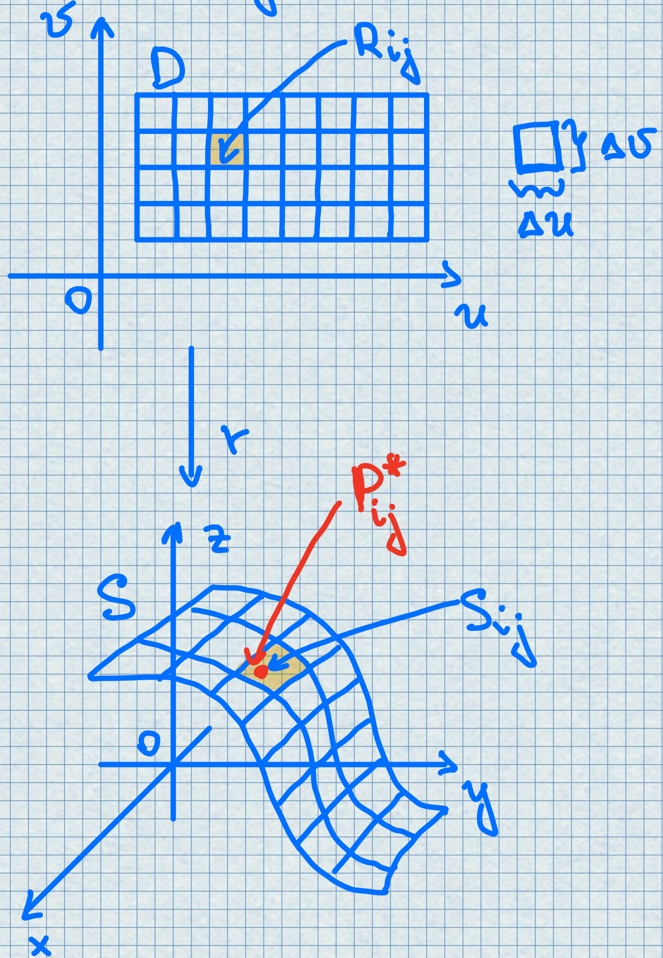
Lecture #35 - Week 12 - Surface Integrals - 16.7

- Parametric Surfaces

Suppose that the surface S has a vector equation

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}, \quad (u,v) \in D$$

Let D be a rectangle.



Then S is divided in S_{ij}.
We evaluate f at P^*_{ij} and multiply by the area ΔS_{ij}.

We form the Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*) \Delta S_{ij}$$

Def.

The surface integral of f over the surface S is

$$\boxed{\iint_S f(x,y,z)\, dS = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*) \Delta S_{ij}}$$

We have that

$$\Delta S_{ij} \approx |r_u \times r_v| \Delta u \Delta v,$$

where

$$r_u = \frac{\partial x}{\partial u} i + \frac{\partial y}{\partial u} j + \frac{\partial z}{\partial u} k$$

$$r_v = \frac{\partial x}{\partial v} i + \frac{\partial y}{\partial v} j + \frac{\partial z}{\partial v} k$$

If the components are continuous, $r_u \neq 0$, $r_v \neq 0$ and $r_u \nparallel r_v$ in the interior of D, then

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_D f(r(u,v)) |r_u \times r_v|\, dA}$$

Applications

If a thin sheet has the shape of a surface S and the density at (x,y,z) is $\rho(x,y,z)$, then the total <u>mass</u> of the sheet is

$$\boxed{m = \iint_S \rho(x,y,z)\, dS}$$

and the <u>center of mass</u> is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m}\iint_S x\,\rho(x,y,z)\, dS \qquad \bar{y} = \frac{1}{m}\iint_S y\,\rho(x,y,z)\, dS$$

$$\bar{z} = \frac{1}{m}\iint_S z\,\rho(x,y,z)\, dS$$

- **Graphs of Functions**

Any S: $z = g(x,y)$ has the parametric form

$$x = x \qquad y = y \qquad z = g(x,y)$$

$$r_x = i + \left(\frac{\partial g}{\partial x}\right) k \qquad r_y = j + \left(\frac{\partial g}{\partial y}\right) k$$

Thus

$$r_x \times r_y = -\frac{\partial g}{\partial x} i - \frac{\partial g}{\partial y} j + k$$

and

$$|r_x \times r_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Therefore,

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\;dA}$$

Similarly, when

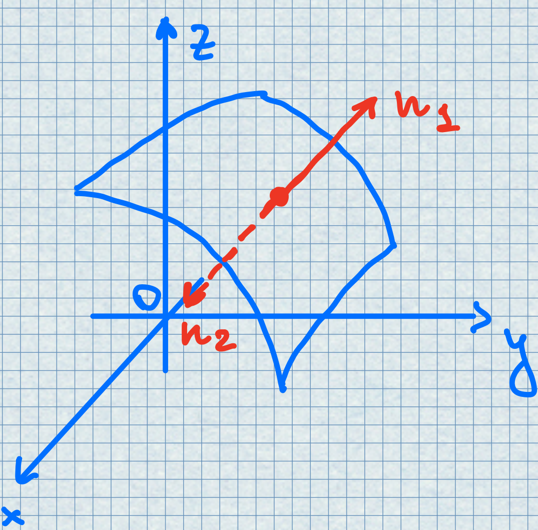
S: $y = h(x,z)$ and D is its projection onto the xz-plane, then

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(x,h(x,z),z)\sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\;dA}$$

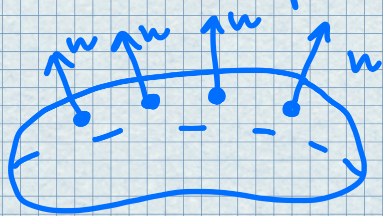
- **Oriented Surfaces**

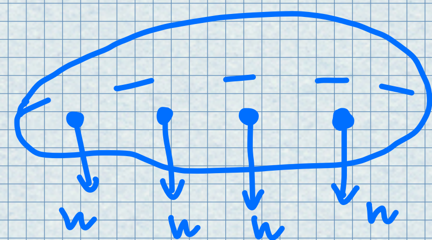
n_1 and n_2 are two unit normal vectors

$n_2 = -n_1$ at (x,y,z).



Def. If it is possible to choose a unit normal vector n at every such point (x,y,z) so that n varies continuously over S, then S is called an <u>oriented surface</u> and the given choice of n provides S with an orientation.





For S: $z = g(x,y)$

$$\boxed{n = \frac{-\frac{\partial g}{\partial x} i - \frac{\partial g}{\partial y} j + k}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}}}$$

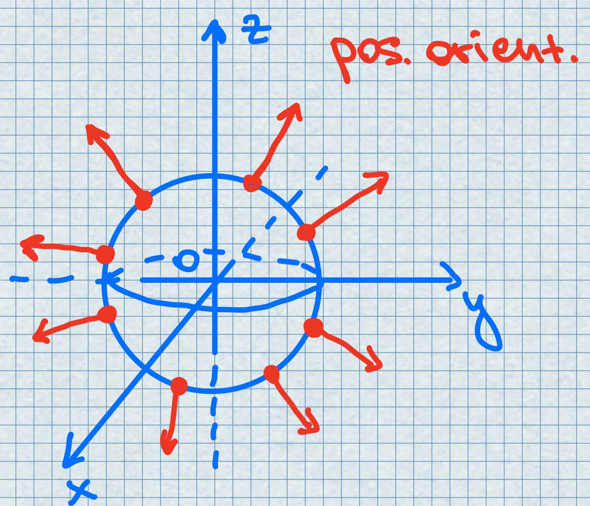
upward orientation
$(1 \cdot k)$
> 0

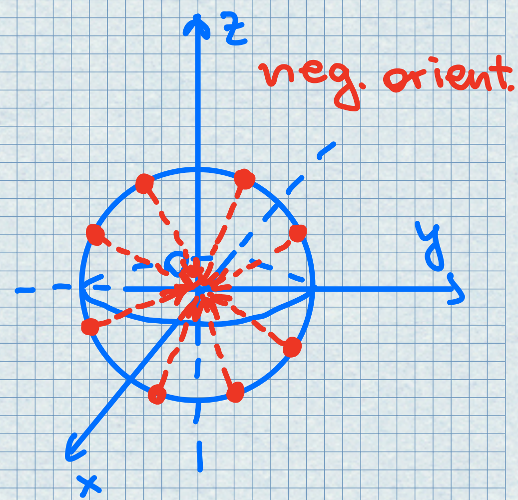
If S is a smooth orientable surface given by $r(u,v)$, then

$$\boxed{n = \frac{r_u \times r_v}{|r_u \times r_v|}}$$

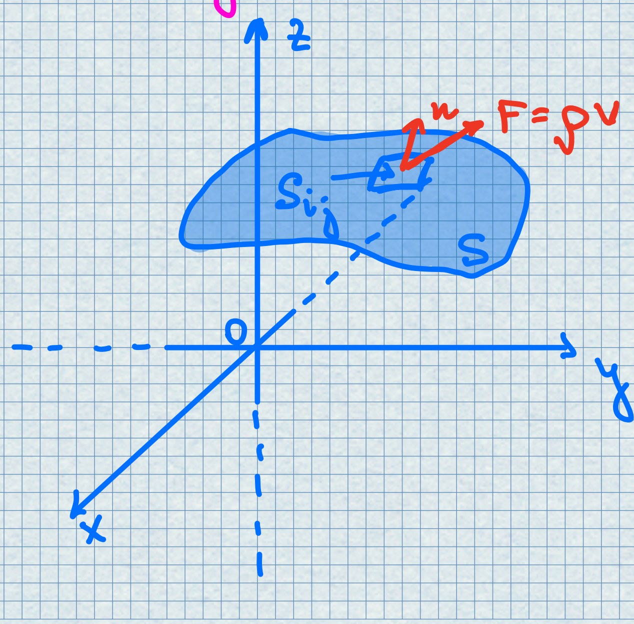
and the opposite orientation is given by $-n$.

Def. For a closed surface, that is, a surface that is the boundary of a solid region E, the convention is that the positive orientation is the one for which the normal vectors point outward from E, and inward-pointing normals give the negative orientation.





- Surface Integrals of Vector Fields



Suppose S is an oriented surface with unit normal vector n, and imagine a fluid with density $\rho(x,y,z)$ and velocity field $v(x,y,z)$ flowing through S.

Then the rate of flow per unit area is ρv.

We divide S into small patches S_{ij}.

Then

$$(\rho v \cdot n)\, A(S_{ij})$$

is the approximate mass of fluid per unit time crossing S_{ij} in n direction.

Thus, by summing these quantities and taking the limit we get:

$$\boxed{\iint_S \rho v \cdot n \, dS = \iint_S \rho(x,y,z)\, v(x,y,z) \cdot n(x,y,z)\, dS}$$

$\uparrow$

<u>rate of flow through S</u>

If $F = \rho v$, then we get

$$\iint_S F \cdot n \, dS$$

Def. If F is a continuous vector field defined on an oriented surface S with unit normal vector n, then the surface integral of F over S is

$$\boxed{\iint_S F \cdot dS = \iint_S F \cdot n \, dS}$$

This integral is also called the flux of F across S.

If S is given by a vector function $r(u,v)$, then

$$n = \frac{r_u \times r_v}{|r_u \times r_v|} \quad \text{and}$$

$$\iint_S F \cdot dS = \iint_S F \cdot \frac{r_u \times r_v}{|r_u \times r_v|} \, dS = \iint_D \left[F(r(u,v)) \cdot \frac{r_u \times r_v}{|r_u \times r_v|} \right] \cdot |r_u \times r_v| \, dA,$$

where D is the parameter domain. Thus we have

$$\boxed{\iint_S F \cdot dS = \iint_D F \cdot (r_u \times r_v) \, dA}$$

If S is given by $z = g(x,y)$, then

$$F\cdot(r_x\times r_y)=(Pi+Qj+Rk)\cdot\left(-\frac{\partial g}{\partial x}i-\frac{\partial g}{\partial y}j+k\right)$$

Thus

$$\boxed{\iint_S F\cdot dS=\iint_D\left(-P\frac{\partial g}{\partial x}-Q\frac{\partial g}{\partial y}+R\right)dA}$$

This formula assumes the upward orientation of S; for a downward orientation we multiply by -1.

Applications

1. If E is an electric field, then the surface integral

$$\iint_S E\cdot dS$$

is called the <u>electric flux of E</u> through S.

Gauss's Law:

$$Q=\varepsilon_0\iint_S E\cdot dS \quad \text{(net charge enclosed by a closed surface } S\text{)}$$

where ε_0 is a constant.

2. Suppose the temperature at a point (x,y,z) in a body is $u(x,y,z)$. Then the

heat flow is defined as the vector field

$$F = -K \nabla u$$

where K is an experimentally determined constant called the conductivity of the substance.

The rate of heat flow across S in the body is then given by

$$\boxed{\iint_S F \cdot dS = -K \iint_S \nabla u \cdot dS}$$

Examples

1. Compute the surface integral $\iint_S x^2\, dS$, where S: $x^2+y^2+z^2=1$.

Solution

$$x=\sin\varphi\cos\theta \qquad y=\sin\varphi\sin\theta \qquad z=\cos\varphi$$

$$0\le\varphi\le\pi \quad , \quad 0\le\theta\le 2\pi$$

$$r(\varphi,\theta)=\sin\varphi\cos\theta\, i+\sin\varphi\sin\theta\, j+\cos\varphi\, k$$

$$|r_\varphi\times r_\theta|=\sin\varphi$$

Therefore,

$$\iint_S x^2\, dS=\iint_D(\sin\varphi\cos\theta)^2\,|r_\varphi\times r_\theta|\, dA=$$

$$=\int_0^{2\pi}\int_0^{\pi}\sin^2\varphi\cos^2\theta\sin\varphi\, d\varphi\, d\theta=$$

$$=\int_0^{2\pi}\cos^2\theta\, d\theta\int_0^{\pi}\sin^3\varphi\, d\varphi=\frac{1}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}\cdot$$

$$\cdot\left(-\cos\varphi+\frac{1}{3}\cos^3\varphi\right)\Big|_0^{\pi}=\frac{4\pi}{3}.$$

2. Evaluate $\iint_S y\, dS$, where S: $z=x+y^2$, $0\le x\le 1$, $0\le y\le 2$.

Solution

$$\frac{\partial z}{\partial x} = 1 \quad , \quad \frac{\partial z}{\partial y} = 2y$$

Then

$$\iint_S y\, dS = \iint_D y \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA =$$

$$= \int_0^1 \int_0^2 y \sqrt{1 + 1 + 4y^2}\, dy\, dx = \int_0^1 dx \, \sqrt{2} \int_0^2 y \sqrt{1 + 2y^2}\, dy =$$

$$= \sqrt{2}\ \frac{1}{4}\ \frac{2}{3} \left(1 + 2y^2\right)^{3/2} \Big|_0^2 = \frac{13\sqrt{2}}{3}$$

$\boxed{3.}$ Find the flux of the vector field $F(x,y,z) = zi + yj + xk$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution

$$r(\varphi, \theta) = \sin\varphi \cos\theta\, i + \sin\varphi \sin\theta\, j + \cos\varphi\, k$$

$$0 \le \varphi \le \pi \qquad 0 \le \theta \le 2\pi$$

Then

$$F(r(\varphi,\theta)) = \cos\varphi\, i + \sin\varphi \sin\theta\, j + \sin\varphi \cos\theta\, k$$

and

$$r_\varphi \times r_\theta = \sin^2\varphi \cos\theta\, i + \sin^2\varphi \sin\theta\, j + \sin\varphi \cos\varphi\, k$$

Therefore

$$F(r(\varphi,\theta)) \cdot (r_\varphi \times r_\theta) = \cos\varphi \sin^2\varphi \cos\theta \ +$$

$+ \sin^3\varphi \sin^2\theta + \sin^2\varphi \cos\varphi \cos\theta$

and the flux is

$$\iint_S F\cdot dS = \iint_D F\cdot(r_\varphi \times r_\theta)\,dA = \int_0^{2\pi}\int_0^{\pi}(2\sin^2\varphi\cos\varphi\cos\theta +$$

$$+ \sin^3\varphi \sin^2\theta)\,d\varphi\,d\theta =$$

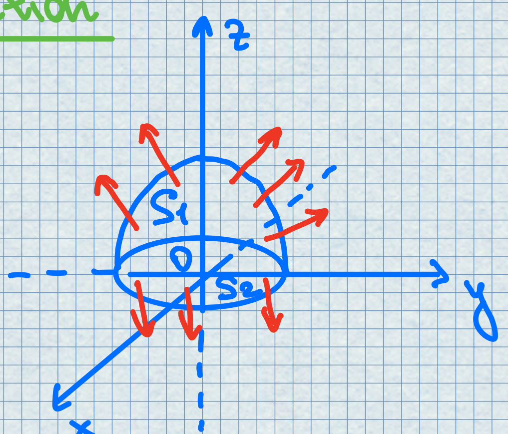
$$= 2\int_0^{\pi}\sin^2\varphi\cos\varphi\,d\varphi\int_0^{2\pi}\cos\theta\,d\theta + \int_0^{\pi}\sin^3\varphi\,d\varphi\int_0^{2\pi}\sin^2\theta\,d\theta =$$

$$= 0 + \int_0^{\pi}\sin^3\varphi\,d\varphi\int_0^{2\pi}\sin^2\theta\,d\theta = \frac{4\pi}{3}$$

▼

4. Evaluate $\iint_S F\cdot dS$, where $F(x,y,z) = yi + xj + zk$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution



$$\left.\begin{aligned} P(x,y,z) &= y \\ Q(x,y,z) &= x \\ R(x,y,z) &= z = 1 - x^2 - y^2 \end{aligned}\right\} \text{on } S_1$$

and $\frac{\partial g}{\partial x} = -2x \qquad \frac{\partial g}{\partial y} = -2y$

Thus

$$\iint_{S_1} F \cdot dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA =$$

$$= \iint_D \left(-y(-2x) - x(-2y) + 1 - x^2 - y^2\right) dA = \iint_D \left(1 + 4xy - x^2 - y^2\right) dA =$$

$$= \int_0^{2\pi}\int_0^1 \left(1 + 4r^2 \cos\theta \sin\theta - r^2\right) r \, dr \, d\theta =$$

$$= \int_0^{2\pi} \left(\frac{1}{4} + \cos\theta \sin\theta\right) d\theta = \frac{1}{4}(2\pi) + 0 = \frac{\pi}{2}.$$

The disk S_2 is oriented downward, so its unit normal vector is $n = -k$ and

$$\iint_{S_2} F \cdot dS = \iint_{S_2} F \cdot (-k) \, dS = \iint_D (-z) \, dA = \iint_D 0 \, dA = 0$$

since $z = 0$ on S_2.

Finally,

$$\iint_S F \cdot dS = \iint_{S_1} F \cdot dS + \iint_{S_2} F \cdot dS = \frac{\pi}{2} + 0 = \frac{\pi}{2}.$$

▼