Linear Algebra Basics

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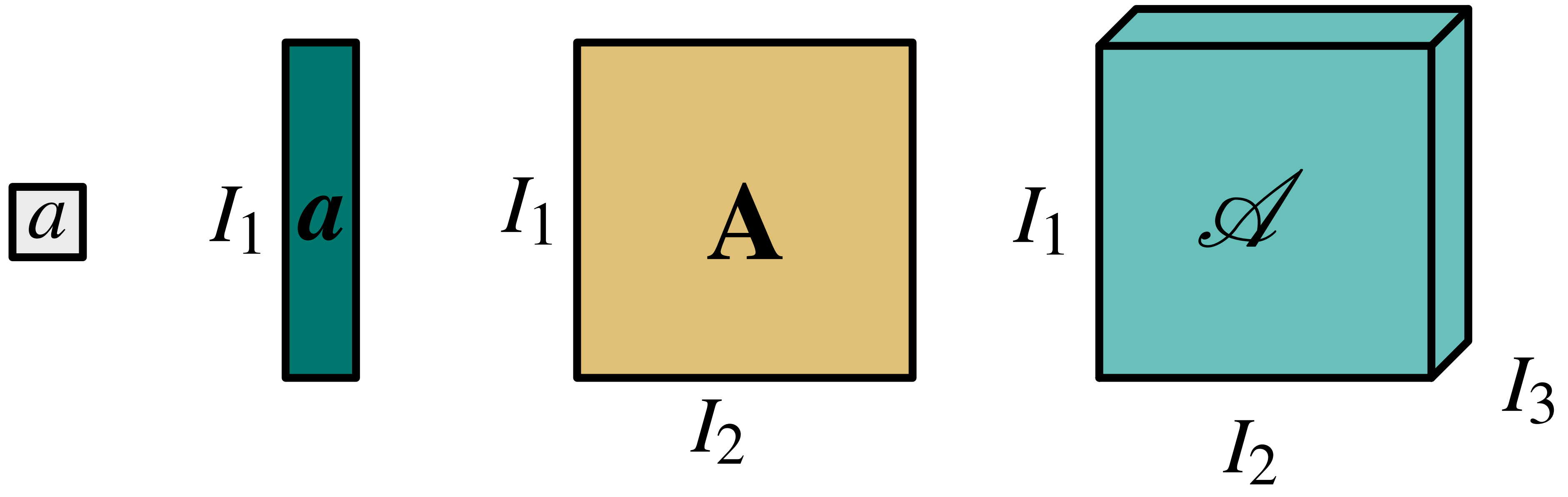
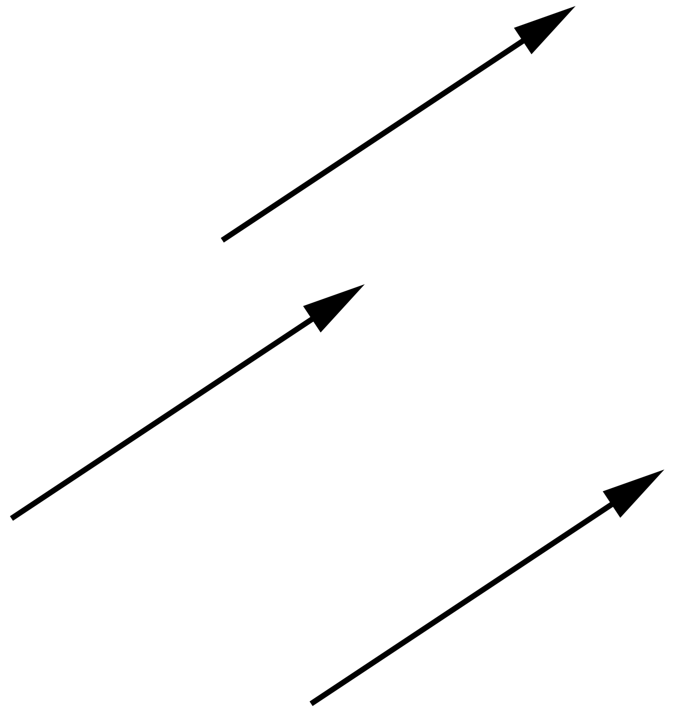
Overview

1. Scalar, points and vectors
2. Vector addition and subtraction
3. Vector space
4. Linear independence
5. Affine space
6. Dot and cross products
7. Matrices

Scalars, Points and Vectors

- Scalars, vectors, matrices and tensors
  - Individual numerical measures or numeric data elements can be combined into data records such as vectors and matrices
- Basic types denoting points and directions in space in N-dimensions
  - $\boldsymbol{N}$-tuple of values, common column vector math notation
    - vector $\boldsymbol{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}$ commonly stored as 1D array a = [a_1, a_2, …, a_N]
- A vector is a quantity with orientation and magnitude
  - High-dimensional data vector (e.g. velocity or force in physics)
- Multiway data arrays for 2D matrices or general d ND data tensors





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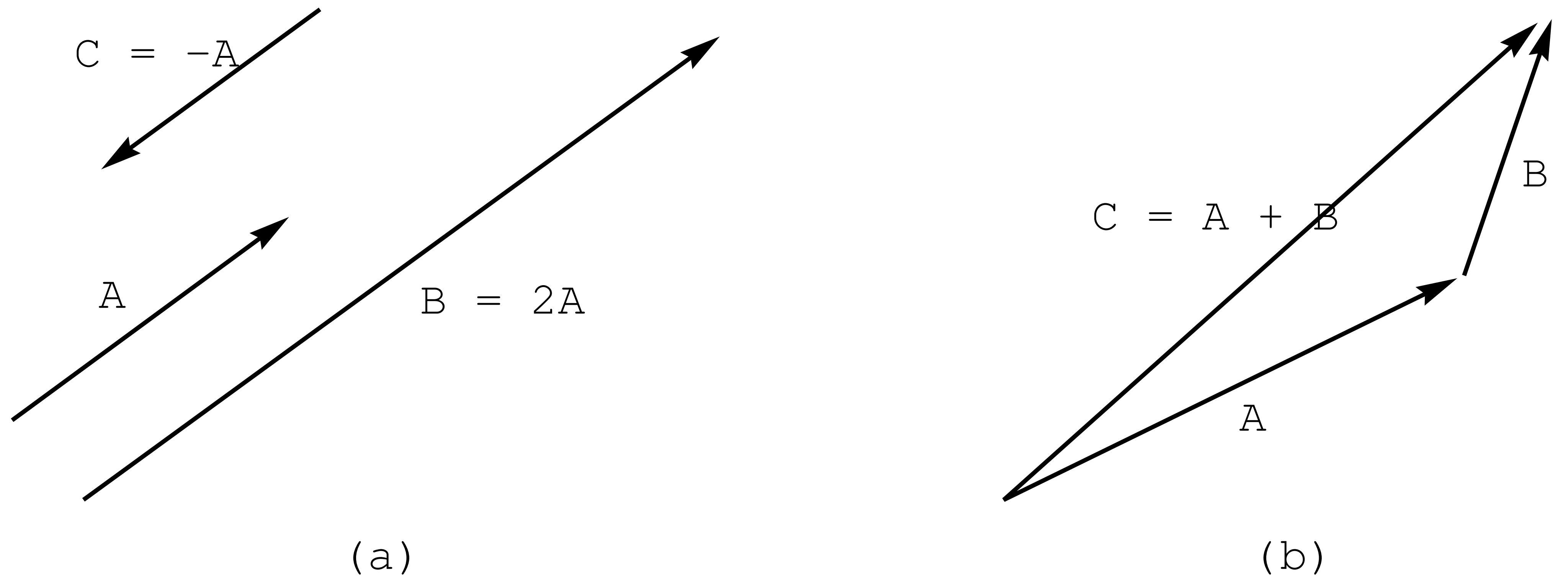
Vector Addition and Subtraction



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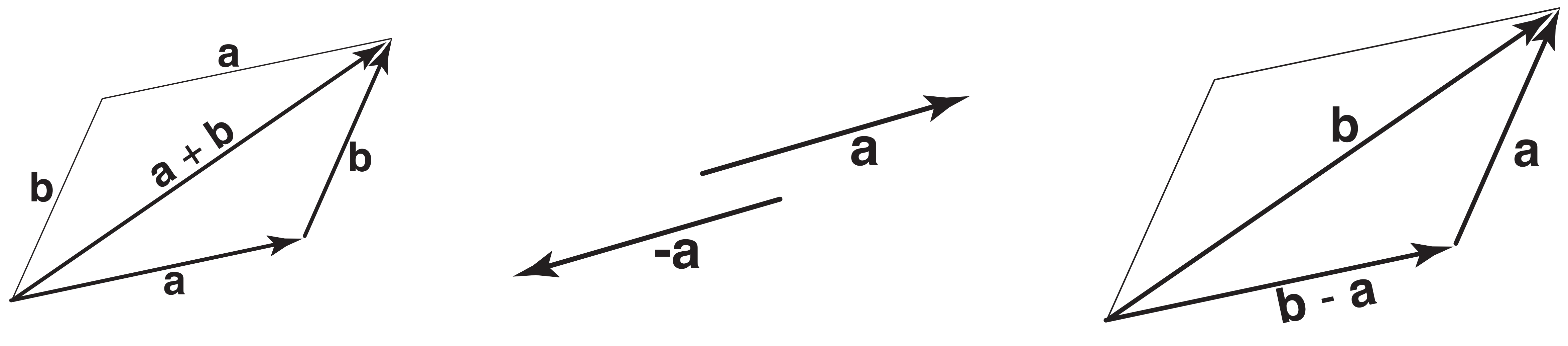

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- Vector addition is diagonal of parallelogram
  - *Head-to-tail* rule of placing vectors
- Scalar multiplication and component-wise vector addition
  - Scalar multiplication $\boldsymbol{b} = s \cdot \boldsymbol{a}$ is equivalent to adding $\boldsymbol{a}$ together s times

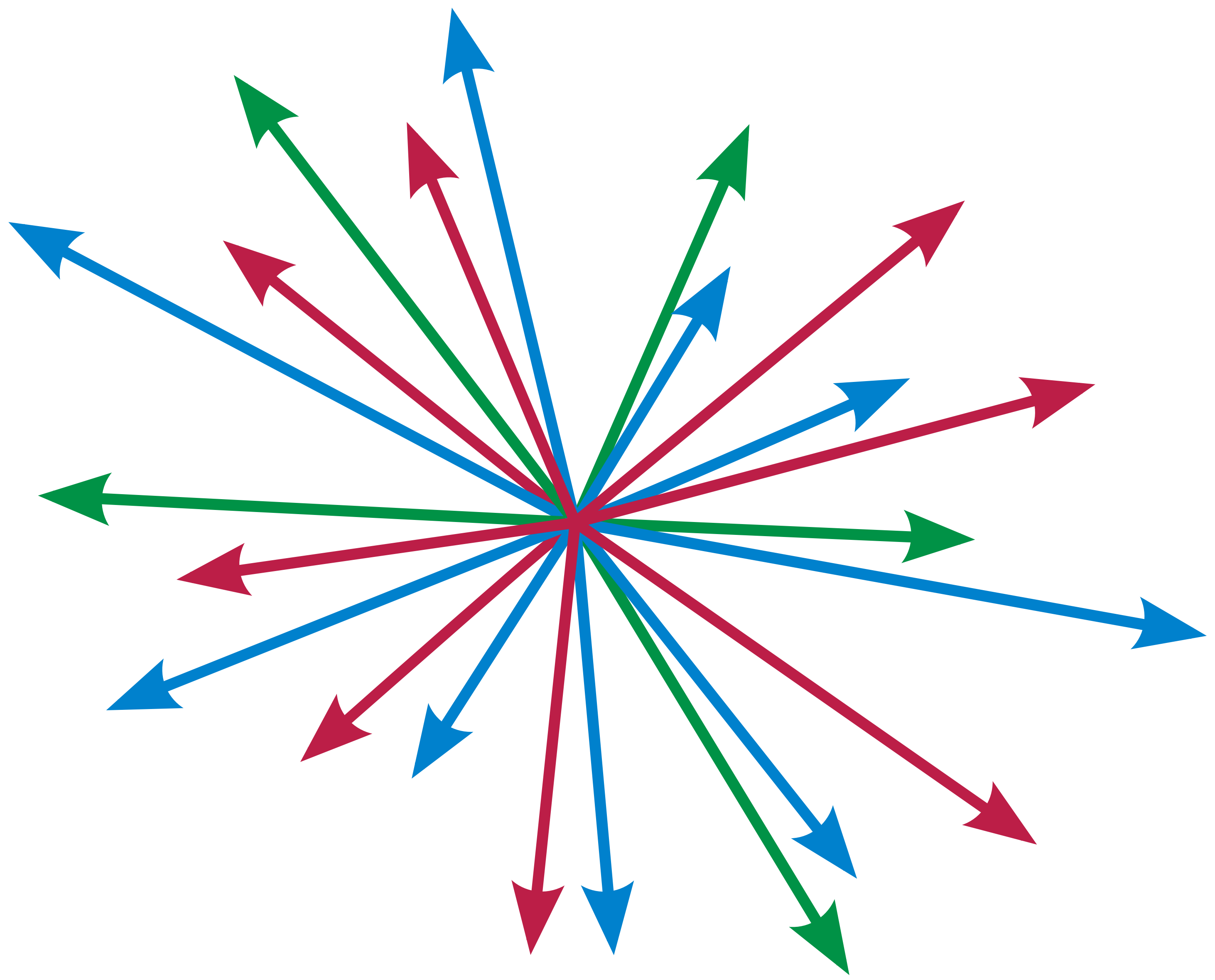
$$\boldsymbol{a} + \boldsymbol{b} = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_N + b_N \end{pmatrix} \qquad \boldsymbol{b} = s \cdot \boldsymbol{a} = \begin{pmatrix} s \cdot a_1 \\ \vdots \\ s \cdot a_N \end{pmatrix}$$

- Subtraction is equivalent to adding a negative vector
  - Vector negation is inversion of direction
- Vectors are equal if of same length and direction
  - Subtract to a zero-vector

$$\boldsymbol{a} - \boldsymbol{b} = \begin{pmatrix} a_1 - b_1 \\ \vdots \\ a_N - b_N \end{pmatrix} = \boldsymbol{a} + (-1 \cdot \boldsymbol{b})$$

Vector Space

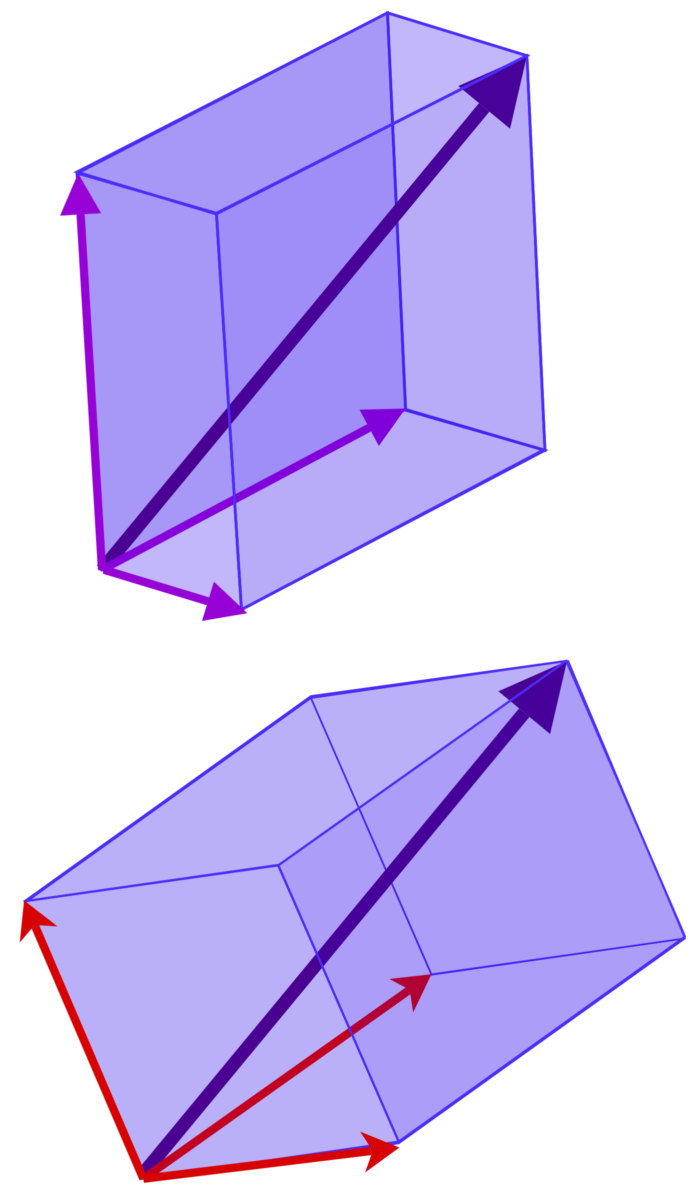
- In a vector space $\mathbf{V}$, addition and multiplication satisfy certain mathematical rules, such as being:
  - Closed: $\boldsymbol{u}+\boldsymbol{v} \in \mathbf{V}, \forall\ \boldsymbol{u},\boldsymbol{v} \in \mathbf{V}$
  - Commutative: $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
  - Associative: $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$
  - Distributive: $\alpha(\boldsymbol{u} + \boldsymbol{v}) = \alpha\boldsymbol{u} + \alpha\boldsymbol{v}$, $(\alpha + \beta)\boldsymbol{u} = \alpha\boldsymbol{u} + \beta\boldsymbol{u}$ for scalars α,β
  - Zero vector: $\boldsymbol{u} + 0 = \boldsymbol{u}$
  - Additive inverse: $\boldsymbol{u} + (-\boldsymbol{u}) = 0$

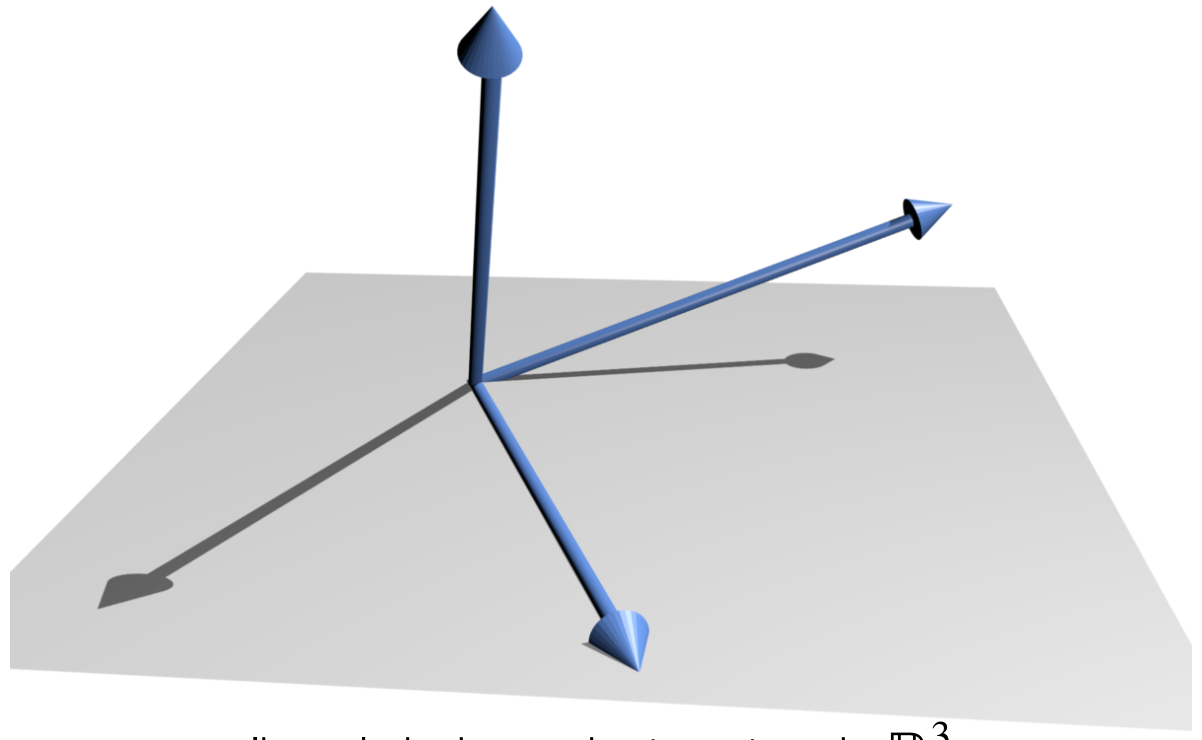


- A vector space only contains vectors as elements!

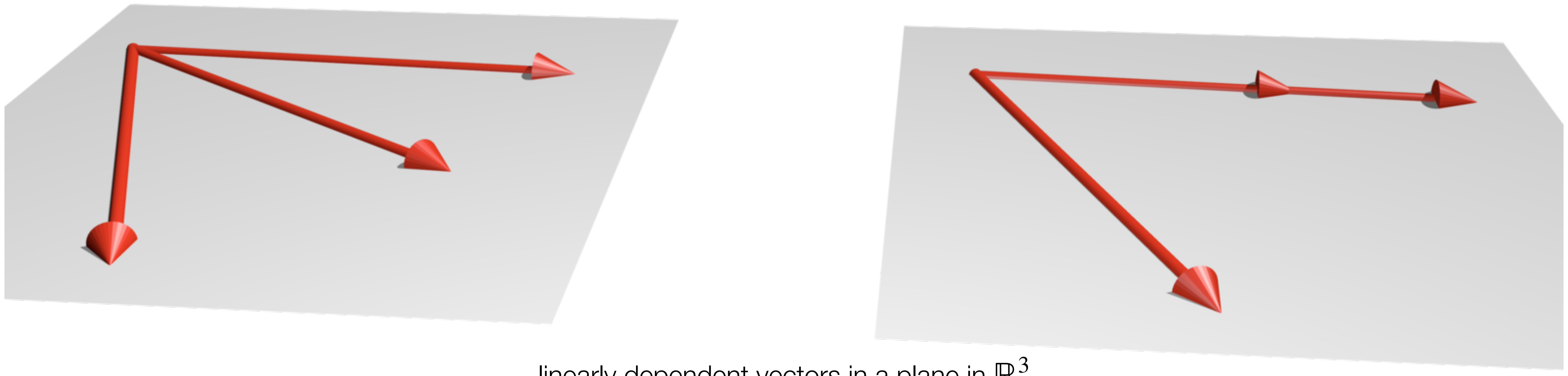
Linear Independence

- A **linear combination** of N vectors $\boldsymbol{v}_1 \ldots \boldsymbol{v}_N$ is defined as $\boldsymbol{u} = \alpha_1 \cdot \boldsymbol{v}_1 + \ldots + \alpha_N \cdot \boldsymbol{v}_N$
    - Linearly independent **if** $\alpha_1 \cdot \boldsymbol{v}_1 + \alpha_2 \cdot \boldsymbol{v}_2 \ldots + \alpha_N \cdot \boldsymbol{v}_N = 0$ **only if all** $\alpha_i = 0$
    - The dimension is defined by the largest number of linearly independent vectors
- N linearly independent vectors $\boldsymbol{v}_1 \ldots \boldsymbol{v}_N$ form a **basis** of an N-dimensional vector space $\mathbf{V}$
    - $\forall \boldsymbol{u} \in \mathbf{V}, \exists \beta_{1 \ldots N} \in \mathbb{R}$ such that $\boldsymbol{u} = \beta_1 \cdot \boldsymbol{v}_1 + \ldots + \beta_N \cdot \boldsymbol{v}_N$





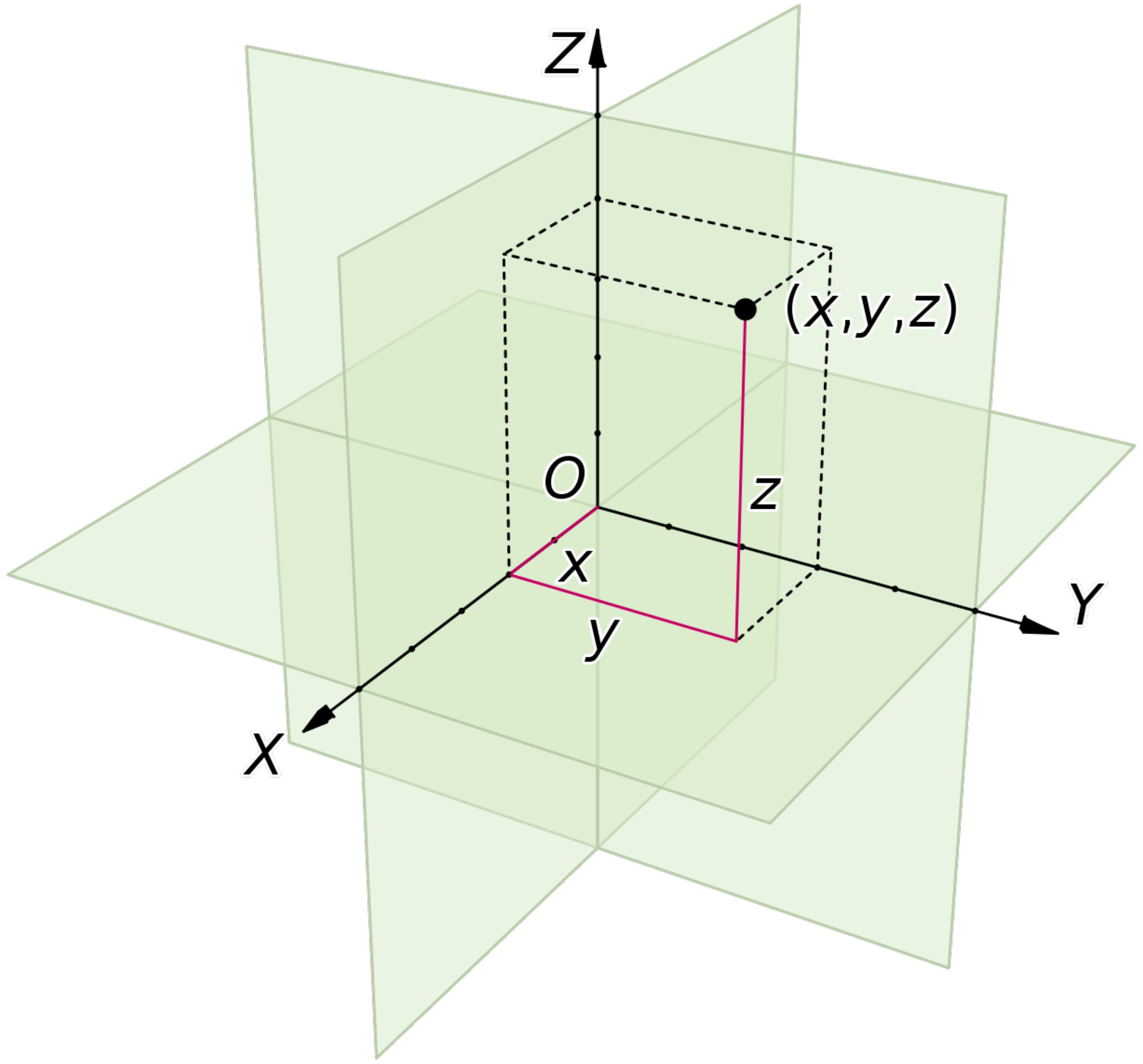
linearly independent vectors in $\mathbb{R}^3$



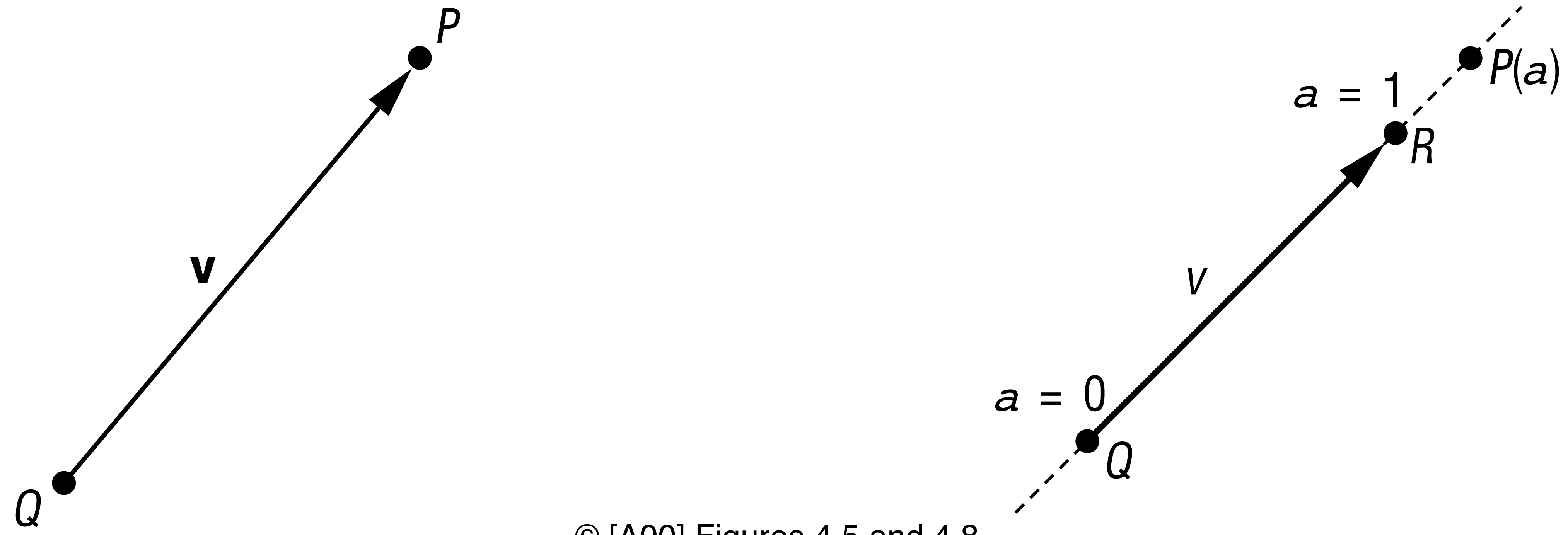
linearly dependent vectors in a plane in $\mathbb{R}^3$

Cartesian Coordinate System

- A Cartesian coordinate system is defined by a set of orthonormal basis vectors
  - Orthonormal basis iff each vector is of unit length and is orthogonal to all others
- Given by the standard Cartesian N-dimensional basis vectors $\boldsymbol{v}_{i=1,\ldots,N} = (\delta_{i1}, \ldots, \delta_{ij}, \ldots, \delta_{iN})^T$ with components δ_{ij}
  - Kronecker delta δ_{ij} is 0 for $i \neq j$ and 1 for $i = j$
  - For 3D we get the expected $\boldsymbol{v}_1 = (1,0,0)$, $\boldsymbol{v}_2 = (0,1,0)$ and $\boldsymbol{v}_3 = (0,0,1)$



Affine Space



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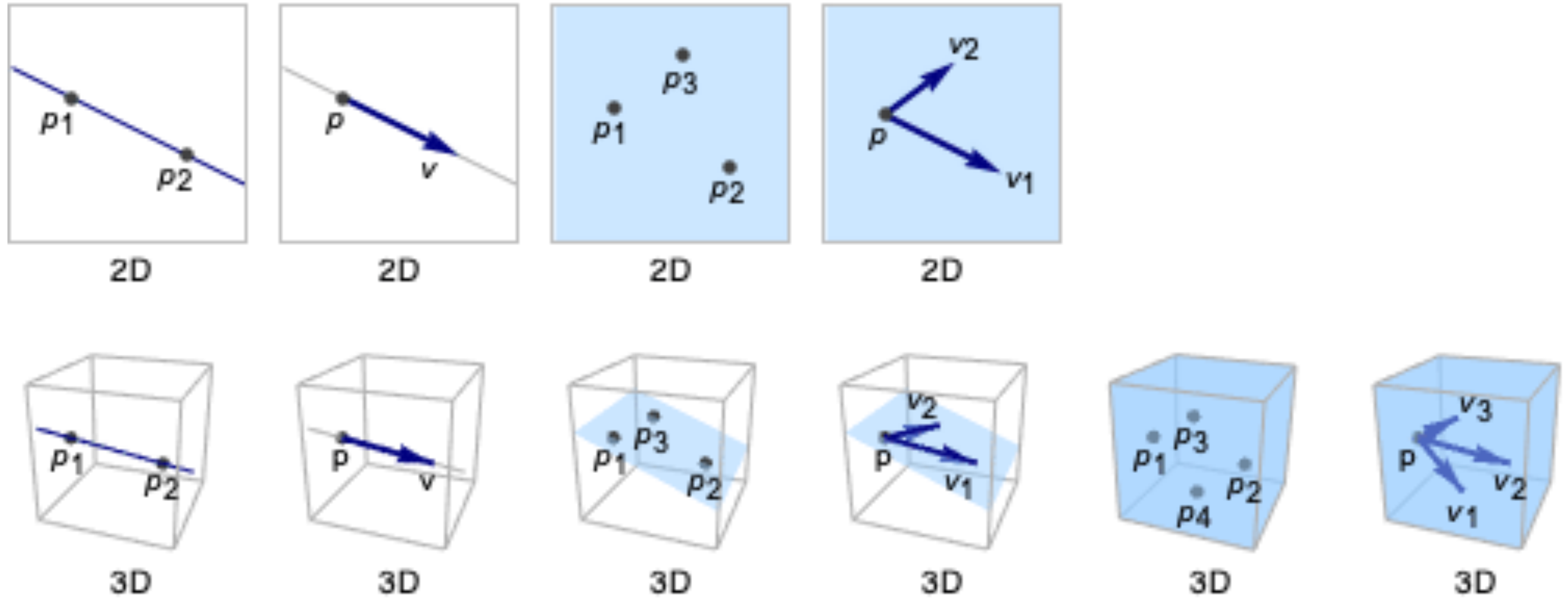


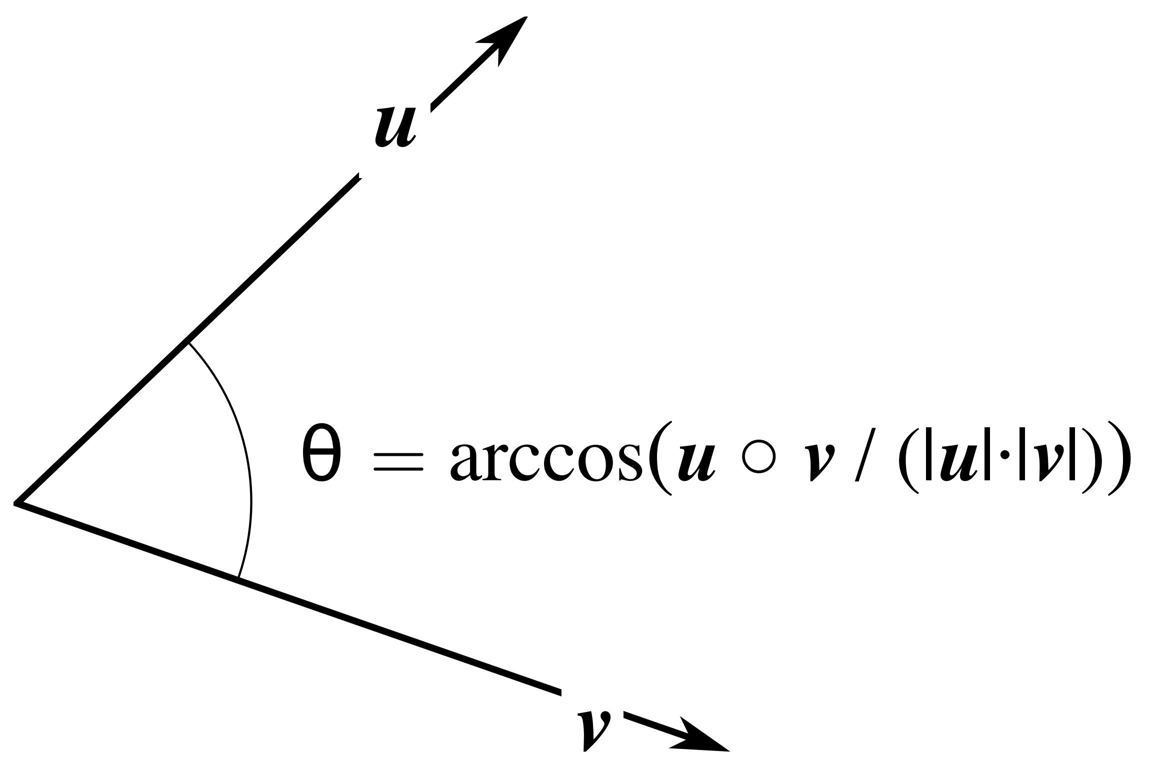
Source: https://reference.wolfram.com/language/ref/AffineSpace.html

- An affine space contains points in addition to scalars and vectors
  - Points are locations and different from vectors
  - Operations: point-vector addition and point-point subtraction
    - add a vector to a point to get a new point
    - subtract two points to get the vector in between them
  - No addition of points or multiplication of points
    - cannot scale a point by scalar multiplication

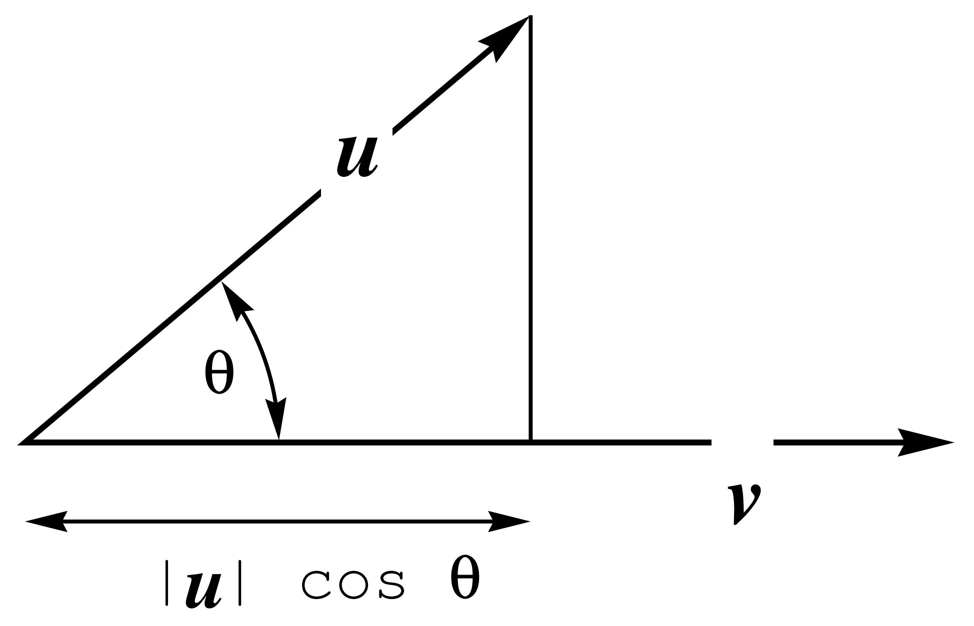
- The line $L(t)$: $P = Q + t \cdot (R - Q)$ is an affine space
  - Linear interpolation $(1-t) \cdot Q + t \cdot R$ (line segment $t \in [0,1]$)
- A line, a plane or a volume in 2D or 3D space represents an affine (sub-) space

The Dot-Product

- Given two N-D vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ the inner- or dot-product '○' is defined as $\boldsymbol{u} \circ \boldsymbol{v} = u_1 v_1 + \ldots + u_N v_N$
  - Operation on two vectors with scalar result
- The dot-product is
  - *Symmetric*: $\boldsymbol{u} \circ \boldsymbol{v} = \boldsymbol{v} \circ \boldsymbol{u}$
  - *Non-degenerate*: $\boldsymbol{v} \circ \boldsymbol{v} = 0 \Leftrightarrow \boldsymbol{v} = \boldsymbol{0}$
  - *Bilinear*: $\boldsymbol{v} \circ (\boldsymbol{u} + \alpha \boldsymbol{w}) = \boldsymbol{v} \circ \boldsymbol{u} + \alpha\, (\boldsymbol{v} \circ \boldsymbol{w})$

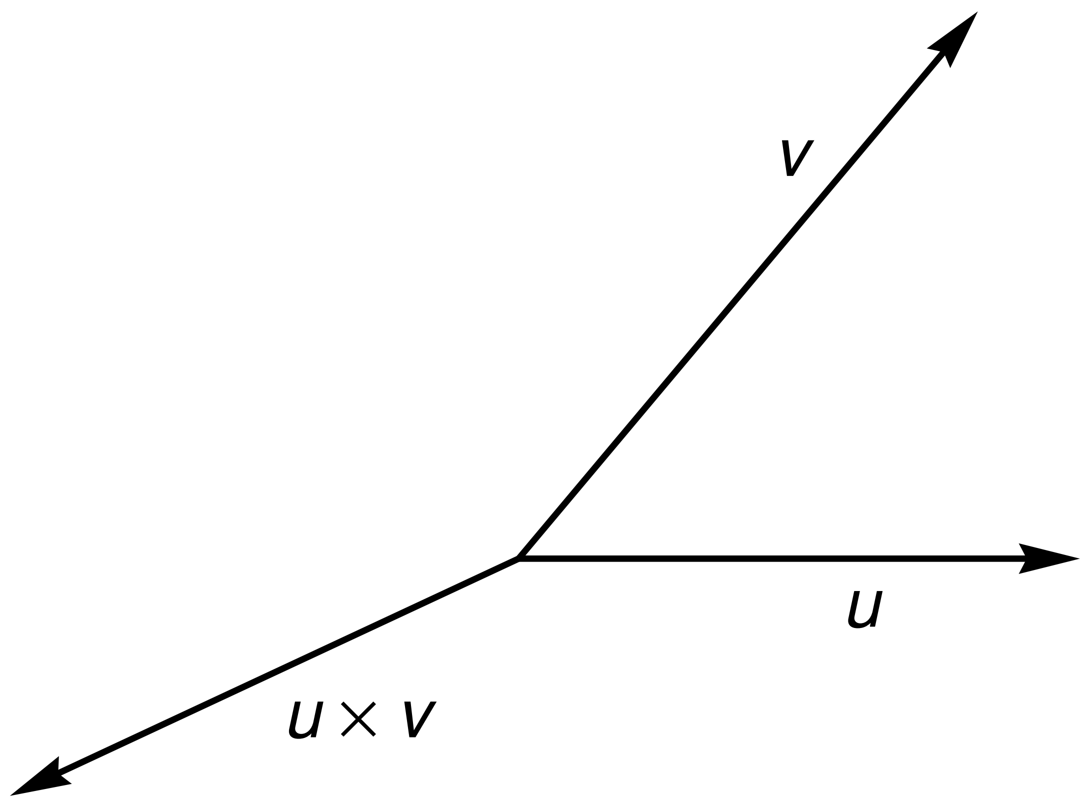


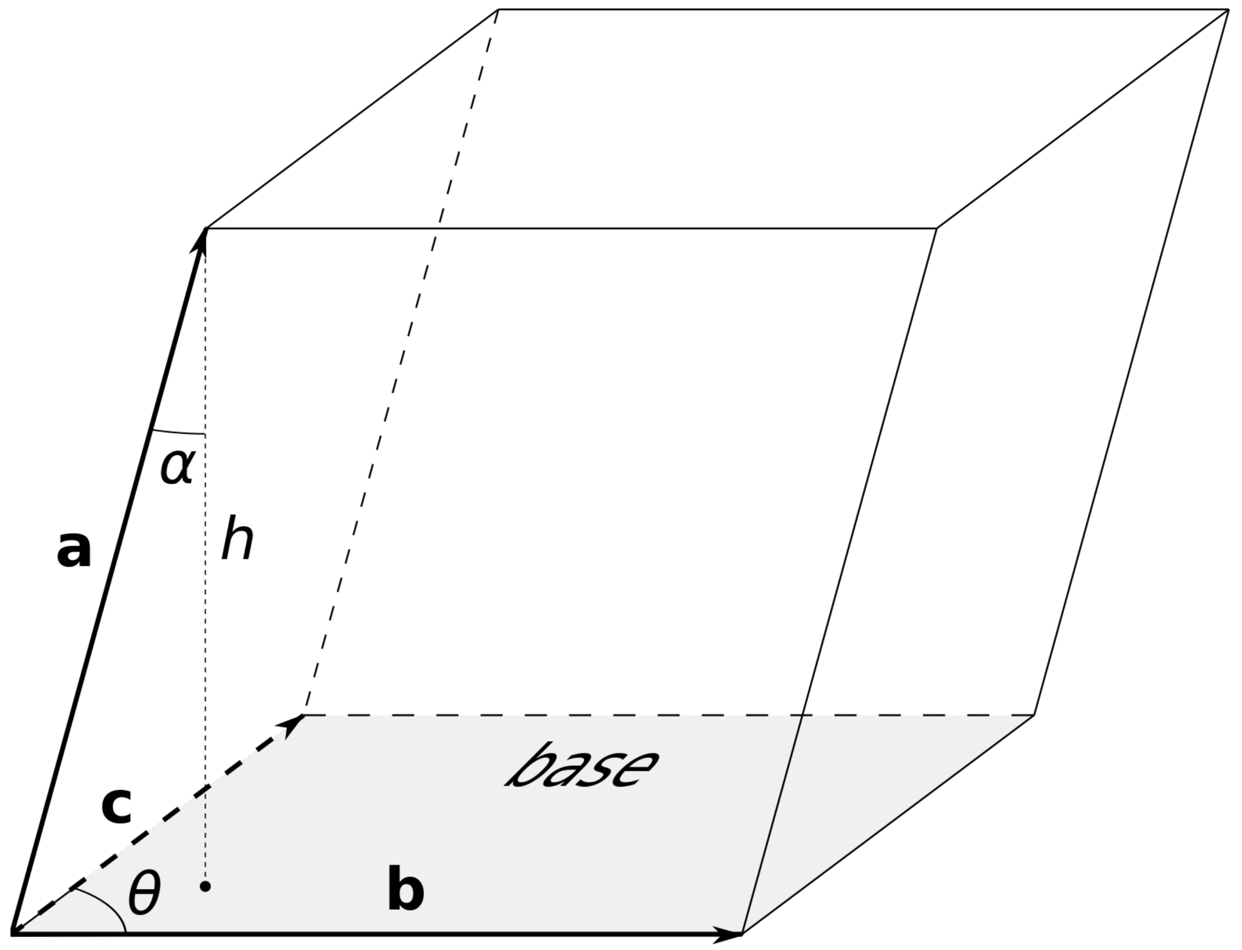
- Relation to angle between vectors: $\boldsymbol{u} \circ \boldsymbol{v} = \cos\theta \cdot |\boldsymbol{u}| \cdot |\boldsymbol{v}|$
  - $\boldsymbol{u}, \boldsymbol{v}$ point to opposite half-spaces if negative
  - Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if $\boldsymbol{u} \circ \boldsymbol{v} = 0$
- Orthogonal projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ given by $|\boldsymbol{u}| \cdot \cos\theta = \boldsymbol{u} \circ \boldsymbol{v} \,/\, |\boldsymbol{v}|$
- Squared length of vector: $|\boldsymbol{u}|^2 = \boldsymbol{u} \circ \boldsymbol{u}$



The Cross-Product

- Given two n-D vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ the cross-product $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$ is defined as
  - $$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$
- The cross-product $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$
  - Defines right-handed vector system
- Relation to angle between vectors: $|\boldsymbol{u} \times \boldsymbol{v}| = |\sin \theta| \cdot |\boldsymbol{u}| \cdot |\boldsymbol{v}|$
- Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are collinear if $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{0}$
- Length $|\boldsymbol{u} \times \boldsymbol{v}|$ is area of parallelogram spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$





Matrices

- A matrix is a two-dimensional array of numbers
  - MxN grid of values organized in M rows and N columns

    - matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & \dots & \dots & a_{2N} \\ & & \vdots & \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{pmatrix}$ commonly stored as 2D array A = [[a_{11}, a_{12}, …, a_{1N}], … , [a_{M1}, a_{M2}, …, a_{MN}]]

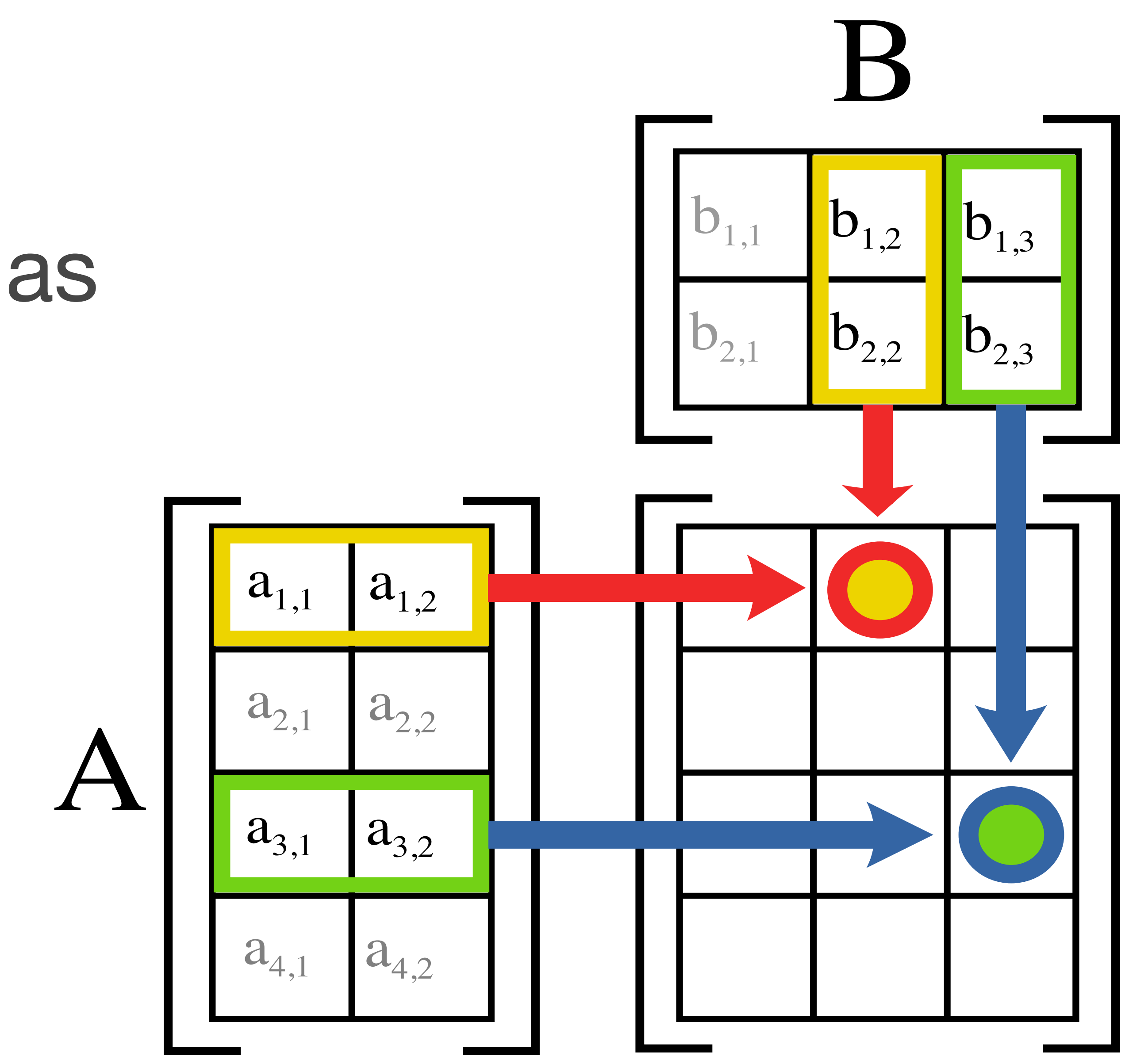
- Special *square*; *diagonal*; *identity*; and *zero* or *null* matrices
  - Having $M=N$; all $a_{ij} = 0$ for $i \neq j$; diagonal and all $a_{ii} = 1$; all entries 0
- Two matrices $\mathbf{A}$ and $\mathbf{B}$ are equal iff $a_{ij} = b_{ij}$ for all i,j
- The sum of two matrices $\mathbf{A}$ and $\mathbf{B}$ is $c_{ij} = a_{ij} + b_{ij}$
  - Analogous for the subtraction/difference
- The scalar multiple $\mathbf{C} = s \cdot \mathbf{A}$ is $c_{ij} = s \cdot a_{ij}$

Matrix Multiplication and Inverse

- Is defined component-wise on its elements
- For an MxN matrix $\mathbf{A}$ and NxP matrix $\mathbf{B}$ we get an MxP matrix $\mathbf{C} = \mathbf{AB}$ as

$$c_{ij} = \sum_{s1=}^{M} a_{is} \cdot b_{sj}$$

  - Element ij of $\mathbf{C}$ is dot-product of j-th column vector from $\mathbf{B}$ with the i-th row vector from $\mathbf{A}$
  - Matrix multiplication is not commutative $\mathbf{AB} \neq \mathbf{BA}$
  - Matrix multiplication is distributive $\mathbf{A(B+C)} = \mathbf{AB} + \mathbf{AC}$
- The inverse $\mathbf{A}^{-1}$ has the property that $\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{A}\,\mathbf{A}^{-1} = \mathrm{I}$
  - I being the identity matrix
  - Can be found by Gaussian elimination in general
  - Inverse only exists for square matrices with non-zero determinant



Transpose

- Transposition $\mathbf{B} = \mathbf{A}^{\mathrm{T}}$ is achieved by mirroring entries about the upper-left to lower-right diagonal
  - Row/column indices and dimensions are swapped $b_{ij} = a_{ji}$
- Matrix $\mathbf{A}$ is said to be symmetric if $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$ is
- An N-dimensional vector $\boldsymbol{a}$ can be considered an $N\times 1$ matrix
  - Called a column vector $\boldsymbol{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}$
  - Its transpose is then a row vector $\boldsymbol{a}^{\mathrm{T}} = (a_1 \ a_2 \ \ldots \ a_N)$
- Matrix notation of dot-product between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\boldsymbol{a} \circ \boldsymbol{b} = \boldsymbol{a}^T \cdot \boldsymbol{b} = (a_1 \ \ldots \ a_N) \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}$$

A

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Matrix-Vector Multiplication and Outer Vector-Product

- Matrix-vector multiplication $\mathbf{A}\boldsymbol{v}$ corresponds to matrix product rule

$$\boldsymbol{u} = \mathbf{A} \cdot \boldsymbol{v} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ & & \vdots & \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$$

  - Column vector $\boldsymbol{v}$ is multiplied (as dot-product) with each row of $\mathbf{A}$
    - number of columns of $\mathbf{A}$ must match rows of $\boldsymbol{v}$
  - Result vector $\boldsymbol{u}$ has M entries (not N !)
    - same as number of rows of $\mathbf{A}$
- Recall dot-product being: $\boldsymbol{u}^{\mathrm{T}} \cdot \boldsymbol{v}$
- Outer product $\boldsymbol{u} \cdot \boldsymbol{v}^{\mathrm{T}}$ of two vectors forms a new matrix

$$\boldsymbol{u} \cdot \boldsymbol{v}^T = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \cdot (v_1 \;\; \dots \;\; v_N) = \begin{pmatrix} u_1 v_1 & u_1 v_2 & \dots & u_1 v_N \\ & & \vdots & \\ u_M v_1 & u_M v_2 & \dots & u_M v_N \end{pmatrix}$$

Recap

- **Scalar points and vectors:** operations on vectors, vector and affine spaces
- **Linear combination:** linear independence, coordinate systems
- **Dot/Cross products:** relation to angles between vectors and areas spanned by vectors
- **Matrices:** properties and operations, inverse and transpose, matrix-vector products, dot-product as matrix product, outer vector product

- Required textbook Chapter(s): 3 & 4