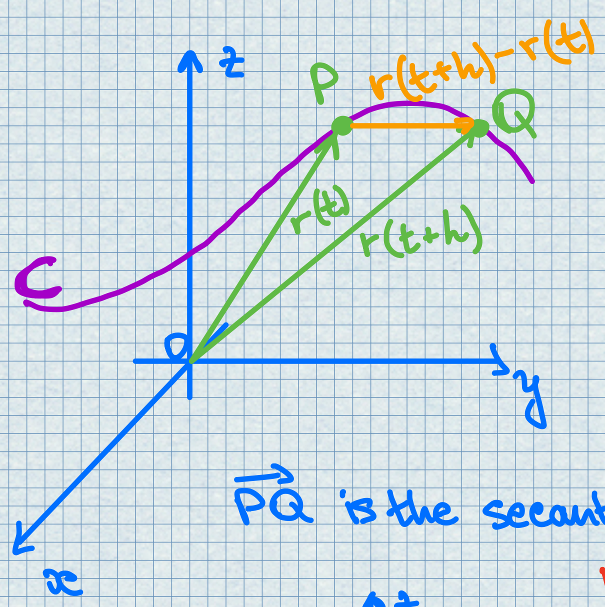
Lecture #8 - Week 3 - Derivatives and Integrals of Vector Functions - 13.2

- **Derivatives**

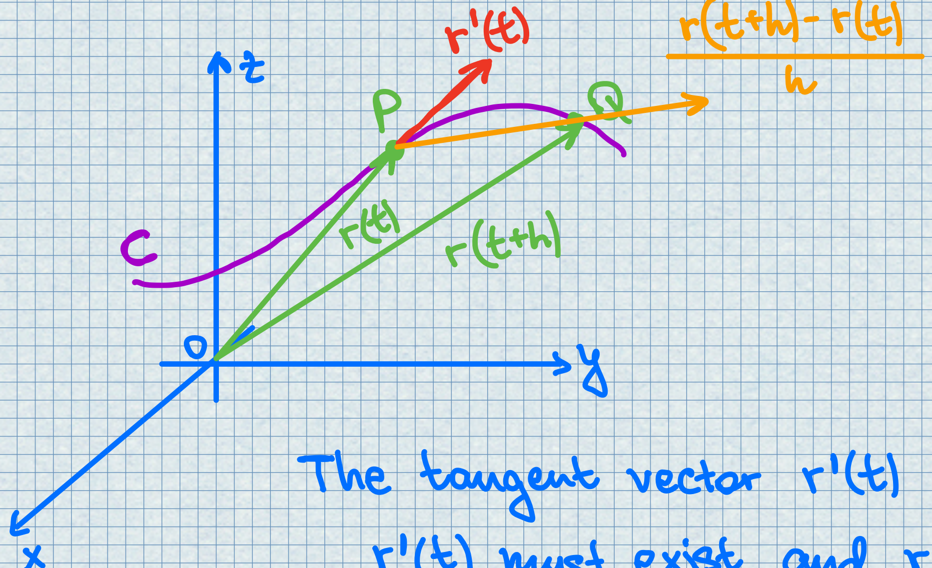
<u>Def.</u> The derivative r' of a vector function r is

$$\boxed{\frac{dr}{dt} = r'(t) = \lim_{h\to 0} \frac{r(t+h) - r(t)}{h}} \qquad (1)$$

if this limit exists.



$\overrightarrow{PQ}$ is the secant vector



The tangent vector $r'(t)$

$r'(t)$ must exist and $r'(t) \neq 0$

The tangent line to C at P is defined to be the line through P parallel to the tangent vector $r'(t)$.

Def. The unit tangent vector

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

Theorem If $r(t) = \langle f(t), g(t), h(t) \rangle = f(t)i + g(t)j + h(t)k$, where f, g, h are differentiable functions, then

$$\boxed{r'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)i + g'(t)j + h'(t)k}$$

- **Differentiation Rules**

Theorem Suppose u and v are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}\left(u(t) + v(t)\right) = u'(t) + v'(t)$
2. $\frac{d}{dt}\left(c\,u(t)\right) = c\,u'(t)$.
3. $\frac{d}{dt}\left(f(t)\,u(t)\right) = f'(t)\,u(t) + f(t)\,u'(t)$
4. $\frac{d}{dt}\left(u(t) \cdot v(t)\right) = u'(t) \cdot v(t) + u(t) \cdot v'(t)$

⑤ $$\frac{d}{dt}\left(u(t)\times v(t)\right) = u'(t)\times v(t) + u(t)\times v'(t)$$

⑥ $$\frac{d}{dt}\left(u(f(t))\right) = f'(t)\,u'(f(t)) \quad \text{(Chain rule)}$$

- Integrals

The <u>definite integral</u> of a continuous vector function $r(t)$ can be defined in much the same way as for real-valued functions except that the integral is a vector.

$$\int_a^b r(t)\,dt = \lim_{n\to\infty}\sum_{i=1}^{n} r(t_i^*)\Delta t =$$

$$= \lim_{n\to\infty}\left[\left(\sum_{i=1}^{n} f(t_i^*)\Delta t\right)i + \left(\sum_{i=1}^{n} g(t_i^*)\Delta t\right)j + \left(\sum_{i=1}^{n} h(t_i^*)\Delta t\right)k\right]$$

and so

$$\boxed{\int_a^b r(t)\,dt = \left(\int_a^b f(t)\,dt\right)i + \left(\int_a^b g(t)\,dt\right)j + \left(\int_a^b h(t)\,dt\right)k}$$

<u>Fundamental Theorem of Calculus</u>

$$\int_a^b r(t)\,dt = R(t)\Big|_a^b = R(b) - R(a), \text{ where } R'(t) = r(t)$$

Examples

1. (a) Find the derivative of $r(t) = (1+t^3)i + t e^{-t} j + \sin 2t\, k$

(b) Find the unit tangent vector at the point where $t=0$.

Solution

(a) $r'(t) = 3t^2 i + (e^{-t} - te^{-t}) j + 2\cos 2t\, k$

(b) $r(0) = i$

$r'(0) = j + 2k$

The unit tangent vector at $(1,0,0)$ is

$$T(0) = \frac{r'(0)}{|r'(0)|} = \frac{j+2k}{\sqrt{1+4}} = \frac{1}{\sqrt{5}} j + \frac{2}{\sqrt{5}} k.$$

2. For the curve $r(t) = \sqrt{t}\, i + (2-t) j$, find $r'(t)$ and sketch the position vector $r(1)$ and the tangent vector $r'(1)$.

Solution

$$r'(t) = \frac{1}{2\sqrt{t}} i - j \qquad r'(1) = \frac{1}{2} i - j$$

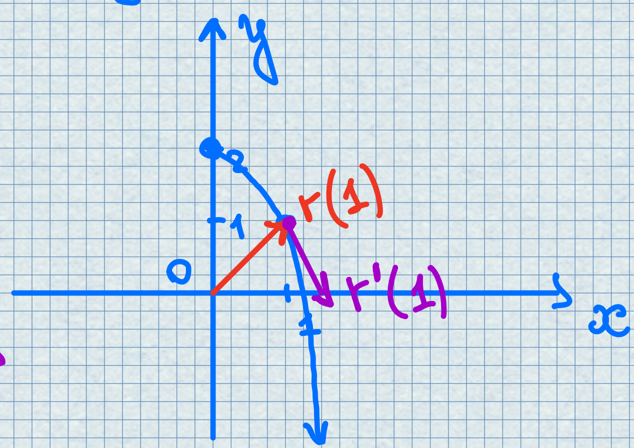
The curve is a plane curve and elimination of the parameter from $x = \sqrt{t}$, $y = 2 - t$ gives

$$y = 2 - x^2, \quad x \geq 0$$

$r(1) = i + j$

$r(1) = \langle 1, 1 \rangle$

$r'(1) = \langle \frac{1}{2}, -1 \rangle$



$\boxed{3.}$ Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $(0, 1, \pi/2)$.

Solution

$$r(t) = \langle 2\cos t, \sin t, t \rangle$$

$$r'(t) = \langle -2\sin t, \cos t, 1 \rangle$$

If $2\cos t = 0$

$\sin t = 1$

$\boxed{t = \frac{\pi}{2}}$

So $r'\left(\frac{\pi}{2}\right) = \langle -2, 0, 1 \rangle$.

The tangent line is the line through $\left(0, 1, \frac{\pi}{2}\right)$ parallel to the vector $\langle -2, 0, 1 \rangle$.

So its parametric equations are

$$x = -2t \qquad y = 1 \qquad z = \frac{\pi}{2} + t.$$

$\boxed{4.}$

If $r(t) = 2\cos t\, i + \sin t\, j + 2t\, k$, then

$$\int r(t)\,dt = i\int 2\cos t\,dt + j\int \sin t\,dt + k\int 2t\,dt =$$

$$= 2\sin t\, i - \cos t\, j + t^2 k + C.$$

$$\int_0^{\frac{\pi}{2}} r(t)\,dt = \left(2\sin t\, i - \cos t\, j + t^2 k\right)\Big|_0^{\frac{\pi}{2}} =$$

$$= 2i + j + \frac{\pi^2}{4}k.$$