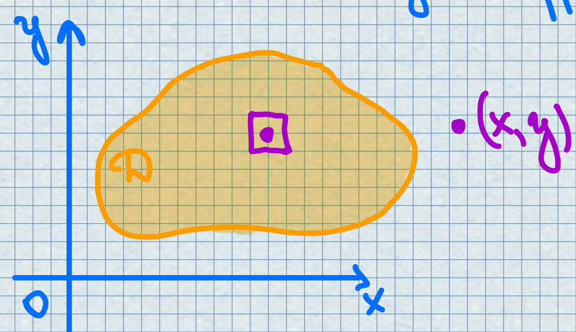
Lecture # 23 - Week 7 - Applications of Double Integrals - 15.4

• Density and Mass

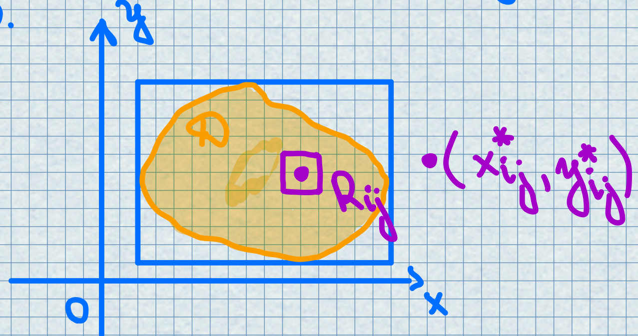
Suppose the lamina occupies a region D of the xy-plane and its <u>density</u> at a point (x,y) in D is given by $\rho(x,y)$, where ρ is a continuous function on D.

$$\rho(x,y) = \lim \frac{\Delta m}{\Delta A}$$

where Δm and ΔA are the mass and area of a small rectangle that contains (x,y) and the limit is taken as the dimensions of the rectangle approach 0.



To find the total mass m of the lamina we divide a rectangle R containing D into subrectangles R_{ij} of the same size and consider $\rho(x,y)$ to be 0 outside D.



$$m \approx \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*) \Delta A$$

$$\boxed{m = \lim_{k,l \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D \rho(x,y)\, dA}$$

For example, if an electric charge is distributed over a region D and the charge density is given by $\sigma(x,y)$ at a point (x,y) in D, then the total charge Q is given by

$$\boxed{Q = \iint_D \sigma(x,y)\, dA}$$

- **Moments and Centers of Mass**

Suppose the lamina occupies a region D and has density function $\rho(x,y)$.

We define the moment of a particle about an axis as a product of its mass and its directed distance from the axis.

The mass of $R_{ij} \approx \rho(x_{ij}^*, y_{ij}^*) \Delta A$, so

the moment of R_{ij} with respect to the x-axis is

$$\boxed{\left(\rho(x_{ij}^*, y_{ij}^*)\,\Delta A\right) y_{ij}^*}$$

The <u>moment</u> of the entire lamina about the x-axis is:

$$\boxed{\begin{aligned} M_x &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^*\,\rho(x_{ij}^*, y_{ij}^*)\,\Delta A \\ &= \iint_D y\,\rho(x,y)\,dA \end{aligned}}$$

The moment about the y-axis is

$$\boxed{M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^*\,\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_D x\,\rho(x,y)\,dA}$$

<u>Statement</u>

The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x,y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\,\rho(x,y)\,dA$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA,$$

where the mass m is given by

$$m = \iint_D \rho(x,y)\,dA.$$

- **Moment of inertia**

The moment of inertia of a particle of mass m about an axis is defined to be $\boxed{mr^2}$, where r is the distance from the particle to the axis.

We extend this concept to a lamina with density function $\rho(x,y)$ and occupying a region D by proceeding as we did for ordinary moments.

The moment of inertia of the lamina about the x-axis:

$$I_x = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D y^2\rho(x,y)\,dA$$

$$I_y = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D x^2\rho(x,y)\,dA$$

The moment of inertia about the origin (polar moment of inertia):

$$\boxed{\begin{aligned} I_0 &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \left((x_{ij}^*)^2 + (y_{ij}^*)^2\right)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \\ &= \iint_D (x^2+y^2)\,\rho(x,y)\,dA \end{aligned}}$$

The <u>radius of gyration of a lamina</u> about an axis is the number R such that

$$\boxed{mR^2 = I}$$

The radius of gyration $\bar{\bar{y}}$ with r. to x-axis and the radius of gyration $\bar{\bar{x}}$ with. r. to y-axis are

$$\boxed{m\bar{\bar{y}}^2 = I_x, \qquad m\bar{\bar{x}}^2 = I_y}$$

- **Probability**

We consider the probability density function f of a continuous random variable X.

Thus, $f(x) \geq 0$ for all x, and $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

And

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

The joint density function of X and Y is a function of two variables s.t. the probability that (X, Y) lies in a region D is

$$\boxed{P((X,Y)\in D) = \iint_D f(x,y)\,dA}$$

In particular,

$$P(a \le X \le b,\ c \le Y \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Since $0 \le p \le 1$, we get

$$f(x,y) \ge 0 \qquad \iint_{\mathbb{R}^2} f(x,y)\,dA = 1$$

Def. Suppose X is a random variable with probability density function $f_1(x)$ and Y is a random variable with density function $f_2(x)$.
Then X and Y are independent random variables if

$$f(x,y) = f_1(x)\,f_2(x)$$

- Expected Values

If X is a random variable with probability density function f, then its mean is

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

If X and Y are random variables with joint density function f, we define the X-mean and Y-mean of X and Y, to be

$$\boxed{\mu_1 = \iint_{\mathbb{R}^2} x f(x,y)\, dA \qquad \mu_2 = \iint_{\mathbb{R}^2} y f(x,y)\, dA}$$

Def. A single random variable is normally distributed if its probability density function is of the form

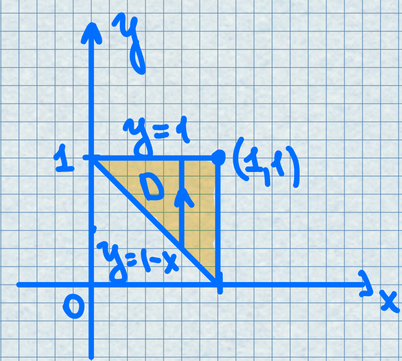
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}$$

where μ is the mean and σ is the standard deviation.

Examples

$\boxed{1.}$ Charge is distributed over the triangular region D so that the charge density at (x,y) is $\sigma(x,y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.

Solution



$$Q = \iint_D \sigma(x,y)\, dA =$$

$$= \int_0^1 \int_{1-x}^1 xy \, dy \, dx =$$

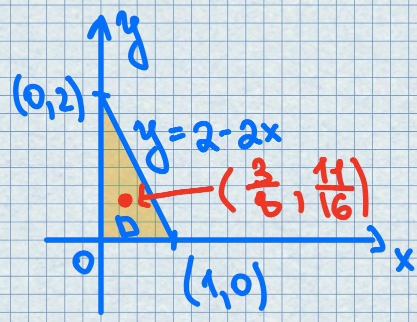
$$= \int_0^1 \left(x \frac{y^2}{2} \right) \Bigg|_{y=1-x}^{y=1} dx = \int_0^1 \frac{x}{2} \left(1 - (1-x)^2 \right) dx =$$

$$= \frac{1}{2} \int_0^1 (2x^2 - x^3)\, dx = \frac{1}{2} \left(\frac{2x^3}{3} - \frac{x^4}{4} \right) \Bigg|_0^1 = \frac{5}{24}$$

$\boxed{2.}$ Find the mass and center of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$, and $(0,2)$ if the density function is $\rho(x,y) = 1 + 3x + y$.

Solution

$$m = \iint_D \rho(x,y)\,dA =$$



$$= \int_0^1 \int_0^{2-2x} (1+3x+y)\,dy\,dx =$$

$$= \int_0^1 \left(y + 3xy + \frac{y^2}{2} \right) \Big|_{y=0}^{y=2-2x} dx = 4\int_0^1 (1-x^2)\,dx =$$

$$= 4\left(x - \frac{x^3}{3} \right) \Big|_0^1 = \frac{8}{3}.$$

$$\bar{x} = \frac{1}{m} \iint_D x\,\rho(x,y)\,dA = \frac{3}{8} \int_0^1 \int_0^{2-2x} (x + 3x^2 + xy)\,dy\,dx =$$

$$= \frac{3}{8} \int_0^1 \left(xy + 3x^2 y + x\frac{y^2}{2} \right) \Big|_{y=0}^{y=2-2x} dx =$$

$$= \frac{3}{2} \int_0^1 (x - x^3)\,dx = \frac{3}{8}.$$

$$\bar{y} = \frac{1}{m} \iint_D y\,\rho(x,y)\,dA = \frac{3}{8} \int_0^1 \int_0^{2-2x} (y + 3xy + y^2)\,dy\,dx =$$

$$= \frac{3}{8} \int_0^1 \left(\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3} \right) \Big|_{y=0}^{y=2-2x} dx =$$

$$= \frac{1}{4} \int_0^1 (7 - 9x - 3x^2 + 5x^3)\,dx = \frac{11}{16}.$$

The center of mass is at the point $\left(\frac{3}{8}, \frac{11}{16}\right)$. ▼

$\boxed{3.}$ Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x,y) = \rho$, center the origin, and radius a.

Solution

The boundary of D is the circle $x^2 + y^2 = a^2$ and in polar coordinates D is:

$0 \leq \theta \leq 2\pi$, $0 \leq r \leq a$.

$$I_0 = \iint_D (x^2+y^2)\rho \, dA = \rho \int_0^{2\pi} \int_0^a r^2 r \, dr \, d\theta =$$

$$= \rho \int_0^{2\pi} d\theta \int_0^a r^3 \, dr = 2\pi\rho \left. \frac{r^4}{4} \right|_0^a = \frac{\pi\rho a^4}{2}$$

$I_x + I_y = I_0$ and $I_x = I_y$. Thus

$$I_x = I_y = \frac{I_0}{2} = \frac{\pi\rho a^4}{4}.$$

▼

$\boxed{4.}$ If the joint density function for X and Y is given by

$$f(x,y) = \begin{cases} C(x+2y) & \text{if } 0 \leq x \leq 10,\ 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

find the value of the constant C. Then find $P(X \le 7, Y \ge 2)$.

Solution

We find the value of C by ensuring that the double integral of f is equal to 1. Because $f(x,y)=0$ outside of $[0,10]\times[0,10]$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^{10}\int_0^{10} C(x+2y)\,dy\,dx =$$

$$= C\int_0^{10} (xy+y^2)\Big|_{y=0}^{y=10} dx = C\int_0^{10}(10x+100)\,dx = 1500\,C$$

Therefore, $1500\,C = 1 \Rightarrow C = \frac{1}{1500}$.

$$P(X \le 7, Y \ge 2) = \int_{-\infty}^{7}\int_{2}^{\infty} f(x,y)\,dy\,dx = \int_0^7\int_2^{10} \frac{1}{1500}(x+2y)\,dy\,dx$$

$$= \frac{1}{1500}\int_0^7 (xy+y^2)\Big|_{y=2}^{y=10} dx = \frac{1}{1500}\int_0^7 (8x+96)\,dx =$$

$$= \frac{868}{1500} \approx 0.5787.$$

▼