Formulas & Definitions: Section 15-6

Definition: The triple integral of f over the box B is

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$$

if this limit exists.

Fubini's Theorem for Triple Integrals: If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

Statement 1: If E a solid region of **type I**, that is,

$$E = \{(x,y,z) \,|\, (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$$

then

$$\iiint_E f(x,y,z)\,dV = \iint_D \Big[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \Big]\,dA$$

Statement 2: If D is a **type II** plane region, then

$$E = \{(x,y,z) \,|\, c \le y \le d, h_1(y) \le x \le h_2(y)), u_1(x,y) \le z \le u_2(x,y)\}$$

and

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

Statement 3: If E a solid region of **type III**, that is,

$$E = \{(x,y,z) \,|\, (x,z) \in D, u_1(x,z) \le z \le u_2(x,z)\}$$

then

$$\iiint_E f(x,y,z)\,dV = \iint_D \Big[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \Big]\,dA$$

Statement: If $f(x,y,z) = 1$ for all points in E, then

$$V(E) = \iiint_E dV.$$

Applications:

- If the density function of a solid object that occupies the region E is $\rho(x,y,z)$ at any given point (x,y,z), then its mass is

$$m = \iiint\limits_E \rho(x,y,z)\,dV$$

and its moments about three coordinate plane are

$$M_{yz} = \iiint\limits_E x\,\rho(x,y,z)\,dV, \quad M_{xz} = \iiint\limits_E y\,\rho(x,y,z)\,dV, \quad M_{xy} = \iiint\limits_E z\,\rho(x,y,z)\,dV$$

- The center of mass is located at the point $(\bar{x},\bar{y},\bar{z})$, where

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}$$

- The moments of inertia about the three coordinate plane axes are

$$I_x = \iiint\limits_E (y^2+z^2)\,\rho(x,y,z)\,dV, \quad I_y = \iiint\limits_E (x^2+z^2)\,\rho(x,y,z)\,dV,_z = \iiint\limits_E (x^2+y^2)\,\rho(x,y,z)\,dV$$

- The total electric charge on a solid object occupying a region E and having density $\sigma(x,y,z)$ is

$$Q = \iiint\limits_E \sigma(x,y,z)\,dV$$

- If we have three continuous random variables X, Y, and Z, their joint density function is a function of three variables such that the probability that (X,Y,Z) lies in E is

$$P((X,Y,Z) \in E) = \iiint\limits_E f(x,y,z)\,dV$$