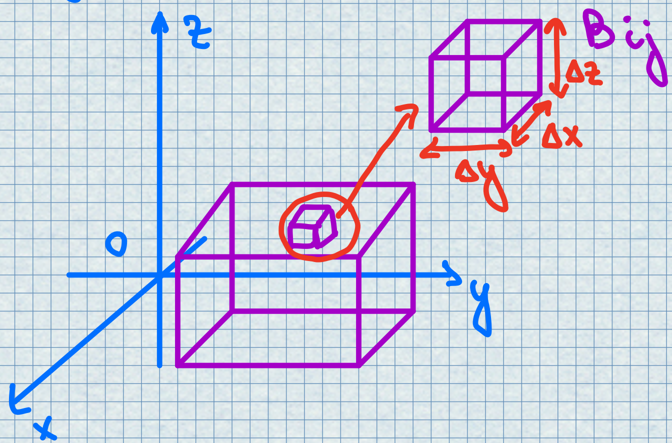
Lecture #25 - Week 9 - Triple Integrals - 15.6

Let $z = f(x,y,z)$ and f is defined on a rectangular box:

$$B = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$$



$$B_{ij} = \Delta x \times \Delta y \times \Delta z$$

$$\Delta V = \Delta x \Delta y \Delta z$$

Then we form the triple Riemann sum

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta V,$$

where $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \in B_{ijk}$.

Def. The triple integral of f over the box B is

$$\iiint_B f(x,y,z)\, dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta V$$

if this limit exists.

Alternatively,

$$\boxed{\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i,y_j,z_k)\,\Delta V}$$

Fubini's Theorem for triple integrals

If f is continuous on the rectangular box $B=[a,b]\times[c,d]\times[r,s]$, then

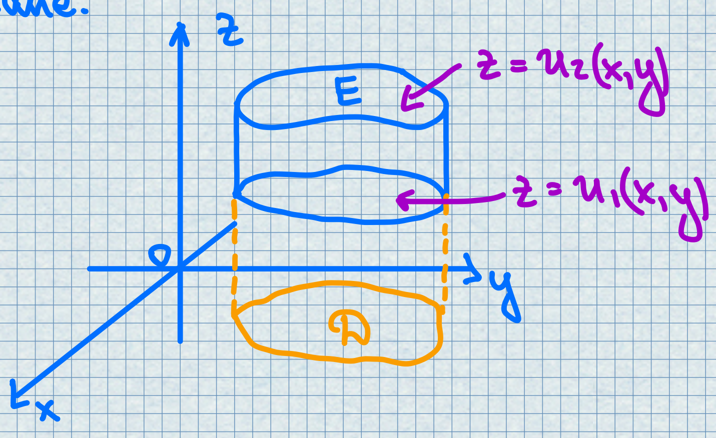
$$\boxed{\iiint_B f(x,y,z)\,dV = \int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz}$$

We restrict our attention to continuous functions f and to certain simple types of regions.

A solid region of type 1:

$$E=\{(x,y,z)\mid (x,y)\in D,\ u_1(x,y)\le z\le u_2(x,y)\},$$

where D is the projection of E onto the xy-plane.

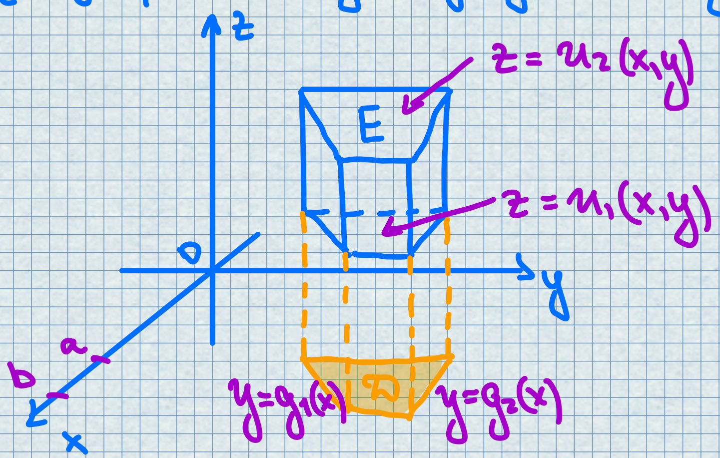


Then,

$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA}$$

In particular, if <u>D is a type I plane region:</u>

$$E = \{(x,y,z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x,y) \le z \le u_2(x,y)\}$$

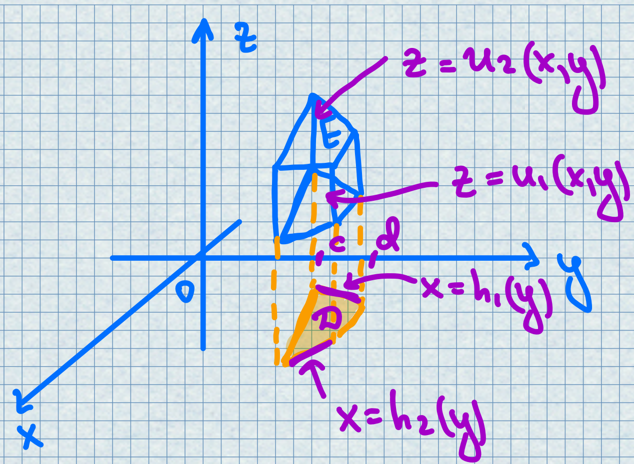


Then

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

<u>If D is a type II plane region</u>

$$E = \{(x,y,z) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y),\ u_1(x,y) \le z \le u_2(x,y)\}$$



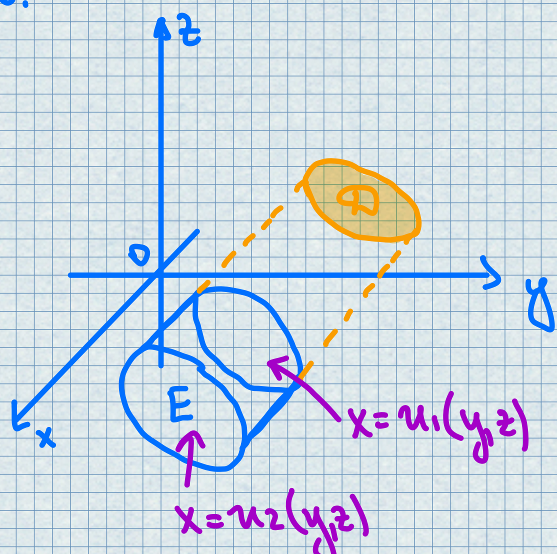
Then

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy}$$

A solid region E of type 2

$$E = \{(x,y,z) \mid (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\},$$

where D is the projection of E onto the yz-plane.



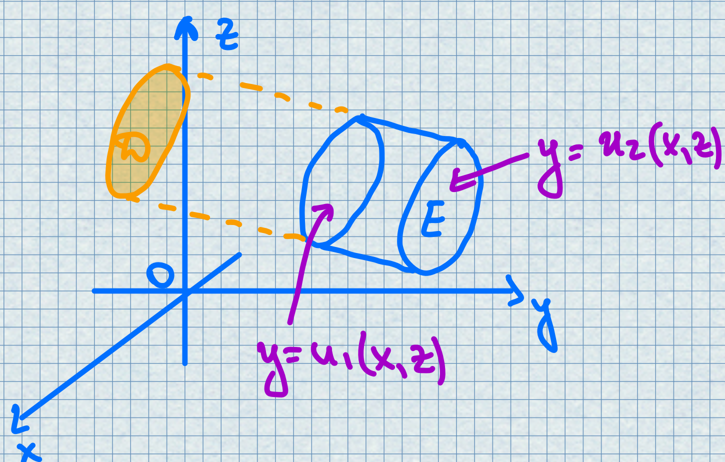
Then

$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx \right) dA}$$

A solid region of type 3

$E = \{(x,y,z) \mid (x,z) \in D,\ u_1(x,z) \le y \le u_2(x,z)\}$,

where D is the projection of E onto the xz-plane.



Then

$$\boxed{\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,z)}^{u_2(x,y)} f(x,y,z)\,dy \right) dA}$$

- Applications of tripple integrals

- $f(x,y,z) = 1$ for all points in E.

Then

$$V(E) = \iiint_E dV$$

From this it follows that

$$\iiint_E 1\, dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} dz \right) dA = \iint_D \left(u_2(x,y) - u_1(x,y) \right) dA$$

- If the density function of a solid object that occupies the region E is $\rho(x,y,z)$, in units of mass per unit volume, at any point (x,y,z), then

$$\boxed{m = \iiint_E \rho(x,y,z)\, dV}$$

and

$$M_{yz} = \iiint_E x\, \rho(x,y,z)\, dV \qquad M_{xz} = \iiint_E y\, \rho(x,y,z)\, dV$$

$$M_{xy} = \iiint_E z\, \rho(x,y,z)\, dV$$

- The <u>center of mass</u> is located at $(\bar{x}, \bar{y}, \bar{z})$, where

$$\boxed{\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}}$$

<u>Def.</u> If $\rho(x,y,z) = \text{const}$, the center of mass of the solid is called the <u>centroid</u> of E.

- The moments of inertia about the tree coordinate axes are

$$\boxed{\begin{aligned} I_x &= \iiint_E (y^2+z^2)\,\rho(x,y,z)\,dV \\ I_y &= \iiint_E (x^2+z^2)\,\rho(x,y,z)\,dV \\ I_z &= \iiint_E (x^2+y^2)\,\rho(x,y,z)\,dV \end{aligned}}$$

- The total electric charge on a solid object occupying a region E and having charge density $\sigma(x,y,z)$ is

$$\boxed{Q = \iiint_E \sigma(x,y,z)\,dV}$$

- If we have three continuous random variables X, Y, and Z, their joint density function is

$$\boxed{P((X,Y,Z)\in E) = \iiint_E f(x,y,z)\,dV}$$

In particular,

$$P(a\le X\le b,\, c\le Y\le d,\, r\le Z\le s) = \int_a^b\int_c^d\int_r^s f(x,y,z)\,dz\,dy\,dx$$

The joint density function satisfies

$$f(x,y,z) \geq 0 \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx = 1.$$

Examples

$\boxed{1.}$ Evaluate the triple integral $\iiint_B xyz^2\,dV$, where B is given by

$$B=\{(x,y,z)\mid 0\le x\le 1, -1\le y\le 2, 0\le z\le 3\}$$

Solution

$$\iiint_B xyz^2\,dV=\int_0^3\int_{-1}^2\int_0^1 xyz^2\,dx\,dy\,dz=$$

$$=\int_0^3\int_{-1}^2 \frac{x^2}{2}yz^2\Big|_0^1\,dy\,dz=\frac{1}{2}\int_0^3\int_{-1}^2 yz^2\,dy\,dz=$$

$$=\int_0^3\frac{3z^2}{4}\,dz=\frac{z^3}{4}\Big|_0^3=\frac{27}{4}.$$

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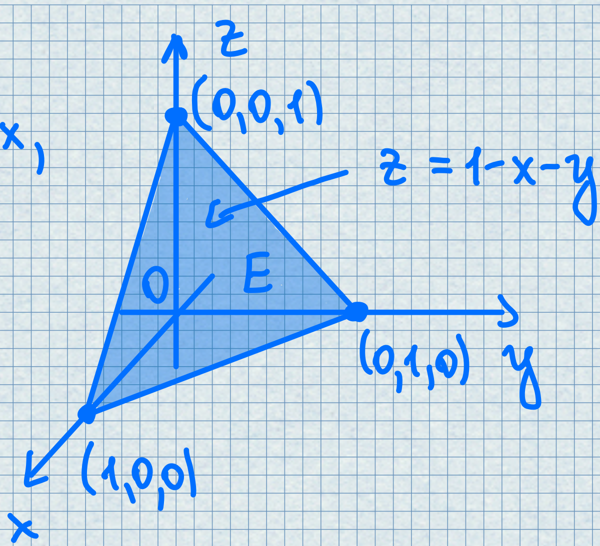
$\boxed{2.}$ Evaluate $\iiint_E z\,dV$, where E is the solid bounded by:

$$x=0,\ y=0,\ z=0,\ x+y+z=1.$$

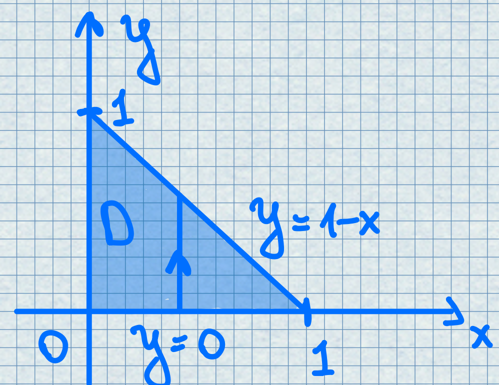
Solution

$$E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y\}$$



$$\iiint_E z\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx =$$

$$= \int_0^1 \int_0^{1-x} \frac{z^2}{2}\Big|_0^{1-x-y} dy\,dx =$$



$$= \frac{1}{2}\int_0^1 \int_0^{1-x} (1-x-y)^2\,dy\,dx =$$

$$= \frac{1}{2}\int_0^1 -\frac{(1-x-y)^3}{3}\Big|_0^{1-x} dx = \frac{1}{6}\int_0^1 (1-x)^3\,dx = \frac{1}{6}\left(-\frac{(1-x)^4}{4}\right)\Big|_0^1 =$$

$$= \frac{1}{24}.$$

$\boxed{3.}$ Express the iterated integral

$\int_0^1 \int_0^{x^2} \int_0^{y} f(x,y,z)\,dz\,dy\,dx$ as a triple integral and then rewrite it as an iterated integral in a different order, integrating first with respect to x, then z, and then y.

Solution

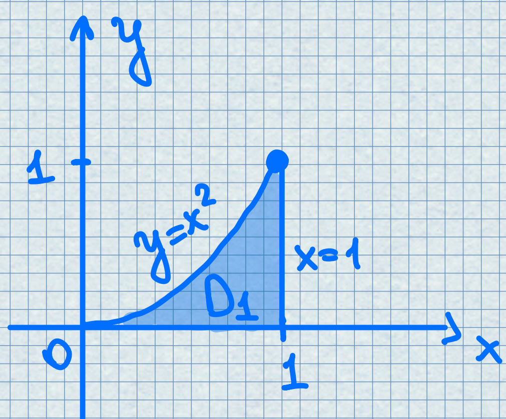
$$\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx = \iiint_E f(x,y,z)\,dV$$

$$E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y\}$$

xy- plane:

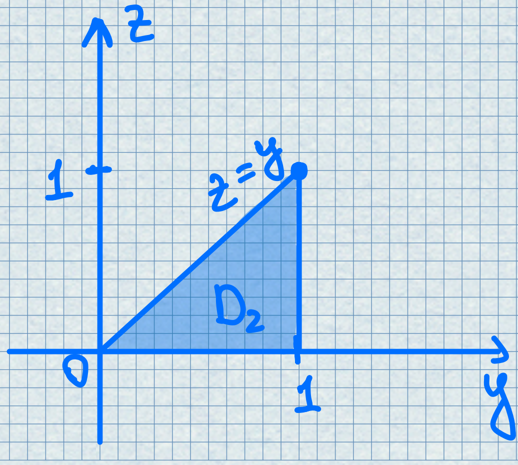
$$D_1 = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x^2\} =$$

$$= \{(x,y) \mid 0 \le y \le 1, \sqrt{y} \le x \le 1\}$$

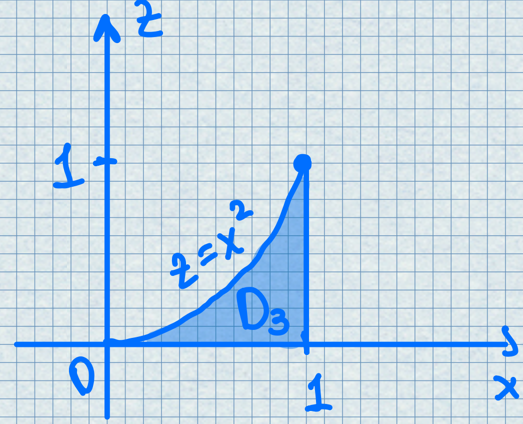


yz - plane:

$$D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y\}$$



xz- plane:



$$D_3 = \{(x,z) \mid 0 \le x \le 1, 0 \le z \le x^2\}$$

$$E = \{(x,y,z) \mid 0 \le y \le 1,\ 0 \le z \le \sqrt{y},\ \sqrt{y} \le x \le 1\}$$

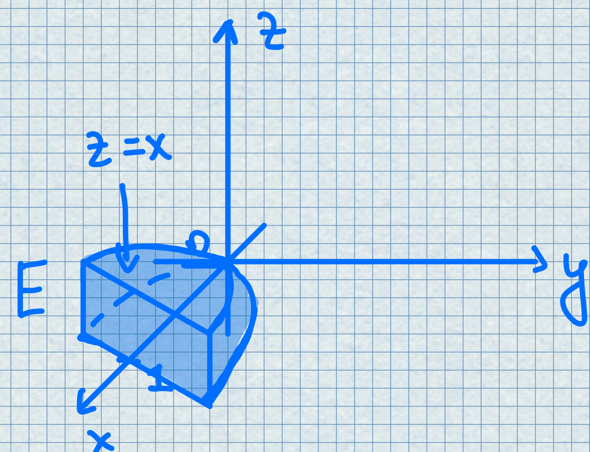
Thus

$$\iiint_E f(x,y,z)\, dV = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\, dx\, dz\, dy$$

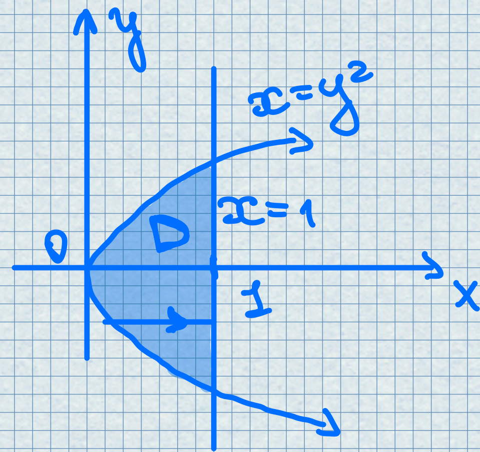
$\boxed{4.}$ Find the center of mass of a solid of constant density that is bounded by $x=y^2$, $x=z$, $z=0$, and $x=1$.

Solution

$$E = \{(x,y,z) \mid -1 \le y \le 1,\ y^2 \le x \le 1,\ 0 \le z \le x\}$$



$\rho(x,y,z) = \rho$



$$m = \iiint_E \rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x \rho \, dz \, dx \, dy =$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} x \, dx \, dy = \rho \int_{-1}^{1} \frac{x^2}{2} \Big|_{y^2}^{1} dy = \frac{\rho}{2} \int_{-1}^{1} (1 - y^4) \, dy =$$

$$= \rho \int_0^1 (1 - y^4) \, dy = \rho \left(y - \frac{y^5}{5} \right) \Big|_0^1 = \frac{4\rho}{5}$$

Because of the symmetry o E and ρ about the xz-plane, we have $M_{xz} = 0$ and therefore $\bar{y} = 0$.

$$M_{yz} = \iiint_E x \rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^1 x \rho \, dz \, dx \, dy =$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} x^2 \, dx \, dy = \rho \int_{-1}^{1} \frac{x^3}{3} \Big|_{y^2}^{1} dy =$$

$$= \frac{2\rho}{3} \int_0^1 (1 - y^6) \, dy = \frac{4\rho}{7}.$$

$$M_{xy} = \iiint_E z\rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x z\rho \, dz \, dx \, dy =$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} \frac{z^2}{2} \Big|_0^x \, dx \, dy = \frac{\rho}{2} \int_{-1}^{1} \int_{y^2}^{1} x^2 \, dx \, dy =$$

$$= \frac{\rho}{3} \int_0^1 (1-y^6) \, dy = \frac{2\rho}{7}.$$

Therefore, the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right) = \left(\frac{5}{7}, 0, \frac{5}{14} \right).$$

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