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Foundations of Computing II

Assignment 4 – Solutions

Non-Context-Freeness, Pushdown Automata

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Upload your solutions to the OLAT system.

4.1 The Pumping Lemma for Context-Free Languages

We have proven the pumping lemma for context-free languages by assuming that the given CFG G is in Chomsky normal form.

Explain how an alternative proof can work without assuming that G is in Chomsky normal form. However, you are allowed to assume that G is normalized besides from that; in particular, its does not contain ε-productions, unit productions, or any useless symbols.

> In essence, a proof similar to the original one is possible, but we need to apply one additional argument. The set P of the rules of G is finite and hence there is a maximum length of the bodies of any of the rules. Let k be this length. Then it follows that every rule creates at most k "new" nonterminals in any derivation step; note that if G were in Chomsky normal form, $k = 2$. It follows that a given parse tree has a degree that is bounded from above by k. Let there be m nonterminals in G. Now let us pick $n_0 = k^{m+1}$ and consider any word z with $|z| \geq n_0$ derived in G. Any parse tree corresponding to z must have one path of length $m + 1$, which thus contains $m + 2$ vertices. On this path, the last vertex is a terminal and the first $m + 1$ vertices are nonterminals. It follows that one of the m nonterminals appears twice on this path. The remainder of the proof can then be done analogously to the original one.

4.2 Non-Context-Freeness

Use the pumping lemma for context-free languages to prove that the following languages are not context-free.

a) $L_1 = \{w \in \{0, 1\}^* \mid w = 1^k 0^{2k} 1^k \text{ for some } k \in \mathbb{N}\}$

Towards contradiction, assume that L_1 were context-free. Let n_0 be the constant from the pumping lemma and consider the word $w = 1^{n_0}0^{2n_0}1^{n_0} \in L_1$. Obviously, $|w| \geq n_0$. Then there is a decomposition $w = uvxyz$ such that

1. $|vxy| \leq n_0$,
2. $|vy| \geq 1$, and
3. $uv^\ell xy^\ell z \in L_1$ for every $\ell \in \mathbb{N}$.

Now consider any decomposition of w that satisfies $|vxy| \leq n_0$ and in which v and y are not both ε. Then it is clear that vxy cannot contain both ones from the beginning and from the end of w, because there are $2n_0$ zeros between them. We distinguish the following cases.

Case 1. Assume that vxy contains only ones from the beginning. Then uv^2xy^2z contains more ones at the beginning than at the end.

Case 2. Assume vxy contains only zeros. Then uv^2xy^2z contains n_0 ones both at the beginning and the end, but strictly more than $2n_0$ zeros in the middle.

Case 3. Assume that vxy contains only ones from the end. Then uv^2xy^2z contains more ones at the end than at the beginning.

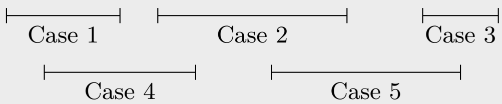
Case 4. Assume that vxy contains both ones from the beginning and zeros from the middle. Then uv^2xy^2z contains more ones at the beginning than at the end, or again more than $2n_0$ zeros in the middle, but n_0 ones at the end.

Case 5. Assume that vxy contains both zeros from the middle and ones from the end. Then uv^2xy^2z contains more ones at the end than at the beginning, or again more than $2n_0$ zeros in the middle, but n_0 ones at the beginning.

In any case, $uv^2xy^2z \notin L_1$, which is a contradiction. Therefore, the pumping lemma does not hold and L_1 cannot be context-free.

As an example, consider the following word and the five cases of where vwx may be located.

$$w = 1111111111000000000000000000001111111111.$$



We have, however, to be careful, because (this is important for cases 4 and 5) either v or y can be ε (but not both).

b) $L_2 = \{ww^{\mathsf{R}}w \mid w \in \{a,b\}^*\}$

Towards contradiction, assume that L_2 were context-free. Let n_0 be the constant from the pumping lemma and consider the word $w = a^{n_0}b^{n_0}b^{n_0}a^{n_0}a^{n_0}b^{n_0} \in L_2$. Obviously, $|w| \geq n_0$. Then there is a decomposition $w = uvxyz$ such that 1., 2., and 3. as above hold. Again consider any decomposition of w that satisfies $|vxy| \leq n_0$ and in which v and y are not both ε. We distinguish the following cases; note that vxy can only contain letters of two consecutive subwords a^{n_0} and b^{n_0}.

Case 1. Assume vxy contains at least one a. If vxy contains an a of the first third of w, it cannot contain any as of the last two thirds. Then uv^2xy^2z contains more as in the first third of the word than half as many as in the last two thirds. Likewise, if vxy contains an a of the second or third third, then it cannot contain any a of the first third. Then uv^0xy^0z contains more as in the first third of the word than half as many as in the last two thirds.

Case 2. Assume vxy contains at least one b. If vxy contains a b of the first or second third of w, then uv^0xy^0z contains more bs in the last third than half as many as in the first two thirds. If vxy contains a b of the last third, uv^2xy^2z contains again more bs in the last third than half as many as in the first two thirds.

The above case distinction is more dense than the one in part a), but also covers all possibilities. In any case, either $uv^0xy^0z \notin L_2$ or $uv^2xy^2z \notin L_2$, which is a contradiction. Therefore, the pumping lemma does not hold and L_2 cannot be context-free.

c) $L_3 = \{0^k 1^{k^2} \mid k \in \mathbb{N}\}$

Towards contradiction, assume that L_3 were context-free. Let n_0 be the constant from the pumping lemma and consider the word $w = 0^{n_0}1^{n_0^2} \in L_3$. Obviously, $|w| \geq n_0$. Then there is a decomposition $w = uvxyz$ such that 1., 2., and 3. as above hold. Again consider any decomposition of w that satisfies $|vxy| \leq n_0$ and in which v and y are not both ε. We distinguish the following cases.

Case 1. Assume vxy contains only zeros. Then uxz contains too few zeros.

Case 2. Assume vxy contains only ones. Then uxz contains too few ones.

Case 3. Assume v or y contains both zeros and ones. Then uv^2xy^2z has an incorrect form since it is not a sequence of zeros followed by a sequence of ones anymore.

Case 4. Assume that v consists of a sequence of zeros, say m zeros while y consists of a sequence of m' ones. Then $uv^\ell xu^\ell z$ is of the form $0^{n_0+(\ell-1)m}1^{n_0^2+(\ell-1)m'}$. According to the pumping lemma, any such word (that is, for any $\ell \in \mathbb{N}$) has to be contained in L_3. This means that

$$(n_0+(\ell-1)m)^2 = n_0^2+(\ell-1)m' \iff 2(\ell-1)mn_0+(\ell-1)^2m^2 = (\ell-1)m'$$

has to be true for every ℓ. Since m, m', and n_0 are constant, this equality cannot be true for every ℓ since the left side grows quadratically in ℓ while the right one grows linearly in ℓ.

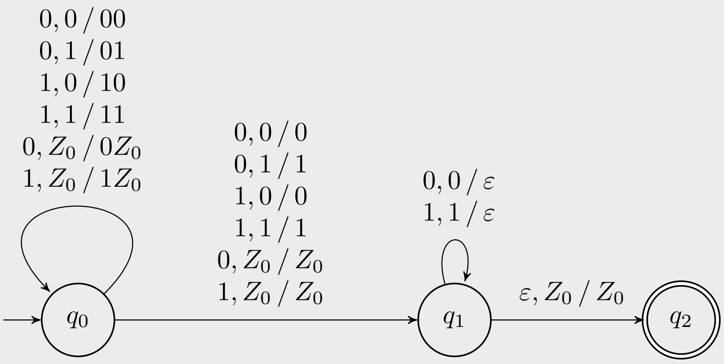
In any case, there is an ℓ for any of the above cases such that $uv^\ell xy^\ell z \notin L_3$, which is a contradiction. Therefore, the pumping lemma does not hold and L_3 cannot be context-free.

4.3 Pushdown Automata

Give pushdown automata that recognize the following languages.

a) $L_4 = \{w \in \{0,1\}^* \mid w = w^{\mathsf{R}} \text{ and the length of } w \text{ is odd}\}$

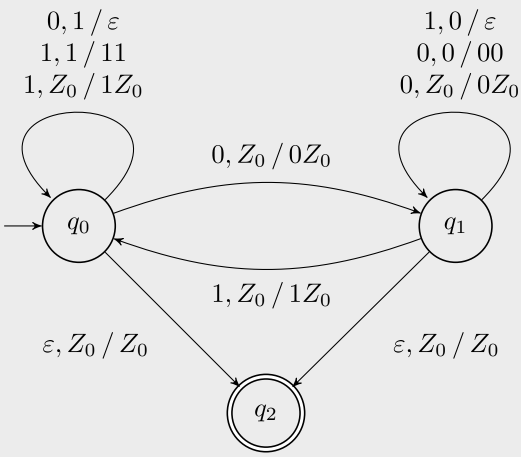
The language L_4 is accepted by the following pushdown automaton P_4.



The idea of P_4 is the following; recall that there is only Z_0 on the stack initially. Every letter that is read gets pushed onto the stack while P_4 is in q_0. If, for some $n \in \mathbb{N}$, the word has length $2n + 1$, P_4 can guess nondeterministically when it has read the first n letters. Reading the $(n+1)$-th letter, it can go to q_1 without changing its stack content. Then it can delete the n letters in reverse order from the stack while reading the last n letters of the word. If the letter currently read and the current top symbol of the stack do not match, P_4 gets stuck. Finally, only when the stack is empty, P_4 can go to the accepting state q_2.

b) $L_5 = \{w \in \{0,1\}^* \mid |w|_0 = |w|_1\}$

The language L_5 is accepted by the following pushdown automaton P_5.



The idea of P_5 is to push ones onto the stack for every one read while staying in q_0. If a zero is read, a one is popped from the stack if there is any. As long as this is done, P_5 has so far read a prefix of the word that contains at least as many ones as zeros. In case that a zero is read that cannot be matched to a previously read one, that is, the stack is empty, P_5 changes to q_2 where the roles of ones and zeros are switched. Here, the prefix read so far contains at least as many zeros as ones. A word can only be accepted if the stack is empty while the complete word is read. In this case, the same number of ones and zeros was read.