Lecture # 23 - Week 10 - Change of variables in multiple integrals - 15.9

In one-dimensional calculus we often use a change of variable (a substitution) to simplify an integral.

$$\boxed{\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du}$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$.

Or

$$\boxed{\int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du}$$

A change of variables can also be useful in double integrals.

For instance,

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

and

$$\boxed{\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.

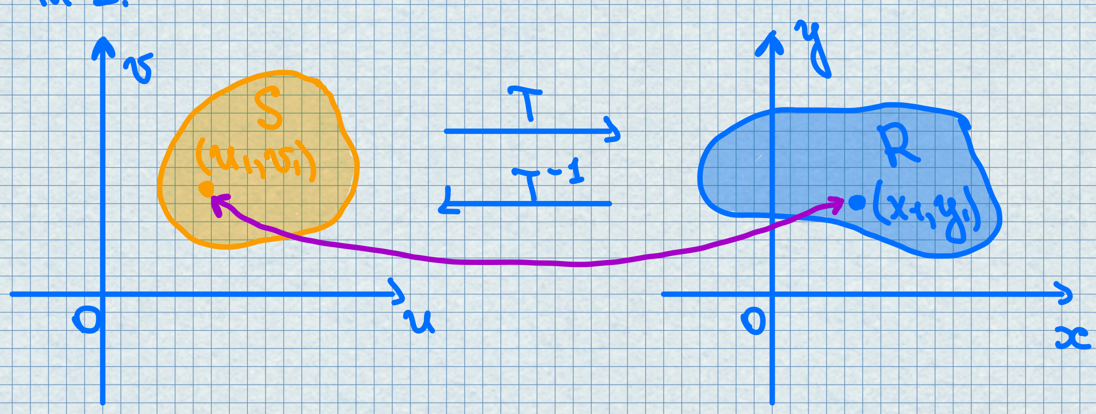
We consider a change of variables that is given by a transformation T from uv-plane to the xy-plane:

$$T(u,v) = (x,y)$$

$$x = g(u,v) \qquad y = h(u,v)$$

We usually assume that T is a C^1 transformation that is $g_u, g_v, h_u, h_v \in C(D)$.

Def.
- If $T(u_1,v_1) = (x_1,y_1)$, then (x_1,y_1) is the image of (u_1,v_1).
- If no points have the same image, then T is one-to-one.
- T transforms S into a region R in the xy-plane called the image of S, consisting of the images of all points in S.

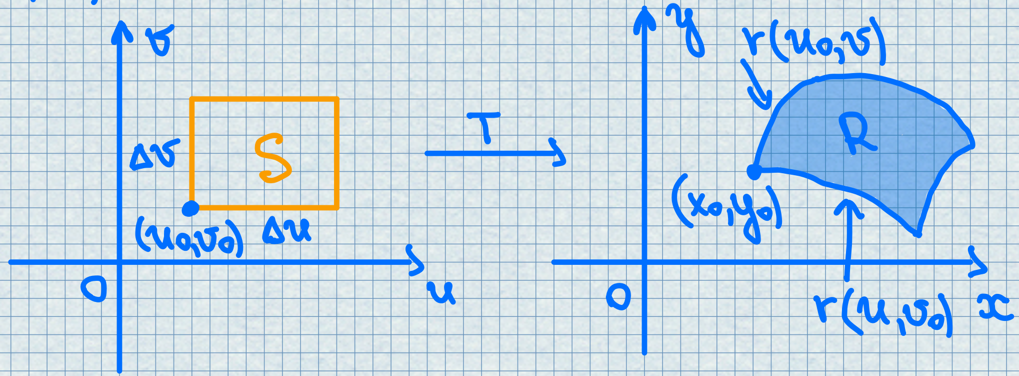


- If T is a one-to-one transformation, then it has an inverse transformation T^{-1} from the xy-plane to the uv-plane and

$$u = G(x,y) \qquad v = H(x,y)$$

- **Change of variables in a double integral.**

Let S be a rectangle in the uv-plane whose lower left corner is the point (u_0, v_0) and whose dimensions are Δu and Δv.



The vector

$$r(u,v) = g(u,v)i + h(u,v)j$$

is the position vector of the image of (u,v).

The image of $v = v_0$ is the curve given by $r(u, v_0)$.

The tangent vector at (x_0, y_0) to this image curve is

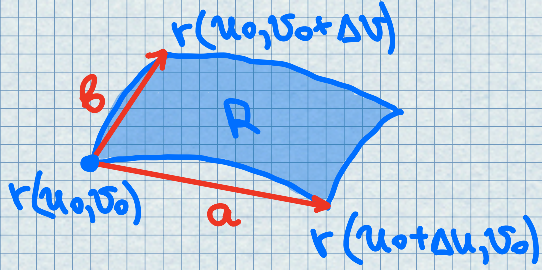
$$r_u = g_u(u_0,v_0)i + h_u(u_0,v_0)j = \frac{\partial x}{\partial u} i + \frac{\partial y}{\partial u} j$$

Similarly,

$$r_v = g_v(u_0,v_0)i + h_v(u_0,v_0)j = \frac{\partial x}{\partial v} i + \frac{\partial y}{\partial v} j$$

We can approximate the image region $R = T(S)$ by the secant vectors

$$a = r(u_0 + \Delta u, v_0) - r(u_0, v_0)$$
$$b = r(u_0, v_0 + \Delta v) - r(u_0, v_0)$$



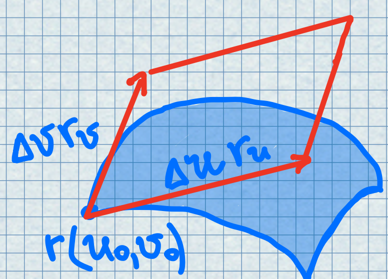
But

$$r_u = \lim_{\Delta u \to 0} \frac{r(u_0 + \Delta u, v_0) - r(u_0, v_0)}{\Delta u}$$

and so

$$r(u_0 + \Delta u, v_0) - r(u_0, v_0) \approx \Delta u\, r_u$$
$$r(u_0, v_0 + \Delta v) - r(u_0, v_0) \approx \Delta v\, r_v$$

This means that we can approximate R by a parallelogram determined by vectors $\Delta u\, r_u$ and $\Delta v\, r_v$.



Therefore,

$$\boxed{|(\Delta u\, r_u) \times (\Delta v\, r_v)| = |r_u \times r_v|\, \Delta u\, \Delta v}$$

From this it follows

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

<u>Def.</u> The <u>Jacobian</u> of the transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is

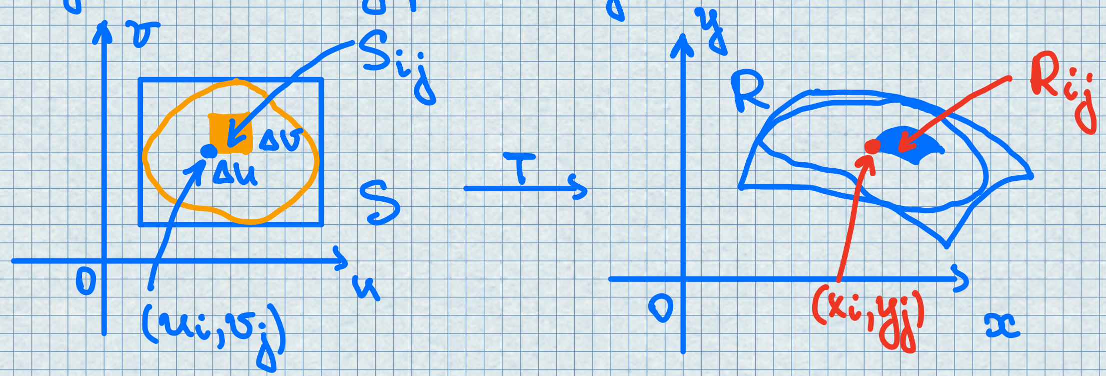
$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}$$

Hence, the approximation of the area ΔA of R is:

$$\Delta A \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \, \Delta v$$

where the Jacobian is evaluated at (u_0, v_0).

Now we divide a region S in the uv-plane into rectangles S_{ij} and call their images in the xy-plane R_{ij}.



$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\,\Delta A$$

$$\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_i,v_j), h(u_i,v_j)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v$$

where the Jacobian is evaluated at (u_i, v_j).

Change of variables in a double integral

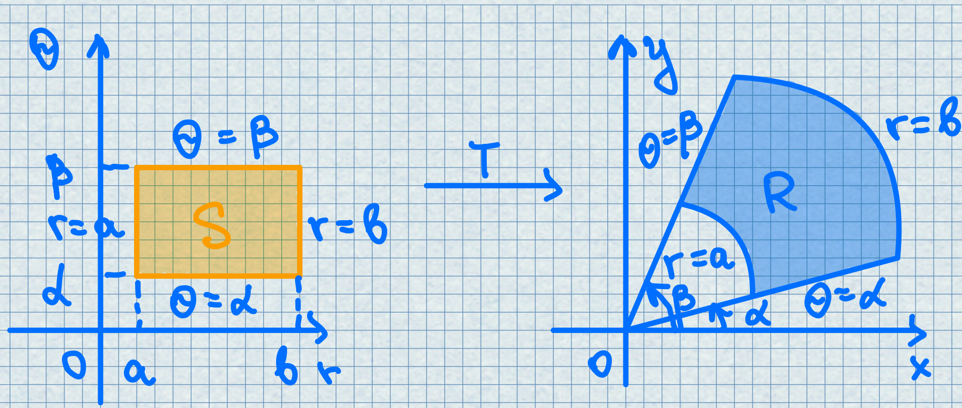
Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\boxed{\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

Now let T be a transformation from the $r\theta$-plane to the xy-plane:

$$x = g(r,\theta) = r\cos\theta \qquad y = h(r,\theta) = r\sin\theta$$

T maps an ordinary rectangle in the $r\theta$-plane to a polar rectangle in the xy-plane.



The Jacobian of T is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r > 0$$

Thus,

$$\boxed{\begin{aligned} \iint_R f(x,y)\,dx\,dy &= \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta \\ &= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta \end{aligned}}$$

- **Triple integrals**

Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations

$$x = g(u,v,w) \qquad y = h(u,v,w)$$
$$z = k(u,v,w)$$

The Jacobian of T is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(x(u,v,w),y(u,v,w),z(u,v,w)) \cdot$$

$$\cdot \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

Examples

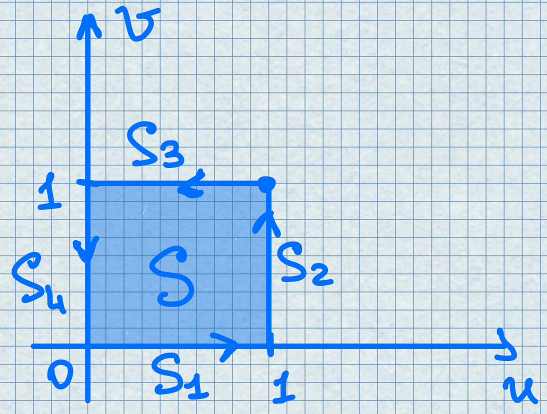
$\boxed{1.}$ A transformation is defined by

$$\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$$

Find the image of the square

$$S = \{(u,v) \mid 0 \leq u \leq 1,\ 0 \leq v \leq 1\}$$

Solution



S_1: $v = 0$

$0 \leq u \leq 1$

$\Downarrow$

$x = u^2$

$y = 0$

$0 \leq x \leq 1$

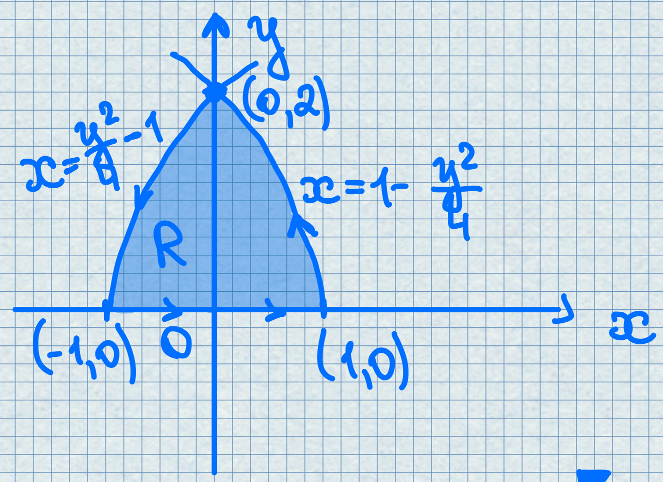
S_2: $\begin{matrix} u = 1 \\ 0 \leq v \leq 1 \end{matrix} \Rightarrow \begin{matrix} x = 1 - v^2 \\ y = 2v \end{matrix} \Rightarrow x = 1 - \frac{y^2}{4}$

$0 \leq x \leq 1$

S_3: $\begin{matrix} v = 1 \\ 0 \leq u \leq 1 \end{matrix} \Rightarrow x = \frac{y^2}{4} - 1$

$-1 \leq x \leq 0$

S_4: $\begin{matrix} u = 0 \\ 0 \leq v \leq 1 \end{matrix} \Rightarrow \begin{matrix} x = -v^2 \\ y = 0 \end{matrix} \quad -1 \leq x \leq 0$



$\boxed{2.}$ Use the change of variables $\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$ to evaluate the integral $\iint_R y \, dA$, where R is a region bounded by the x-axis and the parabolas $y^2 = 4-4x$ and $y^2 = 4+4x$, $y \geq 0$.

Solution

$T(S) = R$

$\boxed{\text{(See figure in Ex. 1)}}$

$S: [0,1] \times [0,1]$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

Therefore,

$$\iint_R y\,dA = \iint_S 2uv\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA = \int_0^1\int_0^1 (2uv)(u^2+v^2)\,du\,dv =$$

$$= 8\int_0^1\int_0^1 (u^3v + uv^3)\,du\,dv = 8\int_0^1 \left(\frac{1}{4}u^4v + \frac{1}{2}u^2v^3\right)\Big|_0^1 dv =$$

$$= \int_0^1 (2v + 4v^3)\,dv = \left(v^2 + v^4\right)\Big|_0^1 = 2.$$

$\boxed{3.}$ Use the formula for triple integrals to derive the formula for triple integration in spherical coordinates.

Solution

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

We compute the Jacobian:

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix} =$$

$$= \cos\varphi\left(-\rho^2\sin\varphi\cos\varphi\sin^2\theta - \rho^2\sin\varphi\cos\varphi\cos^2\theta\right) -$$

$$- \rho\sin\varphi\left(\rho\sin^2\varphi\cos^2\theta + \rho\sin^2\varphi\sin^2\theta\right) =$$

$$= -\rho^2\sin\varphi\cos^2\varphi - \rho^2\sin\varphi\sin^2\varphi = -\rho^2\sin\varphi$$

Since $0 \le \varphi \le \pi$, we have $\sin\varphi \ge 0$.

Therefore,

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = \left|-\rho^2\sin\varphi\right| = \rho^2\sin\varphi$$

and

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) \cdot$$

$$\cdot\ \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi.$$

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