Lecture #33 - Week 11 - Curl and Divergence - 16.5

- Curl

If $F = Pi + Qj + Rk$ is a vector field on $\mathbb{R}^3$ and $P_x, P_y, Q_x, Q_y, R_y, R_x$ exist, then

$$\boxed{\operatorname{curl} F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)k} \quad (1)$$

We have

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

Then

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} =$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)k = \operatorname{curl} F$$

Thus

$$\boxed{\operatorname{curl} F = \nabla \times F} \quad (2)$$

Theorem If f is a function of three variables that has continuous second-order partial derivatives, then

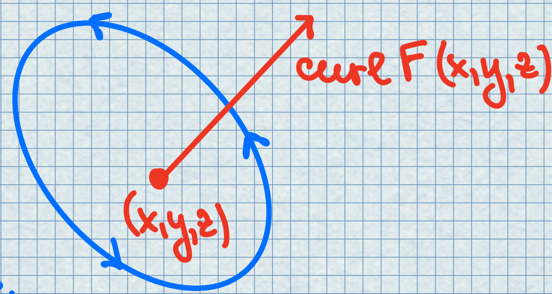
$$\boxed{\operatorname{curl}(\nabla f) = 0}$$

Since a conservative vector field is one for which $F = \nabla f$, then

$$\boxed{\text{If } \mathbf{F} \text{ is conservative, then } \operatorname{curl} \mathbf{F} = \mathbf{0}}$$

Theorem If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl} \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

The reason for the name curl is that the curl vector is associated with rotations.



- **Divergence**

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ exist, then

$$\boxed{\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}} \qquad (1)$$

Note

The $\operatorname{curl} \mathbf{F}$ is a vector field but $\operatorname{div} \mathbf{F}$ is a scalar field.

Also

$$\boxed{\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F}} \qquad (2)$$

Theorem If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\boxed{\text{div curl } F = 0}$$

The reason for the name divergence can be understood in the context of fluid flow.

If $F(x,y,z)$ is the velocity of a fluid, then div $F(x,y,z)$ represents the net rate of change (w.r. to t) of the mass of fluid flowing from the point (x,y,z) per unit volume.

If div $F = 0$, then F is said to be incompressible.

$$\text{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 = \nabla \cdot \nabla$$

∇^2 is called the Laplace operator.

$$\boxed{\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0}$$ Laplace's equation

We can also apply the Laplace operator ∇^2 to a vector field

$$F = Pi + Qj + Rk$$

in terms of its components:

$$\nabla^2 F = \nabla^2 P i + \nabla^2 Q j + \nabla^2 R k.$$

- Vector Forms of Green's Theorem

Let the plane region D, its boundary curve C, and the functions P and Q satisfy the hypotheses of Green's Theorem.

Then we consider $F = Pi + Qj$.

Its line integral is

$$\oint_C F \cdot dr = \oint_C P\,dx + Q\,dy$$

and

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k$$

Therefore,

$$(\operatorname{curl} F) \cdot k = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k \cdot k = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and the Green's Theorem can be written in the vector form

$$\boxed{\oint_C F \cdot dr = \iint_D (\operatorname{curl} F) \cdot k\, dA} \qquad (3)$$

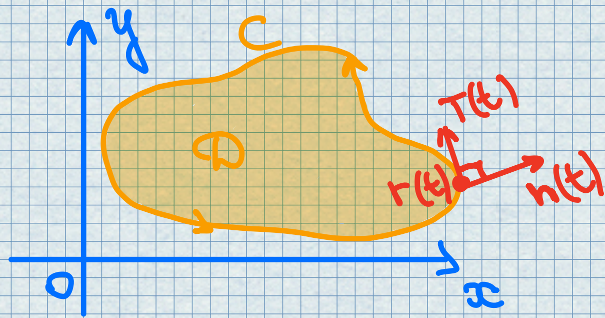
(3) expreses the line integral of the tangential component of F along C as the double integral of the vertical component of curl F over the region D

enclosed by C.

If $C: \ r(t) = x(t)i + y(t)j, \quad a \le t \le b$

then the unit tangent vector is

$$T(t) = \frac{x'(t)}{|r'(t)|} i + \frac{y'(t)}{|r'(t)|} j$$



$$n(t) = \frac{y'(t)}{|r'(t)|} i - \frac{x'(t)}{|r'(t)|} j$$

Then

$$\oint_C F \cdot n \, ds = \int_a^b (F \cdot n)(t) \, |r'(t)| \, dt =$$

$$= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

So we have a second vector form of Green's Theorem:

$$\boxed{\oint_C F \cdot n \, ds = \iint_D \operatorname{div} F(x,y) \, dA} \qquad (4)$$

(4) says that the line integral of the normal component of F along C is equal to the double integral of the divergence of F over the region D enclosed by C.

Examples

$\boxed{1.}$ If $F(x,y,z) = xz\,i + xyz\,j - y^2 k$, find curl F.

Solution

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} =$$

$$= \left(\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right)i - \left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right)j +$$

$$+ \left(\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right)k = (-2y - xy)i - (0 - x)j + (yz - 0)k$$

$$= -y(2+x)i + xj + yzk$$

▼

$\boxed{2.}$ Show that the vector field $F(x,y,z) = xz\,i + xyz\,j - y^2 k$ is not conservative.

Solution

$$\text{curl } F = -y(2+x)i + xj + yzk$$

This shows that curl $F \neq 0$ and so, F is not conservative.

▼

$\boxed{3.}$ (a) Show that

$$F(x,y,z) = y^2 z^3 i + 2xyz^3 j + 3xy^2 z^2 k$$

is a conservative vector field.

(b) Find a function f such that $F = \nabla f$.

Solution

(a)

$$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} =$$

$$= (6xyz^2 - 6xyz^2)i - (3y^2z^2 - 3y^2z^2)j + (2yz^3 - 2yz^3)k =$$
$$= 0.$$

Since $\operatorname{curl} F = 0$ and $\operatorname{Dom}(F) = \mathbb{R}^2$, F is a conservative vector field.

(b)

$$f_x(x,y,z) = y^2z^3 \qquad (1)$$
$$f_y(x,y,z) = 2xyz^3 \qquad (2)$$
$$f_z(x,y,z) = 3xy^2z^2 \qquad (3)$$

We integrate (1) with respect to x:

$$f(x,y,z) = xy^2z^3 + g(y,z) \qquad (4)$$

We differentiate (4) with respect to y:

$$f_y(x,y,z) = 2xyz^3 + g_y(y,z)$$

So, taking into account (2), we get

$$g_y(y,z) = 0 \Rightarrow g(y,z) = h(z)$$

and

$$f_z(x,y,z) = 3xy^2z^2 + h'(z)$$

From (3) : $h'(z)=0$. Therefore

$$f(x,y,z) = xy^2z^3 + K.$$

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$\boxed{4.}$ If $F(x,y,z) = xz\,i + xyz\,j - y^2 k$, find div F.

Solution

$$\operatorname{div} F = \nabla \cdot F = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz.$$

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$\boxed{5.}$ Show that the vector field $F(x,y,z) = xz\,i + xyz\,j - y^2 k$ can't be written as the curl of another vector field, that is, $F \neq \operatorname{curl} G$.

Solution

$$\operatorname{div} F = z + xz$$

and $\operatorname{div} F \neq 0$. If it were true that $F = \operatorname{curl} G$, then

$$\operatorname{div} F = \operatorname{div} \operatorname{curl} G = 0$$

which contradicts $\operatorname{div} F \neq 0$.
Therefore F is not the curl of another vector field.

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