Lecture #7 - Week 3 - Vector Functions and Space Curves - 13.1

Def. A vector-valued function, or vector function, is a function whose domain is a set of real numbers and whose range is a set of vectors.

$$r = r(t) \in V_3$$

$$r(t) = \langle f(t), g(t), h(t) \rangle = f(t)i + g(t)j + h(t)k,$$

where
$f: \mathbb{R} \to \mathbb{R}$
$g: \mathbb{R} \to \mathbb{R}$
$h: \mathbb{R} \to \mathbb{R}$

- **Limits and Continuity**

If $r(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\boxed{\lim_{t \to a} r(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle}$$

provided the limits of the compo

Def. A vector function r is continuous at a if

$$\boxed{\lim_{t \to a} r(t) = r(a)}$$

- **Space curves**

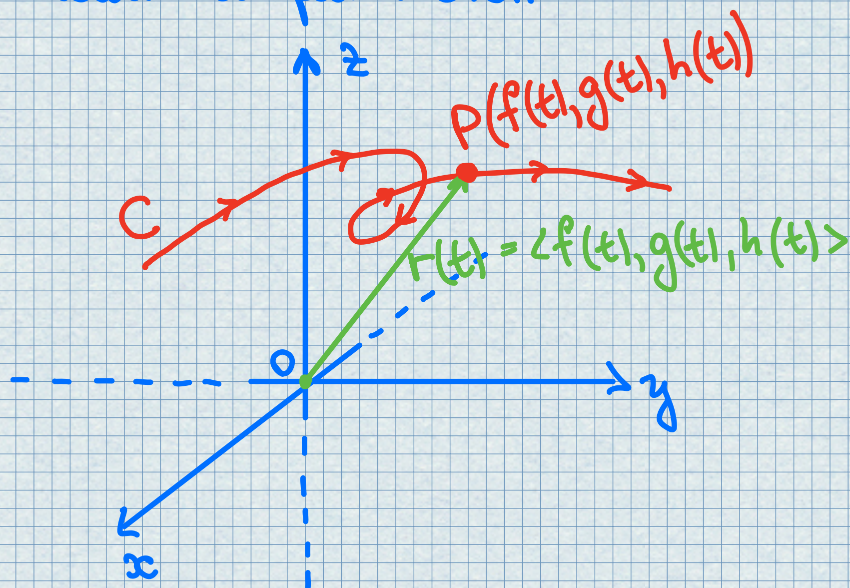
Suppose that f, g, h are continuous real-valued functions on an interval I.

Then the set C of all points (x, y, z) in space, where

$$\boxed{x = f(t), \ y = g(t), \ z = h(t)} \qquad (1)$$

and $t \in I$, is called a <u>space curve</u>

(1) is a parametric equation of C and t is called a parameter.



Example

$$r(t) = \langle 1+t, \ 2+5t, \ -1+6t \rangle$$

$$x(t) = 1+t, \quad y(t) = 2+5t, \quad z(t) = -1+6t$$

$$\boxed{\frac{x-1}{1} = \frac{y-2}{5} = \frac{z+1}{6}} = t$$

line which passes through the point $(1, 2, -1)$ and is $\parallel$ to $\langle 1, 5, 6 \rangle$.

Remark

Plane curves can also be represented in vector notation.

For instance, the curve given by

$$x = t^2 - 2t \quad \text{and} \quad y = t + 1$$

could be described by

$$r(t) = \langle t^2 - 2t, t + 1 \rangle = (t^2 - 2t)i + (t + 1)j$$

Examples

$\boxed{1.}$ If

$$r(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$$

then

$$x(t) = t^3 \qquad y(t) = \ln(3-t) \qquad z(t) = \sqrt{t}$$

$\mathrm{Dom}(y) = \{t \mid 3-t > 0\}$

$\mathrm{Dom}(x) = \mathbb{R}$

$\mathrm{Dom}(z) = \{t \mid t \geq 0\}$

Hence, $\mathrm{Dom}(r) = [0,3)$.

$\boxed{2.}$ Find $\lim_{t \to 0} r(t)$, where $r(t) = (1+t^3)i +$
$+ te^{-t}j + \frac{\sin t}{t}k$.

Solution

$$\lim_{t \to 0} r(t) = \left(\lim_{t \to 0} (1+t^3)\right)i + \left(\lim_{t \to 0} te^{-t}\right)j + \left(\lim_{t \to 0} \frac{\sin t}{t}\right)k = i + k.$$

$\boxed{3.}$ Describe the curve defined by the vector function

$$r(t) = \langle 1+t, 2+5t, -1+6t \rangle$$

Solution

Param. equations:

$$x(t) = t+1 \quad y(t) = 2+5t \quad z(t) = -1+6t$$

This is a line passing through $(1,2,-1)$ and parallel to $\langle 1,5,6 \rangle$.

We can also write $r(t)$ as

$$\boxed{r(t) = r_0 + t\,v, \quad r_0 = \langle 1,2,-1 \rangle \quad v = \langle 1,5,6 \rangle}$$

vector equation of the line

4. Sketch the curve whose vector equation is

$$r(t) = \cos t\, i + \sin t\, j + t k$$

Solution

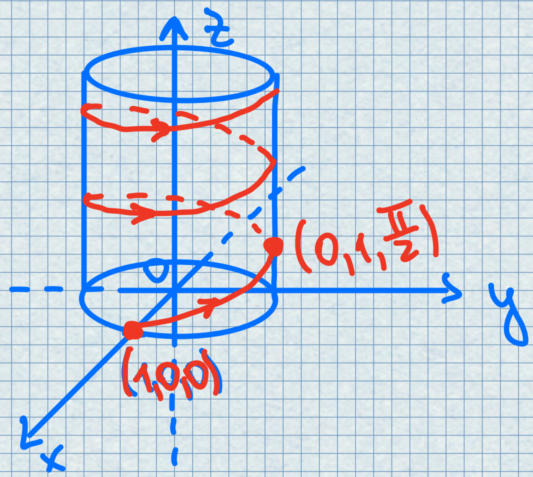
$$x = \cos t \quad y = \sin t \quad z = t$$

Since $x^2 + y^2 = 1$, $t \in \mathbb{R}$, the curve lies on the circular cylinder $x^2 + y^2 = 1$.

Since $z = t$, the curve spirals

upward around the cylinder as t increases.

We obtain a helix curve.



$\boxed{5.}$ Find a vector equation and parametric equations for the line segment that joins the point $P(1,3,-2)$ to the point $Q(2,-1,3)$.

Solution

A vector equation for the line segment that joins the tip of the vector r_0 to the tip of the vector r_1 is:

$$r(t) = (1-t)r_0 + t\,r_1 \,, \qquad 0 \le t \le 1$$

Here $\quad r_0 = \langle 1,3,-2 \rangle$

$\quad\quad\quad r_1 = \langle 2,-1,3 \rangle$

Hence

$$r(t) = (1-t)\langle 1,3,-2\rangle + t\langle 2,-1,3\rangle$$
$$0 \le t \le 1$$

or

$$r(t) = \langle 1+t,\ 3-4t,\ -2+5t\rangle,\quad 0\le t\le 1.$$

The corresponding parametric equations are

$$x = 1+t \qquad y = 3-4t \qquad z = -2+5t\ ,\ 0\le t\le 1.$$