Lecture #13 - Week 5 - Limits and Continuity - 14.2

Def. Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L and we write

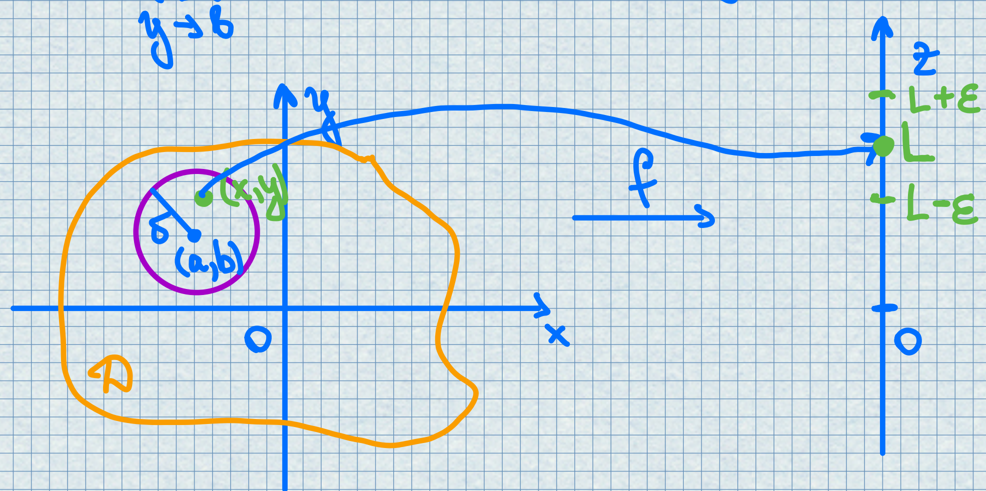
$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x,y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then

$$|f(x,y) - L| < \varepsilon$$

Alternative notations of a limit definition:

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = L \quad \text{and} \quad f(x,y) \to L \text{ as } (x,y)\to(a,b)$$



Proposition If $f(x,y) \longrightarrow L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$, then

$$\lim_{(x,y)\to(a,b)} f(x,y) \quad \boxed{\text{DNE}}$$

- **Continuity**

<u>Def.</u> A function f of two variables is called <u>continuous</u> at (a,b) if

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b)$$

We say f is <u>continuous on D</u> if f is continuous at every point (a,b) in D.

<u>Def.</u> A <u>polynomial function</u> of two variables is a sum of terms of the form $cx^n y^m$, where c is a constant and $m,n \in \mathbb{Z}_+$

A rational function is a ratio of polynomials.

<u>Ex.</u>

$$f(x,y) = \frac{2xy+1}{x^2+y^2}$$

<u>Statement</u>
- All polynomials are continuous on $\mathbb{R}^2$.
- If $f(x,y)$ is continuous and g is continuous and defined on the range of f, then
$$h = g\circ f = g(f(x,y)) \text{ is continuous}$$

- Functions of three or more variables

Limit definition:

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$$

is equivalent to:

for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

if $(x,y,z) \in D$om(f) and $0 < \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} < \delta$

then $$|f(x,y,z) - L| < \varepsilon$$

Def. The function f is <u>continuous</u> at (a,b,c) if

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c)$$

Def. If f is defined on a subset D of $\mathbb{R}^n$, then

$\lim_{x\to a} f(x) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

if $x \in D$ and $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$

Examples

1. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ DNE.

Solution

$$f(x,y) = \frac{x^2-y^2}{x^2+y^2}$$

Approach $(0,0)$ along x-axis: $y=0 \Rightarrow f(x,0) = 1$, for all $x \neq 0$, so

$$f(x,y) \to 1 \text{ as } (x,y) \to (0,0) \text{ along x-axis}$$

Approach $(0,0)$ along y-axis: $x=0 \Rightarrow f(0,y) = -1$, for all $y \neq 0$, so

$$f(x,y) \to -1 \text{ as } (x,y) \to (0,0) \text{ along y-axis.}$$

Since f has two different limits along two different lines, the given limit DNE.

▼

2. If $f(x,y) = \frac{xy^2}{x^2+y^4}$, does $\lim_{\substack{x\to0\\y\to0}} f(x,y)$ exist?

Solution

Let $(x,y) \to (0,0)$ along any line through the origin.

$y = mx$.

$$f(x,y) = f(x, mx) = \frac{m^2 x}{1 + m^4 x^2}$$

So $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along $y = mx$.

If $x = 0$: $f(0,y) = 0$.

So $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along x-axis.

Now, let $y = x^2$.

Then $$f(x,y) = f(x, x^2) = \frac{y^4}{2y^4} = \frac{1}{2}$$

So $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along $y = x^2$.

Hence, the limit DNE. ▼

$\boxed{3.}$ Evaluate

$$\lim_{(x,y) \to (1,2)} (x^2 y^3 - x^3 y^2 + 3x + 2y).$$

Solution

Since $f(x,y) = x^2 y^3 - x^3 y^2 + 3x + 2y$ is a polynomial, it is continuous everywhere.

We use direct substitution:

$$\lim_{\substack{x \to 1 \\ y \to 2}} (x^2 y^3 - x^3 y^2 + 3x + 2y) = 8 - 4 + 3 + 4 = 11.$$

▼

$\boxed{4.}$ Where is the function

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{continuous?}$$

Solution

f is discontinuous at $(0,0)$ because it is not defined there.

Since f is a rational function, it is continuous on its domain

$$D = \{(x,y) \mid (x,y) \neq (0,0)\}.$$

$\boxed{5.}$ Let

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

f is cont. for $(x,y) \neq (0,0)$ since it is equal to a rational function there.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0 = f(0,0).$$

Therefore, f is cont. at $(0,0)$, and so it is continuous on $\mathbb{R}^2$.

$\boxed{6.}$ Where is the function $h(x,y) = \arctan\left(\frac{y}{x}\right)$ continuous?

Solution $f(x,y) = \frac{y}{x}$ is a rational function and continuous except $x=0$. The function $g(t) = \arctan(t)$ is continuous everywhere. So

$$g(f(x,y)) = \arctan\left(\frac{y}{x}\right)$$

is continuous everywhere except where $x=0$.

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