WRH-9-Solutions

15.6: 4, 13, 27
15.7: 3, 17
15.8: 4, 9(a), 23

15.6

(4)
$$\int_0^1\int_y^{2y}\int_0^{x+y} 6xy\,dz\,dx\,dy = \int_0^1\int_y^{2y} 6xyz\Big|_0^{x+y} dx\,dy =$$

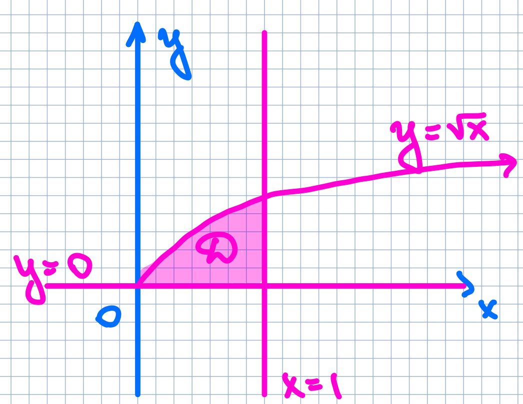
$$= \int_0^1\int_y^{2y} 6xy(x+y)\,dx\,dy = \int_0^1 \left(\frac{6x^3y}{3}+\frac{6x^2y^2}{2}\right)\Big|_y^{2y} dy$$

$$= \int_0^1 \left(2x^3y+3x^2y^2\right)\Big|_y^{2y} dy = \int_0^1 \left(16y^4+12y^4-2y^4-\right.$$

$$\left.-3y^4\right) dy = \int_0^1 23y^4\,dy = \frac{23}{5}y^5\Big|_0^1 = \boxed{\frac{23}{5}}$$

(13)
$$\iiint_E 6xy\,dV$$

E: $z=1+x+y$
$y=\sqrt{x}$
$y=0$
$x=1$



$$\iiint_E 6xy \, dz\, dy\, dx = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy \, dz\, dy\, dx =$$

$$= \int_0^1 \int_0^{\sqrt{x}} 6xyz \Big|_0^{1+x+y} dy\, dx =$$

$$= \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y)\, dy\, dx = \int_0^1 \int_0^{\sqrt{x}} (6xy + 6x^2y + 6xy^2)\, dy\, dx =$$

$$= 6\int_0^1 \left(\frac{xy^2}{2} + \frac{x^2y^2}{2} + \frac{xy^3}{3} \right) \Big|_0^{\sqrt{x}} dx =$$

$$= 6\int_0^1 \left(\frac{x^2}{2} + \frac{x^3}{2} + \frac{x^{5/2}}{3} \right) dx =$$

$$= 6\left(\frac{x^3}{6} + \frac{x^4}{8} + \frac{x^{7/2}}{3 \cdot 7/2} \right) \Big|_0^1 =$$

$$= 6\left(\frac{1}{6} + \frac{1}{8} + \frac{2}{14} \right) = \boxed{\frac{65}{28}}$$

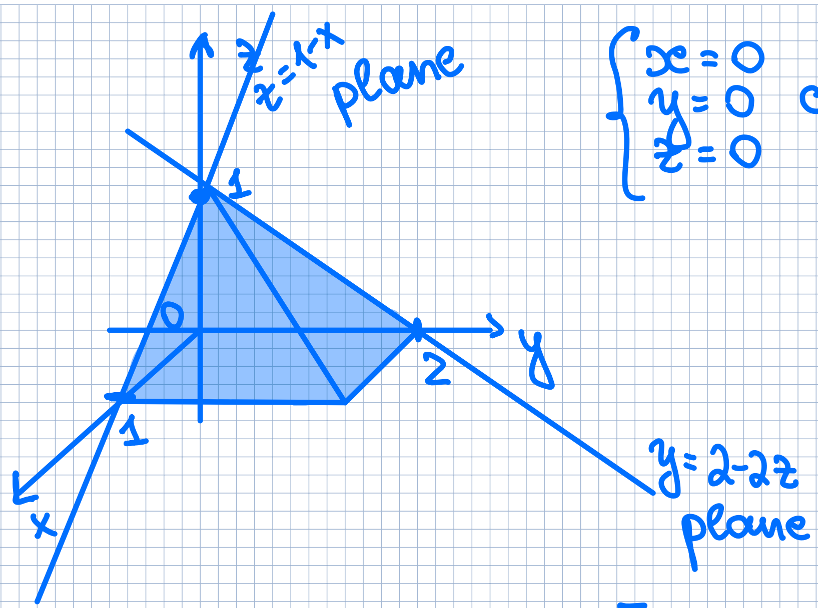
27

$$\int_0^1 \int_0^{1-x} \int_0^{2-2z} dy\, dz\, dx$$

$0 \le x \le 1$

$0 \le y \le 2-2z$

$0 \le z \le 1-x$

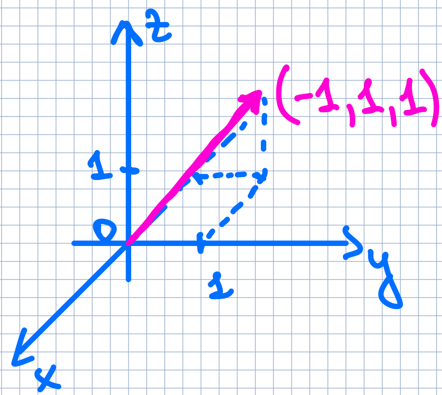


$$\begin{cases} x=0 \\ y=0 \\ z=0 \end{cases}$$ coordinate planes

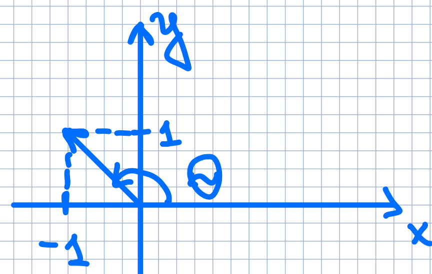
$\boxed{15.7}$

③ (a) $(-1,1,1)$

$$\begin{cases} x(t) = -1 \\ y(t) = 1 \\ z(t) = 1 \end{cases}$$



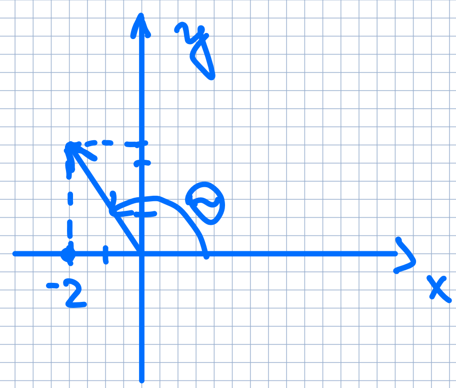
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



$$\begin{cases} 0 \leq r \leq \sqrt{2} \\ 0 \leq \theta \leq \frac{3\pi}{4} \\ 0 \leq z \leq 1 \end{cases}$$

$$\boxed{(r,\theta,z) = \left(\sqrt{2}, \frac{3\pi}{4}, 1\right)}$$

(8) $(-2, 2\sqrt{3}, 3)$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$r = \sqrt{4 + 4 \cdot 3} = 4$$

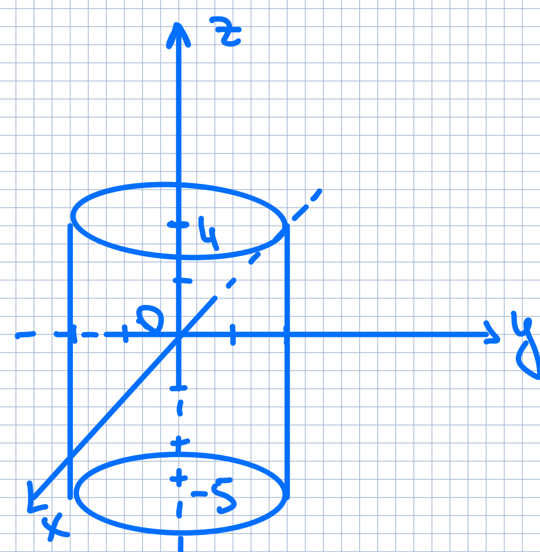
$$\tan\theta = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$$

$$\theta = \frac{2\pi}{3} + 2\pi n$$

$$z = 3$$

Thus, one set of cylindrical coordinates is $\boxed{\left(4, \frac{2\pi}{3}, 3\right)}$

17 $\iiint\limits_E \sqrt{x^2+y^2}\, dV =$



E: $x^2+y^2 = 16$

$z = -5$

$z = 4$

$$= \left[\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array} \qquad \begin{array}{l} x^2+y^2=r^2=16 \\ r=4 \\ \theta \in [0,2\pi] \\ -5 \le z \le 4 \end{array} \right] =$$

$$= \int_0^4 \int_0^{2\pi} \int_{-5}^{4} \sqrt{r^2}\, r \, dz\, d\theta\, dr =$$

$$= \int_0^4 \int_0^{2\pi} r^2 z \Big|_{-5}^{4} d\theta\, dr = \int_0^4 \int_0^{2\pi} 9 r^2 d\theta\, dr =$$

$$= 9 \cdot 2\pi \cdot \frac{r^3}{3} \Big|_0^4 = \frac{\cancel{9}^{3} \cdot 2\pi \cdot 64}{\cancel{3}} = 6\pi \cdot 64 =$$

$$= \boxed{384\pi}$$

▼

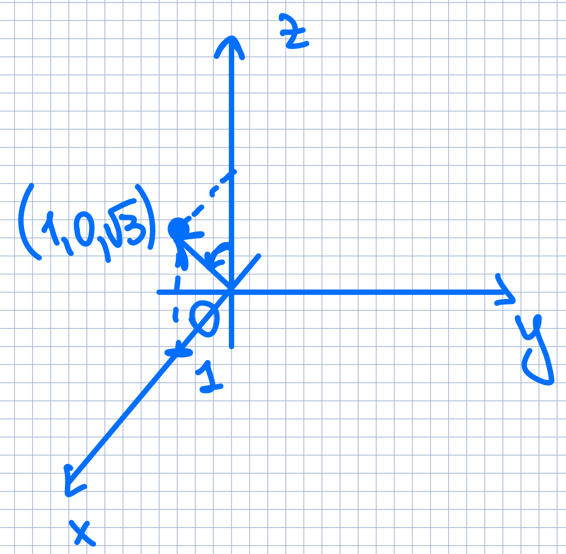
$\boxed{15.8}$

4 (a) $(1,0,\sqrt{3})$

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$$

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{1+0+3} = 2$$

$$\boxed{r=2}$$



$$\cos\varphi = \frac{z}{r} = \frac{\sqrt{3}}{2} \Rightarrow \varphi = \frac{\pi}{6}$$

$$\cos\theta = \frac{x}{r\sin\varphi} = \frac{1}{2\cdot\frac{1}{2}} = 1$$

$$\theta = 0$$

Thus spherical coordinates are

$$\boxed{\left(2, 0, \frac{\pi}{6}\right)}$$

(b) $(\sqrt{3}, -1, 2\sqrt{3})$

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$$

$$r=\sqrt{3+1+4\cdot 3}=4$$

$$\cos\varphi=\frac{2\sqrt{3}}{4}=\frac{\sqrt{3}}{2}$$

$$\varphi=\frac{\pi}{6}$$

$$\cos\theta=\frac{\sqrt{3}}{4\cdot\sin\frac{\pi}{6}}=\frac{\sqrt{3}}{4\cdot\frac{1}{2}}=\frac{\sqrt{3}}{2}$$

$$\theta=\frac{11\pi}{6}\quad(\text{since } y<0)$$

Thus spherical coordinates are $\boxed{\left(4,\frac{11\pi}{6},\frac{\pi}{6}\right).}$

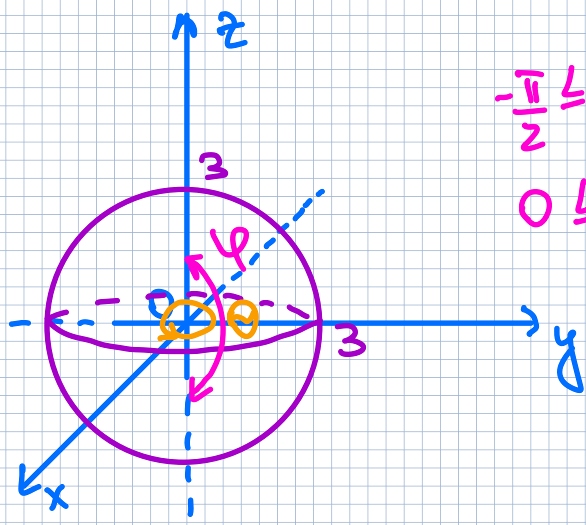
9(a) $x^2+y^2+z^2=9$

$$\begin{cases} x=r\sin\varphi\cos\theta \\ y=r\sin\varphi\sin\theta \\ z=r\cos\varphi \end{cases}$$

$$(r\sin\varphi\cos\theta)^2+(r\sin\varphi\sin\theta)^2+(r\cos\varphi)^2=r^2=9$$

$$\boxed{r=3}$$

$x^2+y^2+z^2=9$ is a sphere of a radius $r=3$ and center $(0,0,0)$.



$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$

$0 \le \theta \le 2\pi$

The equation is $\boxed{r=3}$

23 $\displaystyle\iiint_E (x^2+y^2)\,dV$

$E:\ x^2+y^2+z^2=4$

$x^2+y^2+z^2=9$

$E=\{(r,\theta,\varphi)\mid 2\le r\le 3, 0\le\theta\le 2\pi, 0\le\varphi\le\pi\}$

$x^2+y^2=(r\sin\varphi\cos\theta)^2+(r\sin\varphi\sin\theta)^2=$

$=r^2\sin^2\varphi.$

Thus $\displaystyle\iiint_E (x^2+y^2)\,dV=\int_0^{\pi}\int_0^{2\pi}\int_2^3 r^2\sin^2\varphi\, r^2\sin\varphi\,dr\,d\theta\,d\varphi=$

$$= \int_0^{\pi} \sin^3 \varphi \, d\varphi \int_0^{2\pi} d\theta \int_2^3 r^4 \, dr = \int_0^{\pi} (1-\cos^2\varphi)\sin\varphi \, d\varphi \, \cdot$$

$$\cdot \; \theta \Big|_0^{2\pi} \; \frac{1}{5} r^5 \Big|_2^3 = \boxed{\frac{1688\pi}{15}}$$