1. Consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{y^3 - 4x^2}{y^2 + 2x^2}.$$

Either show it does not exist, or give strong evidence for suspecting it does.

$x=0$: $\lim_{y\to 0} \frac{y^3}{y^2} = \lim_{y\to 0} y = 0$

$y=0$: $\lim_{x\to 0} \frac{-4x^2}{2x^2} = -2$

$-2 \neq 0 \Rightarrow$ the limit DNE

2. For the given function

$$g(x, y) = x^2 \sin(y) - y^2 e^x$$

(a) Use the chain rule to compute $\frac{df}{dt}(0)$, where:

$$f(t) = g(\underbrace{t^2 + 2t}_{x}, \underbrace{2e^t + 1}_{y}).$$

$\frac{dg}{dx} = 2x\sin y - y^2 e^x$

$\frac{dg}{dy} = \cos y \cdot x^2 - 2ye^x$

$\frac{dx}{dt} = 2t + 2$

$\frac{dy}{dt} = 2e^t$

$\frac{df}{dt} = \frac{dg}{dx}\cdot\frac{dx}{dt} + \frac{dg}{dy}\cdot\frac{dy}{dt}$

$\frac{df}{dt} = (2x\sin y - y^2e^x)(2t+2) + (\cos y \cdot x^2 - 2ye^x)\cdot 2e^t$

plug $t=0$

(b) Give an equation for the linear (tangent plane) approximation to g at the point $(0, -1)$, and use it to estimate $f(0.1, -0.8)$.

$L(x,y) = g(0,-1) + g_x(x-0) + g_y(y+1)$

$g(0,-1) = -1$

$g_x = 2x\sin y - y^2 e^x$ $\quad g_x(0,-1) = -1$

$g_y = \cos y \cdot x^2 - 2ye^x$ $\quad g_y(0,-1) = 2$

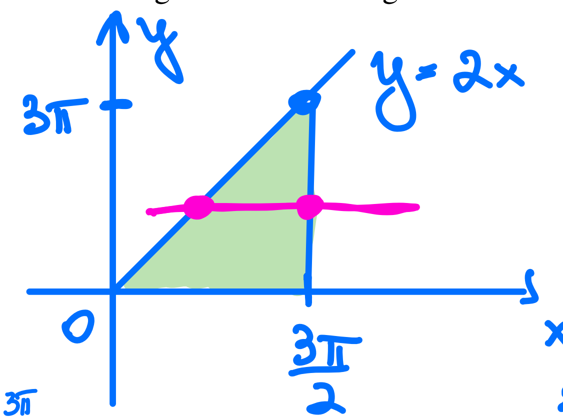
$L(x,y) = -1 - x + 2(y+1) = 1 - x + 2y$

$L(0.1, -0.8) = 1 - 0.1 - 1.6 = -0.7$

3. Evaluate the integral

$$\int_0^{3\pi/2}\int_0^{2x} \cos(y)\,dy\,dx =$$

fully, by first drawing the region of integration, and then reversing the order of integration.



$$= \int_0^{3\pi}\int_{\frac{y}{2}}^{\frac{3\pi}{2}} \cos y\,dx\,dy =$$

$$= \int_0^{3\pi} \sin y \cdot x \Big|_{\frac{y}{2}}^{\frac{3\pi}{2}} dy =$$

$$= \int_0^{3\pi} \sin y\left(\frac{3\pi}{2} - \frac{y}{2}\right)dy = \frac{3\pi}{2}\int_0^{3\pi}\sin y\,dy - \frac{1}{2}\int_0^{3\pi} y\sin y\,dy = \frac{3\pi}{2}\left(-\cos y\right)\Big|_0^{3\pi} -$$

$$- \frac{1}{2}\left(y(-\cos y)\Big|_0^{3\pi} + \int_0^{3\pi}\cos y\,dy\right) = \dots$$

4. Find the classify (using the Second Derivative Test) all critical points of

$$f(x,y) = -x^2 + y + y^2 + 2x.$$

$$\begin{cases} f_x = -2x + 2 = 0 \\ f_y = 2y + 1 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = -\frac{1}{2} \end{cases}$$

CP: $\left(1, -\frac{1}{2}\right)$

$f_{xx} = -2$ $\quad f_{xy} = f_{yx} = 0$

$f_{yy} = 2$

$D = -2 \cdot 2 = -4 < 0$

Since $D < 0$ we have that $\left(1, -\frac{1}{2}\right)$ is a saddle point.

5. Give an equation for the tangent plane to the surface

$$yz + e^z \cos(x + 2y) = 2$$

at the point (2, 1, 1).

$F(x,y,z) = yz + e^z\cos(x+2y) - 2$

$\nabla F = \langle e^z(-\sin(x+2y)), z + e^z(-\sin(x+2y))\cdot 2, y + e^z\cos(x+2y) \rangle$

$\nabla F(2,1,1) = \langle -e\sin 4, 1 - 2e\sin 4, e\cos 4 \rangle$

Hence, the tangent plane equation is

$$-e\sin 4(x-2) + (1-2e\sin 4)(y-1) + e\cos 4(z-1) = 0$$

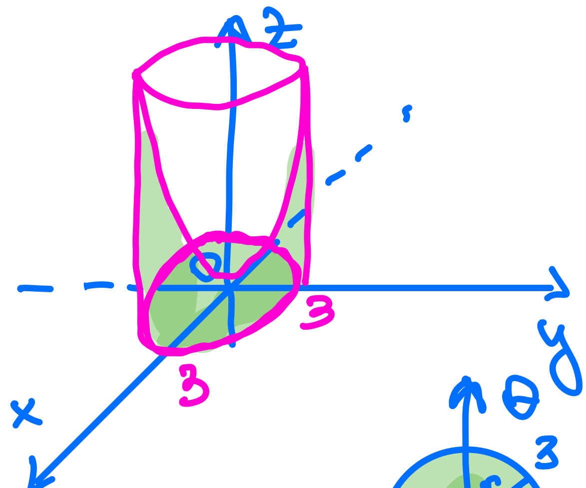
6. Use polar coordinates to find the volume of the solid bounded above by the paraboloid $4z = x^2 + y^2$ and below by the disk $x^2 + y^2 \le 9$.

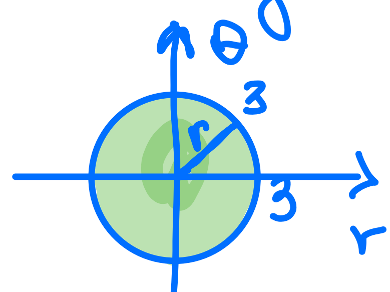
$z = \frac{x^2}{4} + \frac{y^2}{4}$

$x = 2r\cos\theta$

$y = 2r\sin\theta$

$dA = r\,dr\,d\theta$

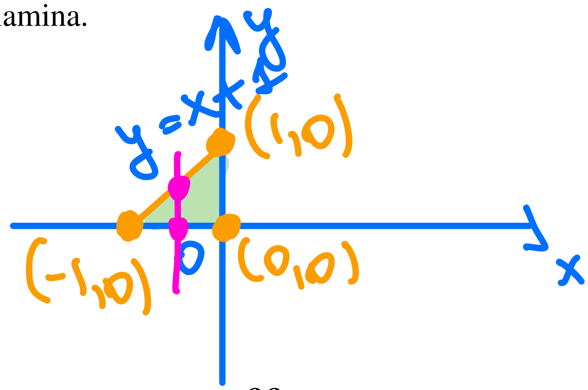




$$V = \iint_D \left(\frac{x^2}{4} + \frac{y^2}{4}\right) dA = \int_0^{2\pi}\int_0^3 r^2\cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^3 d\theta = \frac{3^4}{4}\Big|_0^{2\pi} = \frac{81\pi}{2}$$

7. Find the mass and the center of mass of a triangular lamina with vertices $(-1, 0)$, $(1, 0)$, $(0, 0)$ if the density function is $\rho(x, y) = y + 5$.

   (a) Draw the triangular lamina.



   (b) Use the formula

$$m = \iint_D \rho(x, y)\, dA$$

   to find the mass of the lamina.

$$m = \int_{-1}^{0}\int_{0}^{x+1} (y+5)\,dy\,dx = \int_{-1}^{0}\left(\frac{y^2}{2} + 5y\right)\Big|_0^{x+1} dx =$$

$$= \int_{-1}^{0}\left(\frac{x^2}{2} + x + \frac{1}{2} + 5x + 5\right) dx = \left(\frac{x^3}{6} + \frac{x^2}{2} + \frac{5x^2}{2} + \frac{1}{2}x + 5x\right)\Big|_{-1}^{0} =$$

$$= -\left(-\frac{1}{6} + \cancel{\frac{1}{2}} + \frac{5}{2} - \cancel{\frac{1}{2}} - 5\right) = \frac{1}{6} - \frac{5}{2} + 5$$

   (c) Use formulas

$$\bar{x} = \frac{1}{m}\iint_D x\,\rho(x, y)\, dA, \quad \bar{y} = \frac{1}{m}\iint_D y\,\rho(x, y)\, dA$$

   to set up the expressions for coordinates of the center of mass of the lamina. DO NOT EVALUATE THE INTEGRALS.

$$\bar{x} = \frac{1}{\frac{1}{6} - \frac{5}{2} + 5}\iint_D x(y+5)\,dA$$

$$\bar{y} = \frac{1}{\frac{1}{6} - \frac{5}{2} + 5}\iint_D y(y+5)\,dA$$

8. Use Lagrange multipliers to find the maximum product of two positive numbers satisfying $x^2 + y^2 = 6$.

$x \geq 0$ $\quad f(x,y) = x \cdot y \to \max$

$y \geq 0$ $\quad x^2 + y^2 = 6$

$g(x,y) = x^2 + y^2 - 6$

$$\boxed{\nabla f = \lambda \nabla g}$$

$f_x = y \qquad g_x = 2x$

$f_y = x \qquad g_y = 2y$

$$\begin{cases} y = \lambda 2x \\ x = \lambda \cdot 2y \end{cases} \Rightarrow y = 2\lambda \cdot 2\lambda y = 4\lambda^2 y$$

$y(1 - 4\lambda^2) = 0$

$y = 0$ or $\lambda = \pm \frac{1}{2}$

$y = 0$: $\quad x = 0$

$\lambda = \frac{1}{2}$: $\quad x = y \Rightarrow 2x^2 = 6 \Rightarrow x = \pm\sqrt{3}$ $(\sqrt{3}, \sqrt{3})$

$y = \pm\sqrt{3}$ $(-\sqrt{3}, -\sqrt{3})$

$\lambda = -\frac{1}{2}$: $\quad x = -y \Rightarrow 2x^2 = 6 \Rightarrow x = \pm\sqrt{3}$ $(\sqrt{3}, -\sqrt{3})$

$y = \pm\sqrt{3}$ $(-\sqrt{3}, \sqrt{3})$

$(0,0), (\sqrt{3}, \sqrt{3}), (-\sqrt{3}, -\sqrt{3})$

$(\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3})$

$x \cdot y$ is max when $(x,y) = (\sqrt{3}, \sqrt{3})$ or $(-\sqrt{3}, -\sqrt{3})$

and $x \cdot y = 3$

9. For the given function $f(x, y) = \cos(2x) + 3e^{xy}$, the point $P(-1, 0)$, and the directional vector $u = \langle 4/\sqrt{41}, 5/\sqrt{41} \rangle$

   (a) Find the gradient of f at the point P.

$$\nabla f = \langle -\sin(2x) \cdot 2 + 3ye^{xy}, 3xe^{xy} \rangle$$

$$\nabla f(-1,0) = \langle 2\sin 2, -3 \rangle$$

   (b) Find the rate of change of f at P in the direction of the vector u.

$$\nabla f \cdot u = \langle 2\sin 2, -3 \rangle \cdot \langle 4/\sqrt{41}, 5/\sqrt{41} \rangle =$$

$$= \frac{8\sin 2}{\sqrt{41}} - \frac{15}{\sqrt{41}}$$

   (c) Fully set up bounds and integrand for computing the **surface area** of f over the region $[0, 3] \times [0, 5]$. DO NOT EVALUATE.

$$A(S) = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA =$$

$$= \int_0^3 \int_0^5 \sqrt{1 + (-2\sin(2x) + 3ye^{xy})^2 + (3xe^{xy})^2}\, dy\, dx$$