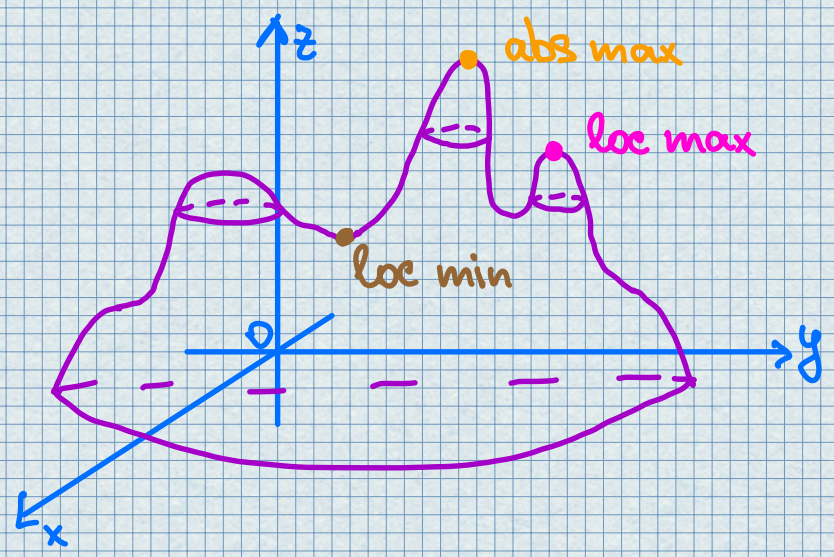
Lecture #13 - Week 6 - Max and Min Values - 14.7



Def. 1 A function of two variables has a <u>local maximum</u> at (a,b) if

$$f(x,y) \le f(a,b) \quad \text{when } (x,y) \text{ is near } (a,b)$$

The number $f(a,b)$ is called a local max value.

If $f(x,y) \ge f(a,b)$ when (x,y) is near (a,b), then f has a local min at (a,b) and $f(a,b)$ is a loc. min value.

Def. 2 If def. 1 holds for all points (x,y) in $\text{dom}(f)$, then f has an absolute max (or absolute min) at (a,b).

Theorem If f has a loc max or min at (a,b) and the first-order partial derivatives of f exists there, then

$$f_x(a,b) = 0 \quad \text{and} \quad f_y(a,b) = 0.$$

Def. A point (a,b) is a critical point (or stationary point) of f if $f_x(a,b)=0$ and $f_y(a,b)=0$, or if one of there partial derivatives DNE.

Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$. Let

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$$

(a) If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a loc. min

(b) If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a loc. max

(c) If $D<0$, then $f(a,b)$ is not a loc. max or min.

Note 1

In case (c) the point (a,b) is called a saddle point of f and the graph of f crosses its tangent plane at (a,b).

Note 2 If $D=0$, the test gives no information: f could have a local maximum or loc. min at (a,b), or (a,b) could be a saddle point of f.

Note 3

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

- **Absolute Maximum and Minimum Values**

Def. A <u>closed set</u> in $\mathbb{R}^2$ is one that contains all its boundary points.
A <u>bounded set</u> in $\mathbb{R}^2$ is one that is contained within some disk.

Extreme Value Theorem for Functions of Two Variables

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the absolute max and min values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of those values is the absolute minimum value.

Examples

1. Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14.$

Then

$$f_x(x,y) = 2x - 2$$
$$f_y(x,y) = 2y - 6$$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 3 \end{cases}$$

CP: $(1,3)$

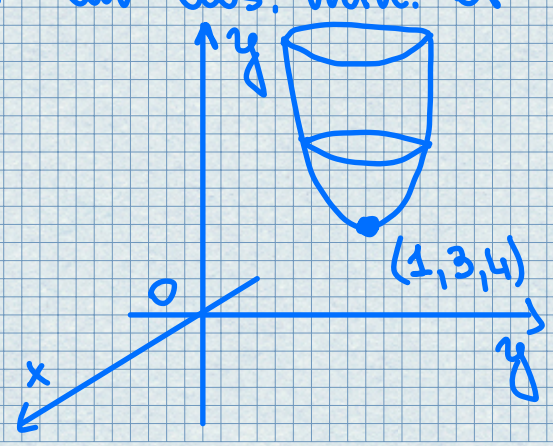
$$f(x,y) = 4 + (x-1)^2 + (y-3)^2$$

Since $(x-1)^2 \geq 0$, $(y-3)^2 \geq 0$, we have that $f(x,y) \geq 4$ for all $x, y \in \mathbb{R}$.

Therefore,

$f(1,3) = 4$ is a loc. min.

and it is an abs. min. of f.



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$\boxed{2.}$ Find the extreme values of

$$f(x,y) = y^2 - x^2.$$

Solution

$$\begin{cases} f_x = -2x = 0 \\ f_y = 2y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

CP: $(0,0)$.

For $y=0$: $f(x,y) = -x^2 < 0$ $(x \neq 0)$

$x=0$: $f(x,y) = y^2 > 0$ $(y \neq 0)$

Thus every disk with center $(0,0)$ contains points where f takes positive values as well as points where $f < 0$.

Therefore, $f(0,0) = 0$ can't be an extreme value for f, so f has no extreme value.

$\boxed{3.}$ Find the loc. minimum and maximum and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.

Solution

$$\begin{cases} f_x = 4x^3 - 4y = 0 \\ f_y = 4y^3 - 4x = 0 \end{cases}$$

$$\begin{cases} x^3 - y = 0 \\ y^3 - x = 0 \end{cases}$$

$$x = y^3$$

$$y^9 - y = 0$$

$$y(y^8 - 1) = 0$$

$$y(y^4 - 1)(y^4 + 1) = 0$$

$$y(y^2 - 1)(y^2 + 1)(y^4 + 1) = 0$$

So, we have real roots : $y = 0, +1, -1$.

Hence, CP: $(0,0), (1,1), (-1,-1)$.

$$f_{xx} = 12x^2 \qquad f_{xy} = -4$$

$$f_{yy} = 12y^2 \qquad f_{yx} = -4$$

$$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

$D(0,0) = -16 < 0$

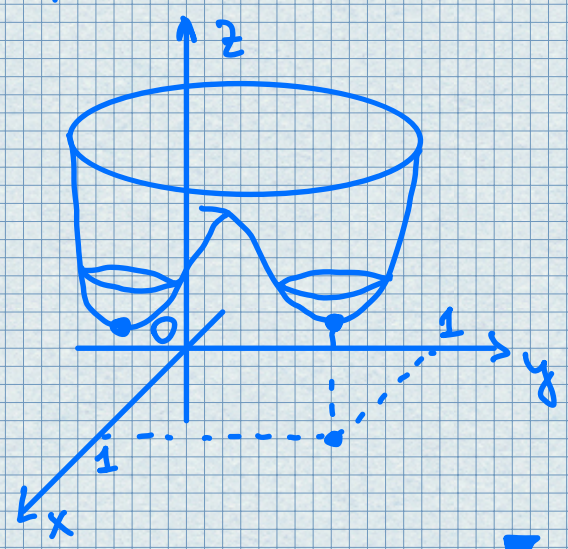
By the second derivatives test, $(0,0)$ is a saddle point.

$D(1,1) = 144-16 = 128 > 0$ and $f_{xx}(1,1) =$
$= 12 > 0$.

By the SDT, $f(1,1) = -1$ is a local minimum.

$D(-1,-1) = 128 > 0$ and $f_{yy}(-1,-1) = 12 > 0$.

So $f(-1,-1) = -1$ is a local minimum.



$\boxed{4.}$ Find the shortest distance from the point $(1,0,-2)$ to the plane $x + 2y + z = 4$.

Solution

(x,y,z) $\qquad$ $(1,0,-2)$

$$d = \sqrt{(x-1)^2 + (y-0)^2 + (z+2)^2}$$

But if (x,y,z) lies on the plane

$x + 2y + z = 4$, then $z = 4 - x - 2y$.

And

$$d = \sqrt{(x-1)^2 + y^2 + (4 - x - 2y + 2)^2}$$

$$d^2 = f(x,y) = (x-1)^2 + y^2 + (6 - x - 2y)^2$$

$$f_x = 2(x-1) - 2(6 - x - 2y) = 4x + 4y - 14 = 0$$

$$f_y = 2y - 4(6 - x - 2y) = 4x + 10y - 24 = 0$$

CP: $\left(\frac{11}{6}, \frac{5}{3}\right)$.

We have that

$$f_{xx} = 4 \qquad f_{xy} = f_{yx} = 4$$

$$f_{yy} = 10$$

$D(x,y) = 40 - 16 = 24 > 0$ and $f_{xx} > 0$.

So, by SDT f has a loc. min. at $\left(\frac{11}{6}, \frac{5}{3}\right)$.

Also, $f\left(\frac{11}{6}, \frac{5}{3}\right)$ is an abs. minimum.

If $x = \frac{11}{6}$, $y = \frac{5}{3}$, then

$$d = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{6}\right)^2} = \frac{5}{6}\sqrt{6}.$$

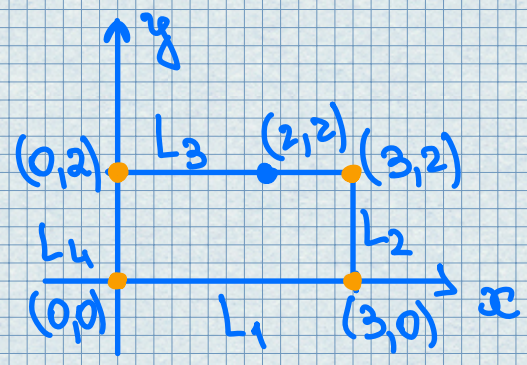
Thus, the shortest distance from $(1,0,-2)$ to the plane $x+2y+z=4$ is $\frac{5}{\sqrt{6}}$.

$\boxed{5.}$ Find the absolute max. and min. values of the function

$$f(x,y) = x^2 - 2xy + 2y \quad \text{on the}$$

rectangle $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$.

Solution



① f is cont. on the closed, bounded rectangle D. Thus f has an abs. min and max.

$$\begin{cases} f_x = 2x - 2y = 0 \\ f_y = -2x + 2 = 0 \end{cases}$$

CP: $(1,1)$

$f(1,1) = 1$.

② Now we look at values of f on the boundary of D, which contains L_1, L_2, L_3, L_4.

On L_1: $y=0$ and

$$f(x,0) = x^2 \qquad 0 \le x \le 3$$

$f_{min}(0,0) = 0.$

$f_{max}(3,0) = 9$

On L_2: $x = 3$ and

$$f(3,y) = 9 - 4y \qquad 0 \le y \le 2$$

$f_{max}(3,0) = 9$

$f_{min}(3,2) = 1$

On L_3: $y = 2$ and

$$f(x,2) = x^2 - 4x + 4 \qquad 0 \le x \le 3$$

$$f(x,2) = (x-2)^2$$

$f_{max}(0,2) = 4$

$f_{min}(2,2) = 0$

On L_4: $x = 0$ and

$$f(0,y) = 2y \qquad 0 \le y \le 2$$

$f_{max}(0,2) = 4$

$f_{min}(0,0) = 0.$

Thus, on the boundary, $f_{min} = 0$ and $f_{max} = 9.$

③ $f(0,0) = f(2,2) = 0$

$f(3,0)=9$

$f(1,1)=1$

Hence, • f has an abs. max on D and $f_{abs.max}(3,0)=9$

• f has an abs. min at $(0,0)$ and $(2,2)$ and $f(0,0)=f(2,2)=0$.

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