Lecture #30 - Week 10 - Line Integrals - 16.2

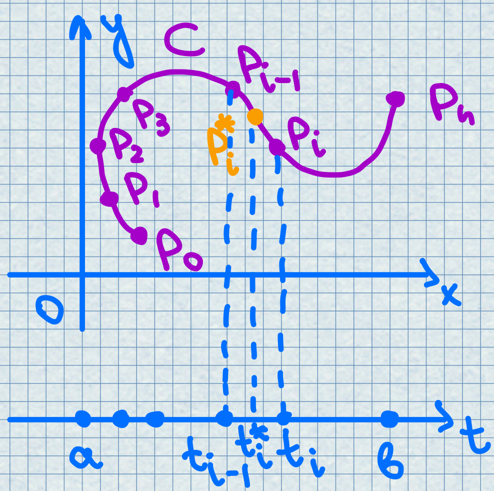
We start with a plane curve C given by parametric equations

$$\boxed{x = x(t) \quad y = y(t) \quad z = z(t)} \quad (1)$$

or

$$r(t) = x(t)\,i + y(t)\,j + z(t)\,k$$

and we assume that C is a smooth curve.



We divide the parameter interval $[a,b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$, $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \dots, \Delta s_n$.

We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc.

Now if f is any function of two variables whose domain includes the curve C, we evaluate f at the point (x_i^*, y_i^*), multiply by the length Δs_i of the subarc, and form the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

Def. 1 If f is defined on a smooth curve C given by (1), then the <u>line integral of f along C</u> is

$$\boxed{\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i} \qquad (2)$$

if this limit exists.

We know that the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

If f is a continuous function, then the limit in Def. 1 always exists and the following formula can be used:

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt} \qquad (3)$$

If $s(t)$ is the length of C between $r(a)$ and $r(t)$, then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

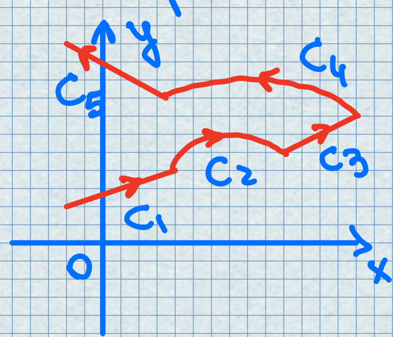
So

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- If $C:\ x = x$, $y = 0$, $a \le x \le b$ then $\displaystyle\int_C f(x,y)\,ds = \int_a^b f(x,0)\,dx$

Def. 2

Suppose that C is a piecewise-smooth curve; that is, C is a union of a finite number of smooth curves $C_1, C_2, \dots, C_n$, where the initial point of C_{i+1} is the terminal point of C_i.



Then we define the integral of f along C as the sum of integrals of f along each of the smooth pieces of C:

$$\boxed{\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \dots + \int_{C_n} f(x,y)\,ds.} \quad (4)$$

- Any phisical interpretation of a line integral $\int_C f(x,y)\,ds$ depends on the physical interpretation of the function f.

Suppose that $\rho(x,y)$ is the linear density at a point (x,y) of a thin wire shaped like a curve C. Then the mass of the part of the wire from P_{i-1} to P_i is approximately $\rho(x_i^*, y_i^*)\Delta s_i$ and so the total mass of the wire is approximately $\sum \rho(x_i^*, y_i^*)\Delta s_i$.

Hence, the <u>mass</u> m of the wire as the limiting value of these approximations is:

$$\boxed{m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*)\Delta s_i = \int_C \rho(x,y)\,ds} \quad (5)$$

The <u>center of mass</u> of the wire with density ρ is located at the point $(\bar{x}, \bar{y})$, where

$$(6) \quad \boxed{\bar{x} = \frac{1}{m}\int_C x\,\rho(x,y)\,ds \qquad \bar{y} = \frac{1}{m}\int_C y\,\rho(x,y)\,ds}$$

Two other line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$.

They are called the line integrals of f along C with respect to x and y:

$$\boxed{\begin{aligned} \int_C f(x,y)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta x_i \\ \int_C f(x,y)\,dy &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta y_i \end{aligned}} \qquad (7)$$

Formulas (7) say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t:

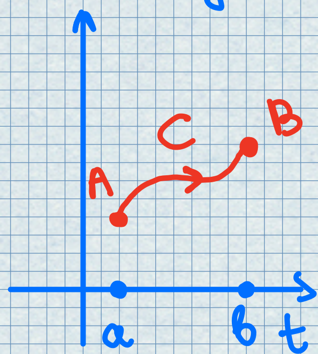
$$x = x(t) \qquad dx = x'(t)\,dt$$
$$y = y(t) \qquad dy = y'(t)\,dt$$

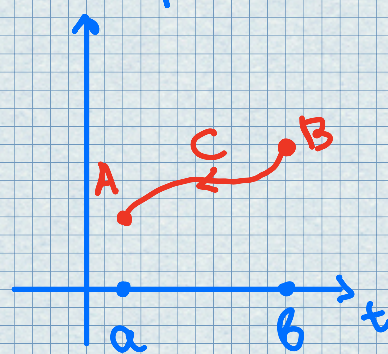
$$\boxed{\begin{aligned} \int_C f(x,y)\,dx &= \int_a^b f(x(t), y(t))\,x'(t)\,dt \\ \int_C f(x,y)\,dy &= \int_a^b f(x(t), y(t))\,y'(t)\,dt \end{aligned}} \qquad (8)$$

Def. The vector representation of the line segment that starts at r_0 and ends at r_1 is given by

$$\boxed{r(t) = (1-t)r_0 + t\,r_1, \quad 0 \le t \le 1}$$

In general, $x = x(t)$, $y = y(t)$, $a \le t \le b$, determines an orientation of a curve C, with the positive direction corresponding to increasing values of the parameter t.





$$\boxed{\int_{-C} f(x,y)\,dx = -\int_{C} f(x,y)\,dx}$$

$$\boxed{\int_{-C} f(x,y)\,dy = -\int_{C} f(x,y)\,dy}$$

* But if we integrate with respect to arc length, the value of the line integral does not change when we reverse the orientation of the curve:

$$\boxed{\int_{-C} f(x,y)\,ds = \int_{C} f(x,y)\,ds}$$

- Line Integrals in Space

We now suppose that C is smooth space curve given by

$$x = x(t) \quad y = y(t) \quad z = z(t) \quad a \le t \le b$$

or by

$$r(t) = x(t)i + y(t)j + z(t)k$$

If $f = f(x,y,z)$ is continuous on some region containing C, then the <u>line integral of f along C</u> is

$$\boxed{\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta s_i}$$

or

$$\boxed{\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt}$$

Line integrals along C with respect to $x, y,$ and z can also be defined.

$$\int_C f(x,y,z)\,dz = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta z_i$$

$$= \int_a^b f(x(t), y(t), z(t))\,z'(t)\,dt$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

$$\boxed{\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz}$$

by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t.

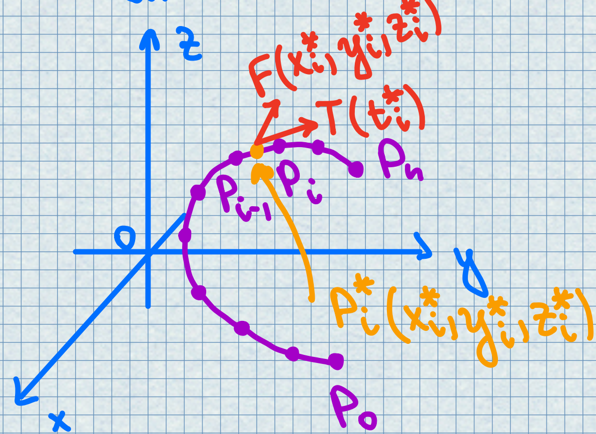
- Line Integrals of Vector Fields

Recall that the work done by a variable force $f(x)$ in moving a particle from a to b along the x-axis is

$$W = \int_a^b f(x)\,dx$$

$$W = F \cdot D, \quad D = \overrightarrow{PQ} \text{ (displacement vector)}$$

Now let $F = Pi + Qj + Rk$ is a continuous force field on $\mathbb{R}^3$.



$C: [P_{i-1}, P_i]$

$|P_{i-1}P_i| = \Delta s_i$

Choose $P_i^*(x_i^*, y_i^*, z_i^*) \in C_i$

If Δs_i is small, then as the particle moves from P_{i-1} to P_i along the curve, it proceeds approximately in the direction of $T(t_i^*)$, the unit tangent vector at P_i^*.

Thus the work done by the force F in moving the particle from P_{i-1} to P_i is ≈

$$\boxed{F(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \, T(t_i^*)] = [F(x_i^*, y_i^*, z_i^*) \cdot T(t_i^*)] \Delta s_i}$$

and the total work along C is

$$\boxed{\sum_{i=1}^{n} [F(x_i^*, y_i^*, z_i^*) \cdot T(x_i^*, y_i^*, z_i^*)] \Delta s_i}$$

where $T(x,y,z)$ is the unit tangent vector at (x,y,z) on C.

We get

$$\boxed{W = \int_C F(x,y,z) \cdot T(x,y,z) \, ds = \int_C F \cdot T \, ds}$$

If

$C: \; r(t) = x(t)i + y(t)j + z(t)k$, then

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

So

$$W = \int_a^b \left[F(r(t)) \cdot \frac{r'(t)}{|r'(t)|} \right] |r'(t)| \, dt$$

$$= \int_a^b F(r(t)) \cdot r'(t) \, dt$$

<u>**Def.**</u> Let F be a continuous vector field defined on a smooth curve C given

by a vector function $r(t)$, $a \le t \le b$.
Then the line integral of F along C is

$$\boxed{\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt = \int_C F \cdot T\, ds}$$

Suppose the vector field F on $\mathbb{R}^3$ is given in component form by the equation

$$F = Pi + Qj + Rk$$

Then the line integral along C is

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) dt = \int_a^b (Pi + Qj + Rk) \cdot$$

$$\cdot (x'(t)i + y'(t)j + z'(t)k)\, dt =$$

$$= \int_a^b \Big(P(x(t), y(t), z(t))\, x'(t) + Q(x(t), y(t), z(t))\, y'(t) + R(x(t), y(t), z(t)) \cdot$$

$$\cdot z'(t) \Big)\, dt$$

or

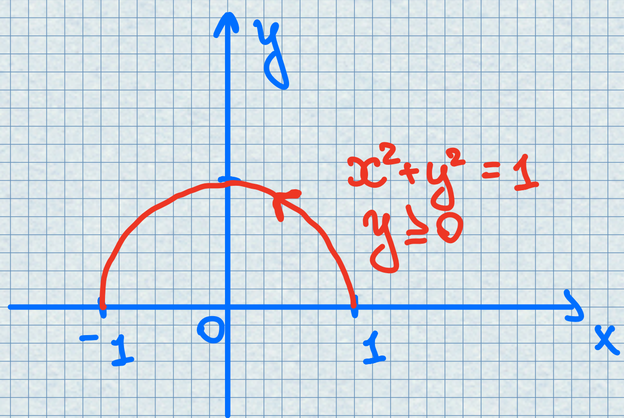
$$\boxed{\int_C F \cdot dr = \int_C P\,dx + Q\,dy + R\,dz \quad \text{where } F = Pi + Qj + Rk}$$

Examples

$\boxed{1.}$ Evaluate $\int_C (2+x^2y)\,ds$, where C is the upper half of the unit circle $x^2+y^2=1$.

Solution

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$



So

$$x=\cos t \qquad y=\sin t$$

$$0\leq t\leq \pi$$

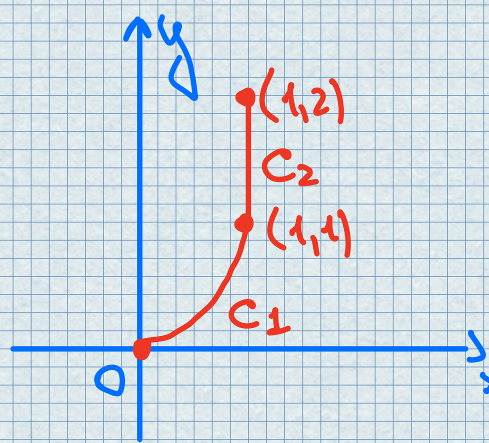
Therefore,

$$\int_C (2+x^2y)\,ds = \int_0^{\pi} (2+\cos^2 t\sin t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt =$$

$$= \int_0^{\pi} (2+\cos^2 t\sin t)\sqrt{\sin^2 t+\cos^2 t}\,dt =$$

$$= \int_0^{\pi} \left(2 + \cos^2 t \sin t \right) dt = \left(2t - \frac{\cos^3 t}{3} \right) \Big|_0^{\pi} = 2\pi + \frac{2}{3}.$$

$\boxed{2.}$ Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of $y = x^2$ from $(0,0)$ to $(1,1)$ followed by a vertical line segment C_2 from $(1,1)$ to $(1,2)$.



Solution

C_1: $\quad x = x$
$\quad y = x^2$
$\quad 0 \le x \le 1$

Therefore

$C = C_1 \cup C_2$

$$\int_{C_1} 2x\,ds = \int_0^1 2x \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2x\sqrt{1+4x^2}\,dx =$$

$$= \frac{1}{4} \cdot \frac{2}{3} \left(1 + 4x^2\right)^{3/2} \Big|_0^1 = \frac{5\sqrt{5} - 1}{6}$$

C_2: $\quad x = 1$
$\quad y = y \quad 1 \le y \le 2$

Therefore

$$\int_{C_2} 2x\,ds = \int_1^2 2\cdot 1\sqrt{\left(\frac{dx}{dy}\right)^2+\left(\frac{dy}{dy}\right)^2}\,dy =$$

$$= \int_1^2 2\,dy = 2.$$

Thus

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds = \frac{5\sqrt{5}-1}{6} + 2.$$

▼

$\boxed{3.}$ Evaluate $\int_C y^2\,dx + x\,dy$, where

(a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$

(b) $C = C_2$ is the arc of the parabola $x = 4-y^2$ from $(-5,-3)$ to $(0,2)$.

Solution

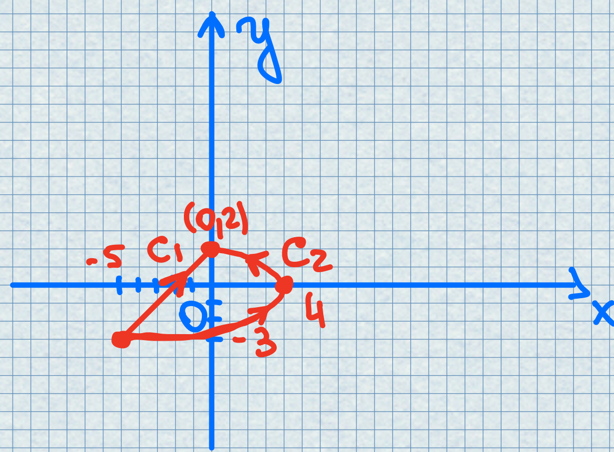
(a) $\quad r(t) = (1-t)r_0 + t r_1 \qquad 0 \le t \le 1$

$r_0 = \langle -5,-3\rangle$

$r_1 = \langle 0,2 \rangle$

$r(t) = (1-t)\langle -5,-3 \rangle + t\langle 0,2 \rangle.$

$x(t) = 5t-5 \qquad y(t) = 5t-3 \qquad 0 \le t \le 1$



$dx = 5\,dt$

$dy = 5\,dt$

$$\int_{C_1} y^2\,dx + x\,dy =$$

$$= \int_0^1 (5t-3)^2(5\,dt) + (5t-5)(5\,dt) = 5\int_0^1 (25t^2-25t+4)\,dt =$$

$$= 5\left(\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right)\Big|_0^1 = -\frac{5}{6}.$$

(b)

$C_2: \quad x = 4 - y^2$

$y = y$

$-3 \le y \le 2$

Then $\quad dx = -2y\,dy$

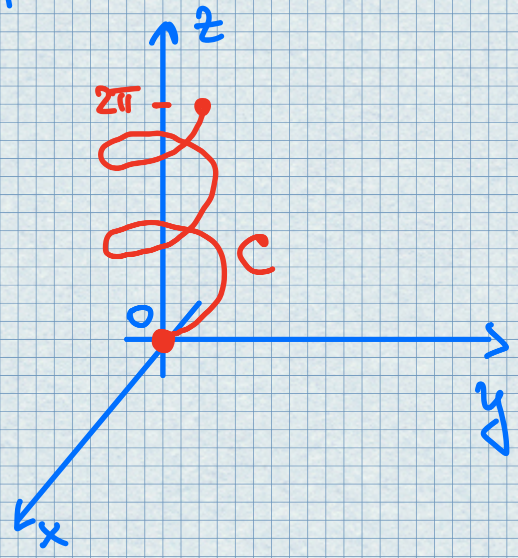
$$\int_{C_2} y^2 dx + x\,dy = \int_{-3}^{2} y^2(-2y)dy + (4-y^2)dy =$$

$$= \int_{-3}^{2} (-2y^3 - y^2 + 4)\,dy = \left(-\frac{y^4}{2} - \frac{y^3}{3} + 4y\right)\Big|_{-3}^{2} = 40\frac{5}{6}.$$

$\boxed{4.}$ Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 2\pi$

Solution

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$



$$\int_C y \sin z \, ds = \int_0^{2\pi} (\sin t) \sin t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt =$$

$$= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1}\, dt = \sqrt{2} \int_0^{2\pi} \frac{1}{2}(1 - \cos 2t)\, dt =$$

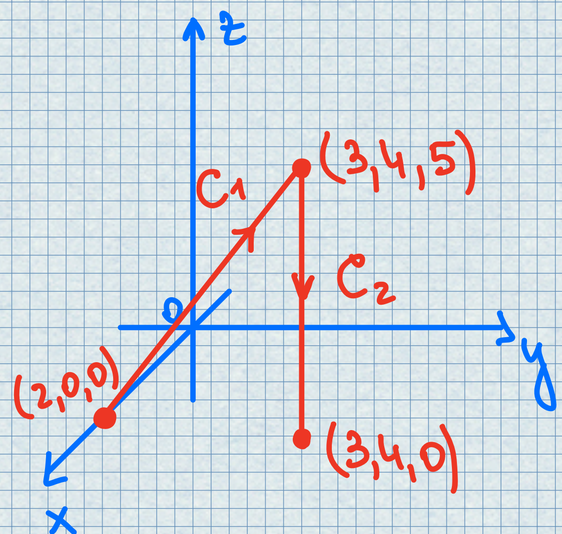
$$= \frac{\sqrt{2}}{2}\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi} = \sqrt{2}\,\pi.$$

▼

$\boxed{5.}$ Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where

C consists of the line segment C_1 from $(2,0,0)$ to $(3,4,5)$, followed by the vertical line segment C_2 from $(3,4,5)$ to $(3,4,0)$.

Solution



C_1: $r(t) = (1-t)\langle 2,0,0\rangle +$

$+ t\langle 3,4,5\rangle =$

$= \langle 2+t, 4t, 5t\rangle$

or

$$\begin{cases} x = 2+t \\ y = 4t \\ z = 5t \end{cases} \qquad 0 \le t \le 1$$

Thus
$$\int_{C_1} y\,dx + z\,dy + x\,dz = \int_0^1 (4t)\,dt + (5t)4\,dt +$$
$$+ (2+t)5\,dt = \int_0^1 (10 + 29t)\,dt = \left(10t + 29\frac{t^2}{2}\right)\Big|_0^1 =$$
$$= 24.5.$$

C_2: $r(t) = (1-t)\langle 3,4,5\rangle + t\langle 3,4,0\rangle =$
$= \langle 3, 4, 5-5t\rangle$

or
$$x = 3 \quad y = 4 \quad z = 5 - 5t \qquad 0 \le t \le 1$$

Then $dx = 0 = dy$, so
$$\int_{C_2} y\,dx + z\,dy + x\,dz = \int_0^1 3(-5)\,dt = -15.$$

Thus
$$\int_C y\,dx + z\,dy + x\,dz = 24.5 - 15 = 9.5$$

▶

$\boxed{6.}$ Find the work done by the force field $F(x,y) = x^2 i - xy j$ in moving a

particle along the quarter-circle
$r(t) = \cos t \, i + \sin t \, j, \quad 0 \le t \le \pi/2.$

Solution

$$x = \cos t \qquad y = \sin t$$

$$F(r(t)) = \cos^2 t \, i - \cos t \sin t \, j$$

$$r'(t) = -\sin t \, i + \cos t \, j$$

Therefore the work done is

$$\int_C F \cdot dr = \int_0^{\pi/2} F(r(t)) \cdot r'(t) \, dt = \int_0^{\pi/2} (-2\cos^2 t \sin t) \, dt =$$

$$= 2 \left. \frac{\cos^3 t}{3} \right|_0^{\pi/2} = -\frac{2}{3}.$$