Lecture #14 - Week 5 - Partial Derivatives - 14.3

Let $z = f(x,y)$.

Let us fix y by setting $y=b$. Then

$$f = f(x,b) = g(x)$$

If g has a derivative at a, then we call it the partial derivative of f with respect to x at (a,b) and denote it by $f_x(a,b)$.

Hence,

$$\boxed{f_x(a,b) = g'(a) \quad \text{where} \quad g(x) = f(x,b)}$$

$$g'(a) = \lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$$

So

$$\boxed{f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}}$$

Similarly,

$$\boxed{f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}}$$

<u>**Def.**</u> If f is a function of two variables, its <u>partial derivatives</u> are f_x, f_y denoted by

- $f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$

- $f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$

Notations for Partial Derivatives

If $z = f(x,y)$, we write

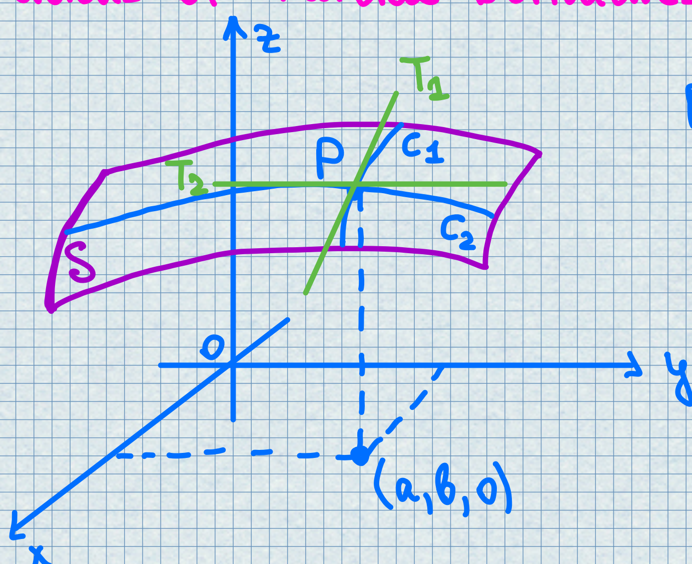
$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for finding Partial Derivatives of $z = f(x,y)$

1. To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

- Interpretations of Partial Derivatives.



$P(a,b,c)$

$z = f(x,y)$ represents a surface S.

If $f(a,b) = c$, then $P(a,b,c)$ lies on S.

By fixing $y = b$, we are restricting

our attention to the curve C_1 in which the vertical plane $y=b$ intersects S.
- " - " - $x=a$ intersects S in a curve C_2.
Both C_1 and C_2 pass through the point P.

The curve C_1 is the graph of the function $g(x) = f(x,b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a,b)$.

The curve C_2 is the graph of the function $G(y) = f(a,y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a,b)$.

Thus, $f_x(a,b)$ and $f_y(a,b)$ are slopes of the tangent lines at $P(a,b,c)$ to the traces C_1 and C_2 of S in the planes $y=b$ and $x=a$.

- If $z = f(x,y)$, then

  $\frac{\partial z}{\partial x}$ is the rate of change of z with respect to x when y is fixed

  $\frac{\partial z}{\partial y}$ is the rate of change of z with respect to y when x is fixed

- Functions of more than two variables.

<u>Def.</u> If $f = f(x,y,z)$, then

$$f_x(x,y,z) = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

If $w = f(x,y,z)$, then $f_x = \frac{\partial w}{\partial x}$ is the rate of change of w with respect to x when y and z are fixed.

In general, if $u = f(x_1, \dots, x_n)$, then

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

- Higher derivatives

Second partial derivatives of $z = f(x,y)$:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

Clairaut's Theorem

Suppose f is defined on disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$\boxed{f_{xy}(a,b) = f_{yx}(a,b)}$$

- Partial Differential Equations

Ex.

① $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0$$ (Laplace's equation)

Solutions of this equation are harmonic functions.

② $$\frac{\partial^2 u}{\partial t^2} = a \frac{\partial^2 u}{\partial x^2}$$ (Wave equation)

This equation describes the motion of a waveform, which could be an ocean wave, a sound wave, etc.

Examples

$\boxed{1.}$ If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$, $f_y(2,1)$.

Solution

$f_x(x,y) = 3x^2 + 2xy^3$

$f_y(x,y) = 3y^2x^2 - 4y$

$f_x(2,1) = 3\cdot 4 + 4 = 16$

$f_y(2,1) = 12 - 4 = 8$

$\boxed{2.}$ If $f(x,y) = 4 - x^2 - 2y^2$, find $f_x(1,1)$ and $f_y(1,1)$ and interpret these numbers as slopes.

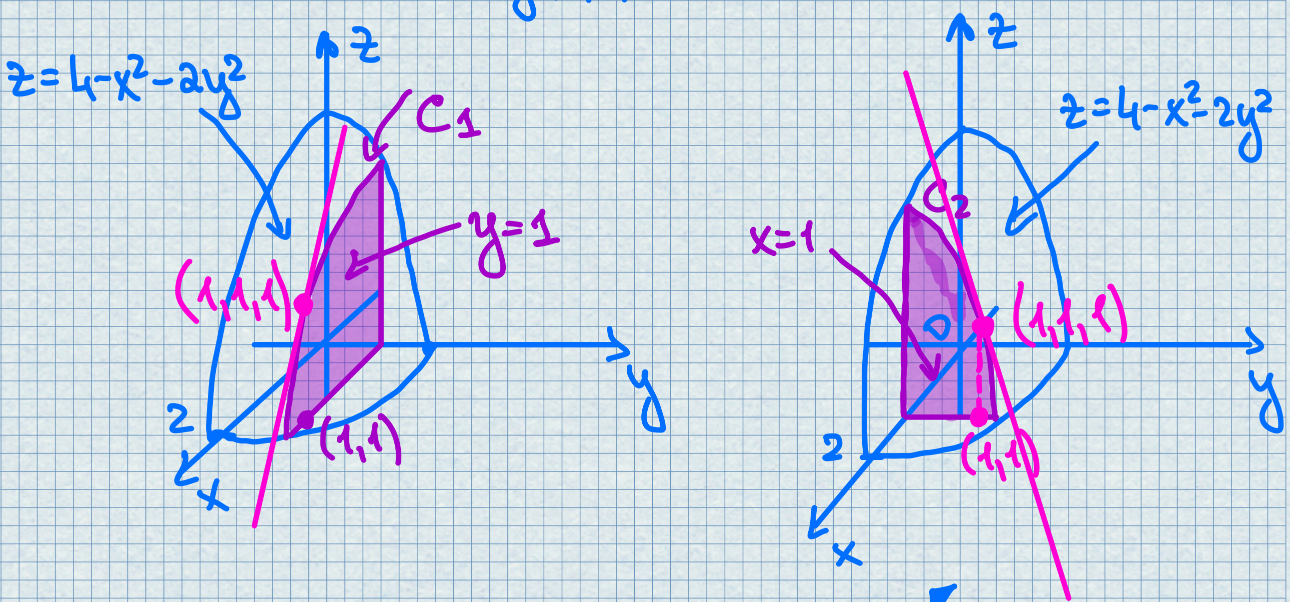
Solution

$f_x(x,y) = -2x \qquad f_x(1,1) = -2$

$f_y(x,y) = -4y \qquad f_y(1,1) = -4$

$z = 4 - x^2 - 2y^2$ is a paraboloid.
$y = 1$ intersects it in the parabola
$z = 2 - x^2$, $y = 1$.
The slope of the tangent line to this parabola at $(1,1,1)$ is $f_x(1,1) = -2$.

Similarly, $x=1$ intersects the paraboloid and we get the parabola $z=3-2y^2$, $x=1$, and the slope of the tangent line at $(1,1,1)$ is $f_y(1,1)=-4$.



$\boxed{3.}$ If $f(x,y)=\sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution

$$\frac{\partial f}{\partial x}=\cos\left(\frac{x}{1+y}\right)\cdot\frac{1}{1+y}$$

$$\frac{\partial f}{\partial y}=\cos\left(\frac{x}{1+y}\right)\cdot x\cdot\frac{-1}{(1+y)^2}.$$

$\boxed{4.}$ Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$x^3 + y^3 + z^3 + 6xyz = 1.$$

Solution

$\frac{\partial z}{\partial x}$: $\quad 3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} =$

$= 0$

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}.$$

Similarly,

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}.$$

▼

$\boxed{5.}$ Find f_x, f_y, f_z if $f(x,y,z) = e^{xy} \ln z$.

Solution

$$f_x = e^{xy} \cdot y \cdot \ln z$$

$$f_y = e^{xy} \cdot x \ln z$$

$$f_z = e^{xy} \frac{1}{z}.$$

▼

$\boxed{6.}$ Find the second partial derivatives of

$$f(x,y) = x^3 + x^2y^3 - 2y^2.$$

Solution

$$f_x = 3x^2 + 2xy^3$$
$$f_y = 3y^2x^2 - 4y$$
$$f_{xx} = 6x + 2y^3$$
$$f_{yy} = 6yx^2 - 4$$
$$f_{xy} = 6xy^2 = f_{yx} = 6y^2x$$

$\boxed{7.}$ Calculate f_{xxyz} if $f(x,y,z) = \sin(3x + yz)$.

Solution

$$f_x = \cos(3x + yz) \cdot 3$$
$$f_{xx} = -\sin(3x + yz) \cdot 9$$
$$f_{xxy} = -\cos(3x + yz) \cdot 9 \cdot z$$
$$f_{xxyz} = \sin(3x + yz) \cdot 9zy - 9\cos(3x + yz).$$

$\boxed{8.}$ Show that the function $u(x,y) = e^x \sin y$

is a solution of Laplace's equation.

Solution

$$u_{xx} + u_{yy} = 0$$ Laplace's equation

$$u_x = e^x \sin y$$

$$u_{xx} = e^x \sin y$$

$$u_y = e^x \cos y$$

$$u_{yy} = -e^x \sin y$$

Hence,

$$u_{xx} + u_{yy} = 0.$$