Reliability Theory

9.1 Introduction

Reliability theory is concerned with determining the probability that a system, possibly consisting of many components, will function. We shall suppose that whether or not the system functions is determined solely from a knowledge of which components are functioning. For instance, a *series* system will function if and only if all of its components are functioning, while a *parallel* system will function if and only if at least one of its components is functioning. In Section 9.2, we explore the possible ways in which the functioning of the system may depend upon the functioning of its components. In Section 9.3, we suppose that each component will function with some known probability (independently of each other) and show how to obtain the probability that the system will function. As this probability often is difficult to explicitly compute, we also present useful upper and lower bounds in Section 9.4. In Section 9.5 we look at a system dynamically over time by supposing that each component initially functions and does so for a random length of time at which it fails. We then discuss the relationship between the distribution of the amount of time that a system functions and the distributions of the component lifetimes. In particular, it turns out that if the amount of time that a component functions has an *increasing failure rate on the average* (IFRA) distribution, then so does the distribution of system lifetime. In Section 9.6 we consider the problem of obtaining the mean lifetime of a system. In the final section we analyze the system when failed components are subjected to repair.

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9.2 Structure Functions

Consider a system consisting of n components, and suppose that each component is either functioning or has failed. To indicate whether or not the ith component is functioning, we define the indicator variable x_i by

$$x_i = \begin{cases} 1, & \text{if the } i\text{th component is functioning} \\ 0, & \text{if the } i\text{th component has failed} \end{cases}$$

The vector $\mathbf{x} = (x_1, \ldots, x_n)$ is called the *state vector*. It indicates which of the components are functioning and which have failed.

We further suppose that whether or not the system as a whole is functioning is completely determined by the state vector $\mathbf{x}$. Specifically, it is supposed that there exists a function $\phi(\mathbf{x})$ such that

$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if the system is functioning when the state vector is } \mathbf{x} \\ 0, & \text{if the system has failed when the state vector is } \mathbf{x} \end{cases}$$

The function $\phi(\mathbf{x})$ is called the *structure function* of the system.

Example 9.1 (The Series Structure) A series system functions if and only if all of its components are functioning. Hence, its structure function is given by

$$\phi(\mathbf{x}) = \min(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i$$

We shall find it useful to represent the structure of a system in terms of a diagram. The relevant diagram for the series structure is shown in Figure 9.1. The idea is that if a signal is initiated at the left end of the diagram then in order for it to successfully reach the right end, it must pass through all of the components; hence, they must all be functioning. ■

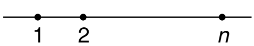

Figure 9.1 A series system.

Example 9.2 (The Parallel Structure) A parallel system functions if and only if at least one of its components is functioning. Hence, its structure function is given by

$$\phi(\mathbf{x}) = \max(x_1, \ldots, x_n)$$

A parallel structure may be pictorially illustrated by Figure 9.2. This follows since a signal at the left end can successfully reach the right end as long as at least one component is functioning. ■

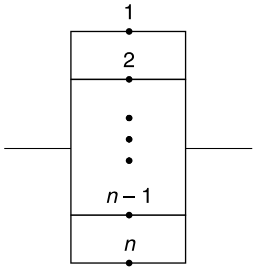

Figure 9.2 A parallel system.

Example 9.3 (The k-out-of-n Structure) The series and parallel systems are both special cases of a k-out-of-n system. Such a system functions if and only if at least k of the n components are functioning. As $\sum_{i=1}^{n} x_i$ equals the number of functioning components, the structure function of a k-out-of-n system is given by

$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} x_i \geqslant k \\ 0, & \text{if } \sum_{i=1}^{n} x_i < k \end{cases}$$

Series and parallel systems are respectively n-out-of-n and 1-out-of-n systems.
The two-out-of-three system may be diagrammed as shown in Figure 9.3. ■

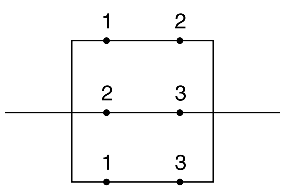

Figure 9.3 A two-out-of-three system.

Example 9.4 (A Four-Component Structure) Consider a system consisting of four components, and suppose that the system functions if and only if components 1 and 2 both function and at least one of components 3 and 4 function. Its structure function is given by

$$\phi(\mathbf{x}) = x_1 x_2 \max(x_3, x_4)$$

Pictorially, the system is as shown in Figure 9.4. A useful identity, easily checked, is that for binary variables,* $x_i, i = 1, \ldots, n$,

$$\max(x_1, \ldots, x_n) = 1 - \prod_{i=1}^{n}(1 - x_i)$$

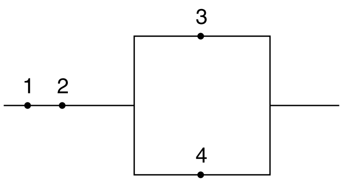

Figure 9.4

When $n = 2$, this yields

$$\max(x_1, x_2) = 1 - (1 - x_1)(1 - x_2) = x_1 + x_2 - x_1 x_2$$

Hence, the structure function in the example may be written as

$$\phi(\mathbf{x}) = x_1 x_2 (x_3 + x_4 - x_3 x_4) \qquad \blacksquare$$

It is natural to assume that replacing a failed component by a functioning one will never lead to a deterioration of the system. In other words, it is natural to assume that the structure function $\phi(\mathbf{x})$ is an increasing function of $\mathbf{x}$, that is, if $x_i \leqslant y_i, i = 1, \ldots, n$, then $\phi(\mathbf{x}) \leqslant \phi(\mathbf{y})$. Such an assumption shall be made in this chapter and the system will be called *monotone*.

9.2.1 Minimal Path and Minimal Cut Sets

In this section we show how any system can be represented both as a series arrangement of parallel structures and as a parallel arrangement of series structures. As a preliminary, we need the following concepts.

A state vector $\mathbf{x}$ is called a *path vector* if $\phi(\mathbf{x}) = 1$. If, in addition, $\phi(\mathbf{y}) = 0$ for all $\mathbf{y} < \mathbf{x}$, then $\mathbf{x}$ is said to be a *minimal path vector*.** If $\mathbf{x}$ is a minimal path vector, then the set $A = \{i\colon\ x_i = 1\}$ is called a *minimal path set*. In other words, a minimal path set is a minimal set of components whose functioning ensures the functioning of the system.

* A binary variable is one that assumes either the value 0 or 1.

** We say that $\mathbf{y} < \mathbf{x}$ if $y_i \leqslant x_i, i = 1, \ldots, n$, with $y_i < x_i$ for some i.

Example 9.5 Consider a five-component system whose structure is illustrated by Figure 9.5. Its structure function equals

$$\begin{aligned}\phi(\mathbf{x}) &= \max(x_1, x_2)\max(x_3x_4, x_5)\\ &= (x_1 + x_2 - x_1x_2)(x_3x_4 + x_5 - x_3x_4x_5)\end{aligned}$$

There are four minimal path sets, namely, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 5\}$, $\{2, 5\}$. ■

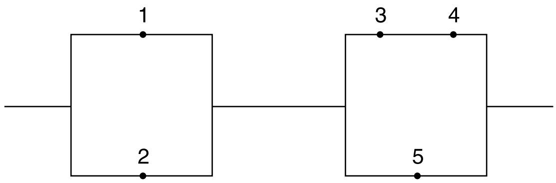

Figure 9.5

Example 9.6 In a k-out-of-n system, there are $\binom{n}{k}$ minimal path sets, namely, all of the sets consisting of exactly k components. ■

Let $A_1, \ldots, A_s$ denote the minimal path sets of a given system. We define $\alpha_j(\mathbf{x})$, the indicator function of the jth minimal path set, by

$$\begin{aligned}\alpha_j(\mathbf{x}) &= \begin{cases}1, & \text{if all the components of } A_j \text{ are functioning}\\ 0, & \text{otherwise}\end{cases}\\ &= \prod_{i \in A_j} x_i\end{aligned}$$

By definition, it follows that the system will function if all the components of at least one minimal path set are functioning; that is, if $\alpha_j(\mathbf{x}) = 1$ for some j. On the other hand, if the system functions, then the set of functioning components must include a minimal path set. Therefore, *a system will function if and only if all the components of at least one minimal path set are functioning*. Hence,

$$\phi(\mathbf{x}) = \begin{cases}1, & \text{if } \alpha_j(\mathbf{x}) = 1 \text{ for some } j\\ 0, & \text{if } \alpha_j(\mathbf{x}) = 0 \text{ for all } j\end{cases}$$

or equivalently,

$$\begin{aligned}\phi(\mathbf{x}) &= \max_j \alpha_j(\mathbf{x})\\ &= \max_j \prod_{i \in A_j} x_i \qquad (9.1)\end{aligned}$$

Since $\alpha_j(\mathbf{x})$ is a series structure function of the components of the jth minimal path set, Equation (9.1) expresses an arbitrary system as a parallel arrangement of series systems.

Example 9.7 Consider the system of Example 9.5. Because its minimal path sets are $A_1 = \{1, 3, 4\}$, $A_2 = \{2, 3, 4\}$, $A_3 = \{1, 5\}$, and $A_4 = \{2, 5\}$, we have by Equation (9.1) that

$$\begin{aligned}\phi(\mathbf{x}) &= \max\{x_1x_3x_4, x_2x_3x_4, x_1x_5, x_2x_5\}\\ &= 1-(1-x_1x_3x_4)(1-x_2x_3x_4)(1-x_1x_5)(1-x_2x_5)\end{aligned}$$

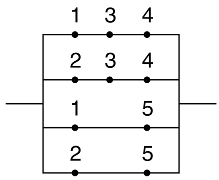

Figure 9.6

You should verify that this equals the value of $\phi(\mathbf{x})$ given in Example 9.5. (Make use of the fact that, since x_i equals 0 or 1, $x_i^2 = x_i$.) This representation may be pictured as shown in Figure 9.6. ■

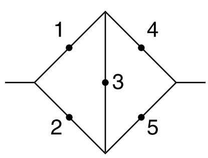

Figure 9.7 The bridge system.

Example 9.8 The system whose structure is as pictured in Figure 9.7 is called the *bridge system*. Its minimal path sets are {1, 4}, {1, 3, 5}, {2, 5}, and {2, 3, 4}. Hence, by Equation (9.1), its structure function may be expressed as

$$\begin{aligned}\phi(\mathbf{x}) &= \max\{x_1x_4, x_1x_3x_5, x_2x_5, x_2x_3x_4\}\\ &= 1-(1-x_1x_4)(1-x_1x_3x_5)(1-x_2x_5)(1-x_2x_3x_4)\end{aligned}$$

This representation $\phi(\mathbf{x})$ is diagrammed as shown in Figure 9.8. ■

A state vector $\mathbf{x}$ is called a *cut vector* if $\phi(\mathbf{x}) = 0$. If, in addition, $\phi(\mathbf{y}) = 1$ for all $\mathbf{y} > \mathbf{x}$, then $\mathbf{x}$ is said to be a *minimal cut vector*. If $\mathbf{x}$ is a minimal cut vector, then

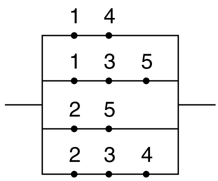

Figure 9.8

the set $C = \{i\colon\ x_i = 0\}$ is called a *minimal cut set*. In other words, a minimal cut set is a minimal set of components whose failure ensures the failure of the system.

Let $C_1, \ldots, C_k$ denote the minimal cut sets of a given system. We define $\beta_j(\mathbf{x})$, the indicator function of the jth minimal cut set, by

$$\beta_j(\mathbf{x}) = \begin{cases} 1, & \text{if at least one component of the } j\text{th minimal} \\ & \text{cut set is functioning} \\ 0, & \text{if all of the components of the } j\text{th minimal} \\ & \text{cut set are not functioning} \end{cases}$$
$$= \max_{i \in C_j} x_i$$

Since a system is not functioning if and only if all the components of at least one minimal cut set are not functioning, it follows that

$$\phi(\mathbf{x}) = \prod_{j=1}^{k} \beta_j(\mathbf{x}) = \prod_{j=1}^{k} \max_{i \in C_j} x_i \tag{9.2}$$

Since $\beta_j(\mathbf{x})$ is a parallel structure function of the components of the jth minimal cut set, Equation (9.2) represents an arbitrary system as a series arrangement of parallel systems.

Example 9.9 The minimal cut sets of the bridge structure shown in Figure 9.9 are {1, 2}, {1, 3, 5}, {2, 3, 4}, and {4, 5}. Hence, from Equation (9.2), we may express $\phi(\mathbf{x})$ by

$$\begin{aligned} \phi(\mathbf{x}) &= \max(x_1, x_2) \max(x_1, x_3, x_5) \max(x_2, x_3, x_4) \max(x_4, x_5) \\ &= [1 - (1 - x_1)(1 - x_2)][1 - (1 - x_1)(1 - x_3)(1 - x_5)] \\ &\quad \times [1 - (1 - x_2)(1 - x_3)(1 - x_4)][1 - (1 - x_4)(1 - x_5)] \end{aligned}$$

This representation of $\phi(\mathbf{x})$ is pictorially expressed as Figure 9.10. ■

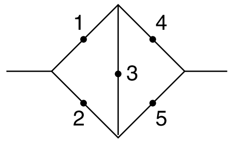

Figure 9.9

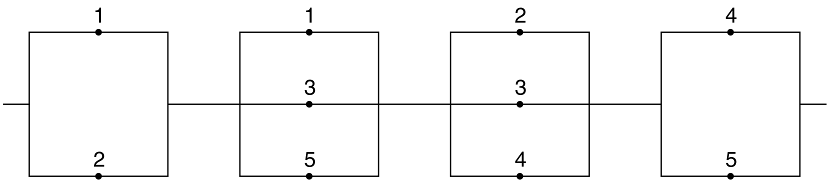

Figure 9.10 Minimal cut representation of the bridge system.

9.3 Reliability of Systems of Independent Components

In this section, we suppose that X_i, the state of the ith component, is a random variable such that

$$P\{X_i = 1\} = p_i = 1 - P\{X_i = 0\}$$

The value p_i, which equals the probability that the ith component is functioning, is called the *reliability* of the ith component. If we define r by

$$r = P\{\phi(\mathbf{X}) = 1\}, \quad \text{where } \mathbf{X} = (X_1, \dots, X_n)$$

then r is called the *reliability* of the system. When the components, that is, the random variables $X_i, i = 1, \dots, n$, are independent, we may express r as a function of the component reliabilities. That is,

$$r = r(\mathbf{p}), \quad \text{where } \mathbf{p} = (p_1, \dots, p_n)$$

The function $r(\mathbf{p})$ is called the *reliability function*. We shall assume throughout the remainder of this chapter that the components are independent.

Example 9.10 (The Series System) The reliability function of the series system of n independent components is given by

$$\begin{aligned} r(\mathbf{p}) &= P\{\phi(\mathbf{X}) = 1\} \\ &= P\{X_i = 1 \text{ for all } i = 1, \dots, n\} \\ &= \prod_{i=1}^{n} p_i \end{aligned}$$ ■

Example 9.11 (The Parallel System) The reliability function of the parallel system of n independent components is given by

$$
\begin{aligned}
r(\mathbf{p}) &= P\{\phi(\mathbf{X}) = 1\} \\
&= P\{X_i = 1 \text{ for some } i = 1, \ldots, n\} \\
&= 1 - P\{X_i = 0 \text{ for all } i = 1, \ldots, n\} \\
&= 1 - \prod_{i=1}^{n}(1 - p_i) \qquad \blacksquare
\end{aligned}
$$

Example 9.12 (The k-out-of-n System with Equal Probabilities) Consider a k-out-of-n system. If $p_i = p$ for all $i = 1, \ldots, n$, then the reliability function is given by

$$
\begin{aligned}
r(p, \ldots, p) &= P\{\phi(\mathbf{X}) = 1\} \\
&= P\left\{\sum_{i=1}^{n} X_i \geqslant k\right\} \\
&= \sum_{i=k}^{n} \binom{n}{i} p^i (1 - p)^{n-i} \qquad \blacksquare
\end{aligned}
$$

Example 9.13 (The Two-out-of-Three System) The reliability function of a two-out-of-three system is given by

$$
\begin{aligned}
r(\mathbf{p}) &= P\{\phi(\mathbf{X}) = 1\} \\
&= P\{\mathbf{X} = (1, 1, 1)\} + P\{\mathbf{X} = (1, 1, 0)\} \\
&\quad + P\{\mathbf{X} = (1, 0, 1)\} + P\{\mathbf{X} = (0, 1, 1)\} \\
&= p_1p_2p_3 + p_1p_2(1 - p_3) + p_1(1 - p_2)p_3 + (1 - p_1)p_2p_3 \\
&= p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3 \qquad \blacksquare
\end{aligned}
$$

Example 9.14 (The Three-out-of-Four System) The reliability function of a three-out-of-four system is given by

$$
\begin{aligned}
r(\mathbf{p}) &= P\{\mathbf{X} = (1, 1, 1, 1)\} + P\{\mathbf{X} = (1, 1, 1, 0)\} + P\{\mathbf{X} = (1, 1, 0, 1)\} \\
&\quad + P\{\mathbf{X} = (1, 0, 1, 1)\} + P\{\mathbf{X} = (0, 1, 1, 1)\} \\
&= p_1p_2p_3p_4 + p_1p_2p_3(1 - p_4) + p_1p_2(1 - p_3)p_4 \\
&\quad + p_1(1 - p_2)p_3p_4 + (1 - p_1)p_2p_3p_4 \\
&= p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4 - 3p_1p_2p_3p_4 \qquad \blacksquare
\end{aligned}
$$

Example 9.15 (A Five-Component System) Consider a five-component system that functions if and only if component 1, component 2, and at least one of the remaining components function. Its reliability function is given by

$$\begin{aligned} r(\mathbf{p}) &= P\{X_1 = 1, X_2 = 1, \max(X_3, X_4, X_5) = 1\} \\ &= P\{X_1 = 1\}P\{X_2 = 1\}P\{\max(X_3, X_4, X_5) = 1\} \\ &= p_1 p_2[1 - (1 - p_3)(1 - p_4)(1 - p_5)] \end{aligned}$$ ■

Since $\phi(\mathbf{X})$ is a 0–1 (that is, a Bernoulli) random variable, we may also compute $r(\mathbf{p})$ by taking its expectation. That is,

$$\begin{aligned} r(\mathbf{p}) &= P\{\phi(\mathbf{X}) = 1\} \\ &= E[\phi(\mathbf{X})] \end{aligned}$$

Example 9.16 (A Four-Component System) A four-component system that functions when both components 1 and 4, and at least one of the other components function has its structure function given by

$$\phi(\mathbf{x}) = x_1 x_4 \max(x_2, x_3)$$

Hence,

$$\begin{aligned} r(\mathbf{p}) &= E[\phi(\mathbf{X})] \\ &= E[X_1 X_4(1 - (1 - X_2)(1 - X_3))] \\ &= p_1 p_4[1 - (1 - p_2)(1 - p_3)] \end{aligned}$$ ■

An important and intuitive property of the reliability function $r(\mathbf{p})$ is given by the following proposition.

Proposition 9.1 If $r(\mathbf{p})$ is the reliability function of a system of independent components, then $r(\mathbf{p})$ is an increasing function of $\mathbf{p}$.

Proof. By conditioning on X_i and using the independence of the components, we obtain

$$\begin{aligned} r(\mathbf{p}) &= E[\phi(\mathbf{X})] \\ &= p_i E[\phi(\mathbf{X}) \mid X_i = 1] + (1 - p_i)E[\phi(\mathbf{X}) \mid X_i = 0] \\ &= p_i E[\phi(1_i, \mathbf{X})] + (1 - p_i)E[\phi(0_i, \mathbf{X})] \end{aligned}$$

where

$$\begin{aligned} (1_i, \mathbf{X}) &= (X_1, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n), \\ (0_i, \mathbf{X}) &= (X_1, \ldots, X_{i-1}, 0, X_{i+1}, \ldots, X_n) \end{aligned}$$

Thus,

$$r(\mathbf{p}) = p_i E[\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X})] + E[\phi(0_i, \mathbf{X})]$$

However, since ϕ is an increasing function, it follows that

$$E[\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X})] \geqslant 0$$

and so the preceding is increasing in p_i for all i. Hence, the result is proven. ■

Let us now consider the following situation: A system consisting of n different components is to be built from a stockpile containing exactly two of each type of component. How should we use the stockpile so as to maximize our probability of attaining a functioning system? In particular, should we build two separate systems, in which case the probability of attaining a functioning one would be

$$\begin{aligned} &P\{\text{at least one of the two systems function}\} \\ &\quad = 1 - P\{\text{neither of the systems function}\} \\ &\quad = 1 - [(1 - r(\mathbf{p}))(1 - r(\mathbf{p}'))] \end{aligned}$$

where $p_i(p_i')$ is the probability that the first (second) number i component functions; or should we build a single system whose ith component functions if at least one of the number i components functions? In this latter case, the probability that the system will function equals

$$r[\mathbf{1} - (\mathbf{1} - \mathbf{p})(\mathbf{1} - \mathbf{p}')]$$

since $1 - (1 - p_i)(1 - p_i')$ equals the probability that the ith component in the single system will function.* We now show that replication at the component level is more effective than replication at the system level.

Theorem 9.1 For any reliability function r and vectors $\mathbf{p}, \mathbf{p}'$,

$$r[\mathbf{1} - (\mathbf{1} - \mathbf{p})(\mathbf{1} - \mathbf{p}')] \geqslant 1 - [1 - r(\mathbf{p})][1 - r(\mathbf{p}')]$$

Proof. Let $X_1, \ldots, X_n, X_1', \ldots, X_n'$ be mutually independent 0–1 random variables with

$$p_i = P\{X_i = 1\}, \quad p_i' = P\{X_i' = 1\}$$

Since $P\{\max(X_i, X_i') = 1\} = 1 - (1 - p_i)(1 - p_i')$, it follows that

$$r[\mathbf{1} - (\mathbf{1} - \mathbf{p})(\mathbf{1} - \mathbf{p}')] = E(\phi[\max(\mathbf{X}, \mathbf{X}')])$$

* Notation: If $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, then $\mathbf{xy} = (x_1 y_1, \ldots, x_n y_n)$. Also, $\max(\mathbf{x}, \mathbf{y}) = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$ and $\min(\mathbf{x}, \mathbf{y}) = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$.

However, by the monotonicity of ϕ, we have that $\phi[\max(\mathbf{X}, \mathbf{X}')]$ is greater than or equal to both $\phi(\mathbf{X})$ and $\phi(\mathbf{X}')$ and hence is at least as large as $\max[\phi(\mathbf{X}), \phi(\mathbf{X}')]$. Hence, from the preceding we have

$$\begin{aligned} r[\mathbf{1} - (\mathbf{1} - \mathbf{p})(\mathbf{1} - \mathbf{p}')] &\geqslant E[\max(\phi(\mathbf{X}), \phi(\mathbf{X}'))] \\ &= P\{\max[\phi(\mathbf{X}), \phi(\mathbf{X}')] = 1\} \\ &= 1 - P\{\phi(\mathbf{X}) = 0, \phi(\mathbf{X}') = 0\} \\ &= 1 - [1 - r(\mathbf{p})][1 - r(\mathbf{p}')] \end{aligned}$$

where the first equality follows from the fact that $\max[\phi(\mathbf{X}), \phi(\mathbf{X}')]$ is a 0–1 random variable and hence its expectation equals the probability that it equals 1. ■

As an illustration of the preceding theorem, suppose that we want to build a series system of two different types of components from a stockpile consisting of two of each of the kinds of components. Suppose that the reliability of each component is $\frac{1}{2}$. If we use the stockpile to build two separate systems, then the probability of attaining a working system is

$$1 - \left(\tfrac{3}{4}\right)^2 = \tfrac{7}{16}$$

while if we build a single system, replicating components, then the probability of attaining a working system is

$$\left(\tfrac{3}{4}\right)^2 = \tfrac{9}{16}$$

Hence, replicating components leads to a higher reliability than replicating systems (as, of course, it must by Theorem 9.1).

9.4 Bounds on the Reliability Function

Consider the bridge system of Example 9.8, which is represented by Figure 9.11. Using the minimal path representation, we have

$$\phi(\mathbf{x}) = 1 - (1 - x_1x_4)(1 - x_1x_3x_5)(1 - x_2x_5)(1 - x_2x_3x_4)$$

Hence,

$$r(\mathbf{p}) = 1 - E[(1 - X_1X_4)(1 - X_1X_3X_5)(1 - X_2X_5)(1 - X_2X_3X_4)]$$

However, since the minimal path sets overlap (that is, they have components in common), the random variables $(1-X_1X_4)$, $(1-X_1X_3X_5)$, $(1-X_2X_5)$, and $(1-X_2X_3X_4)$ are not independent, and thus the expected value of their product is not equal to the product of their expected values. Therefore, in order to

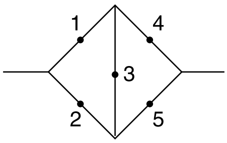

Figure 9.11

compute $r(\mathbf{p})$, we must first multiply the four random variables and then take the expected value. Doing so, using that $X_i^2 = X_i$, we obtain

$$\begin{aligned}
r(\mathbf{p}) &= E[X_1X_4 + X_2X_5 + X_1X_3X_5 + X_2X_3X_4 - X_1X_2X_3X_4 \\
&\quad - X_1X_2X_3X_5 - X_1X_2X_4X_5 - X_1X_3X_4X_5 - X_2X_3X_4X_5 \\
&\quad + 2X_1X_2X_3X_4X_5] \\
&= p_1p_4 + p_2p_5 + p_1p_3p_5 + p_2p_3p_4 - p_1p_2p_3p_4 - p_1p_2p_3p_5 \\
&\quad - p_1p_2p_4p_5 - p_1p_3p_4p_5 - p_2p_3p_4p_5 + 2p_1p_2p_3p_4p_5
\end{aligned}$$

As can be seen by the preceding example, it is often quite tedious to evaluate $r(\mathbf{p})$, and thus it would be useful if we had a simple way of obtaining bounds. We now consider two methods for this.

9.4.1 Method of Inclusion and Exclusion

The following is a well-known formula for the probability of the union of the events $E_1, E_2, \ldots, E_n$:

$$\begin{aligned}
P\left(\bigcup_{i=1}^n E_i\right) &= \sum_{i=1}^n P(E_i) - \sum\sum_{i<j} P(E_iE_j) + \sum\sum\sum_{i<j<k} P(E_iE_jE_k) \\
&\quad - \cdots + (-1)^{n+1} P(E_1E_2\cdots E_n)
\end{aligned} \tag{9.3}$$

A result, not as well known, is the following set of inequalities:

$$\begin{aligned}
P\left(\bigcup_1^n E_i\right) &\leqslant \sum_{i=1}^n P(E_i), \\
P\left(\bigcup_1^n E_i\right) &\geqslant \sum_i P(E_i) - \sum_{i<j} P(E_iE_j), \\
P\left(\bigcup_1^n E_i\right) &\leqslant \sum_i P(E_i) - \sum\sum_{i<j} P(E_iE_j) + \sum\sum\sum_{i<j<k} P(E_iE_jE_k), \\
&\geqslant \cdots \\
&\leqslant \cdots
\end{aligned} \tag{9.4}$$

where the inequality always changes direction as we add an additional term of the expansion of $P(\bigcup_{i=1}^{n} E_i)$.

Equation (9.3) is usually proven by induction on the number of events. However, let us now present another approach that will not only prove Equation (9.3) but also establish Inequalities (9.4).

To begin, define the indicator variables $I_j, j = 1, \ldots, n$, by

$$I_j = \begin{cases} 1, & \text{if } E_j \text{ occurs} \\ 0, & \text{otherwise} \end{cases}$$

Letting

$$N = \sum_{j=1}^{n} I_j$$

then N denotes the number of the $E_j, 1 \leqslant j \leqslant n$, that occur. Also, let

$$I = \begin{cases} 1, & \text{if } N > 0 \\ 0, & \text{if } N = 0 \end{cases}$$

Then, as

$$1 - I = (1 - 1)^N$$

we obtain, upon application of the binomial theorem, that

$$1 - I = \sum_{i=0}^{N} \binom{N}{i} (-1)^i$$

or

$$I = N - \binom{N}{2} + \binom{N}{3} - \cdots \pm \binom{N}{N} \tag{9.5}$$

We now make use of the following combinatorial identity (which is easily established by induction on i):

$$\binom{n}{i} - \binom{n}{i+1} + \cdots \pm \binom{n}{n} = \binom{n-1}{i-1} \geqslant 0, \quad i \leqslant n$$

The preceding thus implies that

$$\binom{N}{i} - \binom{N}{i+1} + \cdots \pm \binom{N}{N} \geqslant 0 \tag{9.6}$$

From Equations (9.5) and (9.6) we obtain

$$\begin{aligned}
&I \leqslant N, && \text{by letting } i = 2 \text{ in (9.6)} \\
&I \geqslant N - \binom{N}{2}, && \text{by letting } i = 3 \text{ in (9.6)} \\
&I \leqslant N - \binom{N}{2} + \binom{N}{3}, && \\
&\quad\vdots &&
\end{aligned} \tag{9.7}$$

and so on. Now, since $N \leqslant n$ and $\binom{m}{i} = 0$ whenever $i > m$, we can rewrite Equation (9.5) as

$$I = \sum_{i=1}^{n} \binom{N}{i} (-1)^{i+1} \tag{9.8}$$

Equation (9.3) and Inequalities (9.4) now follow upon taking expectations of (9.7) and (9.8). This is the case since

$$E[I] = P\{N > 0\} = P\{\text{at least one of the } E_j \text{ occurs}\} = P\left(\bigcup_1^n E_j\right),$$

$$E[N] = E\left[\sum_{j=1}^{n} I_j\right] = \sum_{j=1}^{n} P(E_j)$$

Also,

$$\begin{aligned}
E\left[\binom{N}{2}\right] &= E[\text{number of pairs of the } E_j \text{ that occur}] \\
&= E\left[\sum_{i<j}\sum I_i I_j\right] \\
&= \sum_{i<j}\sum P(E_i E_j)
\end{aligned}$$

and, in general

$$\begin{aligned}
E\left[\binom{N}{i}\right] &= E[\text{number of sets of size } i \text{ that occur}] \\
&= E\left[\sum_{j_1 < j_2 < \cdots < j_i}\sum I_{j_1} I_{j_2} \cdots I_{j_i}\right] \\
&= \sum_{j_1 < j_2 < \cdots < j_i}\sum P(E_{j_1} E_{j_2} \cdots E_{j_i})
\end{aligned}$$

The bounds expressed in (9.4) are commonly called the *inclusion–exclusion bounds*. To apply them in order to obtain bounds on the reliability function, let $A_1 A_2, \dots, A_s$ denote the minimal path sets of a given structure ϕ, and define the events $E_1, E_2, \dots, E_s$ by

$$E_i = \{\text{all components in } A_i \text{ function}\}$$

Now, since the system functions if and only if at least one of the events E_i occur, we have

$$r(\mathbf{p}) = P\left(\bigcup_1^s E_i\right)$$

Applying (9.4) yields the desired bounds on $r(\mathbf{p})$. The terms in the summation are computed thusly:

$$\begin{aligned}
P(E_i) &= \prod_{l \in A_i} p_l, \\
P(E_i E_j) &= \prod_{l \in A_i \cup A_j} p_l, \\
P(E_i E_j E_k) &= \prod_{l \in A_i \cup A_j \cup A_k} p_l
\end{aligned}$$

and so forth for intersections of more than three of the events. (The preceding follows since, for instance, in order for the event $E_i E_j$ to occur, all of the components in A_i and all of them in A_j must function; or, in other words, all components in $A_i \cup A_j$ must function.)

When the p_is are small the probabilities of the intersection of many of the events E_i should be quite small and the convergence should be relatively rapid.

Example 9.17 Consider the bridge structure with identical component probabilities. That is, take p_i to equal p for all i. Letting $A_1 = \{1, 4\}, A_2 = \{1, 3, 5\}, A_3 = \{2, 5\}$, and $A_4 = \{2, 3, 4\}$ denote the minimal path sets, we have

$$\begin{aligned}
P(E_1) &= P(E_3) = p^2, \\
P(E_2) &= P(E_4) = p^3
\end{aligned}$$

Also, because exactly five of the six $= \binom{4}{2}$ unions of A_i and A_j contain four components (the exception being $A_2 \cup A_4$, which contains all five components), we have

$$\begin{aligned}
&P(E_1 E_2) = P(E_1 E_3) = P(E_1 E_4) = P(E_2 E_3) = P(E_3 E_4) = p^4, \\
&P(E_2 E_4) = p^5
\end{aligned}$$

Hence, the first two inclusion–exclusion bounds yield

$$2(p^2 + p^3) - 5p^4 - p^5 \leqslant r(p) \leqslant 2(p^2 + p^3)$$

where $r(p) = r(p, p, p, p, p)$. For instance, when $p = 0.2$, we have

$$0.08768 \leqslant r(0.2) \leqslant 0.09600$$

and, when $p = 0.1$,

$$0.02149 \leqslant r(0.1) \leqslant 0.02200$$

■

Just as we can define events in terms of the minimal path sets whose union is the event that the system functions, so can we define events in terms of the minimal cut sets whose union is the event that the system fails. Let $C_1, C_2, \ldots, C_r$ denote the minimal cut sets and define the events $F_1, \ldots, F_r$ by

$$F_i = \{\text{all components in } C_i \text{ are failed}\}$$

Now, because the system is failed if and only if all of the components of at least one minimal cut set are failed, we have

$$\begin{aligned}
1 - r(\mathbf{p}) &= P\left(\bigcup_1^r F_i\right), \\
1 - r(\mathbf{p}) &\leqslant \sum_i P(F_i), \\
1 - r(\mathbf{p}) &\geqslant \sum_i P(F_i) - \sum_{i<j}\sum P(F_iF_j), \\
1 - r(\mathbf{p}) &\leqslant \sum_i P(F_i) - \sum_{i<j}\sum P(F_iF_j) + \sum_{i<j<k}\sum\sum P(F_iF_jF_k),
\end{aligned}$$

and so on. As

$$\begin{aligned}
P(F_i) &= \prod_{l \in C_i} (1 - p_l), \\
P(F_iF_j) &= \prod_{l \in C_i \cup C_j} (1 - p_l), \\
P(F_iF_jF_k) &= \prod_{l \in C_i \cup C_j \cup C_k} (1 - p_l)
\end{aligned}$$

the convergence should be relatively rapid when the p_is are large.

Example 9.18 (A Random Graph) Let us recall from Section 3.6.2 that a graph consists of a set N of nodes and a set A of pairs of nodes, called arcs. For any two nodes i and j we say that the sequence of arcs $(i, i_1)(i_1, i_2), \ldots, (i_k, j)$ constitutes an i–j path. If there is an i–j path between all the $\binom{n}{2}$ pairs of nodes i and j, $i \neq j$, then the graph is said to be connected. If we think of the nodes of a graph as representing geographical locations and the arcs as representing direct communication links between the nodes, then the graph will be connected if any two nodes can communicate with each other—if not directly, then at least through the use of intermediary nodes.

A graph can always be subdivided into nonoverlapping connected subgraphs called components. For instance, the graph in Figure 9.12 with nodes $N = \{1, 2, 3, 4, 5, 6\}$ and arcs $A = \{(1, 2), (1, 3), (2, 3), (4, 5)\}$ consists of three components (a graph consisting of a single node is considered to be connected).

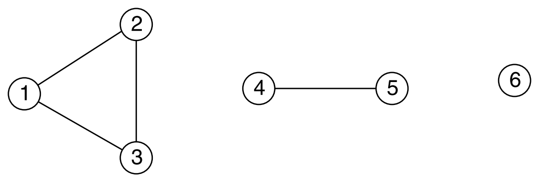

Figure 9.12

Consider now the random graph having nodes $1, 2, \ldots, n$, which is such that there is an arc from node i to node j with probability P_{ij}. Assume in addition that the occurrences of these arcs constitute independent events. That is, assume that the $\binom{n}{2}$ random variables $X_{ij}, i \neq j$, are independent where

$$X_{ij} = \begin{cases} 1, & \text{if } (i, j) \text{ is an arc} \\ 0, & \text{otherwise} \end{cases}$$

We are interested in the probability that this graph will be connected.

We can think of the preceding as being a reliability system of $\binom{n}{2}$ components—each component corresponding to a potential arc. The component is said to work if the corresponding arc is indeed an arc of the network, and the system is said to work if the corresponding graph is connected. As the addition of an arc to a connected graph cannot disconnect the graph, it follows that the structure so defined is monotone.

Let us start by determining the minimal path and minimal cut sets. It is easy to see that a graph will not be connected if and only if the set of nodes can be partitioned into two nonempty subsets X and X^c in such a way that there is no arc connecting a node from X with one from X^c. For instance, if there are six nodes and if there are no arcs connecting any of the nodes 1, 2, 3, 4 with either 5 or 6, then clearly the graph will not be connected. Thus, we see that any partition of the nodes into two nonempty subsets X and X^c corresponds to the minimal cut set defined by

$$\{(i,j):\ i \in X, j \in X^c\}$$

As there are $2^{n-1}-1$ such partitions (there are 2^n-2 ways of choosing a nonempty proper subset X and, as the partition X, X^c is the same as X^c, X, we must divide by 2) there are therefore this number of minimal cut sets.

To determine the minimal path sets, we must characterize a minimal set of arcs that results in a connected graph. The graph in Figure 9.13 is connected but it would remain connected if any one of the arcs from the cycle shown in Figure 9.14 were removed. In fact it is not difficult to see that the minimal path sets are exactly those sets of arcs that result in a graph being connected but not having any cycles (a cycle being a path from a node to itself). Such sets of arcs are called *spanning trees* (Figure 9.15). It is easily verified that any spanning tree contains exactly $n - 1$ arcs, and it is a famous result in graph theory (due to Cayley) that there are exactly n^{n-2} of these minimal path sets.

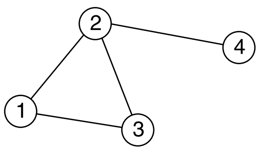

Figure 9.13

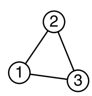

Figure 9.14

Figure 9.15 Two spanning trees (minimal path sets) when $n = 4$.

Because of the large number of minimal path and minimal cut sets (n^{n-2} and $2^{n-1} - 1$, respectively), it is difficult to obtain any useful bounds without making further restrictions. So, let us assume that all the P_{ij} equal the common value p. That is, we suppose that each of the possible arcs exists, independently, with the same probability p. We shall start by deriving a recursive formula for the probability that the graph is connected, which is computationally useful when n is not too large, and then we shall present an asymptotic formula for this probability when n is large.

Let us denote by P_n the probability that the random graph having n nodes is connected. To derive a recursive formula for P_n we first concentrate attention on a single node—say, node 1—and try to determine the probability that node 1 will be part of a component of size k in the resultant graph. Now, for a given set of $k-1$ other nodes these nodes along with node 1 will form a component if

(i) there are no arcs connecting any of these k nodes with any of the remaining $n-k$ nodes;
(ii) the random graph restricted to these k nodes (and $\binom{k}{2}$ potential arcs—each independently appearing with probability p) is connected.

The probability that (i) and (ii) both occur is

$$q^{k(n-k)} P_k$$

where $q = 1-p$. As there are $\binom{n-1}{k-1}$ ways of choosing $k-1$ other nodes (to form along with node 1 a component of size k) we see that

$$\begin{aligned} &P\{\text{node 1 is part of a component of size } k\} \\ &\quad = \binom{n-1}{k-1} q^{k(n-k)} P_k, \qquad k = 1, 2, \ldots, n \end{aligned}$$

Now, since the sum of the foregoing probabilities as k ranges from 1 through n clearly must equal 1, and as the graph is connected if and only if node 1 is part of a component of size n, we see that

$$P_n = 1 - \sum_{k=1}^{n-1} \binom{n-1}{k-1} q^{k(n-k)} P_k, \qquad n = 2, 3, \ldots \tag{9.9}$$

Starting with $P_1 = 1, P_2 = p$, Equation (9.9) can be used to determine P_n recursively when n is not too large. It is particularly suited for numerical computation.

To determine an asymptotic formula for P_n when n is large, first note from Equation (9.9) that since $P_k \leqslant 1$, we have

$$1 - P_n \leqslant \sum_{k=1}^{n-1} \binom{n-1}{k-1} q^{k(n-k)}$$

As it can be shown that for $q < 1$ and n sufficiently large,

$$\sum_{k=1}^{n-1} \binom{n-1}{k-1} q^{k(n-k)} \leqslant (n+1) q^{n-1}$$

we have that for n large

$$1 - P_n \leqslant (n+1) q^{n-1} \tag{9.10}$$

To obtain a bound in the other direction, we concentrate our attention on a particular type of minimal cut set—namely, those that separate one node from all others in the graph. Specifically, define the minimal cut set C_i as

$$C_i = \{(i, j):\ j \neq i\}$$

and define F_i to be the event that all arcs in C_i are not working (and thus, node i is isolated from the other nodes). Now,

$$1 - P_n = P(\text{graph is not connected}) \geqslant P\left(\bigcup_i F_i\right)$$

since, if any of the events F_i occur, then the graph will be disconnected. By the inclusion–exclusion bounds, we have

$$P\left(\bigcup_i F_i\right) \geqslant \sum_i P(F_i) - \sum_{i<j}\sum P(F_i F_j)$$

As $P(F_i)$ and $P(F_i F_j)$ are just the respective probabilities that a given set of $n - 1$ arcs and a given set of $2n - 3$ arcs are not in the graph (why?), it follows that

$$\begin{aligned} P(F_i) &= q^{n-1}, \\ P(F_i F_j) &= q^{2n-3}, \quad i \neq j \end{aligned}$$

and so

$$1 - P_n \geqslant nq^{n-1} - \binom{n}{2} q^{2n-3}$$

Combining this with Equation (9.10) yields that for n sufficiently large,

$$nq^{n-1} - \binom{n}{2} q^{2n-3} \leqslant 1 - P_n \leqslant (n + 1)q^{n-1}$$

and as

$$\binom{n}{2} \frac{q^{2n-3}}{nq^{n-1}} \to 0$$

as $n \to \infty$, we see that, for large n,

$$1 - P_n \approx nq^{n-1}$$

Thus, for instance, when $n = 20$ and $p = \frac{1}{2}$, the probability that the random graph will be connected is approximately given by

$$P_{20} \approx 1 - 20\left(\tfrac{1}{2}\right)^{19} = 0.99998 \qquad \blacksquare$$

9.4.2 Second Method for Obtaining Bounds on r(p)

Our second approach to obtaining bounds on $r(\mathbf{p})$ is based on expressing the desired probability as the probability of the intersection of events. To do so, let $A_1, A_2, \ldots, A_s$ denote the minimal path sets as before, and define the events, $D_i, i = 1, \ldots, s$ by

$$D_i = \{\text{at least one component in } A_i \text{ has failed}\}$$

Now since the system will have failed if and only if at least one component in each of the minimal path sets has failed we have

$$\begin{aligned} 1 - r(\mathbf{p}) &= P(D_1 D_2 \cdots D_s) \\ &= P(D_1)P(D_2 \mid D_1) \cdots P(D_s \mid D_1 D_2 \cdots D_{s-1}) \end{aligned} \tag{9.11}$$

Now it is quite intuitive that the information that at least one component of A_1 is down can only increase the probability that at least one component of A_2 is down (or else leave the probability unchanged if A_1 and A_2 do not overlap). Hence, intuitively

$$P(D_2 \mid D_1) \geqslant P(D_2)$$

To prove this inequality, we write

$$P(D_2) = P(D_2 \mid D_1)P(D_1) + P(D_2 \mid D_1^c)(1 - P(D_1)) \tag{9.12}$$

and note that

$$\begin{aligned} P(D_2 \mid D_1^c) &= P\{\text{at least one failed in } A_2 \mid \text{all functioning in } A_1\} \\ &= 1 - \prod_{\substack{j \in A_2 \\ j \notin A_1}} p_j \\ &\leqslant 1 - \prod_{j \in A_2} p_j \\ &= P(D_2) \end{aligned}$$

Hence, from Equation (9.12) we see that

$$P(D_2) \leqslant P(D_2 \mid D_1)P(D_1) + P(D_2)(1 - P(D_1))$$

or

$$P(D_2 \mid D_1) \geqslant P(D_2)$$

By the same reasoning, it also follows that

$$P(D_i \mid D_1 \cdots D_{i-1}) \geqslant P(D_i)$$

and so from Equation (9.11) we have

$$1 - r(\mathbf{p}) \geqslant \prod_i P(D_i)$$

or, equivalently,

$$r(\mathbf{p}) \leqslant 1 - \prod_i \left(1 - \prod_{j \in A_i} p_j\right)$$

To obtain a bound in the other direction, let $C_1, \ldots, C_r$ denote the minimal cut sets and define the events $U_1, \ldots, U_r$ by

$$U_i = \{\text{at least one component in } C_i \text{ is functioning}\}$$

Then, since the system will function if and only if all of the events U_i occur, we have

$$\begin{aligned} r(\mathbf{p}) &= P(U_1 U_2 \cdots U_r) \\ &= P(U_1)P(U_2 \mid U_1) \cdots P(U_r \mid U_1 \cdots\ U_{r-1}) \\ &\geqslant \prod_i P(U_i) \end{aligned}$$

where the last inequality is established in exactly the same manner as for the D_i. Hence,

$$r(\mathbf{p}) \geqslant \prod_i \left[1 - \prod_{j \in C_i} (1 - p_j)\right]$$

and we thus have the following bounds for the reliability function:

$$\prod_i \left[1 - \prod_{j \in C_i} (1 - p_j)\right] \leqslant r(\mathbf{p}) \leqslant 1 - \prod_i \left(1 - \prod_{j \in A_i} p_j\right) \tag{9.13}$$

It is to be expected that the upper bound should be close to the actual $r(\mathbf{p})$ if there is not too much overlap in the minimal path sets, and the lower bound to be close if there is not too much overlap in the minimal cut sets.

Example 9.19 For the three-out-of-four system the minimal path sets are $A_1 = \{1,2,3\}$, $A_2 = \{1,2,4\}$, $A_3 = \{1,3,4\}$, and $A_4 = \{2,3,4\}$; and the minimal cut sets are $C_1 = \{1,2\}$, $C_2 = \{1,3\}$, $C_3 = \{1,4\}$, $C_4 = \{2,3\}$, $C_5 = \{2,4\}$, and $C_6 = \{3,4\}$. Hence, by Equation (9.13) we have

$$\begin{aligned}&(1 - q_1q_2)(1 - q_1q_3)(1 - q_1q_4)(1 - q_2q_3)(1 - q_2q_4)(1 - q_3q_4)\\ &\quad\leqslant r(\mathbf{p}) \leqslant 1 - (1 - p_1p_2p_3)(1 - p_1p_2p_4)(1 - p_1p_3p_4)(1 - p_2p_3p_4)\end{aligned}$$

where $q_i \equiv 1 - p_i$. For instance, if $p_i = \frac{1}{2}$ for all i, then the preceding yields

$$0.18 \leqslant r\left(\tfrac{1}{2}, \ldots, \tfrac{1}{2}\right) \leqslant 0.59$$

The exact value for this structure is easily computed to be

$$r\left(\tfrac{1}{2}, \ldots, \tfrac{1}{2}\right) = \tfrac{5}{16} = 0.31 \qquad \blacksquare$$

9.5 System Life as a Function of Component Lives

For a random variable having distribution function G, we define $\bar{G}(a) \equiv 1 - G(a)$ to be the probability that the random variable is greater than a.

Consider a system in which the ith component functions for a random length of time having distribution F_i and then fails. Once failed it remains in that state forever. Assuming that the individual component lifetimes are independent, how can we express the distribution of system lifetime as a function of the system reliability function $r(\mathbf{p})$ and the individual component lifetime distributions F_i, $i = 1, \ldots, n$?

To answer this we first note that the system will function for a length of time t or greater if and only if it is still functioning at time t. That is, letting F denote the distribution of system lifetime, we have

$$\begin{aligned}\bar{F}(t) &= P\{\text{system life} > t\}\\ &= P\{\text{system is functioning at time } t\}\end{aligned}$$

But, by the definition of $r(\mathbf{p})$ we have

$$P\{\text{system is functioning at time } t\} = r(P_1(t), \ldots, P_n(t))$$

where

$$\begin{aligned}P_i(t) &= P\{\text{component } i \text{ is functioning at } t\}\\ &= P\{\text{lifetime of } i > t\}\\ &= \bar{F}_i(t)\end{aligned}$$

Hence, we see that

$$\bar{F}(t) = r(\bar{F}_1(t), \ldots, \bar{F}_n(t)) \tag{9.14}$$

Example 9.20 In a series system, $r(\mathbf{p}) = \prod_1^n p_i$ and so from Equation (9.14)

$$\bar{F}(t) = \prod_1^n \bar{F}_i(t)$$

which is, of course, quite obvious since for a series system the system life is equal to the minimum of the component lives and so will be greater than t if and only if all component lives are greater than t. ■

Example 9.21 In a parallel system $r(\mathbf{p}) = 1 - \prod_1^n (1 - p_i)$ and so

$$\bar{F}(t) = 1 - \prod_1^n F_i(t)$$

The preceding is also easily derived by noting that, in the case of a parallel system, the system life is equal to the maximum of the component lives. ■

For a continuous distribution G, we define $\lambda(t)$, the *failure rate function* of G, by

$$\lambda(t) = \frac{g(t)}{\bar{G}(t)}$$

where $g(t) = d/dtG(t)$. In Section 5.2.2, it is shown that if G is the distribution of the lifetime of an item, then $\lambda(t)$ represents the probability intensity that a t-year-old item will fail. We say that G is an *increasing failure rate* (IFR) distribution if $\lambda(t)$ is an increasing function of t. Similarly, we say that G is a *decreasing failure rate* (DFR) distribution if $\lambda(t)$ is a decreasing function of t.

Example 9.22 (The Weibull Distribution) A random variable is said to have the *Weibull* distribution if its distribution is given, for some $\lambda > 0, \alpha > 0$, by

$$G(t) = 1 - e^{-(\lambda t)^\alpha}, \qquad t \geqslant 0$$

The failure rate function for a Weibull distribution equals

$$\lambda(t) = \frac{e^{-(\lambda t)^\alpha}\alpha(\lambda t)^{\alpha-1}\lambda}{e^{-(\lambda t)^\alpha}} = \alpha\lambda(\lambda t)^{\alpha-1}$$

Thus, the Weibull distribution is IFR when $\alpha \geqslant 1$, and DFR when $0 < \alpha \leqslant 1$; when $\alpha = 1, G(t) = 1 - e^{-\lambda t}$, the exponential distribution, which is both IFR and DFR. ■

Example 9.23 (The Gamma Distribution) A random variable is said to have a *gamma* distribution if its density is given, for some $\lambda > 0, \alpha > 0$, by

$$g(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{\alpha-1}}{\Gamma(\alpha)} \qquad \text{for } t \geqslant 0$$

where

$$\Gamma(\alpha) \equiv \int_0^\infty e^{-t} t^{\alpha-1}\, dt$$

For the gamma distribution,

$$\begin{aligned}\frac{1}{\lambda(t)} = \frac{\bar{G}(t)}{g(t)} &= \frac{\int_t^\infty \lambda e^{-\lambda x}(\lambda x)^{\alpha-1} dx}{\lambda e^{-\lambda t}(\lambda t)^{\alpha-1}} \\ &= \int_t^\infty e^{-\lambda(x-t)}\left(\frac{x}{t}\right)^{\alpha-1} dx\end{aligned}$$

With the change of variables $u = x - t$, we obtain

$$\frac{1}{\lambda(t)} = \int_0^\infty e^{-\lambda u}\left(1 + \frac{u}{t}\right)^{\alpha-1} du$$

Hence, G is IFR when $\alpha \geqslant 1$ and is DFR when $0 < \alpha \leqslant 1$. ∎

Suppose that the lifetime distribution of each component in a monotone system is IFR. Does this imply that the system lifetime is also IFR? To answer this, let us at first suppose that each component has the same lifetime distribution, which we denote by G. That is, $F_i(t) = G(t), i = 1, \ldots, n$. To determine whether the system lifetime is IFR, we must compute $\lambda_F(t)$, the failure rate function of F. Now, by definition,

$$\begin{aligned}\lambda_F(t) &= \frac{(d/dt)F(t)}{\bar{F}(t)} \\ &= \frac{(d/dt)[1 - r(\bar{G}(t))]}{r(\bar{G}(t))}\end{aligned}$$

where

$$r(\bar{G}(t)) \equiv r(\bar{G}(t), \ldots, \bar{G}(t))$$

Hence,

$$\lambda_F(t) = \frac{r'(\bar{G}(t))}{r(\bar{G}(t))} G'(t)$$

$$
\begin{aligned}
&= \frac{\bar{G}(t) r'(\bar{G}(t))}{r(\bar{G}(t))} \frac{G'(t)}{\bar{G}(t)} \\
&= \lambda_G(t) \frac{p r'(p)}{r(p)} \bigg|_{p=\bar{G}(t)}
\end{aligned}
\tag{9.15}
$$

Since $\bar{G}(t)$ is a decreasing function of t, it follows from Equation (9.15) that *if each component of a coherent system has the same IFR lifetime distribution, then the distribution of system lifetime will be IFR if $pr'(p)/r(p)$ is a decreasing function of p.*

Example 9.24 (The k-out-of-n System with Identical Components) Consider the k-out-of-n system, which will function if and only if k or more components function. When each component has the same probability p of functioning, the number of functioning components will have a binomial distribution with parameters n and p. Hence,

$$
r(p) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}
$$

which, by continual integration by parts, can be shown to be equal to

$$
r(p) = \frac{n!}{(k-1)!(n-k)!} \int_0^p x^{k-1} (1-x)^{n-k} dx
$$

Upon differentiation, we obtain

$$
r'(p) = \frac{n!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k}
$$

Therefore,

$$
\begin{aligned}
\frac{p r'(p)}{r(p)} &= \left[\frac{r(p)}{p r'(p)} \right]^{-1} \\
&= \left[\frac{1}{p} \int_0^p \left(\frac{x}{p} \right)^{k-1} \left(\frac{1-x}{1-p} \right)^{n-k} dx \right]^{-1}
\end{aligned}
$$

Letting $y = x/p$ yields

$$
\frac{p r'(p)}{r(p)} = \left[\int_0^1 y^{k-1} \left(\frac{1-yp}{1-p} \right)^{n-k} dy \right]^{-1}
$$

Since $(1 - yp)/(1 - p)$ is increasing in p, it follows that $pr'(p)/r(p)$ is decreasing in p. Thus, if a k-out-of-n system is composed of independent, like components having an increasing failure rate, the system itself has an increasing failure rate. ■

It turns out, however, that for a k-out-of-n system, in which the independent components have different IFR lifetime distributions, the system lifetime need not be IFR. Consider the following example of a two-out-of-two (that is, a parallel) system.

Example 9.25 (A Parallel System That Is Not IFR) The life distribution of a parallel system of two independent components, the ith component having an exponential distribution with mean $1/i, i = 1, 2$, is given by

$$\begin{aligned}\bar{F}(t) &= 1 - (1 - e^{-t})(1 - e^{-2t}) \\ &= e^{-2t} + e^{-t} - e^{-3t}\end{aligned}$$

Therefore,

$$\begin{aligned}\lambda(t) &= \frac{f(t)}{\bar{F}(t)} \\ &= \frac{2e^{-2t} + e^{-t} - 3e^{-3t}}{e^{-2t} + e^{-t} - e^{-3t}}\end{aligned}$$

It easily follows upon differentiation that the sign of $\lambda'(t)$ is determined by $e^{-5t} - e^{-3t} + 3e^{-4t}$, which is positive for small values and negative for large values of t. Therefore, $\lambda(t)$ is initially strictly increasing, and then strictly decreasing. Hence, F is not IFR. ■

Remark The result of the preceding example is quite surprising at first glance. To obtain a better feel for it we need the concept of a mixture of distribution functions. The distribution function G is said to be a *mixture* of the distributions G_1 and G_2 if for some $p, 0 < p < 1$,

$$G(x) = pG_1(x) + (1 - p)G_2(x) \tag{9.16}$$

Mixtures occur when we sample from a population made up of two distinct groups. For example, suppose we have a stockpile of items of which the fraction p are type 1 and the fraction $1-p$ are type 2. Suppose that the lifetime distribution of type 1 items is G_1 and of type 2 items is G_2. If we choose an item at random from the stockpile, then its life distribution is as given by Equation (9.16).

Consider now a mixture of two exponential distributions having rates λ_1 and λ_2 where $\lambda_1 < \lambda_2$. We are interested in determining whether or not this mixture distribution is IFR. To do so, we note that if the item selected has survived up to time t, then its distribution of remaining life is still a mixture of the two exponential distributions. This is so since its remaining life will still be exponential with rate λ_1 if it is type 1 or with rate λ_2 if it is a type 2 item. However, the probability that it is a type 1 item is no longer the (prior) probability p but is now a conditional probability given that it has survived to time t. In fact, its probability of being a type 1 is

$$
\begin{aligned}
P\{\text{type } 1 \mid \text{life} > t\} &= \frac{P\{\text{type } 1, \text{life} > t\}}{P\{\text{life} > t\}} \\
&= \frac{pe^{-\lambda_1 t}}{pe^{-\lambda_1 t} + (1-p)e^{-\lambda_2 t}}
\end{aligned}
$$

As the preceding is increasing in t, it follows that the larger t is, the more likely it is that the item in use is a type 1 (the better one, since $\lambda_1 < \lambda_2$). Hence, the older the item is, the less likely it is to fail, and thus the mixture of exponentials far from being IFR is, in fact, DFR.

Now, let us return to the parallel system of two exponential components having respective rates λ_1 and λ_2. The lifetime of such a system can be expressed as the sum of two independent random variables, namely,

$$
\text{system life} = \text{Exp}(\lambda_1 + \lambda_2) + \begin{cases} \text{Exp}(\lambda_1) \text{ with probability } \dfrac{\lambda_2}{\lambda_1 + \lambda_2} \\ \text{Exp}(\lambda_2) \text{ with probability } \dfrac{\lambda_1}{\lambda_1 + \lambda_2} \end{cases}
$$

The first random variable whose distribution is exponential with rate $\lambda_1 + \lambda_2$ represents the time until one of the components fails, and the second, which is a mixture of exponentials, is the additional time until the other component fails. (Why are these two random variables independent?)

Now, given that the system has survived a time t, it is very unlikely when t is large that both components are still functioning, but instead it is far more likely that one of the components has failed. Hence, for large t, the distribution of remaining life is basically a mixture of two exponentials—and so as t becomes even larger its failure rate should decrease (as indeed occurs). ■

Recall that the failure rate function of a distribution $F(t)$ having density $f(t) = F'(t)$ is defined by

$$
\lambda(t) = \frac{f(t)}{1 - F(t)}
$$

By integrating both sides, we obtain

$$
\begin{aligned}
\int_0^t \lambda(s)\, ds &= \int_0^t \frac{f(s)}{1 - F(s)} ds \\
&= -\log \bar{F}(t)
\end{aligned}
$$

Hence,

$$\bar{F}(t) = e^{-\Lambda(t)} \tag{9.17}$$

where

$$\Lambda(t) = \int_0^t \lambda(s)\, ds$$

The function $\Lambda(t)$ is called the *hazard function* of the distribution F.

Definition 9.1 A distribution F is said to have *increasing failure on the average* (IFRA) if

$$\frac{\Lambda(t)}{t} = \frac{\int_0^t \lambda(s)\, ds}{t} \tag{9.18}$$

increases in t for $t \geqslant 0$.

In other words, Equation (9.18) states that the average failure rate up to time t increases as t increases. It is not difficult to show that if F is IFR, then F is IFRA; but the reverse need not be true.

Note that F is IFRA if $\Lambda(s)/s \leqslant \Lambda(t)/t$ whenever $0 \leqslant s \leqslant t$, which is equivalent to

$$\frac{\Lambda(\alpha t)}{\alpha t} \leqslant \frac{\Lambda(t)}{t} \quad \text{for } 0 \leqslant \alpha \leqslant 1, \text{ all } t \geqslant 0$$

But by Equation (9.17) we see that $\Lambda(t) = -\log \bar{F}(t)$, and so the preceding is equivalent to

$$-\log \bar{F}(\alpha t) \leqslant -\alpha \log \bar{F}(t)$$

or equivalently,

$$\log \bar{F}(\alpha t) \geqslant \log \bar{F}^{\alpha}(t)$$

which, since $\log x$ is a monotone function of x, shows that F is *IFRA if and only if*

$$\bar{F}(\alpha t) \geqslant \bar{F}^{\alpha}(t) \qquad \text{for } 0 \leqslant \alpha \leqslant 1, \ \text{all } t \geqslant 0 \tag{9.19}$$

For a vector $\mathbf{p} = (p_1, \ldots, p_n)$ we define $\mathbf{p}^{\alpha} = (p_1^{\alpha}, \ldots, p_n^{\alpha})$. We shall need the following proposition.

Proposition 9.2 Any reliability function $r(\mathbf{p})$ satisfies

$$r(\mathbf{p}^\alpha) \geqslant [r(\mathbf{p})]^\alpha, \qquad 0 \leqslant \alpha \leqslant 1$$

Proof. We prove this by induction on n, the number of components in the system. If $n = 1$, then either $r(p) \equiv 0, r(p) \equiv 1$, or $r(p) \equiv p$. Hence, the proposition follows in this case.

Assume that Proposition 9.2 is valid for all monotone systems of $n-1$ components and consider a system of n components having structure function ϕ. By conditioning upon whether or not the nth component is functioning, we obtain

$$r(\mathbf{p}^\alpha) = p_n^\alpha r(1_n, \mathbf{p}^\alpha) + (1 - p_n^\alpha) r(0_n, \mathbf{p}^\alpha) \tag{9.20}$$

Now consider a system of components 1 through $n-1$ having a structure function $\phi_1(\mathbf{x}) = \phi(1_n, \mathbf{x})$. The reliability function for this system is given by $r_1(\mathbf{p}) = r(1_n, \mathbf{p})$; hence, from the induction assumption (valid for all monotone systems of $n-1$ components), we have

$$r(1_n, \mathbf{p}^\alpha) \geqslant [r(1_n, \mathbf{p})]^\alpha$$

Similarly, by considering the system of components 1 through $n-1$ and structure function $\phi_0(\mathbf{x}) = \phi(0_n, \mathbf{x})$, we obtain

$$r(0_n, \mathbf{p}^\alpha) \geqslant [r(0_n, \mathbf{p})]^\alpha$$

Thus, from Equation (9.20), we obtain

$$r(\mathbf{p}^\alpha) \geqslant p_n^\alpha [r(1_n, \mathbf{p})]^\alpha + (1 - p_n^\alpha)[r(0_n, \mathbf{p})]^\alpha$$

which, by using the lemma to follow (with $\lambda = p_n$, $x = r(1_n, \mathbf{p})$, $y = r(0_n, \mathbf{p})$), implies that

$$\begin{aligned} r(\mathbf{p}^\alpha) &\geqslant [p_n r(1_n, \mathbf{p}) + (1 - p_n) r(0_n, \mathbf{p})]^\alpha \\ &= [r(\mathbf{p})]^\alpha \end{aligned}$$

which proves the result. ■

Lemma 9.3 If $0 \leqslant \alpha \leqslant 1, 0 \leqslant \lambda \leqslant 1$, then

$$h(y) = \lambda^\alpha x^\alpha + (1 - \lambda^\alpha) y^\alpha - (\lambda x + (1 - \lambda) y)^\alpha \geqslant 0$$

for all $0 \leqslant y \leqslant x$.

Proof. The proof is left as an exercise. ■

We are now ready to prove the following important theorem.

Theorem 9.2 For a monotone system of independent components, if each component has an IFRA lifetime distribution, then the distribution of system lifetime is itself IFRA.

Proof. The distribution of system lifetime F is given by

$$\bar{F}(\alpha t) = r(\bar{F}_1(\alpha t), \ldots, \bar{F}_n(\alpha t))$$

Hence, since r is a monotone function, and since each of the component distributions $\bar{F}_i$ is IFRA, we obtain from Equation (9.19)

$$\begin{aligned}\bar{F}(\alpha t) &\geqslant r(\bar{F}_1^{\alpha}(t), \ldots, \bar{F}_n^{\alpha}(t)) \\ &\geqslant [r(\bar{F}_1(t), \ldots, \bar{F}_n(t))]^{\alpha} \\ &= \bar{F}^{\alpha}(t)\end{aligned}$$

which by Equation (9.19) proves the theorem. The last inequality followed, of course, from Proposition 9.2. ■

9.6 Expected System Lifetime

In this section, we show how the mean lifetime of a system can be determined, at least in theory, from a knowledge of the reliability function $r(\mathbf{p})$ and the component lifetime distributions $F_i, i = 1, \ldots, n$.

Since the system's lifetime will be t or larger if and only if the system is still functioning at time t, we have

$$P\{\text{system life} > t\} = r\big(\bar{\mathbf{F}}(t)\big)$$

where $\bar{\mathbf{F}}(t) = (\bar{F}_1(t), \ldots, \bar{F}_n(t))$. Hence, by a well-known formula that states that for any nonnegative random variable X,

$$E[X] = \int_0^{\infty} P\{X > x\}\, dx,$$

we see that*

$$E[\text{system life}] = \int_0^{\infty} r\big(\bar{\mathbf{F}}(t)\big)\, dt \tag{9.21}$$

* That $E[X] = \int_0^{\infty} P\{X > x\}\, dx$ can be shown as follows when X has density f:

$$\int_0^{\infty} P\{X > x\}\, dx = \int_0^{\infty}\int_x^{\infty} f(y)\, dy\, dx = \int_0^{\infty}\int_0^{y} f(y)\, dx\, dy = \int_0^{\infty} y f(y)\, dy = E[X]$$

Example 9.26 (A Series System of Uniformly Distributed Components) Consider a series system of three independent components each of which functions for an amount of time (in hours) uniformly distributed over $(0, 10)$. Hence, $r(\mathbf{p}) = p_1 p_2 p_3$ and

$$F_i(t) = \begin{cases} t/10, & 0 \leqslant t \leqslant 10 \\ 1, & t > 10 \end{cases} \qquad i = 1, 2, 3$$

Therefore,

$$r\big(\bar{\mathbf{F}}(t)\big) = \begin{cases} \left(\dfrac{10-t}{10}\right)^3, & 0 \leqslant t \leqslant 10 \\ 0, & t > 10 \end{cases}$$

and so from Equation (9.21) we obtain

$$\begin{aligned} E[\text{system life}] &= \int_0^{10} \left(\frac{10-t}{10}\right)^3 dt \\ &= 10 \int_0^1 y^3\, dy \\ &= \tfrac{5}{2} \end{aligned}$$

■

Example 9.27 (A Two-out-of-Three System) Consider a two-out-of-three system of independent components, in which each component's lifetime is (in months) uniformly distributed over $(0, 1)$. As was shown in Example 9.13, the reliability of such a system is given by

$$r(\mathbf{p}) = p_1 p_2 + p_1 p_3 + p_2 p_3 - 2 p_1 p_2 p_3$$

Since

$$F_i(t) = \begin{cases} t, & 0 \leqslant t \leqslant 1 \\ 1, & t > 1 \end{cases}$$

we see from Equation (9.21) that

$$\begin{aligned} E[\text{system life}] &= \int_0^1 [3(1-t)^2 - 2(1-t)^3]\, dt \\ &= \int_0^1 (3y^2 - 2y^3)\, dy \\ &= 1 - \tfrac{1}{2} \\ &= \tfrac{1}{2} \end{aligned}$$

■

Example 9.28 (A Four-Component System) Consider the four-component system that functions when components 1 and 2 and at least one of components 3 and 4 functions. Its structure function is given by

$$\phi(\mathbf{x}) = x_1 x_2(x_3 + x_4 - x_3 x_4)$$

and thus its reliability function equals

$$r(\mathbf{p}) = p_1 p_2(p_3 + p_4 - p_3 p_4)$$

Let us compute the mean system lifetime when the ith component is uniformly distributed over $(0, i)$, $i = 1, 2, 3, 4$. Now,

$$\bar{F}_1(t) = \begin{cases} 1 - t, & 0 \leqslant t \leqslant 1 \\ 0, & t > 1 \end{cases}$$
$$\bar{F}_2(t) = \begin{cases} 1 - t/2, & 0 \leqslant t \leqslant 2 \\ 0, & t > 2 \end{cases}$$
$$\bar{F}_3(t) = \begin{cases} 1 - t/3, & 0 \leqslant t \leqslant 3 \\ 0, & t > 3 \end{cases}$$
$$\bar{F}_4(t) = \begin{cases} 1 - t/4, & 0 \leqslant t \leqslant 4 \\ 0, & t > 4 \end{cases}$$

Hence,

$$r(\bar{\mathbf{F}}(t)) = \begin{cases} (1 - t)\left(\dfrac{2 - t}{2}\right)\left[\dfrac{3 - t}{3} + \dfrac{4 - t}{4} - \dfrac{(3 - t)(4 - t)}{12}\right], & 0 \leqslant t \leqslant 1 \\ 0, & t > 1 \end{cases}$$

Therefore,

$$\begin{aligned} E[\text{system life}] &= \frac{1}{24} \int_0^1 (1 - t)(2 - t)(12 - t^2)\, dt \\ &= \frac{593}{(24)(60)} \\ &\approx 0.41 \end{aligned}$$

■

We end this section by obtaining the mean lifetime of a k-out-of-n system of independent identically distributed exponential components. If θ is the mean lifetime of each component, then

$$\bar{F}_i(t) = e^{-t/\theta}$$

Hence, since for a k-out-of-n system,

$$r(p, p, \ldots, p) = \sum_{i=k}^{n} \binom{n}{i} p^i (1 - p)^{n-i}$$

we obtain from Equation (9.21)

$$E[\text{system life}] = \int_0^\infty \sum_{i=k}^{n} \binom{n}{i} (e^{-t/\theta})^i (1 - e^{-t/\theta})^{n-i} dt$$

Making the substitution

$$y = e^{-t/\theta}, \qquad dy = -\frac{1}{\theta} e^{-t/\theta} dt = -\frac{y}{\theta} dt$$

yields

$$E[\text{system life}] = \theta \sum_{i=k}^{n} \binom{n}{i} \int_0^1 y^{i-1} (1 - y)^{n-i} dy$$

Now, it is not difficult to show that*

$$\int_0^1 y^n (1 - y)^m dy = \frac{m!n!}{(m + n + 1)!} \tag{9.22}$$

Thus, the foregoing equals

$$\begin{aligned} E[\text{system life}] &= \theta \sum_{i=k}^{n} \frac{n!}{(n - i)!i!} \frac{(i - 1)!(n - i)!}{n!} \\ &= \theta \sum_{i=k}^{n} \frac{1}{i} \end{aligned} \tag{9.23}$$

Remark Equation (9.23) could have been proven directly by making use of special properties of the exponential distribution. First note that the lifetime of a k-out-of-n system can be written as $T_1 + \cdots + T_{n-k+1}$, where T_i represents the time between the $(i - 1)$st and ith failure. This is true since $T_1 + \cdots + T_{n-k+1}$ equals the time at which the $(n - k + 1)$st component fails, which is also the first time that the number of functioning components is less than k. Now, when all n components are functioning, the rate at which failures occur is n/θ. That is, T_1 is exponentially distributed with mean θ/n. Similarly, since T_i represents the

* Let

$$C(n, m) = \int_0^1 y^n (1 - y)^m dy$$

Integration by parts yields $C(n, m) = [m/(n + 1)]C(n + 1, m - 1)$. Starting with $C(n, 0) = 1/(n + 1)$, Equation (9.22) follows by mathematical induction.

time until the next failure when there are $n-(i-1)$ functioning components, it follows that T_i is exponentially distributed with mean $\theta/(n-i+1)$. Hence, the mean system lifetime equals

$$E[T_1 + \cdots + T_{n-k+1}] = \theta\left[\frac{1}{n} + \cdots + \frac{1}{k}\right]$$

Note also that it follows, from the lack of memory of the exponential, that the $T_i, i = 1, \ldots, n-k+1$, are independent random variables.

9.6.1 An Upper Bound on the Expected Life of a Parallel System

Consider a parallel system of n components, whose lifetimes are not necessarily independent. The system lifetime can be expressed as

$$\text{system life} = \max_i X_i$$

where X_i is the lifetime of component $i, i = 1, \ldots, n$. We can bound the expected system lifetime by making use of the following inequality. Namely, for any constant c

$$\max_i X_i \leqslant c + \sum_{i=1}^{n} (X_i - c)^+ \tag{9.24}$$

where x^+, the positive part of x, is equal to x if $x > 0$ and is equal to 0 if $x \leqslant 0$. The validity of Inequality (9.24) is immediate since if $\max X_i < c$ then the left side is equal to $\max X_i$ and the right side is equal to c. On the other hand, if $X_{(n)} = \max X_i > c$ then the right side is at least as large as $c + (X_{(n)} - c) = X_{(n)}$. It follows from Inequality (9.24), upon taking expectations, that

$$E\left[\max_i X_i\right] \leqslant c + \sum_{i=1}^{n} E[(X_i - c)^+] \tag{9.25}$$

Now, $(X_i - c)^+$ is a nonnegative random variable and so

$$\begin{aligned}
E[(X_i - c)^+] &= \int_0^\infty P\{(X_i - c)^+ > x\}\, dx \\
&= \int_0^\infty P\{X_i - c > x\}\, dx \\
&= \int_c^\infty P\{X_i > y\}\, dy
\end{aligned}$$

Thus, we obtain

$$E\left[\max_i X_i\right] \leqslant c + \sum_{i=1}^{n} \int_c^{\infty} P\{X_i > y\}\, dy \tag{9.26}$$

Because the preceding is true for all c, it follows that we obtain the best bound by letting c equal the value that minimizes the right side of the preceding. To determine that value, differentiate the right side of the preceding and set the result equal to 0, to obtain

$$1 - \sum_{i=1}^{n} P\{X_i > c\} = 0$$

That is, the minimizing value of c is that value c^* for which

$$\sum_{i=1}^{n} P\{X_i > c^*\} = 1$$

Since $\sum_{i=1}^{n} P\{X_i > c\}$ is a decreasing function of c, the value of c^* can be easily approximated and then utilized in Inequality (9.26). Also, it is interesting to note that c^* is such that the expected number of the X_i that exceed c^* is equal to 1 (see Exercise 32). That the optimal value of c has this property is interesting and somewhat intuitive in as much as Inequality (9.24) is an equality when exactly one of the X_i exceeds c.

Example 9.29 Suppose the lifetime of component i is exponentially distributed with rate $\lambda_i, i = 1, \ldots, n$. Then the minimizing value of c is such that

$$1 = \sum_{i=1}^{n} P\{X_i > c^*\} = \sum_{i=1}^{n} e^{-\lambda_i c^*}$$

and the resulting bound of the mean system life is

$$\begin{aligned}
E\left[\max_i X_i\right] &\leqslant c^* + \sum_{i=1}^{n} E[(X_i - c^*)^+] \\
&= c^* + \sum_{i=1}^{n} \big(E[(X_i - c^*)^+ \mid X_i > c^*]P\{X_i > c^*\} \\
&\qquad + E[(X_i - c^*)^+ \mid X_i \leqslant c^*]P\{X_i \leqslant c^*]\big) \\
&= c^* + \sum_{i=1}^{n} \frac{1}{\lambda_i} e^{-\lambda_i c^*}
\end{aligned}$$

In the special case where all the rates are equal, say, $\lambda_i = \lambda, i = 1, \dots, n$, then

$$1 = ne^{-\lambda c^*} \quad \text{or} \quad c^* = \frac{1}{\lambda}\log(n)$$

and the bound is

$$E\Big[\max_i X_i\Big] \leqslant \frac{1}{\lambda}(\log(n) + 1)$$

That is, if $X_1, \dots, X_n$ are identically distributed exponential random variables with rate λ, then the preceding gives a bound on the expected value of their maximum. In the special case where these random variables are also independent, the following exact expression, given by Equation (9.25), is not much less than the preceding upper bound:

$$E\Big[\max_i X_i\Big] = \frac{1}{\lambda}\sum_{i=1}^{n} 1/i \approx \frac{1}{\lambda}\int_1^n \frac{1}{x}\,dx \approx \frac{1}{\lambda}\log(n) \qquad \blacksquare$$

9.7 Systems with Repair

Consider an n-component system having reliability function $r(\mathbf{p})$. Suppose that component i functions for an exponentially distributed time with rate λ_i and then fails; once failed it takes an exponential time with rate μ_i to be repaired, $i = 1, \dots, n$. All components act independently.

Let us suppose that all components are initially working, and let

$$A(t) = P\{\text{system is working at } t\}$$

$A(t)$ is called the *availability* at time t. Since the components act independently, $A(t)$ can be expressed in terms of the reliability function as follows:

$$A(t) = r(A_1(t), \dots, A_n(t)) \tag{9.27}$$

where

$$A_i(t) = P\{\text{component } i \text{ is functioning at } t\}$$

Now the state of component i—either on or off—changes in accordance with a two-state continuous time Markov chain. Hence, from the results of Example 6.12 we have

$$A_i(t) = P_{00}(t) = \frac{\mu_i}{\mu_i + \lambda_i} + \frac{\lambda_i}{\mu_i + \lambda_i}e^{-(\lambda_i+\mu_i)t}$$

Thus, we obtain

$$A(t) = r\left(\frac{\mu}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda} e^{-(\lambda+\mu)t}\right)$$

If we let t approach ∞, then we obtain the limiting availability—call it A—which is given by

$$A = \lim_{t\to\infty} A(t) = r\left(\frac{\mu}{\lambda + \mu}\right)$$

Remarks

(i) If the on and off distribution for component i are arbitrary continuous distributions with respective means $1/\lambda_i$ and $1/\mu_i, i = 1, \ldots, n$, then it follows from the theory of alternating renewal processes (see Section 7.5.1) that

$$A_i(t) \to \frac{1/\lambda_i}{1/\lambda_i + 1/\mu_i} = \frac{\mu_i}{\mu_i + \lambda_i}$$

and thus using the continuity of the reliability function, it follows from (9.27) that the limiting availability is

$$A = \lim_{t\to\infty} A(t) = r\left(\frac{\mu}{\mu + \lambda}\right)$$

Hence, A depends only on the on and off distributions through their means.

(ii) It can be shown (using the theory of regenerative processes as presented in Section 7.5) that A will also equal the long-run proportion of time that the system will be functioning.

Example 9.30 For a series system, $r(\mathbf{p}) = \prod_{i=1}^{n} p_i$ and so

$$A(t) = \prod_{i=1}^{n} \left[\frac{\mu_i}{\mu_i + \lambda_i} + \frac{\lambda_i}{\mu_i + \lambda_i} e^{-(\lambda_i+\mu_i)t}\right]$$

and

$$A = \prod_{i=1}^{n} \frac{\mu_i}{\mu_i + \lambda_i} \qquad \blacksquare$$

Example 9.31 For a parallel system, $r(\mathbf{p}) = 1 - \prod_{i=1}^{n} (1 - p_i)$ and thus

$$A(t) = 1 - \prod_{i=1}^{n} \left[\frac{\lambda_i}{\mu_i + \lambda_i}(1 - e^{-(\lambda_i+\mu_i)t})\right]$$

and

$$A(t) = 1 - \prod_{i=1}^{n} \frac{\lambda_i}{\mu_i + \lambda_i} \qquad \blacksquare$$

The preceding system will alternate between periods when it is up and periods when it is down. Let us denote by U_i and $D_i, i \geqslant 1$, the lengths of the ith up and down period respectively. For instance in a two-out-of-three system, U_1 will be the time until two components are down; D_1, the additional time until two are up; U_2 the additional time until two are down, and so on. Let

$$\bar{U} = \lim_{n\to\infty} \frac{U_1 + \cdots + U_n}{n},$$
$$\bar{D} = \lim_{n\to\infty} \frac{D_1 + \cdots + D_n}{n}$$

denote the average length of an up and down period respectively.*

To determine $\bar{U}$ and $\bar{D}$, note first that in the first n up–down cycles—that is, in time $\sum_{i=1}^{n}(U_i + D_i)$—the system will be up for a time $\sum_{i=1}^{n} U_i$. Hence, the proportion of time the system will be up in the first n up–down cycles is

$$\frac{U_1 + \cdots + U_n}{U_1 + \cdots + U_n + D_1 + \cdots + D_n} = \frac{\sum_{i=1}^{n} U_i/n}{\sum_{i=1}^{n} U_i/n + \sum_{i=1}^{n} D_i/n}$$

As $n \to \infty$, this must converge to A, the long-run proportion of time the system is up. Hence,

$$\frac{\bar{U}}{\bar{U} + \bar{D}} = A = r\left(\frac{\boldsymbol{\mu}}{\boldsymbol{\lambda} + \boldsymbol{\mu}}\right) \tag{9.28}$$

However, to solve for $\bar{U}$ and $\bar{D}$ we need a second equation. To obtain one consider the rate at which the system fails. As there will be n failures in time $\sum_{i=1}^{n}(U_i + D_i)$, it follows that the rate at which the system fails is

$$\begin{aligned} \text{rate at which system fails} &= \lim_{n\to\infty} \frac{n}{\sum_{i}^{n} U_1 + \sum_{1}^{n} D_1} \\ &= \lim_{n\to\infty} \frac{n}{\sum_{1}^{n} U_i/n + \sum_{1}^{n} D_i/n} = \frac{1}{\bar{U} + \bar{D}} \end{aligned} \tag{9.29}$$

That is, the foregoing yields the intuitive result that, on average, there is one failure every $\bar{U} + \bar{D}$ time units. To utilize this, let us determine the rate at which a failure of component i causes the system to go from up to down. Now, the system will go from up to down when component i fails if the states of the other

* It can be shown using the theory of regenerative processes that, with probability 1, the preceding limits will exist and will be constants.

components $x_1, \ldots, x_{i-1}, x_{i-1}, \ldots, x_n$ are such that $\phi(1_i, \mathbf{x}) = 1, \phi(0_i, \mathbf{x}) = 0$. That is, the states of the other components must be such that

$$\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x}) = 1 \tag{9.30}$$

Since component i will, on average, have one failure every $1/\lambda_i + 1/\mu_i$ time units, it follows that the rate at which component i fails is equal to $(1/\lambda_i + 1/\mu_i)^{-1} = \lambda_i\mu_i/(\lambda_i + \mu_i)$. In addition, the states of the other components will be such that (9.30) holds with probability

$$\begin{aligned}
&P\{\phi(1_i, X(\infty)) - \phi(0_i, X(\infty)) = 1\} \\
&\quad = E[\phi(1_i, X(\infty)) - \phi(0_i, X(\infty))] \quad \text{since } \phi(1_i, X(\infty)) - \phi(0_i, X(\infty)) \\
&\quad \text{is a Bernoulli random variable} \\
&\quad = r\left(1_i, \frac{\boldsymbol{\mu}}{\boldsymbol{\lambda} + \boldsymbol{\mu}}\right) - r\left(0_i, \frac{\boldsymbol{\mu}}{\boldsymbol{\lambda} + \boldsymbol{\mu}}\right)
\end{aligned}$$

Hence, putting the preceding together we see that

$$\begin{matrix}\text{rate at which component} \\ i \text{ causes the system to fail}\end{matrix} = \frac{\lambda_i\mu_i}{\lambda_i + \mu_i}\left[r\left(1_i, \frac{\boldsymbol{\mu}}{\boldsymbol{\lambda} + \boldsymbol{\mu}}\right) - r\left(0_i, \frac{\boldsymbol{\mu}}{\boldsymbol{\lambda} + \boldsymbol{\mu}}\right)\right]$$

Summing this over all components i thus gives

$$\text{rate at which system fails} = \sum_i \frac{\lambda_i\mu_i}{\lambda_i + \mu_i}\left[r\left(1_i, \frac{\boldsymbol{\mu}}{\boldsymbol{\lambda} + \boldsymbol{\mu}}\right) - r\left(0_i, \frac{\boldsymbol{\mu}}{\boldsymbol{\lambda} + \boldsymbol{\mu}}\right)\right]$$

Finally, equating the preceding with (9.29) yields

$$\frac{1}{\bar{U} + \bar{D}} = \sum_i \frac{\lambda_i\mu_i}{\lambda_i + \mu_i}\left[r\left(1_i, \frac{\boldsymbol{\mu}}{\boldsymbol{\lambda} + \boldsymbol{\mu}}\right) - r\left(0_i, \frac{\boldsymbol{\mu}}{\boldsymbol{\lambda} + \boldsymbol{\mu}}\right)\right] \tag{9.31}$$

Solving (9.28) and (9.31), we obtain

$$\bar{U} = \frac{r\left(\dfrac{\boldsymbol{\mu}}{\boldsymbol{\lambda} + \boldsymbol{\mu}}\right)}{\displaystyle\sum_{i=1}^{n} \frac{\lambda_i\mu_i}{\lambda_i + \mu_i}\left[r\left(1_i, \frac{\boldsymbol{\mu}}{\boldsymbol{\lambda} + \boldsymbol{\mu}}\right) - r\left(0_i, \frac{\boldsymbol{\mu}}{\boldsymbol{\lambda} + \boldsymbol{\mu}}\right)\right]}, \tag{9.32}$$

$$\bar{D} = \frac{\left[1 - r\left(\dfrac{\boldsymbol{\mu}}{\boldsymbol{\lambda} + \boldsymbol{\mu}}\right)\right]\bar{U}}{r\left(\dfrac{\boldsymbol{\mu}}{\boldsymbol{\lambda} + \boldsymbol{\mu}}\right)} \tag{9.33}$$

Also, (9.31) yields the rate at which the system fails.

Remark In establishing the formulas for $\bar{U}$ and $\bar{D}$, we did not make use of the assumption of exponential on and off times and in fact, our derivation is valid and Equations (9.32) and (9.33) hold whenever $\bar{U}$ and $\bar{D}$ are well defined (a sufficient condition is that all on and off distributions are continuous). The quantities $\lambda_i, \mu_i, i = 1, \ldots, n$, will represent, respectively, the reciprocals of the mean lifetimes and mean repair times.

Example 9.32 For a series system,

$$\bar{U} = \frac{\prod_i \dfrac{\mu_i}{\mu_i + \lambda_i}}{\sum_i \dfrac{\lambda_i \mu_i}{\lambda_i + \mu_i} \prod_{j \neq i} \dfrac{\mu_j}{\mu_j + \lambda_j}} = \frac{1}{\sum_i \lambda_i},$$

$$\bar{D} = \frac{1 - \prod_i \dfrac{\mu_i}{\mu_i + \lambda_i}}{\prod_i \dfrac{\mu_i}{\mu_i + \lambda_i}} \times \frac{1}{\sum_i \lambda_i}$$

whereas for a parallel system,

$$\bar{U} = \frac{1 - \prod_i \dfrac{\lambda_i}{\mu_i + \lambda_i}}{\sum_i \dfrac{\lambda_i \mu_i}{\lambda_i + \mu_i} \prod_{j \neq i} \dfrac{\lambda_j}{\mu_j + \lambda_j}} = \frac{1 - \prod_i \dfrac{\lambda_i}{\mu_i + \lambda_i}}{\prod_j \dfrac{\lambda_j}{\mu_j + \lambda_j}} \times \frac{1}{\sum_i \mu_i},$$

$$\bar{D} = \frac{\prod_i \dfrac{\lambda_i}{\mu_i + \lambda_i}}{1 - \prod_i \dfrac{\lambda_i}{\mu_i + \lambda_i}} \bar{U} = \frac{1}{\sum_i \mu_i}$$

The preceding formulas hold for arbitrary continuous up and down distributions with $1/\lambda_i$ and $1/\mu_i$ denoting respectively the mean up and down times of component $i, i = 1, \ldots, n$. ■

9.7.1 A Series Model with Suspended Animation

Consider a series consisting of n components, and suppose that whenever a component (and thus the system) goes down, repair begins on that component and each of the other components enters a state of suspended animation. That is, after the down component is repaired, the other components resume operation in exactly the same condition as when the failure occurred. If two or more components go down simultaneously, one of them is arbitrarily chosen as being the failed component and repair on that component begins; the others that went down at the same time are considered to be in a state of suspended animation, and they will instantaneously go down when the repair is completed. We suppose that (not counting any time in suspended animation) the distribution of time that component i functions is F_i with mean u_i, whereas its repair distribution is G_i with mean $d_i, i = 1, \ldots, n$.

To determine the long-run proportion of time this system is working, we reason as follows. To begin, consider the time, call it T, at which the system has been up for a time t. Now, when the system is up, the failure times of component i constitute a renewal process with mean interarrival time u_i. Therefore, it follows that

$$\text{number of failures of } i \text{ in time } T \approx \frac{t}{u_i}$$

As the average repair time of i is d_i, the preceding implies that

$$\text{total repair time of } i \text{ in time } T \approx \frac{td_i}{u_i}$$

Therefore, in the period of time in which the system has been up for a time t, the total system downtime has approximately been

$$t \sum_{i=1}^{n} d_i/u_i$$

Hence, the proportion of time that the system has been up is approximately

$$\frac{t}{t + t\sum_{i=1}^{n} d_i/u_i}$$

Because this approximation should become exact as we let t become larger, it follows that

$$\text{proportion of time the system is up} = \frac{1}{1 + \sum_i d_i/u_i} \tag{9.34}$$

which also shows that

$$\begin{aligned}\text{proportion of time the system is down} &= 1 - \text{proportion of time the system is up}\\ &= \frac{\sum_i d_i/u_i}{1 + \sum_i d_i/u_i}\end{aligned}$$

Moreover, in the time interval from 0 to T, the proportion of the repair time that has been devoted to component i is approximately

$$\frac{td_i/u_i}{\sum_i td_i/u_i}$$

Thus, in the long run,

$$\text{proportion of down time that is due to component } i = \frac{d_i/u_i}{\sum_i d_i/u_i}$$

Multiplying the preceding by the proportion of time the system is down gives

$$\text{proportion of time component } i \text{ is being repaired} = \frac{d_i/u_i}{1 + \sum_i d_i/u_i}$$

Also, since component j will be in suspended animation whenever any of the other components is in repair, we see that

$$\text{proportion of time component } j \text{ is in suspended animation} = \frac{\sum_{i \neq j} d_i/u_i}{1 + \sum_i d_i/u_i}$$

Another quantity of interest is the long-run rate at which the system fails. Since component i fails at rate $1/u_i$ when the system is up, and does not fail when the system is down, it follows that

$$\begin{aligned}
\text{rate at which } i \text{ fails} &= \frac{\text{proportion of time system is up}}{u_i} \\
&= \frac{1/u_i}{1 + \sum_i d_i/u_i}
\end{aligned}$$

Since the system fails when any of its components fail, the preceding yields that

$$\text{rate at which the system fails} = \frac{\sum_i 1/u_i}{1 + \sum_i d_i/u_i} \tag{9.35}$$

If we partition the time axis into periods when the system is up and those when it is down, we can determine the average length of an up period by noting that if $U(t)$ is the total amount of time that the system is up in the interval $[0, t]$, and if $N(t)$ is the number of failures by time t, then

$$\begin{aligned}
\text{average length of an up period} &= \lim_{t \to \infty} \frac{U(t)}{N(t)} \\
&= \lim_{t \to \infty} \frac{U(t)/t}{N(t)/t} \\
&= \frac{1}{\sum_i 1/u_i}
\end{aligned}$$

where the final equality used Equations (9.34) and (9.35). Also, in a similar manner it can be shown that

$$\text{average length of a down period} = \frac{\sum_i d_i/u_i}{\sum_i 1/u_i} \tag{9.36}$$

Exercises

1. Prove that, for any structure function ϕ,

$$\phi(\mathbf{x}) = x_i\phi(1_i, \mathbf{x}) + (1 - x_i)\phi(0_i, \mathbf{x})$$

where

$$(1_i, \mathbf{x}) = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n),$$
$$(0_i, \mathbf{x}) = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

2. Show that
   (a) if $\phi(0, 0, \ldots, 0) = 0$ and $\phi(1, 1, \ldots, 1) = 1$, then

$$\min x_i \leqslant \phi(\mathbf{x}) \leqslant \max x_i$$

   (b) $\phi(\max(\mathbf{x}, \mathbf{y})) \geqslant \max(\phi(\mathbf{x}), \phi(\mathbf{y}))$
   (c) $\phi(\min(\mathbf{x}, \mathbf{y})) \leqslant \min(\phi(\mathbf{x}), \phi(\mathbf{y}))$

3. For any structure function, we define the dual structure ϕ^{D} by

$$\phi^{\mathrm{D}}(\mathbf{x}) = 1 - \phi(\mathbf{1} - \mathbf{x})$$

   (a) Show that the dual of a parallel (series) system is a series (parallel) system.
   (b) Show that the dual of a dual structure is the original structure.
   (c) What is the dual of a k-out-of-n structure?
   (d) Show that a minimal path (cut) set of the dual system is a minimal cut (path) set of the original structure.

*4. Write the structure function corresponding to the following:
   (a)

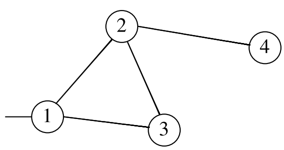

Figure 9.16

   (b)

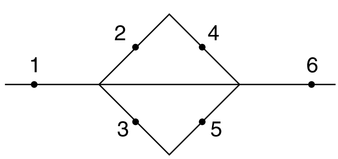

Figure 9.17

(c)

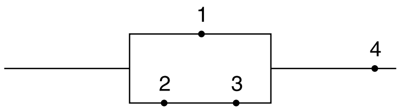

Figure 9.18

5. Find the minimal path and minimal cut sets for:
   (a)

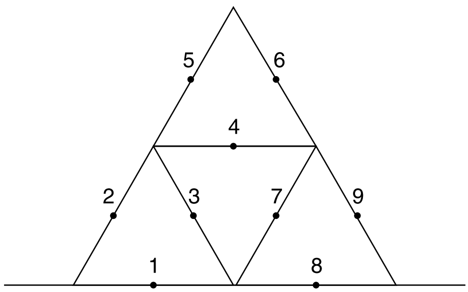

Figure 9.19

(b)

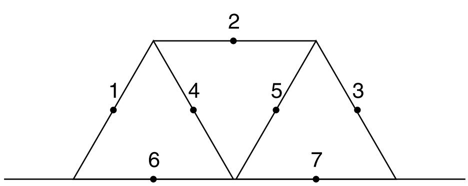

Figure 9.20

*6. The minimal path sets are {1, 2, 4}, {1, 3, 5}, and {5, 6}. Give the minimal cut sets.

7. The minimal cut sets are {1, 2, 3}, {2, 3, 4}, and {3, 5}. What are the minimal path sets?

8. Give the minimal path sets and the minimal cut sets for the structure given by Figure 9.21.

9. Component i is said to be *relevant* to the system if for some state vector $\mathbf{x}$,

$$\phi(1_i, \mathbf{x}) = 1, \qquad \phi(0_i, \mathbf{x}) = 0$$

Otherwise, it is said to be *irrelevant*.

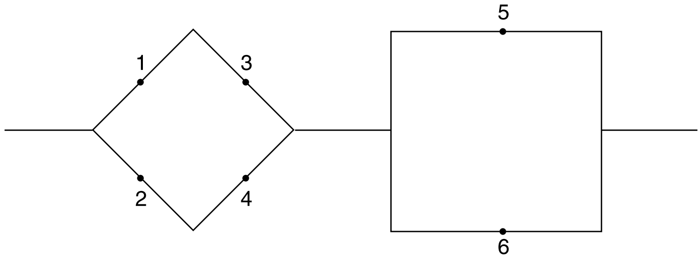

Figure 9.21

(a) Explain in words what it means for a component to be irrelevant.
(b) Let $A_1, \ldots, A_s$ be the minimal path sets of a system, and let S denote the set of components. Show that $S = \bigcup_{i=1}^{s} A_i$ if and only if all components are relevant.
(c) Let $C_1, \ldots, C_k$ denote the minimal cut sets. Show that $S = \bigcup_{i=1}^{k} C_i$ if and only if all components are relevant.

10. Let t_i denote the time of failure of the ith component; let $\tau_\phi(t)$ denote the time to failure of the system ϕ as a function of the vector $\mathbf{t} = (t_1, \ldots, t_n)$. Show that

$$\max_{1 \leqslant j \leqslant s} \min_{i \in A_j} t_i = \tau_\phi(\mathbf{t}) = \min_{1 \leqslant j \leqslant k} \max_{i \in C_j} t_i$$

where $C_1, \ldots, C_k$ are the minimal cut sets, and $A_1, \ldots, A_s$ the minimal path sets.

11. Give the reliability function of the structure of Exercise 8.

*12. Give the minimal path sets and the reliability function for the structure in Figure 9.22.

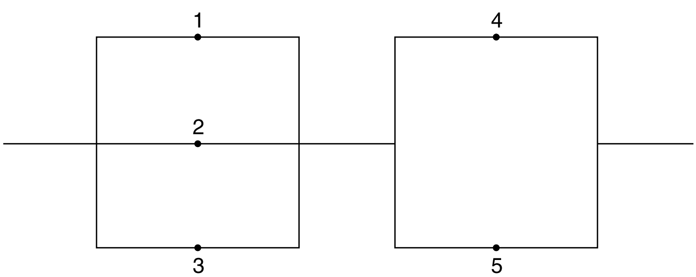

Figure 9.22

13. Let $r(\mathbf{p})$ be the reliability function. Show that

$$r(\mathbf{p}) = p_i r(1_i, \mathbf{p}) + (1 - p_i) r(0_i, \mathbf{p})$$

14. Compute the reliability function of the bridge system (see Figure 9.11) by conditioning upon whether or not component 3 is working.
15. Compute upper and lower bounds of the reliability function (using Method 2) for the systems given in Exercise 4, and compare them with the exact values when $p_i \equiv \frac{1}{2}$.
16. Compute the upper and lower bounds of $r(\mathbf{p})$ using both methods for the
    (a) two-out-of-three system and
    (b) two-out-of-four system.
    (c) Compare these bounds with the exact reliability when
        (i) $p_i \equiv 0.5$
        (ii) $p_i \equiv 0.8$
        (iii) $p_i \equiv 0.2$

*17. Let N be a nonnegative, integer-valued random variable. Show that

$$P\{N > 0\} \geqslant \frac{(E[N])^2}{E[N^2]}$$

and explain how this inequality can be used to derive additional bounds on a reliability function.

Hint:

$$\begin{aligned} E[N^2] &= E[N^2 \mid N > 0]P\{N > 0\} && \text{(Why?)} \\ &\geqslant (E[N \mid N > 0])^2 P\{N > 0\} && \text{(Why?)} \end{aligned}$$

Now multiply both sides by $P\{N > 0\}$.

18. Consider a structure in which the minimal path sets are {1, 2, 3} and {3, 4, 5}.
    (a) What are the minimal cut sets?
    (b) If the component lifetimes are independent uniform $(0, 1)$ random variables, determine the probability that the system life will be less than $\frac{1}{2}$.
19. Let $X_1, X_2, \ldots, X_n$ denote independent and identically distributed random variables and define the order statistics $X_{(1)}, \ldots, X_{(n)}$ by

$$X_{(i)} \equiv i\text{th smallest of } X_1, \ldots, X_n$$

Show that if the distribution of X_j is IFR, then so is the distribution of $X_{(i)}$.

Hint: Relate this to one of the examples of this chapter.

20. Let F be a continuous distribution function. For some positive α, define the distribution function G by

$$\bar{G}(t) = (\bar{F}(t))^{\alpha}$$

Find the relationship between $\lambda_G(t)$ and $\lambda_F(t)$, the respective failure rate functions of G and F.

21. Consider the following four structures:

(i)

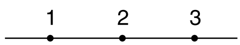

Figure 9.23

(ii)

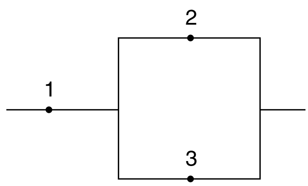

Figure 9.24

(iii)

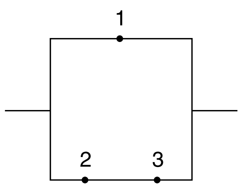

Figure 9.25

(iv)

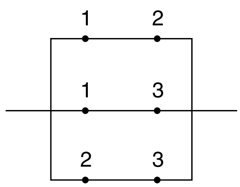

Figure 9.26

Let F_1, F_2, and F_3 be the corresponding component failure distributions; each of which is assumed to be IFR (increasing failure rate). Let F be the system failure distribution. All components are independent.

(a) For which structures is F necessarily IFR if $F_1 = F_2 = F_3$? Give reasons.
(b) For which structures is F necessarily IFR if $F_2 = F_3$? Give reasons.
(c) For which structures is F necessarily IFR if $F_1 \neq F_2 \neq F_3$? Give reasons.

*22. Let X denote the lifetime of an item. Suppose the item has reached the age of t. Let X_t denote its remaining life and define

$$\bar{F}_t(a) = P\{X_t > a\}$$

In words, $\bar{F}_t(a)$ is the probability that a t-year-old item survives an additional time a. Show that

(a) $\bar{F}_t(a) = \bar{F}(t + a)/\bar{F}(t)$ where F is the distribution function of X.
(b) Another definition of IFR is to say that F is IFR if $\bar{F}_t(a)$ decreases in t, for all a. Show that this definition is equivalent to the one given in the text when F has a density.

23. Show that if each (independent) component of a series system has an IFR distribution, then the system lifetime is itself IFR by

(a) showing that

$$\lambda_F(t) = \sum_i \lambda_i(t)$$

where $\lambda_F(t)$ is the failure rate function of the system; and $\lambda_i(t)$ the failure rate function of the lifetime of component i.

(b) using the definition of IFR given in Exercise 22.

24. Show that if F is IFR, then it is also IFRA, and show by counterexample that the reverse is not true.

*25. We say that ζ is a p-percentile of the distribution F if $F(\zeta) = p$. Show that if ζ is a p-percentile of the IFRA distribution F, then

$$\begin{aligned} \bar{F}(x) &\leqslant e^{-\theta x}, \quad x \geqslant \zeta \\ \bar{F}(x) &\geqslant e^{-\theta x}, \quad x \leqslant \zeta \end{aligned}$$

where

$$\theta = \frac{-\log(1-p)}{\zeta}$$

26. Prove Lemma 9.3.

Hint: Let $x = y + \delta$. Note that $f(t) = t^{\alpha}$ is a concave function when $0 \leqslant \alpha \leqslant 1$, and use the fact that for a concave function $f(t+h) - f(t)$ is decreasing in t.

27. Let $r(p) = r(p, p, \ldots, p)$. Show that if $r(p_0) = p_0$, then

$$\begin{aligned} r(p) &\geqslant p \quad \text{for } p \geqslant p_0 \\ r(p) &\leqslant p \quad \text{for } p \leqslant p_0 \end{aligned}$$

Hint: Use Proposition 9.2.

28. Find the mean lifetime of a series system of two components when the component lifetimes are respectively uniform on $(0, 1)$ and uniform on $(0, 2)$. Repeat for a parallel system.

29. Show that the mean lifetime of a parallel system of two components is

$$\frac{1}{\mu_1 + \mu_2} + \frac{\mu_1}{(\mu_1 + \mu_2)\mu_2} + \frac{\mu_2}{(\mu_1 + \mu_2)\mu_1}$$

when the first component is exponentially distributed with mean $1/\mu_1$ and the second is exponential with mean $1/\mu_2$.

*30. Compute the expected system lifetime of a three-out-of-four system when the first two component lifetimes are uniform on $(0, 1)$ and the second two are uniform on $(0, 2)$.

31. Show that the variance of the lifetime of a k-out-of-n system of components, each of whose lifetimes is exponential with mean θ, is given by

$$\theta^2 \sum_{i=k}^{n} \frac{1}{i^2}$$

32. In Section 9.6.1 show that the expected number of X_i that exceed c^* is equal to 1.

33. Let X_i be an exponential random variable with mean $8 + 2i$, for $i = 1, 2, 3$. Use the results of Section 9.6.1 to obtain an upper bound on $E[\max X_i]$, and then compare this with the exact result when the X_i are independent.
34. For the model of Section 9.7, compute for a k-out-of-n structure (i) the average up time, (ii) the average down time, and (iii) the system failure rate.
35. Prove the combinatorial identity

$$\binom{n-1}{i-1} = \binom{n}{i} - \binom{n}{i+1} + \cdots \pm \binom{n}{n}, \quad i \leqslant n$$

   (a) by induction on i
   (b) by a backwards induction argument on i—that is, prove it first for $i = n$, then assume it for $i = k$ and show that this implies that it is true for $i = k - 1$.
36. Verify Equation (9.36).

References

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