Universität Zürich
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Foundations of Computing II
Assignment 6

The Halting Problem, Complexity Theory

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Upload your solutions to the OLAT system.

6.1 The Halting Problem

In the lecture, we reduced the universal language L_{U} to the halting problem L_{H}. Using a similar approach, now reduce L_{H} to L_{U}.

6.2 Closure Properties of Languages in $\mathcal{P}$

Show that the languages in $\mathcal{P}$ are closed under the following operations. Argue on an intuitive, but exact level.

a) **Union**, that is, if $L_1, L_2 \in \mathcal{P}$, then $L_1 \cup L_2 \in \mathcal{P}$,

b) **Intersection**, that is, if $L_1, L_2 \in \mathcal{P}$, then $L_1 \cap L_2 \in \mathcal{P}$,

c) **Complement**, that is, if $L \in \mathcal{P}$, then $\overline{L} \in \mathcal{P}$.

Hint: In the lecture, we have seen that the regular languages are closed under quite a number of operations. This was done by starting with the DFAs for the given languages and then modifying them. Use an analogous approach to answer the above questions.

6.3 Polynomial-Time Reductions

In complexity theory, SAT plays the same role for us as L_{diag} in computability theory. To show that a problem is $\mathcal{NP}$-hard, we can reduce SAT to it.

In the lecture, we have introduced the satisfiability problem (SAT and 3SAT), the independent set problem IND-SET, the clique problem CLIQUE, and the vertex cover problem VC; then we proved SAT $\leq_{\mathsf{p}}$ 3SAT, 3SAT $\leq_{\mathsf{p}}$ IND-SET, IND-SET $\leq_{\mathsf{p}}$ CLIQUE, and IND-SET $\leq_{\mathsf{p}}$ VC, which implies that all of them are $\mathcal{NP}$-hard.

Here, we introduce three other problems which we prove to be $\mathcal{NP}$-hard by polynomial-time reductions.

a) The set cover problem SC is defined as follows. An input is triple $(X, \mathcal{S}, k)$ with

- $X = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}^+$,
- $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ with $S_i \subseteq X$ for some $m \in \mathbb{N}^+$ and $X = \bigcup_{j=1}^{m} S_j$, and
- $k \in \mathbb{N}^+$.

The question is whether there is a set cover of X of size (at most) k, that is, a selection of (at most) k sets from $\mathcal{S}$ such that every element from X is contained in at least one of the selected sets, that is, whether there exist $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$ with $i_j \in \{1, 2, \ldots, m\}$ and

$$X = \bigcup_{j=1}^{k} S_{i_j} \ .$$

Formally,

$$\mathrm{SC} = \{(X, \mathcal{S}, k) \mid X \text{ has a set cover from } \mathcal{S} \text{ of size } k\} \ .$$

As an example, the instance $(X_1, \mathcal{S}_1, 3)$ with $X_1 = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and

$$\mathcal{S}_1 = \{\{1, 3\}, \{1, 2, 5\}, \{1, 4\}, \{3, 4, 6\}, \{5, 6, 8\}, \{5, 7, 8\}\}$$

is a “yes” instance, because there is a set cover

$$\{1, 2, 5\} \cup \{3, 4, 6\} \cup \{5, 7, 8\} = X_1$$

of size 3. Conversely, the instance $(X_2, \mathcal{S}_2, 4)$ with $X_2 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and

$$\mathcal{S}_2 = \{\{1, 2\}, \{2, 3\}, \{2, 4, 6\}, \{4, 5, 6\}, \{6, 7\}, \{7, 9\}, \{8, 9\}\}.$$

is a “no” instance.

Reduce VC to SC.

b) A dominating set in a graph $G = (V, E)$ is a set D of vertices such that every vertex from V is either in D or has an edge to at least one vertex in D; we call such a vertex “dominated.”

An instance of the dominating set problem DOM-SET is a pair (G, k) where G is a graph and the question is whether G contains a dominating set of size $k \in \mathbb{N}^+$ (or smaller).

Formally,

$$\text{DOM-SET} = \{(G, k) \mid G \text{ contains a dominating set of size } k\} \ .$$

Reduce SC to DOM-SET.

c) A half-clique is a clique that contains exactly half of the vertices of a given graph. An instance of HALF-CLIQUE is a graph G and the question is whether G contains a half-clique.

Formally,

$$\text{HALF-CLIQUE} = \{G \mid G \text{ contains a half-clique}\} \ .$$

Note that, if G contains a clique with more than half of its vertices, it also contains a half-clique.

Reduce CLIQUE to HALF-CLIQUE.