Solutions to Starred Exercises

Chapter 1

2. $S = \{(r, g), (r, b), (g, r), (g, b), (b, r), (b, g)\}$ where, for instance, (r, g) means that the first marble drawn was red and the second one green. The probability of each one of these outcomes is $\frac{1}{6}$.

5. $\frac{3}{4}$. If he wins, he only wins \$1; if he loses, he loses \$3.

9. $F = E \cup FE^c$, implying since E and FE^c are disjoint that $P(F) = P(E) + P(FE^c)$.

17.
$$\begin{aligned}
P\{\text{end}\} &= 1 - P\{\text{continue}\} \\
&= 1 - [\text{Prob}(H, H, H) + \text{Prob}(T, T, T)] \\
\text{Fair coin: } P\{\text{end}\} &= 1 - \left[\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right] \\
&= \frac{3}{4} \\
\text{Biased coin: } P\{\text{end}\} &= 1 - \left[\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}\right] \\
&= \frac{9}{16}
\end{aligned}$$

19.
$$E = \text{event at least 1 six}$$
$$P(E) = \frac{\text{number of ways to get } E}{\text{number of sample points}} = \frac{11}{36}$$
$$D = \text{event two faces are different}$$
$$P(D) = 1 - P(\text{two faces the same}) = 1 - \frac{6}{36} = \frac{5}{6}$$
$$P(E|D) = \frac{P(ED)}{P(D)} = \frac{10/36}{5/6} = \frac{1}{3}$$

25. (a) $P\{\text{pair}\} = P\{\text{second card is same denomination as first}\}$
$$= \frac{3}{51}$$

(b)
$$P\{\text{pair} \mid \text{different suits}\} = \frac{P\{\text{pair, different suits}\}}{P\{\text{different suits}\}}$$
$$= \frac{P\{\text{pair}\}}{P\{\text{different suits}\}}$$
$$= \frac{3/51}{39/51} = \frac{1}{13}$$

27.
$$P(E_1) = 1$$
$$P(E_2|E_1) = \frac{39}{51}$$

since 12 cards are in the ace of spades pile and 39 are not.

$$P(E_3|E_1E_2) = \frac{26}{50}$$

since 24 cards are in the piles of the two aces and 26 are in the other two piles.

$$P(E_4|E_1E_2E_3) = \frac{13}{49}$$

So

$$P\{\text{each pile has an ace}\} = \left(\frac{39}{51}\right)\left(\frac{26}{50}\right)\left(\frac{13}{49}\right)$$

30. (a)
$$P\{\text{George} \mid \text{exactly 1 hit}\} = \frac{P\{\text{George, not Bill}\}}{P\{\text{exactly 1}\}}$$
$$= \frac{P\{\text{G, not B}\}}{P\{\text{G, not B}\} + P\{\text{B, not G}\}}$$
$$= \frac{(0.4)(0.3)}{(0.4)(0.3) + (0.7)(0.6)}$$
$$= \frac{2}{9}$$

(b)
$$P\{\text{G} \mid \text{hit}\} = \frac{P\{\text{G, hit}\}}{P\{\text{hit}\}}$$
$$= \frac{P\{\text{G}\}}{P\{\text{hit}\}} = \frac{0.4}{1-(0.3)(0.6)} = \frac{20}{41}$$

32. Let E_i = event person i selects own hat.

$$P\text{ (no one selects hat)}$$
$$= 1 - P(E_1 \cup E_2 \cup \cdots \cup E_n)$$
$$= 1 - \left[\sum_{i_i} P(E_{i_1}) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \cdots + (-1)^{n+1} P(E_1 E_2 \cdots E_n)\right]$$
$$= 1 - \sum_{i_1} P(E_{i_1}) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) - \sum_{i_1<i_2<i_3} P(E_{i_1}E_{i_2}E_{i_3}) + \cdots$$
$$+ (-1)^n P(E_1 E_2 \cdots E_n)$$

Let $k \in \{1, 2, \ldots, n\}$. $P(E_{i_1}E_{i_2}E_{i_k})$ = number of ways k specific men can select own hats ÷ total number of ways hats can be arranged = $(n-k)!/n!$. Number of terms in summation $\sum_{i_1<i_2<\cdots<i_k}$ = number of ways to choose k variables out of n variables = $\binom{n}{k} = n!/k!(n-k)!$. Thus,

$$\sum_{i_1<\cdots<i_k} P(E_{i_1}E_{i_2}\cdots E_{i_k}) = \sum_{i_1<\cdots<i_k} \frac{(n-k)!}{n!}$$
$$= \binom{n}{k}\frac{(n-k)!}{n!} = \frac{1}{k!}$$
$$\therefore P(\text{no one selects own hat}) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$$
$$= \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$$

40. (a) F = event fair coin flipped; U = event two-headed coin flipped.

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)}$$
$$= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2} + 1\cdot\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

(b)
$$P(F|HH) = \frac{P(HH|F)P(F)}{P(HH|F)P(F) + P(HH|U)P(U)}$$
$$= \frac{\frac{1}{4}\cdot\frac{1}{2}}{\frac{1}{4}\cdot\frac{1}{2} + 1\cdot\frac{1}{2}} = \frac{\frac{1}{8}}{\frac{5}{8}} = \frac{1}{5}$$

(c)
$$P(F|HHT) = \frac{P(HHT|F)P(F)}{P(HHT|F)P(F) + P(HHT|U)P(U)}$$
$$= \frac{P(HHT|F)P(F)}{P(HHT|F)P(F) + 0} = 1$$

since the fair coin is the only one that can show tails.

45. Let B_i = event ith ball is black; R_i = event ith ball is red.

$$P(B_1|R_2) = \frac{P(R_2|B_1)P(B_1)}{P(R_2|B_1)P(B_1) + P(R_2|R_1)P(R_1)}$$
$$= \frac{\dfrac{r}{b+r+c}\cdot\dfrac{b}{b+r}}{\dfrac{r}{b+r+c}\cdot\dfrac{b}{b+r} + \dfrac{r+c}{b+r+c}\cdot\dfrac{r}{b+r}}$$
$$= \frac{rb}{rb+(r+c)r}$$
$$= \frac{b}{b+r+c}$$

48. Let C be the event that the randomly chosen family owns a car, and let H be the event that the randomly chosen family owns a house.

$$P(CH^c) = P(C) - P(CH) = 0.6 - 0.2 = 0.4$$

and

$$P(C^cH) = P(H) - P(CH) = 0.3 - 0.2 = 0.1$$

giving the result

$$P(CH^c) + P(C^cH) = 0.5$$

Chapter 2

4. (a) 1, 2, 3, 4, 5, 6.
   (b) 1, 2, 3, 4, 5, 6.
   (c) 2, 3, . . . , 11, 12.
   (d) −5, 4, . . . , 4, 5.

11. $\binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{3}{8}$.

16. $1 - (0.95)^{52} - 52(0.95)^{51}(0.05)$.

23. In order for X to equal n, the first $n-1$ flips must have $r-1$ heads, and then the nth flip must land heads. By independence the desired probability is thus

$$\binom{n-1}{r-1}p^{r-1}(1-p)^{n-r} \times p$$

27. $$\begin{aligned}P\{\text{same number of heads}\} &= \sum_i P\{A = i,\ B = i\}\\ &= \sum_i \binom{k}{i}\left(\frac{1}{2}\right)^k \binom{n-k}{i}\left(\frac{1}{2}\right)^{n-k}\\ &= \sum_i \binom{k}{i}\binom{n-k}{i}\left(\frac{1}{2}\right)^n\\ &= \sum_i \binom{k}{k-i}\binom{n-k}{i}\left(\frac{1}{2}\right)^n\\ &= \binom{n}{k}\left(\frac{1}{2}\right)^n\end{aligned}$$

Another argument is as follows:

$$\begin{aligned}&P\{\#\text{ heads of } A = \#\text{ heads of B}\}\\ &\quad = P\{\#\text{ tails of } A = \#\text{ heads of B}\} \qquad \text{since coin is fair}\\ &\quad = P\{k - \#\text{ heads of } A = \#\text{ heads of B}\}\\ &\quad = P\{k = \text{total }\#\text{ heads}\}\end{aligned}$$

38. $c = 2, \qquad P\{X > 2\} = \int_2^\infty 2e^{-2x}\,dx = e^{-4}$

47. Let X_i be 1 if trial i is a success and 0 otherwise.

(a) The largest value is 0.6. If $X_1 = X_2 = X_3$, then

$$1.8 = E[X] = 3E[X_1] = 3P\{X_1 = 1\}$$

and so $P\{X = 3\} = P\{X_1 = 1\} = 0.6$. That this is the largest value is seen by Markov's inequality, which yields

$$P\{X \geqslant 3\} \leqslant E[X]/3 = 0.6$$

(b) The smallest value is 0. To construct a probability scenario for which $P\{X = 3\} = 0$, let U be a uniform random variable on $(0, 1)$, and define

$$X_1 = \begin{cases}1, & \text{if } U \leqslant 0.6\\ 0, & \text{otherwise}\end{cases}$$

$$X_2 = \begin{cases}1, & \text{if } U \geqslant 0.4\\ 0, & \text{otherwise}\end{cases}$$

$$X_3 = \begin{cases}1, & \text{if either } U \leqslant 0.3 \text{ or } U \geqslant 0.7\\ 0, & \text{otherwise}\end{cases}$$

It is easy to see that

$$P\{X_1 = X_2 = X_3 = 1\} = 0$$

48. If X is a nonnegative random variable, and g is a differentiable function with $g(0) = 0$, then

$$E[g(X)] = \int_0^\infty P(X > t)g'(t)dt$$

Let f be the probability density function of X. One way to prove the result is to integrate by parts ($dv = g'(t)dt, u = P(X > t)$) to obtain

$$\int_0^\infty P(X > t)g'(t)dt = -f(t)g(t)|_0^\infty + \int_0^\infty g(t)f(t)dt = E[g(X)]$$

Another way is to let $I(t)$ be the indicator function for the event that $X > t$. Then,

$$g(X) = \int_0^X g'(t)dt = \int_0^\infty I(t)g'(t)dt$$

Now take expectations of both sides to obtain the result.

49. $E[X^2] - (E[X])^2 = \text{Var}(X) = E[(X - E[X])^2] \geqslant 0$. There is equality when $\text{Var}(X) = 0$, that is, when X is constant.

64. See Section 5.2.3 of Chapter 5. Another way is to use moment generating functions. The moment generating function of the sum of n independent exponentials with rate λ is equal to the product of their moment generating functions. That is, it is $[\lambda/(\lambda - t)]^n$. But this is precisely the moment generating function of a gamma with parameters n and λ.

70. Let X_i be Poisson with mean 1. Then

$$P\left\{\sum_1^n X_i \leqslant n\right\} = e^{-n}\sum_{k=0}^n \frac{n^k}{k!}$$

But for n large $\sum_1^n X_i - n$ has approximately a normal distribution with mean 0, and so the result follows.

72. For the matching problem, letting $X = X_1 + \cdots + X_N$, where

$$X_i = \begin{cases} 1, & \text{if } i\text{th man selects his own hat} \\ 0, & \text{otherwise} \end{cases}$$

we obtain

$$\text{Var}(X) = \sum_{i=1}^N \text{Var}(X_i) + 2\sum\sum_{i<j} \text{Cov}(X_i, X_j)$$

Since $P\{X_i = 1\} = 1/N$, we see

$$\text{Var}(X_i) = \frac{1}{N}\left(1 - \frac{1}{N}\right) = \frac{N-1}{N^2}$$

Also,

$$\mathrm{Cov}(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j]$$

Now,

$$X_i X_j = \begin{cases} 1, & \text{if the } i\text{th and } j\text{th men both select their own hats} \\ 0, & \text{otherwise} \end{cases}$$

and thus

$$\begin{aligned} E[X_i X_j] &= P\{X_i = 1,\ X_j = 1\} \\ &= P\{X_i = 1\}P\{X_j = 1 | X_i = 1\} \\ &= \frac{1}{N}\frac{1}{N-1} \end{aligned}$$

Hence,

$$\mathrm{Cov}(X_i, X_j) = \frac{1}{N(N-1)} - \left(\frac{1}{N}\right)^2 = \frac{1}{N^2(N-1)}$$

and

$$\begin{aligned} \mathrm{Var}(X) &= \frac{N-1}{N} + 2\binom{N}{2}\frac{1}{N^2(N-1)} \\ &= \frac{N-1}{N} + \frac{1}{N} \\ &= 1 \end{aligned}$$

Chapter 3

2. Intuitively it would seem that the first head would be equally likely to occur on any of trials $1, \ldots, n-1$. That is, it is intuitive that

$$P\{X_1 = i \,|\, X_1 + X_2 = n\} = \frac{1}{n-1}, \qquad i = 1, \ldots, n-1$$

Formally,

$$\begin{aligned} P\{X_1 = i \,|\, X_1 + X_2 = n\} &= \frac{P\{X_1 = i, X_1 + X_2 = n\}}{P\{X_1 + X_2 = n\}} \\ &= \frac{P\{X_1 = i, X_2 = n - i\}}{P\{X_1 + X_2 = n\}} \\ &= \frac{p(1-p)^{i-1}p(1-p)^{n-i-1}}{\binom{n-1}{1}p(1-p)^{n-2}p} \\ &= \frac{1}{n-1} \end{aligned}$$

In the preceding, the next to last equality uses the independence of X_1 and X_2 to evaluate the numerator and the fact that $X_1 + X_2$ has a negative binomial distribution to evaluate the denominator.

6.
$$
\begin{aligned}
p_{X|Y}(1\,|\,3) &= \frac{P\{X=1, Y=3\}}{P\{Y=3\}} \\
&= \frac{P\{1 \text{ white, } 3 \text{ black, } 2 \text{ red}\}}{P\{3 \text{ black}\}} \\
&= \frac{\dfrac{6!}{1!3!2!}\left(\dfrac{3}{14}\right)^1\left(\dfrac{5}{14}\right)^3\left(\dfrac{6}{14}\right)^2}{\dfrac{6!}{3!3!}\left(\dfrac{5}{14}\right)^3\left(\dfrac{9}{14}\right)^3} \\
&= \frac{4}{9} \\
p_{X|Y}(0\,|\,3) &= \frac{8}{27} \\
p_{X|Y}(2\,|\,3) &= \frac{2}{9} \\
p_{X|Y}(3\,|\,3) &= \frac{1}{27} \\
E[X\,|\,Y=1] &= \frac{5}{3}
\end{aligned}
$$

13. The conditional density of X given that $X > 1$ is

$$
f_{X|X>1}(X) = \frac{f(x)}{P\{X > 1\}} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda}} \qquad \text{when } x > 1
$$

$$
E[X\,|\,X > 1] = e^{\lambda}\int_1^{\infty} x\lambda\, e^{-\lambda x}\, dx = 1 + 1/\lambda
$$

by integration by parts. This latter result also follows immediately by the lack of memory property of the exponential.

19.
$$
\begin{aligned}
\int E[X\,|\,Y=y]f_Y(y)\,dy &= \iint x f_{X|Y}(x\,|\,y)\,dx f_Y(y)\,dy \\
&= \iint x\frac{f(x,y)}{f_Y(y)}\,dx f_Y(y)\,dy \\
&= \int x \int f(x,y)\,dy\,dx \\
&= \int x f_X(x)\,dx \\
&= E[X]
\end{aligned}
$$

23. Let X denote the first time a head appears. Let us obtain an equation for $E[N \mid X]$ by conditioning on the next two flips after X. This gives

$$E[N \mid X] = E[N \mid X, h, h]p^2 + E[N \mid X, h, t]pq + E[N \mid X, t, h]pq + E[N \mid X, t, t]q^2$$

where $q = 1 - p$. Now

$$E[N \mid X, h, h] = X + 1, \qquad E[N \mid X, h, t] = X + 1$$
$$E[N \mid X, t, h] = X + 2, \qquad E[N \mid X, t, t] = X + 2 + E[N]$$

Substituting back gives

$$E[N|X] = (X + 1)(p^2 + pq) + (X + 2)\, pq + (X + 2 + E[N])\, q^2$$

Taking expectations, and using the fact that X is geometric with mean $1/p$, we obtain

$$E[N] = 1 + p + q + 2pq + q^2/p + 2q^2 + q^2E[N]$$

Solving for $E[N]$ yields

$$E[N] = \frac{2 + 2q + q^2/p}{1 - q^2}$$

42. (a)
$$E[e^{tX^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx^2} e^{-(x-\mu)^2/2} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-(x^2 - 2\mu x + \mu^2 - 2tx^2)/2\} dx$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\mu^2/2} \int_{-\infty}^{\infty} \exp\{-(x^2(1 - 2t) - 2\mu x)/2\} dx$$

Thus, with $\sigma^2 = \frac{1}{1-2t}$

$$E[e^{tX^2}] = \frac{1}{\sqrt{2\pi}} e^{-\mu^2/2} \int_{-\infty}^{\infty} \exp\{-(x^2 - 2\sigma^2\mu x)/2\sigma^2\} dx$$

Using that

$$x^2 - 2\sigma^2\mu x = (x - \sigma^2\mu)^2 - \mu^2\sigma^4$$

we have

$$E[e^{tX^2}] = e^{-\mu^2/2+\mu^2\sigma^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-(x - \sigma^2\mu)^2/2\sigma^2\} dx$$
$$= e^{-(1-\sigma^2)\mu^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-y^2/2\sigma^2\} dy$$
$$= \sigma e^{-(1-\sigma^2)\mu^2/2}$$
$$= (1 - 2t)^{-1/2} \exp\left\{ -\left(1 - \frac{1}{1 - 2t}\right)\mu^2/2 \right\}$$
$$= (1 - 2t)^{-1/2} e^{\frac{t\mu^2}{1-2t}}$$

(b)
$$E\left[\exp\left\{t\sum_{i=1}^{n}X_i^2\right\}\right]=\prod_{i=1}^{n}E\left[e^{tX_i^2}\right]$$
$$=(1-2t)^{-n/2}\exp\left\{\frac{t}{1-2t}\sum_{i=1}^{n}\mu_i^2\right\}$$

(c)
$$\frac{d}{dt}(1-2t)^{-n/2}=n(1-2t)^{-n/2-1}$$
$$\frac{d^2}{dt^2}(1-2t)^{-n/2}=2n(n/2+1)(1-2t)^{-n/2-2}$$

Hence, if χ_n^2 is chi-squared with n degrees of freedom then evaluating the preceding at $t=0$ gives

$$E\left[\chi_n^2\right]=n \quad \mathrm{Var}\left(\chi_n^2\right)=n^2+2n-n^2=2n$$

(d) Conditioning on K yields

$$E\left[e^{tW}\right]=\sum_{k=0}^{\infty}E\left[e^{tW}|K=k\right]e^{-\theta/2}(\theta/2)^k/k!$$
$$=\sum_{k=0}^{\infty}(1-2t)^{-(n+2k)/2}e^{-\theta/2}(\theta/2)^k/k!$$
$$=(1-2t)^{-n/2}e^{-\theta/2}\sum_{k=0}^{\infty}(1-2t)^{-k}(\theta/2)^k/k!$$
$$=(1-2t)^{-n/2}e^{-\theta/2}\sum_{k=0}^{\infty}\left(\frac{\theta}{2(1-2t)}\right)^k\Big/k!$$
$$=(1-2t)^{-n/2}\exp\left\{-\frac{\theta}{2}+\frac{\theta}{2(1-2t)}\right\}$$
$$=(1-2t)^{-n/2}\exp\left\{\frac{t\theta}{1-2t}\right\}$$

Because the preceding is the moment generating function of a noncentral chi-squared random variable with parameters n and θ, and the moment generating function uniquely determines the distribution, the result is proven.

(e) From the preceding, we have

$$E[W|K=k]=E[\chi_{n+2k}^2]=n+2k$$
$$\mathrm{Var}(W|K=k)=\mathrm{Var}(\chi_{n+2k}^2)=2n+4k$$

Hence,

$$E[W]=E[E[W|K]]=E[n+2K]=n+2E[K]=n+\theta$$

and the conditional variance formula yields

$$\mathrm{Var}(W) = E[2n + 4K] + \mathrm{Var}(n + 2K) = 2n + 2\theta + 2\theta = 2n + 4\theta$$

47. $$\begin{aligned} E[X^2Y^2|X] &= X^2E[Y^2|X] \\ &\geqslant X^2(E[Y|X])^2 = X^2 \end{aligned}$$

The inequality follows since for any random variable U, $E[U^2] \geqslant (E[U])^2$ and this remains true when conditioning on some other random variable X. Taking expectations of the preceding shows that

$$E[(XY)^2] \geqslant E[X^2]$$

As

$$E[XY] = E[E[XY \mid X]] = E[XE[Y \mid X]] = E[X]$$

the results follow.

53. $$\begin{aligned} P\{X = n\} &= \int_0^\infty P\{X = n|\lambda\}e^{-\lambda}\, d\lambda \\ &= \int_0^\infty \frac{e^{-\lambda}\lambda^n}{n!} e^{-\lambda}\, d\lambda \\ &= \int_0^\infty e^{-2\lambda}\lambda^n \frac{d\lambda}{n!} \\ &= \int_0^\infty e^{-t}t^n \frac{dt}{n!}\left(\frac{1}{2}\right)^{n+1} \end{aligned}$$

The results follow since $\int_0^\infty e^{-t}t^n\, dt = \Gamma(n + 1) = n!$

60.

(a) Intuitive that $f(p)$ is increasing in p, since the larger p is the greater is the advantage of going first.

(b) 1.

(c) $\frac{1}{2}$ since the advantage of going first becomes nil.

(d) Condition on the outcome of the first flip:

$$\begin{aligned} f(p) &= P\{I \text{ wins} \mid h\}p + P\{I \text{ wins} \mid t\}(1 - p) \\ &= p + [1 - f(p)](1 - p) \end{aligned}$$

Therefore,

$$f(p) = \frac{1}{2 - p}$$

67. Part (a) is proven by noting that a run of j successive heads can occur within the first n flips in two mutually exclusive ways. Either there is a run of j successive heads within the first $n-1$ flips; or there is no run of j successive heads within the first $n-j-1$ flips, flip $n-j$ is not a head, and flips $n-j+1$ through n are all heads.

Let A be the event that a run of j successive heads occurs within the first n, $(n \geqslant j)$, flips. Conditioning on X, the trial number of the first non-head, gives the following

$$
\begin{aligned}
P_j(n) &= \sum_k P(A \mid X=k)p^{k-1}(1-p) \\
&= \sum_{k=1}^{j} P(A \mid X=k)p^{k-1}(1-p) + \sum_{k=j+1}^{\infty} P(A \mid X=k)p^{k-1}(1-p) \\
&= \sum_{i=1}^{j} P_j(n-k)p^{k-1}(1-p) + \sum_{k=j+1}^{\infty} p^{k-1}(1-p) \\
&= \sum_{i=1}^{j} P_j(n-k)p^{k-1}(1-p) + p^j
\end{aligned}
$$

73. Condition on the value of the sum prior to going over 100. In all cases the most likely value is 101. (For instance, if this sum is 98 then the final sum is equally likely to be either 101, 102, 103, or 104. If the sum prior to going over is 95, then the final sum is 101 with certainty.)

93. (a) By symmetry, for any value of $(T_1, \ldots, T_m)$, the random vector $(I_1, \ldots, I_m)$ is equally likely to be any of the $m!$ permutations.

(b)
$$
\begin{aligned}
E[N] &= \sum_{i=1}^{m} E[N|X=i]P\{X=i\} \\
&= \frac{1}{m}\sum_{i=1}^{m} E[N|X=i] \\
&= \frac{1}{m}\left(\sum_{i=1}^{m-1}(E[T_i]+E[N]) + E[T_{m-1}]\right)
\end{aligned}
$$

where the final equality used the independence of X and T_i. Therefore,

$$
E[N] = E[T_{m-1}] + \sum_{i=1}^{m-1} E[T_i]
$$

(c) $E[T_i] = \displaystyle\sum_{j=1}^{i} \frac{m}{m+1-j}$

(d) $$E[N] = \sum_{j=1}^{m-1} \frac{m}{m+1-j} + \sum_{i=1}^{m-1} \sum_{j=1}^{i} \frac{m}{m+1-j}$$

$$= \sum_{j=1}^{m-1} \frac{m}{m+1-j} + \sum_{j=1}^{m-1} \sum_{i=j}^{m-1} \frac{m}{m+1-j}$$

$$= \sum_{j=1}^{m-1} \frac{m}{m+1-j} + \sum_{j=1}^{m-1} \frac{m(m-j)}{m+1-j}$$

$$= \sum_{j=1}^{m-1} \left(\frac{m}{m+1-j} + \frac{m(m-j)}{m+1-j} \right)$$

$$= m(m-1)$$

97. Let X be geometric with parameter p. To compute Var (X), we will use the conditional variance formula, conditioning on the outcome of the first trial. Let I equal 1 if the first trial is a success, and let it equal 0 otherwise. If $I = 1$, then $X = 1$; since the variance of a constant is 0, this gives

$$\mathrm{Var}(X|I = 1) = 0$$

On the other hand, if $I = 0$ then the conditional distribution of X given that $I = 0$ is the same as the unconditional distribution of 1 (the first trial) plus a geometric with parameter p (the number of additional trials needed for a success). Therefore,

$$\mathrm{Var}(X|I = 0) = \mathrm{Var}(X)$$

yielding

$$E[\mathrm{Var}(X|I)] = \mathrm{Var}(X|I = 1)P(I = 1) + \mathrm{Var}(X|I = 0)P(I = 0) = (1 - p)\mathrm{Var}(X)$$

Similarly,

$$E[X|I = 1] = 1, \quad E[X|I = 0] = 1 + E[X] = 1 + \frac{1}{p}$$

which can be written as

$$E[X|I] = 1 + \frac{1}{p}(1 - I)$$

yielding

$$\mathrm{Var}(E[X|I]) = \frac{1}{p^2}\mathrm{Var}(I) = \frac{1}{p^2}p(1 - p) = \frac{1 - p}{p}$$

The conditional variance formula now gives

$$\begin{aligned}\mathrm{Var}(X) &= E[\mathrm{Var}(X|I)] + \mathrm{Var}(E[X|I]) \\ &= (1-p)\mathrm{Var}(X) + \frac{1-p}{p}\end{aligned}$$

or

$$\mathrm{Var}(X) = \frac{1-p}{p^2}$$

Chapter 4

1.
$$\begin{aligned} &P_{01} = 1,\ P_{10} = \tfrac{1}{9}, \quad P_{21} = \tfrac{4}{9}, \quad P_{32} = 1 \\ &P_{11} = \tfrac{4}{9}, \quad P_{22} = \tfrac{4}{9} \\ &P_{12} = \tfrac{4}{9}, \quad P_{23} = \tfrac{1}{9}\end{aligned}$$

4. Let the state space be $S = \{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$, where state $i(\bar{i})$ signifies that the present value is i, and the present day is even (odd).

16. If P_{ij} were (strictly) positive, then P^n_{ji} would be 0 for all n (otherwise, i and j would communicate). But then the process, starting in i, has a positive probability of at least P_{ij} of never returning to i. This contradicts the recurrence of i. Hence $P_{ij} = 0$.

21. The transition probabilities are

$$P_{i,j} = \begin{cases} 1-3\alpha, & \text{if } j = i \\ \alpha, & \text{if } j \neq i \end{cases}$$

By symmetry,

$$P^n_{ij} = \frac{1}{3}(1 - P^n_{ii}), \qquad j \neq i$$

So, let us prove by induction that

$$P^n_{i,j} = \begin{cases} \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n & \text{if } j = i \\ \frac{1}{4} - \frac{1}{4}(1-4\alpha)^n & \text{if } j \neq i \end{cases}$$

As the preceding is true for $n = 1$, assume it for n. To complete the induction proof, we need to show that

$$P^{n+1}_{i,j} = \begin{cases} \frac{1}{4} + \frac{3}{4}(1-4\alpha)^{n+1} & \text{if } j = i \\ \frac{1}{4} - \frac{1}{4}(1-4\alpha)^{n+1} & \text{if } j \neq i \end{cases}$$

Now,

$$\begin{aligned}
P_{i,i}^{n+1} &= P_{i,i}^{n} P_{i,i} + \sum_{j \neq i} P_{i,j}^{n} P_{j,i} \\
&= \left(\frac{1}{4} + \frac{3}{4}(1-4\alpha)^n\right)(1-3\alpha) + 3\left(\frac{1}{4} - \frac{1}{4}(1-4\alpha)^n\right)\alpha \\
&= \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n(1-3\alpha-\alpha) \\
&= \frac{1}{4} + \frac{3}{4}(1-4\alpha)^{n+1}
\end{aligned}$$

By symmetry, for $j \neq i$

$$P_{ij}^{n+1} = \frac{1}{3}(1 - P_{ii}^{n+1}) = \frac{1}{4} - \frac{1}{4}(1-4\alpha)^{n+1}$$

and the induction is complete.

By letting $n \to \infty$ in the preceding, or by using that the transition probability matrix is doubly stochastic, or by just using a symmetry argument, we obtain that $\pi_i = 1/4$, $i = 1, 2, 3, 4$.

27. (a) It is a Markov chain because each individual's state the next period depends only on its current state and not on any information about earlier times.

(b) If i of the N individuals are currently active, then the number of actives in the next period is the sum of two independent random variables; R_i, the number of the i currently active who remain active in the next period; and B_i, the number of the $N - i$ inactives who become active in the next period. Because R_i is binomial (i, α), and B_i is binomial $(N - i, b)$, where $b = 1 - \beta$, we see that

$$E[X_n | X_{n-1} = i] = i\alpha + (N - i)(1 - \beta) = N(1 - \beta) + (\alpha + \beta - 1)i$$

Hence,

$$E[X_n | X_{n-1}] = N(1 - \beta) + (\alpha + \beta - 1)X_{n-1}$$

giving that

$$E[X_n] = N(1 - \beta) + (\alpha + \beta - 1)E[X_{n-1}]$$

Letting $a = N(1 - \beta), b = \alpha + \beta - 1$, the preceding gives

$$\begin{aligned}
E[X_n] &= a + bE[X_{n-1}] \\
&= a + b(a + bE[X_{n-2}]) = a + ba + b^2E[X_{n-2}] \\
&= a + ba + b^2a + b^3E[X_{n-3}]
\end{aligned}$$

Continuing this, we arrive at

$$E[X_n] = a\left(1 + b + \cdots + b^{n-1}\right) + b^nE[X_0]$$

Thus,

$$E[X_n|X_0 = i] = a\left(1 + b + \cdots + b^{n-1}\right) + b^n i$$

Note that

$$\lim_{n\to\infty} E[X_n] = \frac{a}{1-b} = N\frac{1-\beta}{2-\alpha-\beta}$$

(c) With R_i, B_i as previously defined

$$\begin{aligned}
P_{i,j} &= P\big(R_i + B_i = j\big) \\
&= \sum_k P\big(R_i + B_i = j | R_i = k\big) \binom{i}{k} \alpha^i (1-\alpha)^{i-k} \\
&= \sum_k \binom{N-i}{j-k} (1-\beta)^{j-k} \beta^{N-i-j+k} \binom{i}{k} \alpha^i (1-\alpha)^{i-k}
\end{aligned}$$

where $\binom{m}{r} = 0$ if $r < 0$ or $r > m$.

(d) Suppose $N = 1$. Then, with 1 standing for active and 0 for inactive, the limiting probabilities are such that

$$\begin{aligned}
\pi_0 &= \pi_0\beta + \pi_1(1-\alpha) \\
\pi_1 &= \pi_0(1-\beta) + \pi_1\alpha \\
\pi_0 + \pi_1 &= 1
\end{aligned}$$

Solving yields

$$\pi_1 = \frac{1-\beta}{2-\alpha-\beta}, \quad \pi_0 = \frac{1-\alpha}{2-\alpha-\beta}$$

Now consider the case of population size N. Because each member will, in steady state, be active with probability π_1 and because each of the members changes states independently of each other it follows that the steady state number of actives has a binomial (N, π_1) distribution. Hence, the long-run proportion of time that exactly j people are active is

$$\pi_j(N) = \binom{N}{j}\left(\frac{1-\beta}{2-\alpha-\beta}\right)^j \left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{N-j}$$

Note that the steady state expected number of actives is $N\frac{1-\alpha}{2-\alpha-\beta}$, in accord with what we saw in part (b).

32. With the state being the number of on switches this is a three-state Markov chain. The equations for the long-run proportions are

$$\begin{aligned}
\pi_0 &= \frac{9}{16}\pi_0 + \frac{1}{4}\pi_1 + \frac{1}{16}\pi_2, \\
\pi_1 &= \frac{3}{8}\pi_0 + \frac{1}{2}\pi_1 + \frac{3}{8}\pi_2, \\
&\pi_0 + \pi_1 + \pi_2 = 1
\end{aligned}$$

This gives the solution

$$\pi_0 = \frac{2}{7}, \qquad \pi_1 = \frac{3}{7}, \qquad \pi_2 = \frac{2}{7}$$

41. (a) The number of transitions into state i by time n, the number of transitions originating from state i by time n, and the number of time periods the chain is in state i by time n all differ by at most 1. Thus, their long-run proportions must be equal.
(b) $\pi_i P_{ij}$ is the long-run proportion of transitions that go from state i to state j.
(c) $\sum_i \pi_i P_{ij}$ is the long-run proportion of transitions that are into state j.
(d) Since π_j is also the long-run proportion of transitions that are into state j, it follows that $\pi_j = \sum_i \pi_i P_{ij}$.

47. $\{Y_n, n \geqslant 1\}$ is a Markov chain with states (i, j).

$$P_{(i,j),(k,l)} = \begin{cases} 0, & \text{if } j \neq k \\ P_{jl}, & \text{if } j = k \end{cases}$$

where P_{jl} is the transition probability for $\{X_n\}$.

$$\begin{aligned} \lim_{n\to\infty} P\{Y_n = (i, j)\} &= \lim_n P\{X_n = i, X_{n+1} = j\} \\ &= \lim_n [P\{X_n = i\} P_{ij}] \\ &= \pi_i P_{ij} \end{aligned}$$

62. It is easy to verify that the stationary probabilities are $\pi_i = \frac{1}{n+1}$. Hence, the mean time to return to the initial position is $n + 1$.

68. (a) $\displaystyle\sum_i \pi_i Q_{ij} = \sum_i \pi_j P_{ji} = \pi_j \sum_i P_{ji} = \pi_j$

(b) Whether perusing the sequence of states in the forward direction of time or in the reverse direction, the proportion of time the state is i will be the same.

Chapter 5

7. $P\{X_1 < X_2 \mid \min(X_1, X_2) = t\}$

$$= \frac{P\{X_1 < X_2, \min(X_1, X_2) = t\}}{P\{\min(X_1, X_2) = t\}}$$

$$= \frac{P\{X_1 = t, X_2 > t\}}{P\{X_1 = t, X_2 > t\} + P\{X_2 = t, X_1 > t\}}$$

$$= \frac{f_1(t)[1 - F_2(t)]}{f_1(t)[1 - F_2(t)] + f_2(t)[1 - F_1(t)]}$$

Dividing through by $[1 - F_1(t)][1 - F_2(t)]$ yields the result. (Of course, f_i and F_i are the density and distribution function of X_i, $i = 1, 2$.) To make the preceding derivation rigorous, we should replace "$= t$" by $\in (t, t + \varepsilon)$ throughout and then let $\varepsilon \to 0$.

10. (a)
$$\begin{aligned} E[MX|M = X] &= E[M^2|M = X] \\ &= E[M^2] \\ &= \frac{2}{(\lambda + \mu)^2} \end{aligned}$$

(b) By the memoryless property of exponentials, given that $M = Y$, X is distributed as $M + X'$ where X' is an exponential with rate λ that is independent of M. Therefore,

$$\begin{aligned} E[MX|M = Y] &= E[M(M + X')] \\ &= E[M^2] + E[M]E[X'] \\ &= \frac{2}{(\lambda + \mu)^2} + \frac{1}{\lambda(\lambda + \mu)} \end{aligned}$$

(c)
$$\begin{aligned} E[MX] &= E[MX|M = X]\frac{\lambda}{\lambda + \mu} + E[MX|M = Y]\frac{\mu}{\lambda + \mu} \\ &= \frac{2\lambda + \mu}{\lambda(\lambda + \mu)^2} \end{aligned}$$

Therefore,

$$\mathrm{Cov}(X, M) = \frac{\lambda}{\lambda(\lambda + \mu)^2}$$

18. (a) $1/(2\mu)$.
(b) $1/(4\mu^2)$, since the variance of an exponential is its mean squared.
(c) and (d). By the lack of memory property of the exponential it follows that A, the amount by which $X_{(2)}$ exceeds $X_{(1)}$, is exponentially distributed with rate μ and is independent of $X_{(1)}$. Therefore,

$$E[X_{(2)}] = E[X_{(1)} + A] = \frac{1}{2\mu} + \frac{1}{\mu}$$
$$\mathrm{Var}(X_{(2)}) = \mathrm{Var}(X_{(1)} + A) = \frac{1}{4\mu^2} + \frac{1}{\mu^2} = \frac{5}{4\mu^2}$$

23. (a) $\frac{1}{2}$.
(b) $(\frac{1}{2})^{n-1}$. Whenever battery 1 is in use and a failure occurs the probability is $\frac{1}{2}$ that it is not battery 1 that has failed.
(c) $(\frac{1}{2})^{n-i+1}$, $i > 1$.
(d) T is the sum of $n - 1$ independent exponentials with rate 2μ (since each time a failure occurs the time until the next failure is exponential with rate 2μ).
(e) Gamma with parameters $n - 1$ and 2μ.

36.
$$\begin{aligned}
E[S(t)|N(t)=n] &= sE\left[\prod_{i=1}^{N(t)} X_i|N(t)=n\right] \\
&= sE\left[\prod_{i=1}^{n} X_i|N(t)=n\right] \\
&= sE\left[\prod_{i=1}^{n} X_i\right] \\
&= s(E[X])^n \\
&= s(1/\mu)^n
\end{aligned}$$

Thus,

$$\begin{aligned}
E[S(t)] &= s\sum_n (1/\mu)^n e^{-\lambda t}(\lambda t)^n/n! \\
&= se^{-\lambda t}\sum_n (\lambda t/\mu)^n/n! \\
&= se^{-\lambda t+\lambda t/\mu}
\end{aligned}$$

By the same reasoning

$$E[S^2(t)|N(t)=n] = s^2(E[X^2])^n = s^2(2/\mu^2)^n$$

and

$$E[S^2(t)] = s^2 e^{-\lambda t+2\lambda t/\mu^2}$$

40. The easiest way is to use Definition 5.3. It is easy to see that $\{N(t), t \geqslant 0\}$ will also possess stationary and independent increments. Since the sum of two independent Poisson random variables is also Poisson, it follows that $N(t)$ is a Poisson random variable with mean $(\lambda_1 + \lambda_2)t$.

57. (a) e^{-2}.
(b) 2 P.M.
(c) $1 - 5e^{-4}$.

60. (a) $\frac{1}{9}$.
(b) $\frac{5}{9}$.

64. (a) Since, given $N(t)$, each arrival is uniformly distributed on $(0, t)$ it follows that

$$E[X|N(t)] = N(t)\int_0^t (t-s)\frac{ds}{t} = N(t)\frac{t}{2}$$

(b) Let $U_1, U_2, \ldots$ be independent uniform $(0, t)$ random variables. Then

$$\begin{aligned}
\mathrm{Var}(X|N(t)=n) &= \mathrm{Var}\left[\sum_{i=1}^{n}(t-U_i)\right] \\
&= n\,\mathrm{Var}(U_i) = n\frac{t^2}{12}
\end{aligned}$$

(c) By parts (a) and (b) and the conditional variance formula,

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{Var}\left(\frac{N(t)t}{2}\right) + E\left[\frac{N(t)t^2}{12}\right]\\ &= \frac{\lambda t t^2}{4} + \frac{\lambda t t^2}{12} = \frac{\lambda t^3}{3}\end{aligned}$$

79. Consider a Poisson process with rate λ in which an event at time t is counted with probability $\lambda(t)/\lambda$ independently of the past. Clearly such a process will have independent increments. In addition,

$$\begin{aligned}&P\{2 \text{ or more counted events in } (t, t+h)\}\\ &\quad \leqslant P\{2 \text{ or more events in } (t, t+h)\}\\ &\quad = o(h)\end{aligned}$$

and

$$\begin{aligned}&P\{1 \text{ counted event in } (t, t+h)\}\\ &\quad = P\{1 \text{ counted} \mid 1 \text{ event}\}P(1 \text{ event})\\ &\qquad + P\{1 \text{ counted} \mid \geqslant 2 \text{ events}\}P\{\geqslant 2\}\\ &\quad = \int_t^{t+h} \frac{\lambda(s)}{\lambda}\frac{ds}{h}(\lambda h + o(h)) + o(h)\\ &\quad = \frac{\lambda(t)}{\lambda}\lambda h + o(h)\\ &\quad = \lambda(t)h + o(h)\end{aligned}$$

84. There is a record whose value is between t and $t + dt$ if the first X larger than t lies between t and $t + dt$. From this we see that, independent of all record values less than t, there will be one between t and $t + dt$ with probability $\lambda(t)\,dt$ where $\lambda(t)$ is the failure rate function given by

$$\lambda(t) = \frac{f(t)}{1 - F(t)}$$

Since the counting process of record values has, by the preceding, independent increments we can conclude (since there cannot be multiple record values because the X_i are continuous) that it is a nonhomogeneous Poisson process with intensity function $\lambda(t)$. When f is the exponential density, $\lambda(t) = \lambda$ and so the counting process of record values becomes an ordinary Poisson process with rate λ.

91. To begin, note that

$$\begin{aligned}P\left\{X_1 > \sum_2^n X_i\right\} &= P\{X_1 > X_2\}P\{X_1 - X_2 > X_3 | X_1 > X_2\}\\ &\quad \times P\{X_1 - X_2 - X_3 > X_4 | X_1 > X_2 + X_3\}\cdots\\ &\quad \times P\{X_1 - X_2 \cdots - X_{n-1} > X_n | X_1 > X_2 + \cdots + X_{n-1}\}\\ &= \left(\frac{1}{2}\right)^{n-1} \qquad \text{by lack of memory}\end{aligned}$$

Hence,

$$P\left\{M > \sum_{i=1}^{n} X_i - M\right\} = \sum_{i=1}^{n} P\left\{X_i > \sum_{j \neq i} X_j\right\} = \frac{n}{2^{n-1}}$$

Chapter 6

2. Let $N_A(t)$ be the number of organisms in state A and let $N_B(t)$ be the number of organisms in state B. Then $\{N_A(t), N_B(t)\}$ is a continuous-Markov chain with

$$\begin{aligned} \nu_{\{n,m\}} &= \alpha n + \beta m \\ P_{\{n,m\},\{n-1,m+1\}} &= \frac{\alpha n}{\alpha n + \beta m} \\ P_{\{n,m\},\{n+2,m-1\}} &= \frac{\beta m}{\alpha n + \beta m} \end{aligned}$$

4. Let $N(t)$ denote the number of customers in the station at time t. Then $\{N(t)\}$ is a birth and death process with

$$\lambda_n = \lambda \alpha_n, \qquad \mu_n = \mu$$

7. (a) Yes!
   (b) For $\mathbf{n} = (n_1, \ldots, n_i, n_{i+1}, \ldots, n_{k-1})$ let

$$\begin{aligned} S_i(\mathbf{n}) &= (n_1, \ldots, n_i - 1, n_{i+1} + 1, \ldots, n_{k-1}), \qquad i = 1, \ldots, k-2 \\ S_{k-1}(\mathbf{n}) &= (n_1, \ldots, n_i, n_{i+1}, \ldots, n_{k-1} - 1), \\ S_0(\mathbf{n}) &= (n_1 + 1, \ldots, n_i, n_{i+1}, \ldots, n_{k-1}). \end{aligned}$$

   Then

$$\begin{aligned} q_{\mathbf{n},S_i(\mathbf{n})} &= n_i \mu, \qquad i = 1, \ldots, k-1 \\ q_{\mathbf{n},S_0(n)} &= \lambda \end{aligned}$$

11. (b) Follows from the hint about using the lack of memory property and the fact that ε_i, the minimum of $j - (i-1)$ independent exponentials with rate λ, is exponential with rate $(j - i - 1)\lambda$.
    (c) From parts (a) and (b)

$$P\{T_1 + \cdots + T_j \leqslant t\} = P\left\{\max_{1 \leqslant i \leqslant j} X_i \leqslant t\right\} = (1 - e^{-\lambda t})^j$$

    (d) With all probabilities conditional on $X(0) = 1$,

$$\begin{aligned} P_{1j}(t) &= P\{X(t) = j\} \\ &= P\{X(t) \geqslant j\} - P\{X(t) \geqslant j+1\} \\ &= P\{T_1 + \cdots + T_j \leqslant t\} - P\{T_1 + \cdots + T_{j+1} \leqslant t\} \end{aligned}$$

(e) The sum of i independent geometrics, each having parameter $p = e^{-\lambda t}$, is a negative binomial with parameters i, p. The result follows since starting with an initial population of i is equivalent to having i independent Yule processes, each starting with a single individual.

16. Let the state be

2: an acceptable molecule is attached
0: no molecule attached
1: an unacceptable molecule is attached.

Then, this is a birth and death process with balance equations

$$\mu_1 P_1 = \lambda(1-\alpha)P_0$$
$$\mu_2 P_2 = \lambda\alpha P_0$$

Since $\sum_0^2 P_i = 1$, we get

$$P_2 = \left[1 + \frac{\mu_2}{\lambda\alpha} + \frac{1-\alpha}{\alpha}\frac{\mu_2}{\mu_1}\right]^{-1} = \frac{\lambda\alpha\mu_1}{\lambda\alpha\mu_1 + \mu_1\mu_2 + \lambda(1-\alpha)\mu_2}$$

where P_2 is the percentage of time the site is occupied by an acceptable molecule. The percentage of time the site is occupied by an unacceptable molecule is

$$P_1 = \frac{1-\alpha}{\alpha}\frac{\mu_2}{\mu_1}P_1 = \frac{\lambda(1-\alpha)\mu_2}{\lambda\alpha\mu_1 + \mu_1\mu_2 + \lambda(1-\alpha)\mu_2}$$

19. There are four states. Let state 0 mean that no machines are down, state 1 that machine 1 is down and 2 is up, state 2 that machine 1 is up and 2 is down, and state 3 that both machines are down. The balance equations are as follows:

$$\begin{aligned}(\lambda_1 + \lambda_2)P_0 &= \mu_1 P_1 + \mu_2 P_2\\ (\mu_1 + \lambda_2)P_1 &= \lambda_1 P_0\\ (\lambda_1 + \mu_2)P_2 &= \lambda_2 P_0 + \mu_1 P_3\\ \mu_1 P_3 &= \lambda_2 P_1 + \lambda_1 P_2\\ P_0 + P_1 + P_2 + P_3 &= 1\end{aligned}$$

The equations are easily solved and the proportion of time machine 2 is down is $P_2 + P_3$.

24. We will let the state be the number of taxis waiting. Then, we get a birth and death process with $\lambda_n = 1, \mu_n = 2$. This is an $M/M/1$. Therefore:

(a) Average number of taxis waiting $= \dfrac{1}{\mu - \lambda} = \dfrac{1}{2-1} = 1$.

(b) The proportion of arriving customers that gets taxis is the proportion of arriving customers that find at least one taxi waiting. The rate of arrival of such customers is $2(1 - P_0)$. The proportion of such arrivals is therefore

$$\frac{2(1-P_0)}{2} = 1 - P_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{\lambda}{\mu} = \frac{1}{2}$$

28. Let P^x_{ij}, v^x_i denote the parameters of the $X(t)$ and P^y_{ij}, v^y_i of the $Y(t)$ process; and let the limiting probabilities be P^x_i, P^y_i, respectively. By independence we have that for the Markov chain $\{X(t), Y(t)\}$ its parameters are

$$v_{(i,l)} = v^x_i + v^y_l,$$
$$P_{(i,l)(j,l)} = \frac{v^x_i}{v^x_i + v^y_l} P^x_{ij},$$
$$P_{(i,l)(i,k)} = \frac{v^y_l}{v^x_i + v^y_l} P^y_{lk},$$

and

$$\lim_{t\to\infty} P\{(X(t), Y(t)) = (i, j)\} = P^x_i P^y_j$$

Hence, we need to show that

$$P^x_i P^y_l v^x_i P^x_{ij} = P^x_j P^y_l v^x_j P^x_{ji}$$

(That is, the rate from (i, l) to (j, l) equals the rate from (j, l) to (i, l).) But this follows from the fact that the rate from i to j in $X(t)$ equals the rate from j to i; that is,

$$P^x_i v^x_i P^x_{ij} = P^x_j v^x_j P^x_{ji}$$

The analysis is similar in looking at pairs (i, l) and (i, k).

33. Suppose first that the waiting room is of infinite size. Let $X_i(t)$ denote the number of customers at server i, $i = 1, 2$. Then since each of the $M/M/1$ processes $\{X_1(t)\}$ is time reversible, it follows from Exercise 28 that the vector process $\{(X_1(t), (X(t)), t \geqslant 0\}$ is a time reversible Markov chain. Now the process of interest is just the truncation of this vector process to the set of states A where

$$A = \{(0, m): m \leqslant 4\} \cup \{(n, 0): n \leqslant 4\} \cup \{(n, m): nm > 0, n + m \leqslant 5\}$$

Hence, the probability that there are n with server 1 and m with server 2 is

$$P_{n,m} = k\left(\frac{\lambda_1}{\mu_1}\right)^n \left(1 - \frac{\lambda_1}{\mu_1}\right)\left(\frac{\lambda_2}{\mu_2}\right)^m \left(1 - \frac{\lambda_2}{\mu_2}\right)$$
$$= C\left(\frac{\lambda_1}{\mu_1}\right)^n \left(\frac{\lambda_2}{\mu_2}\right)^m, \qquad (n, m) \in A$$

The constant C is determined from

$$\sum P_{n,m} = 1$$

where the sum is over all (n, m) in A.

42. (a) The matrix $\mathbf{P}^*$ can be written as

$$\mathbf{P}^* = \mathbf{I} + \mathbf{R}/v$$

and so P_{ij}^{*n} can be obtained by taking the i, j element of $(\mathbf{I} + \mathbf{R}/v)^n$, which gives the result when $v = n/t$.

(b) Uniformization shows that $P_{ij}(t) = E[P_{ij}^{*N}]$, where N is independent of the Markov chain with transition probabilities P_{ij}^* and is Poisson distributed with mean vt. Since a Poisson random variable with mean vt has standard deviation $(vt)^{1/2}$, it follows that for large values of vt it should be near vt. (For instance, a Poisson random variable with mean 10^6 has standard deviation 10^3 and thus will, with high probability, be within 3000 of 10^6.) Hence, since for fixed i and j, P_{ij}^{*m} should not vary much for values of m about vt where vt is large, it follows that, for large vt,

$$E[P_{ij}^{*N}] \approx P_{ij}^{*n} \qquad \text{where } n = vt$$

Chapter 7

3. By the one-to-one correspondence of $m(t)$ and F, it follows that $\{N(t), t \geqslant 0\}$ is a Poisson process with rate $\frac{1}{2}$. Hence,

$$P\{N(5) = 0\} = e^{-5/2}$$

6. (a) Consider a Poisson process having rate λ and say that an event of the renewal process occurs whenever one of the events numbered $r, 2r, 3r, \ldots$ of the Poisson process occurs. Then

$$\begin{aligned} P\{N(t) \geqslant n\} &= P\{nr \text{ or more Poisson events by } t\} \\ &= \sum_{i=nr}^{\infty} e^{-\lambda t}(\lambda t)^i / i! \end{aligned}$$

(b)
$$\begin{aligned} E[N(t)] &= \sum_{n=1}^{\infty} P\{N(t) \geqslant n\} = \sum_{n=1}^{\infty} \sum_{i=nr}^{\infty} e^{-\lambda t}(\lambda t)^i / i! \\ &= \sum_{i=r}^{\infty} \sum_{n=1}^{[i/r]} e^{-\lambda t}(\lambda t)^i / i! = \sum_{i=r}^{\infty} [i/r] e^{-\lambda t}(\lambda t)^i / i! \end{aligned}$$

8. (a) The number of replaced machines by time t constitutes a renewal process. The time between replacements equals T, if the lifetime of new machine is $\geqslant T$; x, if the lifetime of new machine is $x, x < T$. Hence,

$$E[\text{time between replacements}] = \int_0^T x f(x)\, dx + T[1 - F(T)]$$

and the result follows by Proposition 3.1.

(b) The number of machines that have failed in use by time t constitutes a renewal process. The mean time between in-use failures, $E[F]$, can be calculated by conditioning on the lifetime of the initial machine as $E[F] = E[E[F|\text{ lifetime of initial machine}]]$. Now

$$E[F \mid \text{lifetime of machine is } x] = \begin{cases} x, & \text{if } x \leqslant T \\ T + E[F], & \text{if } x > T \end{cases}$$

Hence,

$$E[F] = \int_0^T x f(x)\,dx + (T + E[F])[1 - F(T)]$$

or

$$E[F] = \frac{\int_0^T x f(x)\,dx + T[1 - F(T)]}{F(T)}$$

and the result follows from Proposition 3.1.

18. We can imagine that a renewal corresponds to a machine failure, and each time a new machine is put in use its life distribution will be exponential with rate μ_1 with probability p, and exponential with rate μ_2 otherwise. Hence, if our state is the index of the exponential life distribution of the machine presently in use, then this is a two-state continuous-time Markov chain with intensity rates

$$q_{1,2} = \mu_1(1-p), \qquad q_{2,1} = \mu_2 p$$

Hence,

$$\begin{aligned} P_{11}(t) = {} & \frac{\mu_1(1-p)}{\mu_1(1-p) + \mu_2 p} \exp\{-[\mu_1(1-p) + \mu_2 p]t\} \\ & + \frac{\mu_2 p}{\mu_1(1-p) + \mu_2 p} \end{aligned}$$

with similar expressions for the other transition probabilities ($P_{12}(t) = 1 - P_{11}(t)$, and $P_{22}(t)$ is the same with $\mu_2 p$ and $\mu_1(1-p)$ switching places). Conditioning on the initial machine now gives

$$\begin{aligned} E[Y(t)] &= pE[Y(t)|X(0) = 1] + (1-p)E[Y(t)|X(0) = 2] \\ &= p\left[\frac{P_{11}(t)}{\mu_1} + \frac{P_{12}(t)}{\mu_2}\right] + (1-p)\left[\frac{P_{21}(t)}{\mu_1} + \frac{P_{22}(t)}{\mu_2}\right] \end{aligned}$$

Finally, we can obtain $m(t)$ from

$$\mu[m(t) + 1] = t + E[Y(t)]$$

where

$$\mu = p/\mu_1 + (1-p)/\mu_2$$

is the mean interarrival time.

22. Cost of a cycle $= C_1 + C_2 I - R(T)(1 - I)$

$$I = \begin{cases} 1, & \text{if } X < T \\ 0, & \text{if } X \geqslant T \end{cases} \quad \text{where } X = \text{life of car}$$

Hence,

$$E[\text{cost of a cycle}] = C_1 + C_2 H(T) - R(T)[1 - H(T)]$$

Also,

$$\begin{aligned} E[\text{time of cycle}] &= \int E[\text{time} \mid X = x] h(x)\, dx \\ &= \int_0^T x h(x)\, dx + T[1 - H(T)] \end{aligned}$$

Thus the average cost per unit time is given by

$$\frac{C_1 + C_2 H(T) - R(T)[1 - H(T)]}{\int_0^T x h(x)\, dx + T[1 - H(T)]}$$

30. $$\begin{aligned} \frac{A(t)}{t} &= \frac{t - S_{N(t)}}{t} \\ &= 1 - \frac{S_{N(t)}}{t} \\ &= 1 - \frac{S_{N(t)}}{N(t)} \frac{N(t)}{t} \end{aligned}$$

The result follows since $S_{N(t)}/N(t) \to \mu$ (by the strong law of large numbers) and $N(t)/t \to 1/\mu$.

35. (a) We can view this as an $M/G/\infty$ system where a satellite launching corresponds to an arrival and F is the service distribution. Hence,

$$P\{X(t) = k\} = e^{-\lambda(t)}[\lambda(t)]^k/k!$$

where $\lambda(t) = \lambda \int_0^t (1 - F(s))\, ds$.

(b) By viewing the system as an alternating renewal process that is on when there is at least one satellite orbiting, we obtain

$$\lim P\{X(t) = 0\} = \frac{1/\lambda}{1/\lambda + E[T]}$$

where T, the on time in a cycle, is the quantity of interest. From part (a)

$$\lim P\{X(t) = 0\} = e^{-\lambda\mu}$$

where $\mu = \int_0^\infty (1 - F(s))\, ds$ is the mean time that a satellite orbits. Hence,

$$e^{-\lambda\mu} = \frac{1/\lambda}{1/\lambda + E[T]}$$

so

$$E[T] = \frac{1 - e^{-\lambda\mu}}{\lambda e^{-\lambda\mu}}$$

42. (a) $F_e(x) = \dfrac{1}{\mu}\displaystyle\int_0^x e^{-y/\mu}\,dy = 1 - e^{-x/\mu}$.

(b) $F_e(x) = \dfrac{1}{c}\displaystyle\int_0^x dy = \dfrac{x}{c}, \qquad 0 \leqslant x \leqslant c$.

(c) You will receive a ticket if, starting when you park, an official appears within one hour. From Example 7.23 the time until the official appears has the distribution F_e which, by part (a), is the uniform distribution on $(0, 2)$. Thus, the probability is equal to $\frac{1}{2}$.

49. Think of each interarrival time as consisting of n independent phases—each of which is exponentially distributed with rate λ—and consider the semi-Markov process whose state at any time is the phase of the present interarrival time. Hence, this semi-Markov process goes from state 1 to 2 to 3 ... to n to 1, and so on. Also the time spent in each state has the same distribution. Thus, clearly the limiting probability of this semi-Markov chain is $P_i = 1/n$, $i = 1, \ldots, n$. To compute $\lim P\{Y(t) < x\}$, we condition on the phase at time t and note that if it is $n - i + 1$, which will be the case with probability $1/n$, then the time until a renewal occurs will be sum of i exponential phases, which will thus have a gamma distribution with parameters i and λ.

Chapter 8

2. This problem can be modeled by an $M/M/1$ queue in which $\lambda = 6$, $\mu = 8$. The average cost rate will be

$10 per hour per machine $\times$ average number of broken machines

The average number of broken machines is just L, which can be computed from Equation (3.2):

$$\begin{aligned} L &= \frac{\lambda}{\mu - \lambda} \\ &= \frac{6}{2} = 3 \end{aligned}$$

Hence, the average cost rate = $30/hour.

7. To compute W for the $M/M/2$, set up balance equations as follows:

$$\begin{aligned} \lambda P_0 &= \mu P_1 \qquad \text{(each server has rate } \mu) \\ (\lambda + \mu)P_1 &= \lambda P_0 + 2\mu P_2 \\ (\lambda + 2\mu)P_n &= \lambda P_{n-1} + 2\mu P_{n+1}, \qquad n \geqslant 2 \end{aligned}$$

These have solutions $P_n = \rho^n/2^{n-1}P_0$ where $\rho = \lambda/\mu$. The boundary condition $\sum_{n=0}^{\infty} P_n = 1$ implies

$$P_0 = \frac{1-\rho/2}{1+\rho/2} = \frac{(2-\rho)}{(2+\rho)}$$

Now we have P_n, so we can compute L, and hence W from $L = \lambda W$:

$$\begin{aligned}
L &= \sum_{n=0}^{\infty} nP_n = \rho P_0 \sum_{n=0}^{\infty} n\left(\frac{\rho}{2}\right)^{n-1} \\
&= 2P_0 \sum_{n=0}^{\infty} n\left(\frac{\rho}{2}\right)^{n} \\
&= 2\frac{(2-\rho)}{(2+\rho)}\frac{(\rho/2)}{(1-\rho/2)^2} \qquad \text{(See derivation of Equation (8.7).)} \\
&= \frac{4\rho}{(2+\rho)(2-\rho)} \\
&= \frac{4\mu\lambda}{(2\mu+\lambda)(2\mu-\lambda)}
\end{aligned}$$

From $L = \lambda W$ we have

$$W = W(M/M/2) = \frac{4\mu}{(2\mu+\lambda)(2\mu-\lambda)}$$

The $M/M/1$ queue with service rate 2μ has

$$W(M/M/1) = \frac{1}{2\mu-\lambda}$$

from Equation (8.8). We assume that in the $M/M/1$ queue, $2\mu > \lambda$ so that the queue is stable. But then $4\mu > 2\mu + \lambda$, or $4\mu/(2\mu+\lambda) > 1$, which implies $W(M/M/2) > W(M/M/1)$. The intuitive explanation is that if one finds the queue empty in the $M/M/2$ case, it would do no good to have two servers. One would be better off with one faster server. Now let $W_Q^1 = W_Q(M/M/1)$ and $W_Q^2 = W_Q(M/M/2)$. Then,

$$\begin{aligned}
W_Q^1 &= W(M/M/1) - 1/2\mu \\
W_Q^2 &= W(M/M/2) - 1/\mu
\end{aligned}$$

So,

$$W_Q^1 = \frac{\lambda}{2\mu(2\mu-\lambda)} \qquad \text{from Equation (8.8)}$$

and

$$W_Q^2 = \frac{\lambda^2}{\mu(2\mu-\lambda)(2\mu+\lambda)}$$

Then,

$$W_Q^1 > W_Q^2 \Leftrightarrow \frac{1}{2} > \frac{\lambda}{(2\mu + \lambda)}$$
$$\lambda < 2\mu$$

Since we assume $\lambda < 2\mu$ for stability in the $M/M/1$ case, $W_Q^2 < W_Q^1$ whenever this comparison is possible, that is, whenever $\lambda < 2\mu$.

13. (a)
$$\begin{aligned} \lambda P_0 &= \mu P_1 \\ (\lambda + \mu)P_1 &= \lambda P_0 + 2\mu P_2 \\ (\lambda + 2\mu)P_n &= \lambda P_{n-1} + 2\mu P_{n+1}, \qquad n \geqslant 2 \end{aligned}$$

These are the same balance equations as for the $M/M/2$ queue and have solution

$$P_0 = \left(\frac{2\mu - \lambda}{2\mu + \lambda}\right), \qquad P_n = \frac{\lambda^n}{2^{n-1}\mu^n} P_0$$

(b) The system goes from 0 to 1 at rate

$$\lambda P_0 = \frac{\lambda(2\mu - \lambda)}{(2\mu + \lambda)}$$

The system goes from 2 to 1 at rate

$$2\mu P_2 = \frac{\lambda^2}{\mu} \frac{(2\mu - \lambda)}{(2\mu + \lambda)}$$

(c) Introduce a new state *cl* to indicate that the stock clerk is checking by himself. The balance equation for P_{cl} is

$$(\lambda + \mu)P_{cl} = \mu P_2$$

Hence,

$$P_{cl} = \frac{\mu}{\lambda + \mu} P_2 = \frac{\lambda^2}{2\mu(\lambda + \mu)} \frac{(2\mu - \lambda)}{(2\mu + \lambda)}$$

Finally, the proportion of time the stock clerk is checking is

$$P_{cl} + \sum_{n=2}^{\infty} P_n = P_{cl} + \frac{2\lambda^2}{\mu(2\mu - \lambda)}$$

21. (a) $\lambda_1 P_{10}$.
(b) $\lambda_2(P_0 + P_{10})$.
(c) $\lambda_1 P_{10}/[\lambda_1 P_{10} + \lambda_2(P_0 + P_{10})]$.

(d) This is equal to the fraction of server 2's customers that are type 1 multiplied by the proportion of time server 2 is busy. (This is true since the amount of time server 2 spends with a customer does not depend on which type of customer it is.) By (c) the answer is thus

$$\frac{(P_{01} + P_{11})\lambda_1 P_{10}}{\lambda_1 P_{10} + \lambda_2(P_0 + P_{10})}$$

24. The states are now $n, n \geqslant 0$, and $n', n \geqslant 1$ where the state is n when there are n in the system and no breakdown, and n' when there are n in the system and a breakdown is in progress. The balance equations are

$$\begin{aligned}
\lambda P_0 &= \mu P_1 \\
(\lambda + \mu + \alpha)P_n &= \lambda P_{n-1} + \mu P_{n+1} + \beta P_{n'}, \qquad n \geqslant 1 \\
(\beta + \lambda)P_{1'} &= \alpha P_1 \\
(\beta + \lambda)P_{n'} &= \alpha P_n + \lambda P_{(n-1)'}, \qquad n \geqslant 2 \\
\sum_{n=0}^{\infty} P_n + \sum_{n=1}^{\infty} P_{n'} &= 1
\end{aligned}$$

In terms of the solution to the preceding,

$$L = \sum_{n=1}^{\infty} n(P_n + P_{n'})$$

and so

$$W = \frac{L}{\lambda_a} = \frac{L}{\lambda}$$

28. If a customer leaves the system busy, the time until the next departure is the time of a service. If a customer leaves the system empty, the time until the next departure is the time until an arrival *plus* the time of a service.

Using moment generating functions we get

$$\begin{aligned}
E\left\{e^{sD}\right\} &= \frac{\lambda}{\mu} E\{e^{sD} \mid \text{system left busy}\} \\
&\quad + \left(1 - \frac{\lambda}{\mu}\right) E\{e^{sD} \mid \text{system left empty}\} \\
&= \left(\frac{\lambda}{\mu}\right)\left(\frac{\mu}{\mu - s}\right) + \left(1 - \frac{\lambda}{\mu}\right) E\{e^{s(X+Y)}\}
\end{aligned}$$

where X has the distribution of interarrival times, Y has the distribution of service times, and X and Y are independent. Then

$$\begin{aligned}
E\left[e^{s(X+Y)}\right] &= E\left[e^{sX}e^{sY}\right] \\
&= E\left[e^{sX}\right]E\left[e^{sY}\right] \qquad \text{by independence} \\
&= \left(\frac{\lambda}{\lambda - s}\right)\left(\frac{\mu}{\mu - s}\right)
\end{aligned}$$

So,

$$E\{e^{sD}\} = \left(\frac{\lambda}{\mu}\right)\left(\frac{\mu}{\mu - s}\right) + \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\lambda - s}\right)\left(\frac{\mu}{\mu - s}\right)$$
$$= \frac{\lambda}{(\lambda - s)}$$

By the uniqueness of generating functions, it follows that D has an exponential distribution with parameter λ.

36. The distributions of the queue size and busy period are the same for all three disciplines; that of the waiting time is different. However, the means are identical. This can be seen by using $W = L/\lambda$, since L is the same for all. The smallest variance in the waiting time occurs under first-come, first-served and the largest under last-come, first-served.

39. (a) $a_0 = P_0$ due to Poisson arrivals. Assuming that each customer pays 1 per unit time while in service the cost identity of Equation (8.1) states that

$$\text{average number in service} = \lambda E[S]$$

or

$$1 - P_0 = \lambda E[S]$$

(b) Since a_0 is the proportion of arrivals that have service distribution G_1 and $1 - a_0$ the proportion having service distribution G_2, the result follows.

(c) We have

$$P_0 = \frac{E[I]}{E[I] + E[B]}$$

and $E[I] = 1/\lambda$ and thus,

$$E[B] = \frac{1 - P_0}{\lambda P_0}$$
$$= \frac{E[S]}{1 - \lambda E[S]}$$

Now from parts (a) and (b) we have

$$E[S] = (1 - \lambda E[S])E[S_1] + \lambda E[S]E[S_2]$$

or

$$E[S] = \frac{E[S_1]}{1 + \lambda E[S_1] + \lambda E[S_2]}$$

Substituting into $E[B] = E[S]/(1 - \lambda E[S])$ now yields the result.

(d) $a_0 = 1/E[C]$, implying that

$$E[C] = \frac{E[S_1] + 1/\lambda - E[S_2]}{1/\lambda - E[S_2]}$$

45. By regarding any breakdowns that occur during a service as being part of that service, we see that this is an $M/G/1$ model. We need to calculate the first two moments of a service time. Now the time of a service is the time T until something happens (either a service completion or a breakdown) plus any additional time A. Thus,

$$\begin{aligned}E[S] &= E[T + A]\\ &= E[T] + E[A]\end{aligned}$$

To compute $E[A]$, we condition upon whether the happening is a service or a breakdown. This gives

$$\begin{aligned}E[A] &= E[A \mid \text{service}]\frac{\mu}{\mu+\alpha} + E[A \mid \text{breakdown}]\frac{\alpha}{\mu+\alpha}\\ &= E[A \mid \text{breakdown}]\frac{\alpha}{\mu+\alpha}\\ &= \left(\frac{1}{\beta} + E[S]\right)\frac{\alpha}{\mu+\alpha}\end{aligned}$$

Since $E[T] = 1/(\alpha + \mu)$ we obtain

$$E[S] = \frac{1}{\alpha+\mu} + \left(\frac{1}{\beta} + E[S]\right)\frac{\alpha}{\mu+\alpha}$$

or

$$E[S] = \frac{1}{\mu} + \frac{\alpha}{\mu\beta}$$

We also need $E[S^2]$, which is obtained as follows:

$$\begin{aligned}E[S^2] &= E[(T + A)^2]\\ &= E[T^2] + 2E[AT] + E[A^2]\\ &= E[T^2] + 2E[A]E[T] + E[A^2]\end{aligned}$$

The independence of A and T follows because the time of the first happening is independent of whether the happening was a service or a breakdown. Now,

$$\begin{aligned}E[A^2] &= E[A^2 \mid \text{breakdown}]\frac{\alpha}{\mu+\alpha}\\ &= \frac{\alpha}{\mu+\alpha}E[(\text{downtime} + S^*)^2]\\ &= \frac{\alpha}{\mu+\alpha}\left\{E[\text{down}^2] + 2E[\text{down}]E[S] + E[S^2]\right\}\\ &= \frac{\alpha}{\mu+\alpha}\left\{\frac{2}{\beta^2} + \frac{2}{\beta}\left[\frac{1}{\mu} + \frac{\alpha}{\mu\beta}\right] + E[S^2]\right\}\end{aligned}$$

Hence,

$$\begin{aligned}E[S^2] = {} & \frac{2}{(\mu+\beta)^2} + 2\left[\frac{\alpha}{\beta(\mu+\alpha)} + \frac{\alpha}{\mu+\alpha}\left(\frac{1}{\mu} + \frac{\alpha}{\mu\beta}\right)\right]\\ & + \frac{\alpha}{\mu+\alpha}\left\{\frac{2}{\beta^2} + \frac{2}{\beta}\left[\frac{1}{\mu} + \frac{\alpha}{\mu\beta}\right] + E[S^2]\right\}\end{aligned}$$

Now solve for $E[S^2]$. The desired answer is

$$W_Q = \frac{\lambda E[S^2]}{2(1 - \lambda E[S])}$$

In the preceding, S^* is the additional service needed after the breakdown is over and S^* has the same distribution as S. The preceding also uses the fact that the expected square of an exponential is twice the square of its mean.

Another way of calculating the moments of S is to use the representation

$$S = \sum_{i=1}^{N}(T_i + B_i) + T_{N+1}$$

where N is the number of breakdowns while a customer is in service, T_i is the time starting when service commences for the ith time until a happening occurs, and B_i is the length of the ith breakdown. We now use the fact that, given N, all of the random variables in the representation are independent exponentials with the T_i having rate $\mu + \alpha$ and the B_i having rate β. This yields

$$E[S|N] = \frac{N+1}{\mu + \alpha} + \frac{N}{\beta},$$
$$\mathrm{Var}(S|N) = \frac{N+1}{(\mu + \alpha)^2} + \frac{N}{\beta^2}$$

Therefore, since $1+N$ is geometric with mean $(\mu+\alpha)/\mu$ (and variance $\alpha(\alpha + \mu)/\mu^2$) we obtain

$$E[S] = \frac{1}{\mu} + \frac{\alpha}{\mu\beta}$$

and, using the conditional variance formula,

$$\mathrm{Var}(S) = \left[\frac{1}{\mu + \alpha} + \frac{1}{\beta}\right]^2 \frac{\alpha(\alpha + \mu)}{\mu^2} + \frac{1}{\mu(\mu + \alpha)} + \frac{\alpha}{\mu\beta^2}$$

52. S_n is the service time of the nth customer; T_n is the time between the arrival of the nth and $(n + 1)$st customer.

Chapter 9

4. (a) $\phi(x) = x_1 \max(x_2, x_3, x_4)x_5$.
   (b) $\phi(x) = x_1 \max(x_2x_4, x_3x_5)x_6$.
   (c) $\phi(x) = \max(x_1, x_2x_3)x_4$.

6. A minimal cut set has to contain at least one component of each minimal path set. There are six minimal cut sets: $\{1, 5\}, \{1, 6\}, \{2, 5\}, \{2, 3, 6\}, \{3, 4, 6\}, \{4, 5\}$.

12. The minimal path sets are $\{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}$. With $q_i = 1 - p_i$, the reliability function is

$$\begin{aligned} r(\mathbf{p}) &= P\{\text{either of } 1, 2, \text{ or } 3 \text{ works}\}P\{\text{either of 4 or 5 works}\} \\ &= (1 - q_1 q_2 q_3)(1 - q_4 q_5) \end{aligned}$$

17. $$\begin{aligned} E[N^2] &= E[N^2 | N > 0]P\{N > 0\} \\ &\geqslant (E[N|N > 0])^2 P\{N > 0\}, \qquad \text{since } E[X^2] \geqslant (E[X])^2 \end{aligned}$$

Thus,

$$\begin{aligned} E[N^2]P\{N > 0\} &\geqslant (E[N|N > 0]P[N > 0])^2 \\ &= (E[N])^2 \end{aligned}$$

Let N denote the number of minimal path sets having all of its components functioning. Then $r(p) = P\{N > 0\}$. Similarly, if we define N as the number of minimal cut sets having all of its components failed, then $1 - r(p) = P\{N > 0\}$. In both cases we can compute expressions for $E[N]$ and $E[N^2]$ by writing N as the sum of indicator (i.e., Bernoulli) random variables. Then we can use the inequality to derive bounds on $r(p)$.

22. (a) $$\begin{aligned} \bar{F}_t(a) &= P\{X > t + a \,|\, X > t\} \\ &= \frac{P[X > t + a\}}{P\{X > t\}} = \frac{\bar{F}(t + a)}{\bar{F}(t)} \end{aligned}$$

(b) Suppose $\lambda(t)$ is increasing. Recall that

$$\bar{F}(t) = e^{-\int_0^t \lambda(s)\,ds}$$

Hence,

$$\frac{\bar{F}(t + a)}{\bar{F}(t)} = \exp\left\{-\int_t^{t+a} \lambda(s)\,ds\right\}$$

which decreases in t since $\lambda(t)$ is increasing. To go the other way, suppose $\bar{F}(t + a)/\bar{F}(t)$ decreases in t. Now when a is small

$$\frac{\bar{F}(t + a)}{\bar{F}(t)} \approx e^{-a\lambda(t)}$$

Hence, $e^{-a\lambda(t)}$ must decrease in t and thus $\lambda(t)$ increases.

25. For $x \geqslant \xi$,

$$1 - p = \bar{F}(\xi) = \bar{F}(x(\xi/x)) \geqslant [\bar{F}(x)]^{\xi/x}$$

since IFRA. Hence, $\bar{F}(x) \leqslant (1 - p)^{x/\xi} = e^{-\theta x}$.

For $x \leqslant \xi$,

$$\bar{F}(x) = \bar{F}(\xi(x/\xi)) \geqslant [\bar{F}(\xi)]^{x/\xi}$$

since IFRA. Hence, $\bar{F}(x) \geqslant (1 - p)^{x/\xi} = e^{-\theta x}$.

30. $r(\boldsymbol{p}) = p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4 - 3p_1p_2p_3p_4$

$$r(\mathbf{1} - \boldsymbol{F}(t)) = \begin{cases} 2(1-t)^2(1-t/2) + 2(1-t)(1-t/2)^2 \\ -3(1-t)^2(1-t/2)^2, & 0 \leqslant t \leqslant 1 \\ 0, & 1 \leqslant t \leqslant 2 \end{cases}$$

$$\begin{aligned} E[\text{lifetime}] &= \int_0^1 \left[2(1-t)^2(1-t/2) + 2(1-t)(1-t/2)^2 \right. \\ &\quad \left. - 3(1-t)^2(1-t/2)^2\right] dt \\ &= \frac{31}{60} \end{aligned}$$

Chapter 10

1. $B(s) + B(t) = 2B(s) + B(t) - B(s)$. Now $2B(s)$ is normal with mean 0 and variance $4s$ and $B(t) - B(s)$ is normal with mean 0 and variance $t - s$. Because $B(s)$ and $B(t) - B(s)$ are independent, it follows that $B(s) + B(t)$ is normal with mean 0 and variance $4s + t - s = 3s + t$.

3. $$\begin{aligned} E[B(t_1)B(t_2)B(t_3)] &= E[E[B(t_1)B(t_2)B(t_3)|B(t_1),\ B(t_2)]] \\ &= E[B(t_1)B(t_2)E[B(t_3) \mid B(t_1), B(t_2)]] \\ &= E[B(t_1)B(t_2)B(t_2)] \\ &= E[E[B(t_1)B^2(t_2) \mid B(t_1)]] \\ &= E[B(t_1)E[B^2(t_2) \mid B(t_1)]] \\ &= E[B(t_1)\{(t_2 - t_1) + B^2(t_1)\}] \qquad (*) \\ &= E[B^3(t_1)] + (t_2 - t_1)E[B(t_1)] \\ &= 0 \end{aligned}$$

where the equality $(*)$ follows since given $B(t_1), B(t_2)$ is normal with mean $B(t_1)$ and variance $t_2 - t_1$. Also, $E[B^3(t)] = 0$ since $B(t)$ is normal with mean 0.

5. $$\begin{aligned} P\{T_1 < T_{-1} < T_2\} &= P\{\text{hit 1 before –1 before 2}\} \\ &= P\{\text{hit 1 before –1}\} \\ &\quad \times P\{\text{hit –1 before 2} \mid \text{hit 1 before –1}\} \\ &= \tfrac{1}{2} P\{\text{down 2 before up 1}\} \\ &= \frac{1}{2}\frac{1}{3} = \frac{1}{6} \end{aligned}$$

The next to last equality follows by looking at the Brownian motion when it first hits 1.

10. (a) Writing $X(t) = X(s) + X(t) - X(s)$ and using independent increments, we see that given $X(s) = c$, $X(t)$ is distributed as $c + X(t) - X(s)$. By stationary increments this has the same distribution as $c + X(t - s)$, and is thus normal with mean $c + \mu(t - s)$ and variance $(t - s)\sigma^2$.

(b) Use the representation $X(t) = \sigma B(t) + \mu t$, where $\{B(t)\}$ is standard Brownian motion. Using Equation (10.4), but reversing s and t, we see that the conditional distribution of $B(t)$ given that $B(s) = (c - \mu s)/\sigma$ is normal with mean $t(c - \mu s)/(\sigma s)$ and variance $t(s - t)/s$. Thus, the conditional distribution of $X(t)$ given that $X(s) = c, s > t$, is normal with mean

$$\sigma\left[\frac{t(c-\mu s)}{\sigma s}\right] + \mu t = \frac{(c-\mu s)t}{s} + \mu t$$

and variance

$$\frac{\sigma^2 t(s-t)}{s}$$

19. Since knowing the value of $Y(t)$ is equivalent to knowing $B(t)$, we have

$$\begin{aligned} E[Y(t) \mid Y(u),\ 0 \leqslant u \leqslant s] &= e^{-c^2t/2}E[e^{cB(t)} \mid B(u), 0 \leqslant u \leqslant s] \\ &= e^{-c^2t/2}E[e^{cB(t)} \mid B(s)] \end{aligned}$$

Now, given $B(s)$, the conditional distribution of $B(t)$ is normal with mean $B(s)$ and variance $t - s$. Using the formula for the moment generating function of a normal random variable we see that

$$\begin{aligned} e^{-c^2t/2}E[e^{cB(t)}|B(s)] &= e^{-c^2t/2}e^{cB(s)+(t-s)c^2/2} \\ &= e^{-c^2s/2}e^{cB(s)} \\ &= Y(s) \end{aligned}$$

Thus $\{Y(t)\}$ is a Martingale.

$$E[Y(t)] = E[Y(0)] = 1$$

20. By the Martingale stopping theorem

$$E[B(T)] = E[B(0)] = 0$$

However, $B(T) = 2 - 4T$ and so $2 - 4E[T] = 0$, or $E[T] = \frac{1}{2}$.

24. It follows from the Martingale stopping theorem and the result of Exercise 18 that

$$E[B^2(T) - T] = 0$$

where T is the stopping time given in this problem and

$$B(t) = \frac{X(t) - \mu t}{\sigma}$$

Therefore,

$$E\left[\frac{(X(T) - \mu T)^2}{\sigma^2} - T\right] = 0$$

However, $X(T) = x$ and so the preceding gives that

$$E[(x - \mu T)^2] = \sigma^2 E[T]$$

But, from Exercise 21, $E[T] = x/\mu$ and so the preceding is equivalent to

$$\mathrm{Var}(\mu T) = \sigma^2 \frac{x}{\mu} \quad \text{or} \quad \mathrm{Var}(T) = \sigma^2 \frac{x}{\mu^3}$$

27. $E[X(a^2t)/a] = (1/a)E[X(a^2t)] = 0$. For $s < t$,

$$\begin{aligned}\mathrm{Cov}\big(Y(s), Y(t)\big) &= \frac{1}{a^2}\mathrm{Cov}\big(X(a^2s), X(a^2t)\big)\\ &= \frac{1}{a^2}a^2 s = s\end{aligned}$$

Because $\{Y(t)\}$ is clearly Gaussian, the result follows.

30. (a) Starting at any time t the continuation of the Poisson process remains a Poisson process with rate λ.
(b)
$$\begin{aligned}E[Y(t)Y(t+s)] &= \int_0^\infty E[Y(t)Y(t+s)|Y(t)=y]\lambda e^{-\lambda y}\,dy\\ &= \int_0^s yE[Y(t+s)|Y(t)=y]\lambda e^{-\lambda y}\,dy\\ &\quad + \int_s^\infty y(y-s)\lambda e^{-\lambda y}\,dy\\ &= \int_0^s y\frac{1}{\lambda}\lambda e^{-\lambda y}\,dy + \int_s^\infty y(y-s)\lambda e^{-\lambda y}\,dy\end{aligned}$$

where the preceding used that

$$E[Y(t)Y(t+s)|Y(t)=y] = \begin{cases} yE(Y(t+s)) = \dfrac{y}{\lambda}, & \text{if } y < s\\ y(y-s), & \text{if } y > s\end{cases}$$

Hence,

$$\mathrm{Cov}(Y(t), Y(t+s)) = \int_0^s ye^{-\lambda y}\,dy + \int_s^\infty y(y-s)\lambda e^{-\lambda y}\,dy - \frac{1}{\lambda^2}$$

Chapter 11

1. (a) Let U be a random number. If $\sum_{j=1}^{i-1} P_j < U \leqslant \sum_{j=1}^{i} P_j$ then simulate from F_i. (In the preceding $\sum_{j=1}^{i-1} P_j \equiv 0$ when $i = 1$.)
(b) Note that

$$F(x) = \frac{1}{3}F_1(x) + \frac{2}{3}F_2(x)$$

where

$$F_1(x) = 1 - e^{2x}, \quad 0 < x < \infty$$

$$F_2(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 < x \end{cases}$$

Hence, using part (a), let $U_1,\ U_2,\ U_3$ be random numbers and set

$$X = \begin{cases} \dfrac{-\log U_2}{2}, & \text{if } U_1 < \frac{1}{3} \\ U_3, & \text{if } U_1 > \frac{1}{3} \end{cases}$$

The preceding uses the fact that $-\log U_2/2$ is exponential with rate 2.

3. If a random sample of size n is chosen from a set of $N + M$ items of which N are acceptable, then X, the number of acceptable items in the sample, is such that

$$P\{X = k\} = \binom{N}{k}\binom{M}{n-k} \Big/ \binom{N+M}{k}$$

To simulate X, note that if

$$I_j = \begin{cases} 1, & \text{if the } j\text{th selection is acceptable} \\ 0, & \text{otherwise} \end{cases}$$

then

$$P\{I_j = 1 | I_1, \ldots, I_{j-1}\} = \frac{N - \sum_1^{j-1} I_i}{N + M - (j-1)}$$

Hence, we can simulate $I_1, \ldots, I_n$ by generating random numbers $U_1, \ldots, U_n$ and then setting

$$I_j = \begin{cases} 1, & \text{if } U_j < \dfrac{N - \sum_1^{j-1} I_i}{N + M - (j-1)} \\ 0, & \text{otherwise} \end{cases}$$

and $X = \sum_{j=1}^{n} I_j$ has the desired distribution.

Another way is to let

$$X_j = \begin{cases} 1, & \text{the } j\text{th acceptable item is in the sample} \\ 0, & \text{otherwise} \end{cases}$$

and then simulate $X_1, \ldots, X_N$ by generating random numbers $U_1, \ldots, U_N$ and then setting

$$X_j = \begin{cases} 1, & \text{if } U_j < \dfrac{n - \sum_{i=1}^{j-1} X_i}{N + M - (j-1)} \\ 0, & \text{otherwise} \end{cases}$$

and $X = \sum_{j=1}^{N} X_j$ then has the desired distribution.

The former method is preferable when $n \leqslant N$ and the latter when $N \leqslant n$.

6. Let

$$c(\lambda) = \max_x \left\{ \frac{f(x)}{\lambda e^{-\lambda x}} \right\} = \frac{2}{\lambda\sqrt{2\pi}} \max_x \left[\exp\left\{ \frac{-x^2}{2} + \lambda x \right\} \right]$$
$$= \frac{2}{\lambda\sqrt{2\pi}} \exp\left\{ \frac{\lambda^2}{2} \right\}$$

Hence,

$$\frac{d}{d\lambda} c(\lambda) = \sqrt{2/\pi} \exp\left\{ \frac{\lambda^2}{2} \right\} \left[1 - \frac{1}{\lambda^2} \right]$$

Hence $(d/d\lambda)c(\lambda) = 0$ when $\lambda = 1$ and it is easy to check that this yields the minimal value of $c(\lambda)$.

16. (a) They can be simulated in the same sequential fashion in which they are defined. That is, first generate the value of a random variable I_1 such that

$$P\{I_1 = i\} = \frac{w_i}{\sum_{j=1}^n w_j}, \qquad i = 1, \ldots, n$$

Then, if $I_1 = k$, generate the value of I_2 where

$$P\{I_2 = i\} = \frac{w_i}{\sum_{j \neq k} w_j}, \qquad i \neq k$$

and so on. However, the approach given in part (b) is more efficient.

(b) Let I_j denote the index of the jth smallest X_i.

23. Let $m(t) = \int_0^t \lambda(s)\, ds$, and let $m^{-1}(t)$ be the inverse function. That is, $m(m^{-1}(t)) = t$.

(a)
$$\begin{aligned} P\{m(X_1) > x\} &= P\{X_1 > m^{-1}(x)\} \\ &= P\{N(m^{-1}(x)) = 0\} \\ &= e^{-m(m^{-1}(x))} \\ &= e^{-x} \end{aligned}$$

(b)
$$\begin{aligned} P\{m(X_i) - m(X_{i-1}) > x | m(X_1), \ldots, m(X_{i-1}) - m(X_{i-2})\} \\ = P\{m(X_i) - m(X_{i-1}) > x | X_1, \ldots, X_{i-1}\} \\ = P\{m(X_i) - m(X_{i-1}) > x | X_{i-1}\} \\ = P\{m(X_i) - m(X_{i-1}) > x | m(X_{i-1})\} \end{aligned}$$

Now,

$$
\begin{aligned}
&P\{m(X_i) - m(X_{i-1}) > x | X_{i-1} = y\} \\
&= P\left\{\int_y^{X_i} \lambda(t)dt > x | X_{i-1} = y\right\} \\
&= P\{X_i > c | X_{i-1} = y\} \qquad \text{where} \int_y^c \lambda(t)\,dt = x \\
&= P\{N(c) - N(y) = 0 \mid X_{i-1} = y\} \\
&= P\{N(c) - N(y) = 0\} \\
&= \exp\left\{-\int_y^c \lambda(t)\,dt\right\} \\
&= e^{-x}
\end{aligned}
$$

32. $\mathrm{Var}[(X+Y)/2] = \frac{1}{4}[\mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)]$

$$= \frac{\mathrm{Var}(X) + \mathrm{Cov}(X,Y)}{2}$$

Now it is always true that

$$\frac{\mathrm{Cov}(V,W)}{\sqrt{\mathrm{Var}(V)\mathrm{Var}(W)}} \leqslant 1$$

and so when X and Y have the same distribution $\mathrm{Cov}(X,Y) \leqslant \mathrm{Var}(X)$.