Lecture # 24 - Week 8 - Surface Area - 15.5

Let S be a surface with $z = f(x,y)$, where f has continuous partial derivatives.

We assume that $f(x,y) \geq 0$ and $\mathrm{Dom}(f) = D$ is a rectangle.

We divide D into small R_{ij} with $\Delta A = \Delta x \Delta y$.

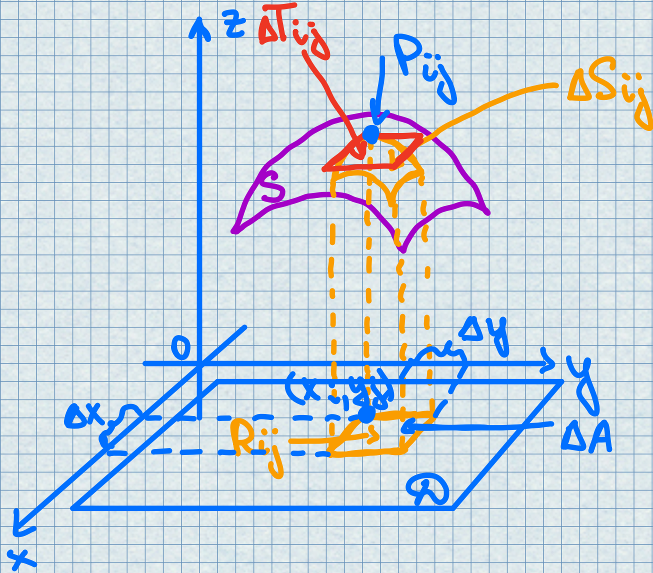
Let $P_{ij}(x_i, y_j, f(x_i, y_j))$ be the point on S directly above the corner of R_{ij}.

The tangent plane to S at P_{ij} is an approximation to S near P_{ij}.

So the area ΔT_{ij} of the part of this tangent plane that lies directly above R_{ij} is an approximation to the area ΔS_{ij} of the part of S that lies directly above R_{ij}. Thus the sum $\sum_i \sum_j \Delta T_{ij}$ is an approximation to the total area of S, and this approximation appears to improve as the number of rectangles increases.

Therefore, the <u>surface area</u> of S is

$$\boxed{A(S) = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}}$$

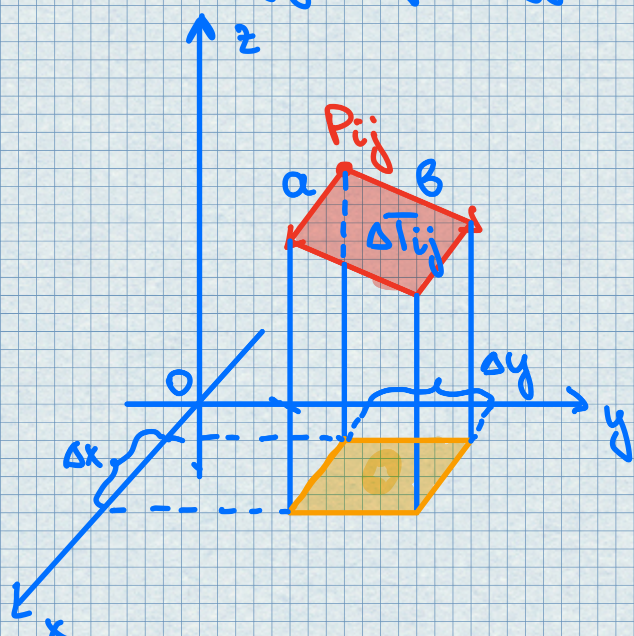


$$\Delta T_{ij} = |a \times b|,$$

where a and b are vectors that start at P_{ij} and lie along the sides of the parallelogram with area ΔT_{ij}.

$$a = \Delta x\, i + f_x(x_i, y_j)\,\Delta x\, k$$

$$b = \Delta y\, j + f_y(x_i, y_j)\,\Delta y\, k$$



$$a \times b = \begin{vmatrix} i & j & k \\ \Delta x & 0 & f_x(x_i, y_j)\Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\Delta y \end{vmatrix} =$$

$$= \left(-f_x(x_i, y_j)i - f_y(x_i, y_j)j + k\right)\Delta A$$

Thus,

$$\Delta T_{ij} = |a \times b| = \sqrt{(f_x(x_i, y_j))^2 + (f_y(x_i, y_j))^2 + 1}\ \Delta A$$

Statement

The area of the surface with equation $z = f(x,y)$, $(x,y) \in D$, where f_x and f_y are continuous, is

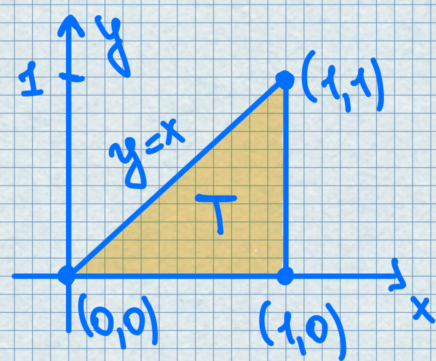
$$\boxed{A(S) = \iint_D \sqrt{(f_x(x,y))^2 + (f_y(x,y))^2 + 1}\ dA}$$

$$\boxed{A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dA}$$

Examples

$\boxed{1.}$ Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region T in the xy-plane with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

Solution



$$T = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x\}$$

$$f(x,y) = x^2 + 2y$$

$$A = \iint_T \sqrt{(2x)^2 + 2^2 + 1}\, dA = \int_0^1 \int_0^x \sqrt{4x^2 + 5}\, dy\, dx =$$

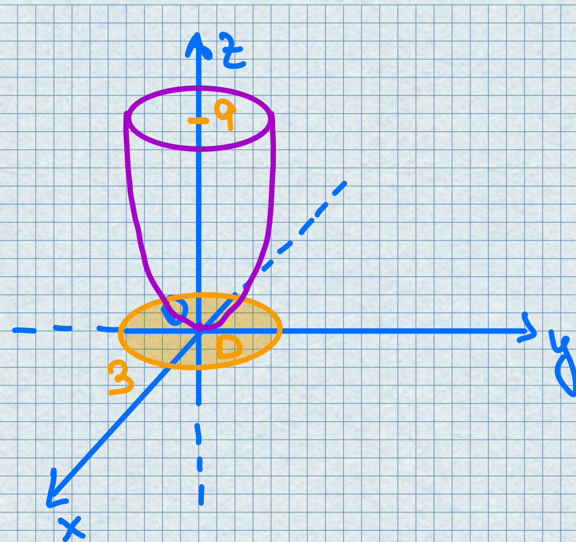
$$= \int_0^1 x\sqrt{4x^2 + 5}\, dx = \frac{1}{8}\,\frac{2}{3}\left(4x^2 + 5\right)^{3/2}\Big|_0^1 = \frac{1}{12}\left(27 - 5\sqrt{5}\right).$$

$\boxed{2.}$ Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$

Solution

$$x^2 + y^2 = 9, \quad z = 9$$

Therefore, the given surface lies above the disk D with center the origin and radius 3.



Thus,

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA =$$

$$= \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA = \iint_D \sqrt{1 + 4(x^2+y^2)}\, dA$$

Converting to polar coordinates, we obtain

$$A = \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 \frac{1}{8}\sqrt{1+4r^2}\,(8r)\,dr$$

$$= 2\pi\left(\frac{1}{8}\right)\frac{2}{3}(1+4r^2)^{3/2}\Big|_0^3 = \frac{\pi}{6}\left(37\sqrt{37} - 1\right).$$

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