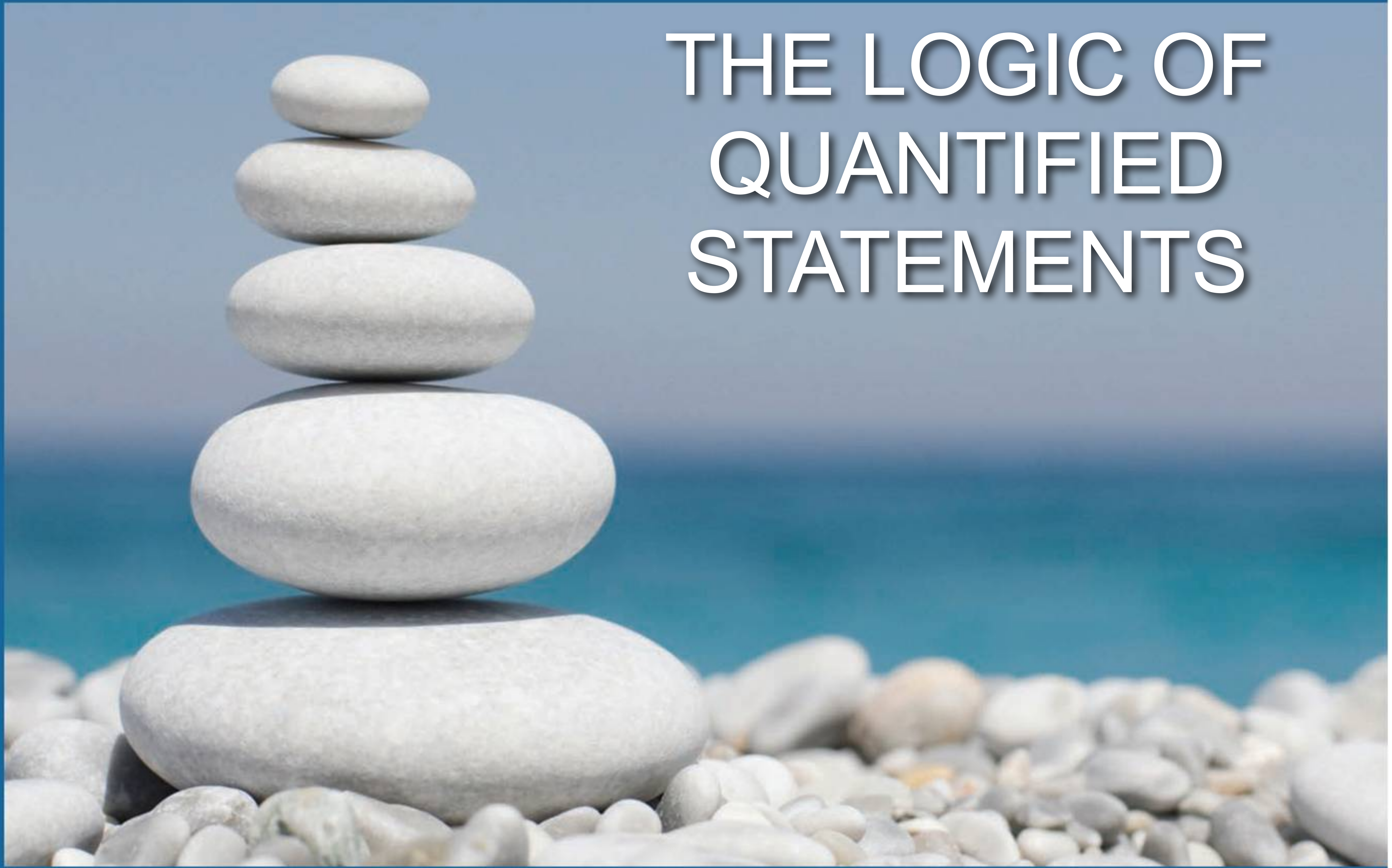
CHAPTER 3

THE LOGIC OF QUANTIFIED STATEMENTS



SECTION 3.1

Predicates and Quantified Statements I

Predicates and Quantified Statements I

In logic, predicates can be obtained by removing some or all of the nouns from a statement.

Example: Let P stand for "is a student at Bedford College" and let Q stand for "is a student at."

→ P and Q are ***predicate symbols***.

The sentences "x is a student at Bedford College" and "x is a student at y" are symbolized as $P(x)$ and as $Q(x, y)$.

→ x and y are ***predicate variables***.

When concrete values are substituted for predicate variables, a statement results.

Predicates and Quantified Statements I

- **Definition**

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

Example: $P(x)$ and $Q(x, y)$ are predicates.

Predicates are sometimes also referred to as **propositional functions** or **open sentences**.

Predicates and Quantified Statements I

- **Definition**

If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}.$$

Example – Finding the truth set:

Let $Q(n)$ be the predicate "*n* is a factor of 8." Find the truth set of $Q(n)$ if the domain of n is the set $\mathbf{Z}^+$ of all positive integers.

Solution:

The truth set is {1, 2, 4, 8}.

The Universal Quantifier: ∀

Quantifiers are words that refer to quantities such as "some" or "all" and tell for which elements a given predicate is true.

The symbol ∀ denotes "for all" and is called the **universal quantifier**.

- **Definition**

Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for every x in D. It is defined to be false if, and only if, $Q(x)$ is false for at least one x in D. A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Exercise – *Truth and Falsity of Universal Statements*

a. Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, x^2 \geq x.$$

Show that this statement is true or false.

b. Consider the statement

$$\forall x \in \mathbf{R}, x^2 \geq x.$$

Show the correctness or find a counterexample to show that this statement is false.

Solution – *Truth and Falsity of Universal Statements*

a. Check if "$x^2 \geq x$" is true for each individual x in D.

$$1^2 \geq 1, \qquad 2^2 \geq 2, \qquad 3^2 \geq 3, \qquad 4^2 \geq 4, \qquad 5^2 \geq 5.$$

Hence "$\forall x \in D, x^2 \geq x$" is true.

b. *Counterexample:* Take $x = {}^1/_2$. Then x is in $\mathbf{R}$ (since ${}^1/_2$ is a real number) and

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \not\geq \frac{1}{2}.$$

Hence "$\forall x \in \mathbf{R}, x^2 \geq x$" is false.

The Universal Quantifier: ∀

The technique used to show the truth of the universal statement in Example **a.** is called the **method of exhaustion:**

Show the truth of the predicate separately for each individual element of the domain.

This method can, in theory, be used whenever the domain of the predicate variable is finite.

The Existential Quantifier: $\exists$

The symbol $\exists$ denotes "there exists" and is called the **existential quantifier.**

For example, the sentence "There is a student in Math 140" can be written as

$\exists$ person p such that p is a student in Math 140,

or, more formally,

$\exists p \in P$ such that p is a student in Math 140,

where P is the set of all people.

The Existential Quantifier: ∃

• Definition

Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for at least one x in D. It is false if, and only if, $Q(x)$ is false for all x in D.

Symbol ∃ can also be read as *there is a*, *we can find a*, *there is at least one*, *for some*, and *for at least one*.

Exercise – *Truth and Falsity of Existential Statements*

a. Consider the statement

$\exists m \in \mathbf{Z}^+$ such that $m^2 = m$.

Show that this statement is true or false.

b. Let $E = \{5, 6, 7, 8\}$ and consider the statement

$\exists m \in E$ such that $m^2 = m$.

Show that this statement is true or false.

Solution – *Truth and Falsity of Existential Statements*

a. $\exists m \in \mathbf{Z}^+$ such that $m^2 = m$

Observe that $1^2 = 1$. Thus "$m^2 = m$" is true for at least one integer m.

Hence "$\exists m \in \mathbf{Z}^+$ such that $m^2 = m$" is true.

b. $\exists m \in E$ such that $m^2 = m$ with $E = \{5, 6, 7, 8\}$

Note that $m^2 = m$ is not true for any integers m from 5 through 8:

$$5^2 = 25 \neq 5, \qquad 6^2 = 36 \neq 6, \qquad 7^2 = 49 \neq 7, \qquad 8^2 = 64 \neq 8.$$

Thus "$\exists m \in E$ such that $m^2 = m$" is false.

Formal Versus Informal Language

It is important to be able to translate from formal to informal language when trying to make sense of mathematical concepts that are new to you.

It is equally important to be able to translate from informal to formal language when thinking out a complicated problem.

Example: Give an informal version of $\forall x \in \mathbf{R}, x^2 \geq 0$.

Solution:

"All real numbers have nonnegative squares."

Or: Every real number has a nonnegative square.

…

Universal Conditional Statements

Arguably the most important form of a statement in mathematics is the **universal conditional statement**:

$\forall x$, if $P(x)$ then $Q(x)$.

Example: $\forall x \in \mathbf{R}$, if $x > 2$ then $x^2 > 4$.

"If a real number is greater than 2 then its square is greater than 4." (informal)

Or: "Whenever a real number is greater than 2, its square is greater than 4." (informal)

Equivalent Forms of Universal and Existential Statements

Observe: Statement "∀ real numbers *x*, if *x* is an integer then *x* is rational" means the same as "∀ integers *x*, *x* is rational", since all integers are real.

In fact, a statement of the form $\forall x \in U$, if $P(x)$ then $Q(x)$

can always be rewritten in the form $\forall x \in D, Q(x)$

by narrowing *U* to be the domain *D* consisting of all values of the variable *x* that make *P*(*x*) true.

Conversely, a statement of the form $\forall x \in D, Q(x)$

can be rewritten as $\forall x$, if x is in D then $Q(x)$.

Equivalent Forms of Universal and Existential Statements

Similarly, a statement of the form

"$\exists x$ such that $P(x)$ and $Q(x)$"

is equivalent to

"$\exists x \in D$ such that $Q(x)$,"

where D is the set of all x for which $P(x)$ is true.

Exercise – *Equivalent Forms for Existential Statements*

A **prime number** is an integer greater than 1 whose only positive integer factors are itself and 1.

Consider the statement

"There is an integer that is both prime and even."

Let Prime(n) be "n is prime" and Even(n) be "n is even." Use the notation Prime(n) and Even(n) to rewrite this statement.

Solution:

$\exists n$ such that Prime(n) $\wedge$ Even(n).

$\exists$ a prime number n such that Even(n)

$\exists$ an even number n such that Prime(n).

Implicit Quantification

Mathematical writing contains many examples of implicitly quantified statements (through the presence of "*a*" or "*an*", or when the context supplies part of its meaning).

For example, in an algebra course in which the letter *x* is always used to indicate a real number, the predicate

If $x > 2$ then $x^2 > 4$

is interpreted to mean the same as the statement

$\forall$ real numbers x, if $x > 2$ then $x^2 > 4$ or

$\forall x \in \boldsymbol{R}$, if $x > 2$ then $x^2 > 4$.

Implicit Quantification

Mathematicians often use a double arrow to indicate implicit quantification symbolically.

For instance, they might express the above statement as

$x > 2 \Rightarrow x^2 > 4$.

• Notation

Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D.

- The notation $\boldsymbol{P(x) \Rightarrow Q(x)}$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently, $\forall x,\ P(x) \to Q(x)$.
- The notation $\boldsymbol{P(x) \Leftrightarrow Q(x)}$ means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently, $\forall x,\ P(x) \leftrightarrow Q(x)$.

Exercise – *Using* $\Rightarrow$ *and* $\Leftrightarrow$

Let $Q(n)$ be "n is a factor of 8,"

$R(n)$ be "n is a factor of 4,"

$S(n)$ be "$n < 5$ and $n \neq 3$,"

and suppose the domain of n is $\mathbf{Z}^+$.

Use the $\Rightarrow$ and $\Leftrightarrow$ symbols to indicate true relationships among $Q(n)$, $R(n)$, and $S(n)$.

Solution:

truth set of $Q(n)$ is $\{1, 2, 4, 8\}$

truth set of $R(n)$ is $\{1, 2, 4\}$

truth set of $S(n)$ is $\{1, 2, 4\}$.

$R(n) \Rightarrow Q(n)$,

$R(n) \Leftrightarrow S(n)$,

$S(n) \Rightarrow Q(n)$

SECTION 3.2

Predicates and Quantified Statements II

Negation of Quantified Statements

The general form of the negation of a universal statement follows immediately from the definitions of negation and of the truth values for universal and existential statements.

Theorem 3.2.1 Negation of a Universal Statement

The negation of a statement of the form

$$\forall x \text{ in } D, Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim Q(x).$$

Symbolically, $\sim(\forall x \in D, Q(x)) \equiv \exists x \in D \text{ such that } \sim Q(x).$

Negation of Quantified Statements

The general form for the negation of an existential statement follows immediately from the definitions of negation and of the truth values for existential and universal statements.

Theorem 3.2.2 Negation of an Existential Statement

The negation of a statement of the form

$$\exists x \text{ in } D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \text{ in } D, \sim Q(x).$$

Symbolically, $\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x).$

Exercise – *Negating Quantified Statements*

Write formal negations for the following statements:

a. $\forall$ primes p, p is odd.

b. $\exists$ a triangle T such that the sum of the angles of T equals $200°$.

Solution:

a. By applying the rule for the negation of a $\forall$ statement, you can see that the answer is

$\exists$ a prime p such that p is not odd.

b. By applying the rule for the negation of a $\exists$ statement, you can see that the answer is

$\forall$ triangles T, the sum of the angles of T does not equal $200°$.

Negations of Universal Conditional Statements

Negations of universal conditional statements are of special importance in mathematics.

The form of such negations can be derived from facts that have already been established.

By definition of the negation of a for all statement,

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } \sim(P(x) \rightarrow Q(x)). \qquad 3.2.1$$

But the negation of an if-then statement is logically equivalent to an and statement. More precisely,

$$\sim(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x). \qquad 3.2.2$$

Negations of Universal Conditional Statements

Substituting (3.2.2) into (3.2.1) gives

$$\sim(\forall x,\ P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } (P(x) \wedge \sim Q(x)).$$

Written less symbolically, this becomes

Negation of a Universal Conditional Statement

$$\sim(\forall x, \text{ if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x).$$

Exercise – *Negating Universal Conditional Statements*

Write a formal negation for statement (a) and an informal negation for statement (b).

a. $\forall$ people p, if p is blond then p has blue eyes.

b. If a computer program has more than 100,000 lines, then it contains a bug.

Solution:

a. $\exists$ a person p such that p is blond and p does not have blue eyes.

b. There is at least one computer program that has more than 100,000 lines and does not contain a bug.

Variants of Universal Conditional Statements

We have known that a conditional statement has a contrapositive, a converse, and an inverse.

The definitions of these terms can be extended to universal conditional statements.

• Definition

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is the statement: $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is the statement: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Exercise – *Contrapositive, Converse, and Inverse of a Universal Conditional Statement*

Write a formal contrapositive, converse, and inverse for the following statement:

$\forall x \in \mathbf{R}$, if $x > 2$ then $x^2 > 4$.

Solution:

Contrapositive: $\forall x \in \mathbf{R}$, if $x^2 \leq 4$ then $x \leq 2$.

Converse: $\forall x \in \mathbf{R}$, if $x^2 > 4$ then $x > 2$.

Inverse: $\forall x \in \mathbf{R}$, if $x \leq 2$ then $x^2 \leq 4$.

Necessary and Sufficient Conditions, Only If

The definitions of necessary, sufficient, and only if can also be extended to apply to universal conditional statements.

• Definition

- "$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x$, if $r(x)$ then $s(x)$."
- "$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x$, if $\sim r(x)$ then $\sim s(x)$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$."
- "$\forall x$, $r(x)$ **only if** $s(x)$" means "$\forall x$, if $\sim s(x)$ then $\sim r(x)$" or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$."

Exercise – *Necessary and Sufficient Conditions*

Rewrite the following statements as quantified conditional statements. Do not use the word necessary or sufficient.

a. Squareness is a sufficient condition for rectangularity.

b. Being at least 35 years old is a necessary condition for being President of the United States.

Solution:

a. $\forall x$, if x is a square, then x is a rectangle.

b. $\forall$ people x, if x is younger than 35, then x cannot be President of the United States.

SECTION 3.3

Statements with Multiple Quantifiers

Statements with Multiple Quantifiers

When a statement contains more than one quantifier, we imagine the actions suggested by the quantifiers as being performed in the order in which the quantifiers occur.

For instance, consider a statement of the form

$\forall x$ in set D, $\exists y$ in set E such that x and y satisfy property $P(x, y)$.

Statements with Multiple Quantifiers

To show that such a statement is true, you must be able to meet the following challenge:

Imagine that someone chooses any element from the set *D*, and gives you that element. Call it *x*.

The challenge for you is to find an element *y* in *E* so that the person's *x* and your *y*, taken together, satisfy property *P*(*x*, *y*).

Example: $\forall x \in \mathbf{Z}^+, \exists y \in \mathbf{R}$ such that $y < x$.

Solution:

For instance, you can always choose $y = -1$. The statement is true.

Statements with Multiple Quantifiers

Now consider a statement where ∃ comes before the ∀:

∃ an x in D such that $\forall y$ in E, x and y satisfy property $P(x, y)$.

To show that a statement of this form is true:

You must find one single element (call it x) in D with the following property:

After you have found your x, someone is allowed to choose any element y from E.

Your job is to show that your x together with the person's y satisfy property $P(x, y)$.

Translating from Informal to Formal Language

Most problems are stated in informal language, but solving them often requires translating them into more formal terms.

Exercise – *Translating Multiply-Quantified Statements from Informal to Formal Language*

The **reciprocal** of a real number a is a real number b such that $ab = 1$.

The following two statements are true. Rewrite them formally using quantifiers and variables:

a. Every nonzero real number has a reciprocal.

b. There is a real number with no reciprocal.

Solution:

a. $\forall$ nonzero real numbers u, $\exists$ a real number v such that $uv = 1$.

b. $\exists$ a real number c such that $\forall$ real numbers d, $cd \neq 1$.

The number 0 has no reciprocal.

Exercise - *Negations of Multiply-Quantified Statements*

You can use the known rules of negation:

$\sim(\forall x \text{ in } D, P(x)) \equiv \exists x \text{ in } D \text{ such that } \sim P(x),$

$\sim(\exists x \text{ in } D \text{ such that } P(x)) \equiv \forall x \text{ in } D, \sim P(x),$

to derive

$\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y))$

$\equiv \exists x \text{ in } D \text{ such that } \sim(\exists y \text{ in } E \text{ such that } P(x, y))$

$\equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y)$

$\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y))$

$\equiv \forall x \text{ in } D, \sim(\forall y \text{ in } E, P(x, y))$

$\equiv \forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y)$

Negations of Multiply-Quantified Statements

Negations of Multiply-Quantified Statements

$$\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)) \equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y).$$

$$\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)) \equiv \forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y).$$

Ambiguous Language

Imagine you are visiting a factory. The factory guide tells you,

> "There is a person supervising every detail of the production process."

This statement contains informal versions of both the existential quantifier *there is* and the universal quantifier *every*.

Ambiguous Language

Which of the following best describes its meaning?

> There is one single person who supervises all the details of the production process.
>
> For any particular production detail, there is a person who supervises that detail, but there might be different supervisors for different details.

Informal statements may be open to multiple interpretations. Therefore, we have to use context to try to interpret them correctly and determine their truth or falsity with the assumed interpretation.

Order of Quantifiers

Consider the following two statements:

"$\forall$ people x, $\exists$ a person y such that x likes y."

"$\exists$ a person y such that $\forall$ people x, x likes y."

Except for the order of the quantifiers, these statements are identical.

The first means that given any person, it is possible to find someone who that person likes.

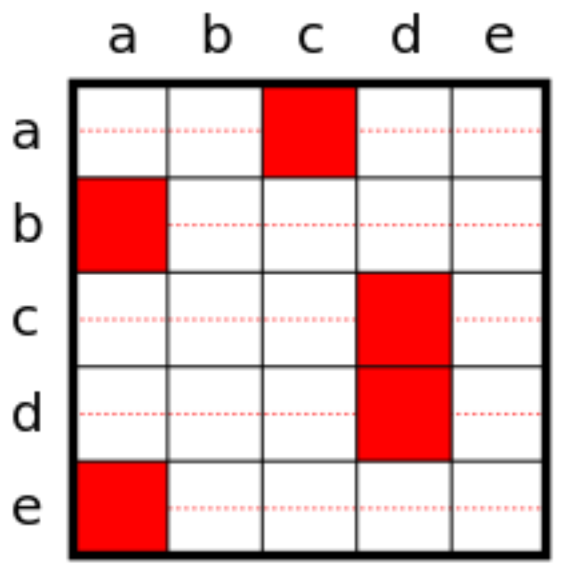
The second means that there is one individual who is liked by all people.

Order of Quantifiers

The two sentences illustrate an important property about multiply-quantified statements:

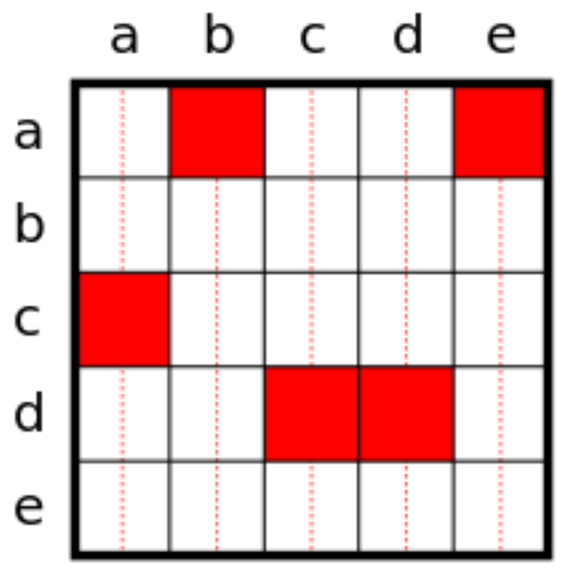
> In a statement containing both $\forall$ and $\exists$, changing the order of the quantifiers usually changes the meaning of the statement.

Interestingly, however, if one quantifier immediately follows another quantifier *of the same type*, then the order of the quantifiers does not affect the meaning.



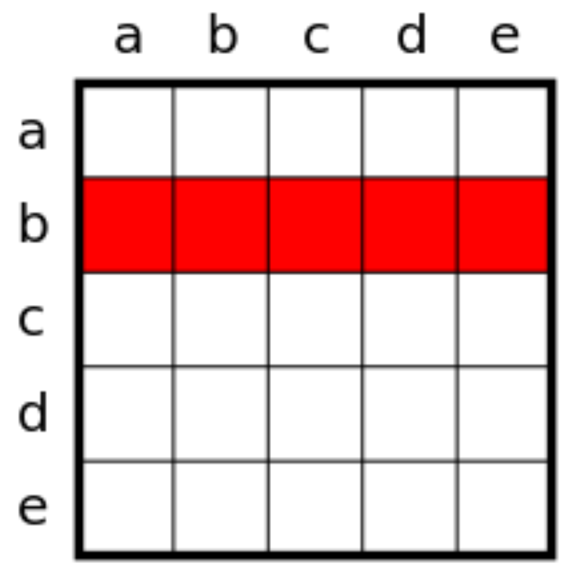
$\forall x \exists y\, L(x,y)$

Everyone likes someone



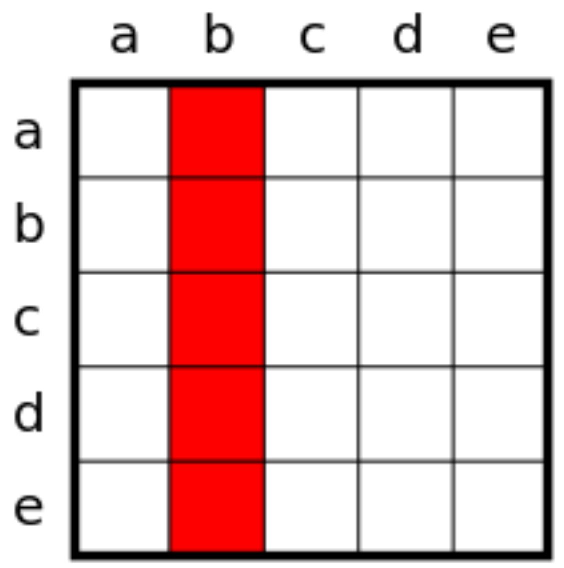
$\forall y \exists x\, L(x,y)$

Everyone is liked by someone



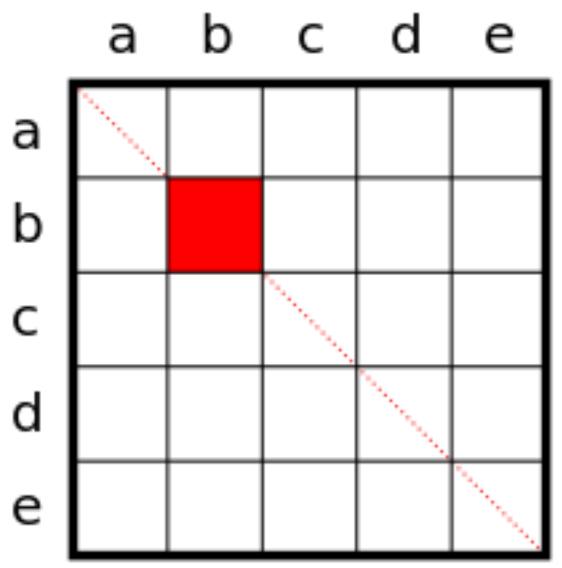
$\exists x \forall y\, L(x,y)$

Someone likes everyone



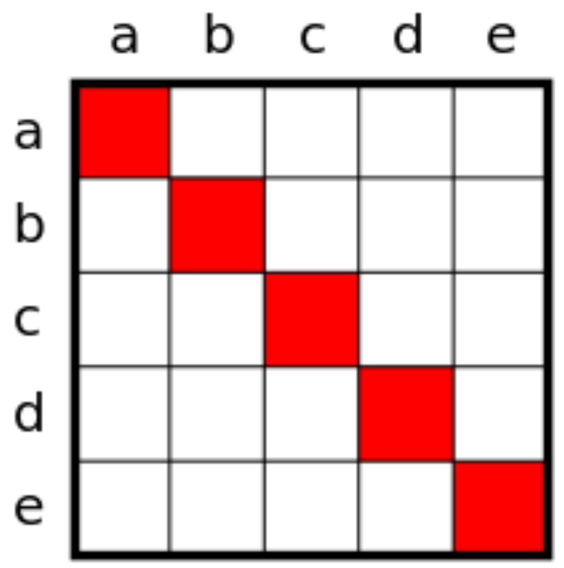
$\exists y \forall x\, L(x,y)$

Someone is liked by everyone



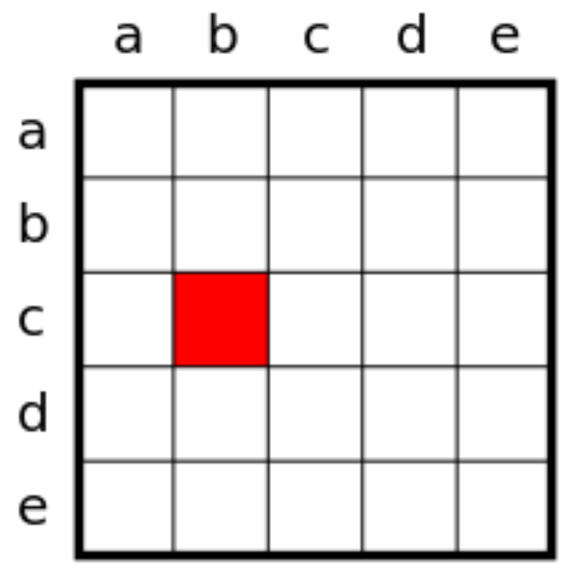
$\exists x\, L(x,x)$

Someone likes her/himself



$\forall x\, L(x,x)$

Everyone likes him/herself

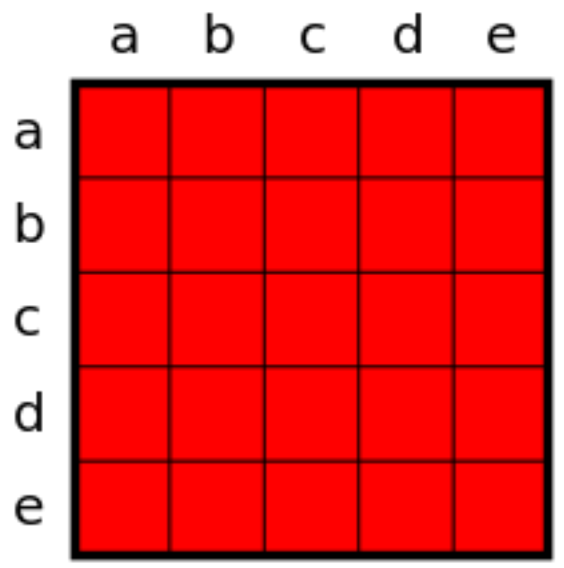


$\exists x \exists y\, L(x,y)$

Someone likes someone

$\exists y \exists x\, L(x,y)$

Someone is liked by someone



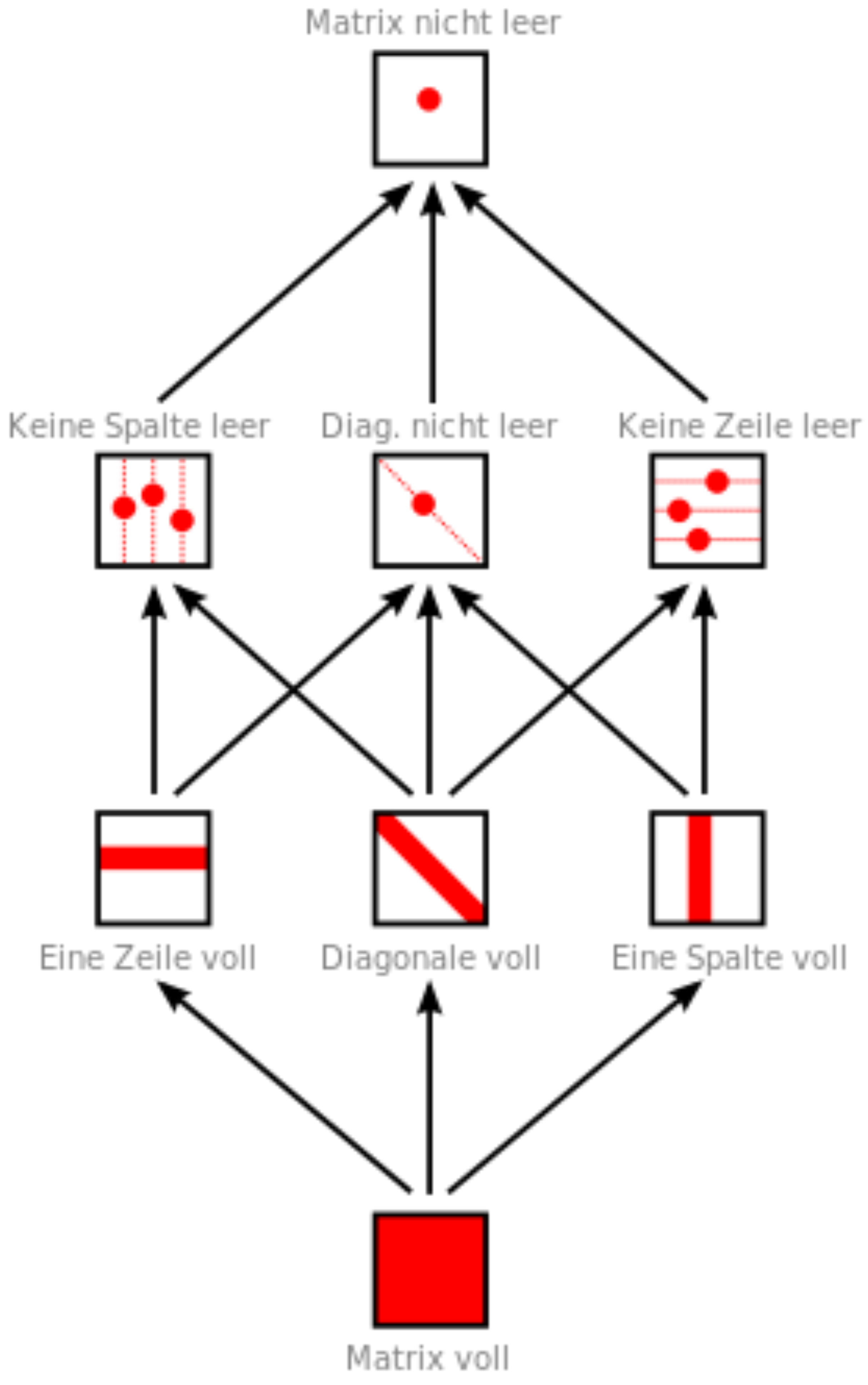
$\forall x \forall y\, L(x,y)$

Everyone likes everyone

$\forall y \forall x\, L(x,y)$

Everyone is liked everyone

$L(\text{row}, \text{column})$ = row x likes column y
= column y is liked by row x



Hasse diagram of implications

Formal Logical Notation

In some areas of computer science, logical statements are expressed in purely symbolic notation.

The notation involves using predicates to describe all properties of variables and omitting the words *such that* in existential statements.

"$\exists x$ in D such that $P(x)$" is written as "$\exists x\ (x \in D \wedge P(x))$".

"$\forall x$ in D, $P(x)$" is written as "$\forall x\ (x \in D \rightarrow P(x))$".

Formal Logical Notation

Taken together, the symbols for quantifiers, variables, predicates, and logical connectives make up what is known as the language of **first-order logic** (predicate logic).

Even though this language is simpler in many respects than the language we use every day, learning it requires the same kind of practice needed to acquire any foreign language.

SECTION 3.4

Arguments with Quantified Statements

Arguments with Quantified Statements

The rule of *universal instantiation* says the following:

> If some property is true of *everything* in a set, then it is true of *any particular* thing in the set.

Universal instantiation is *the* fundamental tool of deductive reasoning.

Mathematical formulas, definitions, and theorems are like general templates that are used over and over in a wide variety of particular situations.

Arguments with Quantified Statements

A given theorem says that such and such is true for all things of a certain type.

If, in a given situation, you have a particular object of that type, then by universal instantiation, you conclude that such and such is true for that particular object.

You may repeat this process 10, 20, or more times in a single proof or problem solution.

Universal Modus Ponens

The rule of universal instantiation can be combined with modus ponens:

Universal Modus Ponens

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$P(a)$ for a particular a.	a makes $P(x)$ true.
$\therefore\ Q(a)$.	$\therefore\ a$ makes $Q(x)$ true.

Exercise – *Recognizing Universal Modus Ponens*

Rewrite the following argument using quantifiers, variables and predicate symbols.

Is this argument valid? Why?

If an integer is even, then its square is even.

k is a particular integer that is even.

$\therefore$ k^2 is even.

Solution:

$\forall x$, if Even(x) then Even(x^2)

Even(k) for a particular k

$\therefore$ Even(k^2)

The argument has the form of modus ponens. Thus it is valid.

Use of Universal Modus Ponens in a Proof

Proof that the sum of any two even integers is even:

Suppose m and n are particular but arbitrarily chosen even integers. Then $m = 2r$ for some integer r,(1) and $n = 2s$ for some integer s.(2)

By definition:
∀ integers x, x is even if, and only if,
∃ an integer k such that $x = 2k$.

Hence

$$m + n = 2r + 2s \quad \text{by substitution}$$
$$= 2(r + s)^{(3)} \quad \text{by factoring out the 2.}$$

Now $r + s$ is an integer,(4) and so $2(r + s)$ is even.(5)

Thus $m + n$ is even.

Use of Universal Modus Ponens in a Proof

(Implicit) expansions of the numbered steps:

(1) If an integer is even, then it equals twice some integer.

m is a particular even integer.

$\therefore$ m equals twice some integer r.

(2) If an integer is even, then it equals twice some integer.

n is a particular even integer.

$\therefore$ n equals twice some integer s.

Use of Universal Modus Ponens in a Proof

(3) If a quantity is an integer, then it is a real number.

r and s are particular integers.

$\therefore$ r and s are real numbers.

For all a, b, and c, if a, b, and c are real numbers, then $ab + ac = a(b + c)$.

2, r, and s are particular real numbers.

$\therefore$ $2r + 2s = 2(r + s)$.

(4) For all u and v, if u and v are integers, then $u + v$ is an integer.

r and s are two particular integers.

$\therefore$ $r + s$ is an integer.

Use of Universal Modus Ponens in a Proof

(5) If a number equals twice some integer, then that number is even.

$2(r + s)$ equals twice the integer $r + s$.

$\therefore$ $2(r + s)$ is even.

Universal Modus Tollens

The rule of universal instantiation can also be combined with modus tollens:

Universal Modus Tollens

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim Q(a)$, for a particular a.	a does not make $Q(x)$ true.
$\therefore\ \sim P(a)$.	$\therefore\ a$ does not make $P(x)$ true.

Universal modus tollens is the heart of proof of contradiction, which is one of the most important methods of mathematical argument.

Exercise – *Recognizing the Form of Universal Modus Tollens*

Give a formal form (with quantifiers and predicates) of the following argument that matches universal modus tollens:

All human beings are mortal.
Zeus is not mortal.
$\therefore$ Zeus is not human.

Solution:

Let H(x) be "x is human". Let M(x) be "x is mortal", and let Z stand for Zeus.

$\forall x$, if H(x) then M(x)

~M(Z)

$\therefore$ ~H(Z)

Proving Validity of Arguments with Quantified Statements

An argument is valid if, and only if, the truth of its conclusion follows *necessarily* from the truth of its premises.

• Definition

To say that an *argument form* is **valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true. An *argument* is called **valid** if, and only if, its form is valid.

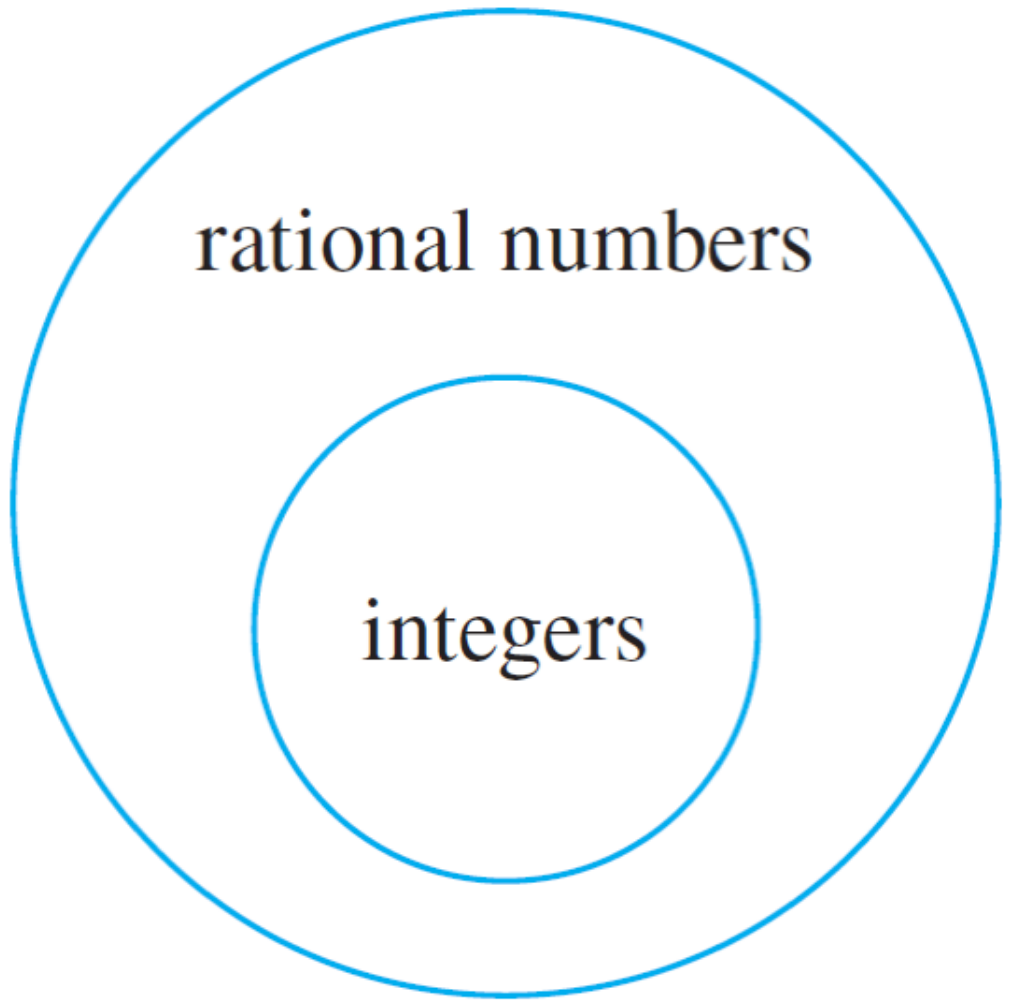
Using Diagrams to Test for Validity

Quantified statements can often be visualized in diagrams for testing validity.

To test validity of an argument:

1. Represent the truth of both premises with diagrams.

2. Analyze the diagrams to see whether they necessarily represent the truth of the conclusion as well.



Example diagram to represent the truth of the statement:

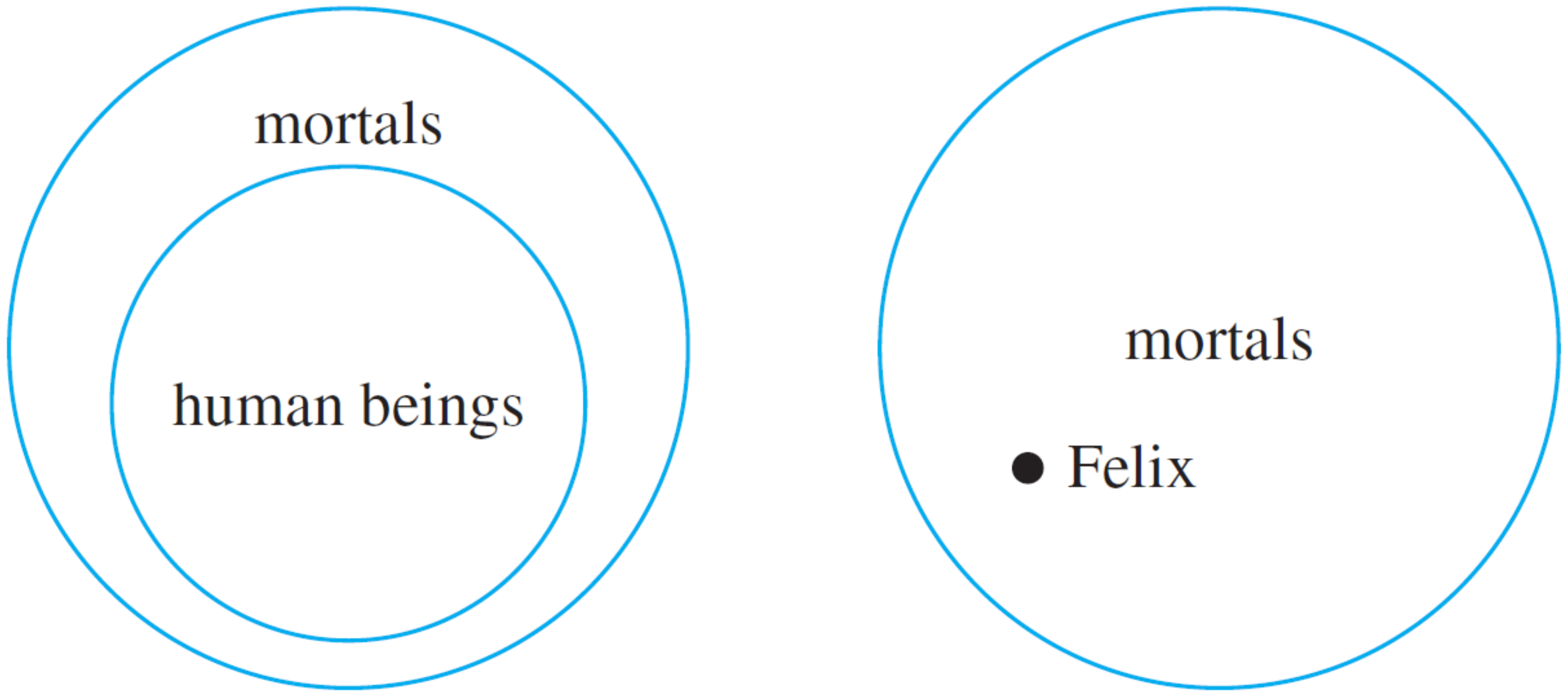
"∀ integers *n*, *n* is a rational number.
(="All integers are rational numbers.")

Exercise – *Using Diagrams to Show Invalidity*

Use a diagram to show the invalidity of the following argument:

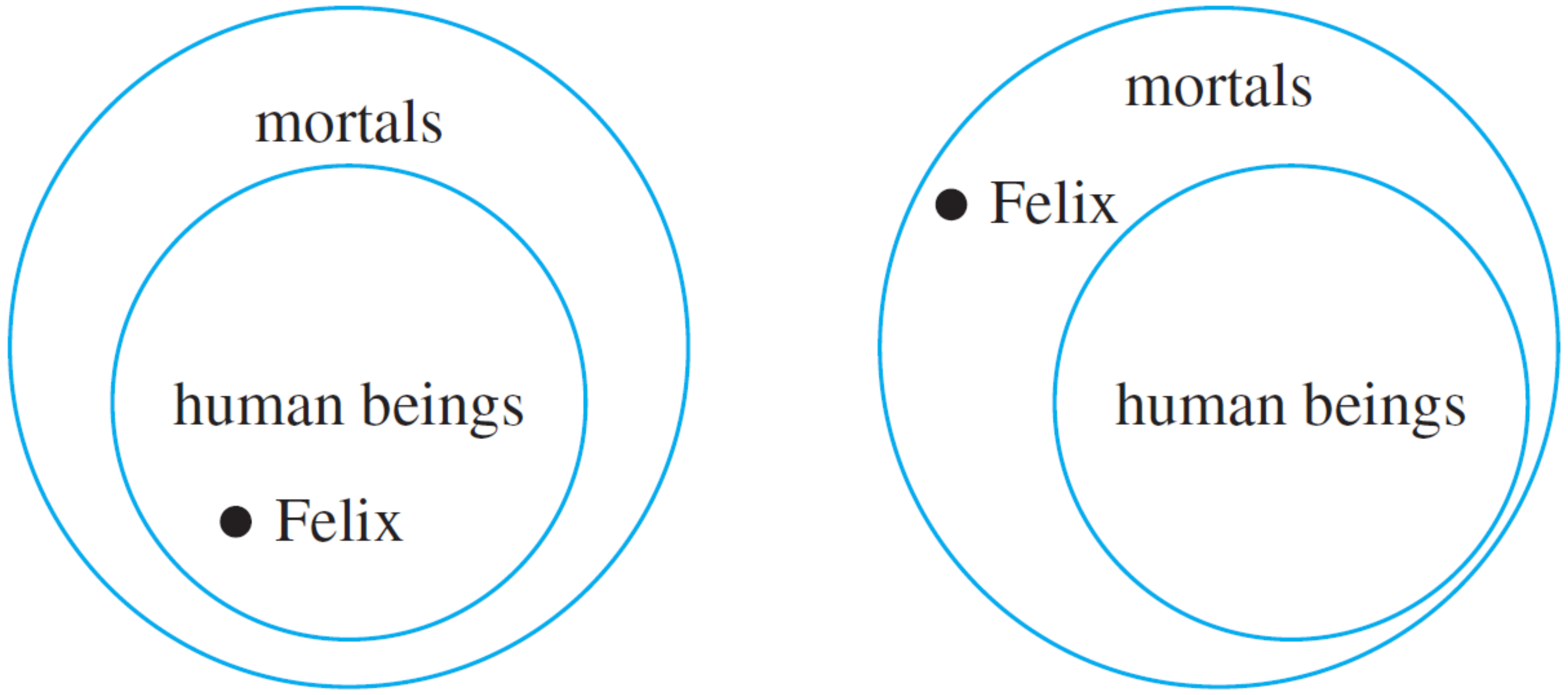
All human beings are mortal.
Felix is mortal.
∴ Felix is a human being.

Solution:



Exercise – *Solution*

After combining the mortals disk, there are two possible cases.



(a) (b)

The conclusion “Felix is a human being” is true in the first case but not in the second (Felix might, for example, be a cat). Because the conclusion does not necessarily follow from the premises, the argument is invalid.

Common Errors

You might have noticed that the example exhibits the classical converse error.

Converse Error (Quantified Form)

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
$\therefore$ $P(a)$. $\leftarrow$ invalid conclusion	$\therefore$ a makes $P(x)$ true. $\leftarrow$ invalid conclusion

Inverse Error (Quantified Form)

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim P(a)$, for a particular a.	a does not make $P(x)$ true.
$\therefore$ $\sim Q(a)$. $\leftarrow$ invalid conclusion	$\therefore$ a does not make $Q(x)$ true. $\leftarrow$ invalid conclusion

Creating Additional Forms of Argument

Universal modus ponens and modus tollens were obtained by combining universal instantiation with modus ponens and modus tollens.

In the same way, additional forms of arguments involving universally quantified statements can be obtained by combining universal instantiation with other of the valid argument forms discussed earlier.

Creating Additional Forms of Argument

Consider the argument of transitivity:

$$p \to q$$
$$q \to r$$
$$\therefore\; p \to r$$

This argument form can be combined with universal instantiation to obtain the following valid argument form.

Universal Transitivity

Formal Version	*Informal Version*
$\forall x\, P(x) \to Q(x)$.	Any x that makes $P(x)$ true makes $Q(x)$ true.
$\forall x\, Q(x) \to R(x)$.	Any x that makes $Q(x)$ true makes $R(x)$ true.
$\therefore\; \forall x\, P(x) \to R(x)$.	$\therefore$ Any x that makes $P(x)$ true makes $R(x)$ true.

Remark on the Converse and Inverse Errors

A variation of the converse error is a useful reasoning tool, e.g. for medical diagnoses, provided that it is used with caution!

It goes like this: If a statement of the form

For all x, if $P(x)$ then $Q(x)$

is true, and if

$Q(a)$ is true, for a particular a,

then follow up on $P(a)$; it might be true and therefore be a cause of phenomenon $Q(a)$.

Remark on the Converse and Inverse Errors

For instance, suppose a doctor knows that

> For all x, if x has pneumonia, then x has a fever and chills, coughs deeply, and feels exceptionally tired and miserable.

And suppose the doctor also knows that

> John has a fever and chills, coughs deeply, and feels exceptionally tired and miserable.

On the basis of these data, the doctor concludes that a diagnosis of pneumonia is a strong possibility, though not a certainty.

Remark on the Converse and Inverse Errors

The doctor will probably attempt to gain further support for this diagnosis through laboratory testing that is specifically designed to detect pneumonia.

Note that the closer a set of symptoms comes to being a necessary and sufficient condition for an illness, the more nearly certain the doctor can be of his or her diagnosis.

This form of reasoning has been named **abduction** by researchers working in artificial intelligence. It is used in certain computer programs, called expert systems, that attempt to duplicate the functioning of an expert in some field of knowledge.

Recap

Predicate statements

- propositional functions
- universal and existential statements, universal conditional statements
- negation of quantified statements

Multiple quantifiers

- negations of multiply-quantified statements, order of multiple quantifiers

Universal argument forms

- universal modus ponens and tollens, universal transitivity
- converse and inverse errors