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Foundations of Computing II
Assignment 6 − Solutions

The Halting Problem, Complexity Theory

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Upload your solutions to the OLAT system.

6.1   The Halting Problem

In the lecture, we reduced the universal language L_U to the halting problem L_H. Using a similar approach, now reduce L_H to L_U.

This time, we suppose that we are given a TM U^* that decides L_U, and we construct a TM H^* that decides L_H using U^*; by definition, both TMs always halt. The following figure is essentially the same as for the reduction from L_U to L_H.
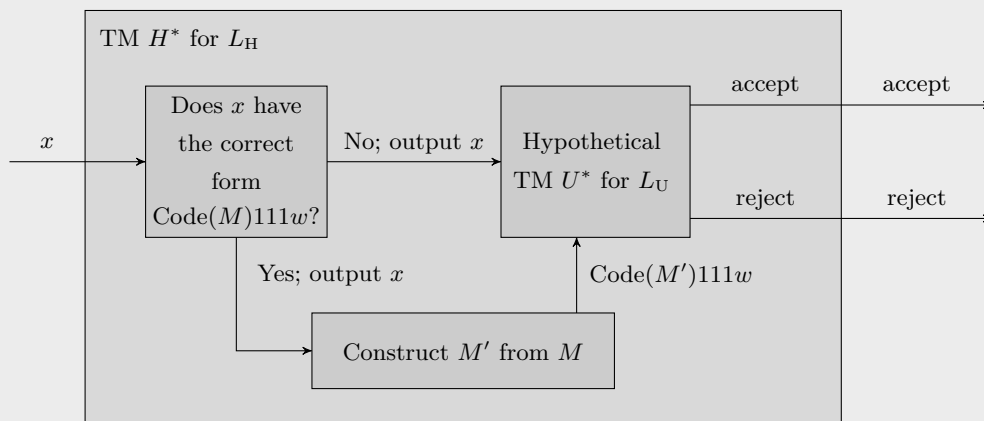
The reduction is indeed very similar to the original one given in the lecture. Only the roles of U^* and H^* have been switched and the TM M' is constructed from the given TM M in a different way, as described in what follows.

This time, we want to know whether M halts, but we can only decide (using U^*) whether M' accepts. The idea is to once more consider all the states in which M may get stuck. In this reduction, there is no new state added, but these "missing" transitions are added

such that they end in an accepting state. We then have the equivalence

$$M \text{ halts on } w \iff M' \text{ accepts } w \,.$$

Therefore, the answer of U^* as to whether $\text{Code}(M')111w$ is in L_{U} can be taken as the correct answer as to whether $\text{Code}(M)111w$ is in L_{H}.

It follows that, if L_{U} were recursive, L_{H} would also be recursive.

6.2 Closure Properties of Languages in $\mathcal{P}$

Show that the languages in $\mathcal{P}$ are closed under the following operations. Argue on an intuitive, but exact level.

a) Union, that is, if $L_1, L_2 \in \mathcal{P}$, then $L_1 \cup L_2 \in \mathcal{P}$,

Let L_1 and L_2 be any two languages such that $L_1, L_2 \in \mathcal{P}$. Now we want to prove that also $L_1 \cup L_2 \in \mathcal{P}$. Let w be any word for which we want to decide in polynomial time whether it is in $L_1 \cup L_2$. In other words, we want to decide whether $w \in L_1$ or $w \in L_2$. By definition, there is a TM M_1 that decides whether w is in L_1, and there is a TM M_2 that decides whether w is in L_2.

A TM M for $L_1 \cup L_2$ first uses M_1 to decide whether $w \in L_1$.

- If $w \in L_1$, M accepts.
- If $w \notin L_1$, M uses M_2 to decide whether $w \in L_2$.
- If $w \in L_2$, M accepts.
- Otherwise M rejects (that is, halts in a non-accepting state).

Since both M_1 and M_2 run in polynomial time, M also runs in polynomial time; thus, $L_1 \cup L_2$ is in $\mathcal{P}$ as a direct consequence of the existence of M.

b) Intersection, that is, if $L_1, L_2 \in \mathcal{P}$, then $L_1 \cap L_2 \in \mathcal{P}$,

A TM M for $L_1 \cap L_2$ works similar to that described in the solution of a). The only difference is that this TM only accepts if M_1 and M_2 both accept the input w; otherwise it rejects.

c) Complement, that is, if $L \in \mathcal{P}$, then $\overline{L} \in \mathcal{P}$.

Let M be a TM for L that runs in polynomial time. A TM $\overline{M}$ for $\overline{L}$ simulates M on the given input.

- If M accepts, $\overline{M}$ rejects.
- If M rejects, $\overline{M}$ accepts.

Clearly, $\overline{M}$ runs in polynomial time and thus $\overline{L} \in \mathcal{P}$.

Hint: In the lecture, we have seen that the regular languages are closed under quite a few

operations. This was done by starting with the DFAs for the given languages and then modifying them. Use an analogous approach to answer the above questions.

6.3 Polynomial-Time Reductions

In complexity theory, SAT plays the same role for us as L_{diag} in computability theory. To show that a problem is $\mathcal{NP}$-hard, we can reduce SAT to it.

In the lecture, we have introduced the satisfiability problem (SAT and 3SAT), the independent set problem IND-SET, the clique problem CLIQUE, and the vertex cover problem VC; then we proved SAT $\leq_{\mathsf{p}}$ 3SAT, 3SAT $\leq_{\mathsf{p}}$ IND-SET, IND-SET $\leq_{\mathsf{p}}$ CLIQUE, and IND-SET $\leq_{\mathsf{p}}$ VC, which implies that all of them are $\mathcal{NP}$-hard.

Here, we introduce three other problems which we prove to be $\mathcal{NP}$-hard by polynomial-time reductions.

a) The set cover problem SC is defined as follows. An input is triple $(X, \mathcal{S}, k)$ with

- $X = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}^+$,
- $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ with $S_i \subseteq X$ for some $m \in \mathbb{N}^+$ and $X = \bigcup_{j=1}^m S_j$, and
- $k \in \mathbb{N}^+$.

The question is whether there is a set cover of X of size (at most) k, that is, a selection of (at most) k sets from $\mathcal{S}$ such that every element from X is contained in at least one of the selected sets, that is, whether there exist $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$ with $i_j \in \{1, 2, \ldots, m\}$ and

$$X = \bigcup_{j=1}^{k} S_{i_j} \; .$$

Formally,

$$\text{SC} = \{(X, \mathcal{S}, k) \mid X \text{ has a set cover from } \mathcal{S} \text{ of size } k\} \; .$$

As an example, the instance $(X_1, \mathcal{S}_1, 3)$ with $X_1 = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and

$$\mathcal{S}_1 = \{\{1, 3\}, \{1, 2, 5\}, \{1, 4\}, \{3, 4, 6\}, \{5, 6, 8\}, \{5, 7, 8\}\}$$

is a "yes" instance, because there is a set cover

$$\{1, 2, 5\} \cup \{3, 4, 6\} \cup \{5, 7, 8\} = X_1$$

of size 3. Conversely, the instance $(X_2, \mathcal{S}_2, 4)$ with $X_2 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and

$$\mathcal{S}_2 = \{\{1, 2\}, \{2, 3\}, \{2, 4, 6\}, \{4, 5, 6\}, \{6, 7\}, \{7, 9\}, \{8, 9\}\}.$$

is a "no" instance (as we can cover at most eight elements with three sets from $\mathcal{S}_2$). Reduce VC to SC.

Consider an instance (G, k) of VC with $G = (V, E)$. We construct an instance $(X, \mathcal{S}, k)$ of SC by mapping the edges to the elements of X since they need to be covered, and we map each vertex to the set of its incident edges, because the vertices are used to cover the edges.

More formally, let $V = \{v_1, v_2, \ldots, v_m\}$ and $E = \{e_1, e_2, \ldots, e_n\}$, where the edges are ordered in an arbitrary but fixed way. We set $X = \{1, 2, \ldots, n\}$; hence, each element of
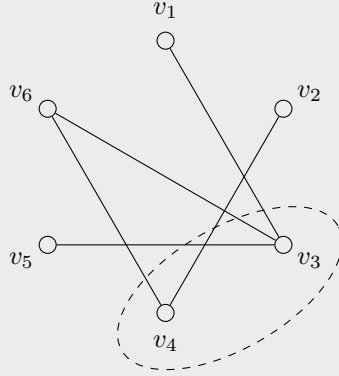
X corresponds to one edge of G. Then we set $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ with

$$S_i = \{j \in \{1, 2, \ldots, n\} \mid \text{the edge } e_j \text{ is incident to } v_i\} .$$

As an example of this reduction, consider the instance $(G, 2)$ of VC where $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and

$$E = \{\{v_1, v_3\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_4, v_6\}\} ,$$

which is shown in the following figure.

For some fixed order, let $\{v_1, v_3\}$ be the first edge e_1, let $\{v_2, v_4\}$ be the second edge e_2 etc.

This is a "yes" instance, since there is a vertex cover $\{v_3, v_4\}$ of size 2, which is circled with a dashed line.

Applying our reduction, we obtain a set cover instance $(X, \mathcal{S}, 2)$ as follows. We set $X = \{1, 2, 3, 4, 5\}$ since there are five edges that need to be covered, and $\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ with

$$S_1 = \{1\}, \ S_2 = \{2\}, \ S_3 = \{1, 3, 4\}, \ S_4 = \{2, 5\}, \ S_5 = \{3\}, \ \text{and } S_6 = \{4, 5\} ,$$

where, for instance, S_3 corresponds to the vertex v_3, which is incident to the edges e_1, e_3, and e_4. We observe that the vertex cover $\{v_3, v_4\}$ translates to a set cover $\{S_3, S_4\}$ covering X and vice versa.

In general, every vertex cover of size k covers all edges of G; the corresponding set cover then covers X and vice versa, which follows directly from the construction.

In conclusion, we have

$$G \text{ contains a vertex cover of size } k \iff X \text{ has a set cover from } \mathcal{S} \text{ of size } k ,$$

and the construction can clearly be done in polynomial time.

b) A dominating set in a graph $G = (V, E)$ is a set D of vertices such that every vertex from V is either in D or has an edge to at least one vertex in D; we call such a vertex "dominated."

An instance of the dominating set problem DOM-SET is a pair (G, k) where G is a graph and the question is whether G contains a dominating set of size $k \in \mathbb{N}^+$ (or

smaller).

Formally,

$$\text{DOM-SET} = \{(G, k) \mid G \text{ contains a dominating set of size } k\} \ .$$

Reduce SC to DOM-SET.

Let $(X, \mathcal{S}, k)$ be an instance of SC with $X = \{1, 2, \ldots, n\}$, $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$, and $k \in \mathbb{N}^+$. We construct an instance (G, k) with $G = (V, E)$ of DOM-SET as follows. The vertices of G can be partitioned into two disjoint sets

$$V_X = \{x_1, x_2, \ldots, x_n\} \quad \text{and} \quad V_{\mathcal{S}} = \{s_1, s_2, \ldots, s_m\} \ ,$$

where V_X corresponds to X and $V_{\mathcal{S}}$ corresponds to $\mathcal{S}$.

The edges E of G are defined as follows. First, there is an edge between any two vertices from $V_{\mathcal{S}}$; that is, $V_{\mathcal{S}}$ is a clique. Second,

$$\{x_i, s_j\} \in E \iff i \in S_j \ ;$$

that is, there is an edge between a vertex from V_X and a vertex from $V_{\mathcal{S}}$ if and only if the element $i \in X$ is covered by $S_j \in \mathcal{S}$.

As an example, consider the instance $(X, \mathcal{S}, 2)$ of SC with $X = \{1, 2, 3, 4, 5, 6\}$ and

$$\mathcal{S} = \{\{1, 3, 4\}, \{2, 4\}, \{2, 5, 6\}, \{4, 6\}, \{5, 6\}\} \ ,$$

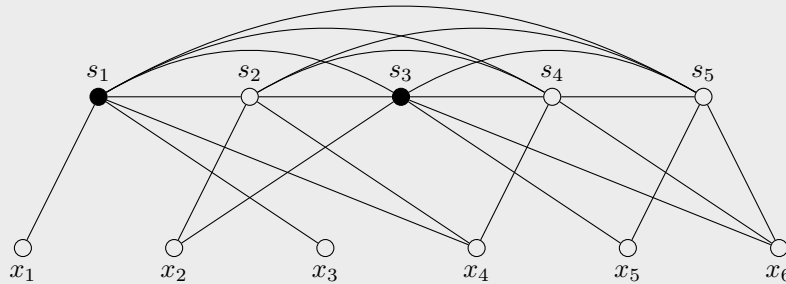where $S_1 = \{1, 3, 4\}$, $S_2 = \{2, 4\}$ etc.

We construct the instance $(G, 2)$ of DOM-SET with $G = (V, E)$ where

$$V = \{x_1, x_2, x_3, x_4, x_5, x_6\} \cup \{s_1, s_2, s_3, s_4, s_5\}$$

and

$$\begin{aligned} E = \{ & \{s_1, x_1\}, \{s_1, x_3\}, \{s_1, x_4\}, \\ & \{s_2, x_2\}, \{s_2, x_4\}, \\ & \{s_3, x_2\}, \{s_3, x_5\}, \{s_3, x_6\} \\ & \{s_4, x_4\}, \{s_4, x_6\}, \\ & \{s_5, x_5\}, \{s_5, x_6\}\} \ , \end{aligned}$$

which is shown in the following figure.

The SC instance $(X, \mathcal{S}, 2)$ is a "yes" instance since there is a set cover $\{S_1, S_3\}$ of size 2, which translates to a dominating set $\{x_1, x_3\}$ in G, which is marked by filled vertices.

In general, consider any set cover of X of size k. Selecting the corresponding vertices from $V_{\mathcal{S}}$ results in a dominating set of G of size k, because

- trivially, as soon as one vertex from $V_{\mathcal{S}}$ is selected, all vertices from $V_{\mathcal{S}}$ are dominated, because $V_{\mathcal{S}}$ is a clique;
- all vertices from V_X are dominated by at least one of the selected vertices from $V_{\mathcal{S}}$, because all elements from X are contained in at least one of the selected sets from $\mathcal{S}$.

Conversely, consider any dominating set D of G of size k. If all vertices from D are from $V_{\mathcal{S}}$, the corresponding selection of sets from $\mathcal{S}$ results in a set cover of X of size k. On the other hand, for each vertex x from D that is not from $V_{\mathcal{S}}$ but from V_X, we can choose an arbitrary adjacent vertex from $V_{\mathcal{S}}$ instead. This does not increase the size of D. Clearly, x will still be dominated afterwards, the other vertices of V_X will still covered, and the vertices from $V_{\mathcal{S}}$ will also definitely be covered since D now contains a vertex from $V_{\mathcal{S}}$.

It follows that

$$X \text{ has a set cover from } \mathcal{S} \text{ of size } k \iff G \text{ contains a dom. set of size } k,$$

and again it is easy to see that the construction can be done in polynomial time.

c) A half-clique is a clique that contains exactly half of the vertices of a given graph. An instance of HALF-CLIQUE is a graph G and the question is whether G contains a half-clique.

Formally,

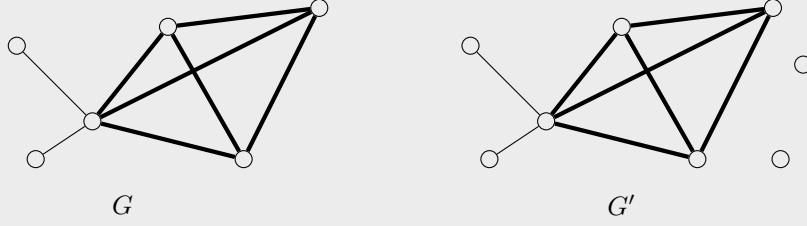$$\text{HALF-CLIQUE} = \{G \mid G \text{ contains a half-clique}\}.$$

Note that, if G contains a clique with more than half of its vertices, it also contains a half-clique.

Reduce CLIQUE to HALF-CLIQUE.

Let (G, k) be an instance of CLIQUE, and let n denote the number of vertices of the graph G. We distinguish two cases.

Case 1. Suppose that k is at least $n/2$, that is, the given CLIQUE instance asks about the existence of a clique that contains more than half of the vertices of G. In this case, we add k' vertices to obtain a new graph G' with $n' = n + k'$ vertices such that $(n + k')/2 = k$ (this means $k' = 2k - n$ vertices are added). The newly added vertices are all isolated, that is, they are not connected to any other vertex and are therefore not part of any clique.

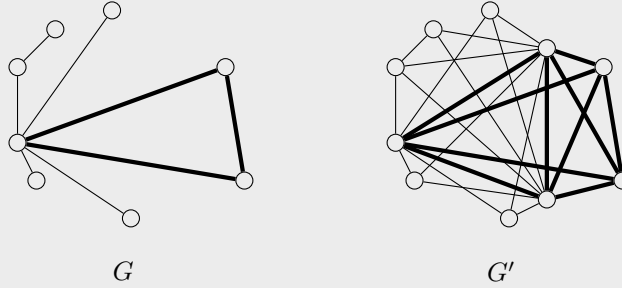The idea is shown in the following figure.

G

G'

G has 6 vertices and contains a clique of size 4, which is marked by thick edges. G' has 8 vertices and a clique of $4 = 8/2$ vertices (that is, a half-clique).

In general, we have the equivalence

$$G \text{ contains a clique of size } k \iff G' \text{ contains a clique of size } k \text{ ,}$$

where k is exactly half of the vertices in G'.

Case 2. Suppose that k is less than $n/2$. Then we again add k' vertices to G to obtain a new graph G'. This time, however, k' is such that $k' + k$ equals exactly half of the vertices from G', that is, $k + k' = (n + k')/2$ (which means that $k' = n - 2k$ vertices are added). All new vertices are connected to each other such that they form a clique of size k' in G'. Moreover, every such new vertex is connected to *all* other vertices from G. The idea is shown in the following figure.

G

G'

Here, G has 8 vertices and contains a clique of size 3, which is again marked by thick edges. G' has 10 vertices and a clique of $5 = (3 + 2)/2$ vertices (that is, a half-clique). Thus, in this case, we have the equivalence

$$G \text{ contains a clique of size } k \iff G' \text{ contains a clique of size } k + k' \text{ ,}$$

where $k + k'$ is exactly half of the vertices in G'.

Hence, we can give the input G' to a TM for HALF-CLIQUE and use the answer to answer the question whether G contains a clique of size k.

Clearly, G' can be constructed from (G, k) in polynomial time in both cases. Moreover, it can easily be checked which of the two cases occurs.