Lecture #2: How Fast Did the Rocket Go?

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New England Sci - Tech

Lecture Series in Elementary Mathematics in Modeling Rocket Flight

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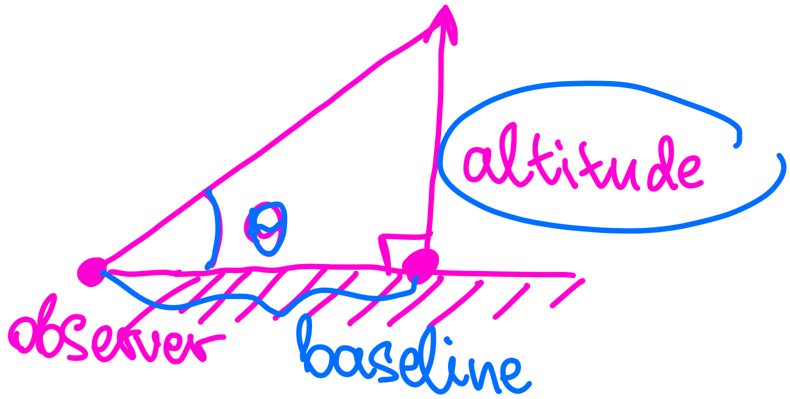


$$\frac{\tan\theta}{1} = \frac{\text{altitude}}{\text{baseline}}$$

$$\tan\theta \cdot \text{baseline} = 1 \cdot \text{altitude}$$

$$\boxed{\text{altitude} = \tan\theta \cdot \text{baseline}}$$

$$\theta = 30^\circ$$

$$\text{baseline} = 10 \text{ meters}$$

$$\text{altitude} = \overset{\sqrt{3}}{\tan 30^\circ} \cdot 10 = \sqrt{3} \cdot 10$$

(m)

Rocket Propulsion

Agenda:

- Describe the application of conservation of momentum when the mass changes with time, as well as the velocity.
- Calculate the speed of a rocket in empty space, at some time, given initial conditions.
- Calculate the speed of a rocket in Earth's gravity field, at some time, given initial conditions.

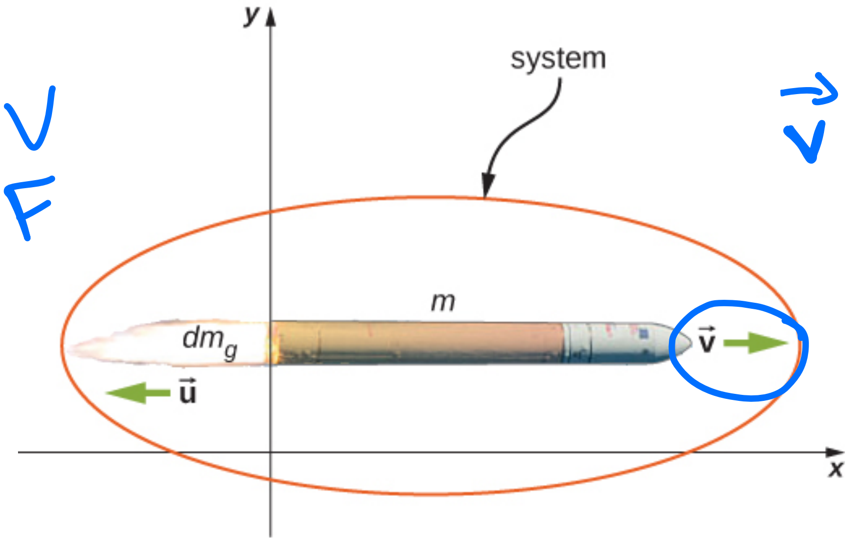
Video of a rocket launch
NASA Website about propulsion

$$\underbrace{F_1}_{\text{action}} = - \underbrace{F_2}_{\text{reaction}}$$

$\vec{V}$ $\vec{F}$

V F

$\vec{v}$ – velocity (vector)



Description: The rocket accelerates to the right due to the expulsion of some of its fuel mass to the left. Conservation of momentum enables us to determine the resulting change of velocity. The mass m is the instantaneous total mass of the rocket (i.e., mass of rocket body plus mass of fuel at that point in time).

$$\boxed{p = m \cdot v}$$

Definition. The conservation of momentum is a fundamental concept of physics along with the conservation of energy and the conservation of mass. Momentum is defined to be the mass of an object multiplied by the velocity of the object. The conservation of momentum states that, within some problem domain, the amount of momentum remains constant; momentum is neither created nor destroyed, but only changed through the action of forces as described by Newton's laws of motion.

The initial momentum of the system is described by

$$\mathbf{p}_i = mv,$$

P_i $P_{initial}$

where m is a mass of the rocket and v is an instantaneous velocity.

Including both the change for the rocket and the change for the exhaust gas, the final momentum of the system is

$$\mathbf{p}_f = p_{rocket} + p_{gas} = \underbrace{\overbrace{(m - dm_g)}^{m}\overbrace{(v + dv)}^{v}}_{P_{rocket}} + \overbrace{dm_g}^{m}\overbrace{(v - u)}^{v},$$

P_{final} P_{gas}

where

- $v - u$ is the velocity of exhaust gas;
- dm_g is a (positive) infinitesimal mass of gas which the engines eject.

Applying conservation of momentum, we obtain

$$p_i = p_f \Longrightarrow mdv = dm_g dv + dm_g u.$$

Now, since $dm_g vdv$ is very small, we omit this term. Therefore,

$$mdv = dm_g u.$$

$$m\,dv = \cancel{dm_g\,dv} + dm_g u$$

$$m\,dv = dm_g u$$

Greek letter $V = ?$

$\Delta v = v_f - v_i = ?$

Delta

$$\frac{\cancel{m}\,dv}{\cancel{m}} = \frac{dm_g u}{m} \quad | : m$$

$$dv = -u \frac{1}{m}\, dm$$

$$\cancel{\int_{\text{initial}}^{\text{final}}} \cancel{d}V = -\int_{\text{initial}}^{\text{final}} \frac{u}{m}\, dm_g$$

$$\boxed{V_f - V_i = -u \cdot \left(\underline{\underline{\ln}}(m_f) - \underline{\underline{\ln}}(m_i) \right)}$$

$$\Delta V = u\left(\ln(m_i) - \ln(m) \right)$$

$$y = \log_a x$$

$$y = \ln x = \log_e x$$

$$e \approx 2.7$$

$$\pi \approx 3.14$$

$$\boxed{V_f - V_i = u \cdot \left(\underline{\underline{\ln}}(m_f) - \underline{\underline{\ln}}(m_i) \right)}$$

$$v_f - v_i = u \ln\left(\frac{m_i}{m_f}\right)$$

$$\boxed{\Delta v = u \cdot \ln\left(\frac{m_i}{m_f}\right)}$$

Since dm_g represents an increase in the mass of ejected gases, it must also represent a decrease of mass of the rocket:

$$dm_g = -dm.$$

Hence,

$$mdv = -dmu.$$

Our final answer is

$$\Delta v = u \ln(m_i/m). \tag{1}$$

Equation (1) is called the **rocket equation**. It was originally derived in 1897. It gives us the change of velocity that the rocket obtains from burning a mass of fuel that decreases the total rocket mass from m_0 down to m.

Rocket in a Gravitational Field

$$\Delta v = u \ln(m_i/m) - g\Delta t. \tag{2}$$

Problem-Solving Strategy: Rocket Propulsion

Strategy:

- To determine the change of velocity, use the rocket equation.
- To determine the acceleration, determine the force by using the impulse-momentum theorem, using the rocket equation to determine the change of velocity.

Practice Problem

Example 2

A spacecraft is moving in gravity-free space along a straight path when its pilot decides to accelerate forward. He turns on the thrusters, and burned fuel is ejected at a constant rate of 2.0×10^2 kg/s, at a speed (relative to the rocket) of 2.5×10^2 m/s. The initial mass of the spacecraft and its unburned fuel is 2.0×10^4 kg, and the thrusters are on for 30 s.

1. What is the thrust (the force applied to the rocket by the ejected fuel) on the spacecraft?
2. What is the spacecraft's acceleration as a function of time?

Solution

1. The momentum of the ejected fuel gas is

   $$p = m_g v.$$

   The ejection velocity $\boxed{v = 2.5 \times 10^2 \text{ m/s}}$ is constant, and therefore the force is

   $$F = -v\frac{dm}{dt}.$$

   Here, dm/dt is the rate of change of mass and is equal to 2.0×10^2 kg/s. Hence,

   50000 N

   $$F = v\frac{dm}{dt} = \underbrace{(2.5 \times 10^2)}_{u}\underbrace{(2.0 \times 10^2)}_{dm/dt} = 5 \times 10^4 (N)$$

2. Above, we defined m to be the combined mass of the empty rocket plus however much unburned fuel it contained: $m = m_g + m_R$. From Newton;s second law,

   $$a = \frac{F}{m_g + m_R}.$$

Solution

$$\frac{F}{m} = \frac{m \cdot a}{m} \Rightarrow a = \frac{F}{m}$$

This gives us

$$a(t) = \frac{F}{M - (dm_g/dt)t}$$

m_R

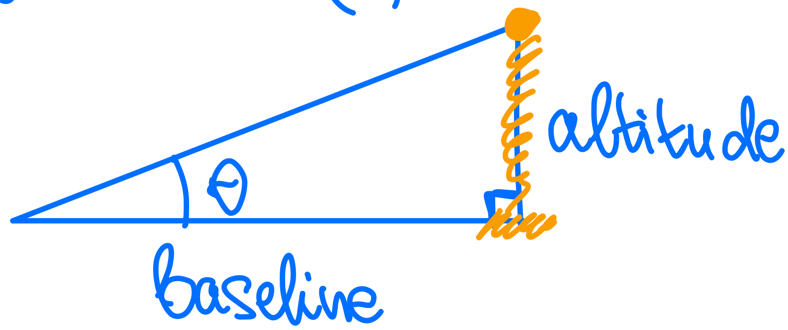
Notice that, as expected, the acceleration is a function of time. Substituting the given numbers:

$$\boxed{a(t) = \frac{5 \times 10^4\ N}{2.0 \times 10^4\ kg - 2.0 \times 10^2\ \frac{kg}{s}\ t}}$$

$t = 0s$: $a(0) = \frac{5 \cdot \cancel{10^4}\ N}{2 \cdot \cancel{10^4}\ kg} = 2.5\ (m/s^2)$

Review:

1. $\text{altitude} = \tan(\theta) \cdot \text{baseline}$

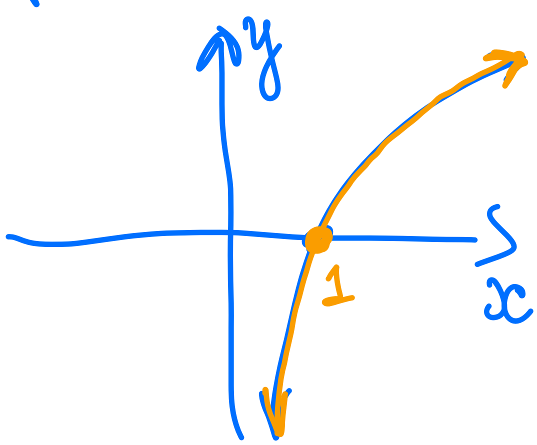


$$\boxed{\tan\theta = \frac{\text{altitude}}{\text{baseline}}}$$

2. $$\boxed{\Delta v = u \cdot \ln\left(\frac{m_i}{m_f}\right)}$$

$\Delta v = v_f - v_i$

Rocket equation



Example 1.

Before a rocket begins to burn fuel, the rocket has a mass of $m_{r,i} = 2.81 \times 10^7\,\text{kg}$, of which the mass of the fuel is $m_{f,i} = 2.46 \times 10^7\,\text{kg}$. The fuel is burned at a constant rate with total burn time is $510\,\text{s}$ and ejected at a speed $u = 3000\,\text{m/s}$ relative to the rocket. If

$$10^7 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 =$$

$$= 10000000$$

the rocket starts from rest in empty space, what is the final speed of the rocket after all the fuel has been burned?

$$m_{r,i} = 2.81 \cdot 10^7\,\text{kg}$$

$$m_{f,i} = 2.46 \cdot 10^7\,\text{kg}$$

$$u = 3000\,\text{m/s}$$

$$t = 510\,\text{s}$$

$$v_f - ?$$

$$\Delta v = u \cdot \ln\left(\frac{m_i}{m_f}\right)$$

$$\Delta v = v_f - v_i$$

$$v_f - v_i = u \cdot \ln\left(\frac{m_i}{m_f}\right)$$

$$v_i = 0\,\text{m/s}$$

$$v_f - v_i = u \cdot \ln\left(\frac{m_i}{m_f}\right)$$

$$v_f = \underbrace{v_i}_{= 0\ m/s} + \underbrace{u}_{= 3000\ m/s} \cdot \ln\left(\frac{m_i}{m_f}\right)$$

$m_i = 2.81 \cdot 10^7$ kg

$m_f = 0.35 \cdot 10^7$ kg

$$m_f = m_i - m_{f,i} = 2.81 \cdot 10^7 - 2.46 \cdot 10^7 =$$

$$= 10^7 (2.81 - 2.46) =$$

$$= 0.35 \cdot 10^7 \text{ (kg)}$$

$$\begin{array}{r} 2.81 \\ -\ 2.46 \\ \hline \boxed{0.35} \end{array}$$

$$v_f = 0 + 3000 \cdot \ln\left(\frac{2.81 \cdot \cancel{10^7}}{0.35 \cdot \cancel{10^7}}\right) =$$

$$= 3000 \cdot \ln\left(\frac{2.81}{0.35}\right) =$$

$$= 3000 \cdot \ln(8.029) =$$

$$\approx 3000 \cdot 2.083 = 6249 \text{ (m/s)}$$

Answer: $\boxed{v_f = 6249 \text{ m/s}}$

Program Code in Python

```
import matplotlib.pyplot as plt
import numpy as np
inputs
xi = float(input("Enter Starting Position: "))
vi = float(input("Enter Initial velocity: "))
u = float(input("Enter exhaust gas velocity: "))
R = float(input("Enter burning rate: "))
mi = float(input("Enter Initial Mass: "))
mf = float(input("Enter Final Mass: "))
N = int(input("Enter Number of Divisions: "))
Find/define flight time, dt, dm
T = -(mf-mi)/R
dt = T/N
t = np.linspace(0,T,N+1)
dm = -R*dt
```

```
Initialize arrays
m = np.empty(N+1)
v = np.empty(N+1)
x = np.empty(N+1)
m[0] = mi
v[0] = vi
x[0] = xi
main loop
for i in range (N):
m[i+1]=m[i]-R*dt
v[i+1]=v[i]-u*dm/m[i]
x[i+1]=x[i]+v[i]*dt
```

```
plots
plt.subplot(1,3,1)
plt.plot(t,m)
plt.title('Mass vs Time Graph')
plt.xlabel('Time(s)')
plt.ylabel('Mass(kg)')
plt.subplot(1,3,2)
plt.plot(t,v)
plt.title('Velocity vs Time Graph')
plt.xlabel('Time(s)')
plt.ylabel('Velocity(m/s)')
plt.subplot(1,3,3)
plt.plot(t,x)
plt.title('Position vs Time Graph')
plt.xlabel('Time(s)')
plt.ylabel('Position (m)')
plt.suptitle('Rocket motion simulation')
plt.tight_layout()
```

Thank you for your attention!