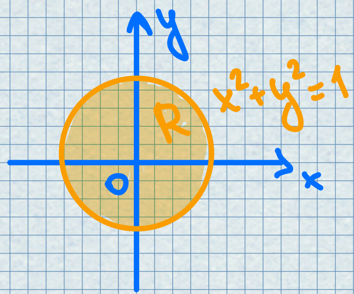
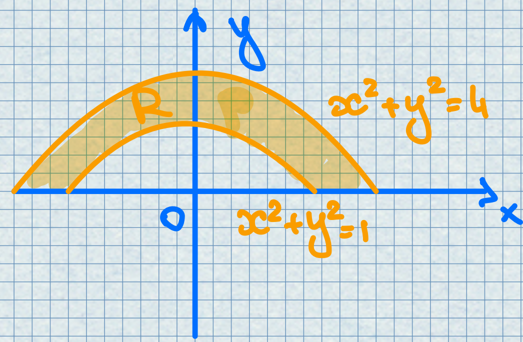
Lecture #22 - Week 7 - Double integrals in polar coordinates - 15.3

Suppose that R is one of the regions shown below



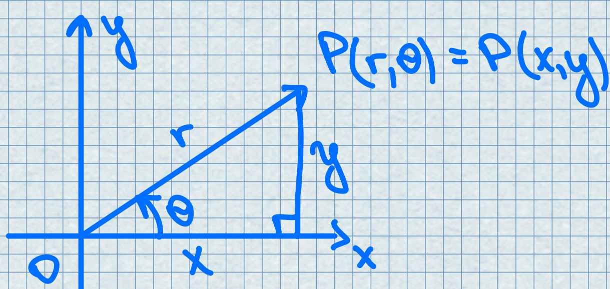


$R=\{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

$R=\{(r,\theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi\}$

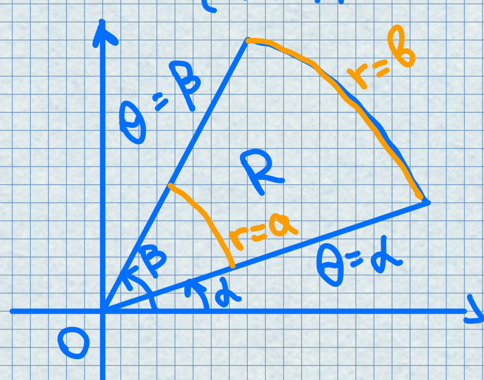
Recall that

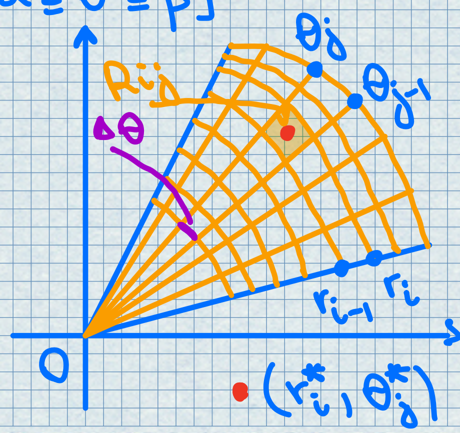
$$\boxed{r^2 = x^2+y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta}$$



Polar Rectangle

$R=\{(r,\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$





We have that

$$\Delta r = (b-a)/m$$
$$\Delta\theta = (\beta-\alpha)/n$$

Then

$$R_{ij} = \{(r,\theta) \mid r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\}$$

and

$$r_i^* = \frac{1}{2}(r_{i-1}+r_i) \qquad \theta_j^* = \frac{1}{2}(\theta_{j-1}+\theta_j)$$

$$\Delta A_i = \frac{1}{2} r_i^2 \Delta\theta - \frac{1}{2} r_{i-1}^2 \Delta\theta = r_i^* \Delta r \Delta\theta$$

The rectangular coordinates of the center of R_{ij} are $(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)$, so the Riemann sum is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) \Delta A_i =$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r \Delta\theta$$

If $g(r,\theta) = r f(r\cos\theta, r\sin\theta)$, then

$$\lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} g(r_i^*, \theta_j^*) \Delta r \Delta\theta =$$

$$= \int_{\alpha}^{\beta} \int_{a}^{b} g(r,\theta)\, dr\, d\theta$$

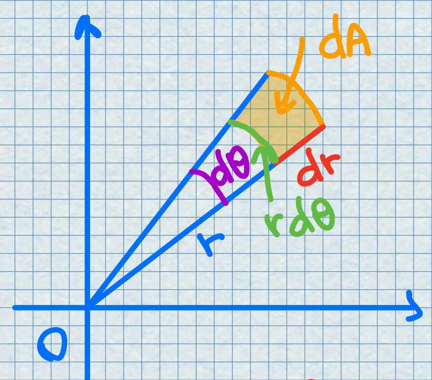
Therefore,

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Change to Polar Coordinates in a Double Integral

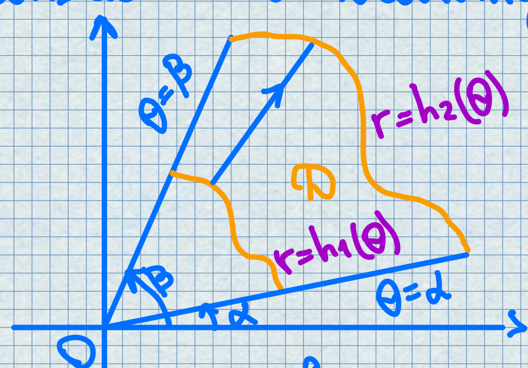
If $f \in C(R)$, where R is given by
$0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, $0 \leq \beta - \alpha \leq 2\pi$, then

$$(1) \quad \iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$



Be careful not to forget the additional factor r on the right side of (1).

Let us consider the following case:



$D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$

Statement If f is continuous on

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta .$$

In particular, if $f(x,y)=1$, $h_1 = 0$ and $h_2 = h$, $\theta=\alpha$, $\theta=\beta$, and $r = h(\theta)$, then

$$A(D) = \iint_D 1\, dA = \int_\alpha^\beta \int_0^{h(\theta)} r\, dr\, d\theta =$$

$$= \frac{1}{2}\int_\alpha^\beta (h(\theta))^2\, d\theta .$$

Examples

1. Evaluate $\iint_R (3x+4y^2)\,dA$, where

R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Solution

$$R=\{(x,y)\mid y\geq 0,\ 1\leq x^2+y^2\leq 4\}$$

In polar coordinates it is given by

$1\leq r\leq 2,\ 0\leq\theta\leq\pi$. Therefore,

$$\iint_R (3x+4y^2)\,dA=\int_0^{\pi}\int_1^2 (3r\cos\theta+4r^2\sin^2\theta)\,r\,dr\,d\theta=$$

$$=\int_0^{\pi}\int_1^2 (3r^2\cos\theta+4r^3\sin^2\theta)\,dr\,d\theta=$$

$$=\int_0^{\pi}\left(r^3\cos\theta+r^4\sin^2\theta\right)\Big|_{r=1}^{r=2}d\theta=\int_0^{\pi}(7\cos\theta+$$

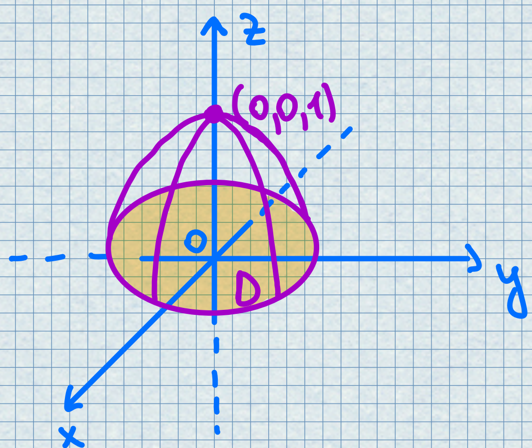
$$+15\sin^2\theta)\,d\theta=$$

$$= \int_0^{\pi} \left(7\cos\theta + \frac{15}{2}\left(1-\cos 2\theta\right)\right) d\theta =$$

$$= \left(7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta\right)\Big|_0^{\pi} = \frac{15\pi}{2}$$

$\boxed{2.}$ Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z = 1-x^2-y^2$.

Solution



$z=0$: $x^2+y^2=1$

D: $x^2+y^2 \le 1$

In polar coordinates:

D: $0 \le r \le 1$
$0 \le \theta \le 2\pi$

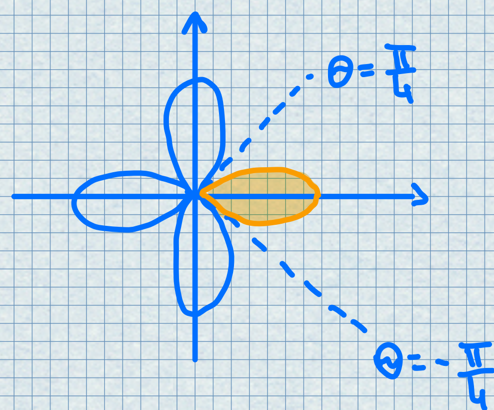
Since $1-x^2-y^2 = 1-r^2$, the volume is

$$V = \iint_D (1-x^2-y^2)\,dA = \int_0^{2\pi}\int_0^1 (1-r^2)r\,dr\,d\theta =$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r-r^3)\,dr = 2\pi\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}$$

3. Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

Solution



$$D = \{(r,\theta) \mid -\pi/4 \le \theta \le \tfrac{\pi}{4},\ 0 \le r \le \cos 2\theta\}$$

So the area is

$$A(D) = \iint_D dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} r\, dr\, d\theta =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \Big|_0^{\cos 2\theta} d\theta = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta =$$

$$= \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + \cos 4\theta)\, d\theta = \frac{1}{4}\left(\theta + \frac{1}{4}\sin 4\theta\right)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{8}.$$