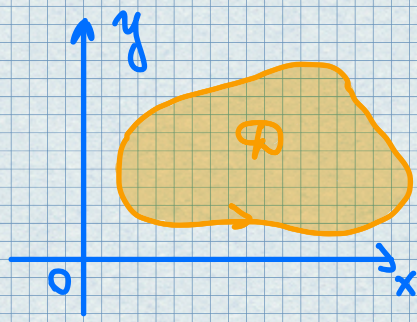
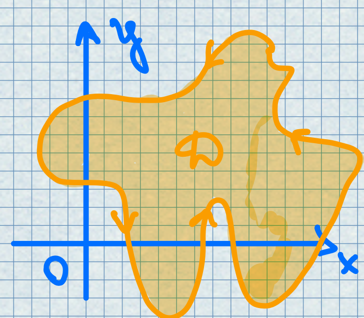
Lecture #32 - Week 11 - Green's Theorem - 16.4

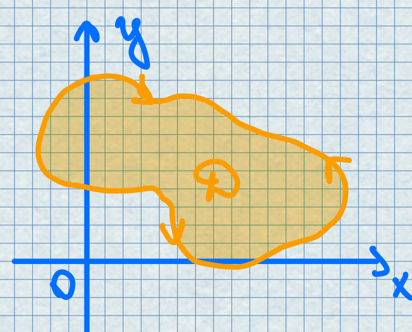


Green's theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.

Def. The positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C.



Positive orientation



Negative orientation

Green's Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C.
If P and Q have continuous partial derivatives on an open region that

contains D, then

$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

Note: The notation

$$\oint_C P\,dx + Q\,dy \quad \text{or} \quad \oint_C P\,dx + Q\,dy$$

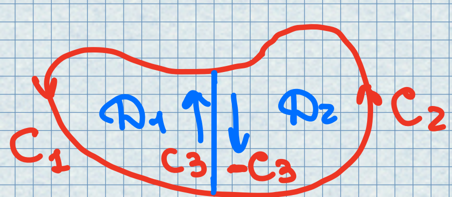
is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C.

Or another notation can be used:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$$

- Extended Versions of Green's Theorem

Let D be a region shown below



$D = D_1 \cup D_2$,

D_1 and D_2 are simple

- The boundary of D_1 is $C_1 \cup C_3$
- The boundary of D_2 is $C_2 \cup (-C_3)$

So, by Green's theorem:

$$\int_{C_1 \cup C_3} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

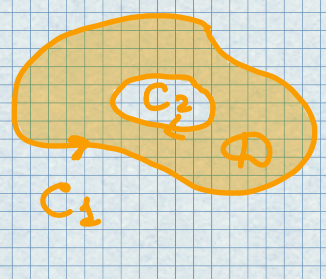
$$\int_{C_2 \cup (-C_3)} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

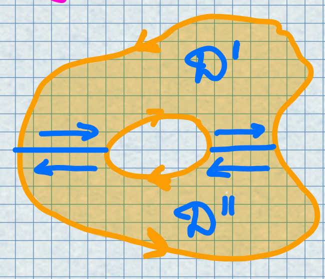
If we add these two equations, we obtain

$$\boxed{\int_{C_1 \cup C_2} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA} \quad (1)$$

(1) is Green's theorem for $D = D_1 \cup D_2$ since $C = C_1 \cup C_2$.

- Not simply-connected regions.





$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy$$

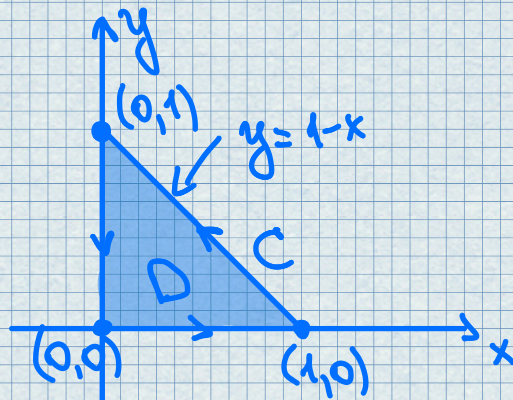
$$\boxed{\begin{aligned} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \\ &= \int_C P\,dx + Q\,dy \end{aligned}}$$

Green's theorem for D

Examples

$\boxed{1.}$ Evaluate $\displaystyle\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.

Solution



$P(x,y) = x^4$

$Q(x,y) = xy$

C is simple and has positive orientation

$$\int_C x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^{1-x} (y-0)\,dy\,dx =$$

$$= \int_0^1 \frac{1}{2} y^2 \Big|_0^{1-x} dx = \frac{1}{2}\int_0^1 (1-x)^2\,dx = -\frac{1}{6}(1-x)^3 \Big|_0^1 = \frac{1}{6}$$

▼

$\boxed{2.}$ Evaluate $\displaystyle\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4+1}\right)dy,$

where C is the circle $x^2+y^2=9$.

Solution

The region D bounded by C is the disk $x^2+y^2 \leq 9$, so we will switch to polar coordinates after applying Green's Theorem:

$$\oint_C \left(3y-e^{\sin x}\right)dx+\left(7x+\sqrt{y^4+1}\right)dy =$$

$$= \iint_D \left(\frac{\partial}{\partial x}\left(7x+\sqrt{y^4+1}\right)-\frac{\partial}{\partial y}\left(3y-e^{\sin x}\right)\right) dA =$$

$$= \int_0^{2\pi}\int_0^3 (7-3)\, r\, dr\, d\theta = 4\int_0^{2\pi} d\theta \int_0^3 r\, dr = 36\pi$$

▼

$\boxed{3.}$ Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution

We will be using the following formula:

$$\boxed{A = \frac{1}{2}\oint_C x\,dy - y\,dx}$$

The ellipse has parametric equations:

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \quad 0 \le t \le 2\pi$$

Hence,

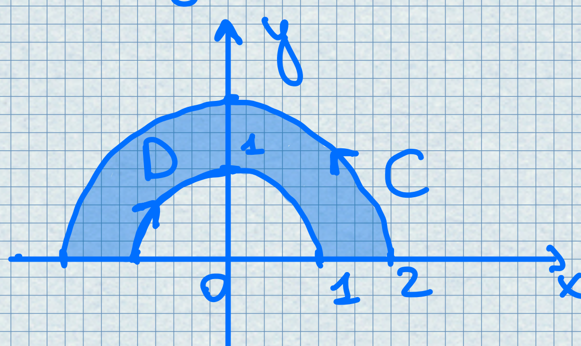
$$A = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\cdot$$

$$\cdot\, dt - (b\sin t)(-a\sin t)\,dt = \frac{ab}{2}\int_0^{2\pi} dt = \pi ab.$$

$\boxed{4.}$ Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of D in the upper half-plane between $x^2+y^2=1$ and $x^2+y^2=4$.

Solution

D is not simple, the y-axis divides it



into two simple regions.

In polar coordinates:

$$D = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$$

Therefore, by Green's Theorem

$$\oint_C y^2\,dx + 3xy\,dy = \iint_D \left(\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\right) dA =$$

$$= \iint_D y\,dA = \int_0^\pi \int_1^2 (r\sin\theta)\, r\,dr\,d\theta =$$

$$= \int_0^\pi \sin\theta\,d\theta \int_1^2 r^2\,dr = (-\cos\theta)\Big|_0^\pi \cdot \frac{1}{3}r^3\Big|_1^2 = \frac{14}{3}.$$

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