Proposition 1 *Soit X une variable aléatoire obéissant à une loi géométrique telle que*

$$\Pr(X=k)=\begin{cases} p(1-p)^{k-1}, \ k=\mathbf{1},2,3,... \\ 0, \ \textit{ailleurs.}\end{cases}$$

Montrer que $e_X(d)=E[X-d\,|X>d]=E[X]$.

Proof.

$$\begin{aligned}
E[X-d\,|X>d] &= \sum_{k=1}^{\infty}(k-d)\Pr(X=k\,|X>d) \\
&= \sum_{k=1}^{\infty}(k-d)\frac{\Pr(X=k,X>d)}{\Pr(X>d)} \\
&= \sum_{k=d+1}^{\infty}(k-d)\frac{p(1-p)^{k-1}}{(1-p)^d} \\
&= \frac{1}{(1-p)^d}\left(\sum_{k=d+1}^{\infty}kp(1-p)^{k-1}-d\sum_{k=d+1}^{\infty}p(1-p)^{k-1}\right) \\
&= \frac{1}{(1-p)^d}\left(\sum_{k=d+1}^{\infty}kp(1-p)^{k-1}-d\,(1-p)^d\right)
\end{aligned}$$

Posons $y=k-d$ et donc $k=y+d$. Alors,

$$\begin{aligned}
E[X-d\,|X>d] &= \frac{1}{(1-p)^d}\left(\sum_{y=1}^{\infty}(y+d)\,p(1-p)^{(y+d)-1}-d\,(1-p)^d\right) \\
&= \frac{1}{(1-p)^d}\left(\sum_{y=1}^{\infty}yp(1-p)^{(y+d)-1}+\sum_{y=1}^{\infty}dp(1-p)^{(y+d)-1}-d\,(1-p)^d\right) \\
&= \frac{1}{(1-p)^d}\left((1-p)^d\sum_{y=1}^{\infty}yp(1-p)^{y-1}+d\,(1-p)^d\sum_{y=1}^{\infty}p(1-p)^{y-1}-d\,(1-p)^d\right) \\
&= \frac{1}{(1-p)^d}\left((1-p)^d\,E[X]+d\,(1-p)^d-d\,(1-p)^d\right) \\
&= E[X].
\end{aligned}$$

■

Proposition 2 *Soit Xune variable aléatoire obéissant à une loi géométrique telle que*

$$\Pr(X=k)=\begin{cases} p(1-p)^{k}, \ k=\mathbf{0},1,2,3,... \\ 0, \ \textit{ailleurs.}\end{cases}$$

Montrer que $e_X(d)=E[X-d\,|X>d]=E[X]+1$.

Proof.

$$
\begin{aligned}
E\left[X-d\,|X>d\right] &= \sum_{k=0}^{\infty}(k-d)\Pr\left(X=k\,|X>d\right)\\
&= \sum_{k=0}^{\infty}(k-d)\,\frac{\Pr\left(X=k,X>d\right)}{\Pr\left(X>d\right)}\\
&= \sum_{k=d+1}^{\infty}(k-d)\,\frac{p(1-p)^{k}}{(1-p)^{d+1}}\\
&= \frac{1}{(1-p)^{d+1}}\left(\sum_{k=d+1}^{\infty}kp(1-p)^{k}-d\sum_{k=d+1}^{\infty}p(1-p)^{k}\right)\\
&= \frac{1}{(1-p)^{d+1}}\left(\sum_{k=d+1}^{\infty}kp(1-p)^{k}-d\,(1-p)^{d+1}\right)
\end{aligned}
$$

Posons $y=k-(d+1)$ et donc $k=y+(d+1)$. Alors,

$$
\begin{aligned}
E\left[X-d\,|X>d\right] &= \frac{1}{(1-p)^{d+1}}\left(\sum_{y=0}^{\infty}(y+(d+1))\,p(1-p)^{y+(d+1)}-d\,(1-p)^{d+1}\right)\\
&= \frac{1}{(1-p)^{d+1}}\left(\sum_{y=0}^{\infty}yp(1-p)^{y+(d+1)}+\sum_{y=0}^{\infty}(d+1)\,p(1-p)^{y+(d+1)}-d\,(1-p)^{d+1}\right)\\
&= \frac{1}{(1-p)^{d+1}}\left((1-p)^{d+1}\sum_{y=0}^{\infty}yp(1-p)^{y}+(1-p)^{d+1}\sum_{y=0}^{\infty}(d+1)\,p(1-p)^{y}-d\,(1-p)^{d+1}\right)\\
&= \frac{1}{(1-p)^{d+1}}\left((1-p)^{d+1}E\left[X\right]+(1-p)^{d+1}(d+1)-d\,(1-p)^{d+1}\right)\\
&= E\left[X\right]+1
\end{aligned}
$$

$\blacksquare$