Name:

Instructions. (100 points) You have two hours. The exam is closed book, closed notes, and only simple calculators are allowed. Show all your work in order to receive full credit.

1. **[6 points]** Consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2}.$$

Either show it does not exist, or give strong evidence for suspecting it does.

For $x=0$: $\lim_{y\to 0} \frac{-4y^2}{2y^2} = \boxed{-2}$

For $y=0$: $\lim_{x\to 0} \frac{x^4}{x^2} = \lim_{x\to 0} x^2 = \boxed{0}$

$0 \neq -2 \Rightarrow$ The limit DNE.

2. **[10 points]** For the given function

$$f(x, y) = x^2 y - y^2 x$$

(a) (5 pts) Use the chain rule to compute $\frac{dg}{dt}(0)$, where:

$$g(t) = f(\underbrace{t^2 + e^{2t}}_{x}, \underbrace{2t + 1}_{y}).$$

$\frac{dg}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt}$

$\frac{dg}{dt} = (2xy - y^2)\cdot(2t + 2e^{2t}) + (x^2 - 2xy)\cdot 2$

$\frac{dg}{dt}(0) = (2\cdot 1 - 1)(2) + (1-2)\cdot 2 = 2 - 2 = \boxed{0}$

(b) (5 pts) Give an equation for the linear (tangent plane) approximation to f at te point $(1, -1)$, and use it to estimate $f(1.1, -0.9)$.

$L(x,y) = f(1,-1) + f_x(1,-1)(x-1) + f_y(1,-1)(y+1)$

$f_x = 2xy - y^2$ $\quad f_x(1,-1) = -2$

$f_y = x^2 - 2xy$ $\quad f_y(1,-1) = 3$

$f(1,-1) = -1 - 1 = -2$

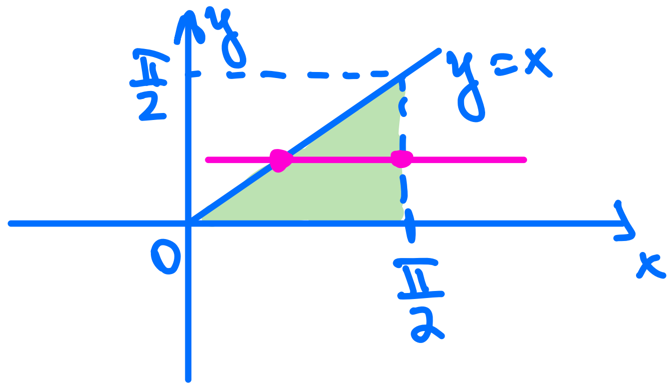
$L(x,y) = -2 - 2(x-1) + 3(y+1) = \boxed{-2x + 3y + 3}$

$f(1.1, -0.9) = -2.2 - 2.7 + 3 \approx \boxed{-1.9}$

3. **[12 points]** Evaluate the integral

$$\int_0^{\pi/2}\int_0^{x} x\sin(y)\,dy\,dx$$

fully, by first drawing the region of integration, and then reversing the order of integration.



$$\int_0^{\frac{\pi}{2}}\int_y^{\frac{\pi}{2}} \sin y\,dx\,dy = \int_0^{\frac{\pi}{2}} x\cdot \sin y\Big|_y^{\frac{\pi}{2}} dy =$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2}-y\right)\sin y\,dy = -\frac{\pi}{2}\cos y\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} y\sin y\,dy =$$

$$= \frac{\pi}{2} + y\cdot\cos y\Big|_0^{\frac{\pi}{2}} - \sin y\Big|_0^{\frac{\pi}{2}} = \boxed{\frac{\pi}{2}-1}$$

4. **[12 points]** Find the classify (using the Second Derivative Test) all critical points of

$$f(x,y) = x^2 + xy + y^2 + y.$$

$$\begin{cases} f_x = 2x + y = 0 \Rightarrow y = -2x \\ f_y = 2y + x + 1 = 0 \Rightarrow 2(-2x) + x + 1 = 0 \end{cases}$$

$$-4x + x + 1 = 0$$

$$-3x = -1 \Rightarrow x = \frac{1}{3}$$

$$y = -\frac{2}{3}$$

C.P.: $\left(\frac{1}{3}, -\frac{2}{3}\right)$

$f_{xx} = 2 > 0$

$f_{yy} = 2$ $\Rightarrow$ $D = 2\cdot 2 - 1 = 3 > 0$

$f_{xy} = f_{yx} = 1$ $\quad f_{xx} > 0$ for all $(x,y) \in Dom(f)$

Hence, $f\left(\frac{1}{3}, -\frac{2}{3}\right)$ is a loc. minimum value.

5. **[8 points]** Give an equation for the tangent plane to the surface

$$\frac{xy}{z} + e^x \ln(z + 2y) = 2$$

at the point $(2, 1, \cancel{0}\ 1)$.

$$F(x,y,z) = \frac{xy}{z} + e^x \ln(z+2y) - 2$$

$$\nabla F = \left\langle \frac{y}{z} + e^x \ln(z+2y),\ \frac{x}{z} + e^x \frac{1}{z+2y}\cdot 2,\ -\frac{xy}{z^2} + e^x \frac{1}{z+2y} \right\rangle$$

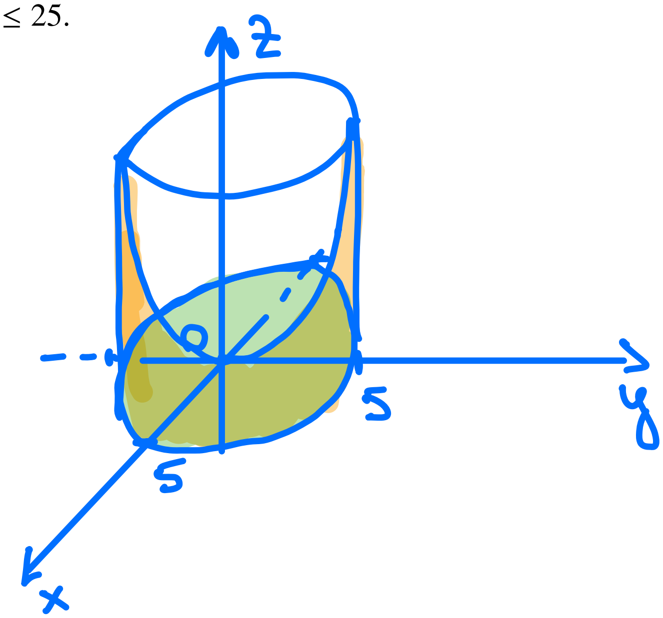
$$\nabla F(2,1,1) = \left\langle 1 + e^2 \ln 3,\ 2 + e^2 \cdot \frac{1}{3}\cdot 2,\ -\frac{2}{1} + e^2 \frac{1}{3} \right\rangle =$$

$$= \left\langle 1 + e^2 \ln 3,\ 2 + \frac{2}{3}e^2,\ -2 + \frac{1}{3}e^2 \right\rangle$$

Then, the TP:

$$\boxed{(1 + e^2 \ln 3)(x-2) + \left(2 + \frac{2}{3}e^2\right)(y-1) + \left(-2 + \frac{1}{3}e^2\right)(z-1) = 0}$$

6. **[10 points]** Use polar coordinates to find the volume of the solid bounded above by the paraboloid $z = x^2 + y^2$ and below by the disk $x^2 + y^2 \leq 25$.



$$V = \iint_D f(x,y)\, dA =$$

$$V = \iint_D (x^2 + y^2)\, dy\, dx$$

$0 \leq r \leq 5$

$0 \leq \theta \leq 2\pi$

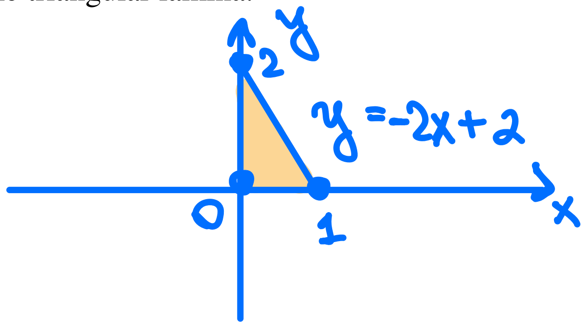
$dA = r\, dr\, d\theta$

$x = r\cos\theta$ $y = r\sin\theta$

$$= \int_0^{2\pi}\int_0^5 r^2 \cdot r\, dr\, d\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^5 d\theta = \frac{5^4}{4}\cdot 2\pi = \boxed{\frac{625\pi}{2}}$$

7. **[16 points]** Find the mass and the center of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$, $(0,2)$ if the density function is $\rho(x,y) = x\cancel{y} + 2$.

(a) **[4 points]** Draw the triangular lamina.



(b) **[6 points]** Use the formula

$$m = \iint_D \rho(x,y)\,dA$$

to find the mass of the lamina.

$$m = \int_0^1 \int_0^{-2x+2} (x+2)\,dy\,dx = \int_0^1 (xy+2y)\Big|_0^{-2x+2} dx =$$

$$= \int_0^1 \left(x(-2x+2) - 4x + 4\right) dx = \int_0^1 (-2x^2 - 2x + 4)\,dx =$$

$$= \left(-\frac{2}{3}x^3 - x^2 + 4x\right)\Big|_0^1 = -\frac{2}{3} - 1 + 4 = \boxed{\frac{8}{3}}$$

(c) **[6 points]** Use formulas

$$\bar{x} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA, \quad \bar{y} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA$$

to find the coordinates of the center of mass of the lamina.

$$\bar{x} = \frac{3}{8}\int_0^1 \int_0^{-2x+2} x(x+2)\,dy\,dx$$

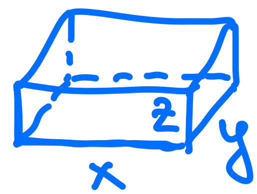
$$\bar{y} = \frac{3}{8}\int_0^1 \int_0^{-2x+2} y(x+2)\,dy\,dx$$

8. **[10 points]** Use Lagrange multipliers to find the maximum ~~and minimum values of a rectangular box whose~~ surface area ~~is 1500 cm^2 and~~ whose total edge length is 200 cm.

of the box

$f(x,y,z) = A = 2xy + 2zy + 2xz$

$4x + 4y + 4z = 200$



$A(x,y,z) = 2xy + 2zy + 2zx$

Subj. to: $x + y + z = 50$

$\begin{cases} \nabla f = \lambda \nabla g \\ g = k \end{cases}$

$\nabla f = 2\langle y+z, x+z, x+y \rangle$

$\nabla g = \langle 1, 1, 1 \rangle$

$\begin{cases} y + z = \frac{\lambda}{2} \\ x + z = \frac{\lambda}{2} \\ x + y = \frac{\lambda}{2} \\ x + y + z = 50 \end{cases}$ +

$2y + 2x + 2z = \frac{3\lambda}{2}$

$50 = \frac{3}{4}\lambda \Rightarrow \boxed{\lambda = \frac{200}{3}}$

$x + y + 2z = \lambda$

$50 - z + 2z = \frac{200}{3}$

$z = \frac{200}{3} - 50 = \frac{200-150}{3} = \frac{50}{3}$

$y = \frac{100}{3} - \frac{50}{3} = \frac{50}{3}$

$x = \frac{50}{3}$

$\boxed{P\left(\frac{50}{3}, \frac{50}{3}, \frac{50}{3}\right)}$

$0 \le x \le \frac{50}{3}$

$0 \le y \le \frac{50}{3}$

$0 \le z \le \frac{50}{3}$

Hence, the maximum surface area is $A = 2 \cdot \cancel{3} \cdot \frac{50^2}{\cancel{9}_3} = \frac{2 \cdot 25 \cdot 100}{3} = \boxed{\frac{5000}{3}\ cm^2}$

9. **[16 points]** For the given function $f(x, y) = y^2 e^{xy}$, the point $P(0, 1)$, and the directional vector $u = \langle 3/5, 4/5 \rangle$

(a) **[5 points]** Find the gradient of f at the point P.

$$\nabla f = \langle y^3 e^{xy}, 2y e^{xy} + x y^2 e^{xy} \rangle$$

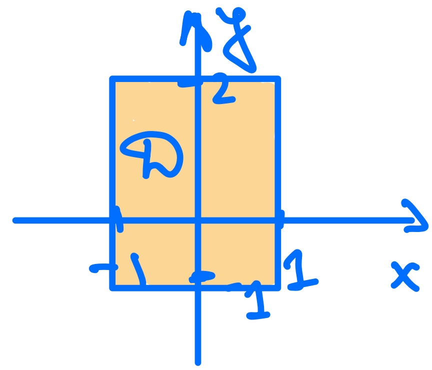
$$\nabla f(0,1) = \langle 1, 1 \rangle$$

(b) **[5 points]** Find the rate of change of f at P in the direction of the vector u.

$$\nabla f \cdot u = \langle 1, 1 \rangle \cdot \langle 3/5, 4/5 \rangle = \boxed{\frac{7}{5}}$$

(c) **[6 points]** Fully set up bounds and integrand for computing the **surface area** of f over the region $[-1, 1] \times [-1, 2]$. DO NOT EVALUATE.

$$A(S) = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA$$



$$f_x = y^3 e^{xy}$$

$$f_y = 2y e^{xy} + x y^2 e^{xy}$$

$$A(S) = \int_{-1}^{1} \int_{-1}^{2} \sqrt{(y^3 e^{xy})^2 + (2y e^{xy} + x y^2 e^{xy})^2 + 1}\, dA.$$