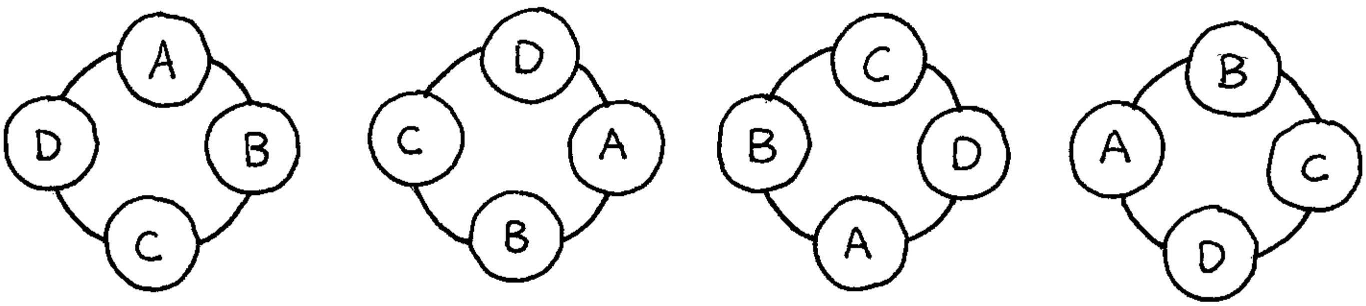
Question 1

a) On utilise la formule $(n-1)!$

Exemple avec $n = 4$



Puisqu'on s'intéresse à la position relative des uns par rapport aux autres, ces quatre arrangements sont identiques.
Le nombre total d'arrangements différents serait donc égal à $\frac{4!}{4} = (4-1)! \Rightarrow (n-1)!$

En a), on obtient

$$\underbrace{(8-1)!}_{\text{Total}} - \underbrace{(7-1)!\,2!}_{\text{A et B côte à côte}} = 3600$$

b)

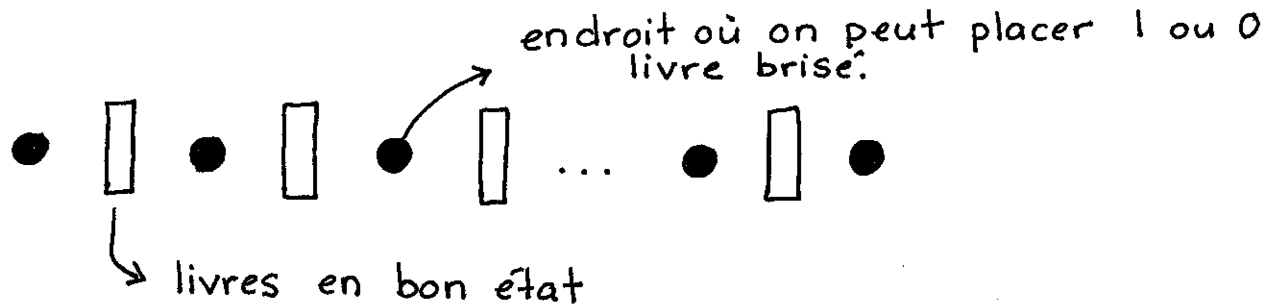
$$(6-1)!\ 2!\ 2! = 480$$

Question 2

n : nombre total de livres

m : nombre de livres brisés

$n-m$: nombre de livres en bon état



$$\underbrace{\frac{n!}{m!\,(n-m)!}}_{\text{Nombre total de cas possibles}} \quad - \quad \underbrace{\binom{n-m+1}{m}}_{\text{Nombre de cas où il n'y a \underline{PAS} de livres brisés consécutifs}}$$

Dans l'illustration ci-haut, on comptera $(n-m)$ ▯ et $(n+m+1)$ ●, c'est pourquoi on choisit m ● parmi les $(n-m+1)$ pour placer les m livres brisés.

■

Question 3

a) $\binom{10}{4} 4! = \boxed{5040}$

b)

i) $\binom{1}{1}\binom{9}{3} 3!$ ☺ A

$+ \binom{1}{0}\binom{9}{4} 4!$ ☹ A

$= \boxed{3528}$

ii) $\binom{1}{1}\binom{1}{1}\binom{8}{1} 2! \binom{7}{1}$ ☺ A ☺ B

$+ \binom{1}{0}\binom{1}{1}\binom{8}{1} 2! \binom{7}{2} 2!$ ☹ A ☺ B

$+ \binom{1}{0}\binom{1}{0}\binom{8}{4} 4!$ ☹ A ☹ B

$+ \binom{1}{1}\binom{1}{0}\binom{8}{3} 3!$ ☺ A ☹ B

$= \boxed{2800}$

Note : lorsqu'on choisit l'enfant B, on doit aussi choisir un autre enfant pour permuter la guitare et le camion.

Question 3 (Suite)

b) iii) Nous allons repartir du résultat obtenu en ii) et y soustraire les cas où C et D ont tous les deux un jouet.

Cas où C et D ont tous les deux un jouet

$\binom{1}{1}\binom{1}{1}\binom{2}{1}2!\binom{1}{1}$ ☺ A ☺ B C ou D permutte avec B.

$+$ $\binom{1}{0}\binom{1}{1}\binom{2}{1}2!\binom{1}{1}\binom{6}{1}2!$ ☹ A ☺ B C ou D permutte avec B.

$+$ $\binom{1}{0}\binom{1}{0}\binom{2}{2}\binom{6}{2}4!$ ☹ A ☹ B

$+$ $\binom{1}{1}\binom{1}{0}\binom{2}{2}\binom{6}{1}3!$ ☺ A ☹ B

$+$ $\binom{1}{0}\binom{1}{1}\binom{6}{1}2!\binom{2}{2}2!$ ☹ A ☺ B C ou D ne permutte pas avec B.

$= 472$

Réponse : $2800 - 472 = \boxed{2328}$

Question 4

a) $b \times b \times b \times \ldots \times b = b^j$

b) $\binom{j}{n_1}(b-1)^{j-n_1}$

On choisit n_1 jetons et on les place dans la première boîte. Chacun des $(j-n_1)$ jetons restants peut aller dans une des $(b-1)$ boîtes restantes.

Les jetons et les boîtes sont tous numérotés, ils sont donc tous distincts.

c) $\binom{j}{n_1}\binom{j-n_1}{n_b}(b-2)^{j-n_1-n_b}$

d) $\binom{j}{n_1}\binom{j-n_1}{n_2} \cdots \binom{j-n_1-\ldots-n_{b-2}}{n_{b-1}}\binom{j-n_1-\ldots-n_{b-1}}{n_b}$

$$= \frac{j!}{n_1!\, n_2! \ldots n_b!}$$

e) $b \times (b-1) \times \ldots \times (b-j+1)$

Note : Il serait pertinent de refaire cette question avec des chiffres.

Question 5

a)

i) $\binom{n-1}{r-1} = \binom{6}{3} = 20$

ii) $\binom{n+r-1}{r-1} = \binom{10}{3} = 120$

b)

i) $\underbrace{\binom{6}{4}} + \binom{6}{3} = 35$

5e urne = balles non distribuées

ii) $\binom{5+7-1}{5-1} = \binom{11}{4} = 330$

⚠ Formules à utiliser lorsqu'on distribue des objets indissociables parmi des "urnes" distinctes.

■

Question 7

T_i : ne pas rencontrer le taxi i , $i = 1, 2, 3$

Nous cherchons donc $\left(Pr(T_1 \cup T_2 \cup T_3) \right)^c$

$$= 1 - Pr\{T_1 \cup T_2 \cup T_3\}$$

$$Pr(T_1 \cup T_2 \cup T_3) = \sum_{i=1}^{3} Pr(T_i) - \sum_{i<j} Pr(T_i \cap T_j) + \sum_{i<j<k} Pr(T_i \cap T_j \cap T_k)$$

$$Pr(T_1) = Pr(T_2) = Pr(T_3) = \left(\frac{2}{3}\right)^6$$

$$Pr(T_1 \cap T_2) = Pr(T_2 \cap T_3) = Pr(T_1 \cap T_3) = \left(\frac{1}{3}\right)^6$$

$Pr(T_1 \cap T_2 \cap T_3) = 0$ ... car on a forcément rencontré au moins un taxi

$$Pr(T_1 \cup T_2 \cup T_3) = 3\left(\frac{2}{3}\right)^6 - 3\left(\frac{1}{3}\right)^6 = \frac{7}{27}$$

Réponse : $1 - \frac{7}{27} = \boxed{\frac{20}{27}}$

■

Question 8

B : Barbie obtient 6 points

K : Ken obtient 7 points

$$Pr\{B\} = p_B = \frac{5}{36}$$

$$Pr\{K\} = p_K = \frac{6}{36}$$

B_i : Barbie gagne à son i^e lancer

$$Pr\{B_1\} = 5/36$$

$$Pr\{B_2\} = (1-p_B)(1-p_K)\,p_B$$

$$Pr\{B_i\} = \left((1-p_B)(1-p_K)\right)^{i-1} p_B$$

a) On cherche

$$\sum_{i=1}^{\infty} Pr(B_i) = \sum_{i=1}^{\infty} \left((1-p_B)(1-p_K)\right)^{i-1} p_B$$

On pose $j = i-1$

$$\sum_{j=0}^{\infty} \left((1-p_B)(1-p_K)\right)^{j} p_B = \frac{p_B}{1-(1-p_B)(1-p_K)} = \boxed{\frac{30}{61}}$$

b) $$\left((1-p_B)(1-p_K)\right)^{10} = \boxed{0.03621}$$

Question 9

G : gagner à pile ou face

T : être un tricheur

On nous donne $Pr(T) = p$

$Pr(G \mid T) = 1$

On sait également que $Pr(G \mid \bar{T}) = 1/2$.

$$Pr(T \mid G) = \frac{Pr(G \mid T)\, Pr(T)}{Pr(G)}$$

$$= \frac{Pr(G \mid T)\, Pr(T)}{Pr(G \mid T)\, Pr(T) + Pr(G \mid \bar{T})\, Pr(\bar{T})}$$

$$= \frac{(1)p}{(1)p + \frac{1}{2}(1-p)}$$

$$= \frac{2p}{p+1} \leq 2p$$

car $(p+1) \geq 1$.

Question 10

On doit avoir que $\sum_{x=1}^{6} Pr(X=x) = 1$

$$= \sum_{x=1}^{5} Pr(X=x) + Pr(X=6) = 1$$

$$= \sum_{x=1}^{5} c(1-\theta)\max(x,\ x^2-2x) + \theta = 1$$

$$= c(1-\theta)\sum_{x=1}^{5}\max(x,\ x^2-2x) = 1-\theta$$

$$= c\sum_{x=1}^{5}\max(x,\ x^2-2x) = 1$$

$$c = \frac{1}{\sum_{x=1}^{5}\max(x,\ x^2-2x)}$$

x	1	2	3	4	5	Σ
x^2-2x	-1	0	3	8	15	
$\max(x,\ x^2-2x)$	1	2	3	8	15	29

$$c = \boxed{\frac{1}{29}}$$

■

Question 11

J : nombre de jours avant de piger (sans remise) 2 fois la même couleur de robe.

On sait qu'il y a $8n$ robes et que

$$\Pr\{J=2\} + \Pr\{J=3\} = \frac{99}{355}$$

$$\Pr\{J=2\} = \underbrace{\left(\frac{8n}{8n}\right)}_{\text{Jour 1}}\underbrace{\left(\frac{7}{8n-1}\right)}_{\text{Jour 2}}$$

$$\Pr\{J=3\} = \underbrace{\left(\frac{8n}{8n}\right)}_{\text{Jour 1}}\underbrace{\left(\frac{8n-8}{8n-1}\right)}_{\text{Jour 2}}\underbrace{\left(\frac{14}{8n-2}\right)}_{\text{Jour 3}}$$

$$\Pr(J=2) + \Pr(J=3) = \frac{7(8n-2) + 14(8n-8)}{(8n-1)(8n-2)}$$

$$= \frac{168n - 126}{64n^2 - 24n + 2} = \frac{99}{355}$$

On isole n avec la formule quadratique.

On trouve $n = 9$.

Réponse : $(8)(9) = \boxed{72}$ robes.

■

Question 12

a) $N \in \{0, 1, 2, 3\}$

$$\Pr\{N=0\} = \underbrace{\left(\frac{5}{9}\right)^3}_{\circ\,\circ\,\circ} = 0.1715$$

$$\Pr\{N=1\} = \underbrace{\left(\frac{4}{9}\right)\left(\frac{5}{8}\right)\left(\frac{5}{8}\right)}_{\bullet\ \circ\ \circ} + \underbrace{\left(\frac{5}{9}\right)\left(\frac{4}{9}\right)\left(\frac{5}{8}\right)}_{\circ\ \bullet\ \circ} + \underbrace{\left(\frac{5}{9}\right)\left(\frac{5}{9}\right)\left(\frac{4}{9}\right)}_{\circ\ \circ\ \bullet} = 0.4651$$

$$\Pr\{N=2\} = \underbrace{\left(\frac{4}{9}\right)\left(\frac{3}{8}\right)\left(\frac{5}{7}\right)}_{\bullet\ \bullet\ \circ} + \underbrace{\left(\frac{4}{9}\right)\left(\frac{5}{8}\right)\left(\frac{3}{8}\right)}_{\bullet\ \circ\ \bullet} + \underbrace{\left(\frac{5}{9}\right)\left(\frac{4}{9}\right)\left(\frac{3}{8}\right)}_{\circ\ \bullet\ \bullet} = 0.3158$$

$$\Pr\{N=3\} = \underbrace{\left(\frac{4}{9}\right)\left(\frac{3}{8}\right)\left(\frac{2}{7}\right)}_{\bullet\ \bullet\ \bullet} = 0.0476$$

$$\Pr\{N=n\} = \begin{cases} 0.1715 \,, & n=0 \\ 0.4651 \,, & n=1 \\ 0.3158 \,, & n=2 \\ 0.0476 \,, & n=3 \\ 0 \,, & \text{ailleurs} \end{cases}$$

Question 12 (suite)

b) $$E\left[\frac{100}{N+1}\right] = 100 \sum_{n=0}^{3} \frac{1}{n+1} \Pr(N=n)$$

$$= 100 \left[\frac{1}{1}\cdot(0.1715) + \frac{1}{2}\cdot(0.4651) + \ldots + \frac{1}{4}(0.0476) \right]$$

$$= 0.5212\ (100)$$

$$= \boxed{52.12}$$

c) $$Y = (N \mid N \geq 2)$$

$$\Pr(Y=y) = \frac{\Pr(N=y)}{\Pr(N \geq 2)}, \quad y = 2 \text{ ou } 3$$

$$\Pr\{N \geq 2\} = 0.3158 + 0.0476 = 0.3634$$

$$\Pr(Y=2) = \frac{0.3158}{0.3634} = 0.8690$$

$$\Pr(Y=3) = \frac{0.0476}{0.3634} = 0.1310$$

$$E[Y] = 2(0.8690) + 3(0.1310)$$

$$= \boxed{2.131}$$

■

Question 13

X : v.a. du nombre de personnes qui achètent un billet d'avion

B : le prix du billet est inférieur à 1000 \$

On a donc
$$(X \mid B) \sim Bin(n=8,\ p=0.75)$$
$$(X \mid \overline{B}) \sim Bin(n=8,\ p=0.35)$$
$$Pr(B) = p$$

$$Pr(\text{manquer de places}) = Pr(X \geq 6) = 0.40$$

$$= \left[Pr(X=6 \mid B) + Pr(X=7 \mid B) + Pr(X=8 \mid B)\right] Pr(B)$$
$$+ \left[Pr(X=6 \mid \overline{B}) + Pr(X=7 \mid \overline{B}) + Pr(X=8 \mid \overline{B})\right] Pr(\overline{B})$$

$$= \left[\binom{8}{6} 0.75^6\, 0.55^2 + \binom{8}{7} 0.75^7\, 0.55 + \binom{8}{8} 0.75^8\right] p$$
$$+ \left[\binom{8}{6} 0.35^6\, 0.65^2 + \binom{8}{7} 0.35^7\, 0.65 + \binom{8}{8} 0.35^8\right] (1-p)$$

$$= 0.678543\, p + 0.0253175\,(1-p) = 0.40$$

On trouve $p = 0.5736 = Pr(B)$

On cherche
$$Pr(X=2 \mid X \leq 5) = \frac{Pr(X=2)}{Pr(X \leq 5)}$$

$$= \frac{Pr(X=2 \mid B)\, Pr(B) + Pr(X=2 \mid \overline{B})\, Pr(\overline{B})}{0.6}$$

$$= \frac{\left[\binom{8}{2} 0.75^2\, 0.25^6\right] 0.5736 + \left[\binom{8}{2} 0.35^2\, 0.65^6\right] 0.4264}{0.6} = \boxed{0.1875}$$

Question 14

$$E[Y] = M_Y'(t) \Big|_{t=0}$$

$$= 3\, M_X'(3t)\, e^{-2t} + M_X(3t)\, e^{-2t}(-2) \Big|_{t=0}$$

$$= 3\, M_X'(0) - 2\, M_X(0)$$

$$= 3\, E[X] - 2\, M_X(0)$$

Rappel : $M_X(t) = 1 + t\,E[X] + \dfrac{t^2 E[X^2]}{2!} + \ldots$

$M_X(0) = 1$

Donc, $E[Y] = 3(5) - 2(1)$

$= \boxed{13}$

■

Question 15

$X \sim Pois(1.5)$

$$N = \begin{cases} 0 , & x = 0 \\ 3(X-1) , & x \geq 1 \end{cases}$$

$$E[N] = \sum_{x=1}^{\infty} 3(x-1) Pr(X=x)$$

$$= \sum_{x=0}^{\infty} 3(x-1) Pr(X=x) - [3(-1) Pr(X=0)]$$

$$= 3 \sum_{x=0}^{\infty} x \, Pr(X=x) - 3 \sum_{x=0}^{\infty} Pr(X=x) + 3e^{-1.5}$$

$$= 3E[X] - 3(1) + 3e^{-1.5}$$

$$= 3(1.5) - 3 + 3e^{-1.5}$$

$$= \boxed{2.16939} \quad \blacksquare$$

Question 16

$X \sim Bin(n, p)$

$$Pr(X=n) = \binom{n}{n} p^n (1-p)^0 = p^n = 0.00032$$

$$Pr(X=n-1) = \binom{n}{n-1} p^{n-1}(1-p)$$

$$= n\, p^{n-1}(1-p)$$

$$= n(p^{n-1} - p^n) = 0.00128n$$

$$\Rightarrow \quad p^{n-1} - p^n = 0.00128$$

$$p^{n-1} - 0.00032 = 0.00128$$

$$p^{n-1} = 0.0016$$

$$\frac{p^n}{p^{n-1}} = p = \frac{0.00032}{0.0016} = 0.20$$

$$p^n = 0.2^n = 0.00032$$

$$n = 5$$

$$E[X] = np = (5)(0.2) = \boxed{1}$$

■

Question 17

$$F_T(x) = \int_0^x \frac{2\,000\,000}{(1000+x)^3}\,dx$$

$$= 1 - \frac{1\,000\,000}{(1000+x)^2}$$

On cherche

$$Pr(X \geq 2) = 1 - Pr(X=0) - Pr(X=1)$$

où $X \sim Bin(n=12,\ p)$

$$p = \frac{Pr(1000 < T < 1400)}{Pr(T > 900)}$$

$$= \frac{F_T(1400) - F_T(1000)}{1 - F_T(900)}$$

$$= 0.27576$$

Donc,

$$Pr(X \geq 2) = 1 - \binom{12}{0} 0.27576^0\, 0.72424^{12} - \binom{12}{1} 0.27576 \cdot 0.72424^{11}$$

$$= \boxed{0.88402}$$

Question 18

a) $F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy =$

$$\left[\int_{-\infty}^{x} 0\,dy\right] 1\{x < t_1\} + \left[\int_{-\infty}^{t_1} 0\,dy + \int_{t_1}^{x} \ell\,dy\right] 1\{t_1 \leq x < t_2\}$$

$$+ \left[\int_{-\infty}^{t_1} 0\,dy + \int_{t_1}^{t_2} \ell\,dy + \int_{t_2}^{x} \theta\,dy\right] 1\{t_2 \leq x < t_3\}$$

$$+ \left[\int_{-\infty}^{t_1} 0\,dy + \int_{t_1}^{t_2} \ell\,dy + \int_{t_2}^{t_3} \theta\,dy + \int_{t_3}^{x} \omega\,dy\right] 1\{t_3 \leq x < t_4\}$$

$$+ 1\{x \geq t_4\}$$

$$F_X(x) = \begin{cases} 0 & , \quad x < t_1 \\ \ell(x - t_1) & , \quad t_1 \leq x < t_2 \\ \ell(t_2 - t_1) + \theta(x - t_2) & , \quad t_2 \leq x < t_3 \\ \ell(t_2 - t_1) + \theta(t_3 - t_2) + \omega(x - t_3) & , \quad t_3 \leq x < t_4 \\ 1 & , \quad x \geq t_4 \end{cases}$$

b) $E[X^K] = \int_{-\infty}^{\infty} x^K f_X(x)\,dx$

On sépare $K = -1$ et $K \neq -1$

$$E[X^K] = \begin{cases} \dfrac{\ell}{K+1}\left(t_2^{K+1} - t_1^{K+1}\right) + \dfrac{\theta}{K+1}\left(t_3^{K+1} - t_2^{K+1}\right) \\ \qquad \cdots + \dfrac{\omega}{K+1}\left(t_4^{K+1} - t_3^{K+1}\right) & , \quad K \neq -1 \\ \ell \ln\left(\dfrac{t_2}{t_1}\right) + \theta \ln\left(\dfrac{t_3}{t_2}\right) + \omega \ln\left(\dfrac{t_4}{t_3}\right) & , \quad K = -1 \end{cases}$$

Question 19

Technique de la fonction de répartition

$$\Pr\{Y \le y\} = \Pr\left\{-\frac{\ln(X)}{3} \le y\right\}$$

$$= \Pr\{X \ge e^{-3y}\}$$

$$= 1 - F_X(e^{-3y})$$

$$= 1 - e^{-3y}$$

On trouve $Y \sim Exp(3)$

Donc, $Var(Y) = \boxed{\frac{1}{9}}$

Technique FGM

$$E[e^{tY}] = E\left[e^{-\frac{\ln(X)}{3}t}\right]$$

$$= E[X^{-t/3}]$$

$$= \int_0^1 X^{-t/3}(1)\,dx$$

$$= \left.\frac{X^{-t/3+1}}{-t/3+1}\right|_0^1 = \frac{3}{3-t}$$

On trouve $Y \sim Exp(3)$

Donc, $Var(Y) = \frac{1}{9}$

Question 20

a) $M_X(t) = e^{3(e^t-1)}$

$X \sim Pois(3)$

$$F_X(x) = \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= \begin{cases} 0 & , \quad x < 0 \\ 0.0498 & , \quad 0 \leq x < 1 \\ 0.1991 & , \quad 1 \leq x < 2 \\ 0.4232 & , \quad 2 \leq x < 3 \\ 0.6472 & , \quad 3 \leq x < 4 \\ \cdots & \end{cases}$$

On trouve $F_X^{-1}(0.5) = \boxed{3}$

b) $$M_Y(t) = \left(\left(\frac{1}{3e^{-t} - 2} \right)^2 \right)^{3/2}$$

$$= \left(\frac{1}{3e^{-t} - 2} \right)^3$$

$$= \left(\frac{\frac{1}{3} e^t}{1 - \frac{2}{3} e^t} \right)^3$$

On trouve $Y \sim BinNeg(r = 3, \ p = 1/3)$

$$Pr(Y < 5) = \sum_{k=3}^{4} \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

$$= \boxed{\frac{1}{9}}$$

Question 20 (Suite)

c) $Z \sim \text{Uni}[0, 5]$

$$\Pr\left\{ \left|Z - \frac{5}{2}\right| < t \right\} = 0{,}7$$

$$= \Pr\left\{ \frac{5}{2} - t < Z < \frac{5}{2} + t \right\}$$

$$= \int_{\frac{5}{2}-t}^{\frac{5}{2}+t} \frac{1}{5}\, dz$$

$$= \frac{1}{5}\left[\frac{5}{2} + t - \frac{5}{2} + t \right]$$

$$= \frac{2t}{5}$$

$$\frac{2t}{5} = 0{,}7$$

On trouve $t = \boxed{\frac{7}{4}}$ ■

Question 21

$$E[Y] = \sum_{y=0}^{\infty} y \Pr(Y=y)$$

$$= \sum_{y=0}^{\infty} y \Pr(\lfloor X \rfloor = y)$$

$$= \sum_{y=0}^{\infty} y \Pr(y \leq X < y+1)$$

$$= \sum_{y=0}^{\infty} y \left(F_X(y+1) - F_X(y) \right)$$

$$= \sum_{y=0}^{\infty} y \left(e^{-1/2\, y} - e^{-1/2\,(y+1)} \right)$$

$$= \sum_{y=0}^{\infty} y\, e^{-1/2\, y} (1-e^{-1/2}) \text{ , } \quad \text{posons } y^* = y+1$$

$$= \sum_{y^*=1}^{\infty} (y^*-1)(e^{-1/2})^{y^*-1}(1-e^{-1/2})$$

$$= \sum_{y^*=1}^{\infty} y^*(1-p)^{y^*-1} p - \sum_{y^*=1}^{\infty} (1-p)^{y^*-1} p$$

où $p = 1 - e^{-1/2}$

$$= \frac{1}{1-e^{-1/2}} - 1$$

$$= \boxed{1.541}$$

■

Question 22

$45\% \times 10\,000 = 4500$ voteront pour A
$40\% \times 10\,000 = 4000$ voteront pour B
$15\% \times 10\,000 = 1500$ sont indécis

X : nombre de votes pour le candidat B

$X \in \{4000, 4001, \ldots, 5499, 5500\}$

$X = 4000 + Y$ où $Y \sim \text{Bin}(n = 1500, p = 2/3)$

$$\Pr(X = 4000 + k) = \binom{1500}{k}\left(\frac{2}{3}\right)^{k}\left(\frac{1}{3}\right)^{1500-k}$$

$$\begin{aligned} E[X] &= 4000 + E[Y] \\ &= 4000 + (1500)\left(\frac{2}{3}\right) \\ &= \boxed{5000} \end{aligned}$$

■

Question 23

$X \sim LN(\mu, \sigma^2)$

Deux équations, deux inconnus

$\Pr\{X < 95\} = 0.2358$

$\Pr\{X < 110\} = 0.6026$

$$\Pr\left\{Z < \frac{\ln(95) - \mu}{\sigma}\right\} = 0.2358$$

① $$\frac{\ln(95) - \mu}{\sigma} = -0.72$$

$$\Pr\left\{Z < \frac{\ln(110) - \mu}{\sigma}\right\} = 0.6026$$

② $$\frac{\ln(110) - \mu}{\sigma} = 0.26$$

On trouve $\sigma = 0.1496$

$\mu = 4.6616$

$$E[X] = e^{\mu + \sigma^2/2}$$

$$= \boxed{106.9959}$$

Question 24

$X \sim U[200, \theta]$

18 \$ de pourboire : 12 verres

N : v.a. du nombre de verres remplis

$N \sim BinNeg(r = 12 \,;\, p)$

$$p = \frac{\theta - 300}{\theta - 200}$$

On veut que $\frac{r}{p} = 16$

$$\frac{12(\theta - 200)}{\theta - 300} = 16$$

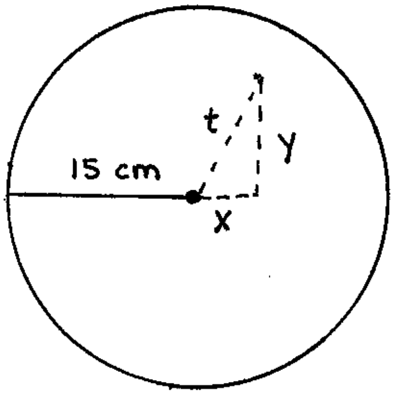
$$12\theta - 2400 = 16\theta - 4800$$

$$2400 = 4\theta$$

$$\theta = \boxed{600}$$

■

Question 25



A : atteindre la cible

T : v.a. distance (en cm) entre fléchette et centre

G : v.a. gain

$$\begin{aligned} E[G] &= E[G \mid A]\Pr(A) + E[G \mid \bar{A}]\Pr(\bar{A}) \\ &= 0.75\,E[G \mid A] \\ &= 0.75\,E[T] \end{aligned}$$

$$\begin{aligned} \Pr(T < t) &= \Pr(X^2 + Y^2 < t^2) \\ &= \frac{\text{Aire du cercle de rayon } t}{\text{Aire du cercle de rayon } 15} \\ &= \frac{\pi t^2}{\pi\, 15^2} \\ &= \frac{t^2}{225}, \quad 0 \leq t \leq 15 \end{aligned}$$

$$f_T(t) = \frac{d}{dt} F_T(t) = \frac{d}{dt}\left(\frac{t^2}{225}\right) = \frac{2t}{225}$$

$$E[T] = \int_0^{15} t \cdot \frac{2t}{225}\, dt = \left.\frac{2t^3}{3(225)}\right|_0^{15} = 10$$

$$E[G] = 0.75(10) = \boxed{7.50}$$

$\blacksquare$

Question 26

$$\begin{aligned} Var(Y) &= E[Y^2] - E^2[Y] \\ &= E[X^2 e^{-2x}] - E^2[X e^{-x}] \end{aligned}$$

On a $\frac{\alpha}{\lambda} = 5$ et $\frac{\alpha}{\lambda^2} = \frac{5}{\lambda} = 10$

On trouve $\lambda = 0.5$
$\alpha = 2.5$

$$E[X^K e^{-Kx}] = \int_0^\infty x^K e^{-Kx} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{(K+\alpha)-1} e^{-(K+\lambda)x} dx$$

$$= \frac{\lambda^\alpha \Gamma(K+\alpha)}{(\lambda+K)^{\alpha+K}\Gamma(\alpha)} \int_0^\infty \frac{x^{(\alpha+K)-1} e^{-(K+\lambda)x} (\lambda+K)^{\alpha+K}}{\Gamma(\alpha+K)} dx$$

$$= \frac{\lambda^\alpha \Gamma(\alpha+K)}{(\lambda+K)^{\alpha+K}\Gamma(\alpha)}$$

$$E[Xe^{-x}] = \frac{\lambda^\alpha \Gamma(\alpha+1)}{(\lambda+1)^{\alpha+1}\Gamma(\alpha)} = \frac{0.5^{2.5}(2.5)}{(1.5)^{3.5}} = 0.10692$$

$$E[X^2 e^{-2x}] = \frac{\lambda^\alpha \Gamma(\alpha+2)}{(\lambda+2)^{\alpha+2}\Gamma(\alpha)} = \frac{0.5^{2.5}(3.5)(2.5)}{(2.5)^{4.5}} = 0.02504$$

$$Var(Y) = 0.02504 - 0.10692^2 = \boxed{0.01361}$$

■

Question 27

a) $X \sim Bin(n = 300,\ p = 1/65)$

b) $E[X] = 300\left(\frac{1}{65}\right) = 4.6154$

$Var(X) = 300\left(\frac{1}{65}\right)\left(\frac{64}{65}\right) = \boxed{4.5444}$

Question 28

a) $X \sim Pois(\lambda)$

$(Y \mid X=n) \sim Bin(n, p)$

$$Pr(Y=k \mid X=n) = \binom{n}{k} p^k (1-p)^{n-k}$$

b) $$Pr(Y=k, X=n) = Pr(Y=k \mid X=n)\, Pr(X=n)$$

$$Pr(Y=k, X=n) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} \dfrac{e^{-\lambda}\lambda^n}{n!}, & k=0,1,\dots n \\ & n=k, k+1, \dots \\ 0, & \text{ailleurs.} \end{cases}$$

$$Pr(Y=k) = \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{e^{-\lambda}\lambda^n}{n!}$$

$$= \frac{e^{-\lambda} p^k\, n!}{(1-p)^k\, k!\, n!} \sum_{n=k}^{\infty} \frac{(1-p)^n \lambda^n}{(n-k)!}$$

$$= \frac{e^{-\lambda} p^k \lambda^k}{k!} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k} \lambda^{n-k}}{(n-k)!}$$

Posons $i = n-k$

$$= \frac{e^{-\lambda} p^k \lambda^k}{k!} \underbrace{\sum_{i=0}^{\infty} \frac{(1-p)^i \lambda^i}{i!}}$$

$$= \frac{e^{-\lambda} p^k \lambda^k}{k!} \left(e^{(1-p)\lambda} \right)$$

$$= \frac{e^{-p\lambda} (p\lambda)^k}{k!}$$

On trouve $Y \sim Pois(\lambda^* = p\lambda)$ ■

Question 29

a) $(Y \mid X = x) \sim Bin(x,\ p = 1/4)$

$$Pr(X = x,\ Y = y) = Pr(Y = y \mid X = x)\,Pr(X = x)$$

$$= \binom{x}{y}\left(\frac{1}{4}\right)^{y}\left(\frac{3}{4}\right)^{x-y} \cdot \left(\frac{1}{4}\right)$$

$$= \binom{x}{y}\frac{1}{4^{x+1}} \cdot 3^{x-y}$$

b)

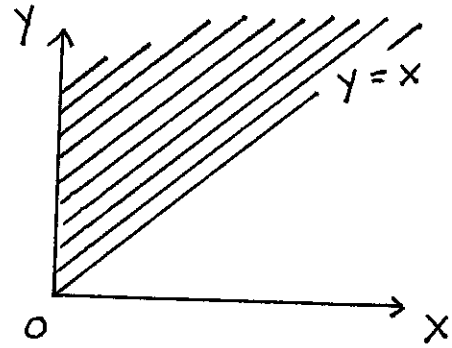
X \ Y	0	1	2	3	4
1	$3/16$	$1/16$	0	0	0
2	$9/64$	$6/64$	$1/64$	0	0
3	$27/256$	$27/256$	$9/256$	$1/256$	0
4	$81/1024$	$108/1024$	$54/1024$	$12/1024$	$1/1024$
Total	$\frac{525}{1024}$	$\frac{376}{1024}$	$\frac{106}{1024}$	$\frac{16}{1024}$	$\frac{1}{1024}$

$$E[Y] = 0 \cdot \frac{525}{1024} + 1 \cdot \frac{376}{1024} + 2 \cdot \frac{106}{1024} + 3 \cdot \frac{16}{1024} + 4 \cdot \frac{1}{1024}$$

$$= \boxed{0.625}$$

■

Question 30



a) $$\int_0^\infty \int_x^\infty K e^{-\theta y}\, dy\, dx = 1$$

$$= \int_0^\infty \frac{-K e^{-\theta y}}{\theta} \Big|_x^\infty dx$$

$$= \int_0^\infty \frac{K e^{-\theta x}}{\theta}\, dx$$

$$= \frac{-K e^{-\theta x}}{\theta^2} \Big|_0^\infty = \frac{K}{\theta^2} = 1$$

Donc, $K = \theta^2$

b) $$f_X(x) = \int_x^\infty \theta^2 e^{-\theta y}\, dy$$

$$= -\theta^2 e^{-\theta y} \Big|_x^\infty$$

$$= \theta e^{-\theta x}, \quad x > 0$$

$$f_Y(y) = \int_0^y \theta^2 e^{-\theta y}\, dx$$

$$= x\theta^2 e^{-\theta y} \Big|_0^y$$

$$= y\theta^2 e^{-\theta y}, \quad y > 0$$

Note : $f_X(x)\, f_Y(y) \neq f_{X,Y}(x, y)$

Donc, X et Y ne sont pas indépendants

Question 30 (Suite)

c) $\Pr(X \leq 1 \mid Y \leq 1) = \dfrac{\Pr(X \leq 1,\ Y \leq 1)}{\Pr(Y \leq 1)}$

$$\Pr(X \leq 1,\ Y \leq 1) = \int_0^1 \int_x^1 \theta^2 e^{-\theta y}\, dy\, dx$$

$$= \int_0^1 -\theta e^{-\theta y} \Big|_x^1 dx$$

$$= \int_0^1 \theta \left(e^{-\theta x} - e^{-\theta}\right) dx$$

$$= 1 - (1+\theta) e^{-\theta}$$

$$\Pr(Y \leq 1) = \int_0^1 y \theta^2 e^{-\theta y}\, dy \qquad \text{(par parties)}$$

$$= \left(\frac{-y e^{-\theta y}}{\theta} \Big|_0^1 + \int_0^1 \frac{e^{-\theta y}}{\theta}\, dy \right) \theta^2$$

$$= 1 - e^{-\theta}(1+\theta)$$

$$\frac{\Pr(X \leq 1,\ Y \leq 1)}{\Pr(Y \leq 1)} = \frac{1 - e^{-\theta}(1+\theta)}{1 - e^{-\theta}(1+\theta)} = 1$$

Puisque $X \leq Y$, $Y \leq 1$ implique $X \leq 1$ ■

Question 31

a) $$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} K \, dy\,dx = 1$$

$$= \int_{-1}^{1} 2\sqrt{1-x^2}\,dx$$

$$= \sqrt{1-x^2}\,x + \sin^{-1}(x) \Big|_{-1}^{1}$$

$$= K\pi = 1$$

$$K = \frac{1}{\pi}$$

b) $$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy$$

$$= \frac{2}{\pi}\sqrt{1-x^2}\ 1\{-1 \leq x \leq 1\}$$

$$f_Y(x) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\,dx =$$

$$= \frac{2}{\pi}\sqrt{1-y^2}\ 1\{-1 \leq y \leq 1\}$$

$$f_X(x)\,f_Y(y) = 4\pi^{-2}\sqrt{1-x^2}\sqrt{1-y^2}$$

$$\neq f_{X.Y}(x.y)$$

X et Y ne sont pas indépendants.

Question 31 (Suite)

c) $$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{1/\pi}{\frac{2}{\pi}\sqrt{1-y^2}}$$

- $$\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x \; f_{X|Y}(x)\,dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x \frac{1}{2\sqrt{1-y^2}}\,dx =$$

$$= \frac{x^2}{4\sqrt{1-y^2}}\Bigg|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = \frac{(1-y^2)-(1-y^2)}{4\sqrt{1-y^2}} = 0$$

- $$\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2 \, f_{X|Y}(x)\,dx = \frac{x^3}{6\sqrt{1-y^2}}\Bigg|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}$$

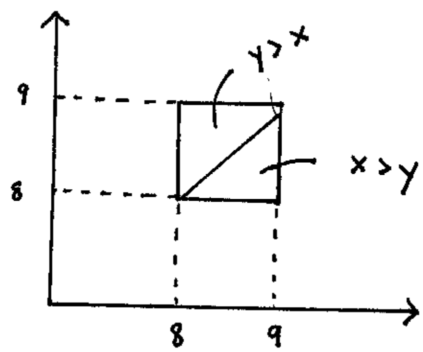
$$= \frac{(1-y^2)^{3/2} + (1-y^2)^{3/2}}{6\sqrt{1-y^2}}$$

$$= \frac{1-y^2}{3}$$

$$Var(X|Y) = \boxed{\frac{1-y^2}{3}}$$

■

Question 32



$$E[|X-Y|] = \int_8^9 \int_8^9 |x-y| \, f_{X,Y}(x,y)$$

$$f_{X,Y}(x,y) \underset{\text{indep}}{=} f_X(x) \cdot f_Y(y)$$

$$= \frac{1}{9-8} \cdot \frac{1}{9-8}$$

$$= 1$$

On sépare en deux intégrales

$$E[|X-Y|] = \int_8^9 \int_8^x x-y \, dy\,dx + \int_8^9 \int_x^9 y-x \, dy\,dx$$

$$= \int_8^9 \left(xy - \frac{y^2}{2}\right)\Big|_8^x dx + \int_8^9 \left(-xy + \frac{y^2}{2}\right)\Big|_x^9 dx$$

$$= \int_8^9 \frac{x^2}{2} - 8x + 32 \, dx + \int_8^9 \frac{x^2}{2} - 9x + \frac{81}{2} \, dx$$

$$= \ldots$$

$$= \boxed{\frac{1}{3}}$$

■

Question 33

$X \sim N(1,1)$

$(Y \mid X=x) \sim N(2x, 4)$

$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$

- $Var(X) = 1$

- $$\begin{aligned} Var(Y) &= E[Var(Y|X)] + Var(E[Y|X]) \\ &= E[4] + Var(2X) \\ &= 4 + 4\,Var(X) \\ &= 8 \end{aligned}$$

- $$\begin{aligned} Cov(X,Y) &= E[XY] - E[X]E[Y] \\ &= E[E[XY|X]] - (1)\,E[E[Y|X]] \\ &= E[X\,E[Y|X]] - E[2X] \\ &= E[2X^2] - 2E[X] \\ &= 2\left[Var(X) + E^2[X]\right] - 2E[X] \\ &= 2(2) - 2(1) \\ &= 2 \end{aligned}$$

- $$\begin{aligned} Var(X+Y) &= 1 + 8 + 2(2) \\ &= \boxed{13} \end{aligned}$$

Question 34

$(X \mid \Lambda = \lambda) \sim \text{Exp}(\lambda)$

$\Lambda \sim \text{Gamma}(\alpha = 4,\ \beta = 5)$

$$E[X] = E[\,E[X \mid \Lambda]\,]$$

$$= E\left[\frac{1}{\lambda}\right]$$

$$= \int_0^\infty \frac{1}{\lambda} \cdot \frac{\lambda^{\alpha-1} e^{-\beta\lambda} \beta^\alpha}{\Gamma(\alpha)}\, d\lambda$$

$$= \int_0^\infty \frac{\lambda^{\alpha-2} e^{-\beta\lambda} \beta^\alpha}{\Gamma(\alpha)}\, d\lambda$$

$$= \frac{\beta^\alpha\, \Gamma(\alpha-1)}{\Gamma(\alpha)\, \beta^{\alpha-1}} \int_0^\infty \frac{\lambda^{(\alpha-1)-1} e^{-\beta\lambda} \beta^{\alpha-1}}{\Gamma(\alpha-1)}\, d\lambda$$

$$= \frac{\beta\,(\alpha-2)!}{(\alpha-1)!}$$

$$= \frac{\beta}{(\alpha-1)}$$

$$= \boxed{5/3}$$

$$\text{Var}(X) = E[\,\text{Var}(X \mid \Lambda)\,] + \text{Var}(E[X \mid \Lambda])$$

$$= E\left[\frac{1}{\lambda^2}\right] + \text{Var}\left(\frac{1}{\lambda}\right)$$

$$= E\left[\frac{1}{\lambda^2}\right] + E\left[\frac{1}{\lambda^2}\right] - E^2\left[\frac{1}{\lambda}\right]$$

Question 34 (Suite)

$$E\left[\frac{1}{\lambda^2}\right] = \int_0^\infty \frac{1}{\lambda^2} \frac{\lambda^{\alpha-1} e^{-\beta\lambda} \beta^\alpha}{\Gamma(\alpha)} \, d\lambda$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \lambda^{\alpha-3} e^{-\beta\lambda} \, d\lambda$$

$$= \frac{\beta^\alpha \, \Gamma(\alpha-2)}{\Gamma(\alpha)\, \beta^{\alpha-2}} \int_0^\infty \frac{\lambda^{(\alpha-2)-1} e^{-\beta\lambda} \beta^{\alpha-2}}{\Gamma(\alpha-2)} \, d\lambda$$

$$= \frac{\beta^2 (\alpha-3)!}{(\alpha-1)!}$$

$$= \frac{\beta^2}{(\alpha-1)(\alpha-2)}$$

$$= \frac{5^2}{(3)(2)} = \frac{25}{6}$$

$$Var(X) = 2\left(\frac{25}{6}\right) - \left(\frac{5}{3}\right)^2$$

$$= \boxed{\frac{50}{9}}$$

■

Question 35

Y : Aire du cercle

$$Y = \pi \left(\frac{X}{2}\right)^2$$

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

$$X = \sqrt{\frac{4}{\pi} Y} \Rightarrow \frac{dX}{dY} = 0.5 \sqrt{\frac{4}{\pi}} \, Y^{-0.5}$$

$$f_Y(y) = 100 \cdot 0.5 \cdot \sqrt{\frac{4}{\pi}} \, y^{-0.5}$$

$$= 56.419 \, y^{-0.5}$$

Domaine de Y : $\frac{\pi}{4} \leq y \leq \frac{1.01^2 \cdot \pi}{4}$

$$0.7854 \leq y \leq 0.8012$$

■

Question 36

$$f_X(x) = \int_x^{\infty} x e^{-x(y-x)} dy$$

$$= \int_x^{\infty} x e^{-xy} e^{x^2} dy$$

$$= -e^{x^2} e^{-xy} \Big|_x^{\infty} = 1$$

$$f_Y(y \mid X = x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = x e^{-x(y-x)}$$

$$E[Y \mid X = x] = \int_x^{\infty} y \cdot x e^{-x(y-x)} dy$$

$$= x e^{x^2} \int_x^{\infty} y e^{-xy} dy$$

À faire par parties... $= x + \frac{1}{x}$

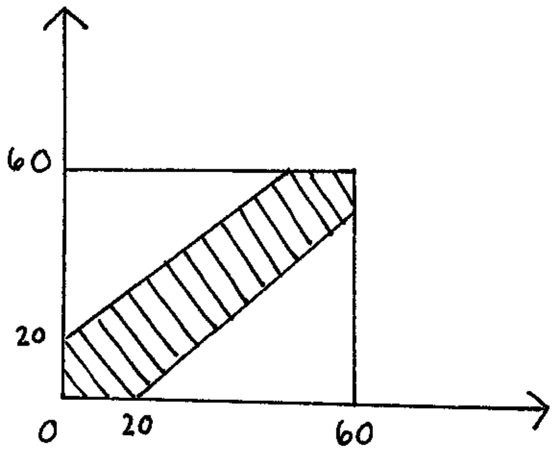
$$\Pr\left(Y > x + \frac{1}{x} \mid X = x\right) = \int_{x+\frac{1}{x}}^{\infty} x e^{-x(y-x)} dy$$

$$= e^{-1}$$

$$= \boxed{0.36788}$$

■

Question 37



$$Pr(|Y-X| \leq 20) = Pr(-20 \leq Y-X \leq 20)$$

$$= Pr(X-20 \leq Y \leq X+20)$$

Aire du carreé : $60 \times 60 = 3600$

Aire des triangles : $\frac{40 \cdot 40}{2} = 800$

$$Pr(|Y-X| \leq 20) = \frac{3600 - 2(800)}{3600}$$ Ratio d'aires.

$$= \boxed{\frac{5}{9}}$$

■

Question 38

$f_{X,Y}(x,y) = c(4-x)$

Lorsque $x = 1$, $y = 0$

$x = 2$, $y = 0$ ou 1

$x = 3$, $y = 0$ ou 1 ou 2

$f(1,0) + f(2,0) + f(2,1) + f(3,0) + f(3,1) + f(3,2) = 1$

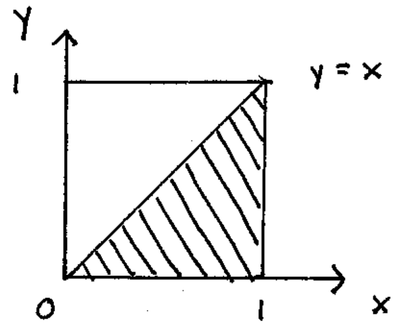
$3c + 2c + 2c + c + c + c = 1$

$10c = 1$

On trouve $c = \boxed{1/10}$

Question 39

a) $F_{X,Y}(0.5,\ 0.75) = F_{X,Y}(0.5,\ 0.5)$



$$= \int_0^{0.5}\int_0^{x} \sqrt{\frac{x}{y}}\,dy\,dx$$

$$= \int_0^{0.5} 2\sqrt{x}\,y^{1/2}\Big|_0^x\,dx$$

$$= \int_0^{0.5} 2x\,dx$$

$$= \boxed{0.25}$$

b) $$f_Y(y \mid X=0.8) = \frac{f_{X,Y}(0.8,\ y)}{f_X(0.8)}$$

$$f_X(x) = \int_0^x \sqrt{x}\,y^{-1/2}\,dy = 2\sqrt{x}\,y^{1/2}\Big|_0^x = 2x$$

$f_X(0.8) = 1.6$

$$f_Y(y \mid X=0.8) = \frac{\sqrt{\frac{0.8}{y}}}{1.6} = \frac{\sqrt{0.8}}{1.6\sqrt{y}}$$

$$E[Y \mid X=0.8] = \int_0^{0.8} y\,\frac{\sqrt{0.8}}{1.6\sqrt{y}}\,dy$$

$$= \boxed{\frac{4}{15}}$$

■

Question 40

a) $Pr(Y=4) = Pr(Y=4 \mid D=4)Pr(D=4) + Pr(Y=4 \mid D=5)Pr(D=5)$
$\quad \dots + Pr(Y=4 \mid D=6)Pr(D=6)$

On a $Pr(Y=4 \mid D=i) = 0$, $i = 1, \dots, 3$

$$Pr(Y=4) = \frac{1}{6}p^4 + \frac{1}{6}\binom{5}{4}p^4(1-p) + \frac{1}{6}\binom{6}{4}p^4(1-p)^2$$

$$= \frac{1}{6}p^4\left[1 + 5(1-p) + 15(1-p)^2\right] = p^4$$

$$= 1 + 5(1-p) + 15(1-p)^2 = 6$$

$$\Rightarrow \quad 15(1-p)^2 + 5(1-p) - 5 = 0$$

$$\frac{-5 \pm \sqrt{5^2 - 4(15)(-5)}}{2(15)} = 0.43426 \text{ ou } -0.76759$$
(à rejeter)

$$\Rightarrow \quad (1-p) = 0.43426$$

On trouve $\boxed{p = 0.5657}$

Question 40 (Suite)

b)

$$\Pr(D=1 \mid Y=0) = \frac{\Pr(D=1 \cap Y=0)}{\Pr(Y=0)}$$

- $\Pr(D=1 \cap Y=0) = \frac{1}{6}(1-p) = 0.07238$

- $\Pr(Y=0) = \sum_{i=1}^{6} \Pr(Y=0 \mid D=i)\Pr(D=i)$

$$= \frac{1}{6} \sum_{i=1}^{6} \binom{i}{0} p^0 (1-p)^i$$

$$= \frac{1}{6} \sum_{i=1}^{6} (1-p)^i$$

$$= \frac{1}{6} \left[\sum_{i=0}^{6} (1-p)^i - 1 \right]$$

$$= \frac{1}{6} \left[\frac{1-(1-p)^7}{p} - 1 \right]$$

$$= 0.12707$$

$$\Pr(D=1 \mid Y=0) = \frac{0.07238}{0.12707} = \boxed{0.5696}$$

■

Question 41

$$E[XY] = \sum_{j=0}^{1}\sum_{i=0}^{1} i \cdot j \; Pr(X=i,\ Y=j)$$

$$= Pr(X=1,\ Y=1) = 4/9$$

↳ Nombre pair, supérieur à 3
= 4 ou 6

$Pr(\text{Nb pair}) = 2/9$

$\frac{2}{9}(3) + 3x = 1 \quad \Rightarrow \quad x = Pr(\text{Nb impair}) = 1/9$

$Pr(X=1,\ Y=1) = 4/9$ (avoir 4 ou 6)

$Pr(X=0,\ Y=1) = 1/9$ (avoir 5)

$Pr(X=1,\ Y=0) = 2/9$ (avoir 2)

$Pr(X=0,\ Y=0) = 2/9$ (avoir 1 ou 3)

a)

$$Pr(X=x,\ Y=y) = \begin{cases} 4/9, & x=1,\ y=1 \\ 1/9, & x=0,\ y=1 \\ 2/9, & x=1,\ y=0 \\ 2/9, & x=0,\ y=0 \\ 0, & \text{ailleurs.} \end{cases}$$

b)

$$\frac{12!}{2^6}\left(\frac{2}{9}\right)^2\left(\frac{2}{9}\right)^2\left(\frac{2}{9}\right)^2\left(\frac{1}{9}\right)^2\left(\frac{1}{9}\right)^2\left(\frac{1}{9}\right)^2$$

$$= \boxed{0.001696}$$

Question 42

X : v.a. prix action A

Y : v.a. prix action B

$X \sim Uni[0, a]$

$(Y \mid X = x) \sim Uni[0, x]$

$$Cov(X, Y) = E[XY] - E[X]E[Y] = 6$$

- $E[X] = \dfrac{a-0}{2} = \dfrac{a}{2}$

- $E[Y] = E[E[Y|X]] = E\left[\dfrac{x}{2}\right] = \displaystyle\int_0^a \dfrac{x}{2}\,\dfrac{1}{a}\,dx$

$$= \frac{x^2}{4a}\Big|_0^a = \frac{a}{4}$$

- $E[XY] = \displaystyle\int_0^a \int_0^x xy\, f_x(x) f(y|x)\, dy\, dx$

$$= \int_0^a \int_0^x xy\left(\frac{1}{a}\right)\left(\frac{1}{x}\right) dy\, dx$$

$$= \frac{a^2}{6}$$

Donc, $\left(\dfrac{a^2}{6}\right) - \left(\dfrac{a}{2}\right)\left(\dfrac{a}{4}\right) = 6$

$$\frac{a^2}{6} - \frac{a^2}{8} = 6$$

$$\ldots \quad \boxed{a = 12}$$

■

Question 43

$$f_{X,Y}(x,y) = (2e^{-2y})(e^{-x}) \quad , \quad x>0 \; ; \; y>0$$

$\rightarrow$ X et Y indépendants

$$f(y \mid Y>3) = \frac{f(y)}{Pr(Y>3)} = \frac{2e^{-2y}}{e^{-6}} = 2e^{6-2y} \quad , \quad y>3$$

- $$E[Y \mid Y>3] = \int_3^\infty y \; 2e^{6-2y} \, dy$$

Posons $u = y-3$

$$= \int_0^\infty (u+3)e^{-2u} \, du = 3.5$$

- $$E[Y^2 \mid Y>3] = \int_3^\infty y^2 \cdot 2e^{6-2y} \, dy$$

$$= 12.5$$

$$Var(Y \mid Y>3) = 12.5 - 3.5^2$$

$$= \boxed{0.25}$$