Logarithmic Functions These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions, and we discuss these functions in Section 1.6. Figure 1.23 shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

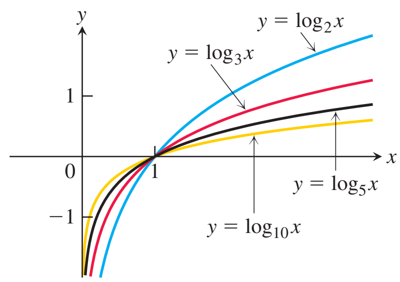


FIGURE 1.23 Graphs of four logarithmic functions.

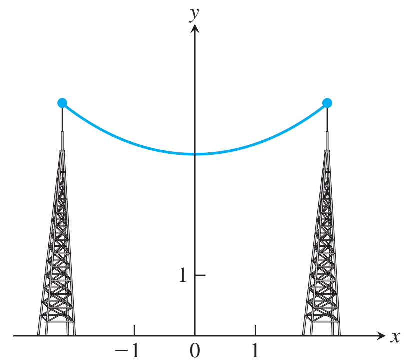


FIGURE 1.24 Graph of a catenary or hanging cable. (The Latin word *catena* means "chain.")

Transcendental Functions These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well. A particular example of a transcendental function is a **catenary**. Its graph has the shape of a cable, like a telephone line or electric cable, strung from one support to another and hanging freely under its own weight (Figure 1.24). The function defining the graph is discussed in Section 7.3.

Exercises 1.1

Functions

In Exercises 1–6, find the domain and range of each function.

1. $f(x) = 1 + x^2$

2. $f(x) = 1 - \sqrt{x}$

3. $F(x) = \sqrt{5x + 10}$

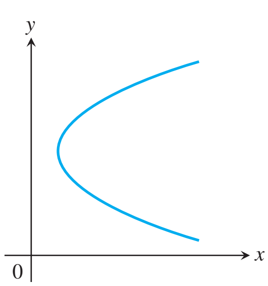
4. $g(x) = \sqrt{x^2 - 3x}$

5. $f(t) = \dfrac{4}{3 - t}$

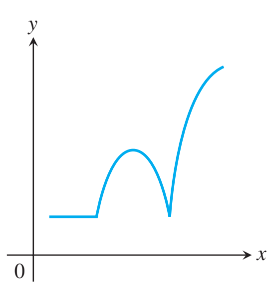
6. $G(t) = \dfrac{2}{t^2 - 16}$

In Exercises 7 and 8, which of the graphs are graphs of functions of x, and which are not? Give reasons for your answers.

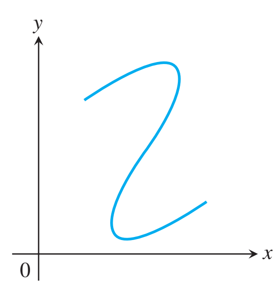
7. a.



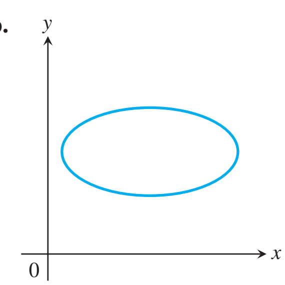
b.



8. a.



b.



Finding Formulas for Functions

9. Express the area and perimeter of an equilateral triangle as a function of the triangle's side length x.

10. Express the side length of a square as a function of the length d of the square's diagonal. Then express the area as a function of the diagonal length.

11. Express the edge length of a cube as a function of the cube's diagonal length d. Then express the surface area and volume of the cube as a function of the diagonal length.

12. A point P in the first quadrant lies on the graph of the function $f(x) = \sqrt{x}$. Express the coordinates of P as functions of the slope of the line joining P to the origin.

13. Consider the point (x, y) lying on the graph of the line $2x + 4y = 5$. Let L be the distance from the point (x, y) to the origin $(0, 0)$. Write L as a function of x.

14. Consider the point (x, y) lying on the graph of $y = \sqrt{x - 3}$. Let L be the distance between the points (x, y) and $(4, 0)$. Write L as a function of y.

Functions and Graphs

Find the natural domain and graph the functions in Exercises 15–20.

15. $f(x) = 5 - 2x$ **16.** $f(x) = 1 - 2x - x^2$

17. $g(x) = \sqrt{|x|}$ **18.** $g(x) = \sqrt{-x}$

19. $F(t) = t/|t|$ **20.** $G(t) = 1/|t|$

21. Find the domain of $y = \dfrac{x + 3}{4 - \sqrt{x^2 - 9}}$.

22. Find the range of $y = 2 + \dfrac{x^2}{x^2 + 4}$.

23. Graph the following equations and explain why they are not graphs of functions of x.

a. $|y| = x$ **b.** $y^2 = x^2$

24. Graph the following equations and explain why they are not graphs of functions of x.

a. $|x| + |y| = 1$ **b.** $|x + y| = 1$

Piecewise-Defined Functions

Graph the functions in Exercises 25–28.

25. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

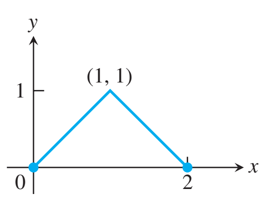
26. $g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

27. $F(x) = \begin{cases} 4 - x^2, & x \le 1 \\ x^2 + 2x, & x > 1 \end{cases}$

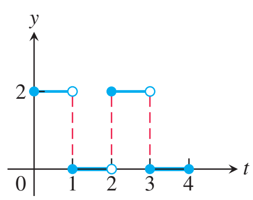
28. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \le x \end{cases}$

Find a formula for each function graphed in Exercises 29–32.

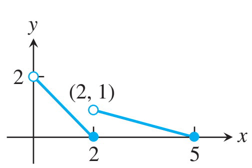
29. a.



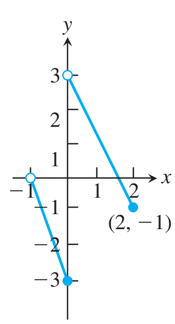
b.



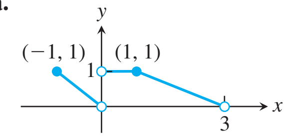
30. a.



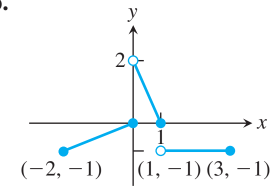
b.



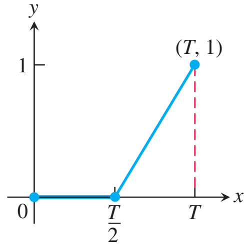
31. a.



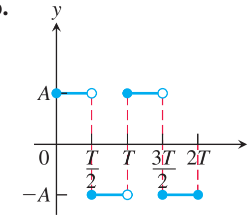
b.



32. a.



b.



The Greatest and Least Integer Functions

33. For what values of x is

a. $\lfloor x \rfloor = 0$? **b.** $\lceil x \rceil = 0$?

34. What real numbers x satisfy the equation $\lfloor x \rfloor = \lceil x \rceil$?

35. Does $\lceil -x \rceil = -\lfloor x \rfloor$ for all real x? Give reasons for your answer.

36. Graph the function

$$f(x) = \begin{cases} \lfloor x \rfloor, & x \ge 0 \\ \lceil x \rceil, & x < 0. \end{cases}$$

Why is $f(x)$ called the *integer part* of x?

Increasing and Decreasing Functions

Graph the functions in Exercises 37–46. What symmetries, if any, do the graphs have? Specify the intervals over which the function is increasing and the intervals where it is decreasing.

37. $y = -x^3$ **38.** $y = -\dfrac{1}{x^2}$

39. $y = -\dfrac{1}{x}$ **40.** $y = \dfrac{1}{|x|}$

41. $y = \sqrt{|x|}$ **42.** $y = \sqrt{-x}$

43. $y = x^3/8$ **44.** $y = -4\sqrt{x}$

45. $y = -x^{3/2}$ **46.** $y = (-x)^{2/3}$

Even and Odd Functions

In Exercises 47–58, say whether the function is even, odd, or neither. Give reasons for your answer.

47. $f(x) = 3$ **48.** $f(x) = x^{-5}$

49. $f(x) = x^2 + 1$ **50.** $f(x) = x^2 + x$

51. $g(x) = x^3 + x$ **52.** $g(x) = x^4 + 3x^2 - 1$

53. $g(x) = \dfrac{1}{x^2 - 1}$ **54.** $g(x) = \dfrac{x}{x^2 - 1}$

55. $h(t) = \dfrac{1}{t - 1}$ **56.** $h(t) = |t^3|$

57. $h(t) = 2t + 1$ **58.** $h(t) = 2|t| + 1$

Theory and Examples

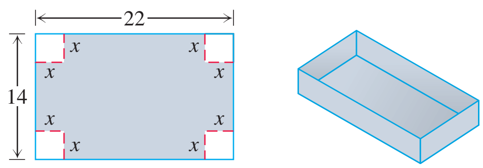
59. The variable s is proportional to t, and $s = 25$ when $t = 75$. Determine t when $s = 60$.

60. Kinetic energy The kinetic energy K of a mass is proportional to the square of its velocity v. If $K = 12{,}960$ joules when $v = 18\text{ m/sec}$, what is K when $v = 10\text{ m/sec}$?

61. The variables r and s are inversely proportional, and $r = 6$ when $s = 4$. Determine s when $r = 10$.

62. Boyle's Law Boyle's Law says that the volume V of a gas at constant temperature increases whenever the pressure P decreases, so that V and P are inversely proportional. If $P = 14.7\text{ lb/in}^2$ when $V = 1000\text{ in}^3$, then what is V when $P = 23.4\text{ lb/in}^2$?

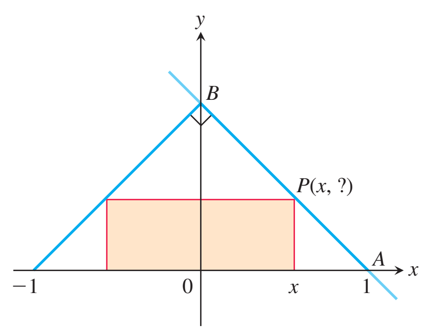
63. A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 14 in. by 22 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x.



64. The accompanying figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.

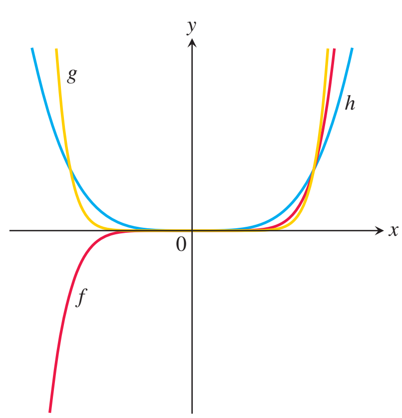
a. Express the y-coordinate of P in terms of x. (You might start by writing an equation for the line AB.)

b. Express the area of the rectangle in terms of x.

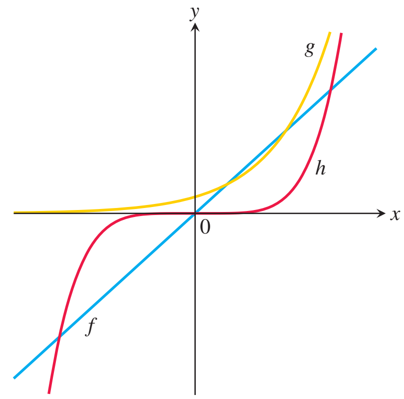


In Exercises 65 and 66, match each equation with its graph. Do not use a graphing device, and give reasons for your answer.

65. a. $y = x^4$ **b.** $y = x^7$ **c.** $y = x^{10}$



66. a. $y = 5x$ **b.** $y = 5^x$ **c.** $y = x^5$



T **67. a.** Graph the functions $f(x) = x/2$ and $g(x) = 1 + (4/x)$ together to identify the values of x for which

$$\frac{x}{2} > 1 + \frac{4}{x}.$$

b. Confirm your findings in part (a) algebraically.

T **68. a.** Graph the functions $f(x) = 3/(x - 1)$ and $g(x) = 2/(x + 1)$ together to identify the values of x for which

$$\frac{3}{x - 1} < \frac{2}{x + 1}.$$

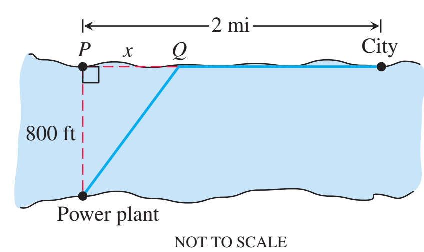
b. Confirm your findings in part (a) algebraically.

69. For a curve to be *symmetric about the x-axis*, the point (x, y) must lie on the curve if and only if the point $(x, -y)$ lies on the curve. Explain why a curve that is symmetric about the x-axis is not the graph of a function, unless the function is $y = 0$.

70. Three hundred books sell for \$40 each, resulting in a revenue of (300)(\$40) = \$12,000. For each \$5 increase in the price, 25 fewer books are sold. Write the revenue R as a function of the number x of \$5 increases.

71. A pen in the shape of an isosceles right triangle with legs of length x ft and hypotenuse of length h ft is to be built. If fencing costs \$5/ft for the legs and \$10/ft for the hypotenuse, write the total cost C of construction as a function of h.

72. Industrial costs A power plant sits next to a river where the river is 800 ft wide. To lay a new cable from the plant to a location in the city 2 mi downstream on the opposite side costs \$180 per foot across the river and \$100 per foot along the land.



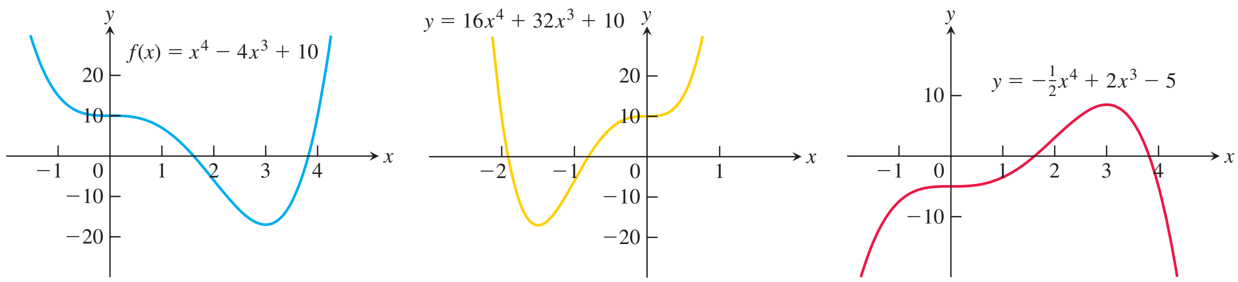
a. Suppose that the cable goes from the plant to a point Q on the opposite side that is x ft from the point P directly opposite the plant. Write a function $C(x)$ that gives the cost of laying the cable in terms of the distance x.

b. Generate a table of values to determine if the least expensive location for point Q is less than 2000 ft or greater than 2000 ft from point P.

EXAMPLE 5 Given the function $f(x) = x^4 - 4x^3 + 10$ (Figure 1.35a), find formulas to

(a) compress the graph horizontally by a factor of 2 followed by a reflection across the y-axis (Figure 1.35b).

(b) compress the graph vertically by a factor of 2 followed by a reflection across the x-axis (Figure 1.35c).



(a) (b) (c)

FIGURE 1.35 (a) The original graph of f. (b) The horizontal compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the y-axis. (c) The vertical compression of $y = f(x)$ in part (a) by a factor of 2, followed by a reflection across the x-axis (Example 5).

Solution

(a) We multiply x by 2 to get the horizontal compression, and by -1 to give reflection across the y-axis. The formula is obtained by substituting $-2x$ for x in the right-hand side of the equation for f:

$$\begin{aligned} y = f(-2x) &= (-2x)^4 - 4(-2x)^3 + 10 \\ &= 16x^4 + 32x^3 + 10. \end{aligned}$$

(b) The formula is

$$y = -\frac{1}{2}f(x) = -\frac{1}{2}x^4 + 2x^3 - 5.$$

Exercises 1.2

Algebraic Combinations

In Exercises 1 and 2, find the domains and ranges of f, g, $f + g$, and $f \cdot g$.

1. $f(x) = x, \quad g(x) = \sqrt{x - 1}$
2. $f(x) = \sqrt{x + 1}, \quad g(x) = \sqrt{x - 1}$

In Exercises 3 and 4, find the domains and ranges of f, g, f/g, and g/f.

3. $f(x) = 2, \quad g(x) = x^2 + 1$
4. $f(x) = 1, \quad g(x) = 1 + \sqrt{x}$

Composites of Functions

5. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.
   - **a.** $f(g(0))$ **b.** $g(f(0))$
   - **c.** $f(g(x))$ **d.** $g(f(x))$
   - **e.** $f(f(-5))$ **f.** $g(g(2))$
   - **g.** $f(f(x))$ **h.** $g(g(x))$
6. If $f(x) = x - 1$ and $g(x) = 1/(x + 1)$, find the following.
   - **a.** $f(g(1/2))$ **b.** $g(f(1/2))$
   - **c.** $f(g(x))$ **d.** $g(f(x))$
   - **e.** $f(f(2))$ **f.** $g(g(2))$
   - **g.** $f(f(x))$ **h.** $g(g(x))$

In Exercises 7–10, write a formula for $f \circ g \circ h$.

7. $f(x) = x + 1, \quad g(x) = 3x, \quad h(x) = 4 - x$
8. $f(x) = 3x + 4, \quad g(x) = 2x - 1, \quad h(x) = x^2$
9. $f(x) = \sqrt{x + 1}, \quad g(x) = \dfrac{1}{x + 4}, \quad h(x) = \dfrac{1}{x}$
10. $f(x) = \dfrac{x + 2}{3 - x}, \quad g(x) = \dfrac{x^2}{x^2 + 1}, \quad h(x) = \sqrt{2 - x}$

Let $f(x) = x - 3$, $g(x) = \sqrt{x}$, $h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 11 and 12 as a composite involving one or more of f, g, h, and j.

11. **a.** $y = \sqrt{x} - 3$ **b.** $y = 2\sqrt{x}$
c. $y = x^{1/4}$ **d.** $y = 4x$
e. $y = \sqrt{(x-3)^3}$ **f.** $y = (2x-6)^3$

12. **a.** $y = 2x - 3$ **b.** $y = x^{3/2}$
c. $y = x^9$ **d.** $y = x - 6$
e. $y = 2\sqrt{x-3}$ **f.** $y = \sqrt{x^3 - 3}$

13. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$x - 7$	$\sqrt{x}$	?
b.	$x + 2$	$3x$	?
c.	?	$\sqrt{x-5}$	$\sqrt{x^2-5}$
d.	$\dfrac{x}{x-1}$	$\dfrac{x}{x-1}$	?
e.	?	$1 + \dfrac{1}{x}$	x
f.	$\dfrac{1}{x}$	?	x

14. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$\dfrac{1}{x-1}$	$\lvert x \rvert$	?
b.	?	$\dfrac{x-1}{x}$	$\dfrac{x}{x+1}$
c.	?	$\sqrt{x}$	$\lvert x \rvert$
d.	$\sqrt{x}$	?	$\lvert x \rvert$

15. Evaluate each expression using the given table of values:

x	-2	-1	0	1	2
$f(x)$	1	0	-2	1	2
$g(x)$	2	1	0	-1	0

a. $f(g(-1))$ **b.** $g(f(0))$ **c.** $f(f(-1))$
d. $g(g(2))$ **e.** $g(f(-2))$ **f.** $f(g(1))$

16. Evaluate each expression using the functions

$$f(x) = 2 - x, \quad g(x) = \begin{cases} -x, & -2 \le x < 0 \\ x - 1, & 0 \le x \le 2. \end{cases}$$

a. $f(g(0))$ **b.** $g(f(3))$ **c.** $g(g(-1))$
d. $f(f(2))$ **e.** $g(f(0))$ **f.** $f(g(1/2))$

In Exercises 17 and 18, **(a)** write formulas for $f \circ g$ and $g \circ f$ and find the **(b)** domain and **(c)** range of each.

17. $f(x) = \sqrt{x+1}$, $g(x) = \dfrac{1}{x}$

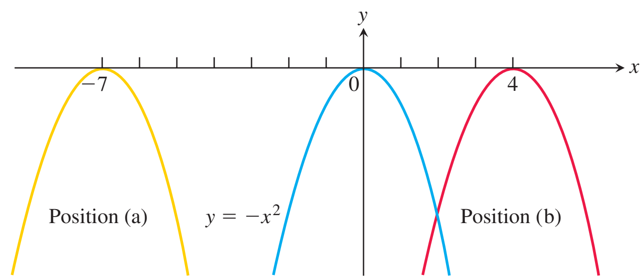
18. $f(x) = x^2$, $g(x) = 1 - \sqrt{x}$

19. Let $f(x) = \dfrac{x}{x-2}$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x$.

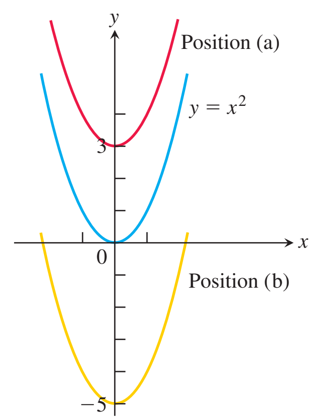
20. Let $f(x) = 2x^3 - 4$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x + 2$.

Shifting Graphs

21. The accompanying figure shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.

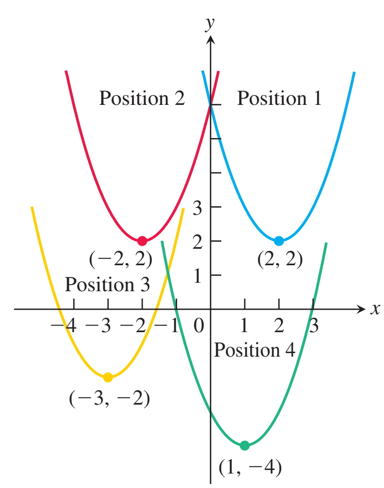


22. The accompanying figure shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.

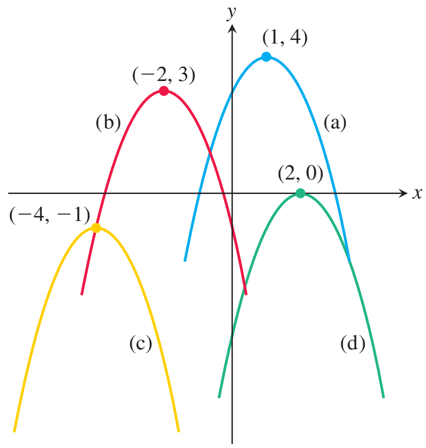


23. Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

a. $y = (x-1)^2 - 4$ **b.** $y = (x-2)^2 + 2$
c. $y = (x+2)^2 + 2$ **d.** $y = (x+3)^2 - 2$



24. The accompanying figure shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



Exercises 25–34 tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation.

25. $x^2 + y^2 = 49$ Down 3, left 2

26. $x^2 + y^2 = 25$ Up 3, left 4

27. $y = x^3$ Left 1, down 1

28. $y = x^{2/3}$ Right 1, down 1

29. $y = \sqrt{x}$ Left 0.81

30. $y = -\sqrt{x}$ Right 3

31. $y = 2x - 7$ Up 7

32. $y = \frac{1}{2}(x + 1) + 5$ Down 5, right 1

33. $y = 1/x$ Up 1, right 1

34. $y = 1/x^2$ Left 2, down 1

Graph the functions in Exercises 35–54.

35. $y = \sqrt{x + 4}$

36. $y = \sqrt{9 - x}$

37. $y = |x - 2|$

38. $y = |1 - x| - 1$

39. $y = 1 + \sqrt{x - 1}$

40. $y = 1 - \sqrt{x}$

41. $y = (x + 1)^{2/3}$

42. $y = (x - 8)^{2/3}$

43. $y = 1 - x^{2/3}$

44. $y + 4 = x^{2/3}$

45. $y = \sqrt[3]{x - 1} - 1$

46. $y = (x + 2)^{3/2} + 1$

47. $y = \dfrac{1}{x - 2}$

48. $y = \dfrac{1}{x} - 2$

49. $y = \dfrac{1}{x} + 2$

50. $y = \dfrac{1}{x + 2}$

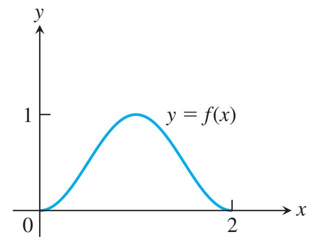
51. $y = \dfrac{1}{(x - 1)^2}$

52. $y = \dfrac{1}{x^2} - 1$

53. $y = \dfrac{1}{x^2} + 1$

54. $y = \dfrac{1}{(x + 1)^2}$

55. The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.



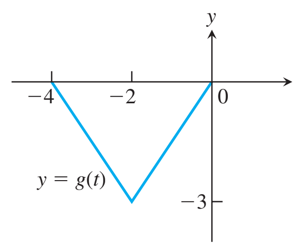
a. $f(x) + 2$ **b.** $f(x) - 1$

c. $2f(x)$ **d.** $-f(x)$

e. $f(x + 2)$ **f.** $f(x - 1)$

g. $f(-x)$ **h.** $-f(x + 1) + 1$

56. The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions, and sketch their graphs.



a. $g(-t)$ **b.** $-g(t)$

c. $g(t) + 3$ **d.** $1 - g(t)$

e. $g(-t + 2)$ **f.** $g(t - 2)$

g. $g(1 - t)$ **h.** $-g(t - 4)$

Vertical and Horizontal Scaling

Exercises 57–66 tell by what factor and direction the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph.

57. $y = x^2 - 1$, stretched vertically by a factor of 3

58. $y = x^2 - 1$, compressed horizontally by a factor of 2

59. $y = 1 + \dfrac{1}{x^2}$, compressed vertically by a factor of 2

60. $y = 1 + \dfrac{1}{x^2}$, stretched horizontally by a factor of 3

61. $y = \sqrt{x + 1}$, compressed horizontally by a factor of 4

62. $y = \sqrt{x + 1}$, stretched vertically by a factor of 3

63. $y = \sqrt{4 - x^2}$, stretched horizontally by a factor of 2

64. $y = \sqrt{4 - x^2}$, compressed vertically by a factor of 3

65. $y = 1 - x^3$, compressed horizontally by a factor of 3

66. $y = 1 - x^3$, stretched horizontally by a factor of 2

Graphing

In Exercises 67–74, graph each function, not by plotting points, but by starting with the graph of one of the standard functions presented in Figures 1.14–1.17 and applying an appropriate transformation.

67. $y = -\sqrt{2x+1}$

68. $y = \sqrt{1 - \dfrac{x}{2}}$

69. $y = (x-1)^3 + 2$

70. $y = (1-x)^3 + 2$

71. $y = \dfrac{1}{2x} - 1$

72. $y = \dfrac{2}{x^2} + 1$

73. $y = -\sqrt[3]{x}$

74. $y = (-2x)^{2/3}$

75. Graph the function $y = |x^2 - 1|$.

76. Graph the function $y = \sqrt{|x|}$.

Combining Functions

77. Assume that f is an even function, g is an odd function, and both f and g are defined on the entire real line $(-\infty, \infty)$. Which of the following (where defined) are even? odd?

a. fg **b.** f/g **c.** g/f

d. $f^2 = ff$ **e.** $g^2 = gg$ **f.** $f \circ g$

g. $g \circ f$ **h.** $f \circ f$ **i.** $g \circ g$

78. Can a function be both even and odd? Give reasons for your answer.

T **79.** (*Continuation of Example 1.*) Graph the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$ together with their (a) sum, (b) product, (c) two differences, (d) two quotients.

T **80.** Let $f(x) = x - 7$ and $g(x) = x^2$. Graph f and g together with $f \circ g$ and $g \circ f$.

1.3 Trigonometric Functions

This section reviews radian measure and the basic trigonometric functions.

Angles

Angles are measured in degrees or radians. The number of **radians** in the central angle $A'CB'$ within a circle of radius r is defined as the number of "radius units" contained in the arc s subtended by that central angle. If we denote this central angle by θ when measured in radians, this means that $\theta = s/r$ (Figure 1.36), or

$$s = r\theta \qquad (\theta \text{ in radians}). \tag{1}$$

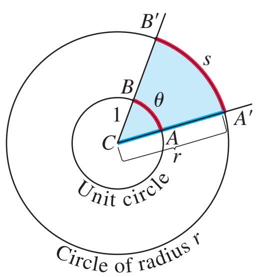


FIGURE 1.36 The radian measure of the central angle $A'CB'$ is the number $\theta = s/r$. For a unit circle of radius $r = 1$, θ is the length of arc AB that central angle ACB cuts from the unit circle.

If the circle is a unit circle having radius $r = 1$, then from Figure 1.36 and Equation (1), we see that the central angle θ measured in radians is just the length of the arc that the angle cuts from the unit circle. Since one complete revolution of the unit circle is 360° or 2π radians, we have

$$\pi \text{ radians} = 180° \tag{2}$$

and

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57.3) \text{ degrees} \qquad \text{or} \qquad 1 \text{ degree} = \frac{\pi}{180} (\approx 0.017) \text{ radians}.$$

Table 1.1 shows the equivalence between degree and radian measures for some basic angles.

TABLE 1.1 Angles measured in degrees and radians

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Q	Slope of $PQ = \Delta p/\Delta t$ (flies / day)
(45, 340)	$\dfrac{340-150}{45-23} \approx 8.6$
(40, 330)	$\dfrac{330-150}{40-23} \approx 10.6$
(35, 310)	$\dfrac{310-150}{35-23} \approx 13.3$
(30, 265)	$\dfrac{265-150}{30-23} \approx 16.4$

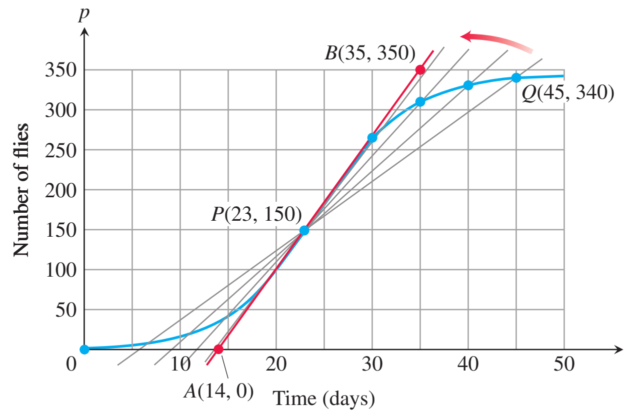


FIGURE 2.6 The positions and slopes of four secants through the point P on the fruit fly graph (Example 5).

The values in the table show that the secant slopes rise from 8.6 to 16.4 as the t-coordinate of Q decreases from 45 to 30, and we would expect the slopes to rise slightly higher as t continued on toward 23. Geometrically, the secants rotate counterclockwise about P and seem to approach the red tangent line in the figure. Since the line appears to pass through the points (14, 0) and (35, 350), it has slope

$$\frac{350-0}{35-14} = 16.7 \text{ flies/day (approximately).}$$

On day 23 the population was increasing at a rate of about 16.7 flies / day.

The instantaneous rates in Example 2 were found to be the values of the average speeds, or average rates of change, as the time interval of length h approached 0. That is, the instantaneous rate is the value the average rate approaches as the length h of the interval over which the change occurs approaches zero. The average rate of change corresponds to the slope of a secant line; the instantaneous rate corresponds to the slope of the tangent line as the independent variable approaches a fixed value. In Example 2, the independent variable t approached the values $t = 1$ and $t = 2$. In Example 3, the independent variable x approached the value $x = 2$. So we see that instantaneous rates and slopes of tangent lines are closely connected. We investigate this connection thoroughly in the next chapter, but to do so we need the concept of a *limit*.

Exercises 2.1

Average Rates of Change

In Exercises 1–6, find the average rate of change of the function over the given interval or intervals.

1. $f(x) = x^3 + 1$

a. $[2, 3]$ **b.** $[-1, 1]$

2. $g(x) = x^2 - 2x$

a. $[1, 3]$ **b.** $[-2, 4]$

3. $h(t) = \cot t$

a. $[\pi/4, 3\pi/4]$ **b.** $[\pi/6, \pi/2]$

4. $g(t) = 2 + \cos t$

a. $[0, \pi]$ **b.** $[-\pi, \pi]$

5. $R(\theta) = \sqrt{4\theta + 1}$; $[0, 2]$

6. $P(\theta) = \theta^3 - 4\theta^2 + 5\theta$; $[1, 2]$

Slope of a Curve at a Point

In Exercises 7–14, use the method in Example 3 to find **(a)** the slope of the curve at the given point P, and **(b)** an equation of the tangent line at P.

7. $y = x^2 - 5$, $P(2, -1)$

8. $y = 7 - x^2$, $P(2, 3)$

9. $y = x^2 - 2x - 3$, $P(2, -3)$

10. $y = x^2 - 4x$, $P(1, -3)$

11. $y = x^3$, $P(2, 8)$

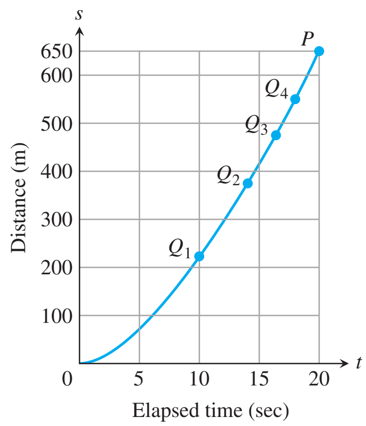
12. $y = 2 - x^3, \quad P(1, 1)$

13. $y = x^3 - 12x, \quad P(1, -11)$

14. $y = x^3 - 3x^2 + 4, \quad P(2, 0)$

Instantaneous Rates of Change

15. Speed of a car The accompanying figure shows the time-to-distance graph for a sports car accelerating from a standstill.



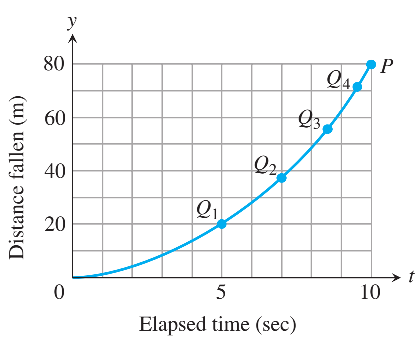
a. Estimate the slopes of secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in order in a table like the one in Figure 2.6. What are the appropriate units for these slopes?

b. Then estimate the car's speed at time $t = 20$ sec.

16. The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80 m to the surface of the moon.

a. Estimate the slopes of the secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in a table like the one in Figure 2.6.

b. About how fast was the object going when it hit the surface?



T **17.** The profits of a small company for each of the first five years of its operation are given in the following table:

Year	Profit in \$1000s
2010	6
2011	27
2012	62
2013	111
2014	174

a. Plot points representing the profit as a function of year, and join them by as smooth a curve as you can.

b. What is the average rate of increase of the profits between 2012 and 2014?

c. Use your graph to estimate the rate at which the profits were changing in 2012.

T **18.** Make a table of values for the function $F(x) = (x + 2)/(x - 2)$ at the points $x = 1.2$, $x = 11/10$, $x = 101/100$, $x = 1001/1000$, $x = 10001/10000$, and $x = 1$.

a. Find the average rate of change of $F(x)$ over the intervals $[1, x]$ for each $x \neq 1$ in your table.

b. Extending the table if necessary, try to determine the rate of change of $F(x)$ at $x = 1$.

T **19.** Let $g(x) = \sqrt{x}$ for $x \geq 0$.

a. Find the average rate of change of $g(x)$ with respect to x over the intervals $[1, 2]$, $[1, 1.5]$ and $[1, 1 + h]$.

b. Make a table of values of the average rate of change of g with respect to x over the interval $[1, 1 + h]$ for some values of h approaching zero, say $h = 0.1, 0.01, 0.001, 0.0001, 0.00001$, and 0.000001.

c. What does your table indicate is the rate of change of $g(x)$ with respect to x at $x = 1$?

d. Calculate the limit as h approaches zero of the average rate of change of $g(x)$ with respect to x over the interval $[1, 1 + h]$.

T **20.** Let $f(t) = 1/t$ for $t \neq 0$.

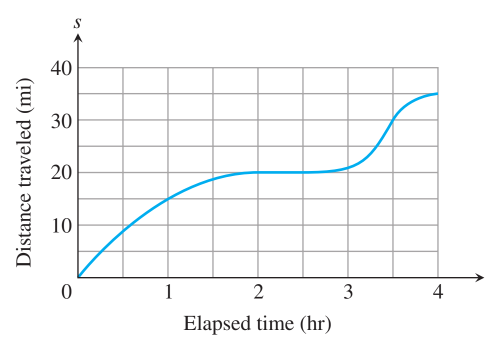
a. Find the average rate of change of f with respect to t over the intervals (i) from $t = 2$ to $t = 3$, and (ii) from $t = 2$ to $t = T$.

b. Make a table of values of the average rate of change of f with respect to t over the interval $[2, T]$, for some values of T approaching 2, say $T = 2.1, 2.01, 2.001, 2.0001, 2.00001$, and 2.000001.

c. What does your table indicate is the rate of change of f with respect to t at $t = 2$?

d. Calculate the limit as T approaches 2 of the average rate of change of f with respect to t over the interval from 2 to T. You will have to do some algebra before you can substitute $T = 2$.

21. The accompanying graph shows the total distance s traveled by a bicyclist after t hours.

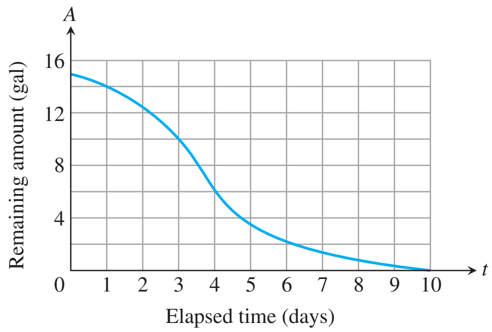


a. Estimate the bicyclist's average speed over the time intervals $[0, 1]$, $[1, 2.5]$, and $[2.5, 3.5]$.

b. Estimate the bicyclist's instantaneous speed at the times $t = \frac{1}{2}$, $t = 2$, and $t = 3$.

c. Estimate the bicyclist's maximum speed and the specific time at which it occurs.

22. The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for t days.



a. Estimate the average rate of gasoline consumption over the time intervals $[0, 3]$, $[0, 5]$, and $[7, 10]$.

b. Estimate the instantaneous rate of gasoline consumption at the times $t = 1$, $t = 4$, and $t = 8$.

c. Estimate the maximum rate of gasoline consumption and the specific time at which it occurs.

2.2 Limit of a Function and Limit Laws

In Section 2.1 we saw that limits arise when finding the instantaneous rate of change of a function or the tangent to a curve. Here we begin with an informal definition of *limit* and show how we can calculate the values of limits. A precise definition is presented in the next section.

HISTORICAL ESSAY
Limits

Limits of Function Values

Frequently when studying a function $y = f(x)$, we find ourselves interested in the function's behavior *near* a particular point c, but not *at* c. This might be the case, for instance, if c is an irrational number, like π or $\sqrt{2}$, whose values can only be approximated by "close" rational numbers at which we actually evaluate the function instead. Another situation occurs when trying to evaluate a function at c leads to division by zero, which is undefined. We encountered this last circumstance when seeking the instantaneous rate of change in y by considering the quotient function $\Delta y/h$ for h closer and closer to zero. Here's a specific example in which we explore numerically how a function behaves near a particular point at which we cannot directly evaluate the function.

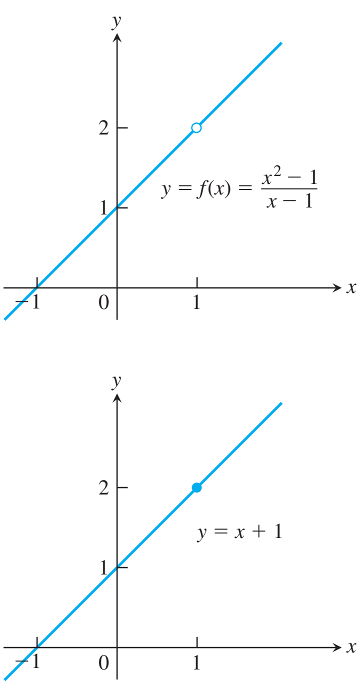


FIGURE 2.7 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

EXAMPLE 1 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution The given formula defines f for all real numbers x except $x = 1$ (we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \qquad \text{for} \qquad x \neq 1.$$

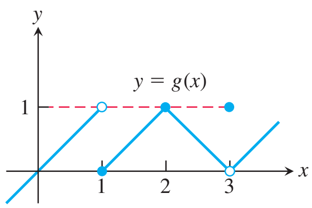
The graph of f is the line $y = x + 1$ with the point (1, 2) *removed*. This removed point is shown as a "hole" in Figure 2.7. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ *as close as we want* to 2 by choosing x *close enough* to 1 (Table 2.2). ■

Exercises 2.2

Limits from Graphs

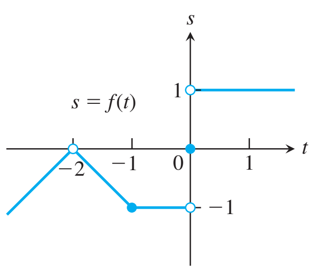
1. For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

   a. $\lim_{x\to 1} g(x)$ **b.** $\lim_{x\to 2} g(x)$ **c.** $\lim_{x\to 3} g(x)$ **d.** $\lim_{x\to 2.5} g(x)$



2. For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.

   a. $\lim_{t\to -2} f(t)$ **b.** $\lim_{t\to -1} f(t)$ **c.** $\lim_{t\to 0} f(t)$ **d.** $\lim_{t\to -0.5} f(t)$



3. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

   a. $\lim_{x\to 0} f(x)$ exists.

   b. $\lim_{x\to 0} f(x) = 0$

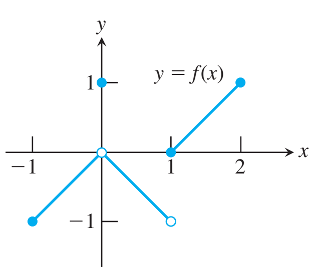
   c. $\lim_{x\to 0} f(x) = 1$

   d. $\lim_{x\to 1} f(x) = 1$

   e. $\lim_{x\to 1} f(x) = 0$

   f. $\lim_{x\to c} f(x)$ exists at every point c in $(-1, 1)$.

   g. $\lim_{x\to 1} f(x)$ does not exist.



4. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

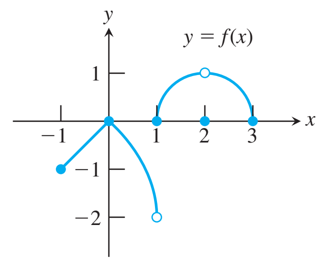
   a. $\lim_{x\to 2} f(x)$ does not exist.

   b. $\lim_{x\to 2} f(x) = 2$

   c. $\lim_{x\to 1} f(x)$ does not exist.

   d. $\lim_{x\to c} f(x)$ exists at every point c in $(-1, 1)$.

   e. $\lim_{x\to c} f(x)$ exists at every point c in $(1, 3)$.



Existence of Limits

In Exercises 5 and 6, explain why the limits do not exist.

5. $\lim_{x\to 0} \dfrac{x}{|x|}$

6. $\lim_{x\to 1} \dfrac{1}{x-1}$

7. Suppose that a function $f(x)$ is defined for all real values of x except $x = c$. Can anything be said about the existence of $\lim_{x\to c} f(x)$? Give reasons for your answer.

8. Suppose that a function $f(x)$ is defined for all x in $[-1, 1]$. Can anything be said about the existence of $\lim_{x\to 0} f(x)$? Give reasons for your answer.

9. If $\lim_{x\to 1} f(x) = 5$, must f be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude *anything* about the values of f at $x = 1$? Explain.

10. If $f(1) = 5$, must $\lim_{x\to 1} f(x)$ exist? If it does, then must $\lim_{x\to 1} f(x) = 5$? Can we conclude *anything* about $\lim_{x\to 1} f(x)$? Explain.

Calculating Limits

Find the limits in Exercises 11–22.

11. $\lim_{x\to -3} (x^2 - 13)$

12. $\lim_{x\to 2} (-x^2 + 5x - 2)$

13. $\lim_{t\to 6} 8(t-5)(t-7)$

14. $\lim_{x\to -2} (x^3 - 2x^2 + 4x + 8)$

15. $\lim_{x\to 2} \dfrac{2x+5}{11-x^3}$

16. $\lim_{s\to 2/3} (8-3s)(2s-1)$

17. $\lim_{x\to -1/2} 4x(3x+4)^2$

18. $\lim_{y\to 2} \dfrac{y+2}{y^2+5y+6}$

19. $\lim_{y\to -3} (5-y)^{4/3}$

20. $\lim_{z\to 4} \sqrt{z^2-10}$

21. $\lim_{h\to 0} \dfrac{3}{\sqrt{3h+1}+1}$

22. $\lim_{h\to 0} \dfrac{\sqrt{5h+4}-2}{h}$

Limits of quotients Find the limits in Exercises 23–42.

23. $\lim_{x\to 5} \dfrac{x-5}{x^2-25}$

24. $\lim_{x\to -3} \dfrac{x+3}{x^2+4x+3}$

25. $\lim_{x\to -5} \dfrac{x^2+3x-10}{x+5}$

26. $\lim_{x\to 2} \dfrac{x^2-7x+10}{x-2}$

27. $\lim_{t\to 1} \dfrac{t^2+t-2}{t^2-1}$

28. $\lim_{t\to -1} \dfrac{t^2+3t+2}{t^2-t-2}$

29. $\lim_{x\to -2} \dfrac{-2x-4}{x^3+2x^2}$

30. $\lim_{y\to 0} \dfrac{5y^3+8y^2}{3y^4-16y^2}$

31. $\lim_{x\to1}\dfrac{x^{-1}-1}{x-1}$

32. $\lim_{x\to0}\dfrac{\frac{1}{x-1}+\frac{1}{x+1}}{x}$

33. $\lim_{u\to1}\dfrac{u^4-1}{u^3-1}$

34. $\lim_{v\to2}\dfrac{v^3-8}{v^4-16}$

35. $\lim_{x\to9}\dfrac{\sqrt{x}-3}{x-9}$

36. $\lim_{x\to4}\dfrac{4x-x^2}{2-\sqrt{x}}$

37. $\lim_{x\to1}\dfrac{x-1}{\sqrt{x+3}-2}$

38. $\lim_{x\to-1}\dfrac{\sqrt{x^2+8}-3}{x+1}$

39. $\lim_{x\to2}\dfrac{\sqrt{x^2+12}-4}{x-2}$

40. $\lim_{x\to-2}\dfrac{x+2}{\sqrt{x^2+5}-3}$

41. $\lim_{x\to-3}\dfrac{2-\sqrt{x^2-5}}{x+3}$

42. $\lim_{x\to4}\dfrac{4-x}{5-\sqrt{x^2+9}}$

Limits with trigonometric functions Find the limits in Exercises 43–50.

43. $\lim_{x\to0}(2\sin x-1)$

44. $\lim_{x\to\pi/4}\sin^2 x$

45. $\lim_{x\to0}\sec x$

46. $\lim_{x\to\pi/3}\tan x$

47. $\lim_{x\to0}\dfrac{1+x+\sin x}{3\cos x}$

48. $\lim_{x\to0}(x^2-1)(2-\cos x)$

49. $\lim_{x\to-\pi}\sqrt{x+4}\cos(x+\pi)$

50. $\lim_{x\to0}\sqrt{7+\sec^2 x}$

Using Limit Rules

51. Suppose $\lim_{x\to0} f(x)=1$ and $\lim_{x\to0} g(x)=-5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to0}\frac{2f(x)-g(x)}{(f(x)+7)^{2/3}}=\frac{\lim_{x\to0}(2f(x)-g(x))}{\lim_{x\to0}(f(x)+7)^{2/3}}\qquad(a)$$

$$=\frac{\lim_{x\to0}2f(x)-\lim_{x\to0}g(x)}{\left(\lim_{x\to0}(f(x)+7)\right)^{2/3}}\qquad(b)$$

$$=\frac{2\lim_{x\to0}f(x)-\lim_{x\to0}g(x)}{\left(\lim_{x\to0}f(x)+\lim_{x\to0}7\right)^{2/3}}\qquad(c)$$

$$=\frac{(2)(1)-(-5)}{(1+7)^{2/3}}=\frac{7}{4}$$

52. Let $\lim_{x\to1} h(x)=5$, $\lim_{x\to1} p(x)=1$, and $\lim_{x\to1} r(x)=2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to1}\frac{\sqrt{5h(x)}}{p(x)(4-r(x))}=\frac{\lim_{x\to1}\sqrt{5h(x)}}{\lim_{x\to1}(p(x)(4-r(x)))}\qquad(a)$$

$$=\frac{\sqrt{\lim_{x\to1}5h(x)}}{\left(\lim_{x\to1}p(x)\right)\left(\lim_{x\to1}(4-r(x))\right)}\qquad(b)$$

$$=\frac{\sqrt{5\lim_{x\to1}h(x)}}{\left(\lim_{x\to1}p(x)\right)\left(\lim_{x\to1}4-\lim_{x\to1}r(x)\right)}\qquad(c)$$

$$=\frac{\sqrt{(5)(5)}}{(1)(4-2)}=\frac{5}{2}$$

53. Suppose $\lim_{x\to c} f(x)=5$ and $\lim_{x\to c} g(x)=-2$. Find

a. $\lim_{x\to c} f(x)g(x)$ b. $\lim_{x\to c} 2f(x)g(x)$

c. $\lim_{x\to c}(f(x)+3g(x))$ d. $\lim_{x\to c}\dfrac{f(x)}{f(x)-g(x)}$

54. Suppose $\lim_{x\to4} f(x)=0$ and $\lim_{x\to4} g(x)=-3$. Find

a. $\lim_{x\to4}(g(x)+3)$ b. $\lim_{x\to4} xf(x)$

c. $\lim_{x\to4}(g(x))^2$ d. $\lim_{x\to4}\dfrac{g(x)}{f(x)-1}$

55. Suppose $\lim_{x\to b} f(x)=7$ and $\lim_{x\to b} g(x)=-3$. Find

a. $\lim_{x\to b}(f(x)+g(x))$ b. $\lim_{x\to b} f(x)\cdot g(x)$

c. $\lim_{x\to b} 4g(x)$ d. $\lim_{x\to b} f(x)/g(x)$

56. Suppose that $\lim_{x\to-2} p(x)=4$, $\lim_{x\to-2} r(x)=0$, and $\lim_{x\to-2} s(x)=-3$. Find

a. $\lim_{x\to-2}(p(x)+r(x)+s(x))$

b. $\lim_{x\to-2} p(x)\cdot r(x)\cdot s(x)$

c. $\lim_{x\to-2}(-4p(x)+5r(x))/s(x)$

Limits of Average Rates of Change

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

occur frequently in calculus. In Exercises 57–62, evaluate this limit for the given value of x and function f.

57. $f(x)=x^2,\quad x=1$

58. $f(x)=x^2,\quad x=-2$

59. $f(x)=3x-4,\quad x=2$

60. $f(x)=1/x,\quad x=-2$

61. $f(x)=\sqrt{x},\quad x=7$

62. $f(x)=\sqrt{3x+1},\quad x=0$

Using the Sandwich Theorem

63. If $\sqrt{5-2x^2}\le f(x)\le\sqrt{5-x^2}$ for $-1\le x\le 1$, find $\lim_{x\to0} f(x)$.

64. If $2-x^2\le g(x)\le 2\cos x$ for all x, find $\lim_{x\to0} g(x)$.

65. a. It can be shown that the inequalities

$$1-\frac{x^2}{6}<\frac{x\sin x}{2-2\cos x}<1$$

hold for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x\to0}\frac{x\sin x}{2-2\cos x}?$$

Give reasons for your answer.

T **b.** Graph $y=1-(x^2/6)$, $y=(x\sin x)/(2-2\cos x)$, and $y=1$ together for $-2\le x\le 2$. Comment on the behavior of the graphs as $x\to0$.

66. **a.** Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero. (They do, as you will see in Section 9.9.) What, if anything, does this tell you about

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2}?$$

Give reasons for your answer.

T **b.** Graph the equations $y = (1/2) - (x^2/24)$, $y = (1 - \cos x)/x^2$, and $y = 1/2$ together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

Estimating Limits

T You will find a graphing calculator useful for Exercises 67–76.

67. Let $f(x) = (x^2 - 9)/(x + 3)$.

a. Make a table of the values of f at the points $x = -3.1, -3.01, -3.001$, and so on as far as your calculator can go. Then estimate $\lim_{x\to -3} f(x)$. What estimate do you arrive at if you evaluate f at $x = -2.9, -2.99, -2.999, \ldots$ instead?

b. Support your conclusions in part (a) by graphing f near $c = -3$ and using Zoom and Trace to estimate y-values on the graph as $x \to -3$.

c. Find $\lim_{x\to -3} f(x)$ algebraically, as in Example 7.

68. Let $g(x) = (x^2 - 2)/(x - \sqrt{2})$.

a. Make a table of the values of g at the points $x = 1.4, 1.41, 1.414$, and so on through successive decimal approximations of $\sqrt{2}$. Estimate $\lim_{x\to\sqrt{2}} g(x)$.

b. Support your conclusion in part (a) by graphing g near $c = \sqrt{2}$ and using Zoom and Trace to estimate y-values on the graph as $x \to \sqrt{2}$.

c. Find $\lim_{x\to\sqrt{2}} g(x)$ algebraically.

69. Let $G(x) = (x + 6)/(x^2 + 4x - 12)$.

a. Make a table of the values of G at $x = -5.9, -5.99, -5.999$, and so on. Then estimate $\lim_{x\to -6} G(x)$. What estimate do you arrive at if you evaluate G at $x = -6.1, -6.01, -6.001, \ldots$ instead?

b. Support your conclusions in part (a) by graphing G and using Zoom and Trace to estimate y-values on the graph as $x \to -6$.

c. Find $\lim_{x\to -6} G(x)$ algebraically.

70. Let $h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$.

a. Make a table of the values of h at $x = 2.9, 2.99, 2.999$, and so on. Then estimate $\lim_{x\to 3} h(x)$. What estimate do you arrive at if you evaluate h at $x = 3.1, 3.01, 3.001, \ldots$ instead?

b. Support your conclusions in part (a) by graphing h near $c = 3$ and using Zoom and Trace to estimate y-values on the graph as $x \to 3$.

c. Find $\lim_{x\to 3} h(x)$ algebraically.

71. Let $f(x) = (x^2 - 1)/(|x| - 1)$.

a. Make tables of the values of f at values of x that approach $c = -1$ from above and below. Then estimate $\lim_{x\to -1} f(x)$.

b. Support your conclusion in part (a) by graphing f near $c = -1$ and using Zoom and Trace to estimate y-values on the graph as $x \to -1$.

c. Find $\lim_{x\to -1} f(x)$ algebraically.

72. Let $F(x) = (x^2 + 3x + 2)/(2 - |x|)$.

a. Make tables of values of F at values of x that approach $c = -2$ from above and below. Then estimate $\lim_{x\to -2} F(x)$.

b. Support your conclusion in part (a) by graphing F near $c = -2$ and using Zoom and Trace to estimate y-values on the graph as $x \to -2$.

c. Find $\lim_{x\to -2} F(x)$ algebraically.

73. Let $g(\theta) = (\sin\theta)/\theta$.

a. Make a table of the values of g at values of θ that approach $\theta_0 = 0$ from above and below. Then estimate $\lim_{\theta\to 0} g(\theta)$.

b. Support your conclusion in part (a) by graphing g near $\theta_0 = 0$.

74. Let $G(t) = (1 - \cos t)/t^2$.

a. Make tables of values of G at values of t that approach $t_0 = 0$ from above and below. Then estimate $\lim_{t\to 0} G(t)$.

b. Support your conclusion in part (a) by graphing G near $t_0 = 0$.

75. Let $f(x) = x^{1/(1-x)}$.

a. Make tables of values of f at values of x that approach $c = 1$ from above and below. Does f appear to have a limit as $x \to 1$? If so, what is it? If not, why not?

b. Support your conclusions in part (a) by graphing f near $c = 1$.

76. Let $f(x) = (3^x - 1)/x$.

a. Make tables of values of f at values of x that approach $c = 0$ from above and below. Does f appear to have a limit as $x \to 0$? If so, what is it? If not, why not?

b. Support your conclusions in part (a) by graphing f near $c = 0$.

Theory and Examples

77. If $x^4 \le f(x) \le x^2$ for x in $[-1, 1]$ and $x^2 \le f(x) \le x^4$ for $x < -1$ and $x > 1$, at what points c do you automatically know $\lim_{x\to c} f(x)$? What can you say about the value of the limit at these points?

78. Suppose that $g(x) \le f(x) \le h(x)$ for all $x \ne 2$ and suppose that

$$\lim_{x\to 2} g(x) = \lim_{x\to 2} h(x) = -5.$$

Can we conclude anything about the values of f, g, and h at $x = 2$? Could $f(2) = 0$? Could $\lim_{x\to 2} f(x) = 0$? Give reasons for your answers.

79. If $\displaystyle\lim_{x\to 4} \frac{f(x) - 5}{x - 2} = 1$, find $\displaystyle\lim_{x\to 4} f(x)$.

80. If $\displaystyle\lim_{x\to -2} \frac{f(x)}{x^2} = 1$, find

a. $\displaystyle\lim_{x\to -2} f(x)$ **b.** $\displaystyle\lim_{x\to -2} \frac{f(x)}{x}$

81. **a.** If $\displaystyle\lim_{x\to 2} \frac{f(x) - 5}{x - 2} = 3$, find $\displaystyle\lim_{x\to 2} f(x)$.

b. If $\displaystyle\lim_{x\to 2} \frac{f(x) - 5}{x - 2} = 4$, find $\displaystyle\lim_{x\to 2} f(x)$.

82. If $\lim_{x\to 0} \dfrac{f(x)}{x^2} = 1$, find

a. $\lim_{x\to 0} f(x)$

b. $\lim_{x\to 0} \dfrac{f(x)}{x}$

T **83.** **a.** Graph $g(x) = x\sin(1/x)$ to estimate $\lim_{x\to 0} g(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

T **84.** **a.** Graph $h(x) = x^2\cos(1/x^3)$ to estimate $\lim_{x\to 0} h(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

85. $\lim_{x\to 2} \dfrac{x^4 - 16}{x - 2}$

86. $\lim_{x\to -1} \dfrac{x^3 - x^2 - 5x - 3}{(x+1)^2}$

87. $\lim_{x\to 0} \dfrac{\sqrt[3]{1+x} - 1}{x}$

88. $\lim_{x\to 3} \dfrac{x^2 - 9}{\sqrt{x^2+7} - 4}$

89. $\lim_{x\to 0} \dfrac{1 - \cos x}{x\sin x}$

90. $\lim_{x\to 0} \dfrac{2x^2}{3 - 3\cos x}$

COMPUTER EXPLORATIONS

Graphical Estimates of Limits

In Exercises 85–90, use a CAS to perform the following steps:

a. Plot the function near the point c being approached.

b. From your plot guess the value of the limit.

2.3 The Precise Definition of a Limit

We now turn our attention to the precise definition of a limit. We replace vague phrases like "gets arbitrarily close to" in the informal definition with specific conditions that can be applied to any particular example. With a precise definition, we can avoid misunderstandings, prove the limit properties given in the preceding section, and establish many important limits.

To show that the limit of $f(x)$ as $x \to c$ equals the number L, we need to show that the gap between $f(x)$ and L can be made "as small as we choose" if x is kept "close enough" to c. Let us see what this would require if we specified the size of the gap between $f(x)$ and L.

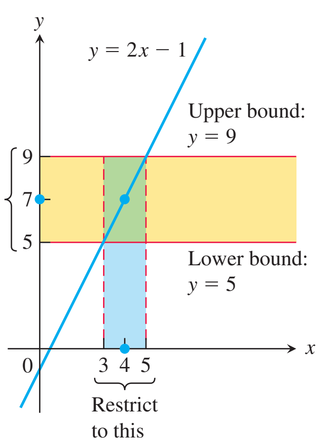


FIGURE 2.15 Keeping x within 1 unit of $x = 4$ will keep y within 2 units of $y = 7$ (Example 1).

EXAMPLE 1 Consider the function $y = 2x - 1$ near $x = 4$. Intuitively it appears that y is close to 7 when x is close to 4, so $\lim_{x\to 4}(2x - 1) = 7$. However, how close to $x = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

Solution We are asked: For what values of x is $|y - 7| < 2$? To find the answer we first express $|y - 7|$ in terms of x:

$$|y - 7| = |(2x - 1) - 7| = |2x - 8|.$$

The question then becomes: what values of x satisfy the inequality $|2x - 8| < 2$? To find out, we solve the inequality:

$$\begin{aligned} |2x - 8| &< 2 \\ -2 < 2x - 8 &< 2 \\ 6 < 2x &< 10 \\ 3 < x &< 5 && \text{Solve for } x. \\ -1 < x - 4 &< 1. && \text{Solve for } x - 4. \end{aligned}$$

Keeping x within 1 unit of $x = 4$ will keep y within 2 units of $y = 7$ (Figure 2.15). ■

Therefore, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|(g(x) - f(x)) - (M - L)| < \epsilon \qquad \text{whenever} \quad 0 < |x - c| < \delta.$$

Since $L - M > 0$ by hypothesis, we take $\epsilon = L - M$ in particular and we have a number $\delta > 0$ such that

$$|(g(x) - f(x)) - (M - L)| < L - M \qquad \text{whenever} \quad 0 < |x - c| < \delta.$$

Since $a \le |a|$ for any number a, we have

$$(g(x) - f(x)) - (M - L) < L - M \qquad \text{whenever} \quad 0 < |x - c| < \delta$$

which simplifies to

$$g(x) < f(x) \qquad \text{whenever} \quad 0 < |x - c| < \delta.$$

But this contradicts $f(x) \le g(x)$. Thus the inequality $L > M$ must be false. Therefore $L \le M$. ■

Exercises 2.3

Centering Intervals About a Point

In Exercises 1–6, sketch the interval (a, b) on the x-axis with the point c inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - c| < \delta \Rightarrow a < x < b$.

1. $a = 1, \quad b = 7, \quad c = 5$

2. $a = 1, \quad b = 7, \quad c = 2$

3. $a = -7/2, \quad b = -1/2, \quad c = -3$

4. $a = -7/2, \quad b = -1/2, \quad c = -3/2$

5. $a = 4/9, \quad b = 4/7, \quad c = 1/2$

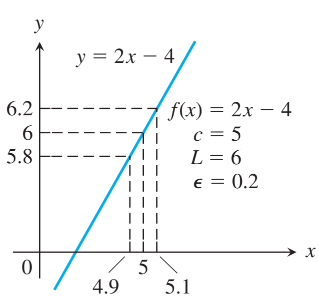
6. $a = 2.7591, \quad b = 3.2391, \quad c = 3$

Finding Deltas Graphically

In Exercises 7–14, use the graphs to find a $\delta > 0$ such that for all x

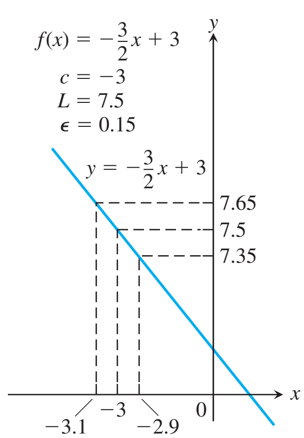
$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

7.



NOT TO SCALE

8.



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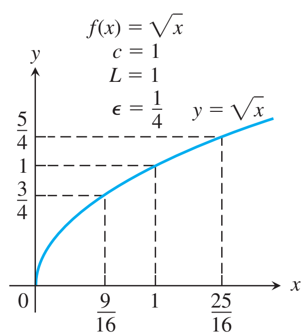

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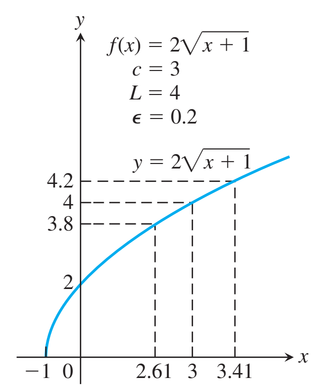

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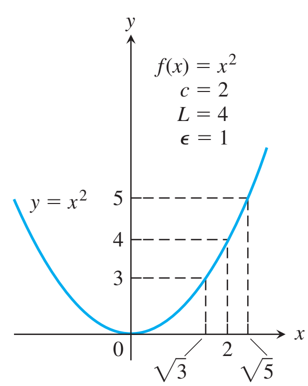

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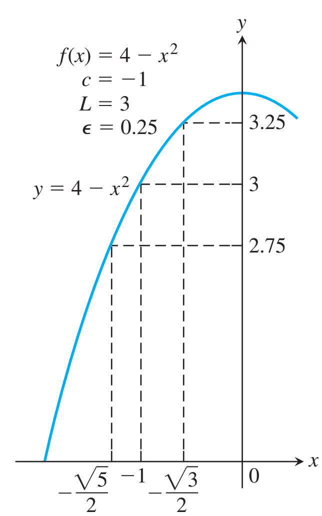

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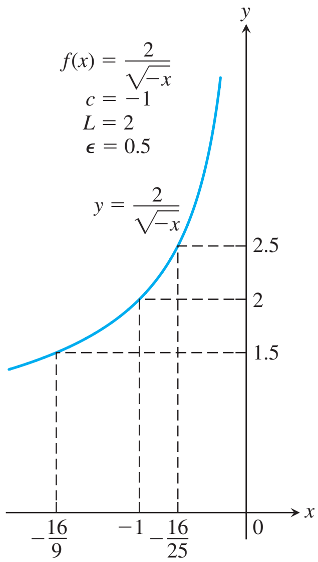

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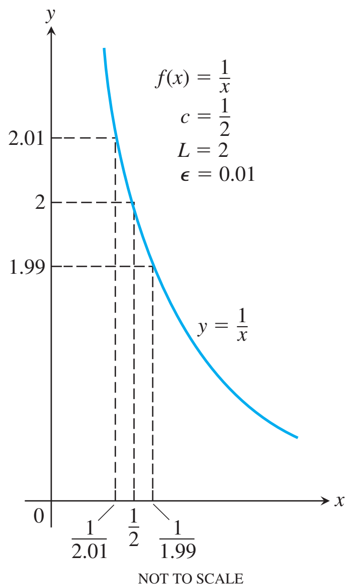


Finding Deltas Algebraically

Each of Exercises 15–30 gives a function $f(x)$ and numbers L, c, and $\epsilon > 0$. In each case, find an open interval about c on which the inequality $|f(x) - L| < \epsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$ the inequality $|f(x) - L| < \epsilon$ holds.

15. $f(x) = x + 1, \quad L = 5, \quad c = 4, \quad \epsilon = 0.01$

16. $f(x) = 2x - 2, \quad L = -6, \quad c = -2, \quad \epsilon = 0.02$

17. $f(x) = \sqrt{x + 1}, \quad L = 1, \quad c = 0, \quad \epsilon = 0.1$

18. $f(x) = \sqrt{x}, \quad L = 1/2, \quad c = 1/4, \quad \epsilon = 0.1$

19. $f(x) = \sqrt{19 - x}, \quad L = 3, \quad c = 10, \quad \epsilon = 1$

20. $f(x) = \sqrt{x - 7}, \quad L = 4, \quad c = 23, \quad \epsilon = 1$

21. $f(x) = 1/x, \quad L = 1/4, \quad c = 4, \quad \epsilon = 0.05$

22. $f(x) = x^2, \quad L = 3, \quad c = \sqrt{3}, \quad \epsilon = 0.1$

23. $f(x) = x^2, \quad L = 4, \quad c = -2, \quad \epsilon = 0.5$

24. $f(x) = 1/x, \quad L = -1, \quad c = -1, \quad \epsilon = 0.1$

25. $f(x) = x^2 - 5, \quad L = 11, \quad c = 4, \quad \epsilon = 1$

26. $f(x) = 120/x, \quad L = 5, \quad c = 24, \quad \epsilon = 1$

27. $f(x) = mx, \quad m > 0, \quad L = 2m, \quad c = 2, \quad \epsilon = 0.03$

28. $f(x) = mx, \quad m > 0, \quad L = 3m, \quad c = 3, \quad \epsilon = c > 0$

29. $f(x) = mx + b, \quad m > 0, \quad L = (m/2) + b, \quad c = 1/2, \quad \epsilon = c > 0$

30. $f(x) = mx + b, \quad m > 0, \quad L = m + b, \quad c = 1, \quad \epsilon = 0.05$

Using the Formal Definition

Each of Exercises 31–36 gives a function $f(x)$, a point c, and a positive number ϵ. Find $L = \lim_{x \to c} f(x)$. Then find a number $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

31. $f(x) = 3 - 2x, \quad c = 3, \quad \epsilon = 0.02$

32. $f(x) = -3x - 2, \quad c = -1, \quad \epsilon = 0.03$

33. $f(x) = \dfrac{x^2 - 4}{x - 2}, \quad c = 2, \quad \epsilon = 0.05$

34. $f(x) = \dfrac{x^2 + 6x + 5}{x + 5}, \quad c = -5, \quad \epsilon = 0.05$

35. $f(x) = \sqrt{1 - 5x}, \quad c = -3, \quad \epsilon = 0.5$

36. $f(x) = 4/x, \quad c = 2, \quad \epsilon = 0.4$

Prove the limit statements in Exercises 37–50.

37. $\lim_{x \to 4} (9 - x) = 5$

38. $\lim_{x \to 3} (3x - 7) = 2$

39. $\lim_{x \to 9} \sqrt{x - 5} = 2$

40. $\lim_{x \to 0} \sqrt{4 - x} = 2$

41. $\lim_{x \to 1} f(x) = 1 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$

42. $\lim_{x \to -2} f(x) = 4 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq -2 \\ 1, & x = -2 \end{cases}$

43. $\lim_{x \to 1} \dfrac{1}{x} = 1$

44. $\lim_{x \to \sqrt{3}} \dfrac{1}{x^2} = \dfrac{1}{3}$

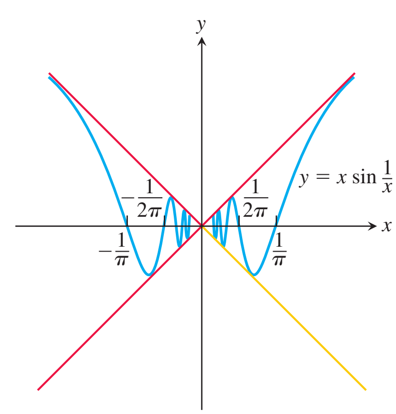
45. $\lim_{x \to -3} \dfrac{x^2 - 9}{x + 3} = -6$

46. $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} = 2$

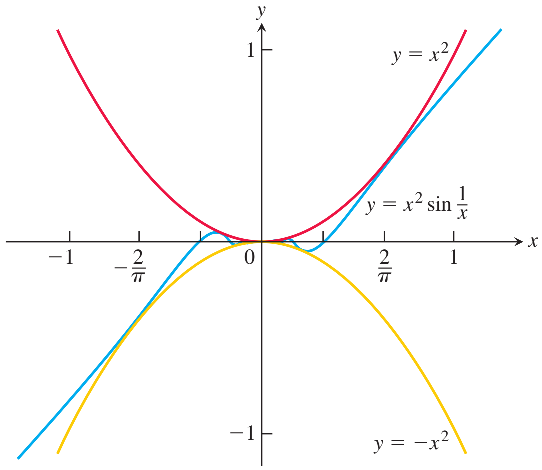
47. $\lim_{x \to 1} f(x) = 2 \quad \text{if} \quad f(x) = \begin{cases} 4 - 2x, & x < 1 \\ 6x - 4, & x \geq 1 \end{cases}$

48. $\lim_{x \to 0} f(x) = 0 \quad \text{if} \quad f(x) = \begin{cases} 2x, & x < 0 \\ x/2, & x \geq 0 \end{cases}$

49. $\lim_{x \to 0} x \sin \dfrac{1}{x} = 0$



50. $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$



Theory and Examples

51. Define what it means to say that $\lim_{x\to 0} g(x) = k$.

52. Prove that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{h\to 0} f(h + c) = L$.

53. A wrong statement about limits Show by example that the following statement is wrong.

> The number L is the limit of $f(x)$ as x approaches c if $f(x)$ gets closer to L as x approaches c.

Explain why the function in your example does not have the given value of L as a limit as $x \to c$.

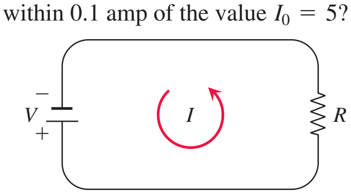
54. Another wrong statement about limits Show by example that the following statement is wrong.

> The number L is the limit of $f(x)$ as x approaches c if, given any $\epsilon > 0$, there exists a value of x for which $|f(x) - L| < \epsilon$.

Explain why the function in your example does not have the given value of L as a limit as $x \to c$.

T **55. Grinding engine cylinders** Before contracting to grind engine cylinders to a cross-sectional area of 9 in^2, you need to know how much deviation from the ideal cylinder diameter of $c = 3.385$ in. you can allow and still have the area come within 0.01 in^2 of the required 9 in^2. To find out, you let $A = \pi(x/2)^2$ and look for the interval in which you must hold x to make $|A - 9| \le 0.01$. What interval do you find?

56. Manufacturing electrical resistors Ohm's law for electrical circuits like the one shown in the accompanying figure states that $V = RI$. In this equation, V is a constant voltage, I is the current in amperes, and R is the resistance in ohms. Your firm has been asked to supply the resistors for a circuit in which V will be 120 volts and I is to be 5 ± 0.1 amp. In what interval does R have to lie for I to be within 0.1 amp of the value $I_0 = 5$?



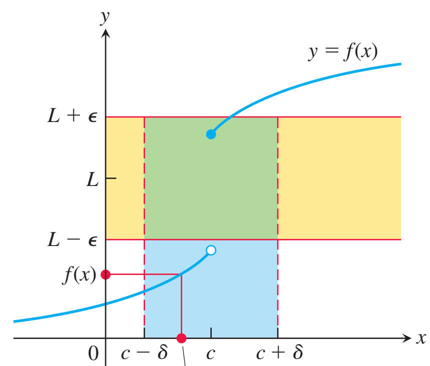
When Is a Number L Not the Limit of $f(x)$ as $x \to c$?

Showing L is not a limit We can prove that $\lim_{x\to c} f(x) \ne L$ by providing an $\epsilon > 0$ such that no possible $\delta > 0$ satisfies the condition

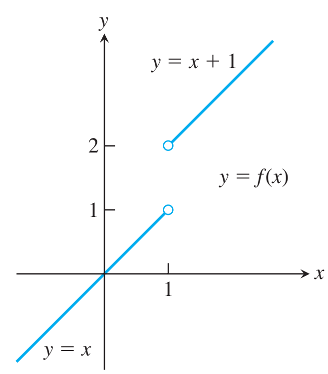
$$\text{for all } x, \quad 0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

We accomplish this for our candidate ϵ by showing that for each $\delta > 0$ there exists a value of x such that

$$0 < |x - c| < \delta \quad \text{and} \quad |f(x) - L| \ge \epsilon.$$



57. Let $f(x) = \begin{cases} x, & x < 1 \\ x + 1, & x > 1. \end{cases}$



a. Let $\epsilon = 1/2$. Show that no possible $\delta > 0$ satisfies the following condition:

$$\text{For all } x, \quad 0 < |x - 1| < \delta \quad \Rightarrow \quad |f(x) - 2| < 1/2.$$

That is, for each $\delta > 0$ show that there is a value of x such that

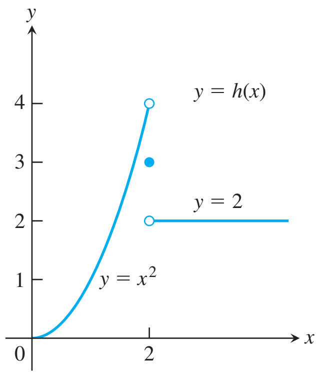
$$0 < |x - 1| < \delta \quad \text{and} \quad |f(x) - 2| \ge 1/2.$$

This will show that $\lim_{x\to 1} f(x) \ne 2$.

b. Show that $\lim_{x\to 1} f(x) \ne 1$.

c. Show that $\lim_{x\to 1} f(x) \ne 1.5$.

58. Let $h(x) = \begin{cases} x^2, & x < 2 \\ 3, & x = 2 \\ 2, & x > 2. \end{cases}$



Show that

a. $\lim_{x \to 2} h(x) \neq 4$

b. $\lim_{x \to 2} h(x) \neq 3$

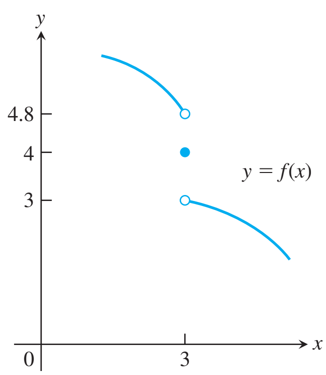
c. $\lim_{x \to 2} h(x) \neq 2$

59. For the function graphed here, explain why

a. $\lim_{x \to 3} f(x) \neq 4$

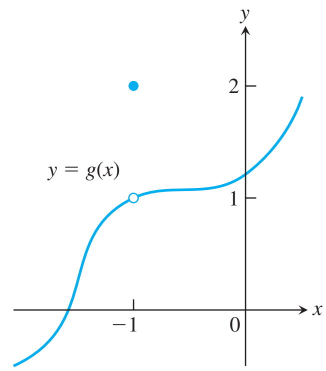
b. $\lim_{x \to 3} f(x) \neq 4.8$

c. $\lim_{x \to 3} f(x) \neq 3$



60. a. For the function graphed here, show that $\lim_{x \to -1} g(x) \neq 2$.

b. Does $\lim_{x \to -1} g(x)$ appear to exist? If so, what is the value of the limit? If not, why not?



COMPUTER EXPLORATIONS

In Exercises 61–66, you will further explore finding deltas graphically. Use a CAS to perform the following steps:

a. Plot the function $y = f(x)$ near the point c being approached.

b. Guess the value of the limit L and then evaluate the limit symbolically to see if you guessed correctly.

c. Using the value $\epsilon = 0.2$, graph the banding lines $y_1 = L - \epsilon$ and $y_2 = L + \epsilon$ together with the function f near c.

d. From your graph in part (c), estimate a $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

Test your estimate by plotting f, y_1, and y_2 over the interval $0 < |x - c| < \delta$. For your viewing window use $c - 2\delta \leq x \leq c + 2\delta$ and $L - 2\epsilon \leq y \leq L + 2\epsilon$. If any function values lie outside the interval $[L - \epsilon, L + \epsilon]$, your choice of δ was too large. Try again with a smaller estimate.

e. Repeat parts (c) and (d) successively for $\epsilon = 0.1, 0.05$, and 0.001.

61. $f(x) = \dfrac{x^4 - 81}{x - 3}, \quad c = 3$

62. $f(x) = \dfrac{5x^3 + 9x^2}{2x^5 + 3x^2}, \quad c = 0$

63. $f(x) = \dfrac{\sin 2x}{3x}, \quad c = 0$

64. $f(x) = \dfrac{x(1 - \cos x)}{x - \sin x}, \quad c = 0$

65. $f(x) = \dfrac{\sqrt[3]{x} - 1}{x - 1}, \quad c = 1$

66. $f(x) = \dfrac{3x^2 - (7x + 1)\sqrt{x} + 5}{x - 1}, \quad c = 1$

2.4 One-Sided Limits

In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number c from the left-hand side (where $x < c$) or the right-hand side ($x > c$) only.

Approaching a Limit from One Side

To have a limit L as x approaches c, a function f must be defined on *both sides* of c and its values $f(x)$ must approach L as x approaches c from either side. That is, f must be defined in some open interval about c, but not necessarily at c. Because of this, ordinary limits are called **two-sided**.

Solution

(a) Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned}\lim_{h\to 0}\frac{\cos h - 1}{h} &= \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h}\\ &= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta && \text{Let } \theta = h/2.\\ &= -(1)(0) = 0. && \text{Eq. (1) and Example 11a in Section 2.2}\end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{5x} &= \lim_{x\to 0}\frac{(2/5)\cdot\sin 2x}{(2/5)\cdot 5x}\\ &= \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x} && \text{Now, Eq. (1) applies with } \theta = 2x.\\ &= \frac{2}{5}(1) = \frac{2}{5}\end{aligned}$$

EXAMPLE 6 Find $\displaystyle\lim_{t\to 0}\frac{\tan t\sec 2t}{3t}$.

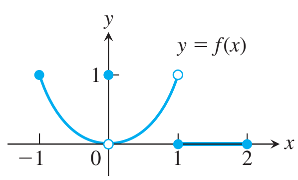
Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\begin{aligned}\lim_{t\to 0}\frac{\tan t\sec 2t}{3t} &= \lim_{t\to 0}\frac{1}{3}\cdot\frac{1}{t}\cdot\frac{\sin t}{\cos t}\cdot\frac{1}{\cos 2t}\\ &= \frac{1}{3}\lim_{t\to 0}\frac{\sin t}{t}\cdot\frac{1}{\cos t}\cdot\frac{1}{\cos 2t}\\ &= \frac{1}{3}(1)(1)(1) = \frac{1}{3}. && \text{Eq. (1) and Example 11b in Section 2.2}\end{aligned}$$

Exercises 2.4

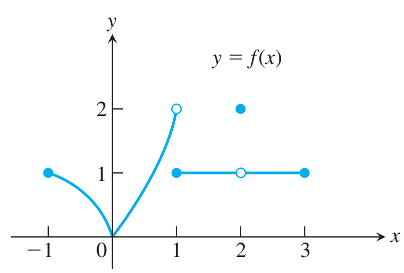
Finding Limits Graphically

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



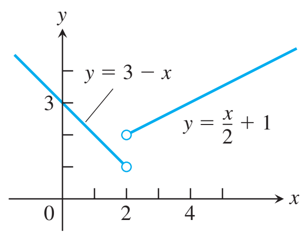
a. $\lim_{x\to -1^+} f(x) = 1$ **b.** $\lim_{x\to 0^-} f(x) = 0$
c. $\lim_{x\to 0^-} f(x) = 1$ **d.** $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$
e. $\lim_{x\to 0} f(x)$ exists. **f.** $\lim_{x\to 0} f(x) = 0$
g. $\lim_{x\to 0} f(x) = 1$ **h.** $\lim_{x\to 1} f(x) = 1$
i. $\lim_{x\to 1} f(x) = 0$ **j.** $\lim_{x\to 2^-} f(x) = 2$
k. $\lim_{x\to -1^-} f(x)$ does not exist. **l.** $\lim_{x\to 2^+} f(x) = 0$

2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x\to -1^+} f(x) = 1$ **b.** $\lim_{x\to 2} f(x)$ does not exist.
c. $\lim_{x\to 2} f(x) = 2$ **d.** $\lim_{x\to 1^-} f(x) = 2$
e. $\lim_{x\to 1^+} f(x) = 1$ **f.** $\lim_{x\to 1} f(x)$ does not exist.
g. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$
h. $\lim_{x\to c} f(x)$ exists at every c in the open interval $(-1, 1)$.
i. $\lim_{x\to c} f(x)$ exists at every c in the open interval $(1, 3)$.
j. $\lim_{x\to -1^-} f(x) = 0$ **k.** $\lim_{x\to 3^+} f(x)$ does not exist.

3. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2. \end{cases}$



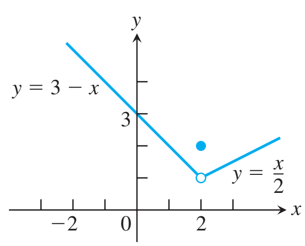
a. Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$.

b. Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$.

d. Does $\lim_{x\to 4} f(x)$ exist? If so, what is it? If not, why not?

4. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \dfrac{x}{2}, & x > 2. \end{cases}$



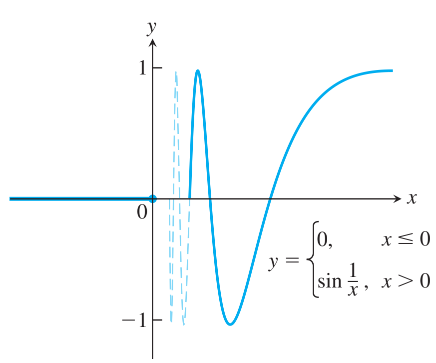
a. Find $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $f(2)$.

b. Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$.

d. Does $\lim_{x\to -1} f(x)$ exist? If so, what is it? If not, why not?

5. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\dfrac{1}{x}, & x > 0. \end{cases}$

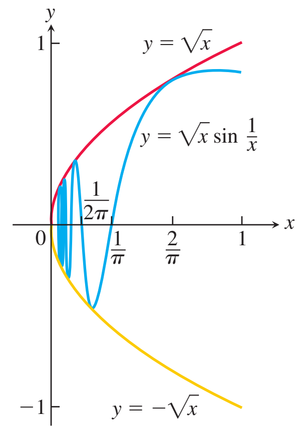


a. Does $\lim_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

6. Let $g(x) = \sqrt{x}\sin(1/x)$.



a. Does $\lim_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

7. a. Graph $f(x) = \begin{cases} x^3, & x \ne 1 \\ 0, & x = 1. \end{cases}$

b. Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

c. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

8. a. Graph $f(x) = \begin{cases} 1 - x^2, & x \ne 1 \\ 2, & x = 1. \end{cases}$

b. Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.

c. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

Graph the functions in Exercises 9 and 10. Then answer these questions.

a. What are the domain and range of f?

b. At what points c, if any, does $\lim_{x\to c} f(x)$ exist?

c. At what points does only the left-hand limit exist?

d. At what points does only the right-hand limit exist?

9. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

10. $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$

Finding One-Sided Limits Algebraically

Find the limits in Exercises 11–18.

11. $\displaystyle\lim_{x\to -0.5^-} \sqrt{\frac{x+2}{x+1}}$

12. $\displaystyle\lim_{x\to 1^+} \sqrt{\frac{x-1}{x+2}}$

13. $\displaystyle\lim_{x\to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right)$

14. $\displaystyle\lim_{x\to 1^-} \left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right)$

15. $\displaystyle\lim_{h\to 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}$

16. $\lim_{h\to 0^-} \dfrac{\sqrt{6} - \sqrt{5h^2 + 11h + 6}}{h}$

17. a. $\lim_{x\to -2^+} (x + 3)\dfrac{|x + 2|}{x + 2}$ b. $\lim_{x\to -2^-} (x + 3)\dfrac{|x + 2|}{x + 2}$

18. a. $\lim_{x\to 1^+} \dfrac{\sqrt{2x}(x - 1)}{|x - 1|}$ b. $\lim_{x\to 1^-} \dfrac{\sqrt{2x}(x - 1)}{|x - 1|}$

Use the graph of the greatest integer function $y = \lfloor x \rfloor$, Figure 1.10 in Section 1.1, to help you find the limits in Exercises 19 and 20.

19. a. $\lim_{\theta\to 3^+} \dfrac{\lfloor \theta \rfloor}{\theta}$ b. $\lim_{\theta\to 3^-} \dfrac{\lfloor \theta \rfloor}{\theta}$

20. a. $\lim_{t\to 4^+} (t - \lfloor t \rfloor)$ b. $\lim_{t\to 4^-} (t - \lfloor t \rfloor)$

Using $\lim_{\theta\to 0} \dfrac{\sin \theta}{\theta} = 1$

Find the limits in Exercises 21–42.

21. $\lim_{\theta\to 0} \dfrac{\sin \sqrt{2}\theta}{\sqrt{2}\theta}$

22. $\lim_{t\to 0} \dfrac{\sin kt}{t}$ (k constant)

23. $\lim_{y\to 0} \dfrac{\sin 3y}{4y}$

24. $\lim_{h\to 0^-} \dfrac{h}{\sin 3h}$

25. $\lim_{x\to 0} \dfrac{\tan 2x}{x}$

26. $\lim_{t\to 0} \dfrac{2t}{\tan t}$

27. $\lim_{x\to 0} \dfrac{x \csc 2x}{\cos 5x}$

28. $\lim_{x\to 0} 6x^2(\cot x)(\csc 2x)$

29. $\lim_{x\to 0} \dfrac{x + x\cos x}{\sin x \cos x}$

30. $\lim_{x\to 0} \dfrac{x^2 - x + \sin x}{2x}$

31. $\lim_{\theta\to 0} \dfrac{1 - \cos \theta}{\sin 2\theta}$

32. $\lim_{x\to 0} \dfrac{x - x\cos x}{\sin^2 3x}$

33. $\lim_{t\to 0} \dfrac{\sin(1 - \cos t)}{1 - \cos t}$

34. $\lim_{h\to 0} \dfrac{\sin(\sin h)}{\sin h}$

35. $\lim_{\theta\to 0} \dfrac{\sin \theta}{\sin 2\theta}$

36. $\lim_{x\to 0} \dfrac{\sin 5x}{\sin 4x}$

37. $\lim_{\theta\to 0} \theta \cos \theta$

38. $\lim_{\theta\to 0} \sin \theta \cot 2\theta$

39. $\lim_{x\to 0} \dfrac{\tan 3x}{\sin 8x}$

40. $\lim_{y\to 0} \dfrac{\sin 3y \cot 5y}{y \cot 4y}$

41. $\lim_{\theta\to 0} \dfrac{\tan \theta}{\theta^2 \cot 3\theta}$

42. $\lim_{\theta\to 0} \dfrac{\theta \cot 4\theta}{\sin^2 \theta \cot^2 2\theta}$

Theory and Examples

43. Once you know $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ at an interior point of the domain of f, do you then know $\lim_{x\to a} f(x)$? Give reasons for your answer.

44. If you know that $\lim_{x\to c} f(x)$ exists, can you find its value by calculating $\lim_{x\to c^+} f(x)$? Give reasons for your answer.

45. Suppose that f is an odd function of x. Does knowing that $\lim_{x\to 0^+} f(x) = 3$ tell you anything about $\lim_{x\to 0^-} f(x)$? Give reasons for your answer.

46. Suppose that f is an even function of x. Does knowing that $\lim_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim_{x\to -2^-} f(x)$ or $\lim_{x\to -2^+} f(x)$? Give reasons for your answer.

Formal Definitions of One-Sided Limits

47. Given $\epsilon > 0$, find an interval $I = (5, 5 + \delta)$, $\delta > 0$, such that if x lies in I, then $\sqrt{x - 5} < \epsilon$. What limit is being verified and what is its value?

48. Given $\epsilon > 0$, find an interval $I = (4 - \delta, 4)$, $\delta > 0$, such that if x lies in I, then $\sqrt{4 - x} < \epsilon$. What limit is being verified and what is its value?

Use the definitions of right-hand and left-hand limits to prove the limit statements in Exercises 49 and 50.

49. $\lim_{x\to 0^-} \dfrac{x}{|x|} = -1$

50. $\lim_{x\to 2^+} \dfrac{x - 2}{|x - 2|} = 1$

51. **Greatest integer function** Find **(a)** $\lim_{x\to 400^+} \lfloor x \rfloor$ and **(b)** $\lim_{x\to 400^-} \lfloor x \rfloor$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x\to 400} \lfloor x \rfloor$? Give reasons for your answer.

52. **One-sided limits** Let $f(x) = \begin{cases} x^2 \sin(1/x), & x < 0 \\ \sqrt{x}, & x > 0. \end{cases}$

Find **(a)** $\lim_{x\to 0^+} f(x)$ and **(b)** $\lim_{x\to 0^-} f(x)$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x\to 0} f(x)$? Give reasons for your answer.

2.5 Continuity

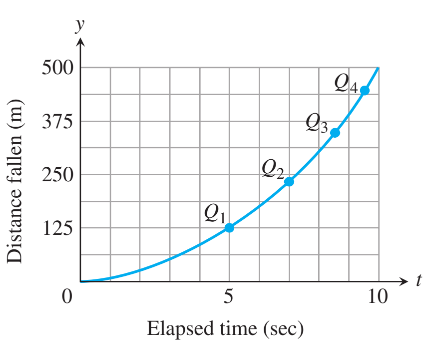


FIGURE 2.34 Connecting plotted points by an unbroken curve from experimental data $Q_1, Q_2, Q_3, \ldots$ for a falling object.

When we plot function values generated in a laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the points we did not measure (Figure 2.34). In doing so, we are assuming that we are working with a *continuous function*, so its outputs vary regularly and consistently with the inputs, and do not jump abruptly from one value to another without taking on the values in between. Intuitively, any function $y = f(x)$ whose graph can be sketched over its domain in one unbroken motion is an example of a continuous function. Such functions play an important role in the study of calculus and its applications.

Continuity at a Point

To understand continuity, it helps to consider a function like that in Figure 2.35, whose limits we investigated in Example 2 in the last section.

is equal to $f(x)$ for $x \neq 2$, but is continuous at $x = 2$, having there the value of $5/4$. Thus F is the continuous extension of f to $x = 2$, and

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} f(x) = \frac{5}{4}.$$

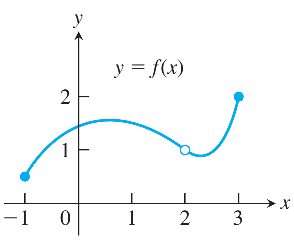
The graph of f is shown in Figure 2.48. The continuous extension F has the same graph except with no hole at $(2, 5/4)$. Effectively, F is the function f with its point of discontinuity at $x = 2$ removed. ■

Exercises 2.5

Continuity from Graphs

In Exercises 1–4, say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?

1.



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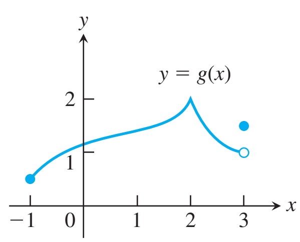

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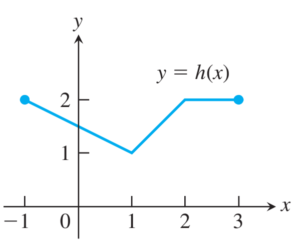

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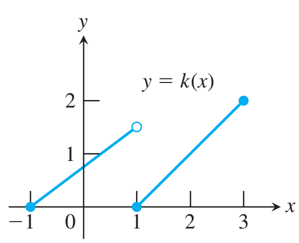
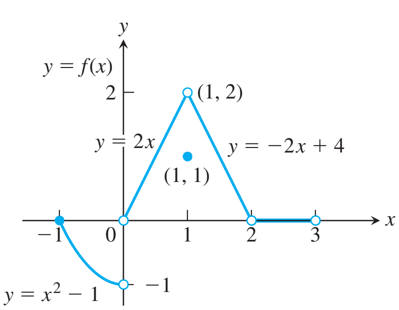


Exercises 5–10 refer to the function

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

graphed in the accompanying figure.



The graph for Exercises 5–10.

5. **a.** Does $f(-1)$ exist?
  b. Does $\lim_{x \to -1^+} f(x)$ exist?
  c. Does $\lim_{x \to -1^+} f(x) = f(-1)$?
  d. Is f continuous at $x = -1$?

6. **a.** Does $f(1)$ exist?
  b. Does $\lim_{x \to 1} f(x)$ exist?
  c. Does $\lim_{x \to 1} f(x) = f(1)$?
  d. Is f continuous at $x = 1$?

7. **a.** Is f defined at $x = 2$? (Look at the definition of f.)
  b. Is f continuous at $x = 2$?

8. At what values of x is f continuous?

9. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

10. To what new value should $f(1)$ be changed to remove the discontinuity?

Applying the Continuity Test

At which points do the functions in Exercises 11 and 12 fail to be continuous? At which points, if any, are the discontinuities removable? Not removable? Give reasons for your answers.

11. Exercise 1, Section 2.4

12. Exercise 2, Section 2.4

At what points are the functions in Exercises 13–30 continuous?

13. $y = \dfrac{1}{x - 2} - 3x$

14. $y = \dfrac{1}{(x + 2)^2} + 4$

15. $y = \dfrac{x + 1}{x^2 - 4x + 3}$

16. $y = \dfrac{x + 3}{x^2 - 3x - 10}$

17. $y = |x - 1| + \sin x$

18. $y = \dfrac{1}{|x| + 1} - \dfrac{x^2}{2}$

19. $y = \dfrac{\cos x}{x}$

20. $y = \dfrac{x + 2}{\cos x}$

21. $y = \csc 2x$

22. $y = \tan \dfrac{\pi x}{2}$

23. $y = \dfrac{x \tan x}{x^2 + 1}$

24. $y = \dfrac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$

25. $y = \sqrt{2x + 3}$

26. $y = \sqrt[4]{3x - 1}$

27. $y = (2x - 1)^{1/3}$

28. $y = (2 - x)^{1/5}$

29. $g(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & x \neq 3 \\ 5, & x = 3 \end{cases}$

30. $f(x) = \begin{cases} \dfrac{x^3 - 8}{x^2 - 4}, & x \neq 2, x \neq -2 \\ 3, & x = 2 \\ 4, & x = -2 \end{cases}$

Limits Involving Trigonometric Functions

Find the limits in Exercises 31–38. Are the functions continuous at the point being approached?

31. $\lim_{x \to \pi} \sin(x - \sin x)$

32. $\lim_{t \to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right)$

33. $\lim_{y \to 1} \sec(y \sec^2 y - \tan^2 y - 1)$

34. $\lim_{x \to 0} \tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right)$

35. $\lim_{t \to 0} \cos\left(\frac{\pi}{\sqrt{19 - 3\sec 2t}}\right)$

36. $\lim_{x \to \pi/6} \sqrt{\csc^2 x + 5\sqrt{3}\tan x}$

37. $\lim_{x \to 0^+} \sin\left(\frac{\pi}{2}e^{\sqrt{x}}\right)$

38. $\lim_{x \to 1} \cos^{-1}\left(\ln\sqrt{x}\right)$

Continuous Extensions

39. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9)/(x - 3)$ to be continuous at $x = 3$.

40. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.

41. Define $f(1)$ in a way that extends $f(s) = (s^3 - 1)/(s^2 - 1)$ to be continuous at $s = 1$.

42. Define $g(4)$ in a way that extends

$$g(x) = (x^2 - 16)/(x^2 - 3x - 4)$$

to be continuous at $x = 4$.

43. For what value of a is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x?

44. For what value of b is

$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$

continuous at every x?

45. For what values of a is

$$f(x) = \begin{cases} a^2x - 2a, & x \geq 2 \\ 12, & x < 2 \end{cases}$$

continuous at every x?

46. For what value of b is

$$g(x) = \begin{cases} \dfrac{x - b}{b + 1}, & x < 0 \\ x^2 + b, & x > 0 \end{cases}$$

continuous at every x?

47. For what values of a and b is

$$f(x) = \begin{cases} -2, & x \leq -1 \\ ax - b, & -1 < x < 1 \\ 3, & x \geq 1 \end{cases}$$

continuous at every x?

48. For what values of a and b is

$$g(x) = \begin{cases} ax + 2b, & x \leq 0 \\ x^2 + 3a - b, & 0 < x \leq 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuous at every x?

T In Exercises 49–52, graph the function f to see whether it appears to have a continuous extension to the origin. If it does, use Trace and Zoom to find a good candidate for the extended function's value at $x = 0$. If the function does not appear to have a continuous extension, can it be extended to be continuous at the origin from the right or from the left? If so, what do you think the extended function's value(s) should be?

49. $f(x) = \dfrac{10^x - 1}{x}$

50. $f(x) = \dfrac{10^{|x|} - 1}{x}$

51. $f(x) = \dfrac{\sin x}{|x|}$

52. $f(x) = (1 + 2x)^{1/x}$

Theory and Examples

53. A continuous function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Why does the equation $f(x) = 0$ have at least one solution between $x = 0$ and $x = 1$? Illustrate with a sketch.

54. Explain why the equation $\cos x = x$ has at least one solution.

55. **Roots of a cubic** Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

56. **A function value** Show that the function $F(x) = (x - a)^2 \cdot (x - b)^2 + x$ takes on the value $(a + b)/2$ for some value of x.

57. **Solving an equation** If $f(x) = x^3 - 8x + 10$, show that there are values c for which $f(c)$ equals **(a)** π; **(b)** $-\sqrt{3}$; **(c)** 5,000,000.

58. Explain why the following five statements ask for the same information.

   a. Find the roots of $f(x) = x^3 - 3x - 1$.

   b. Find the x-coordinates of the points where the curve $y = x^3$ crosses the line $y = 3x + 1$.

   c. Find all the values of x for which $x^3 - 3x = 1$.

   d. Find the x-coordinates of the points where the cubic curve $y = x^3 - 3x$ crosses the line $y = 1$.

   e. Solve the equation $x^3 - 3x - 1 = 0$.

59. **Removable discontinuity** Give an example of a function $f(x)$ that is continuous for all values of x except $x = 2$, where it has a removable discontinuity. Explain how you know that f is discontinuous at $x = 2$, and how you know the discontinuity is removable.

60. **Nonremovable discontinuity** Give an example of a function $g(x)$ that is continuous for all values of x except $x = -1$, where it has a nonremovable discontinuity. Explain how you know that g is discontinuous there and why the discontinuity is not removable.

61. A function discontinuous at every point

a. Use the fact that every nonempty interval of real numbers contains both rational and irrational numbers to show that the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point.

b. Is f right-continuous or left-continuous at any point?

62. If functions $f(x)$ and $g(x)$ are continuous for $0 \le x \le 1$, could $f(x)/g(x)$ possibly be discontinuous at a point of $[0, 1]$? Give reasons for your answer.

63. If the product function $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$, must $f(x)$ and $g(x)$ be continuous at $x = 0$? Give reasons for your answer.

64. Discontinuous composite of continuous functions Give an example of functions f and g, both continuous at $x = 0$, for which the composite $f \circ g$ is discontinuous at $x = 0$. Does this contradict Theorem 9? Give reasons for your answer.

65. Never-zero continuous functions Is it true that a continuous function that is never zero on an interval never changes sign on that interval? Give reasons for your answer.

66. Stretching a rubber band Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? Give reasons for your answer.

67. A fixed point theorem Suppose that a function f is continuous on the closed interval $[0, 1]$ and that $0 \le f(x) \le 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$ (c is called a **fixed point** of f).

68. The sign-preserving property of continuous functions Let f be defined on an interval (a, b) and suppose that $f(c) \neq 0$ at some c where f is continuous. Show that there is an interval $(c - \delta, c + \delta)$ about c where f has the same sign as $f(c)$.

69. Prove that f is continuous at c if and only if

$$\lim_{h \to 0} f(c + h) = f(c).$$

70. Use Exercise 69 together with the identities

$$\sin(h + c) = \sin h \cos c + \cos h \sin c,$$

$$\cos(h + c) = \cos h \cos c - \sin h \sin c$$

to prove that both $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every point $x = c$.

Solving Equations Graphically

T Use the Intermediate Value Theorem in Exercises 71–78 to prove that each equation has a solution. Then use a graphing calculator or computer grapher to solve the equations.

71. $x^3 - 3x - 1 = 0$

72. $2x^3 - 2x^2 - 2x + 1 = 0$

73. $x(x - 1)^2 = 1$ (one root)

74. $x^x = 2$

75. $\sqrt{x} + \sqrt{1 + x} = 4$

76. $x^3 - 15x + 1 = 0$ (three roots)

77. $\cos x = x$ (one root). Make sure you are using radian mode.

78. $2 \sin x = x$ (three roots). Make sure you are using radian mode.

2.6 Limits Involving Infinity; Asymptotes of Graphs

In this section we investigate the behavior of a function when the magnitude of the independent variable x becomes increasingly large, or $x \to \pm\infty$. We further extend the concept of limit to *infinite limits*, which are not limits as before, but rather a new use of the term limit. Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large in magnitude. We use these limit ideas to analyze the graphs of functions having *horizontal* or *vertical asymptotes*.

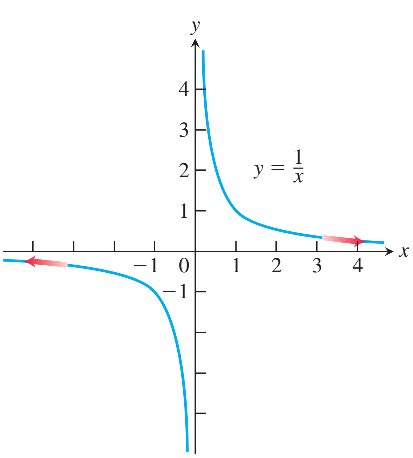


FIGURE 2.49 The graph of $y = 1/x$ approaches 0 as $x \to \infty$ or $x \to -\infty$.

Finite Limits as $x \to \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, the function $f(x) = 1/x$ is defined for all $x \neq 0$ (Figure 2.49). When x is positive and becomes increasingly large, $1/x$ becomes increasingly small. When x is negative and its magnitude becomes increasingly large, $1/x$ again becomes small. We summarize these observations by saying that $f(x) = 1/x$ has limit 0 as $x \to \infty$ or $x \to -\infty$, or that 0 is a *limit of* $f(x) = 1/x$ *at infinity and negative infinity*. Here are precise definitions.

Limit Laws of Theorem 1 and in the Sandwich Theorem, all of which are proved from the precise definition of the limit. We saw that these computational rules also apply to one-sided limits and to limits at infinity. Moreover, we can sometimes apply these rules when calculating limits of simple transcendental functions, as illustrated by our examples or in cases like the following:

$$\lim_{x\to 0}\frac{e^x-1}{e^{2x}-1}=\lim_{x\to 0}\frac{e^x-1}{(e^x-1)(e^x+1)}=\lim_{x\to 0}\frac{1}{e^x+1}=\frac{1}{1+1}=\frac{1}{2}.$$

However, calculating more complicated limits involving transcendental functions such as

$$\lim_{x\to 0}\frac{x}{e^{2x}-1},\qquad \lim_{x\to 0}\frac{\ln x}{x},\quad\text{and}\quad \lim_{x\to 0}\left(1+\frac{1}{x}\right)^x$$

requires more than simple algebraic techniques. The *derivative* is exactly the tool we need to calculate limits such as these (see Section 4.5), and this notion is the main subject of our next chapter.

Exercises 2.6

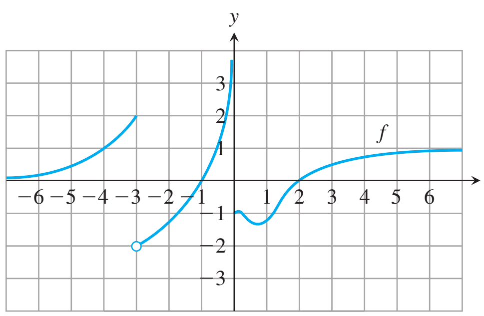
Finding Limits

1. For the function f whose graph is given, determine the following limits.

a. $\lim_{x\to 2} f(x)$ **b.** $\lim_{x\to -3^+} f(x)$ **c.** $\lim_{x\to -3^-} f(x)$

d. $\lim_{x\to -3} f(x)$ **e.** $\lim_{x\to 0^+} f(x)$ **f.** $\lim_{x\to 0^-} f(x)$

g. $\lim_{x\to 0} f(x)$ **h.** $\lim_{x\to \infty} f(x)$ **i.** $\lim_{x\to -\infty} f(x)$



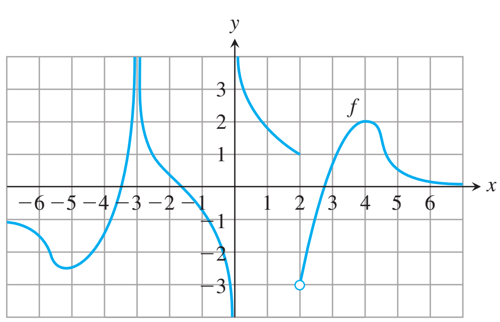
2. For the function f whose graph is given, determine the following limits.

a. $\lim_{x\to 4} f(x)$ **b.** $\lim_{x\to 2^+} f(x)$ **c.** $\lim_{x\to 2^-} f(x)$

d. $\lim_{x\to 2} f(x)$ **e.** $\lim_{x\to -3^+} f(x)$ **f.** $\lim_{x\to -3^-} f(x)$

g. $\lim_{x\to -3} f(x)$ **h.** $\lim_{x\to 0^+} f(x)$ **i.** $\lim_{x\to 0^-} f(x)$

j. $\lim_{x\to 0} f(x)$ **k.** $\lim_{x\to \infty} f(x)$ **l.** $\lim_{x\to -\infty} f(x)$



In Exercises 3–8, find the limit of each function **(a)** as $x\to\infty$ and **(b)** as $x\to-\infty$. (You may wish to visualize your answer with a graphing calculator or computer.)

3. $f(x)=\dfrac{2}{x}-3$

4. $f(x)=\pi-\dfrac{2}{x^2}$

5. $g(x)=\dfrac{1}{2+(1/x)}$

6. $g(x)=\dfrac{1}{8-(5/x^2)}$

7. $h(x)=\dfrac{-5+(7/x)}{3-(1/x^2)}$

8. $h(x)=\dfrac{3-(2/x)}{4+(\sqrt{2}/x^2)}$

Find the limits in Exercises 9–12.

9. $\lim_{x\to\infty}\dfrac{\sin 2x}{x}$

10. $\lim_{\theta\to-\infty}\dfrac{\cos\theta}{3\theta}$

11. $\lim_{t\to-\infty}\dfrac{2-t+\sin t}{t+\cos t}$

12. $\lim_{r\to\infty}\dfrac{r+\sin r}{2r+7-5\sin r}$

Limits of Rational Functions

In Exercises 13–22, find the limit of each rational function **(a)** as $x\to\infty$ and **(b)** as $x\to-\infty$.

13. $f(x)=\dfrac{2x+3}{5x+7}$

14. $f(x)=\dfrac{2x^3+7}{x^3-x^2+x+7}$

15. $f(x)=\dfrac{x+1}{x^2+3}$

16. $f(x)=\dfrac{3x+7}{x^2-2}$

17. $h(x)=\dfrac{7x^3}{x^3-3x^2+6x}$

18. $h(x)=\dfrac{9x^4+x}{2x^4+5x^2-x+6}$

19. $g(x)=\dfrac{10x^5+x^4+31}{x^6}$

20. $g(x)=\dfrac{x^3+7x^2-2}{x^2-x+1}$

21. $f(x)=\dfrac{3x^7+5x^2-1}{6x^3-7x+3}$

22. $h(x)=\dfrac{5x^8-2x^3+9}{3+x-4x^5}$

Limits as $x\to\infty$ or $x\to-\infty$

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x: Divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 23–36.

23. $\lim_{x\to\infty}\sqrt{\dfrac{8x^2-3}{2x^2+x}}$

24. $\lim_{x\to-\infty}\left(\dfrac{x^2+x-1}{8x^2-3}\right)^{1/3}$

25. $\lim_{x\to-\infty}\left(\frac{1-x^3}{x^2+7x}\right)^5$

26. $\lim_{x\to\infty}\sqrt{\frac{x^2-5x}{x^3+x-2}}$

27. $\lim_{x\to\infty}\frac{2\sqrt{x}+x^{-1}}{3x-7}$

28. $\lim_{x\to\infty}\frac{2+\sqrt{x}}{2-\sqrt{x}}$

29. $\lim_{x\to-\infty}\frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$

30. $\lim_{x\to\infty}\frac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}}$

31. $\lim_{x\to\infty}\frac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}}$

32. $\lim_{x\to-\infty}\frac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4}$

33. $\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x+1}$

34. $\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x+1}$

35. $\lim_{x\to\infty}\frac{x-3}{\sqrt{4x^2+25}}$

36. $\lim_{x\to-\infty}\frac{4-3x^3}{\sqrt{x^6+9}}$

Infinite Limits

Find the limits in Exercises 37–48.

37. $\lim_{x\to0^+}\frac{1}{3x}$

38. $\lim_{x\to0^-}\frac{5}{2x}$

39. $\lim_{x\to2^-}\frac{3}{x-2}$

40. $\lim_{x\to3^+}\frac{1}{x-3}$

41. $\lim_{x\to-8^+}\frac{2x}{x+8}$

42. $\lim_{x\to-5^-}\frac{3x}{2x+10}$

43. $\lim_{x\to7}\frac{4}{(x-7)^2}$

44. $\lim_{x\to0}\frac{-1}{x^2(x+1)}$

45. **a.** $\lim_{x\to0^+}\frac{2}{3x^{1/3}}$ **b.** $\lim_{x\to0^-}\frac{2}{3x^{1/3}}$

46. **a.** $\lim_{x\to0^+}\frac{2}{x^{1/5}}$ **b.** $\lim_{x\to0^-}\frac{2}{x^{1/5}}$

47. $\lim_{x\to0}\frac{4}{x^{2/5}}$

48. $\lim_{x\to0}\frac{1}{x^{2/3}}$

Find the limits in Exercises 49–52.

49. $\lim_{x\to(\pi/2)^-}\tan x$

50. $\lim_{x\to(-\pi/2)^+}\sec x$

51. $\lim_{\theta\to0^-}(1+\csc\theta)$

52. $\lim_{\theta\to0}(2-\cot\theta)$

Find the limits in Exercises 53–58.

53. $\lim\frac{1}{x^2-4}$ as

a. $x\to2^+$ **b.** $x\to2^-$
c. $x\to-2^+$ **d.** $x\to-2^-$

54. $\lim\frac{x}{x^2-1}$ as

a. $x\to1^+$ **b.** $x\to1^-$
c. $x\to-1^+$ **d.** $x\to-1^-$

55. $\lim\left(\frac{x^2}{2}-\frac{1}{x}\right)$ as

a. $x\to0^+$ **b.** $x\to0^-$
c. $x\to\sqrt[3]{2}$ **d.** $x\to-1$

56. $\lim\frac{x^2-1}{2x+4}$ as

a. $x\to-2^+$ **b.** $x\to-2^-$
c. $x\to1^+$ **d.** $x\to0^-$

57. $\lim\frac{x^2-3x+2}{x^3-2x^2}$ as

a. $x\to0^+$ **b.** $x\to2^+$
c. $x\to2^-$ **d.** $x\to2$
e. What, if anything, can be said about the limit as $x\to0$?

58. $\lim\frac{x^2-3x+2}{x^3-4x}$ as

a. $x\to2^+$ **b.** $x\to-2^+$
c. $x\to0^-$ **d.** $x\to1^+$
e. What, if anything, can be said about the limit as $x\to0$?

Find the limits in Exercises 59–62.

59. $\lim\left(2-\frac{3}{t^{1/3}}\right)$ as

a. $t\to0^+$ **b.** $t\to0^-$

60. $\lim\left(\frac{1}{t^{3/5}}+7\right)$ as

a. $t\to0^+$ **b.** $t\to0^-$

61. $\lim\left(\frac{1}{x^{2/3}}+\frac{2}{(x-1)^{2/3}}\right)$ as

a. $x\to0^+$ **b.** $x\to0^-$
c. $x\to1^+$ **d.** $x\to1^-$

62. $\lim\left(\frac{1}{x^{1/3}}-\frac{1}{(x-1)^{4/3}}\right)$ as

a. $x\to0^+$ **b.** $x\to0^-$
c. $x\to1^+$ **d.** $x\to1^-$

Graphing Simple Rational Functions

Graph the rational functions in Exercises 63–68. Include the graphs and equations of the asymptotes and dominant terms.

63. $y=\frac{1}{x-1}$

64. $y=\frac{1}{x+1}$

65. $y=\frac{1}{2x+4}$

66. $y=\frac{-3}{x-3}$

67. $y=\frac{x+3}{x+2}$

68. $y=\frac{2x}{x+1}$

Inventing Graphs and Functions

In Exercises 69–72, sketch the graph of a function $y=f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)

69. $f(0)=0,\ f(1)=2,\ f(-1)=-2,\ \lim_{x\to-\infty}f(x)=-1$, and $\lim_{x\to\infty}f(x)=1$

70. $f(0)=0,\ \lim_{x\to\pm\infty}f(x)=0,\ \lim_{x\to0^+}f(x)=2$, and $\lim_{x\to0^-}f(x)=-2$

71. $f(0)=0,\ \lim_{x\to\pm\infty}f(x)=0,\ \lim_{x\to1^-}f(x)=\lim_{x\to-1^+}f(x)=\infty$, $\lim_{x\to1^+}f(x)=-\infty$, and $\lim_{x\to-1^-}f(x)=-\infty$

72. $f(2)=1,\ f(-1)=0,\ \lim_{x\to\infty}f(x)=0,\ \lim_{x\to0^+}f(x)=\infty$, $\lim_{x\to0^-}f(x)=-\infty$, and $\lim_{x\to-\infty}f(x)=1$

In Exercises 73–76, find a function that satisfies the given conditions and sketch its graph. (The answers here are not unique. Any function that satisfies the conditions is acceptable. Feel free to use formulas defined in pieces if that will help.)

73. $\lim_{x\to\pm\infty} f(x) = 0$, $\lim_{x\to 2^-} f(x) = \infty$, and $\lim_{x\to 2^+} f(x) = \infty$

74. $\lim_{x\to\pm\infty} g(x) = 0$, $\lim_{x\to 3^-} g(x) = -\infty$, and $\lim_{x\to 3^+} g(x) = \infty$

75. $\lim_{x\to-\infty} h(x) = -1$, $\lim_{x\to\infty} h(x) = 1$, $\lim_{x\to 0^-} h(x) = -1$, and $\lim_{x\to 0^+} h(x) = 1$

76. $\lim_{x\to\pm\infty} k(x) = 1$, $\lim_{x\to 1^-} k(x) = \infty$, and $\lim_{x\to 1^+} k(x) = -\infty$

77. Suppose that $f(x)$ and $g(x)$ are polynomials in x and that $\lim_{x\to\infty} (f(x)/g(x)) = 2$. Can you conclude anything about $\lim_{x\to-\infty} (f(x)/g(x))$? Give reasons for your answer.

78. Suppose that $f(x)$ and $g(x)$ are polynomials in x. Can the graph of $f(x)/g(x)$ have an asymptote if $g(x)$ is never zero? Give reasons for your answer.

79. How many horizontal asymptotes can the graph of a given rational function have? Give reasons for your answer.

Finding Limits of Differences When $x \to \pm\infty$

Find the limits in Exercises 80–86.

80. $\lim_{x\to\infty} \left(\sqrt{x+9} - \sqrt{x+4}\right)$

81. $\lim_{x\to\infty} \left(\sqrt{x^2+25} - \sqrt{x^2-1}\right)$

82. $\lim_{x\to-\infty} \left(\sqrt{x^2+3} + x\right)$

83. $\lim_{x\to-\infty} \left(2x + \sqrt{4x^2+3x-2}\right)$

84. $\lim_{x\to\infty} \left(\sqrt{9x^2-x} - 3x\right)$

85. $\lim_{x\to\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-2x}\right)$

86. $\lim_{x\to\infty} \left(\sqrt{x^2+x} - \sqrt{x^2-x}\right)$

Using the Formal Definitions

Use the formal definitions of limits as $x \to \pm\infty$ to establish the limits in Exercises 87 and 88.

87. If f has the constant value $f(x) = k$, then $\lim_{x\to\infty} f(x) = k$.

88. If f has the constant value $f(x) = k$, then $\lim_{x\to-\infty} f(x) = k$.

Use formal definitions to prove the limit statements in Exercises 89–92.

89. $\lim_{x\to 0} \dfrac{-1}{x^2} = -\infty$

90. $\lim_{x\to 0} \dfrac{1}{|x|} = \infty$

91. $\lim_{x\to 3} \dfrac{-2}{(x-3)^2} = -\infty$

92. $\lim_{x\to -5} \dfrac{1}{(x+5)^2} = \infty$

93. Here is the definition of **infinite right-hand limit**.

> We say that $f(x)$ approaches infinity as x approaches c from the right, and write
>
> $$\lim_{x\to c^+} f(x) = \infty,$$
>
> if, for every positive real number B, there exists a corresponding number $\delta > 0$ such that for all x
>
> $$c < x < c + \delta \quad \Rightarrow \quad f(x) > B.$$

Modify the definition to cover the following cases.

a. $\lim_{x\to c^-} f(x) = \infty$

b. $\lim_{x\to c^+} f(x) = -\infty$

c. $\lim_{x\to c^-} f(x) = -\infty$

Use the formal definitions from Exercise 93 to prove the limit statements in Exercises 94–98.

94. $\lim_{x\to 0^+} \dfrac{1}{x} = \infty$

95. $\lim_{x\to 0^-} \dfrac{1}{x} = -\infty$

96. $\lim_{x\to 2^-} \dfrac{1}{x-2} = -\infty$

97. $\lim_{x\to 2^+} \dfrac{1}{x-2} = \infty$

98. $\lim_{x\to 1^-} \dfrac{1}{1-x^2} = \infty$

Oblique Asymptotes

Graph the rational functions in Exercises 99–104. Include the graphs and equations of the asymptotes.

99. $y = \dfrac{x^2}{x-1}$

100. $y = \dfrac{x^2+1}{x-1}$

101. $y = \dfrac{x^2-4}{x-1}$

102. $y = \dfrac{x^2-1}{2x+4}$

103. $y = \dfrac{x^2-1}{x}$

104. $y = \dfrac{x^3+1}{x^2}$

Additional Graphing Exercises

T Graph the curves in Exercises 105–108. Explain the relationship between the curve's formula and what you see.

105. $y = \dfrac{x}{\sqrt{4-x^2}}$

106. $y = \dfrac{-1}{\sqrt{4-x^2}}$

107. $y = x^{2/3} + \dfrac{1}{x^{1/3}}$

108. $y = \sin\left(\dfrac{\pi}{x^2+1}\right)$

T Graph the functions in Exercises 109 and 110. Then answer the following questions.

a. How does the graph behave as $x \to 0^+$?

b. How does the graph behave as $x \to \pm\infty$?

c. How does the graph behave near $x = 1$ and $x = -1$?

Give reasons for your answers.

109. $y = \dfrac{3}{2}\left(x - \dfrac{1}{x}\right)^{2/3}$

110. $y = \dfrac{3}{2}\left(\dfrac{x}{x-1}\right)^{2/3}$