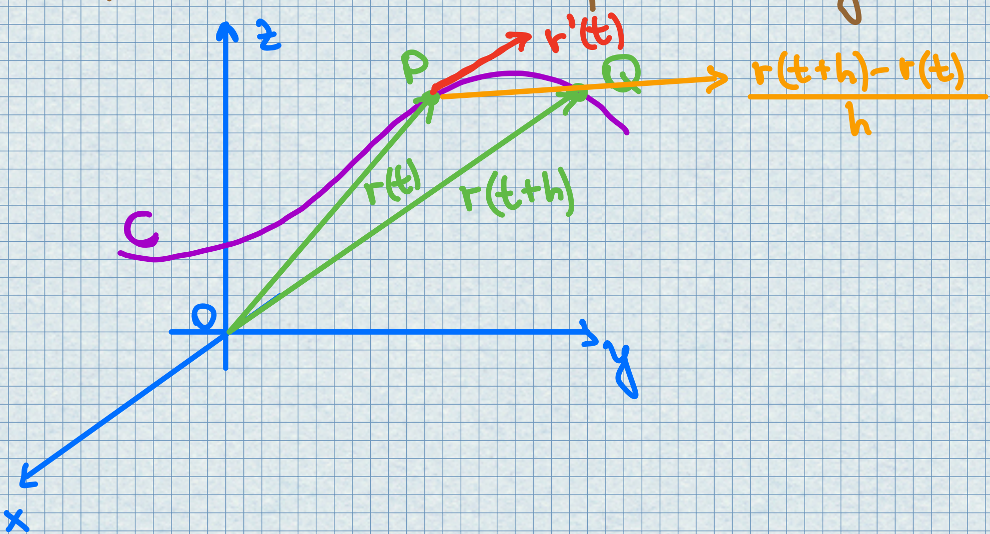
Lecture #10 - Week 4 - Motion in Space: Velocity - 13.4



Suppose that a particle moves through space so that its position vector at time t is $r(t)$.
Then, for small h

$$\frac{r(t+h) - r(t)}{h}$$

approximates the direction of the particle along $r(t)$.

The <u>velocity vector</u> $v(t)$ at time t:

$$\boxed{v(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h} = r'(t)}$$

$v(t)$ is also the tangent vector and points in the direction of the tangent line.

Def. The speed of a particle $\upsilon(t)$ is

$$\boxed{\upsilon(t) = |v(t)|}$$

$$\upsilon(t) = |r'(t)| = \frac{ds}{dt}$$

The acceleration of the particle is

$$\boxed{a(t) = v'(t) = r''(t)}$$

In general, vector integrals allow us to recover velocity when acceleration is known and position when velocity is known:

$$\boxed{v(t) = v_0 + \int_{t_0}^{t} a(u)\,du}$$

$$\boxed{r(t) = r_0 + \int_{t_0}^{t} v(u)\,du}$$

If the force acts on a particle, then by the Newton's Second Law of Motion:

$$\boxed{F(t) = m\,a(t)}$$

Examples

$\boxed{1.}$ The position vector of an object moving in a plane is given by

$$r(t) = t^3 i + t^2 j.$$

Find its velocity, speed, and acceleration when $t=1$ and illustrate geometrically.

Solution

$$v(t) = r'(t) = 3t^2 i + 2t j$$

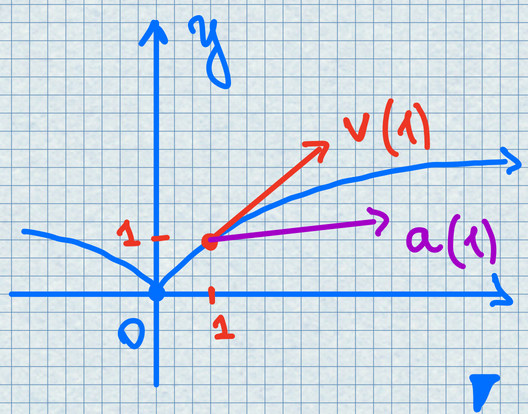
$$a(t) = r''(t) = 6t i + 2j$$

$$\text{speed} = |v(t)| = \sqrt{9t^4 + 4t^2}$$

When $t=1$: $v(1) = 3i + 2j$

$a(1) = 6i + 2j$

$|v(1)| = \sqrt{13}$.



$\boxed{2.}$ Find the velocity, acceleration, and speed of a particle with position vector

$r(t) = \langle t^2, e^t, te^t \rangle.$

Solution

$$v(t) = r'(t) = \langle 2t, e^t, e^t + te^t \rangle$$

$$a(t) = r''(t) = \langle 2, e^t, 2e^t + te^t \rangle$$

$$|v(t)| = \sqrt{4t^2 + e^{2t} + (1+t)^2 e^{2t}}.$$

$\boxed{3.}$ A moving particle starts at an initial position $r(0) = \langle 1,0,0 \rangle$ with initial velocity $v(0) = i - j + k$. Its acceleration is $a(t) = 4ti + 6tj + k$. Find its velocity and position at time t.

Solution

$$a(t) = v'(t)$$

$$v(t) = \int a(t)\,dt = \int (4ti + 6tj + k)\,dt =$$

$$= 2t^2 i + 3t^2 j + tk + C$$

$$v(0) = i - j + k$$

$$v(0) = C, \quad \text{so} \quad C = i - j + k$$

$$v(t) = 2t^2 i + 3t^2 j + tk + i - j + k =$$

$$= (2t^2 + 1)i + (3t^2 - 1)j + (t+1)k.$$

$$v(t) = r'(t)$$

$$r(t) = \int v(t)\,dt = \int \left((2t^2+1)i + (3t^2-1)j + (t+1)k\right)dt = \left(\frac{2}{3}t^3 + t\right)i + (t^3 - t)j + \left(\frac{t^2}{2} + t\right)k + D$$

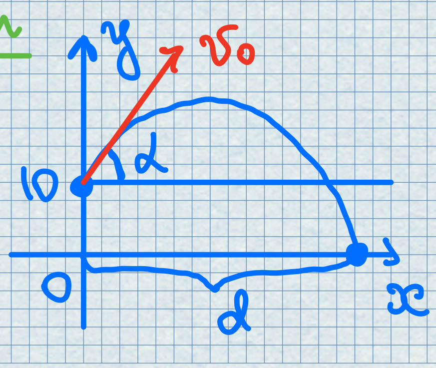
$$t = 0: \quad D = r(0) = i$$

So

$$r(t) = \left(\frac{2}{3}t^3 + t + 1\right)i + (t^3 - t)j + \left(\frac{1}{2}t^2 + t\right)k.$$

$\boxed{4.}$ A projectile is fired with muzzle speed 150 m/s and angle of elevation 45° from a position 10 m above ground level. Where does the projectile hit the ground, and with what speed?

Solution



Initial position : $(0,10)$

$v_0 = 150$ m/s

$\alpha = 45^\circ$

$g = 9.8$ m/s^2

So $x = 150\cos(45^\circ)t = 75\sqrt{2}\,t$

$y = 10 + 150\sin(45^\circ)t - \frac{1}{2}(9.8)t^2 =$

$= 10 + 75\sqrt{2}\,t - 4.9t^2$

Impact occurs when $y = 0$.

$4.9t^2 - 75\sqrt{2}\,t - 10 = 0$

$$t = \frac{75\sqrt{2} + \sqrt{11250 + 196}}{9.8} \approx 21.74\ (s)$$

$$x \approx 75\sqrt{2}\,(21.74) \approx 2306\ (m)$$

$$v(t) = r'(t) = 75\sqrt{2}\,i + (75\sqrt{2} - 9.8t)\,j$$

$$|v(21.74)| = \sqrt{(75\sqrt{2})^2 + (75\sqrt{2} - 9.8\cdot 21.74)^2} \approx 151 \text{ m/s}.$$