

Name:

**Instructions.** (100 points) You have two hours. The exam is closed book, closed notes, and only simple calculators are allowed. Show all your work in order to receive full credit.

1. [6 points] Consider the limit

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}.$$

Either show it does not exist, or give strong evidence for suspecting it does.

$$\text{For } x=0: \lim_{y \rightarrow 0} \frac{-4y^2}{2y^2} = \boxed{-2}$$

$$\text{For } y=0: \lim_{x \rightarrow 0} \frac{x^4}{x^2} = \lim_{x \rightarrow 0} x^2 = \boxed{0}$$

$0 \neq -2 \Rightarrow$  The limit DNE.

2. [10 points] For the given function

$$f(x, y) = x^2y - y^2x$$

- (a) (5 pts) Use the chain rule to compute  $\frac{dg}{dt}(0)$ , where:

$$g(t) = f(\underbrace{t^2 + e^{2t}}_x, \underbrace{2t + 1}_y).$$

$$\frac{dg}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt}$$

$$\frac{dg}{dt} = (2xy - y^2) \cdot (2t + 2e^{2t}) + (x^2 - 2xy) \cdot 2$$

$$\frac{dg}{dt}(0) = (2 \cdot 1 - 1)(2) + (1 - 2) \cdot 2 = 2 - 2 = \boxed{0}$$

- (b) (5 pts) Give an equation for the linear (tangent plane) approximation to  $f$  at the point  $(1, -1)$ , and use it to estimate  $f(1.1, -0.9)$ .

$$L(x, y) = f(1, -1) + f_x(1, -1)(x-1) + f_y(1, -1)(y+1)$$

$$f_x = 2xy - y^2$$

$$f_y = x^2 - 2xy$$

$$f(1, -1) = -1 - 1 = -2$$

$$f_x(1, -1) = -3$$

$$f_y(1, -1) = 3$$

$$L(x, y) = -2 - 3(x-1) + 3(y+1) = -3x + 3y + 4$$

$$f(1.1, -0.9) \approx -3.3 - 2.7 + 4 = \boxed{-2}$$

3. [12 points] Evaluate the integral

$$\int_0^{\pi/2} \int_0^x \sin(y) dy dx$$

fully, by first drawing the region of integration, and then reversing the order of integration.

Handwritten solution for problem 3:

$$\begin{aligned} & \int_0^{\pi/2} \int_0^x \sin(y) dy dx = \\ &= \int_0^{\pi/2} x \cdot \sin y \Big|_0^{\pi/2} dy = \\ &= \int_0^{\pi/2} \left( \frac{\pi}{2} - y \right) \sin y dy = -\frac{\pi}{2} \cos y \Big|_0^{\pi/2} - \int_0^{\pi/2} y \sin y dy = \\ &= \frac{\pi}{2} + y \cdot \cos y \Big|_0^{\pi/2} - \sin y \Big|_0^{\pi/2} = \boxed{\frac{\pi}{2} - 1} \end{aligned}$$

4. [12 points] Find the classify (using the Second Derivative Test) all critical points of

$$f(x, y) = x^2 + xy + y^2 + y.$$

$$\begin{aligned} \begin{cases} f_x = 2x + y = 0 \\ f_y = 2y + x + 1 = 0 \end{cases} & \Rightarrow y = -2x \\ & \Rightarrow 2(-2x) + x + 1 = 0 \\ & \Rightarrow -4x + x + 1 = 0 \\ & \Rightarrow -3x = -1 \Rightarrow x = \frac{1}{3} \\ & \Rightarrow y = -\frac{2}{3} \end{aligned}$$

$$\text{C.P.: } \left( \frac{1}{3}, -\frac{2}{3} \right)$$

$$f_{xx} = 2 > 0$$

$$f_{yy} = 2$$

$$f_{xy} = f_{yx} = 1$$

$$\Rightarrow D = 2 \cdot 2 - 1 = 3 > 0$$

$$f_{xx} > 0 \text{ for all } (x, y) \in \text{Dom}(f)$$

Hence,  $f\left(\frac{1}{3}, -\frac{2}{3}\right)$  is a loc. minimum value.

5. [8 points] Give an equation for the tangent plane to the surface

$$\frac{xy}{z} + e^x \ln(z + 2y) = 2$$

at the point  $(2, 1, \frac{1}{3})$ .

$$F(x, y, z) = \frac{xy}{z} + e^x \ln(z + 2y) - 2$$

$$\nabla F = \left\langle \frac{y}{z} + e^x \ln(z + 2y), \frac{x}{z} + e^x \frac{1}{z + 2y} \cdot 2, -\frac{xy}{z^2} + e^x \frac{1}{z + 2y} \right\rangle$$

$$\begin{aligned} \nabla F(2, 1, \frac{1}{3}) &= \left\langle 1 + e^2 \ln 3, 2 + e^2 \cdot \frac{1}{\frac{1}{3} + 2} \cdot 2, -\frac{2}{\frac{1}{9}} + e^2 \frac{1}{\frac{1}{3} + 2} \right\rangle = \\ &= \left\langle 1 + e^2 \ln 3, 2 + \frac{2}{3}e^2, -2 + \frac{1}{3}e^2 \right\rangle \end{aligned}$$

Then, the TP:

$$(1 + e^2 \ln 3)(x - 2) + (2 + \frac{2}{3}e^2)(y - 1) + (-2 + \frac{1}{3}e^2)(z - \frac{1}{3}) = 0$$

6. [10 points] Use polar coordinates to find the volume of the solid bounded above by the paraboloid  $z = x^2 + y^2$  and below by the disk  $x^2 + y^2 \leq 25$ .

$$V = \iint_D f(x, y) dA \quad \text{①}$$

$$V = \iint_D (x^2 + y^2) dy dx$$

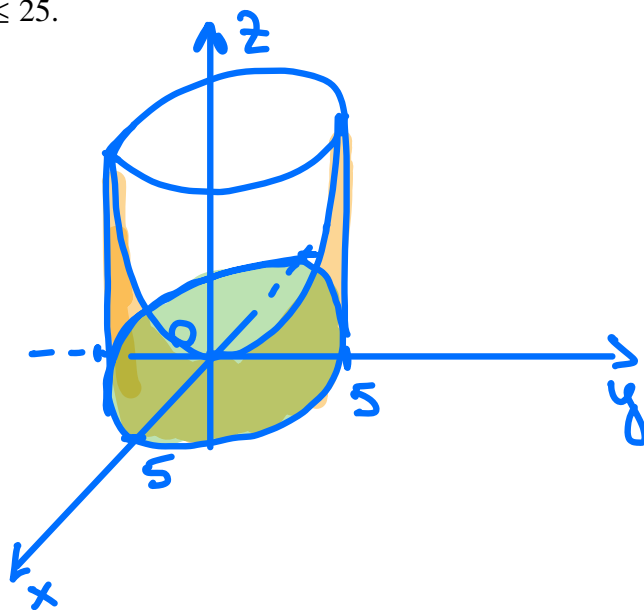
$$0 \leq r \leq 5$$

$$0 \leq \theta \leq 2\pi$$

$$dA = r dr d\theta$$

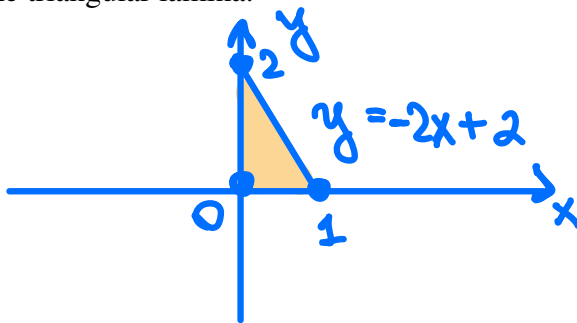
$$x = r \cos \theta \quad y = r \sin \theta$$

$$\text{②} \quad \int_0^{2\pi} \int_0^5 r^2 \cdot r dr d\theta = \int_0^{2\pi} \left[ \frac{r^4}{4} \right]_0^5 d\theta = \frac{5^4}{4} \cdot 2\pi = \boxed{\frac{625\pi}{2}}$$



7. [16 points] Find the mass and the center of mass of a triangular lamina with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 2)$  if the density function is  $\rho(x, y) = x + 2$ .

(a) [4 points] Draw the triangular lamina.



(b) [6 points] Use the formula

$$m = \iint_D \rho(x, y) dA$$

to find the mass of the lamina.

$$\begin{aligned} m &= \int_0^1 \int_0^{-2x+2} (x+2) dy dx = \int_0^1 (xy + 2y) \Big|_0^{-2x+2} dx = \\ &= \int_0^1 (x(-2x+2) - 4x+4) dx = \int_0^1 (-2x^2 - 2x + 4) dx = \\ &= \left( -\frac{2}{3}x^3 - x^2 + 4x \right) \Big|_0^1 = -\frac{2}{3} - 1 + 4 = \boxed{\frac{7}{3}} \end{aligned}$$

(c) [6 points] Use formulas

$$\bar{x} = \frac{1}{m} \iint_D x \rho(x, y) dA, \quad \bar{y} = \frac{1}{m} \iint_D y \rho(x, y) dA$$

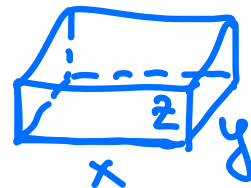
to find the coordinates of the center of mass of the lamina.

$$\begin{aligned} \bar{x} &= \frac{3}{7} \int_0^1 \int_0^{-2x+2} x(x+2) dy dx \\ \bar{y} &= \frac{3}{7} \int_0^1 \int_0^{-2x+2} y(x+2) dy dx \end{aligned}$$

8. [10 points] Use Lagrange multipliers to find the maximum and minimum values of ~~rectangular box whose surface area is 1500 cm<sup>2</sup>~~ <sup>of the box</sup> whose total edge length is 200 cm.

$$f(x, y, z) = A = 2xy + 2zy + 2xz$$

$$4x + 4y + 4z = 200$$



$$A(x, y, z) = 2xy + 2zy + 2zx$$

$$\text{Subj. to: } x + y + z = 50$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = k \end{cases}$$

$$\nabla f = 2\langle y+z, x+z, x+y \rangle$$

$$\nabla g = \langle 1, 1, 1 \rangle$$

$$\begin{cases} y+z = \frac{\lambda}{2} \\ x+z = \frac{\lambda}{2} \\ x+y = \frac{\lambda}{2} \\ x+y+z = 50 \end{cases}$$

$$2y + 2x + 2z = \frac{3\lambda}{2}$$

$$50 = \frac{3}{4}\lambda \Rightarrow \lambda = \frac{200}{3}$$

$$x + y + 2z = \lambda$$

$$50 - z + 2z = \frac{200}{3}$$

$$z = \frac{200}{3} - 50 = \frac{200 - 150}{3} = \frac{50}{3}$$

$$y = \frac{100}{3} - \frac{50}{3} = \frac{50}{3}$$

$$x = \frac{50}{3}$$

$$P\left(\frac{50}{3}, \frac{50}{3}, \frac{50}{3}\right)$$

$$0 \leq x \leq \frac{50}{3}$$

$$0 \leq y \leq \frac{50}{3}$$

$$0 \leq z \leq \frac{50}{3}$$

Hence, the maximum surface

area is  $A = 2 \cdot \frac{50^2}{3} = \frac{2 \cdot 25 \cdot 100}{3} = \frac{5000}{3} \text{ cm}^2$

9. [16 points] For the given function  $f(x, y) = y^2 e^{xy}$ , the point  $P(0, 1)$ , and the directional vector  $u = \langle 3/5, 4/5 \rangle$

(a) [5 points] Find the gradient of  $f$  at the point  $P$ .

$$\nabla f = \langle y^3 e^{xy}, 2y e^{xy} + xy^2 e^{xy} \rangle$$

$$\nabla f(0, 1) = \langle 1, 2 \rangle$$

(b) [5 points] Find the rate of change of  $f$  at  $P$  in the direction of the vector  $u$ .

$$\nabla f \cdot u = \langle 1, 2 \rangle \cdot \langle 3/5, 4/5 \rangle = \boxed{\frac{11}{5}}$$

(c) [6 points] Fully set up bounds and integrand for computing the **surface area** of  $f$  over the region  $[-1, 1] \times [-1, 2]$ . DO NOT EVALUATE.

$$A(S) = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1} \, dA$$

$$f_x = y^3 e^{xy}$$

$$f_y = 2y e^{xy} + xy^2 e^{xy}$$

$$A(S) = \int_{-1}^1 \int_{-1}^2 \sqrt{(y^3 e^{xy})^2 + (2y e^{xy} + xy^2 e^{xy})^2 + 1} \, dA.$$
