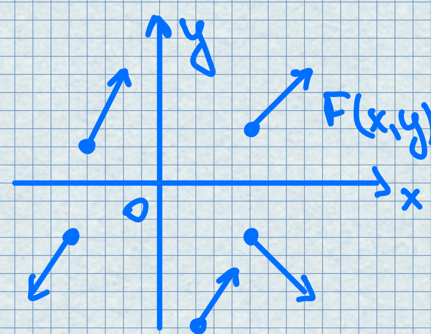
Lecture #29 - Week 10 - Vector Fields - 16.1

In general, a vector field is a function whose domain is a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and whose range is a set of vectors in V_2 (or V_3).

Def.

Let D be a set in $\mathbb{R}^2$.
A vector field on $\mathbb{R}^2$ is a function F that assigns to each point (x,y) in D a two-dimensional vector $F(x,y)$.



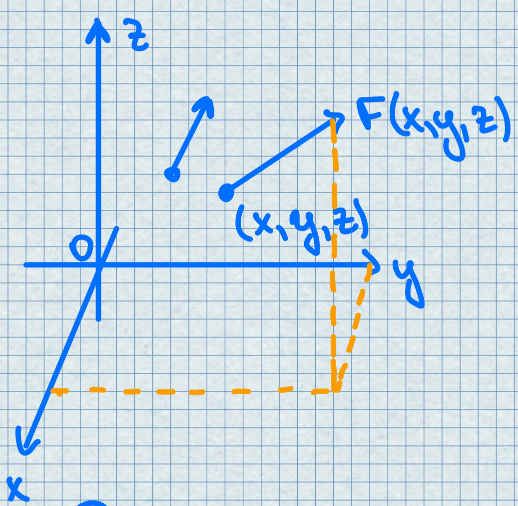
Since $F(x,y)$ is a two-dimensional vector, we can write it in terms of its component functions P and Q as follows:

$$F(x,y) = P(x,y)\,i + Q(x,y)\,j = \langle P(x,y), Q(x,y)\rangle =$$
$$= P\,i + Q\,j$$

Note P and Q are scalar functions of two variables and are sometimes called scalar fields.

Def. Let E be a set of $\mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a function F that assigns to each point (x,y,z) in E

a three-dimensional vector $F(x,y,z)$.



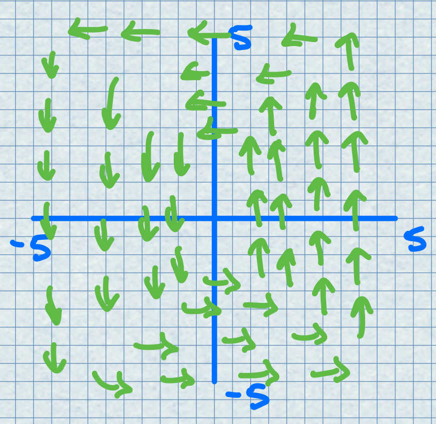
A vector field F on $\mathbb{R}^3$ we can express as

$$\boxed{F(x,y,z) = P(x,y,z)\,i + Q(x,y,z)\,j + R(x,y,z)\,k}$$

We sometimes identify a point (x,y,z) with its position vector $x = \langle x,y,z \rangle$ and write $F(x)$ instead of $F(x,y,z)$. Then F becomes a function that assigns a vector $F(x)$ to a vector x.

Example

$F(x,y) = \langle -y, x \rangle$



Example

Imagine a fluid flowing steadily along a pipe and let $V(x,y,z)$ be the velocity vector at a point (x, y, z).

Then V assigns a vector to each point (x, y, z) in a certain domain E and so V is a vector field on $\mathbb{R}^3$ called a velocity field.

Example

Newton's Law of Gravitation states that

$$|F| = \frac{mMG}{r^2},$$

where r is the distance between the objects and G is the gravitational constant.

The gravitational force acting on the object at $x = \langle x, y, z \rangle$ is

$$\boxed{F(x) = -\frac{mMG}{|x|^3} x} \qquad (1)$$

(1) is the gravitational field.

Equivalently,

$x = xi + yj + zk$ and $|x| = \sqrt{x^2 + y^2 + z^2}$:

$$F(x, y, z) = \frac{-mMGx}{(x^2+y^2+z^2)^{3/2}} i + \frac{-mMGy}{(x^2+y^2+z^2)^{3/2}} j + \frac{-mMGz}{(x^2+y^2+z^2)^{3/2}} k.$$

Example.

Suppose an electric charge Q is located at the origin.

According to Coulomb's Law, the electric force $F(x)$ exerted by this charge on a charge q located at a point (x,y,z) with position vector $x = \langle x,y,z \rangle$ is

$$\boxed{F(x) = \frac{\varepsilon q Q}{|x|^3}\, x} \qquad (2)$$

where ε is a constant.

(2) is a <u>force field.</u>

Instead of considering the electric force F, physicists often consider the force per unit charge:

$$\boxed{E(x) = \frac{1}{q}F(x) = \frac{\varepsilon Q}{|x|^3}\, x} \qquad (3)$$

(3) is a vector field on $\mathbb{R}^3$ called the <u>electric field</u> of Q.

- **Gradient Fields**

If f is a scalar function of two variables, then its gradient ∇f is defined by

$$\boxed{\nabla f(x,y) = f_x(x,y)\,i + f_y(x,y)\,j}$$

Therefore ∇f is a vector field and is called a <u>gradient vector field.</u>

If f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\boxed{\nabla f(x,y,z) = f_x(x,y,z)\,i + f_y(x,y,z)\,j + f_z(x,y,z)\,k}$$

A vector field F is called a <u>conservative vector field</u> if it is the gradient of some scalar function, that is, if there exists a function f s.t.

$$F = \nabla f.$$

In this situation f is called a <u>potential function</u> for F.

For instance, the gravitational field F is conservative because if

$$f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$$

then

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = F(x,y,z).$$

Examples

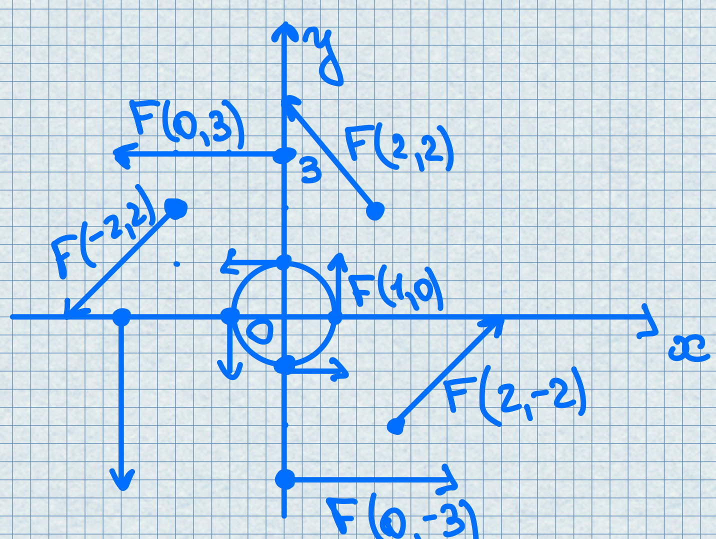
1. A vector field on $\mathbb{R}^2$ is defined by $F(x,y) = -yi + xj$. Describe F by sketching some of the vectors $F(x,y)$.

Solution

$F(1,0) = j$

$F(0,1) = -i$

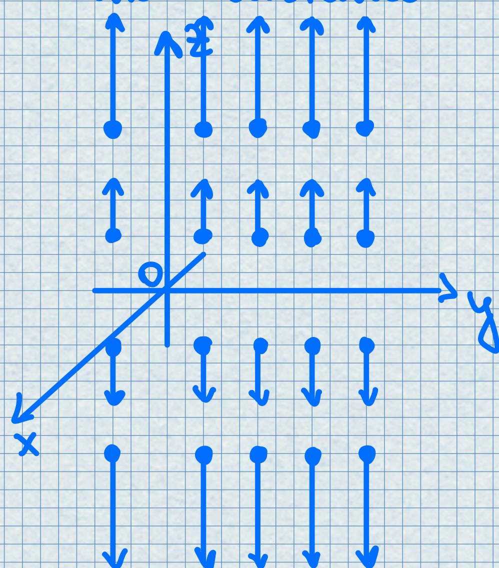
(x,y)	$F(x,y)$	(x,y)	$F(x,y)$
$(1,0)$	$\langle 0,1 \rangle$	$(-1,0)$	$\langle 0,-1 \rangle$
$(2,2)$	$\langle -2,2 \rangle$	$(-2,-2)$	$\langle 2,-2 \rangle$
$(3,0)$	$\langle 0,3 \rangle$	$(-3,0)$	$\langle 0,-3 \rangle$
$(0,1)$	$\langle -1,0 \rangle$	$(0,-1)$	$\langle 1,0 \rangle$
$(-2,2)$	$\langle -2,-2 \rangle$	$(2,-2)$	$\langle 2,2 \rangle$
$(0,3)$	$\langle -3,0 \rangle$	$(0,-3)$	$\langle 3,0 \rangle$



2. Sketch the vector field on $\mathbb{R}^3$ given by $F(x,y,z) = z\,k$.

Solution

All vectors are vertical and point upward above the xy-plane or downward below it. The magnitude increases with the distance from xy-plane.



3. Suppose an electric charge Q is located at the origin. According to Coulomb's Law, the electric force $F(x)$ exerted by this charge on a charge q located at a point (x,y,z) with position vector $x = \langle x,y,z \rangle$ is

$$F(x) = \frac{\varepsilon q Q}{|x|^3}\, x$$

where ε = constant.

For like charges, we have $qQ > 0$ and the force is repulsive;
For unlike charges, we have $qQ < 0$ and the force is attractive.
Both vector fields are examples of <u>force fields</u>.

Instead of considering the electric force F, physicists often consider the force per unit charge:

$$E(x) = \frac{1}{q} F(x) = \frac{\varepsilon Q}{|x|^3} x$$

Then E is a vector field on $\mathbb{R}^3$ called the <u>electric field</u> of Q.

▼

$\boxed{4.}$ Find the gradient vector field of

$$f(x,y) = x^2 y - y^3.$$

<u>Solution</u>

$$\nabla f(x,y) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j = 2xy\, i + (x^2 - 3y^2) j$$

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