Lecture #12 - Week 5 - Functions of Several Variables

14.1

• Functions of two variables

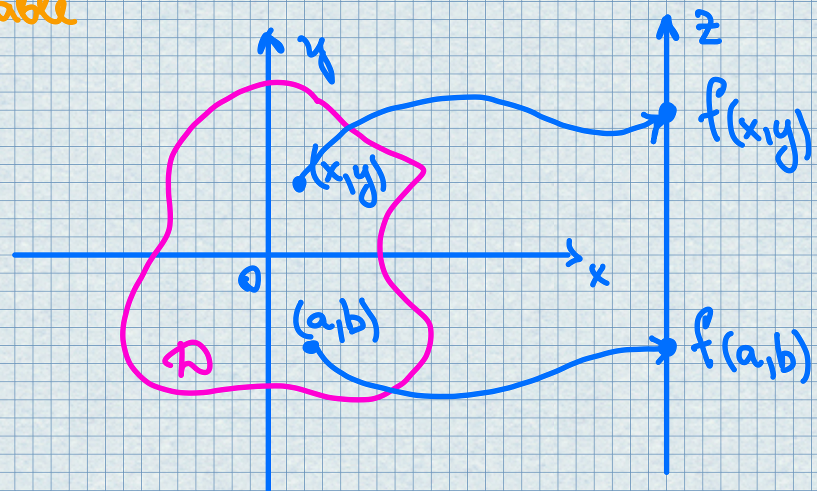
Def. A function f of two variables is a rule that assigns to each ordered pair of real numbers $(x,y) \in D$ a unique real number $f(x,y)$.

The set D is the domain of f and its range is the set of values that f takes on, that is, $\{f(x,y) \mid (x,y) \in D\}$.

$$z = f(x,y)$$

z: dependent variable

x, y: independent variables

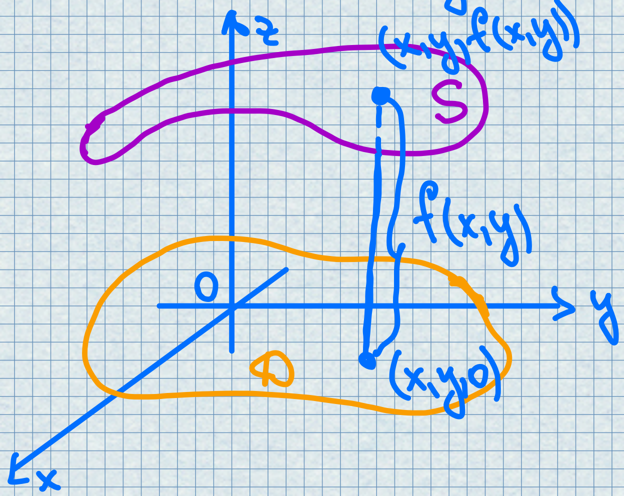


• Graphs

Def. If f is a function of two variables with domain D, then the graph of f is the set of all points $(x,y,z) \in \mathbb{R}^3$

such that

$$z = f(x,y) \quad \text{and} \quad (x,y) \in \mathcal{D}$$



Def. The function

$$f(x,y) = ax + by + c$$

is called a <u>linear function</u>.
Then the graph of such a function has the equation

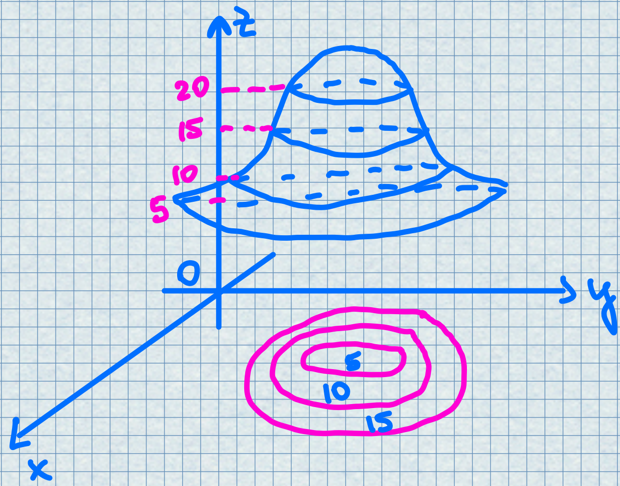
$$z = ax + by + c \quad \text{or} \quad ax + by - z + c = 0$$

and is a plane.

- **Level curves.**

Def. The level curves of a function f of two variables are the curves with equations

$$f(x,y) = k, \text{ where } k \text{ is a constant}$$



- Functions of three or more variables.

Def. A function of tree variables, f, is a rule that assigns to each ordered triple (x,y,z) in $D \subset \mathbb{R}^3$ a unique real number $f(x,y,z)$.

Also, $f(x,y,z)=k$ are level surfaces, where k is a constant.

Def. A function of n variables is a rule that assigns a number $z=f(x_1,x_2,\dots,x_n)$ to an n-tuple $(x_1,x_2,\dots,x_n)$ of real numbers.

We denote by $\mathbb{R}^n$ the set of all such n-tuples.

Examples

1. For the following function, evaluate $f(3,2)$ and find and sketch the domain.

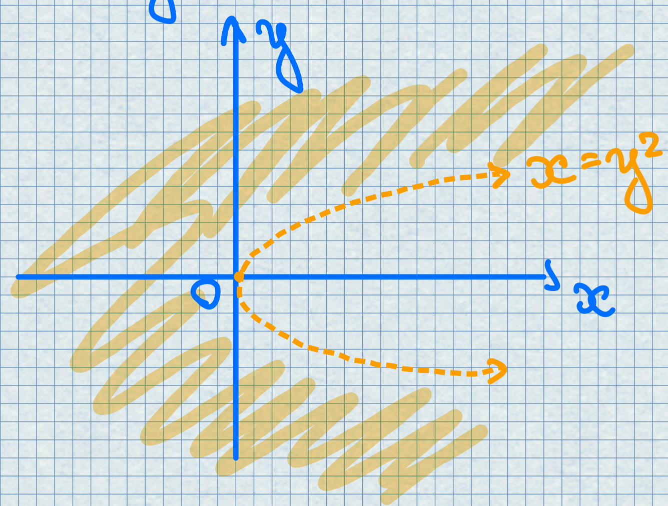
$$f(x,y) = x\ln(y^2 - x)$$

Solution

$$f(3,2) = 3\ln(4-3) = 0$$

Domain: $y^2 - x > 0 \Rightarrow y^2 > x.$

$D = \{(x,y) \mid x < y^2\}.$



2. Find the domain and range of

$$g(x,y) = \sqrt{9 - x^2 - y^2}$$

Solution

$$D = \{(x,y) \mid 9 - x^2 - y^2 \geq 0\}$$

D is a disk with center $(0,0)$ and radius $r=3$.

Range of g is $\{z \mid z = \sqrt{9-x^2-y^2}\,,\ (x,y)\in D\}$

Since z is a positive square root, $z \geq 0$.
Also, because $9-x^2-y^2 \leq 9$, we have

$$\sqrt{9-x^2-y^2} \leq 3$$

So

$$\operatorname{Ran}(g) = \{z \mid 0 \leq z \leq 3\} = [0,3].$$

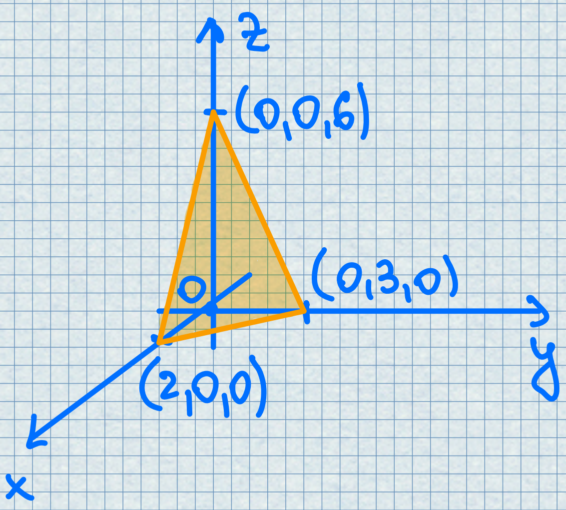
$\boxed{3.}$ Sketch the graph of the function

$$f(x,y) = 6-3x-2y$$

Solution

$z = 6-3x-2y$ is a plane in $\mathbb{R}^3$.

for $x=y=0$: $z=6$
for $x=z=0$: $y=3$
for $y=z=0$: $x=2$



4. Sketch the graph of $g(x,y) = \sqrt{9-x^2-y^2}$.

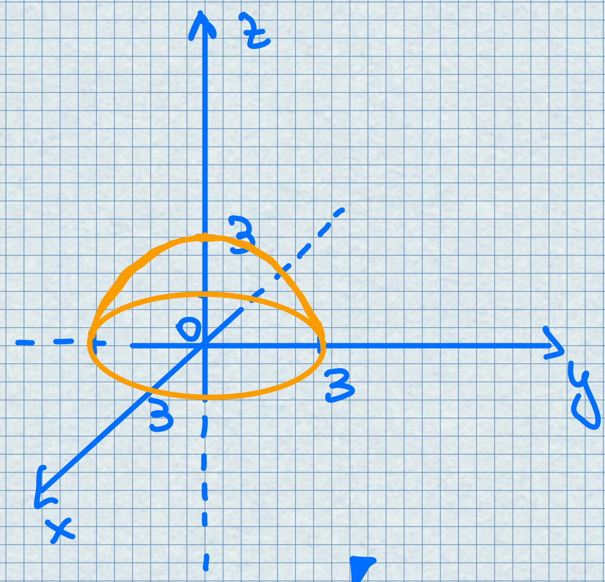
Solution

$z = \sqrt{9-x^2-y^2}$ is an upper half of a sphere in $\mathbb{R}^3$.

$z^2 + x^2 + y^2 = 9$

center $(0,0,0)$

radius $r = 3$



5. Sketch the level curves of the function $f(x,y) = 6-3x-2y$ for $k = -6, 0, 6, 12$.

Solution

The level curves are

$$6-3x-2y = k$$

or

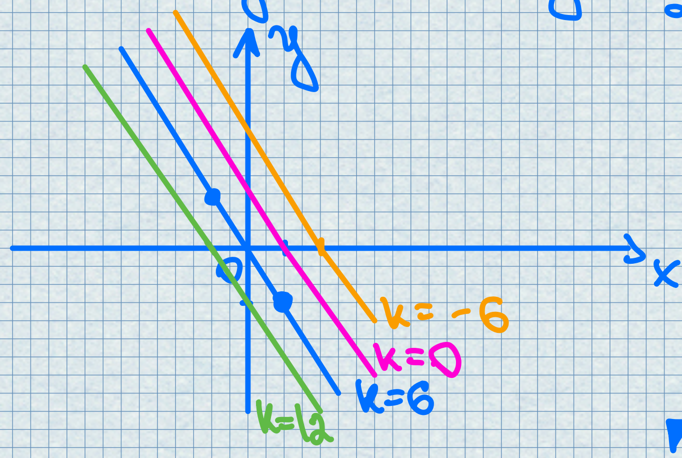
$$3x+2y+k-6 = 0$$

$k=-6$: $\quad 3x+2y = 12 \Rightarrow y = -\frac{3}{2}x + 6$

$k=0$: $\quad 3x+2y = 6 \Rightarrow y = -\frac{3}{2}x + 3$

$k=6$: $\quad 3x+2y=0 \Rightarrow y=-\frac{3}{2}x$

$k=12$: $\quad 3x+2y=-6 \Rightarrow y=-\frac{3}{2}x-3$



$\boxed{6.}$ Sketch the level curves of the function

$$g(x,y)=\sqrt{9-x^2-y^2}\,, \quad k=0,1,2,3$$

Solution

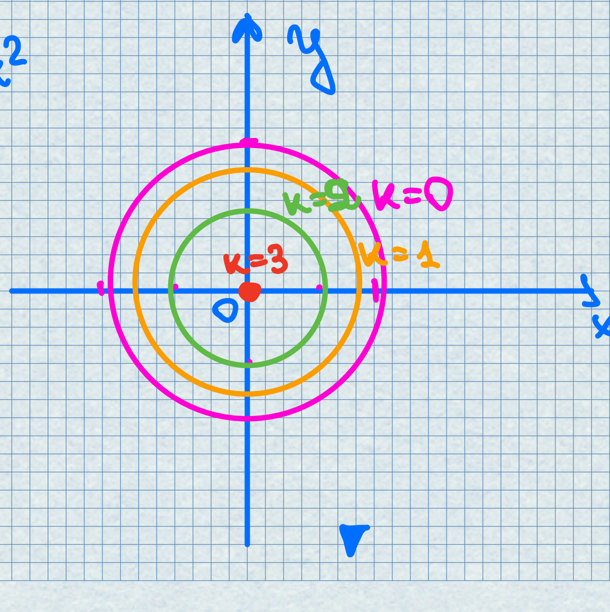
$$z=\sqrt{9-x^2-y^2}$$
$$k^2=9-x^2-y^2$$
$$x^2+y^2=9-k^2$$

$k=0$: $\quad x^2+y^2=9$

$k=1$: $\quad x^2+y^2=8$

$k=2$: $\quad x^2+y^2=5$

$k=3$: $\quad x^2+y^2=0$



$\boxed{7.}$ Find the domain of f if

$$f(x,y,z) = \ln(z-y) + xy\sin z$$

Solution

$$z - y > 0$$

$$D = \{(x,y,z) \in \mathbb{R}^3 \mid z > y\}$$

This is a half-plane consisting of all points that lie above the plane $z = y$.