Interpolation and Data Fitting

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Overview

1. Piecewise constant and linear interpolation
2. Interpolation in 2D
3. Smooth polynomial interpolation
4. Polynomial least squares interpolation

Piecewise Constant and Linear Interpolation

Interpolation

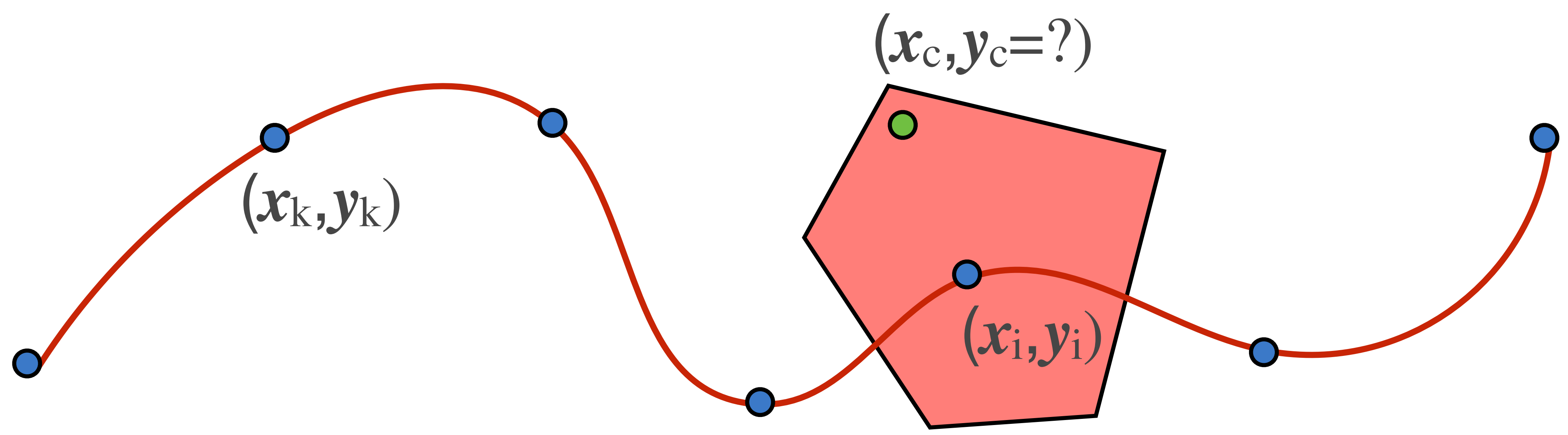
- Interpolation is a method to estimate, construct new data points from a given set of discrete known data points
  - Over a bounded range of the independent variable attribute(s) of the input data points
    - extrapolation used to estimate data points outside the range of the known data points
  - An interpolation function exactly passes through all known data points
  - Different from an approximation which only approximates them, thus passes nearby instead
- Transformation of data from a given (source) distribution to another (target) resolution is a frequent task of data preprocessing
  - Filling-in of continuous data values in between scattered sample data points
    - e.g. temporal domain, filling in intermediate values of temperature between observed values
    - e.g. spatial domain, image resizing or resampling (up-/down-)

Piecewise Constant Interpolation

- Available data points often represent a discrete sampling of a continuous phenomenon
  - Observed **output values** y_i are given for a few known **input data points** $x_i \in X$
- Given a distance metric $|x_b - x_a|$ between any two data points a and b in the input domain allows a generic *nearest neighbor interpolation* for any **new** data point c with input parameters x_c as:

$$y_c = y_i \text{ where } \forall x_{k \neq i} \in X: |x_c - x_i| < |x_c - x_k|$$

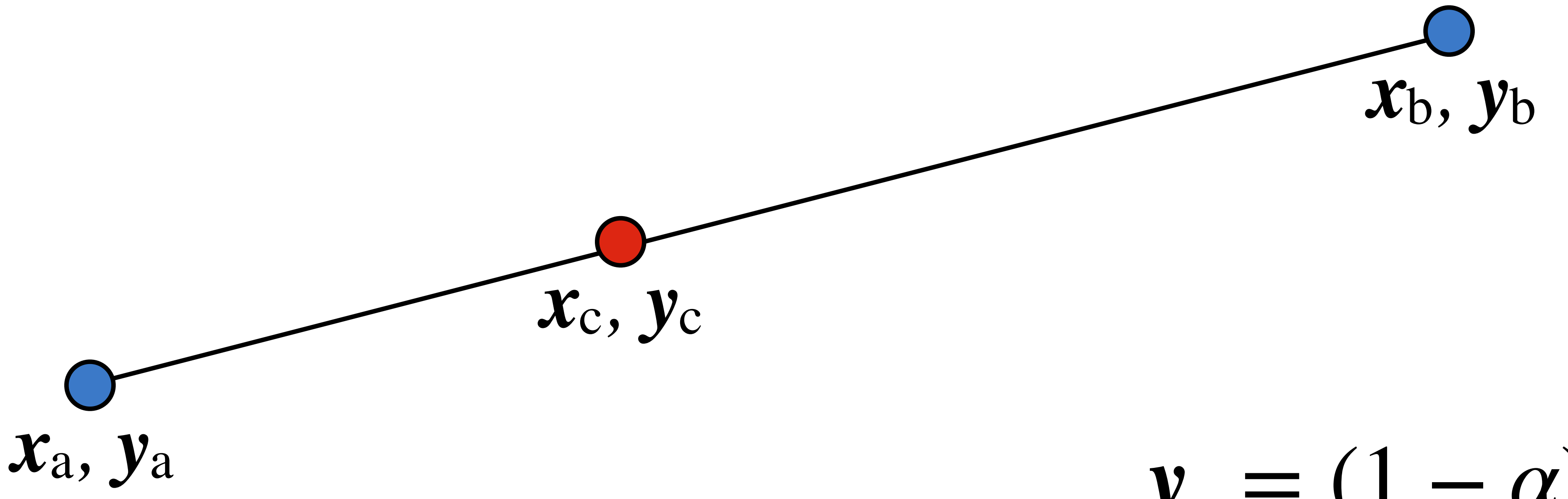
  - New unknown value is inherited from nearest available data point in any dimensions (e.g. $x_i \in \mathbb{R}^d$)
- Continuous interpolation requires (piecewise) linear or higher-order interpolation functions
  - Linear interpolations exhibit only C_0 continuity



Piecewise Linear Interpolation

- Prediction of unknown values from nearby sample locations by interpolation
  - Linear interpolation of new **data value** y_c at a **position** x_c from given data values y_b and y_b on a straight line between two known sample input data points x_a and x_b
    - x_a, x_b and x_c are all on a line $\in \mathbb{R}^D$

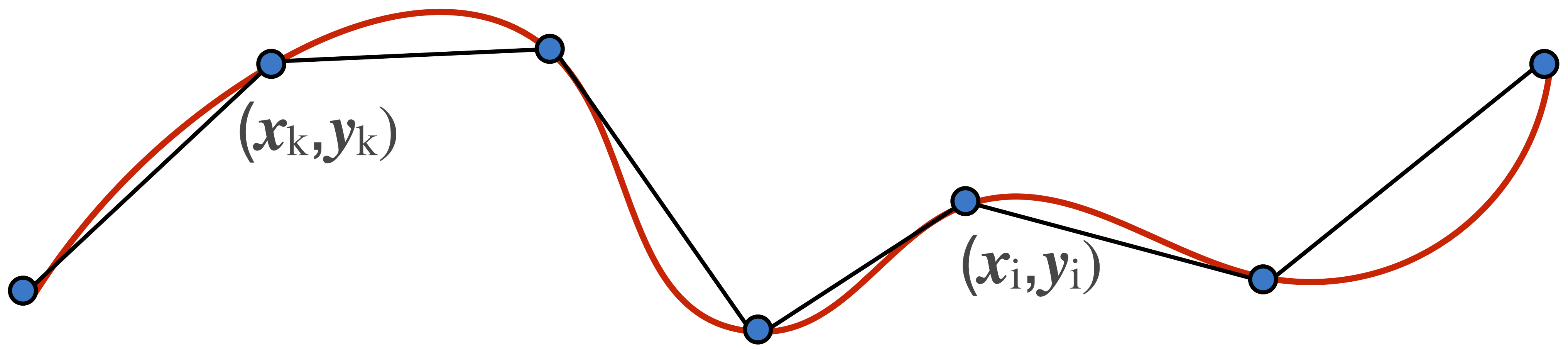
$$y_c = y_a + (y_b - y_a) \cdot \frac{|x_c - x_a|}{|x_b - x_a|}$$



$$\alpha = \frac{|x_c - x_a|}{|x_b - x_a|}$$

$$y_c = (1 - \alpha) \cdot y_a + \alpha \cdot y_b$$

- Linear interpolations exhibit only C_0 continuity
  - Discontinuity in the slope, first and higher derivatives

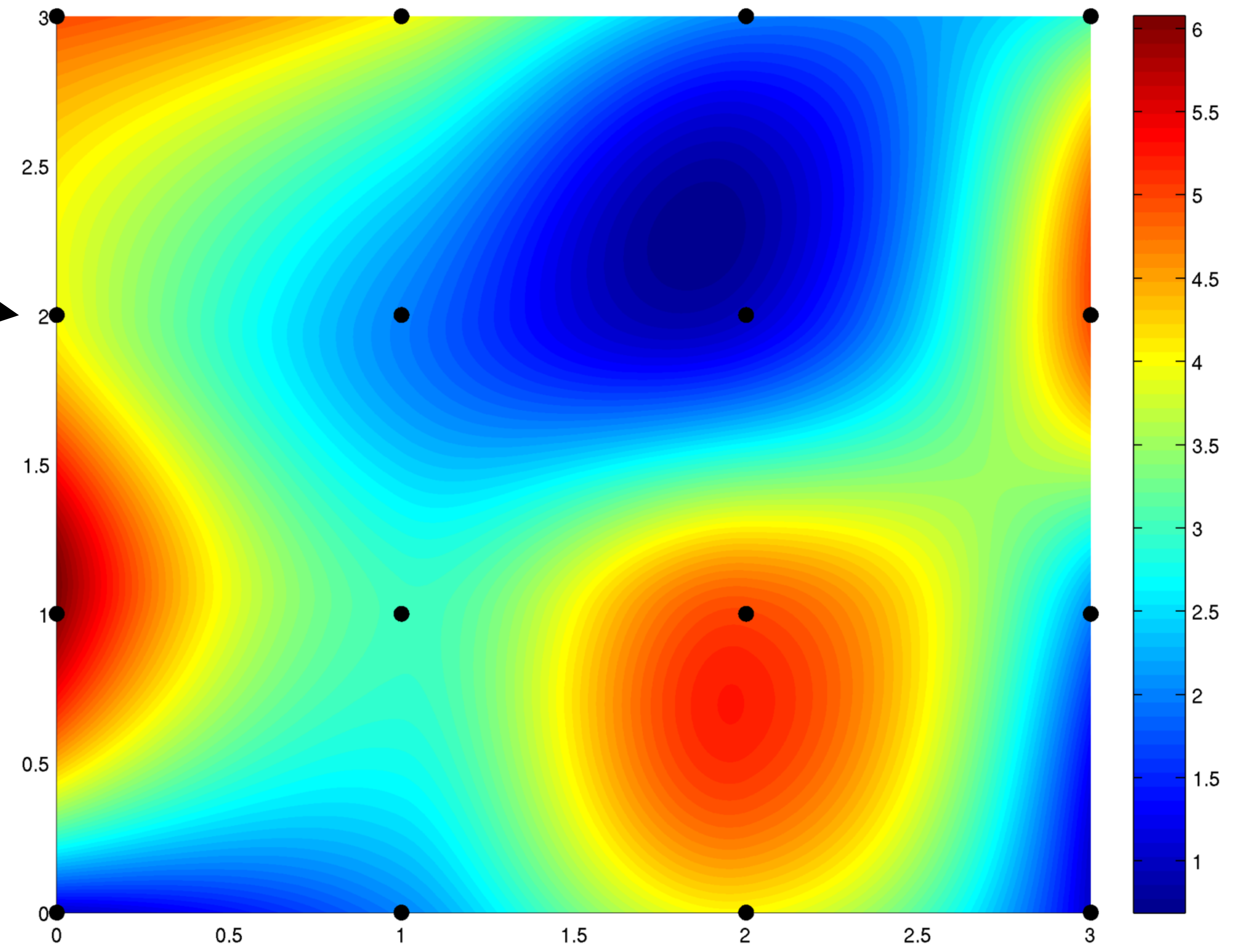


Interpolation in 2D

What Does “2D” Mean?

We always refer to the input’s dimensionality

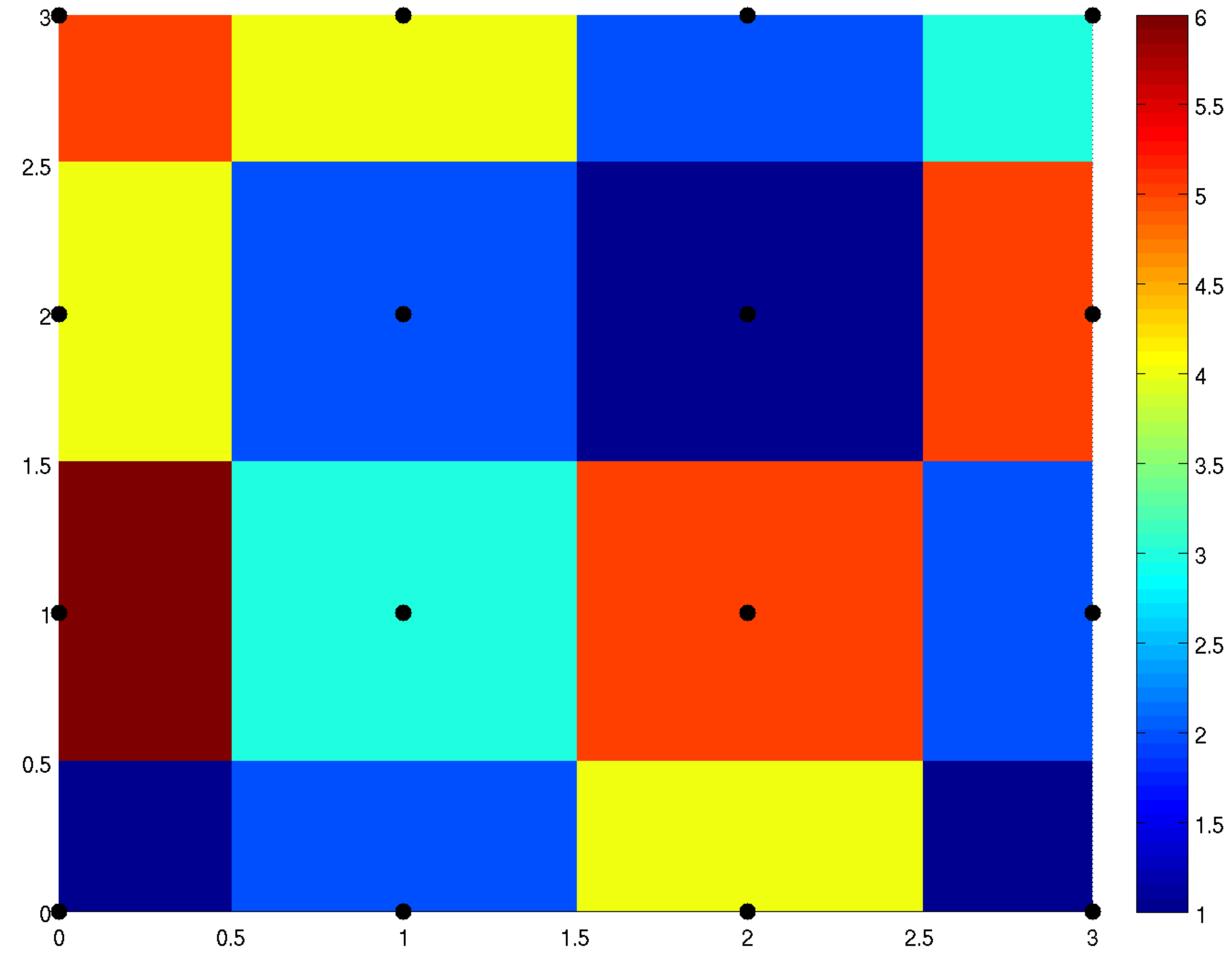
- 1D: one input x → one output y
- 2D: two inputs x, y → one output z
- *N*-dimensions: *N* inputs → still one output

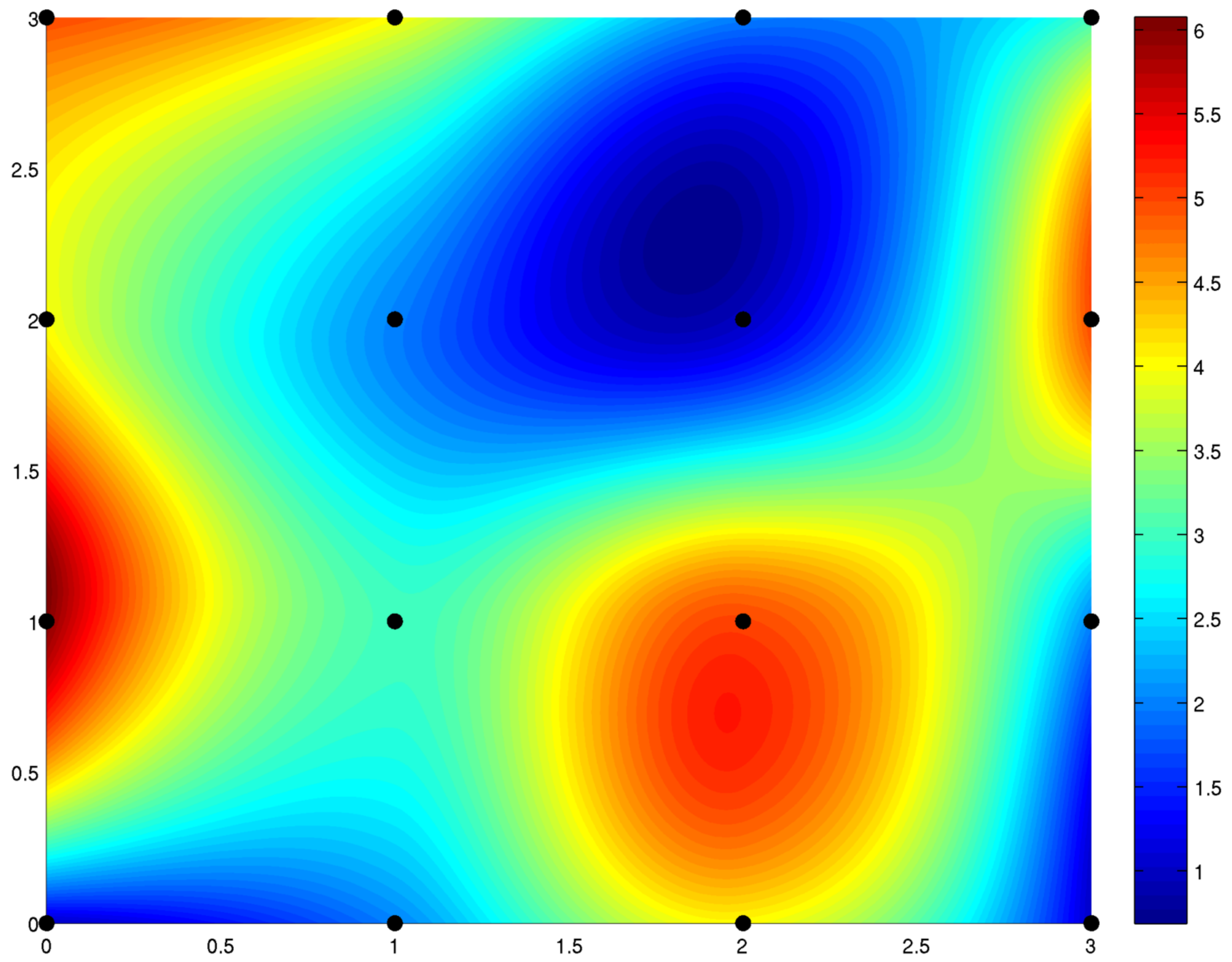


For visualization, we can map the z as e.g. color

Piecewise Constant Interpolation in 2D

- Simple piecewise constant nearest neighbor interpolation clearly exhibits sparse sample distribution
    - Patches of constant values in contrast to smooth distribution of values





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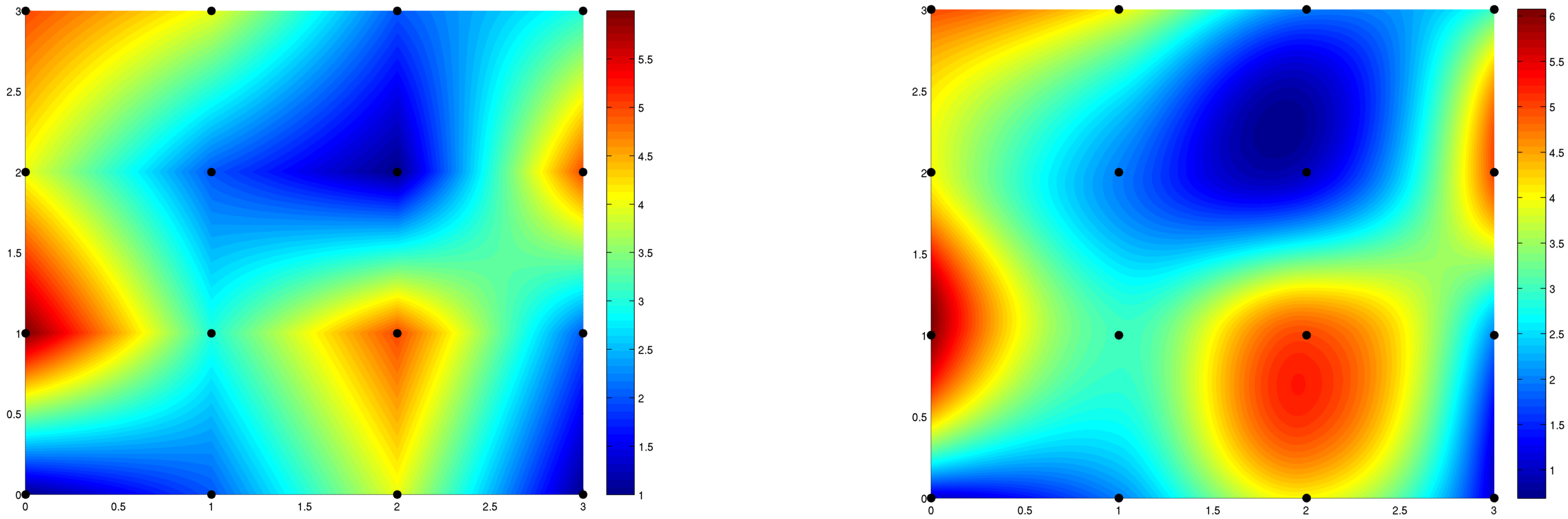
Bilinear Interpolation in 2D

- A grid of data values can easily be interpolated using *bilinear* interpolation
- Only C_0 continuous
  - Discontinuous change in slopes



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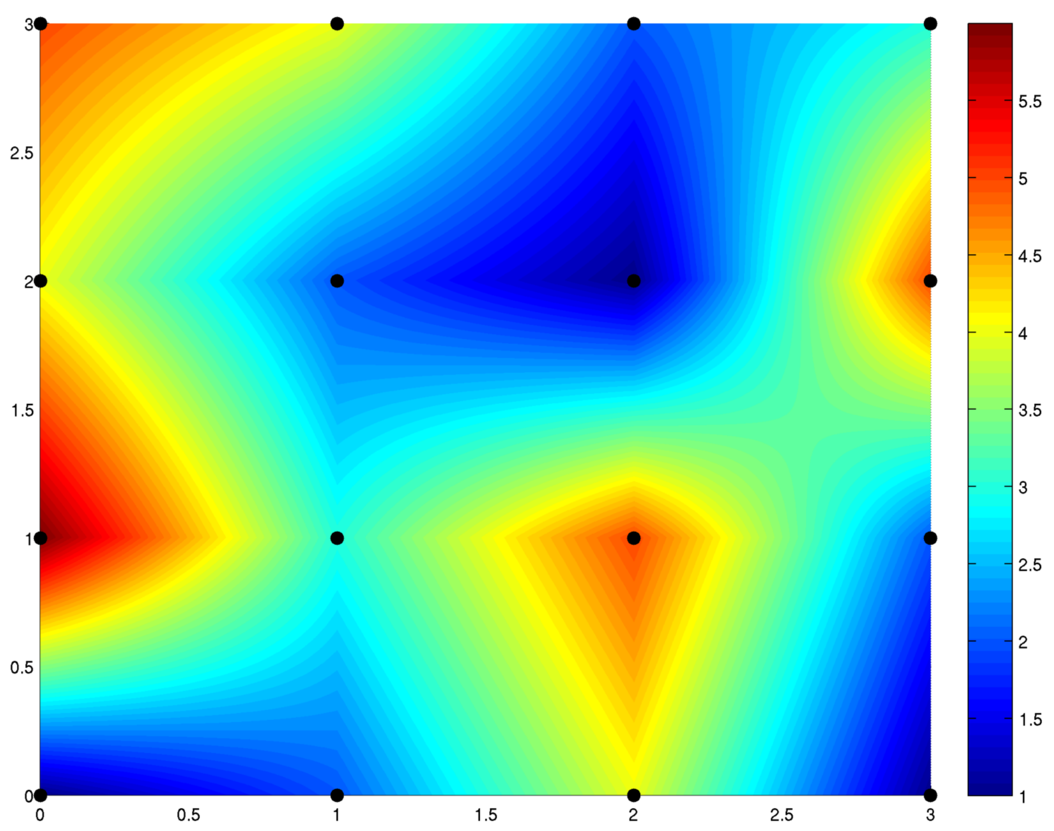
Bilinear Interpolation in 2D

- **Bi**linear because of the linear interpolations along each dimension separately
  - Two linear interpolations are performed along two opposite boundaries of a grid cell in one dimension, e.g. horizontally
  - One final interpolation is performed across the grid cell in the other dimension, e.g. vertically
- Non-smooth interpolation, only $\mathbf{C_0}$ but not $\mathbf{C_1}$ continuity



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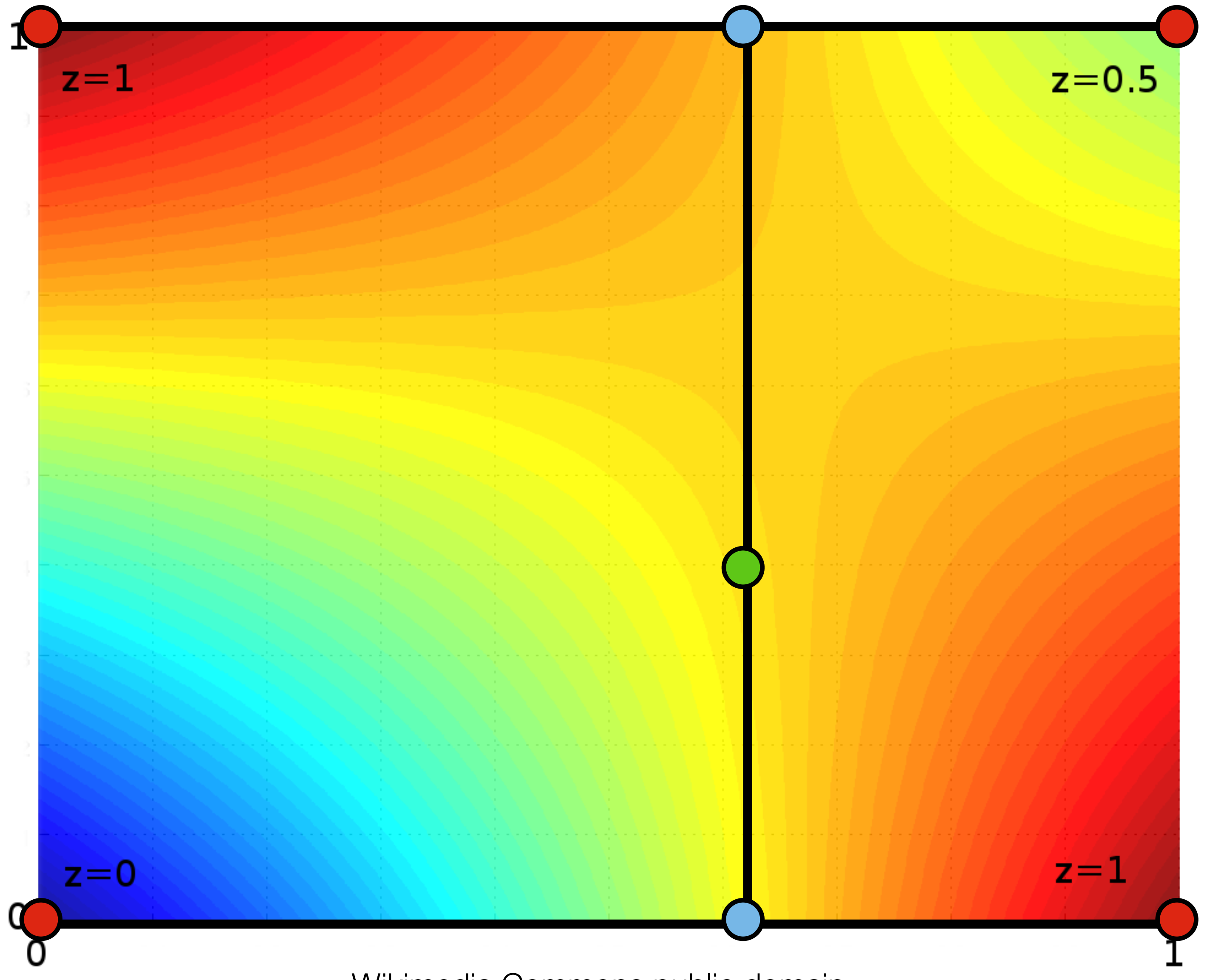

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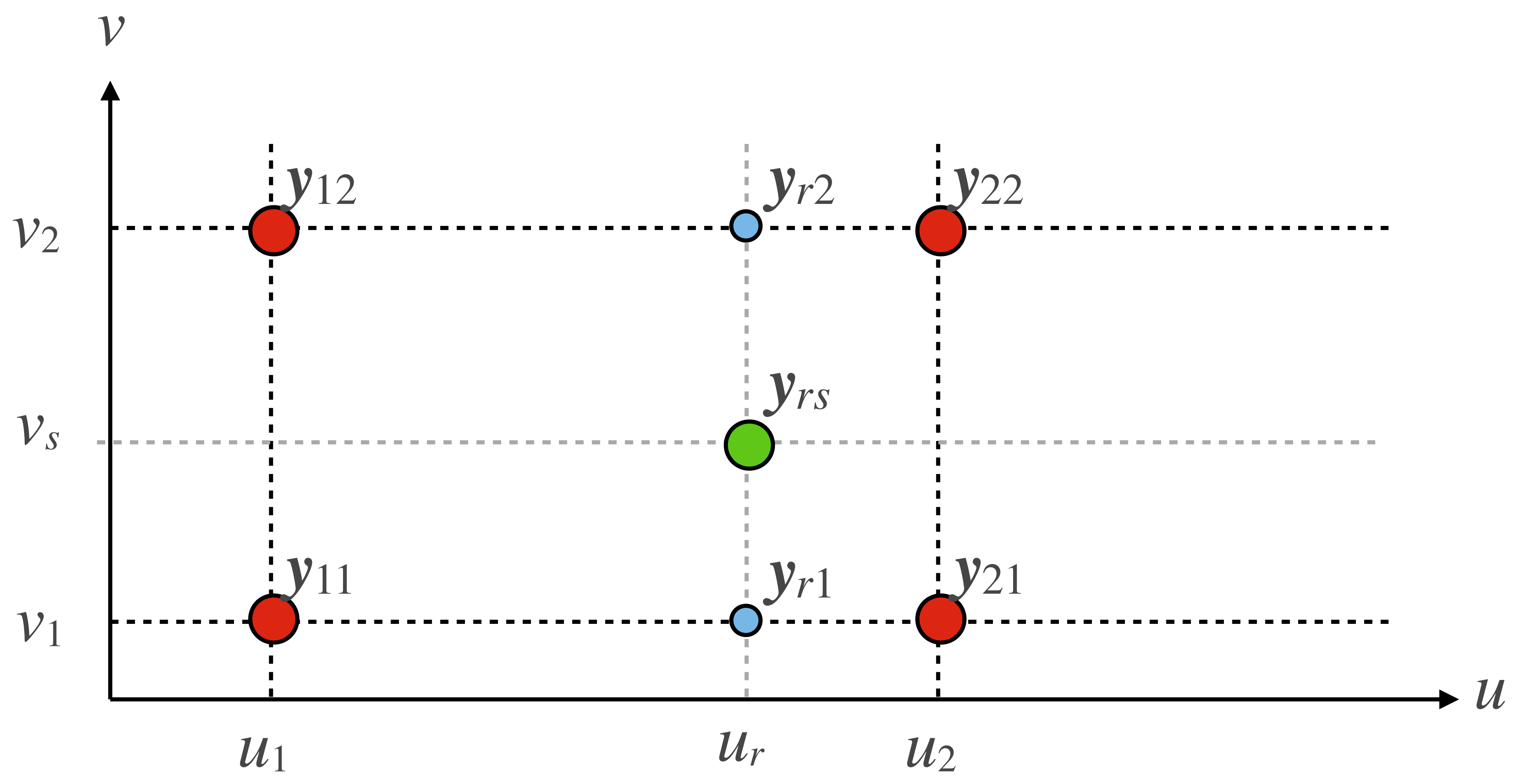
Bilinear Interpolation in 2D

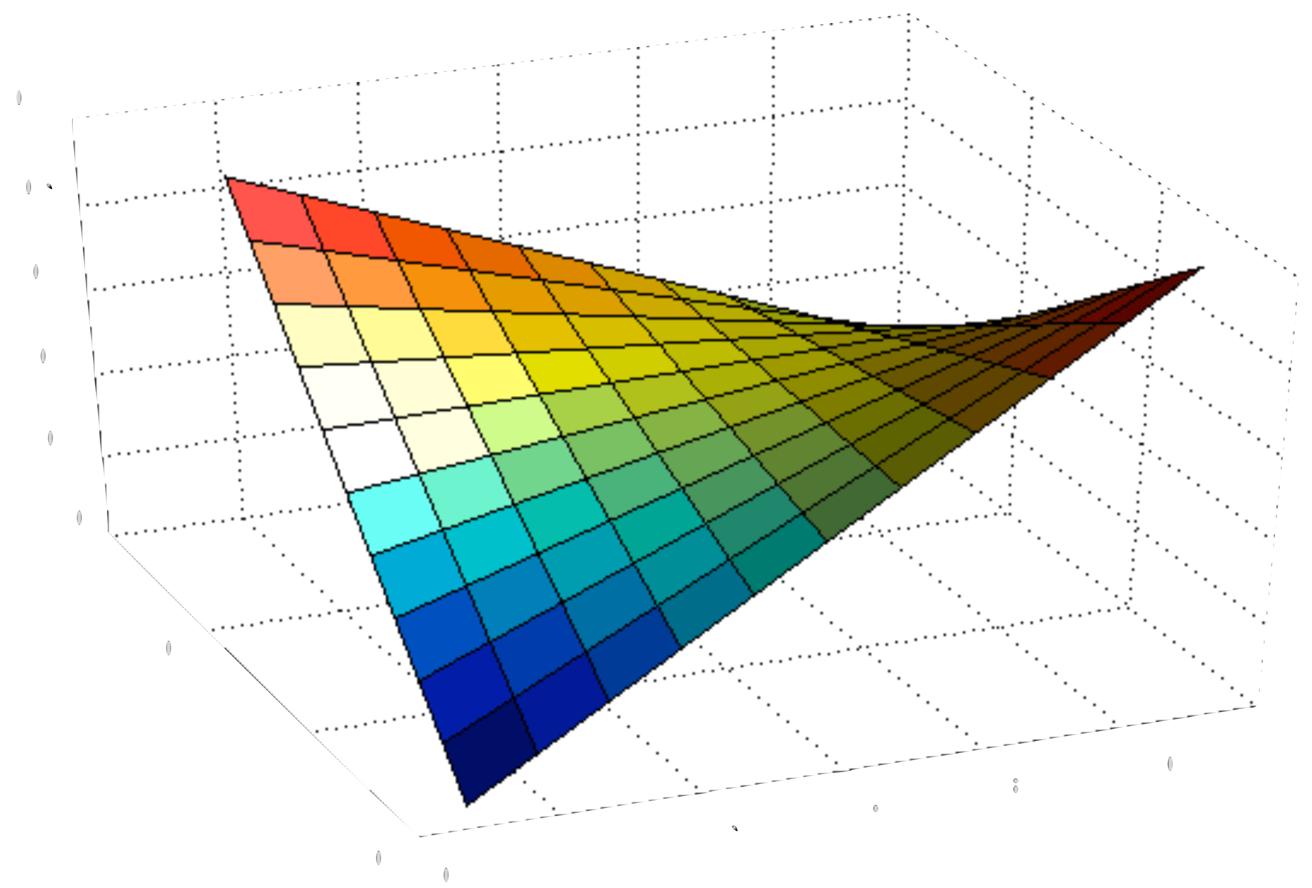
- Interpolation between four grid corners (u_1, v_1), (u_2, v_1), (u_1, v_2) and (u_2, v_2)
  - Input data points $\boldsymbol{x}_{ij}$ are parametrized in two dimensions, hence $\boldsymbol{x}_{ij} \rightarrow (u_i, v_j)$
  - Corresponding output values given by $\boldsymbol{y}_{ij}$
- New data point at parameters (u_r, v_s) with output value $\boldsymbol{y}_{rs}$ given by two successive linear interpolations

$$\boldsymbol{y}_{r2} = \boldsymbol{y}_{12} + (\boldsymbol{y}_{22} - \boldsymbol{y}_{12}) \cdot \frac{u_r - u_1}{u_2 - u_1}$$



$$\boldsymbol{y}_{r1} = \boldsymbol{y}_{11} + (\boldsymbol{y}_{21} - \boldsymbol{y}_{11}) \cdot \frac{u_r - u_1}{u_2 - u_1}$$

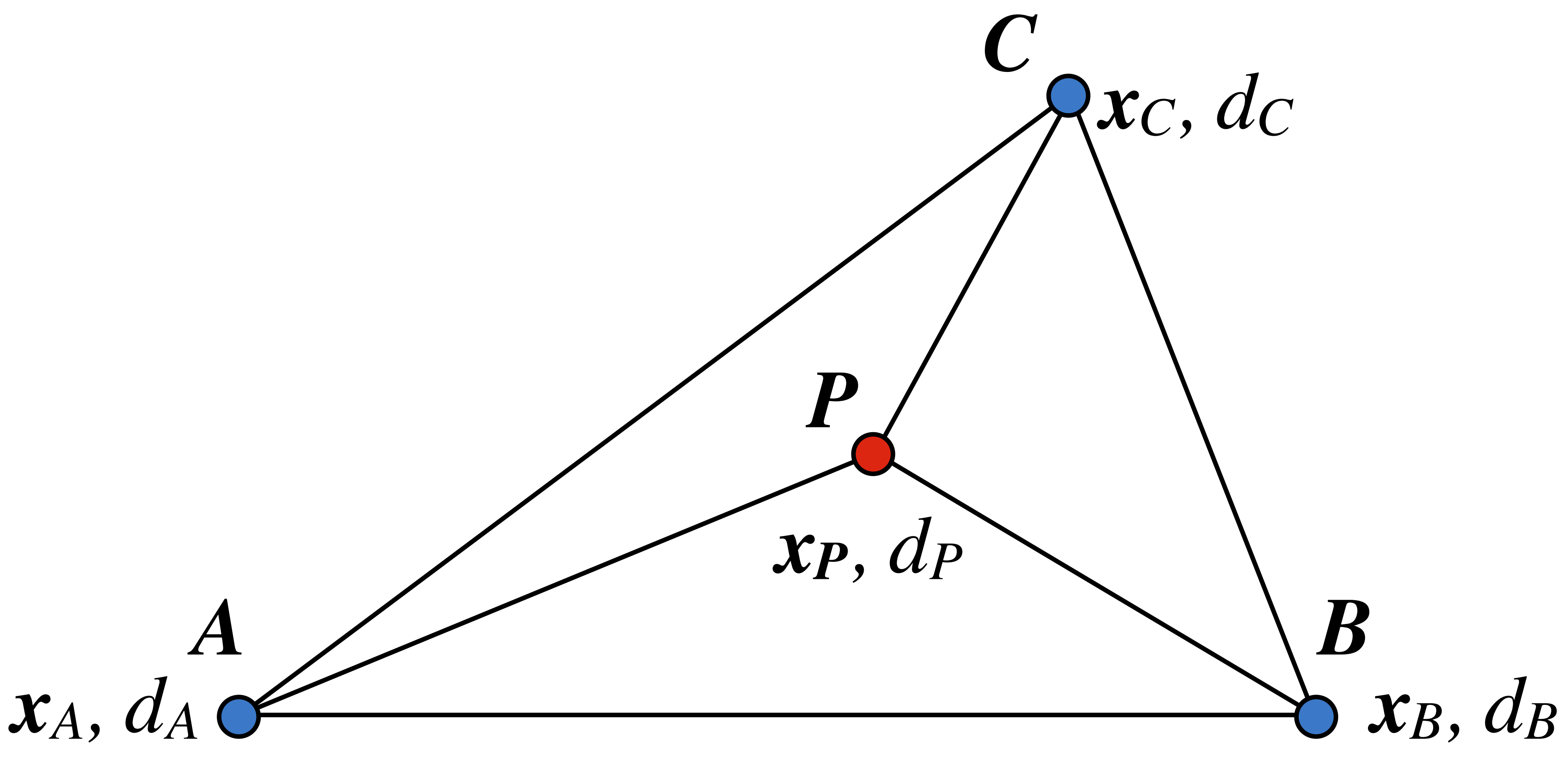
$$\boldsymbol{y}_{rs} = \boldsymbol{y}_{r1} + (\boldsymbol{y}_{r2} - \boldsymbol{y}_{r1}) \cdot \frac{v_s - v_1}{v_2 - v_1}$$



Piecewise Linear Interpolation of Irregular Sample Points in 2D

- Barycentric coordinates $\boldsymbol{\lambda}$ proportional to the signed triangle areas formed by $\boldsymbol{P}$ and corners $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$
  - $\boldsymbol{x_P} = \boldsymbol{\lambda_1\, x_A + \lambda_2\, x_B + \lambda_3\, x_C}$
  - with $\boldsymbol{\lambda_1 + \lambda_2 + \lambda_3 = 1.0}$

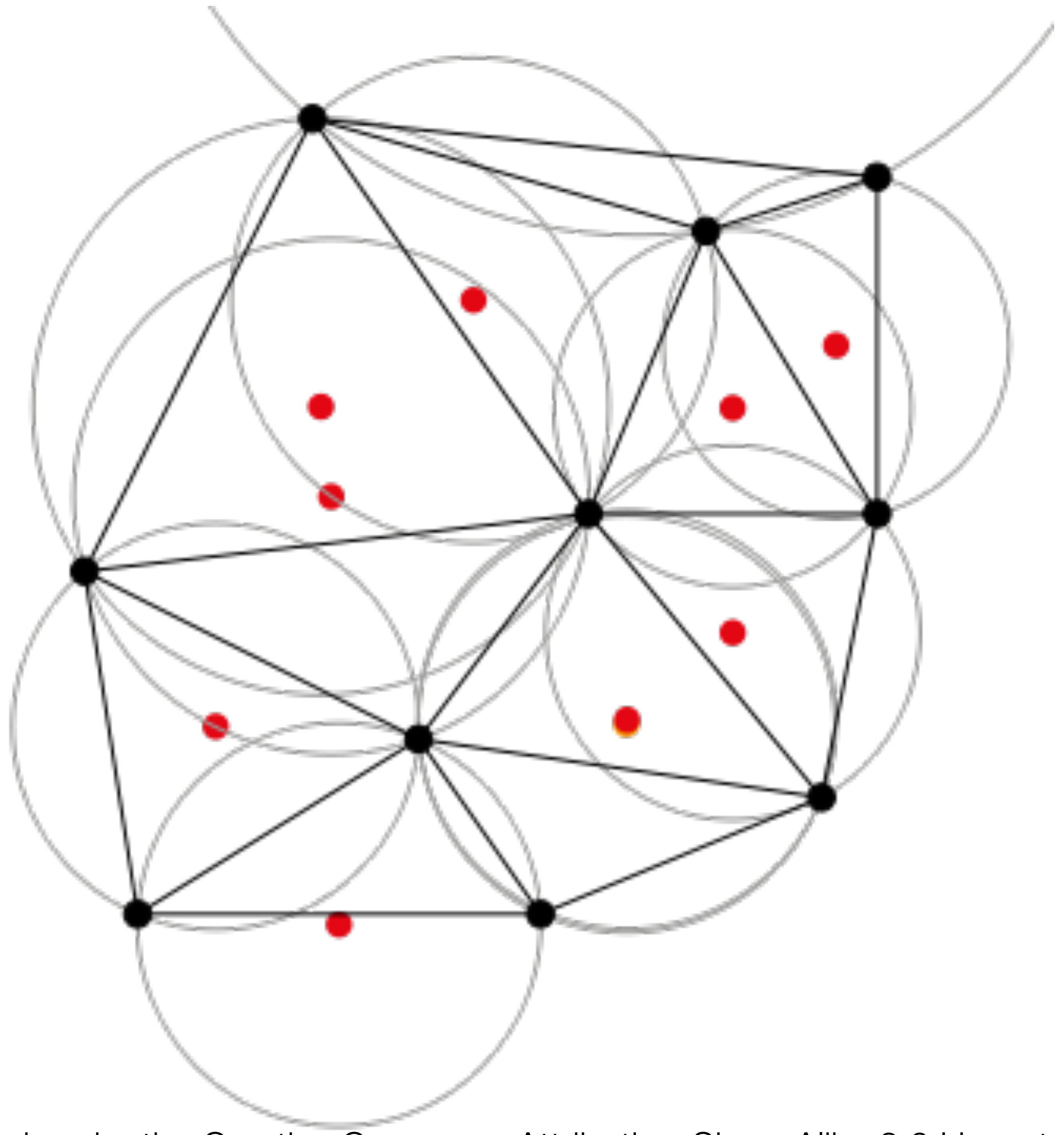
$$\vec{\lambda} = f(\boldsymbol{x}_A, \boldsymbol{x}_B, \boldsymbol{x}_C)$$



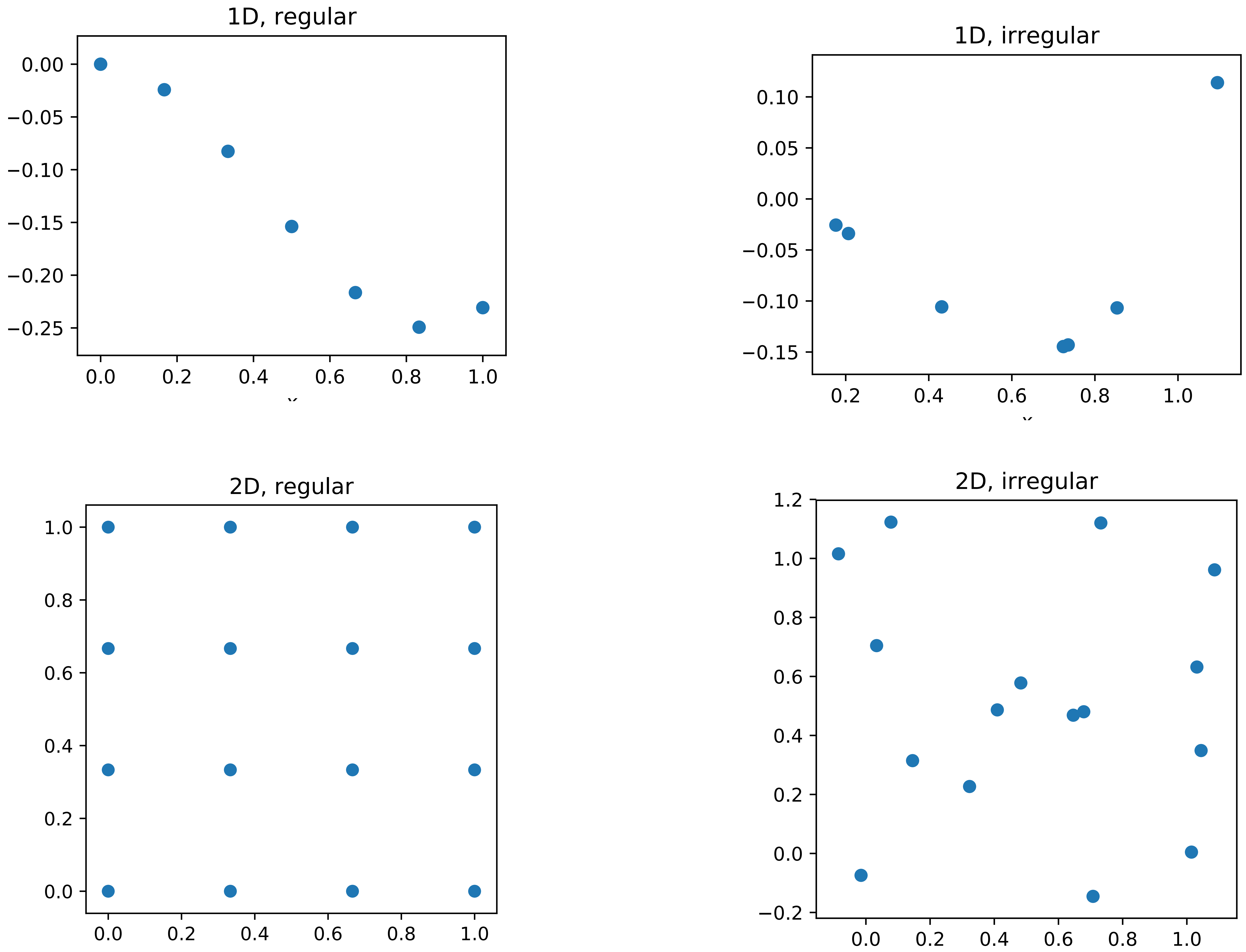
$$d_P = \lambda_A \cdot d_A + \lambda_B \cdot d_B + \lambda_C \cdot d_C$$

- Linear interpolation of any corner attribute d within a triangle defined by barycentric interpolation

- Scattered 2D points can be triangulated and linearly interpolated
  - e.g. using Delaunay triangulation
    - extension to higher dimensions as well



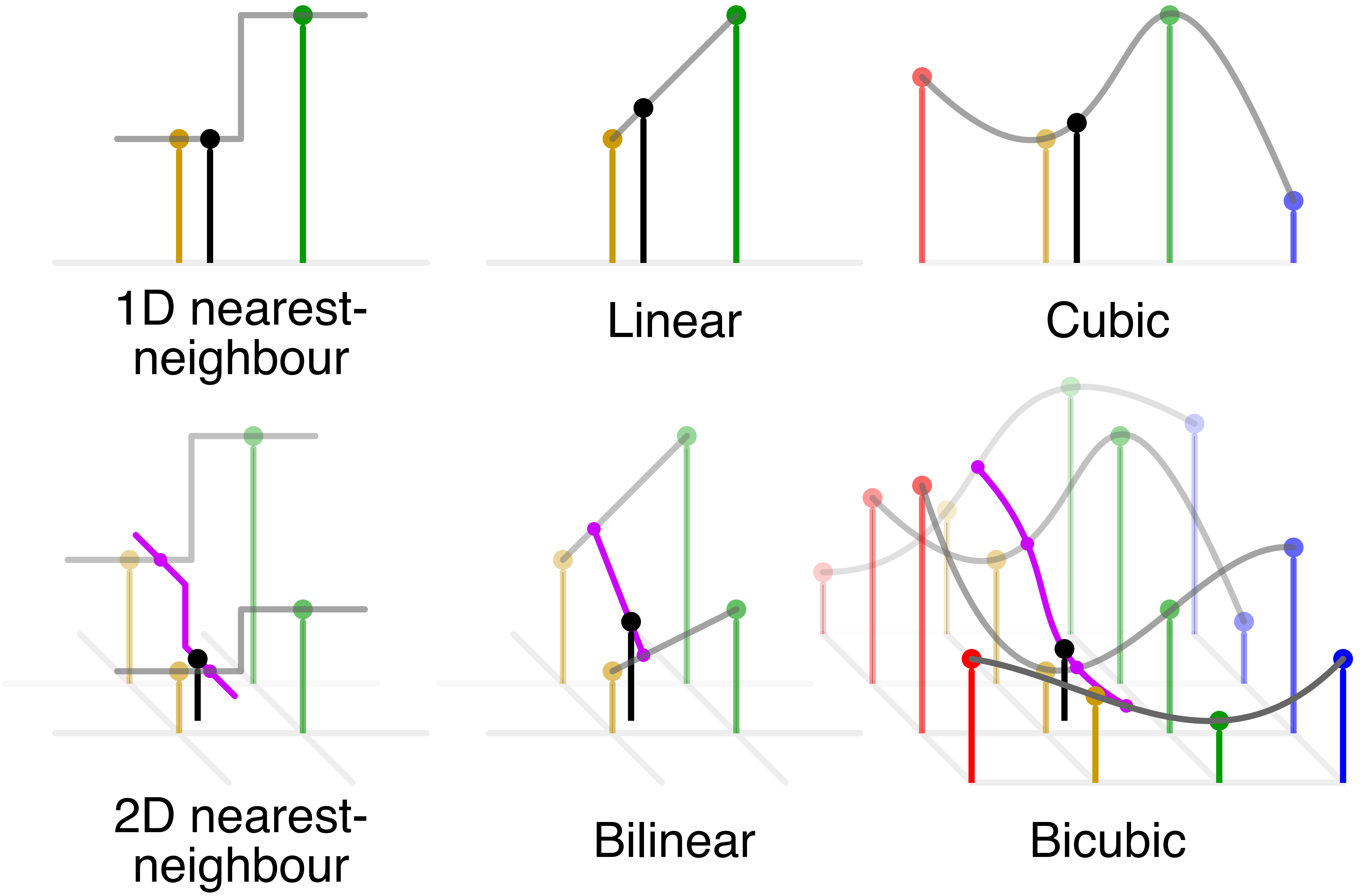
Regular and Irregular 1D and 2D Cases



Smooth Interpolation

Smooth Interpolation

- Continuity may not only be important for the interpolated function value but also with respect to the function's slopes
- Higher order polynomial interpolation or approximation methods support (any) desired continuity levels
    - Interpolated data distribution can be derived analytically
        - for analyzing slope, curvature, edges etc.

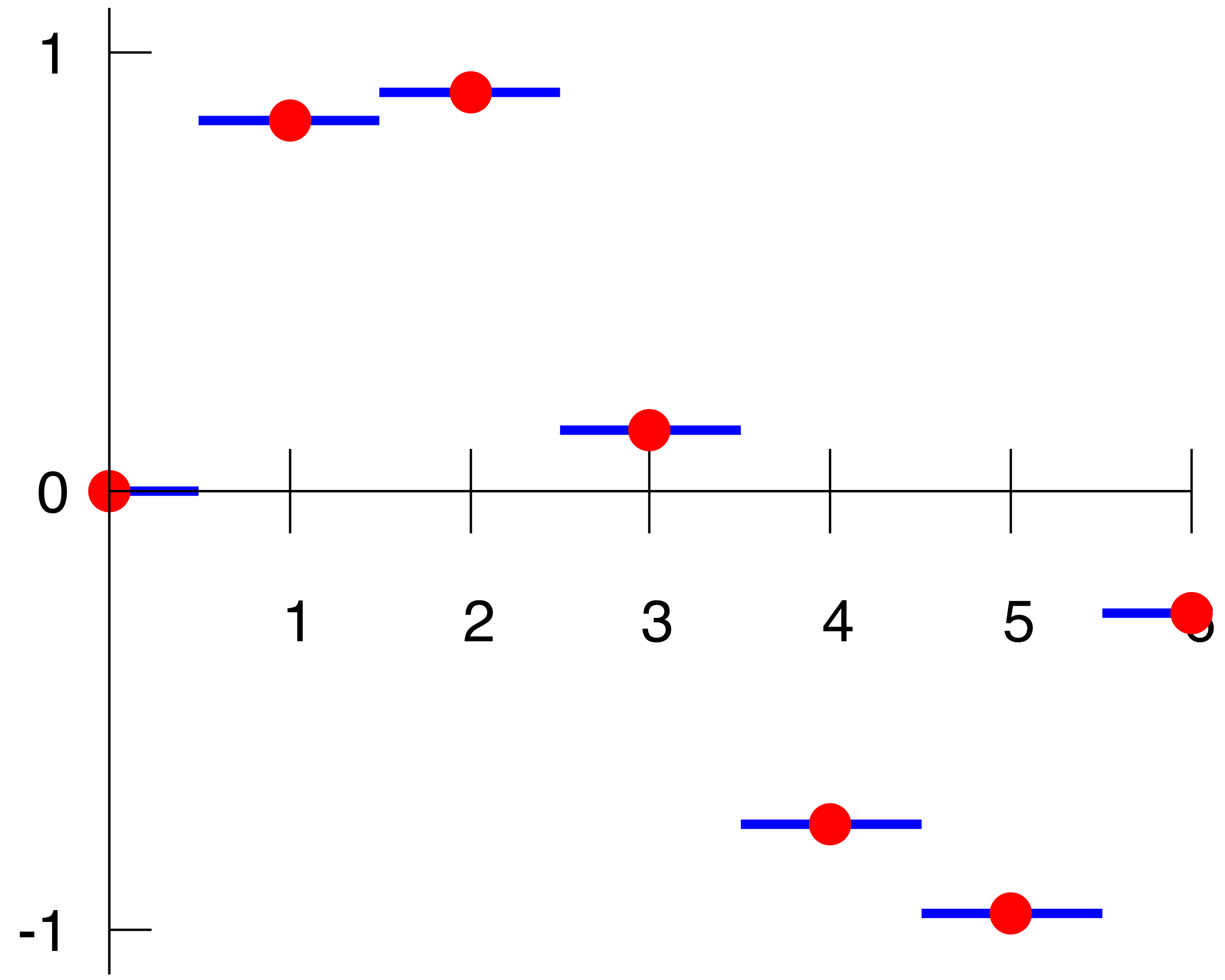


Discontinuity is bad: e.g. infinitely high frequencies appear (remember lecture on sampling & quantization)

Review: Piecewise Constant or Linear 1D Interpolation

- Piecewise constant interpolation sets $f(\boldsymbol{x})$ to the value $\boldsymbol{y}_i$ of nearest element i, with smallest distance $|\,\boldsymbol{x} - \boldsymbol{x}_i\,|$

- In 1D, linear interpolation computes $f(\boldsymbol{x})$ from nearby values $\boldsymbol{y}_i$ and $\boldsymbol{y}_{i+1}$ for $\boldsymbol{x}_i \leq \boldsymbol{x} \leq \boldsymbol{x}_{i+1}$ as



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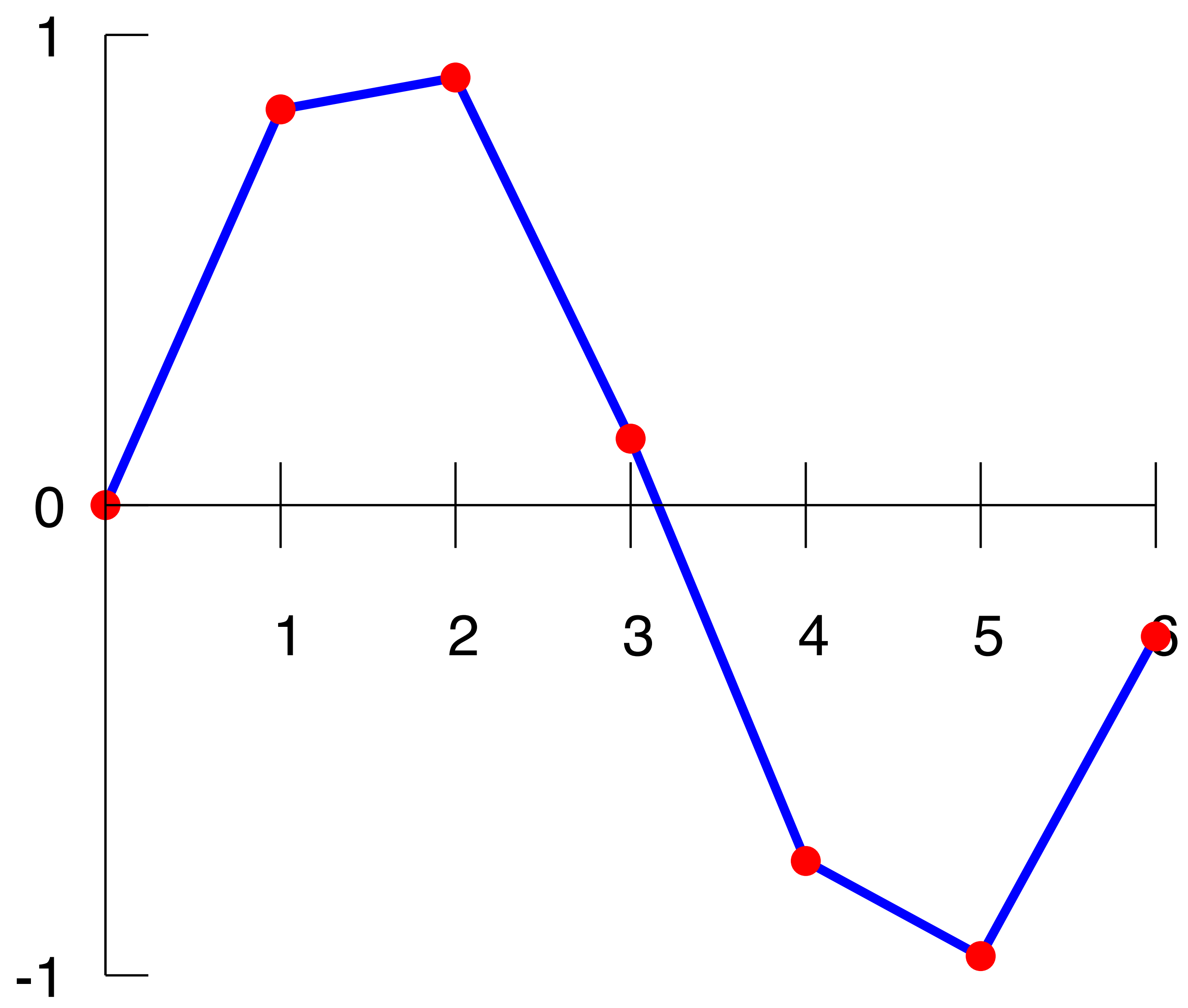


Smooth Polynomial Interpolation in 1D

- Find polynomial $p(\boldsymbol{x})$ such that:

$$p(\boldsymbol{x}_0) = \boldsymbol{y}_0$$

$$p(\boldsymbol{x}_1) = \boldsymbol{y}_1$$

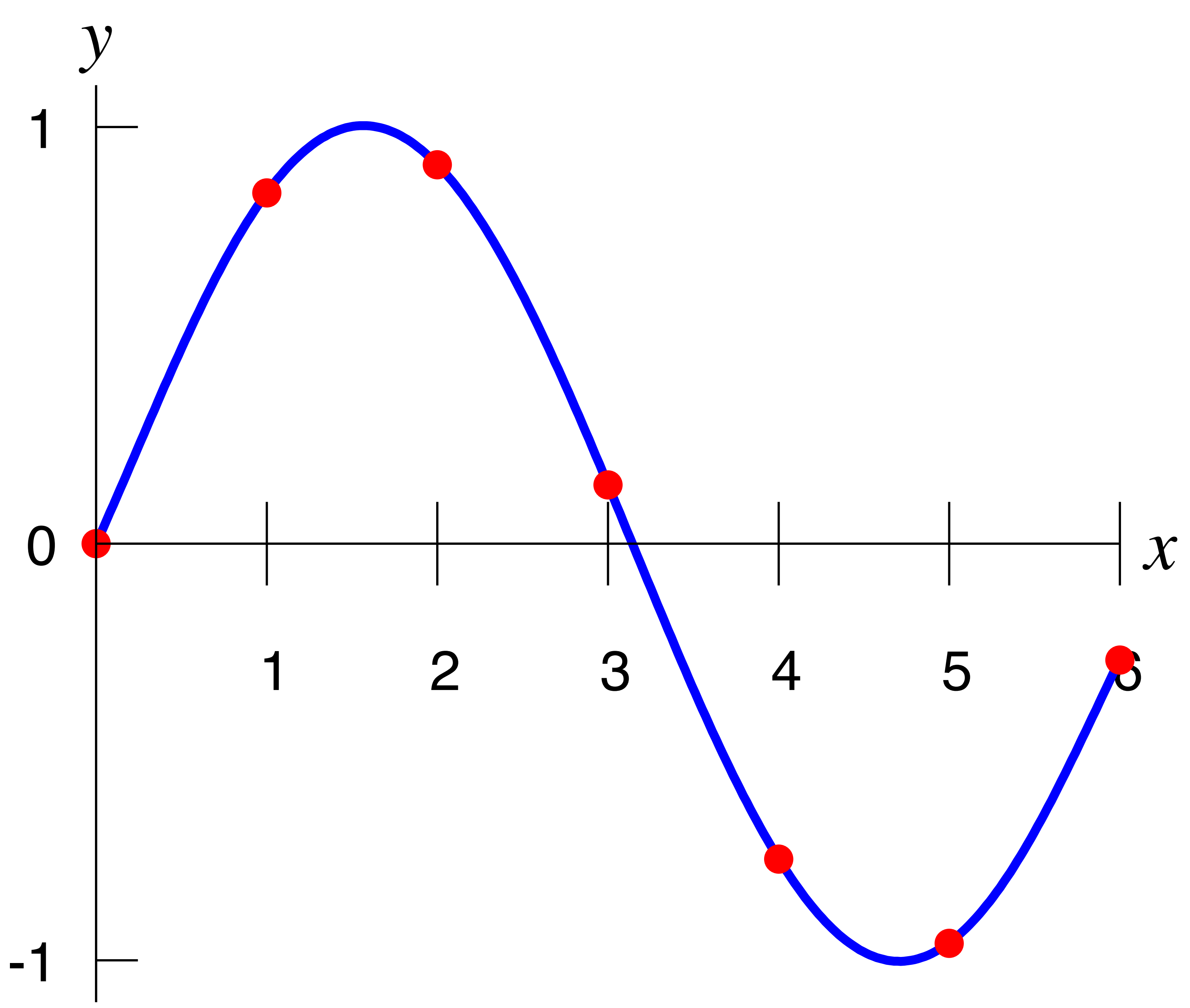
$$p(\boldsymbol{x}_2) = \boldsymbol{y}_2$$

$$p(\boldsymbol{x}_3) = \boldsymbol{y}_3$$

- E.g. cubic polynomial $p(\boldsymbol{x}) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$
  - Several equations of the form $\boldsymbol{y_i} = a_0 + a_1 x_{\boldsymbol{i}} + a_2 x_{\boldsymbol{i}}^2 + a_3 x_{\boldsymbol{i}}^3$
  - Solve system of four equations with four unknowns a_i and four given pairs (x_i, y_i)
- Matrix form:

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \boldsymbol{y} = \mathbf{X} \cdot \boldsymbol{a} \qquad \Rightarrow \boldsymbol{a} = \mathbf{X}^{-1} \cdot \boldsymbol{y}$$

Vandermonde matrix



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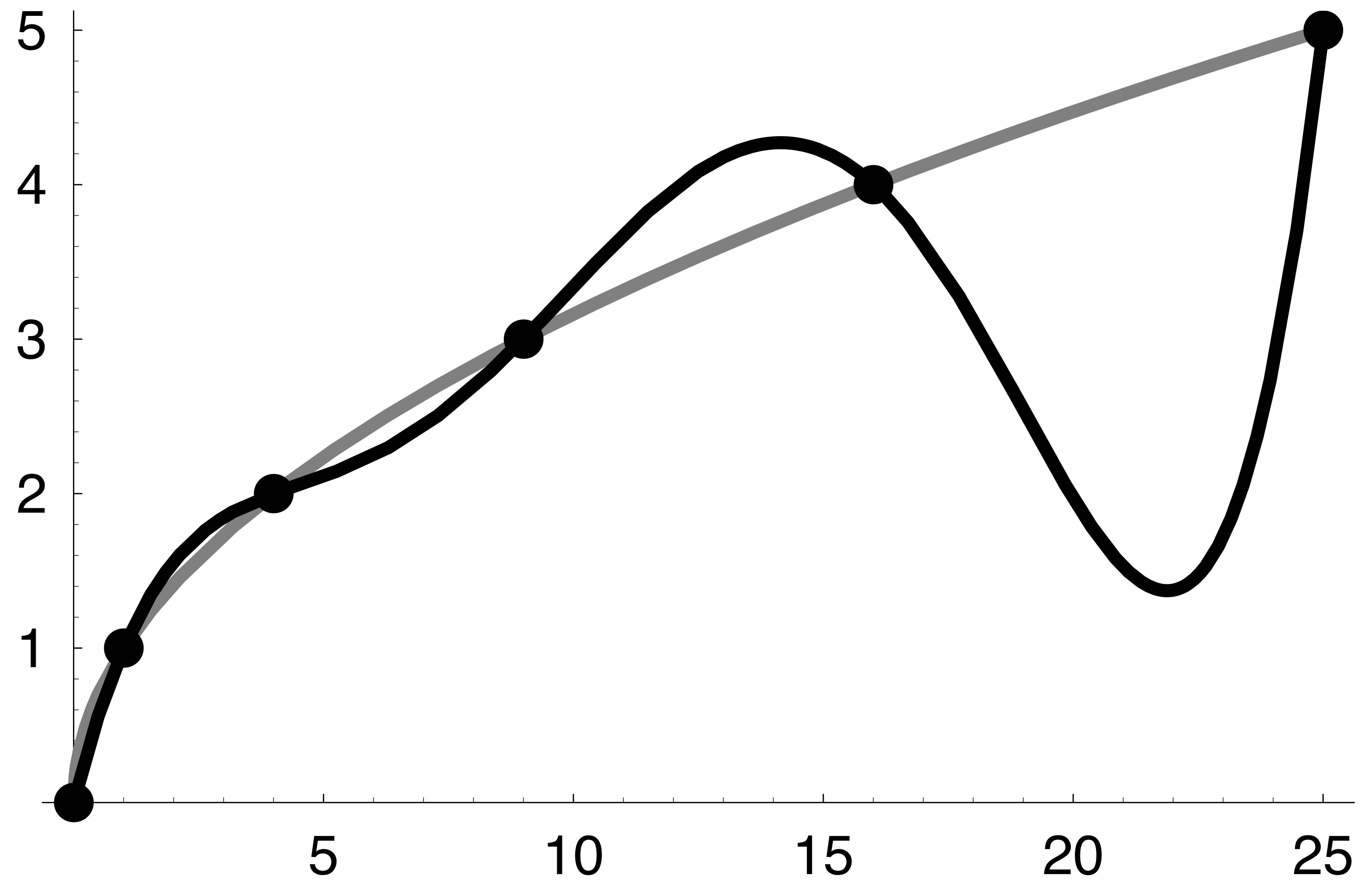
Overfitting

- Given any number of n data points (x_i, y_i) a degree n-1 polynomial can be fitted to exactly interpolate them
- Forcing strict interpolation can be dangerous
  - Overfitting can lead to unwanted and fatal oscillations
- Hence careful controlled approximation may be more useful than strict interpolation



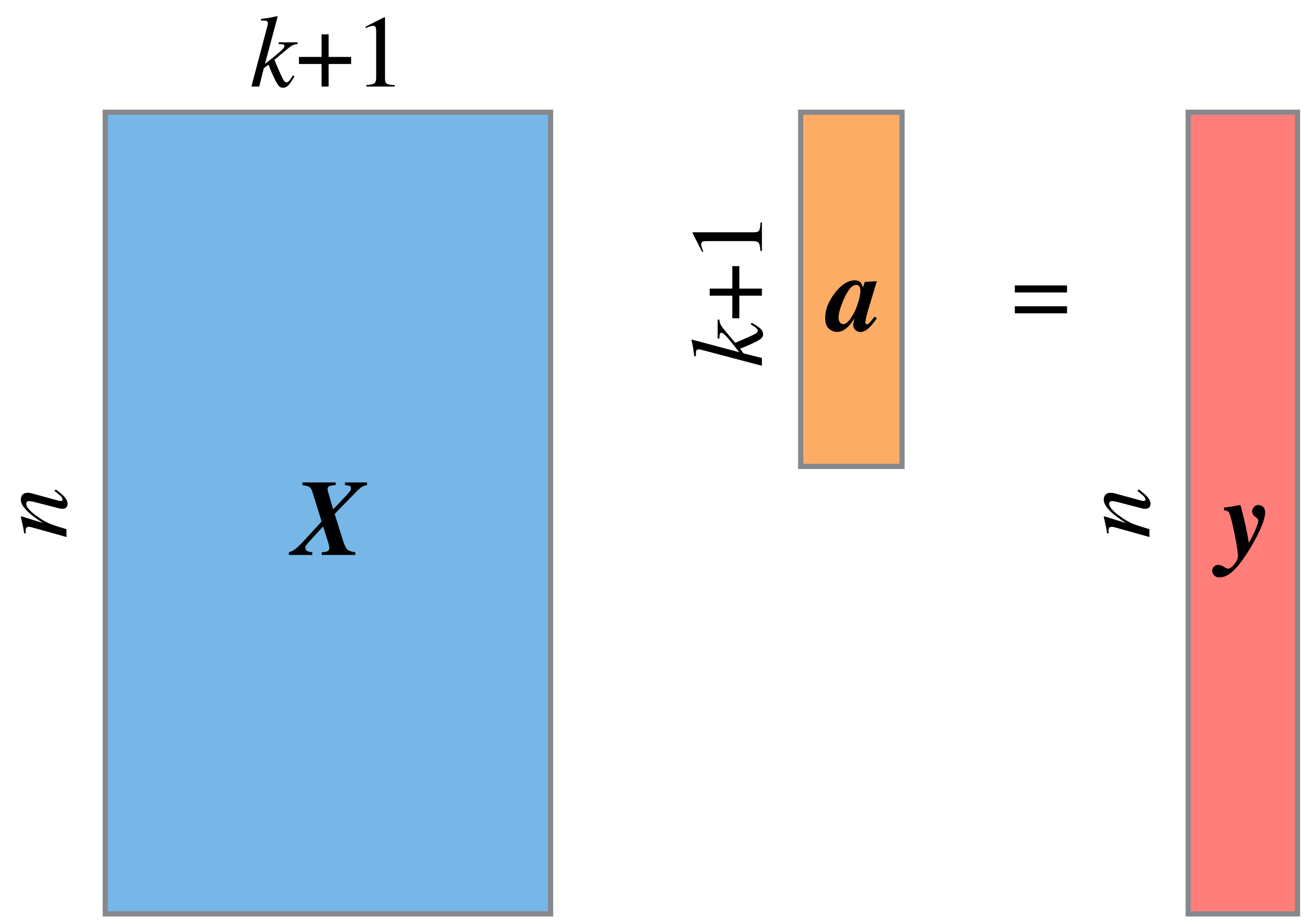
Fitting degree-4 polynomial to $\sqrt{x}$

Polynomial Least-Squares Approximation

- In contrast to interpolation, polynomial approximation is more reliable and typically leads to robust data fitting results
- Task: Given n data points $(\boldsymbol{x_i}, \boldsymbol{y_i})$, find a fixed degree k polynomial $p(\boldsymbol{x})$ that for each $p(\boldsymbol{x_i})$ as close as possible to $\boldsymbol{y_i}$
- Leads to overdetermined system of equations $\boldsymbol{X} \cdot \boldsymbol{a} = \boldsymbol{y}$
  - Tall matrix $\boldsymbol{X}$ and value vector $\boldsymbol{y}$
- Normal equations $\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X} \cdot \boldsymbol{a} = \boldsymbol{X}^{\mathrm{T}}\boldsymbol{y}$ can be solved efficiently
  - Using Gaussian elimination or SVD
- Resulting polynomial is the best least squares approximation

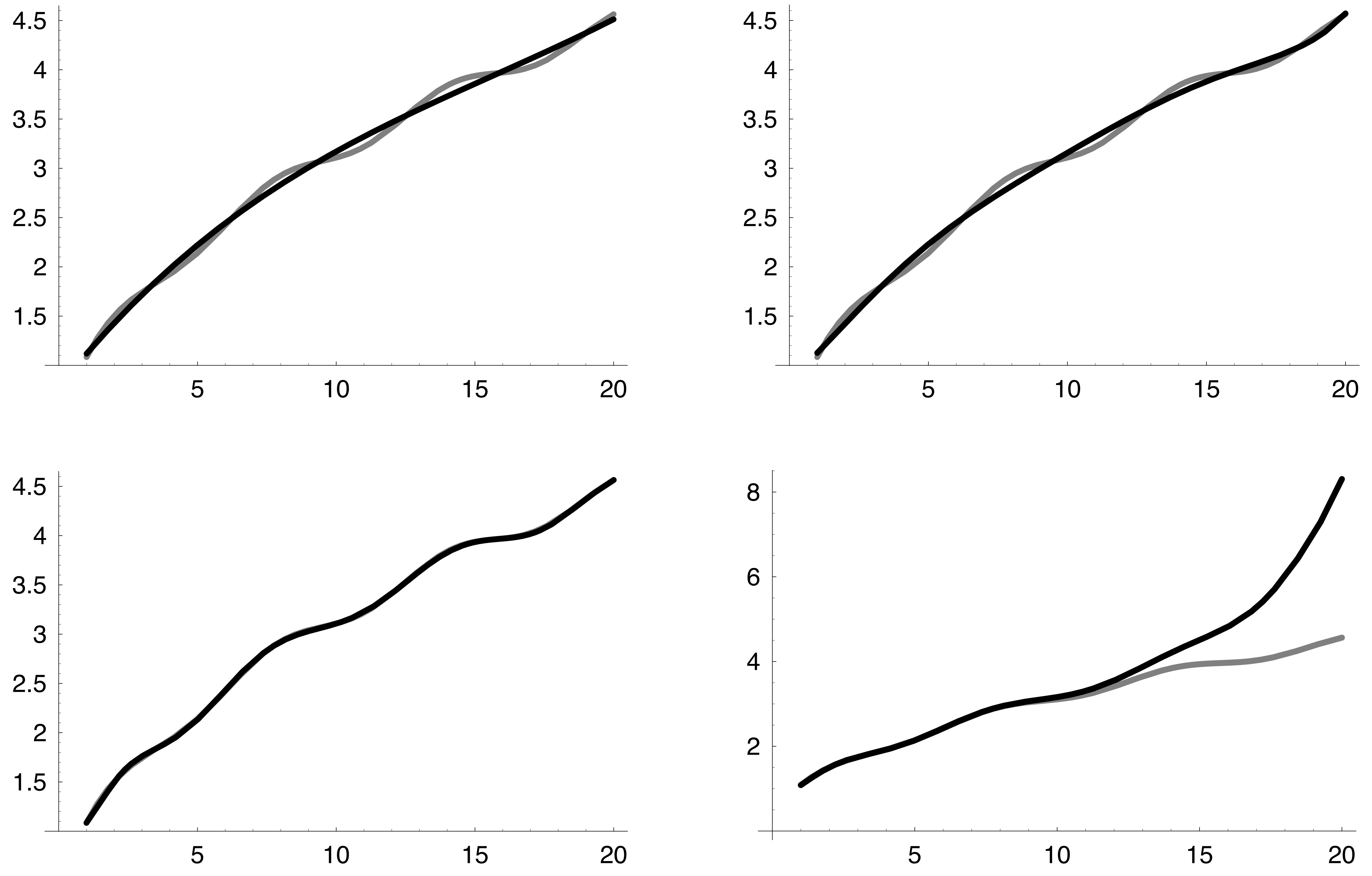
$$\varepsilon = \sum_{i=0}^{n-1} (y_i - p(x_i))^2$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ & \vdots & \vdots & \\ 1 & x_{n-2} & x_{n-2}^2 & x_{n-2}^3 \\ 1 & x_{n-1} & x_{n-1}^2 & x_{n-1}^3 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix}$$



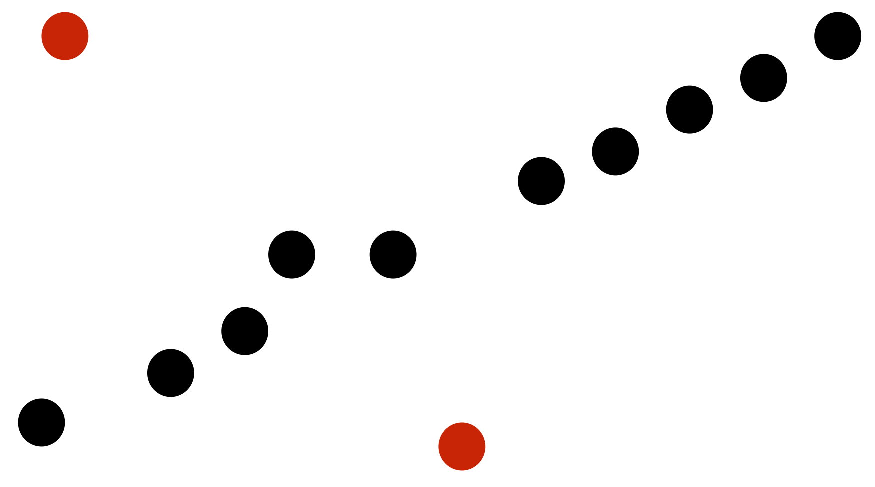
Approximation Problems

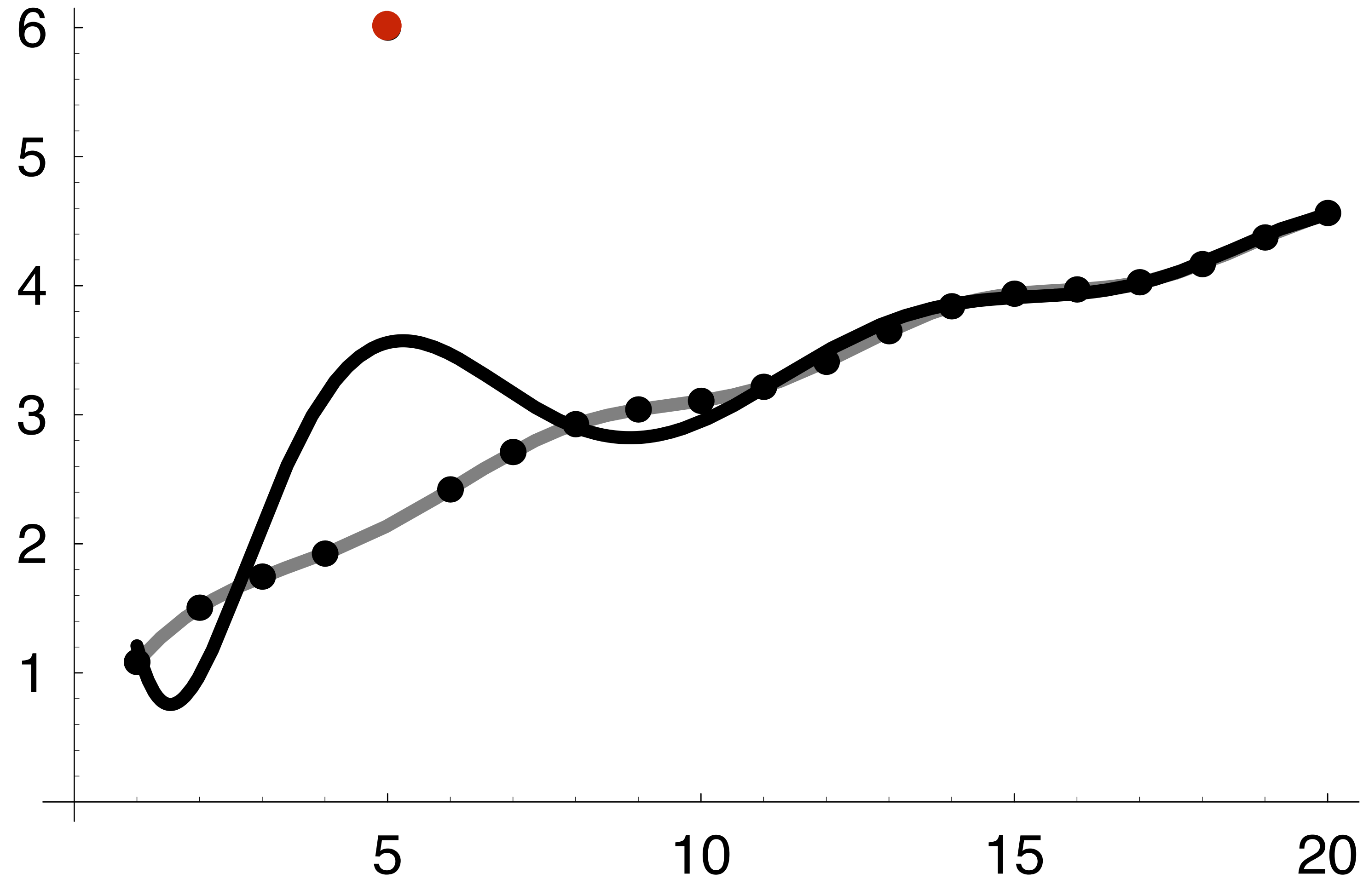
- Approximation of $\sqrt{x} + 0.1\,\sin(x)$
  - For degree $n=3$, $n=6$, $n=10$ and $n=11$
  - Instabilities may occur
    - due to high condition number of $\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}$



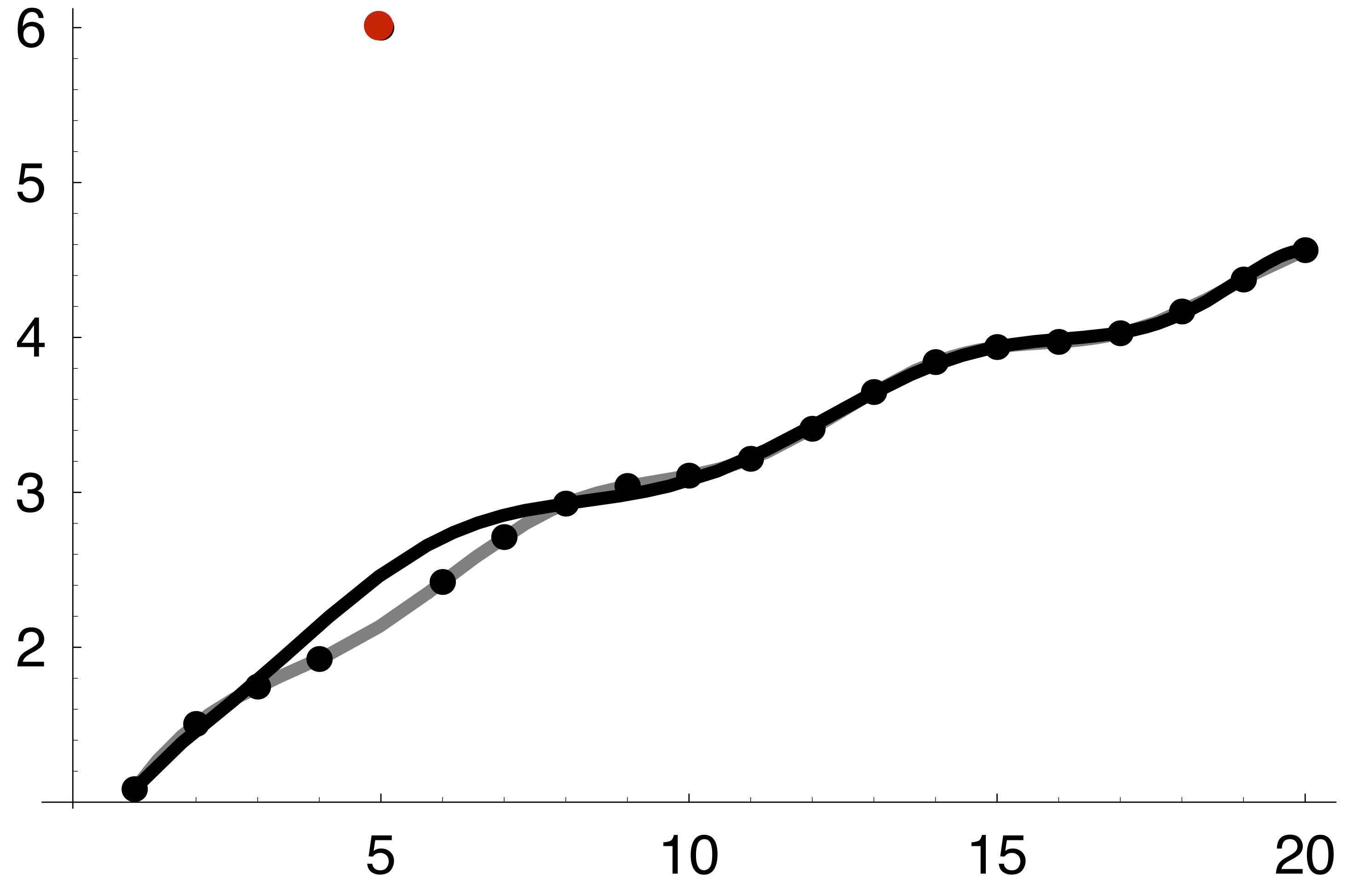
Outliers and Weighted Interpolation

- An outlier is a measurement that is less reliable or accurate
- We may know beforehand what samples are outliers
  - If we do not know, there are techniques to estimate if a sample is an inlier or outlier
- Outliers may severely affect data fitting
  - Weighted least-squares tries to follow the inliers more





Unweighted



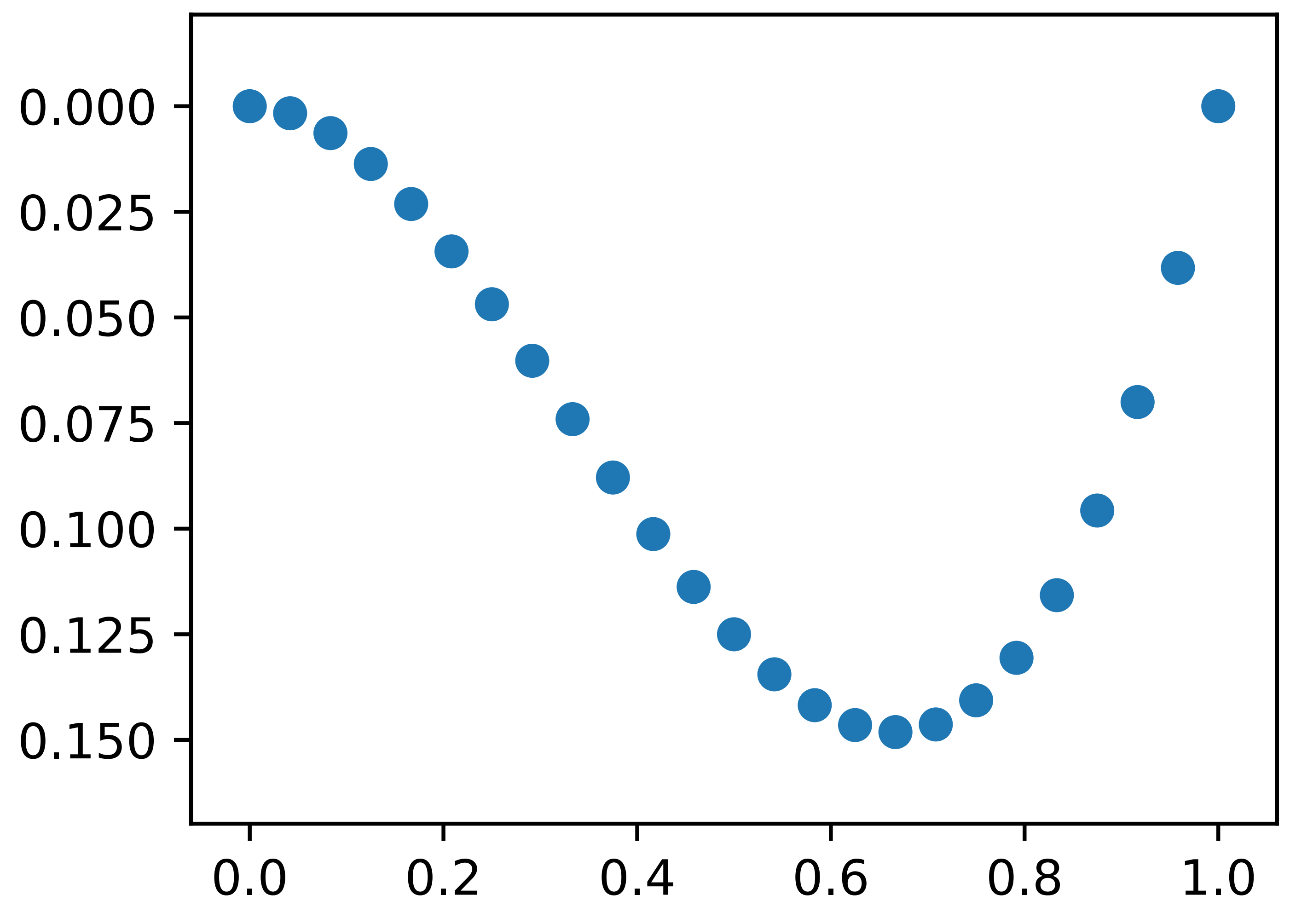
Weighted

Important

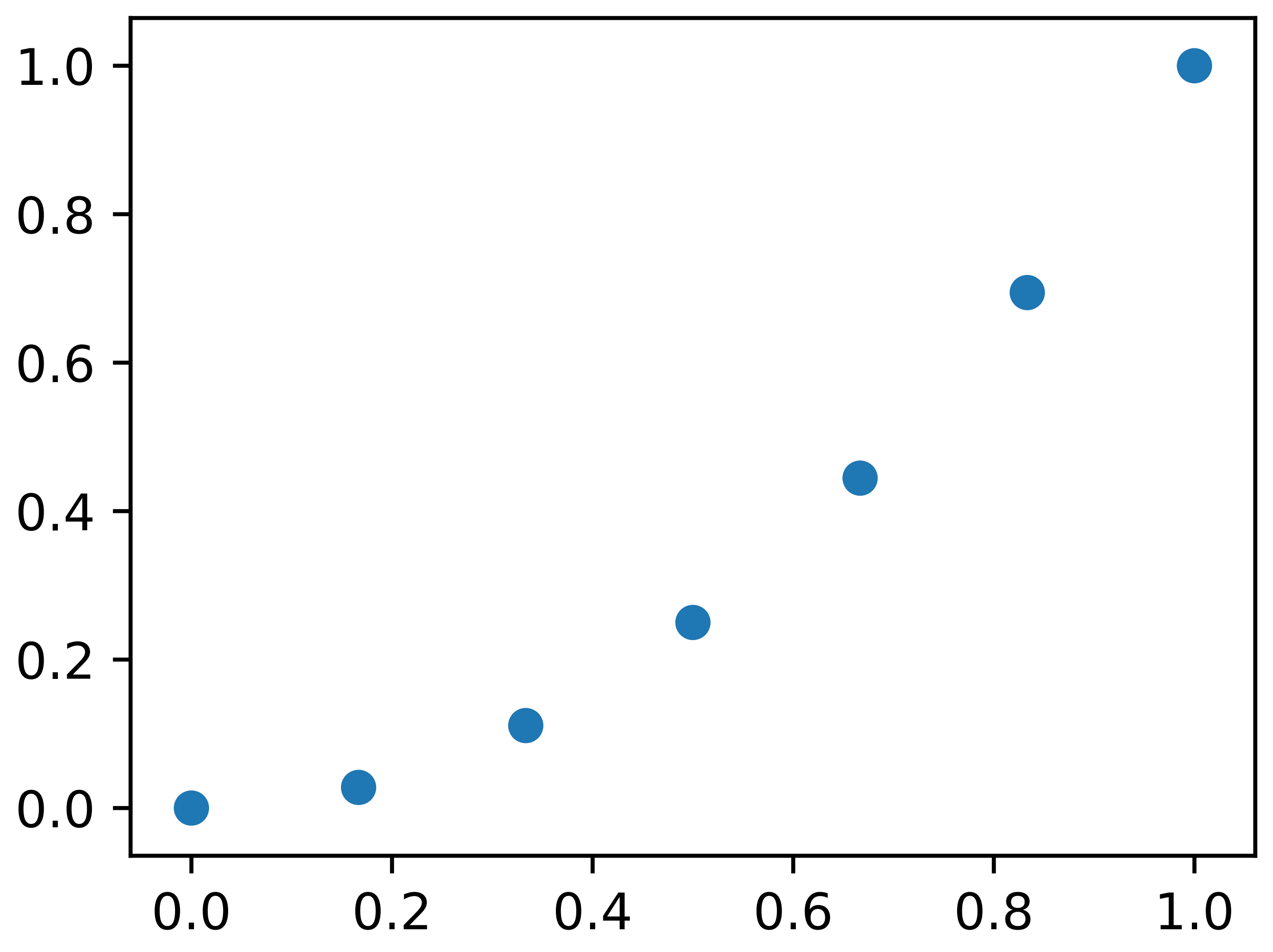
Use the right tool for the right problem!

- One has to know what interpolation fits what type of data
- Piecewise-constant interpolation: simple; no overfitting; good if enough samples available
- Low-degree polynomials: smoother; faster; less variance; less overfitting; less fitting
- High-degree polynomials: bumpier; slower; more variance; more overfitting; more fitting
- **Make sure you understand each technique's advantages, disadvantages and risks**

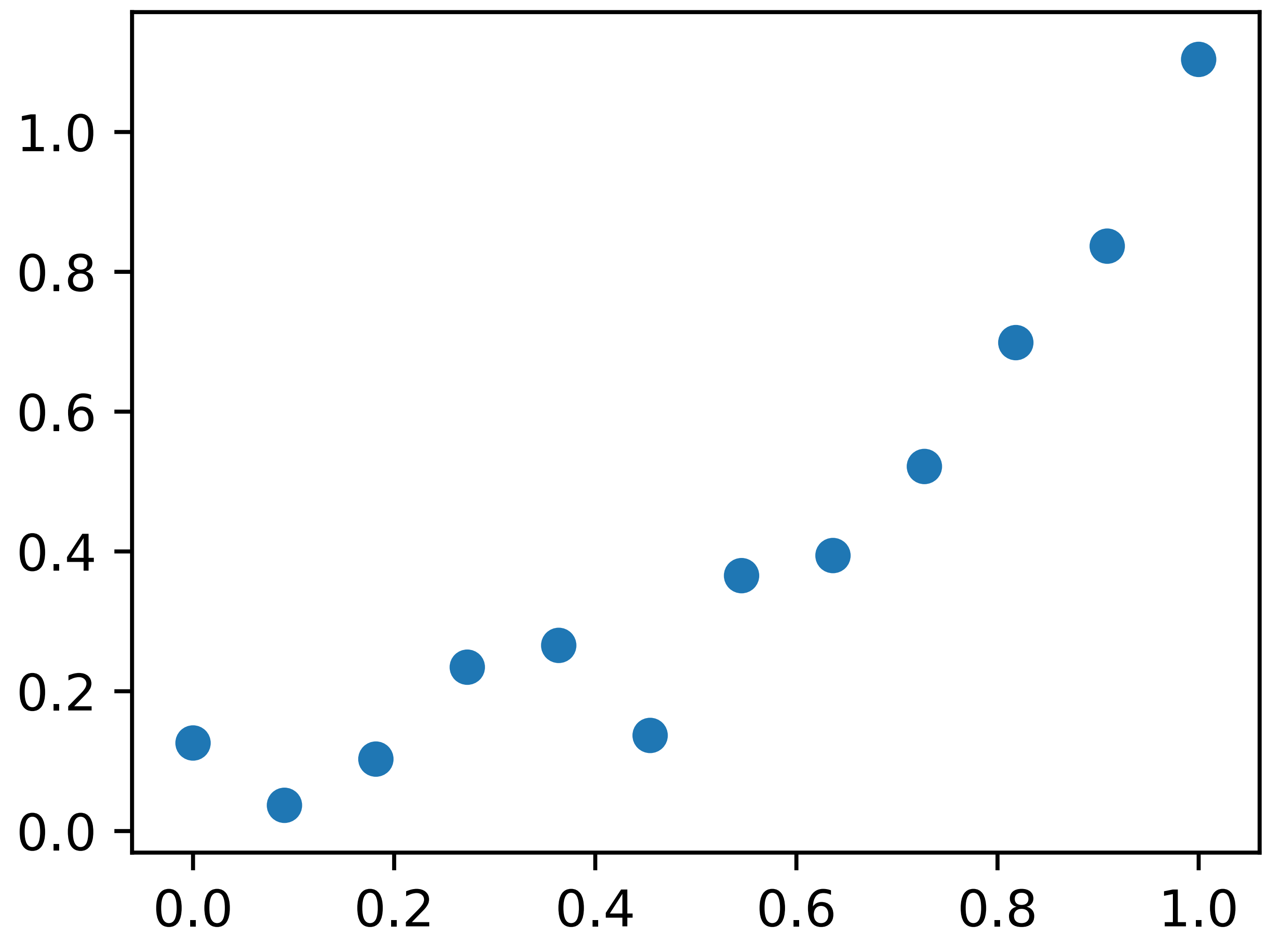
Examples



Piecewise would work



Low-degree exact



Low-degree least-squares

Recap

- **Sparse data:** set of discrete, scattered sample points (x, y)
- **Piecewise interpolation:** interpolation depends only on neighboring samples
- **Smooth interpolation:** function’s derivative (optionally high-order derivatives too) are continuous
- **Exact interpolation:** #points = #unknowns → interpolator passes *through* the given points
- **Least-squares interpolation:** #points > #unknowns → interpolator passes *near* the given points
- **Overfitting:** when a too complex model fits too poorly the true model and too well the samples
- **Outlier:** isolated noisy/unreliable measurement

- Required textbook Chapter(s): 8.1 - 8.5