Lecture #31 - Week 10 - The fundamental theorem for line integrals - 16.3

Recall, FTC

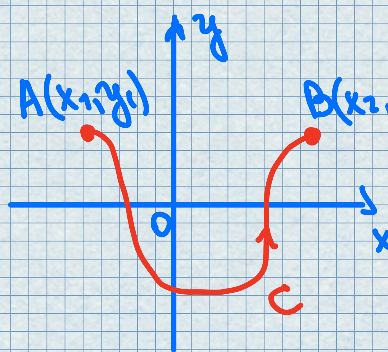
$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

where $F' \in C[a,b]$.

Theorem Let C be a smooth curve given by the vector function $r(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

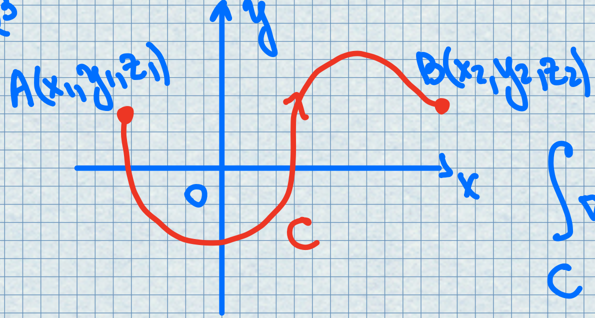
$$\boxed{\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))}$$

$\mathbb{R}^2$



$$\int_C \nabla f \cdot dr = f(x_2, y_2) - f(x_1, y_1)$$

$\mathbb{R}^3$



$$\int_C \nabla f \cdot dr = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

- Independence of path

Suppose C_1 and C_2 are two piecewise-smooth curves (paths) that have the same initial point A and terminal point B.

$$\int_{C_1} F \cdot dr \neq \int_{C_2} F \cdot dr$$

But

$$\boxed{\int_{C_1} \nabla f \cdot dr = \int_{C_2} \nabla f \cdot dr}$$

whenever ∇f is continuous.

In other words, the line integral of a conservative vector field depends only on the initial point and terminal point of a curve.

- In general, if F is a continuous vector field with domain D,

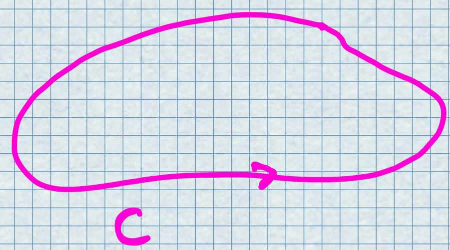
the line integral $\int_C F \cdot dr$ is independent of path

if

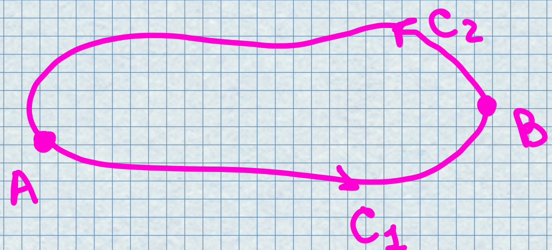
$$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr \quad \text{for any two paths}$$

C_1 and C_2 in D that have the same A and B points.

Def. A curve is closed if its terminal point coincides with its initial point ($r(b) = r(a)$).



If $\int_C F\cdot dr$ is independent of path in D and C is any closed path in D, then



$$\int_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr = \int_{C_1} F\cdot dr - \int_{-C_2} F\cdot dr = 0$$

since C_1 and $-C_2$ have the same initial and terminal points.

Conversely,

if $\int_C F\cdot dr = 0$ whenever C is closed path in D,

then

$$0 = \int_C F\cdot dr = \int_{C_1} F\cdot dr + \int_{-C_2} F\cdot dr = \int_{C_1} F\cdot dr - \int_{C_2} F\cdot dr$$

and so
$$\int_{C_1} F\cdot dr = \int_{C_2} F\cdot dr$$

Theorem $\int_C F\cdot dr$ is independent of path in D if and only if $\int_C F\cdot dr = 0$ for every closed path C in D.

Def. The region D is open if for every point P in D there is a disk with center P that lies entirely in D.

The region D is connected if any two points in D can be joined by a path that lies in D.

Theorem Suppose F is a vector field that is continuous on an open connected region D. If $\int_C F\cdot dr$ is independent of path in D, then F is a conservative vector field on D; that is, there exists a function f such that
$$\nabla f = F.$$

Question: How is it possible to determine whether or not a vector field F is conservative?

Suppose $F = Pi + Qj$ is conservative.
Then there is a function f such that

$$F = \nabla f \iff P = \frac{\partial f}{\partial x} \quad Q = \frac{\partial f}{\partial y}$$

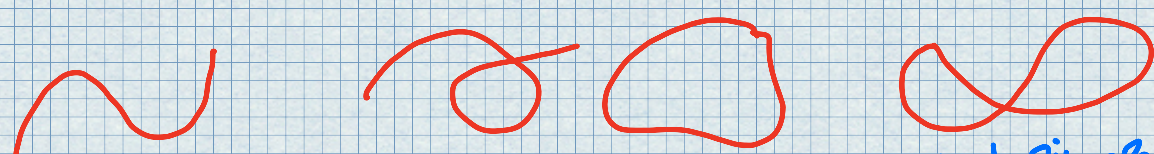
Therefore,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

Theorem If $F(x,y) = P(x,y)i + Q(x,y)j$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

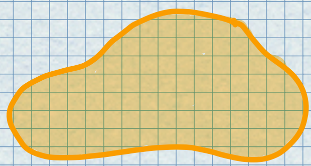
$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

Def. A curve is simple if it doesn't intersect itself anywhere between its endpoints.

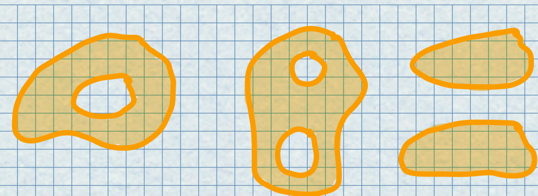


simple, not closed — not simple, not closed — simple, closed — not simple, closed

Def. A simply-connected region in the plane is a connected region D s.t. every simple closed curve in D encloses only points that are in D.



simply-connected region



not simply-connected regions

Theorem

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D.$$

Then $\mathbf{F}$ is conservative.

- **Conservation of Energy**

Let $\mathbf{F}$ be a continuous force field that moves an object along a path C given by $\mathbf{r}(t)$, $a \le t \le b$, where $\mathbf{r}(a) = A$, $\mathbf{r}(b) = B$.

By Newton's Second Law of Motion:

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

So

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt =$$

$$= \frac{m}{2}\int_a^b \frac{d}{dt}\left(\mathbf{r}'(t) \cdot \mathbf{r}'(t)\right)dt$$

$$= \frac{m}{2}\left(|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2\right)$$

Therefore,

$$\boxed{W = \frac{1}{2} m |v(b)|^2 - \frac{1}{2} m |v(a)|^2}$$

$$\boxed{W = K(B) - K(A)}$$

kinetic energy

Now let's assume that F is a conservative force field : $F = \nabla f$.

The <u>potential energy</u> of an object at (x, y, z) is

$$P(x,y,z) = -f(x,y,z).$$

So

$$F = -\nabla P.$$

Then

$$\boxed{W = \int_C F \cdot dr = -\int_C \nabla P \cdot dr = P(A) - P(B)}$$

Hence,

$$\boxed{P(A) + K(A) = P(B) + K(B)}$$

Law of Conservation of energy

Examples

$\boxed{1.}$ Determine whether or not the vector field

$$F(x,y) = (x-y)i + (x-2)j$$

is conservative.

Solution

$F = Pi + Qj$ is conservative if

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$ in D (open and simply-connected)

$P(x,y) = x-y$ and $Q(x,y) = x-2$.

$$\frac{\partial P}{\partial y} = -1 \qquad \frac{\partial Q}{\partial x} = 1$$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, F is not conservative.

$\boxed{2.}$ Determine whether or not the vector field

$$F(x,y) = (3 + 2xy)i + (x^2 - 3y^2)j$$

is conservative.

Solution

$$P(x,y) = 3 + 2xy \qquad Q(x,y) = x^2 - 3y^2$$

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$$

$Dom(F) = \mathbb{R}^2$, which is open and simply-connected. Therefore, F is conservative.

$\boxed{3.}$ If $F(x,y,z) = y^2 i + (2xy + e^{3z}) j + 3ye^{3z} k$, find a function f such that $\nabla f = F$.

Solution

If there is such a function, then

$$f_x(x,y,z) = y^2$$
$$f_y(x,y,z) = 2xy + e^{3z}$$
$$f_z(x,y,z) = 3ye^{3z}$$

$$\int f_x \, dx = xy^2 + g(y,z) = f(x,y,z)$$

$$f_y(x,y,z) = 2xy + g_y(y,z) = 2xy + e^{3z}$$

Hence, $$g_y(y,z) = e^{3z}$$

Thus $$g(y,z) = ye^{3z} + h(z) \text{ and}$$

$$f(x,y,z) = xy^2 + ye^{3z} + h(z)$$

Then,

$$f_z = h'(z) + 3ye^{3z} = 3ye^{3z}$$

$$h'(z) = 0 \Rightarrow h(z) = \text{const} = K$$

The desired function is

$$\boxed{f(x,y,z) = xy^2 + ye^{3z} + K}$$

and

$\nabla f = F$ holds.

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