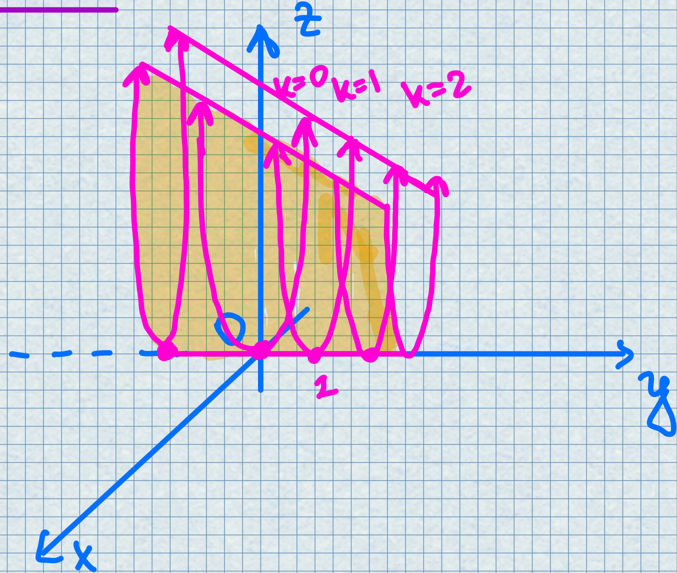
Lecture #6 - Week 2 - Quadratic Surfaces - 12.6

- **Cylinders**

A cylinder is a surface that consists of all lines (rulings) that are parallel to a given line and pass through a given plane curve.

Example Sketch the graph of the surface $z = x^2$.

Solution



$z = x^2$ ← parabola in xz-plane

Since there is no y included in $z = x^2$, we have that $\boxed{y = k}$, where k is a parameter

- **Quadratic Surfaces**

Def. A quadratic surface is the graph of a second-degree equation in three variables x, y, z.

In general, it is described by equation

$$\boxed{Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0}$$

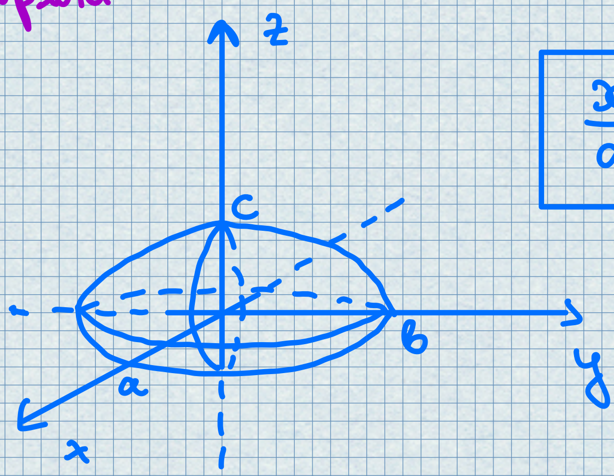
where $A, B, C, \dots, J$ are constants.

By translation and rotation this equation can get the form

$$\boxed{Ax^2 + By^2 + Cz^2 + J = 0} \quad \text{or} \quad \boxed{Ax^2 + By^2 + Iz = 0}$$

Graphs of Quadratic Surfaces

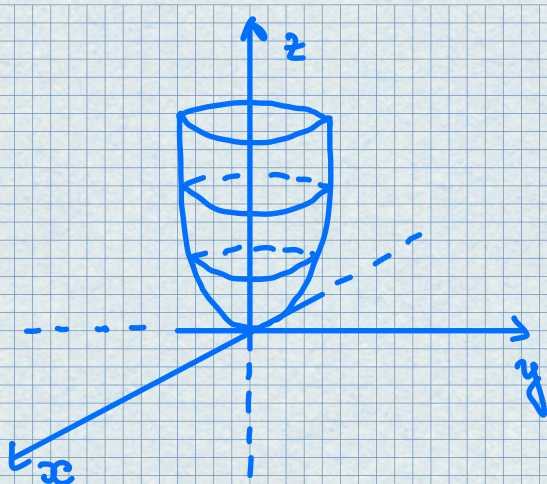
① Ellipsoid



$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$$

All traces are ellipses.
If $a=b=c$, the ellipsoid is a sphere.

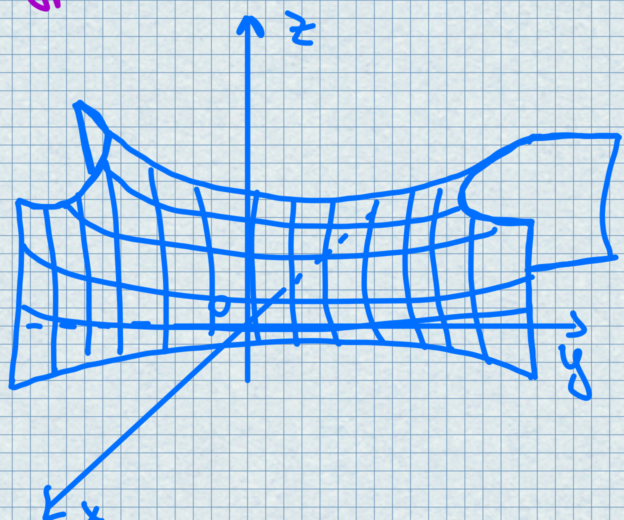
② Elliptic Paraboloid



$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Horizontal traces are ellipses.
Vertical traces are parabolas.
The variable raised to the first power indicates the axis of the paraboloid.

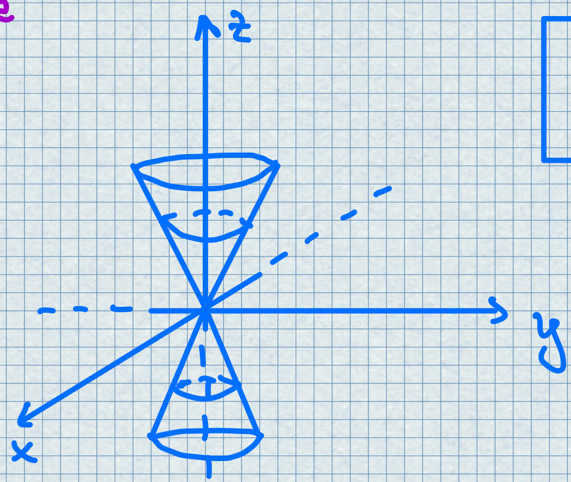
③ Hyperbolic Paraboloid.



$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Horizontal traces are hyperbolas.
Vertical traces are parabolas.
The case where $\boxed{c<0}$ is illustrated.

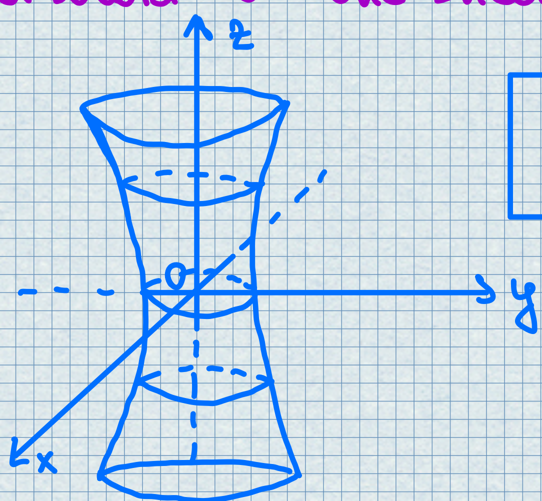
④ Cone



$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Horizontal traces are ellipses.
Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.

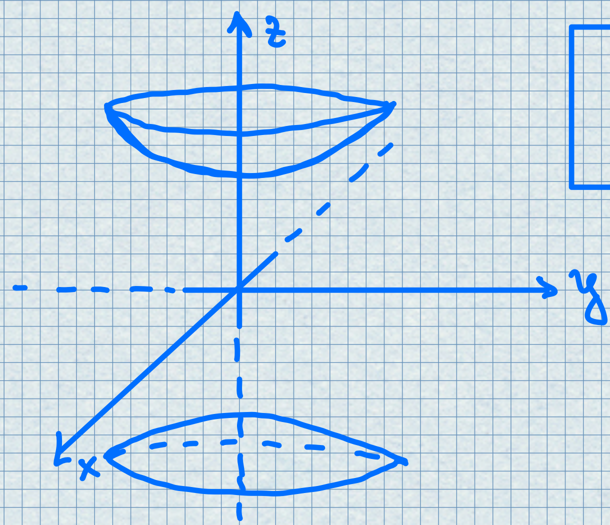
⑤ Hyperboloid of One Sheet



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Horizontal traces are ellipses.
Vertical traces are hyperbolas.
The axis of symmetry corresponds to the variable whose coefficient is negative.

⑥ Hyperboloid of Two Sheets



$$\boxed{-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$$

Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$.

Vertical traces are hyperbolas.

The two minus signs indicate two sheets.

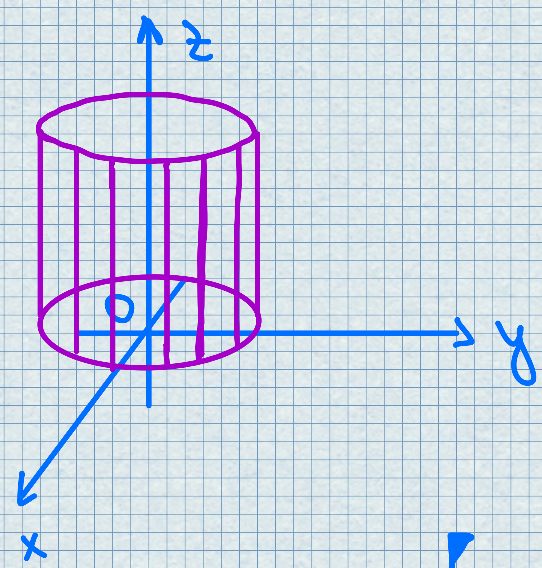
Examples

$\boxed{1.}$ Identify and sketch the surface

$$x^2 + y^2 = 1$$

Solution

Since z is missing and equations $x^2+y^2=1$, $z=k$ represent a circle with $r=1$ in the plane $z=k$, the surface $x^2+y^2=1$ is a circular cylinder whose axis is the z-axis.



$\boxed{2.}$ Use traces to sketch the quadric surface with equation

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1.$$

Solution

$z=0$: $\quad x^2 + \frac{y^2}{9} = 1 \quad$ (ellipse)

The horizontal trace in the plane $z=k$ is

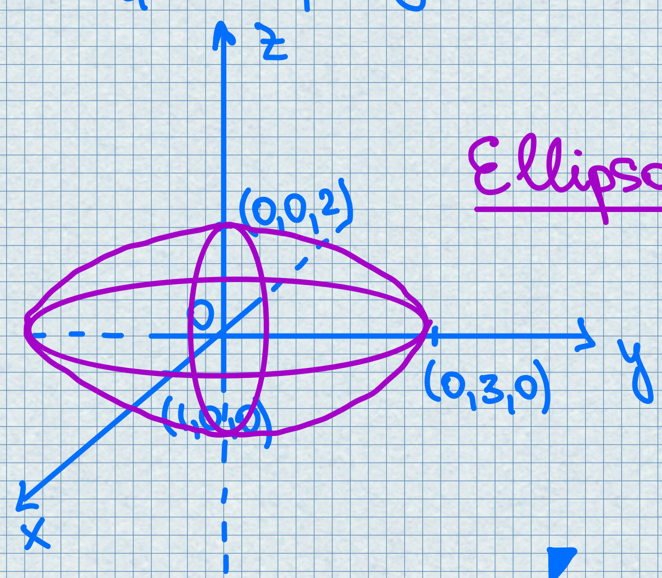
$$x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4} \qquad z=k$$

which is an ellipse for $k^2 < 4$, $-2<k<2$.

Similarly,

$x=0$: $\quad \frac{y^2}{9} + \frac{z^2}{4} = 1-k^2 \quad x=k \quad (-1<k<1)$

$y=0$: $\quad x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9} \quad y=k \quad (-3<k<3)$



Ellipsoid

$\boxed{3.}$ Sketch the surface $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1.$

Solution

$z=k$: $\quad \dfrac{x^2}{4}+y^2=1+\dfrac{k^2}{4} \quad z=k$

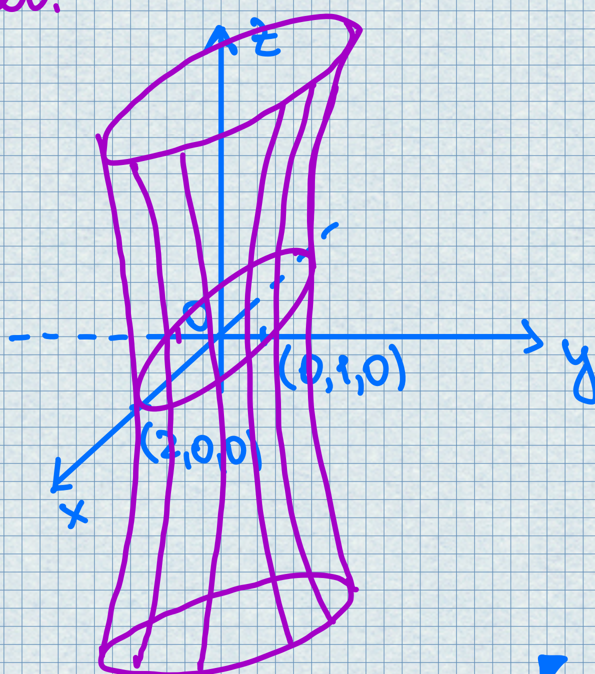
but for

$x=0$: $\quad y^2-\dfrac{z^2}{4}=1$

are the hyperbolas.

$y=0$: $\quad \dfrac{x^2}{4}-\dfrac{z^2}{4}=1$

This surface is called a hyperboloid of one sheet.



$\boxed{4.}$ Classify the quadric surface

$$x^2+2z^2-6x-y+10=0.$$

Solution

By completing a square we get

$$\boxed{y-1=(x-3)^2+2z^2}$$

elliptic paraboloid

The axis of the paraboloid is parallel to the y-axis, and it has been shifted. The vertex is $(3,1,0)$.

For $y=k$ $(k>1)$: $\underline{(x-3)^2+2z^2=k-1}$, $y=k$

ellipses

The trace in the xy-plane is the parabola with equation $y=1+(x-3)^2$, $z=0$.