Content of the Course

Foundations of Computing II

Sven Seuken and Dennis Komm

September 14, 2020

Goal

The central questions are

- What **cannot** be done using a computer?
- What **cannot** be done efficiently using a computer?

Problems in Math

Problems in Math

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Gottfried Wilhelm Leibniz (1646 – 1716)

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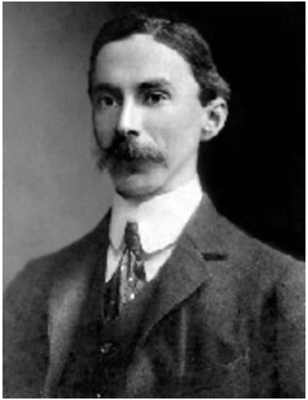
"Let us calculate, without further ado, to see who is right."

Bertrand Russell (1872 – 1970)



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- Let S be the set containing all sets that do not contain themselves, that is, $S = \{X \mid X \text{ is a set and } X \notin X\}$
- Does S contain itself, that is, $S \in S$ or $S \notin S$?
- This leads to a contradiction
- As a consequence, S cannot exist

Bertrand Russell (1872 – 1970)

Suppose we have $S \in S$

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- ⇨ Thus, S is a set that contains itself
- ⇨ But then S cannot be in S, because S is defined such that it does not contain any set that contains itself

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Contradiction

David Hilbert (1862 – 1943)

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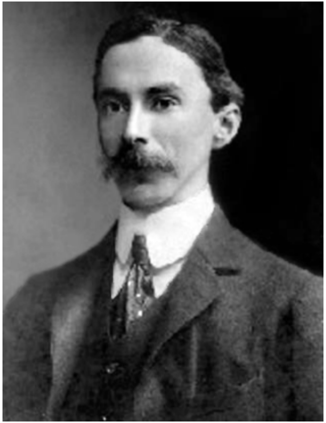
Hilbert Program

Create a system of axioms in which every true statement can be proven, and that does not contain any paradoxes and inconsistencies

B. Russell, Alfred N. Whitehead (1861 – 1947)



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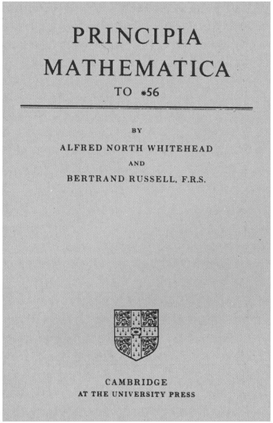
Unknown



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Kurt Gödel (1862 – 1943)

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Gödel's Incompleteness Theorem

If a system is consistent (there are no paradoxes) and powerful enough, then there are true statements in this system that cannot be proven (it is incomplete)

Conclusion

Today we know (thanks to Gödel) that ...

- Mathematics is incomplete
- There are statements that we can neither prove nor disprove
- This will stay that way

Conclusion

Today we know (thanks to Gödel) that . . .

- Mathematics is incomplete
- There are statements that we can neither prove nor disprove
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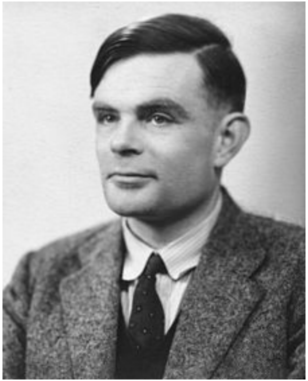
Around the same time, an English mathematician asked similar questions

- What can be automated?
- Where lies the border to what cannot be automated?

Alan M. Turing (1912 – 1954)



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There are provable statements that we cannot prove automatically (that is, we cannot compute the answer)

The idea is very similar to the one of Gödel's Incompleteness Theorem

Infinity times Infinity

Infinity times Infinity

Infinity times Infinity

Walt Disney Pictures

Infinity times Infinity

- A shepherd has black and white sheep
- Does he have more black ones or more white ones?
- He can only count to three

Infinity times Infinity

- A shepherd has black and white sheep
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⇨ Pair one black sheep with a white sheep and bring them to another meadow

Juraj Hromkovič, *Berechenbarkeit*, Vieweg Teubner

Infinity times Infinity

Theorem (of Cantor and Bernstein)

Two sets A and B have the same size if there is a bijection between them $\square$

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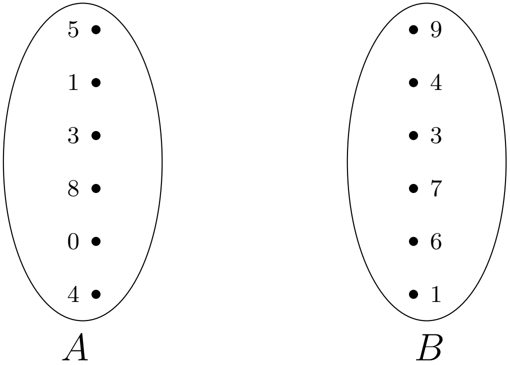

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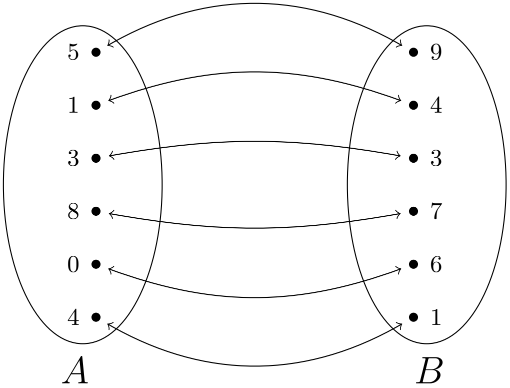

Infinity times Infinity

This is also true for infinite sets

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ (natural numbers)
- $\mathbb{N}_{\text{even}} = \{0, 2, 4, 6, \dots\}$ (even numbers)
- $\mathbb{N}_{\text{odd}} = \{1, 3, 5, 7, \dots\}$ (odd numbers)

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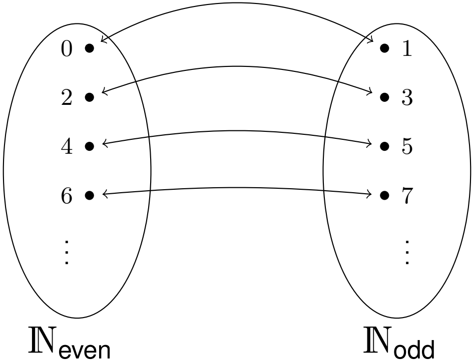

Infinity times Infinity

⇨ Thus we have $|\mathbb{N}_{\text{even}}| = |\mathbb{N}_{\text{odd}}|$

- How about $\mathbb{N}^+ = \{1, 2, 3, \dots\}$? (positive natural numbers)
- Intuitively, $|\mathbb{N}^+| < |\mathbb{N}|$ should be true

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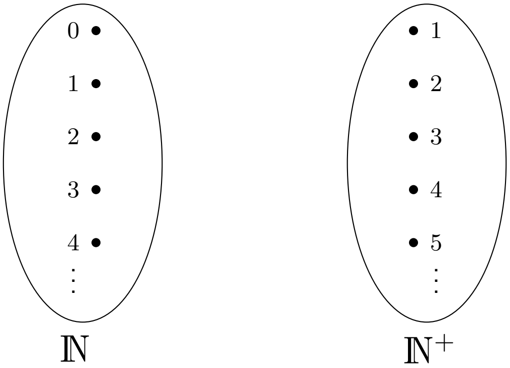

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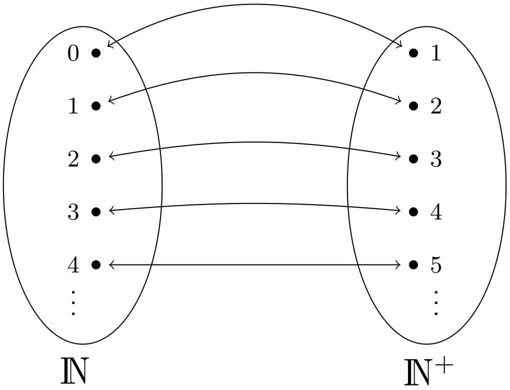

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- How about $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$? (integers)
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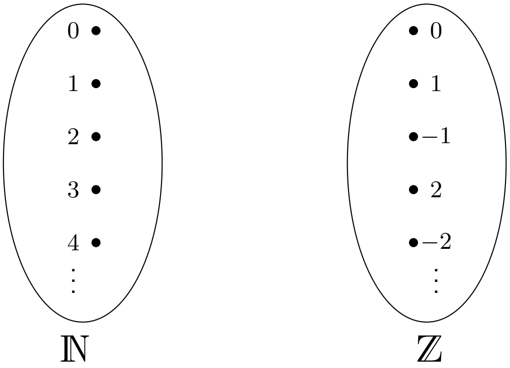

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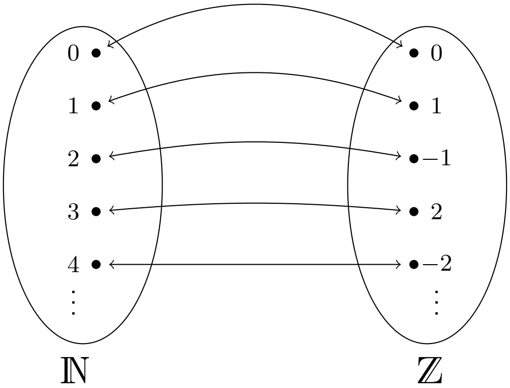

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An infinite set A is called **countable** if it has the same size as $\mathbb{N}$

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- ⇨ Every element of A is assigned to a natural number
- ⇨ We can **enumerate** (count) the elements of A

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- $\mathbb{Q}^+ = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \ldots\right\}$ (positive rational numbers)
- $\mathbb{Q} \ = \left\{\ldots, -\frac{1}{3}, -\frac{1}{2}, -\frac{1}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ (rational numbers)

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Theorem

$\mathbb{Q}^+$ and $\mathbb{Q}$ are both countable

David Hilbert (1862 – 1943)

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The Hilbert Hotel

- Infinite number of rooms
- There is one guest in every room

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- Infinite number of rooms
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- A new guest arrives

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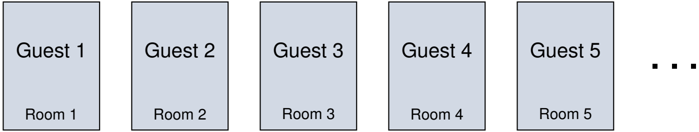
The Hilbert Hotel

- Infinite number of rooms
- There is one guest in every room
- A new guest arrives
- Can we assign this guest to a room without kicking anyone out?

A new guest arrives. . .



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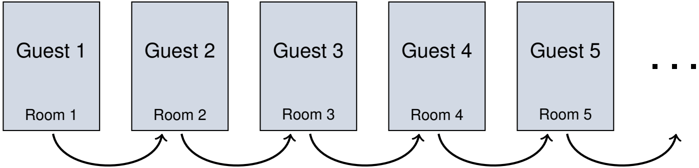

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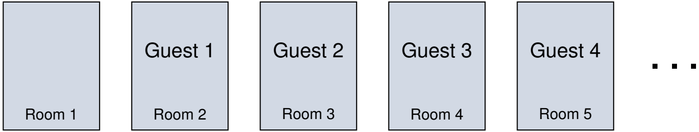

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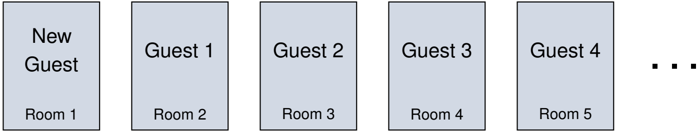

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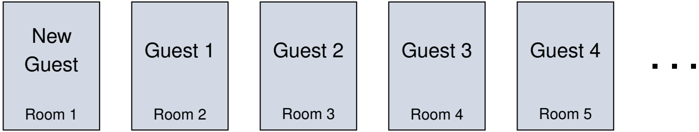
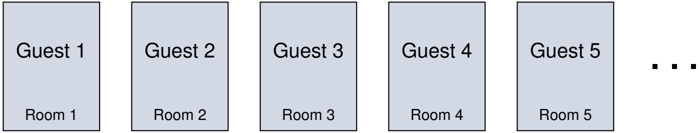


Infinitely many new guests arrive...



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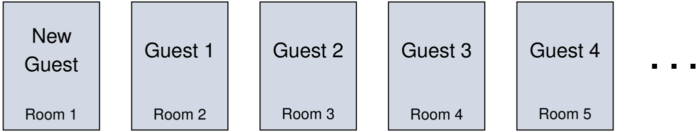

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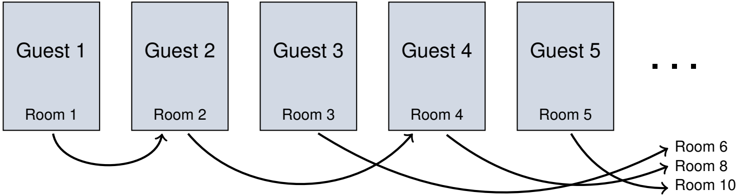

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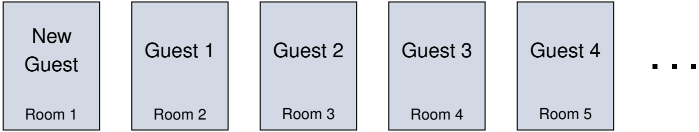

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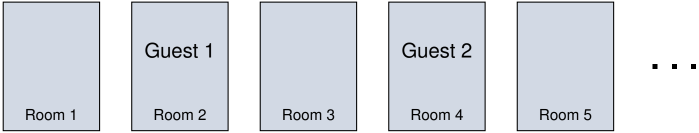

David Hilbert (1862 – 1943)

A new guest arrives...

New Guest	Guest 1	Guest 2	Guest 3	Guest 4	...
Room 1	Room 2	Room 3	Room 4	Room 5	

Infinitely many new guests arrive...

New Guest 1	Guest 1	New Guest 2	Guest 2	New Guest 3	...
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Uncountability of $\mathbb{R}$

- Are there sets that are actually larger than $\mathbb{N}$?
- Consider $\mathbb{R}$ (real numbers)
- Recall that $\pi, \sqrt{2} \in \mathbb{R}$, but $\pi, \sqrt{2} \notin \mathbb{Q}$

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$\mathbb{R}$ is not countable (we say **uncountable**)

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- Again, we prove the claim by **contradiction**
- Suppose the opposite were true (that is, "$\mathbb{R}$ is countable")
- We show that this leads to a contradiction

Uncountability of $\mathbb{R}$

- So suppose $\mathbb{R}$ were countable
- We even only look at the real numbers between 0 and 1, that is, $[0, 1]_{\mathbb{R}}$
- We suppose that we can enumerate (count) these numbers

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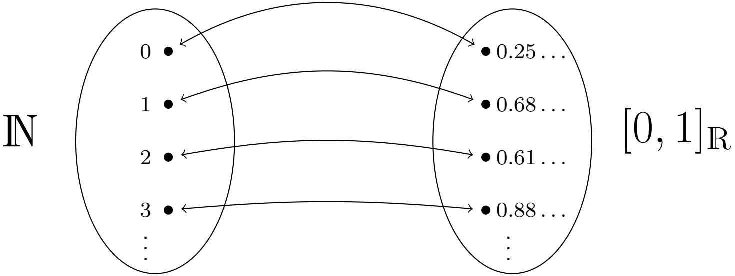
⇨ Then we get a bijection:



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⇨ Then we get a bijection, resp. a table as follows:

Number	Real number
0	$0.\ 2\ 5\ 6\ 5\ 1\ 4\ 0\ 5\ \ldots$
1	$0.\ 6\ 8\ 0\ 0\ 7\ 1\ 4\ 3$
2	$0.\ 6\ 1\ 7\ 3\ 9\ 0\ 1\ 9$
3	$0.\ 8\ 8\ 7\ 4\ 0\ 8\ 4\ 8$
$\vdots$	$\vdots \qquad \ddots$

Uncountability of $\mathbb{R}$

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0	$0.$	2	5	6	5	1	4	0	5	
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2	$0.$	6	1	7	3	9	0	1	9	
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- Consider the ith decimal place of the ith number

Uncountability of $\mathbb{R}$

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- Consider the ith decimal place of the ith number
- We construct a real number x
- x is different from the ith number at the ith decimal place
- Here, for instance $x = 0.1763\dots$

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Contradiction

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⋮				⋮						⋱

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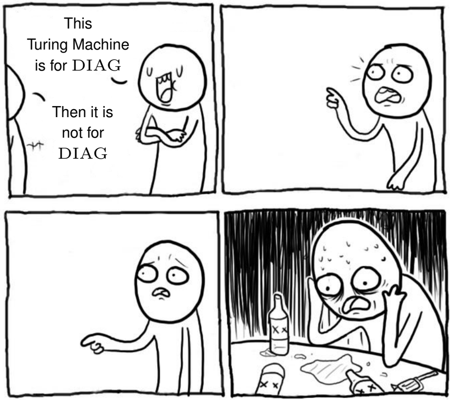
⇨ This method is called **diagonalization**

Contradiction

The Turing Machine

The Turing Machine

The Turing Machine



The Turing Machine

Now we want to show that there are problems that we cannot solve automatically (that is, algorithmically); the basic idea is...

- There is a countable number of algorithms
- But there is an uncountable number of problems

The Turing Machine

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- There is a countable number of algorithms
- But there is an uncountable number of problems

However, before that we need to meditate...

- What is a problem?
- What is an algorithm?

⇨ We need a precise mathematical definition

What is a Problem?

- An **alphabet** is a finite set of symbols (letters)
- A **word** is a string of these symbols
- $\mathrm{DEC} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is the decimal alphabet
- $x = 13817$ is a word “over” DEC

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Definition

A **decision problem** L is to decide, for a given word x, whether it is in the set L, that is, whether $x \in L$ or $x \notin L$

- The answer is always YES or NO
- As a matter of fact, this modelling is very general

What is a Problem?

- $\mathrm{PRIMES} = \{y \mid y \text{ is a prime number}\}$ (y is over DEC)
- Is x a prime number?

⇨ Is $x \in \mathrm{PRIMES}$?

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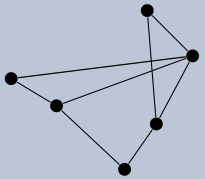
- $\mathrm{HC} = \{y \mid y \text{ is a graph with a Hamiltonian cycle}\}$
- The graph



is in HC

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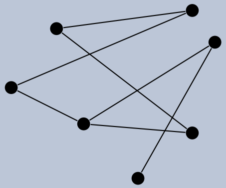

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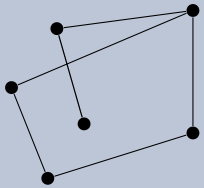

neither

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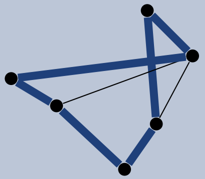
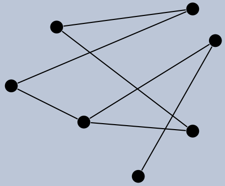
- $\mathrm{HC} = \{y \mid y$ is a graph with a Hamiltonian cycle$\}$
- The graph



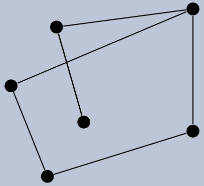
is in HC

- The graph



is not in HC

- The graph



neither

What is an Algorithm?

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Definition

An **algorithm** is a finite step-by-step method to solve every instance of a given problem in finite time

What is an Algorithm?

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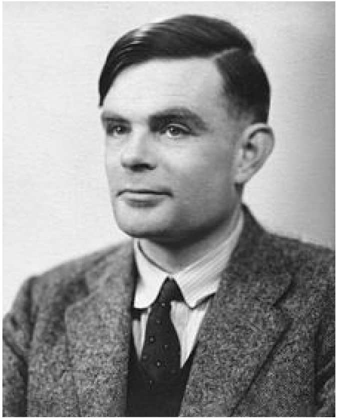
Definition

An **algorithm** is a finite step-by-step method to solve every instance of a given problem in finite time

- A computer program that checks whether a given number is a prime number
- Recipe, directions, manual, ...

⇨ No mathematical object

What is an Algorithm?



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230 A. M. Turing [Nov. 12,

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

By A. M. Turing.

[Received 28 May, 1936.—Read 12 November, 1936.]

The "computable" numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means. Although the subject of this paper is ostensibly the computable *numbers*, it is almost equally easy to define and investigate computable functions of an integral variable or a real or computable variable, computable predicates, and so forth. The fundamental problems involved are, however, the same in each case, and I have chosen the computable numbers for explicit treatment as involving the least cumbrous technique. I hope shortly to give an account of the relations of the computable numbers, functions, and so forth to one another. This will include a development of the theory of functions of a real variable expressed in terms of computable numbers. According to my definition, a number is computable if its decimal can be written down by a machine.

In §§ 9, 10 I give some arguments with the intention of showing that the computable numbers include all numbers which could naturally be regarded as computable. In particular, I show that certain large classes of numbers are computable. They include, for instance, the real parts of all algebraic numbers, the real parts of the zeros of the Bessel functions, the numbers π, e, etc. The computable numbers do not, however, include all definable numbers, and an example is given of a definable number

Oxford University Press

1936.] On computable numbers. 231

have valuable applications. In particular, it is shown (§ 11) that the Hilbertian Entscheidungsproblem can have no solution.

In a recent paper Alonzo Church† has introduced an idea of "effective calculability", which is equivalent to my "computability", but is very differently defined. Church also reaches similar conclusions about the Entscheidungsproblem‡. The proof of equivalence between "computability" and "effective calculability" is outlined in an appendix to the present paper.

1. *Computing machines.*

We have said that the computable numbers are those whose decimals are calculable by finite means. This requires rather more explicit definition. No real attempt will be made to justify the definitions given until we reach § 9. For the present I shall only say that the justification lies in the fact that the human memory is necessarily limited.

We may compare a man in the process of computing a real number to a machine which is only capable of a finite number of conditions $q_1, q_2, \ldots, q_R$ which will be called "m-configurations". The machine is supplied with a "tape" (the analogue of paper) running through it, and divided into sections (called "squares") each capable of bearing a "symbol". At any moment there is just one square, say the r-th, bearing the symbol $\mathfrak{S}(r)$ which is "in the machine". We may call this square the "scanned square". The symbol on the scanned square may be called the "scanned symbol". The "scanned symbol" is the only one of which the machine is, so to speak, "directly aware". However, by altering its m-configuration the machine can effectively remember some of the symbols which it has "seen" (scanned) previously. The possible behaviour of the machine at any moment is determined by the m-configuration q_n and the scanned symbol $\mathfrak{S}(r)$. This pair $q_n, \mathfrak{S}(r)$ will be called the "configuration": thus the configuration determines the possible behaviour of the machine. In some of the configurations in which the scanned square is blank (*i.e.*

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What is an Algorithm?

A mathematician has the following things available

- A pen
- An arbitrary number of sheets of checkered paper
- A finite number of rules in his head (arithmetic, logic, ...)

What is an Algorithm?

A mathematician has the following things available

- A pen
- An arbitrary number of sheets of checkered paper
- A finite number of rules in his head (arithmetic, logic, . . .)

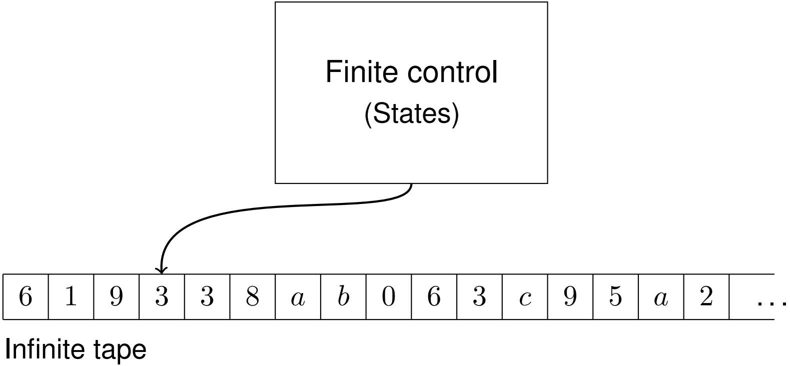
Input: String (word) written on the paper

Procedure: The mathematician. . .

- Looks a symbol (letter) to which the pen currently points
- Moves pen depending on this and writes on paper
- Eventually ends her/his work (“halts”)

Output: String (word) written on the paper

The Turing Machine



Intermission

Related Models of Computation

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Turing machine that...

- has read-only access on tape
- reads tape once from left to right

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- is allowed to make nondeterministic decisions

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Turing machine that...

- uses a stack instead of a tape
- is allowed to make nondeterministic decisions

⇨ **Nondeterministic pushdown automaton**

- (equivalent to context-free grammars)

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The Chomsky Hierarchy

Hierarchy of decision problems (in this context called “languages”) according to which model of computing (equivalently type of grammar) can decide them

But back to the Turing Machine. . .

Alonzo Church (1903 – 1995)

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Church Turing Thesis

Turing machines (TMs) that halt can compute exactly what any algorithm can compute

Alonzo Church (1903 – 1995)

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Church Turing Thesis

Turing machines (TMs) that halt can compute exactly what any algorithm can compute

The other way around

The things that TMs cannot compute can also not be computed by any algorithm (no matter whether it is written in Assembler, C, C++, Java, Python, or Ruby on Rails)

Undecidability

- TMs solve decision problems
- Is $x \in \mathrm{PRIMES}$?
- $\mathrm{PRIMES} = \{y \mid y \text{ is a prime number}\}$

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- Let M_{PRIMES} be a TM for PRIMES
- If $x \in \mathrm{PRIMES}$, then M_{PRIMES} "answers" YES
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⇨ $\mathrm{PRIMES} = \{2, 3, 5, 7, 11, 13, 17, 19, \ldots\}$

⇨ Decision problems are infinite sets of strings (words)

Undecidability

Theorem (Turing)

There are decision problems, which cannot be solved by any TM (they are therefore called **undecidable**)

Undecidability

Theorem (Turing)

There are decision problems, which cannot be solved by any TM (they are therefore called **undecidable**)

- Enumerate all TMs: $M_1, M_2, M_3, \ldots$
- Enumerate all words (over fixed alphabet): $w_1, w_2, w_3, \ldots$
- Assume every TM M_i solves some decision problem
- For every word w_j the TM M_i answers either YES or NO

Undecidability

- Again construct table
- Rows corresond to TMs, columns correspond to words

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	$\dots$
M_1										
M_2										
M_3										
M_4										
$\vdots$										

Undecidability

- Again construct table
- Rows corresond to TMs, columns correspond to words

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	$\dots$
M_1	1	0	0	1	1	1	1	0	0	
M_2	0	0	1	0	0	0	1	0	1	$\dots$
M_3	1	0	0	1	0	1	0	0	0	
M_4	0	0	1	1	0	0	1	0	0	
$\vdots$				$\vdots$						$\ddots$

- 1 in cell (i, j) if M_i answers YES for w_j
- 0 in cell (i, j) if M_i answers NO for w_j

Undecidability

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	$\dots$
M_1	1	0	1	1	1	1	1	0	0	
M_2	0	0	0	0	0	0	1	0	1	$\dots$
M_3	1	0	0	1	0	1	0	0	0	
M_4	0	0	1	1	0	0	1	0	0	
$\vdots$				$\vdots$						$\ddots$

Undecidability

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	$\dots$
M_1	$\boxed{1}$	0	1	1	1	1	1	0	0	
M_2	0	0	0	0	0	0	1	0	1	$\dots$
M_3	1	0	0	1	0	1	0	0	0	
M_4	0	0	1	1	0	0	1	0	0	
$\vdots$				$\vdots$						$\ddots$

- M_1 answers YES for w_1

Undecidability

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	$\dots$
M_1	$\boxed{1}$	0	1	1	1	1	1	0	0	
M_2	0	$\boxed{0}$	0	0	0	0	1	0	1	$\dots$
M_3	1	0	0	1	0	1	0	0	0	
M_4	0	0	1	1	0	0	1	0	0	
$\vdots$				$\vdots$						$\ddots$

- M_1 answers YES for w_1
- M_2 answers NO for w_2

Undecidability

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	$\dots$
M_1	$\boxed{1}$	0	1	1	1	1	1	0	0	
M_2	0	$\boxed{0}$	0	0	0	0	1	0	1	$\dots$
M_3	1	0	$\boxed{0}$	1	0	1	0	0	0	
M_4	0	0	1	1	0	0	1	0	0	
$\vdots$				$\vdots$						$\ddots$

- M_1 answers YES for w_1
- M_2 answers NO for w_2
- M_3 answers NO for w_3

Undecidability

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	$\dots$
M_1	$\boxed{1}$	0	1	1	1	1	1	0	0	
M_2	0	$\boxed{0}$	0	0	0	0	1	0	1	$\dots$
M_3	1	0	$\boxed{0}$	1	0	1	0	0	0	
M_4	0	0	1	$\boxed{1}$	0	0	1	0	0	
$\vdots$				$\vdots$						$\ddots$

- M_1 answers YES for w_1
- M_2 answers NO for w_2
- M_3 answers NO for w_3
- M_4 answers YES for w_4

Undecidability

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	$\dots$
M_1	$\boxed{1}$	0	1	1	1	1	1	0	0	
M_2	0	$\boxed{0}$	0	0	0	0	1	0	1	$\dots$
M_3	1	0	$\boxed{0}$	1	0	1	0	0	0	
M_4	0	0	1	$\boxed{1}$	0	0	1	0	0	
$\vdots$				$\vdots$						$\ddots$

- M_1 answers YES for w_1
- M_2 answers NO for w_2
- M_3 answers NO for w_3
- M_4 answers YES for w_4
- Define decision problem DIAG (**Diagonalization**)
- w_i is in DIAG if M_i answers NO for w_i
- In this example, $\mathrm{DIAG} = \{w_2, w_3, \dots\}$

Undecidability

The word w_i is in DIAG if and only if M_i answers NO for w_i

Undecidability

The word w_i is in DIAG if and only if M_i answers NO for w_i

- Suppose there is a TM M for *DIAG*

⇨ M must appear in the enumeration of all TMs

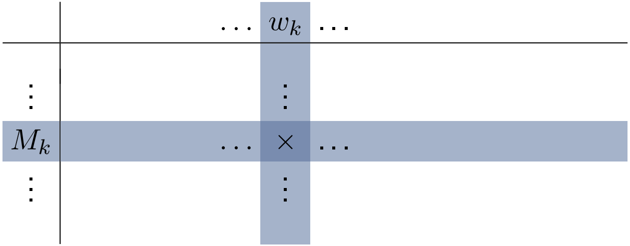
⇨ Suppose M is the kth TM, that is, M_k

The word w_i is in DIAG if and only if M_i answers NO for w_i

- Suppose there is a TM M for *DIAG*

⇨ M must appear in the enumeration of all TMs

⇨ Suppose M is the kth TM, that is, M_k; consider cell (k, k)



- Does this cell contain a 1 or a 0?

Undecidability

The word w_i is in DIAG if and only if M_i answers NO for w_i

Suppose cell (k, k) contains a 1

⇨ This means that M_k answers YES for w_k

Undecidability

The word w_i is in DIAG if and only if M_i answers NO for w_i

Suppose cell (k, k) contains a 1

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- ⇨ By construction, w_k is not DIAG

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- But M_k is a TM for DIAG

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- ⇨ This means that M_k answers YES for w_k
- ⇨ By construction, w_k is not DIAG
- But M_k is a TM for DIAG
- ⇨ Thus, M_k answers NO for w_k

Oh well, then suppose cell (k, k) contains a 0

- ⇨ This means that M_k answers NO for w_k

Undecidability

The word w_i is in DIAG if and only if M_i answers NO for w_i

Suppose cell (k, k) contains a 1

⇨ This means that M_k answers YES for w_k

⇨ By construction, w_k is not DIAG

- But M_k is a TM for DIAG

⇨ Thus, M_k answers NO for w_k

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- But M_k is a TM for DIAG

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- But M_k is a TM for DIAG

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Undecidability

The word w_i is in DIAG if and only if M_i answers NO for w_i

Suppose cell (k, k) contains a 1

- ⇨ This means that M_k answers YES for w_k
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- ⇨ This means that M_k answers NO for w_k
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  - But M_k is a TM for DIAG
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Undecidability

The word w_i is in DIAG if and only if M_i answers NO for w_i

Suppose cell (k, k) contains a 1

- ⇨ This means that M_k answers YES for w_k
- ⇨ By construction, w_k is not DIAG
- But M_k is a TM for DIAG
- ⇨ Thus, M_k answers NO for w_k

Oh well, then suppose cell (k, k) contains a 0

- ⇨ This means that M_k answers NO for w_k
- ⇨ By construction, w_k is in DIAG
- But M_k is a TM for DIAG
- ⇨ Thus, M_k answers YES on w_k

Contradiction

The Halting Problem

```python
print("Hello, world.")
exit()
```

The Halting Problem

```
print("Hello, world.")
exit()
```

```
while True:
    i = 1
print("Hello, world.")
exit()
```

The Halting Problem

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print("Hello, world.")
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```
while True:
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```

```
n = 4
p = 1
while True:
    p = p + 1
    if isprime(p):
        q = n - p
        if isprime(q):
            print n + "=" + p + "+" + q
            n = n + 2
            p = 1
    if p > n/2:
        exit()
```

The Halting Problem

```
print("Hello, world.")
exit()
```

```
while True:
    i = 1
print("Hello, world.")
exit()
```

```
n = 4                                                   # Initialisation
p = 1
while True:
    p = p + 1
    if isprime(p):
        q = n - p
        if isprime(q):                                  # p and q are primes and n = p + q
            print n + "=" + p + "+" + q
            n = n + 2
            p = 1
    if p > n/2:                                         # No primes p and q with n = p + q found
        exit()
```

The Halting Problem

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            print n + "=" + p + "+" + q
            n = n + 2
            p = 1
    if p > n/2:                   # No primes p and q with n = p + q found
        exit()
```

Goldbach Conjecture

Intermission

Other Things due to Turing

Other Things due to Turing

The Turing test

- Interaction (dialog) between human and computer (algorithm)
- An interrogator I talks to both of them
  (no audio visual contact)
- If I cannot tell which of them
  is the machine, computer passes test

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- An interrogator I talks to both of them (no audio visual contact)
- If I cannot tell which of them is the machine, computer passes test

Deciphering the ENIGMA

- German encryption device used in WW2
- Turing designed the **bombe**
- German communication could be read
- Probably war-deciding

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But again back to the Turing Machine. . .

Fast Turing Machines


Additional ABI support: linux.
Starting cron.
Local package initialization:.
Additional TCP options:.
Starting moused:.
Starting background file system checks in 60 seconds.
Fri Aug 5 18:58:36 CEST 2005
FreeBSD/i386 (unixbox) (ttyv0)
login: root
Password:
Login incorrect
SAMSUNG
1.25 Liter
Bruteforce Cat 2.0

Running Time Analysis

- Now consider decision problems that can be solved by TMs

⇨ For example, PRIMES

Running Time Analysis

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- How fast can this be done?
- We are interested in the number of operations

Running Time Analysis

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- How fast can this be done?
- We are interested in the number of operations
- For sure, $x \in \mathrm{PRIMES}$ can be decided faster for $x = 5$ than for $x = 10\,000\,000\,013$
- In general, the "running time" increases with the input size n
- Roughly, we have $n = \log_2 x$

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- In general, the "running time" increases with the input size n
- Roughly, we have $n = \log_2 x$

⇨ But how fast "grows" the running time?

Running Time Analysis

n	10	50	100	300	$10\,000$
$10n$	100	500	$1\,000$	$3\,000$	$100\,000$
$4n^2$	400	$10\,000$	$40\,000$	$360\,000$	$400\,000\,000$
n^3	$1\,000$	$125\,000$	$1\,000\,000$	$27\,000\,000$	13 digits
2^n	1024	16 digits	31 digits	91 digits	$3\,011$ digits
3^n	$59\,049$	24 digits	48 digits	143 digits	$4\,772$ digits

Running Time Analysis

n	10	50	100	300	10 000
$10n$	100	500	1 000	3 000	100 000
$4n^2$	400	10 000	40 000	360 000	400 000 000
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Definition

A function f is in $\mathcal{O}(g)$ for a function g if (for sufficiently large n) f only grows constantly faster than g

Running Time Analysis

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$10n$	100	500	1 000	3 000	100 000
$4n^2$	400	10 000	40 000	360 000	400 000 000
n^3	1 000	125 000	1 000 000	27 000 000	13 digits
2^n	1024	16 digits	31 digits	91 digits	3 011 digits
3^n	59 049	24 digits	48 digits	143 digits	4 772 digits

Definition

A function f is in $\mathcal{O}(g)$ for a function g if (for sufficiently large n) f only grows constantly faster than g

- $4n^2 \in \mathcal{O}(n^2)$
- $100n^3 \in \mathcal{O}(n^3)$
- $100n^3 + 50n \in \mathcal{O}(n^3)$
- $n^3 \notin \mathcal{O}(n^2)$
- $5n \in \mathcal{O}(n^2)$
- $2^n \notin \mathcal{O}(n^2)$
- $\pi n^5 \in \mathcal{O}(n^5)$
- $n \notin \mathcal{O}(\sqrt{n})$
- $\log n \in \mathcal{O}(n)$

Running Time Analysis

Consider a simple test whether a given number is a prime number

```
test = 2
while test < num:
    res = num % test
    if res == 0:
        print("NO")
        exit()
    test += 1
print("YES")
exit()
```

Running Time Analysis

Consider a simple test whether a given number is a prime number

```
test = 2
while test < num:
    res = num % test
    if res == 0:
        print("NO")
        exit()
    test += 1
print("YES")
exit()
```

If `num` is prime, then the loop is executed roughly `num` $\approx 2^n$ times

Running Time Analysis

Consider a simple test whether a given number is a prime number

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test = 2
while test < num:
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test = 2
root_num = sqrt(num)
while test <= root_num:
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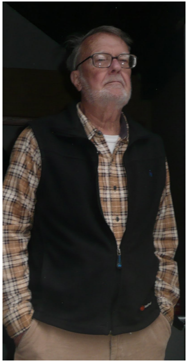
It is sufficient to test until the square root of num; loop is executed roughly $\sqrt{\texttt{num}} \approx 1.41^n$ times



Unknown

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Thesis of C. and E.

Efficient algorithms are those that run in polynomial time

This means the running time is in $\mathcal{O}(n^k)$ for some fixed $k \in \mathbb{N}$

Efficient Algorithms

- The term is independent of the concrete computing model

⇨ If a “real algorithm” can compute something in $\mathcal{O}(n^k)$, then there is a TM that can do it in $\mathcal{O}(n^{k'})$

- Therefore, we call the class of polynomials **robust**

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Definition

The class $\mathcal{P}$ contains all decision problems that can be solved efficiently by TMs

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- We are interested in the running time in the **worst case**

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- How about decision problems that are not in $\mathcal{P}$?

$\mathcal{NP}$-Completeness

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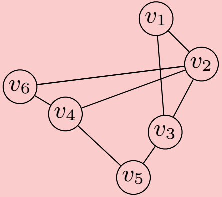

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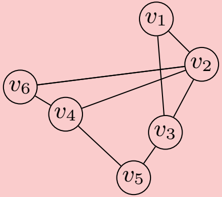

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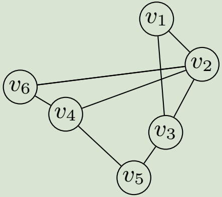

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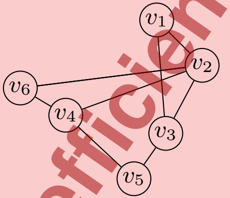

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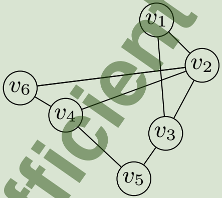

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$\mathcal{P}$ versus $\mathcal{NP}$

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- But what about the opposite direction?

⇨ Until today, we do not know whether $\mathcal{P} = \mathcal{NP}$ or $\mathcal{P} \subsetneq \mathcal{NP}$

$\mathcal{P}$ versus $\mathcal{NP}$

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Definition

A decision problem A in $\mathcal{NP}$ is called **$\mathcal{NP}$-complete** if the existence of an efficient TM for A implies the existence of efficient TMs for all problems in $\mathcal{NP}$

Stephen A. Cook (*1939)

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Cook's Theorem

There is an $\mathcal{NP}$-complete decision problem

Richard M. Karp (*1935)

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Karp's 21 Problems

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Today we know thousands such problems, but for none of them we can actually prove that there is no efficient TM

Polynomial-Time Reduction

Assumption

- Let A and B be two decision problems in $\mathcal{NP}$

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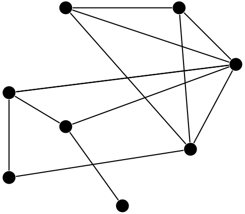
- We "reduce" solving A efficiently to solving B efficiently

⇨ This is called a **polynomial-time reduction**

Definition

A **clique** in a graph is a subset of vertices that are all pairwise connected by edges



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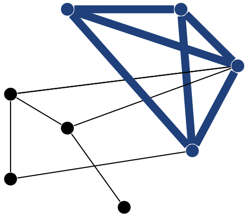

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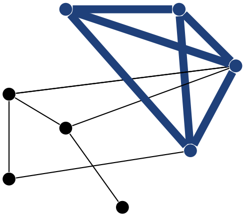

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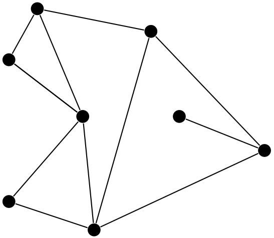
An **independent set** in a graph is a subset of vertices such that there is no edge between any of them



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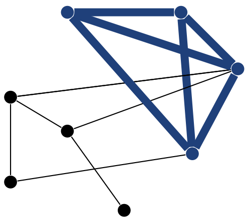

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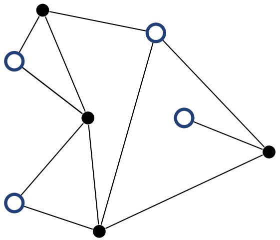

Polynomial-Time Reduction

Consider the following decision problems

- $\mathrm{CLIQUE} = \{(y,k) \mid y$ is a graph with a clique of size $k\}$
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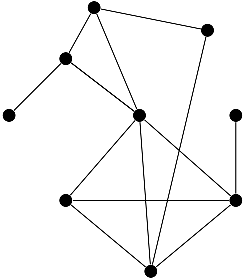
⇨ If we could solve $\mathrm{IND\text{-}SET}$ efficiently, then also CLIQUE

Polynomial-Time Reduction

We make the following observation



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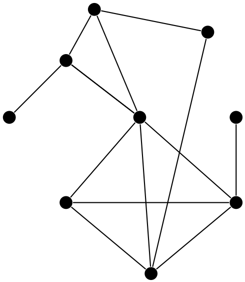

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(that is, delete edges and
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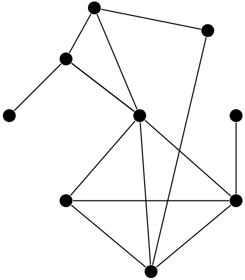

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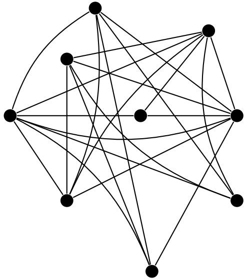

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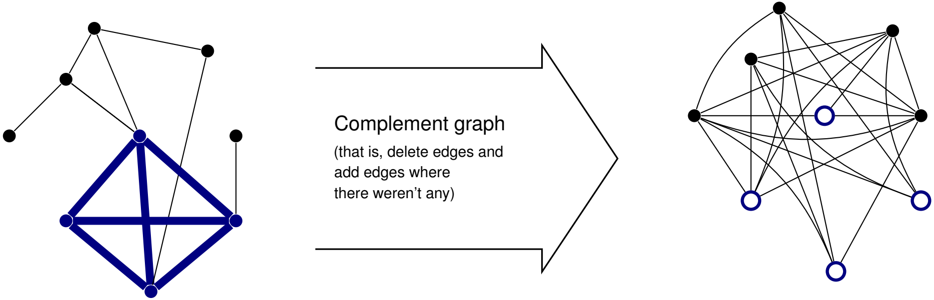

Clique of size k becomes independent set of size k

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Hypothetical
TM that solves
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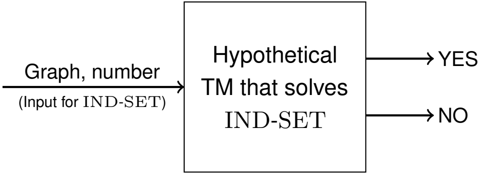

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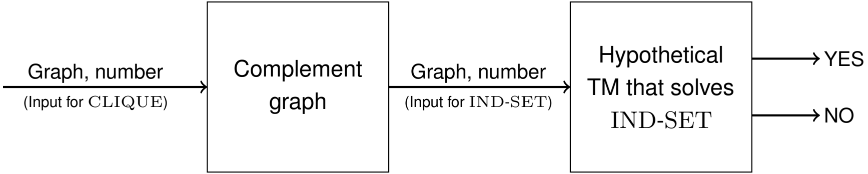

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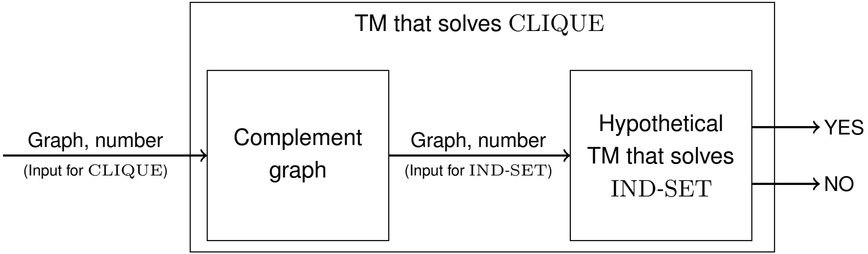

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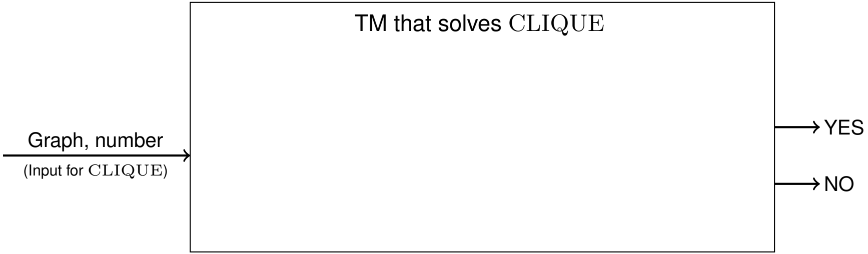

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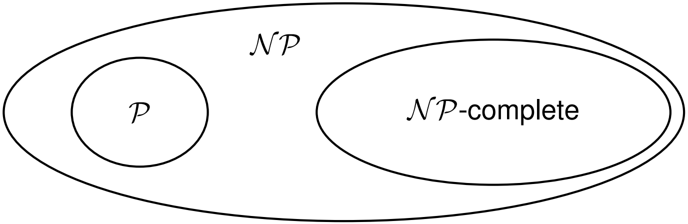

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- $\mathcal{P}$ versus $\mathcal{NP}$ problem open for decades
- One of seven “millennium problems” ($1\,000\,000$ USD reward)

Thanks for the Attention