Lecture # 16 - Week 6 - Chain Rule - 14.5

If $y = f(x)$ and $x = g(t)$, then

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$$

If $z = f(x,y)$ and $x = g(t)$, $y = h(t)$, then

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

The Chain Rule (Case 1)

Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

or

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

Now, let $z = f(x,y)$, where

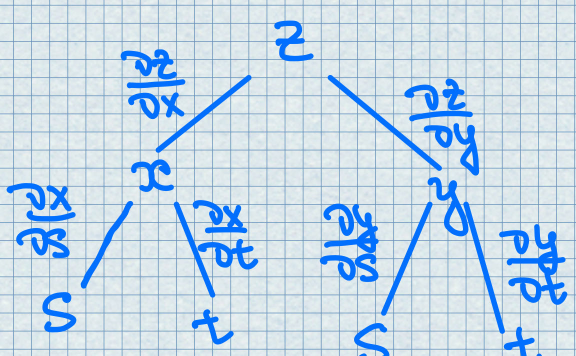
$$x = g(t,s)$$
$$y = h(t,s)$$

We get
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

The Chain Rule (Case 2)

Suppose that $z=f(x,y)$ is a differentiable function of x and y, where $x=g(s,t)$ and $y=h(s,t)$ are differentiable functions of s and t. Then

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}}$$



The Chain Rule (General version)

Suppose that u is a differentiable function of the n variables $x_1,\dots,x_n$ and each x_j is a differentiable function of the m variables $t_1,t_2,\dots,t_m$. Then u is a function of $t_1,\dots,t_m$ and

$$\boxed{\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}, \quad i=1,\dots,m}$$

- Implicit Differentiation

We suppose that

$$F(x,y)=0$$

defines y implicitly as a differ. function of x, that is, $y=f(x)$, $F(x,f(x))=0$ for all x in the domain of f.

If F is differentiable, we obtain

$$\boxed{\frac{\partial F}{\partial x}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0}$$

So, if $\frac{\partial F}{\partial y}\neq 0$, we obtain

$$\boxed{\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\frac{F_x}{F_y}} \qquad (1)$$

Now, we suppose that

$$z=f(x,y)$$

is given implicitly by an equation of the form

$$F(x,y,z)=0.$$

$$F(x,y,f(x,y)) = 0 \text{ for all } (x,y) \in \mathrm{Dom}(f).$$

If F and f are differentiable, then

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

But $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0.$

Hence,

$$\boxed{\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0}$$

If $\frac{\partial F}{\partial z} \neq 0$, then

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}} \qquad (2)$$

Similarly,

$$\boxed{\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}}$$

Implicit Function Theorem:

if F is defined on a disk containing (a,b), where $F(a,b) = 0$, $F_y(a,b) \neq 0$, and F_x and F_y

are continuous on the disk, then $F(x,y)=0$ defines y as a function of x near the point (a,b) and the derivative of this function is given by (1).

Implicit Function Theorem:

If F is defined within a sphere containing (a,b,c), where $F(a,b,c)=0$, $F_z(a,b,c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then $F(x,y,z)=0$ defines z as a function of x and y near the point (a,b,c) and this function is differentiable, with partial derivatives given by (2).

Examples

$\boxed{1.}$ If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{dz}{dt}$ when $t=0$.

Solution

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{\partial z}{\partial x} = 2xy + 3y^4 \qquad \frac{dx}{dt} = 2\cos 2t$$

$$\frac{\partial z}{\partial y} = x^2 + 12xy^3 \qquad \frac{dy}{dt} = -\sin t$$

$$\frac{dz}{dt} = (2xy + 3y^4)\,2\cos 2t + (x^2 + 12xy^3)(-\sin t)$$

$$\left.\frac{dz}{dt}\right|_{t=0} = (0+3)(2\cdot\cos 0) + (0+0)(-\sin 0) = 6.$$

$\boxed{2.}$ If $z = e^x \sin y$, where

$x = st^2$

$y = s^2t$

find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial s} = e^x \sin y \cdot t^2 + e^x \cos y \cdot 2st =$$

$$= e^{st^2}\sin(s^2t)t^2 + e^{st^2}\cos(s^2t)\,2st$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial t} = e^x \sin y \cdot 2st + e^x \cos y \cdot s^2 =$$

$$= e^{st^2}\sin(s^2t)\,2st + e^{st^2}\cos(s^2t)s^2.$$

$\boxed{3.}$ If $u = x^4 y + y^2 z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, $z = r^2 s \sin t$, find the value of $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$, $t = 0$.

Solution

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} =$$

$$= (4x^3y)\, re^t + (x^4 + 2yz^3)\,2rse^{-t} +$$

$$+ 3z^2y^2 \cdot r^2 \sin t$$

$$\left.\frac{\partial u}{\partial s}\right|_{r=2,s=1,t=0} = (4\cdot 2^3\cdot 2)2\cdot 1 + (2^4 + 2\cdot 2\cdot 0)\cdot$$

$$\cdot\, 2\cdot 2\cdot 1\cdot 1 + 3\cdot 0\cdot 2^2\cdot 2^2\cdot 0 =$$

$$= 8\cdot 8\cdot 2 + 16\cdot 4 = 64\cdot 2 + 64 = 64\cdot 3 = 192.$$

$\boxed{4.}$ If $g(s,t) = f(\overbrace{s^2-t^2}^{x}, \overbrace{t^2-s^2}^{y})$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

Solution

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$\frac{\partial x}{\partial s} = 2s \qquad \frac{\partial x}{\partial t} = -2t$$

$$\frac{\partial y}{\partial s} = -2s \qquad \frac{\partial y}{\partial t} = 2t$$

$$t\cdot\left(\frac{\partial f}{\partial x}\cdot 2s+\frac{\partial f}{\partial y}\cdot(-2s)\right)+s\left(\frac{\partial f}{\partial x}(-2t)+\right.$$

$$\left.+\frac{\partial f}{\partial y}\cdot 2t\right)=0$$

$\boxed{5.}$ Find y' if $x^3+y^3=6xy$.

Solution

$$F(x,y)=x^3+y^3-6xy$$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{3x^2-6y}{3y^2-6x}=-\frac{x^2-2y}{y^2-2x}.$$

$\boxed{6.}$ Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$x^3+y^3+z^3+6xyz=1.$$

Solution

$$F(x,y,z)=x^3+y^3+z^3+6xyz-1$$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{3x^2+6yz}{3z^2+6xy}=-\frac{x^2+2yz}{z^2+2xy}.$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2+6xz}{3z^2+6xy} = -\frac{y^2+2xz}{z^2+2xy}.$$