Lecture #3 - Week 1 - The Dot Product - 12.3

Def. If $a = \langle a_1, a_2, a_3 \rangle$ and $b = \langle b_1, b_2, b_3 \rangle$, then the dot product of a and b is the number $a \cdot b$ given by

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

In $\mathbb{R}^2$:

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

Properties of the Dot Product

If a, b and c are vectors in V_3 and α is a scalar, then

1. $a \cdot a = |a|^2$
2. $a \cdot b = b \cdot a$
3. $a \cdot (b + c) = a \cdot b + a \cdot c$
4. $(\alpha a) \cdot b = \alpha (a \cdot b) = a \cdot (\alpha b)$
5. $\vec{0} \cdot a = 0$

Theorem

If θ is the angle between the vectors a and b, then

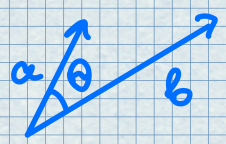
$$a \cdot b = |a|\,|b| \cos\theta$$

Corollary If θ is the angle between the nonzero vectors a and b, then

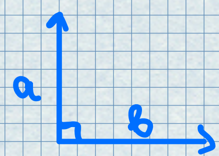
$$\cos\theta = \frac{a \cdot b}{|a|\,|b|}$$

Statement Two vectors a and b are orthogonal if and only if

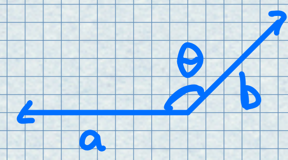
$$a \cdot b = 0$$



$a \cdot b > 0$
θ acute



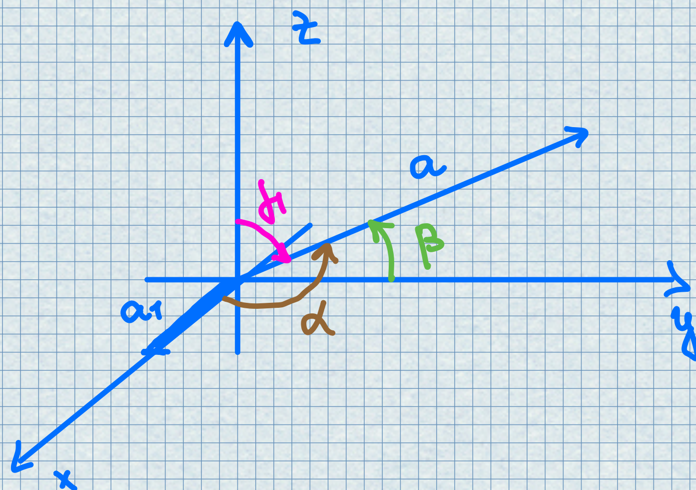
$a \cdot b = 0$
$\theta = \frac{\pi}{2}$



$a \cdot b < 0$
θ obtuse

- **Direction Angles and direction cosines**

Def. The direction angles of $a \neq 0$ are the angles $\alpha, \beta, \gamma \in [0, \pi]$ that a makes with the positive x-, y-, z-axes.



Def. $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are direction cosines of a vector a.

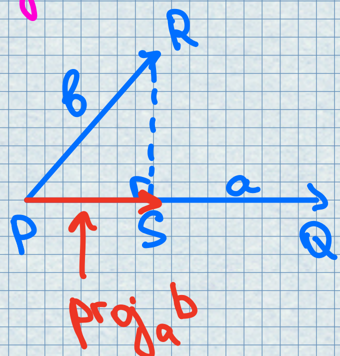
$$\cos\alpha = \frac{a \cdot i}{|a| \cdot |i|} = \frac{a_1}{|a|}$$

$$\cos\beta = \frac{a_2}{|a|} \qquad \cos\gamma = \frac{a_3}{|a|}$$

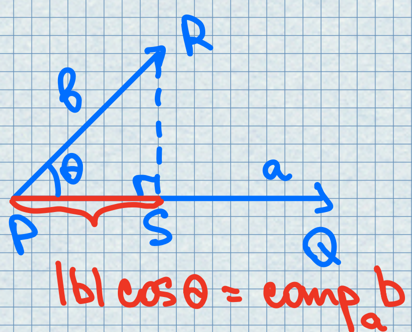
$$\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

$$\boxed{\langle \cos\alpha, \cos\beta, \cos\gamma \rangle = \frac{1}{|a|}\, a}$$

- **Projections**



$\overrightarrow{PS}$ is a <u>vector projection</u> of b onto a ($\mathrm{proj}_a b$).

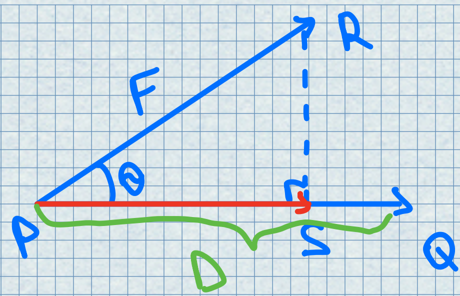


The <u>scalar projection</u> of b onto a (component of b onto a) is $|b|\cos\theta$, where $\theta = \angle(a,b)$.

Scalar proj. of b onto a : $\quad \mathrm{comp}_a b = \dfrac{a \cdot b}{|a|}$

Vector proj. of b onto a : $\quad \mathrm{proj}_a b = \dfrac{a \cdot b}{|a|^2}\, a$

- **Application**



$F = \overrightarrow{PR}$ (force vector)

$D = \overrightarrow{PQ}$ (displacement vector)

W - work

$$\boxed{W = (|F| \cos\theta)\,|D|}$$

$$\boxed{W = |F|\,|D| \cos\theta = F \cdot D}$$

Examples

$\boxed{1.}$ $\langle 2,4\rangle \cdot \langle 3,-1\rangle = 2\cdot 3 + 4(-1) = 6-4 = 2$

$(i+2j-3k)\cdot(2j-k) = 1(0) + 2(2) + (-3)(-1) = 7$

$\boxed{2.}$ If $|a| = 4$

$|b| = 6$

$\angle(a,b) = \varphi = \frac{\pi}{3}$

find $a\cdot b$.

Solution

$$a\cdot b = |a||b| \cos\left(\frac{\pi}{3}\right) = 4\cdot 6\cdot \frac{1}{2} = 12.$$

$\boxed{3.}$ Show that $2i+2j-k$ is perpendicular to $5i-4j+2k$.

Solution

$$(2i+2j-k)\cdot(5i-4j+2k) = 2\cdot 5 + 2\cdot(-4) + (-1)2 = 0.$$

$\boxed{4.}$ Find the direction angles of the vector $a = \langle 1,2,3\rangle$.

Solution

$$|a| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

$$\cos\alpha = \frac{1}{\sqrt{14}} \qquad \cos\beta = \frac{2}{\sqrt{14}} \qquad \cos\gamma = \frac{3}{\sqrt{14}}$$

So

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) \approx 14^\circ \qquad \gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) \approx 37^\circ$$

$$\beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) \approx 58^\circ$$

$\boxed{5.}$ Find the scalar projection and vector projection of $b = \langle 1,1,2 \rangle$, $a = \langle -2,3,1 \rangle$.

Solution

$$|a| = \sqrt{4+3^2+1^2} = \sqrt{14}$$

$$\text{comp}_a b = \frac{a \cdot b}{|a|} = \frac{(-2)\cdot 1 + 3 \cdot 1 + 1 \cdot 2}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

$$\text{proj}_a b = \frac{3}{\sqrt{14}} \frac{a}{|a|} = \frac{3}{14} a = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle . \blacktriangledown$$

$\boxed{6.}$ A force is given by a vector $F = 3i + 4j + 5k$ and moves a particle from the point $P(2,1,0)$ to the point $Q(4,6,2)$. Find the work done.

Solution

$$D = \overrightarrow{PQ} = \langle 2,5,2 \rangle \quad \text{- displacement vector.}$$

$$W = F \cdot D = \langle 3,4,5 \rangle \cdot \langle 2,5,2 \rangle = 6 + 20 + 10 = 36.$$

If the unit of length is meters and the magnitude of the force is measured in newtons, then the work done is 36 J.

$\blacktriangledown$