Renewal Theory and Its Applications

7.1 Introduction

We have seen that a Poisson process is a counting process for which the times between successive events are independent and identically distributed exponential random variables. One possible generalization is to consider a counting process for which the times between successive events are independent and identically distributed with an arbitrary distribution. Such a counting process is called a *renewal process*.

Let $\{N(t),\ t \geqslant 0\}$ be a counting process and let X_n denote the time between the $(n-1)$st and the nth event of this process, $n \geqslant 1$.

Definition 7.1 If the sequence of nonnegative random variables $\{X_1, X_2, \ldots\}$ is independent and identically distributed, then the counting process $\{N(t),\ t \geqslant 0\}$ is said to be a *renewal process*.

Thus, a renewal process is a counting process such that the time until the first event occurs has some distribution F, the time between the first and second event has, independently of the time of the first event, the same distribution F, and so on. When an event occurs, we say that a renewal has taken place.

For an example of a renewal process, suppose that we have an infinite supply of lightbulbs whose lifetimes are independent and identically distributed. Suppose also that we use a single lightbulb at a time, and when it fails we immediately replace it with a new one. Under these conditions, $\{N(t),\ t \geqslant 0\}$ is a renewal process when $N(t)$ represents the number of lightbulbs that have failed by time t.

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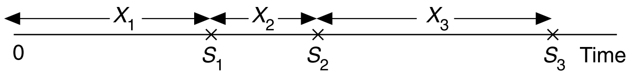

Figure 7.1 Renewal and interarrival times.

For a renewal process having interarrival times $X_1, X_2, \ldots$, let

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, \quad n \geqslant 1$$

That is, $S_1 = X_1$ is the time of the first renewal; $S_2 = X_1 + X_2$ is the time until the first renewal plus the time between the first and second renewal, that is, S_2 is the time of the second renewal. In general, S_n denotes the time of the nth renewal (see Figure 7.1).

We shall let F denote the interarrival distribution and to avoid trivialities, we assume that $F(0) = P\{X_n = 0\} < 1$. Furthermore, we let

$$\mu = E[X_n], \quad n \geqslant 1$$

be the mean time between successive renewals. It follows from the nonnegativity of X_n and the fact that X_n is not identically 0 that $\mu > 0$.

The first question we shall attempt to answer is whether an infinite number of renewals can occur in a finite amount of time. That is, can $N(t)$ be infinite for some (finite) value of t? To show that this cannot occur, we first note that, as S_n is the time of the nth renewal, $N(t)$ may be written as

$$N(t) = \max\{n : \; S_n \leqslant t\} \tag{7.1}$$

To understand why Equation (7.1) is valid, suppose, for instance, that $S_4 \leqslant t$ but $S_5 > t$. Hence, the fourth renewal had occurred by time t but the fifth renewal occurred after time t; or in other words, $N(t)$, the number of renewals that occurred by time t, must equal 4. Now by the strong law of large numbers it follows that, with probability 1,

$$\frac{S_n}{n} \to \mu \quad \text{as } n \to \infty$$

But since $\mu > 0$ this means that S_n must be going to infinity as n goes to infinity. Thus, S_n can be less than or equal to t for at most a finite number of values of n, and hence by Equation (7.1), $N(t)$ must be finite.

However, though $N(t) < \infty$ for each t, it is true that, with probability 1,

$$N(\infty) \equiv \lim_{t \to \infty} N(t) = \infty$$

This follows since the only way in which $N(\infty)$, the total number of renewals that occur, can be finite is for one of the interarrival times to be infinite. Therefore,

$$\begin{aligned} P\{N(\infty) < \infty\} &= P\{X_n = \infty \text{ for some } n\} \\ &= P\left\{\bigcup_{n=1}^{\infty}\{X_n = \infty\}\right\} \\ &\leqslant \sum_{n=1}^{\infty} P\{X_n = \infty\} \\ &= 0 \end{aligned}$$

7.2 Distribution of $N(t)$

The distribution of $N(t)$ can be obtained, at least in theory, by first noting the important relationship that *the number of renewals by time t is greater than or equal to n if and only if the nth renewal occurs before or at time t*. That is,

$$N(t) \geqslant n \quad \Leftrightarrow \quad S_n \leqslant t \tag{7.2}$$

From Equation (7.2) we obtain

$$\begin{aligned} P\{N(t) = n\} &= P\{N(t) \geqslant n\} - P\{N(t) \geqslant n+1\} \\ &= P\{S_n \leqslant t\} - P\{S_{n+1} \leqslant t\} \end{aligned} \tag{7.3}$$

Now, since the random variables X_i, $i \geqslant 1$, are independent and have a common distribution F, it follows that $S_n = \sum_{i=1}^{n} X_i$ is distributed as F_n, the n-fold convolution of F with itself (Section 2.5). Therefore, from Equation (7.3) we obtain

$$P\{N(t) = n\} = F_n(t) - F_{n+1}(t)$$

Example 7.1 Suppose that $P\{X_n = i\} = p(1-p)^{i-1}$, $i \geqslant 1$. That is, suppose that the interarrival distribution is geometric. Now $S_1 = X_1$ may be interpreted as the number of trials necessary to get a single success when each trial is independent and has a probability p of being a success. Similarly, S_n may be interpreted as the number of trials necessary to attain n successes, and hence has the negative binomial distribution

$$P\{S_n = k\} = \begin{cases} \dbinom{k-1}{n-1} p^n (1-p)^{k-n}, & k \geqslant n \\ 0, & k < n \end{cases}$$

Thus, from Equation (7.3) we have that

$$P\{N(t)=n\} = \sum_{k=n}^{[t]} \binom{k-1}{n-1} p^n (1-p)^{k-n} - \sum_{k=n+1}^{[t]} \binom{k-1}{n} p^{n+1} (1-p)^{k-n-1}$$

Equivalently, since an event independently occurs with probability p at each of the times $1, 2, \ldots$

$$P\{N(t)=n\} = \binom{[t]}{n} p^n (1-p)^{[t]-n}$$ ■

Another expression for $P(N(t)=n)$ can be obtained by conditioning on S_n. This yields

$$P\left(N(t)=n\right) = \int_0^\infty P\left(N(t)=n|S_n=y\right) f_{S_n}(y)dy$$

Now, if the nth event occurred at time $y > t$, then there would have been less than n events by time t. On the other hand, if it occurred at a time $y \leqslant t$, then there would be exactly n events by time t if the next interarrival exceeds $t-y$. Consequently,

$$\begin{aligned} P\left(N(t)=n\right) &= \int_0^t P\left(X_{n+1} > t-y|S_n=y\right) f_{S_n}(y)dy \\ &= \int_0^t \bar{F}(t-y) f_{S_n}(y)dy \end{aligned}$$

where $\bar{F} = 1 - F$.

Example 7.2 If $F(x) = 1 - e^{\lambda x}$ then S_n, being the sum of n independent exponentials with rate λ, will have a gamma (n, λ) distribution. Consequently, the preceding identity gives

$$\begin{aligned} P\left(N(t)=n\right) &= \int_0^t e^{-\lambda(t-y)} \frac{\lambda e^{-\lambda y} \left(\lambda y\right)^{n-1}}{(n-1)!} dy \\ &= \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \int_0^t y^{n-1} dy \\ &= e^{-\lambda t} \frac{(\lambda t)^n}{n!} \end{aligned}$$ ■

By using Equation (7.2) we can calculate $m(t)$, the mean value of $N(t)$, as

$$\begin{aligned}
m(t) &= E[N(t)] \\
&= \sum_{n=1}^{\infty} P\{N(t) \geqslant n\} \\
&= \sum_{n=1}^{\infty} P\{S_n \leqslant t\} \\
&= \sum_{n=1}^{\infty} F_n(t)
\end{aligned}$$

where we have used the fact that if X is nonnegative and integer valued, then

$$\begin{aligned}
E[X] &= \sum_{k=1}^{\infty} kP\{X = k\} = \sum_{k=1}^{\infty} \sum_{n=1}^{k} P\{X = k\} \\
&= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P\{X = k\} = \sum_{n=1}^{\infty} P\{X \geqslant n\}
\end{aligned}$$

The function $m(t)$ is known as the *mean-value* or the *renewal function*.

It can be shown that the mean-value function $m(t)$ uniquely determines the renewal process. Specifically, there is a one-to-one correspondence between the interarrival distributions F and the mean-value functions $m(t)$.

Another interesting result that we state without proof is that

$$m(t) < \infty \quad \text{for all } t < \infty$$

Remarks

(i) Since $m(t)$ uniquely determines the interarrival distribution, it follows that the Poisson process is the only renewal process having a linear mean-value function.

(ii) Some readers might think that the finiteness of $m(t)$ should follow directly from the fact that, with probability 1, $N(t)$ is finite. However, such reasoning is not valid; consider the following: Let Y be a random variable having the following probability distribution:

$$Y = 2^n \text{ with probability } \left(\tfrac{1}{2}\right)^n, \quad n \geqslant 1$$

Now,

$$P\{Y < \infty\} = \sum_{n=1}^{\infty} P\{Y = 2^n\} = \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n = 1$$

But

$$E[Y] = \sum_{n=1}^{\infty} 2^n P\{Y = 2^n\} = \sum_{n=1}^{\infty} 2^n \left(\tfrac{1}{2}\right)^n = \infty$$

Hence, even when Y is finite, it can still be true that $E[Y] = \infty$.

An integral equation satisfied by the renewal function can be obtained by conditioning on the time of the first renewal. Assuming that the interarrival distribution F is continuous with density function f this yields

$$m(t) = E[N(t)] = \int_0^{\infty} E[N(t)|X_1 = x] f(x)\, dx \tag{7.4}$$

Now suppose that the first renewal occurs at a time x that is less than t. Then, using the fact that a renewal process probabilistically starts over when a renewal occurs, it follows that the number of renewals by time t would have the same distribution as 1 plus the number of renewals in the first $t - x$ time units. Therefore,

$$E[N(t)|X_1 = x] = 1 + E[N(t - x)] \quad \text{if } x < t$$

Since, clearly

$$E[N(t)|X_1 = x] = 0 \quad \text{when } x > t$$

we obtain from Equation (7.4) that

$$\begin{aligned} m(t) &= \int_0^t [1 + m(t - x)] f(x)\, dx \\ &= F(t) + \int_0^t m(t - x) f(x)\, dx \end{aligned} \tag{7.5}$$

Equation (7.5) is called the *renewal equation* and can sometimes be solved to obtain the renewal function.

Example 7.3 One instance in which the renewal equation can be solved is when the interarrival distribution is uniform—say, uniform on (0, 1). We will now present a solution in this case when $t \leqslant 1$. For such values of t, the renewal function becomes

$$\begin{aligned} m(t) &= t + \int_0^t m(t - x)\, dx \\ &= t + \int_0^t m(y)\, dy \quad \text{by the substitution } y = t - x \end{aligned}$$

Differentiating the preceding equation yields

$$m'(t) = 1 + m(t)$$

Letting $h(t) = 1 + m(t)$, we obtain

$$h'(t) = h(t)$$

or

$$\log h(t) = t + C$$

or

$$h(t) = Ke^t$$

or

$$m(t) = Ke^t - 1$$

Since $m(0) = 0$, we see that $K = 1$, and so we obtain

$$m(t) = e^t - 1, \quad 0 \leqslant t \leqslant 1 \qquad \blacksquare$$

7.3 Limit Theorems and Their Applications

We have shown previously that, with probability 1, $N(t)$ goes to infinity as t goes to infinity. However, it would be nice to know the rate at which $N(t)$ goes to infinity. That is, we would like to be able to say something about $\lim_{t\to\infty} N(t)/t$.

As a prelude to determining the rate at which $N(t)$ grows, let us first consider the random variable $S_{N(t)}$. In words, just what does this random variable represent? Proceeding inductively suppose, for instance, that $N(t) = 3$. Then $S_{N(t)} = S_3$ represents the time of the third event. Since there are only three events that have occurred by time t, S_3 also represents the time of the last event prior to (or at) time t. This is, in fact, what $S_{N(t)}$ represents—namely, the time of the last renewal *prior to or at* time t. Similar reasoning leads to the conclusion that $S_{N(t)+1}$ represents the time of the first renewal *after* time t (see Figure 7.2). We now are ready to prove the following.

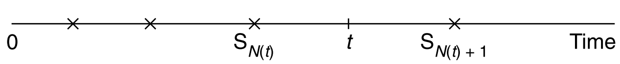

Figure 7.2

Proposition 7.1 With probability 1,

$$\frac{N(t)}{t} \to \frac{1}{\mu} \quad \text{as } t \to \infty$$

Proof. Since $S_{N(t)}$ is the time of the last renewal prior to or at time t, and $S_{N(t)+1}$ is the time of the first renewal after time t, we have

$$S_{N(t)} \leqslant t < S_{N(t)+1}$$

or

$$\frac{S_{N(t)}}{N(t)} \leqslant \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)} \tag{7.6}$$

However, since $S_{N(t)}/N(t) = \sum_{i=1}^{N(t)} X_i/N(t)$ is the average of $N(t)$ independent and identically distributed random variables, it follows by the strong law of large numbers that $S_{N(t)}/N(t) \to \mu$ as $N(t) \to \infty$. But since $N(t) \to \infty$ when $t \to \infty$, we obtain

$$\frac{S_{N(t)}}{N(t)} \to \mu \quad \text{as } t \to \infty$$

Furthermore, writing

$$\frac{S_{N(t)+1}}{N(t)} = \left(\frac{S_{N(t)+1}}{N(t)+1}\right)\left(\frac{N(t)+1}{N(t)}\right)$$

we have that $S_{N(t)+1}/(N(t)+1) \to \mu$ by the same reasoning as before and

$$\frac{N(t)+1}{N(t)} \to 1 \quad \text{as } t \to \infty$$

Hence,

$$\frac{S_{N(t)+1}}{N(t)} \to \mu \quad \text{as } t \to \infty$$

The result now follows by Equation (7.6) since $t/N(t)$ is between two random variables, each of which converges to μ as $t \to \infty$. ■

Remarks

(i) The preceding propositions are true even when μ, the mean time between renewals, is infinite. In this case, we interpret $1/\mu$ to be 0.

(ii) The number $1/\mu$ is called the *rate* of the renewal process.
(iii) Because the average time between renewals is μ, it is quite intuitive that the average rate at which renewals occur is 1 per every μ time units. ■

Example 7.4 Beverly has a radio that works on a single battery. As soon as the battery in use fails, Beverly immediately replaces it with a new battery. If the lifetime of a battery (in hours) is distributed uniformly over the interval (30, 60), then at what rate does Beverly have to change batteries?

Solution: If we let $N(t)$ denote the number of batteries that have failed by time t, we have by Proposition 7.1 that the rate at which Beverly replaces batteries is given by

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} = \frac{1}{45}$$

That is, in the long run, Beverly will have to replace one battery every 45 hours. ■

Example 7.5 Suppose in Example 7.4 that Beverly does not keep any surplus batteries on hand, and so each time a failure occurs she must go and buy a new battery. If the amount of time it takes for her to get a new battery is uniformly distributed over (0, 1), then what is the average rate that Beverly changes batteries?

Solution: In this case the mean time between renewals is given by

$$\mu = E[U_1] + E[U_2]$$

where U_1 is uniform over (30, 60) and U_2 is uniform over (0, 1). Hence,

$$\mu = 45 + \tfrac{1}{2} = 45\tfrac{1}{2}$$

and so in the long run, Beverly will be putting in a new battery at the rate of $\frac{2}{91}$. That is, she will put in two new batteries every 91 hours. ■

Example 7.6 Suppose that potential customers arrive at a single-server bank in accordance with a Poisson process having rate λ. However, suppose that the potential customer will enter the bank only if the server is free when he arrives. That is, if there is already a customer in the bank, then our arriver, rather than entering the bank, will go home. If we assume that the amount of time spent in the bank by an entering customer is a random variable having distribution G, then

(a) what is the rate at which customers enter the bank?
(b) what proportion of potential customers actually enter the bank?

Solution: In answering these questions, let us suppose that at time 0 a customer has just entered the bank. (That is, we define the process to start when the first customer enters the bank.) If we let μ_G denote the mean service time, then, by the memoryless property of the Poisson process, it follows that the mean time between entering customers is

$$\mu = \mu_G + \frac{1}{\lambda}$$

Hence, the rate at which customers enter the bank will be given by

$$\frac{1}{\mu} = \frac{\lambda}{1 + \lambda\mu_G}$$

On the other hand, since potential customers will be arriving at a rate λ, it follows that the proportion of them entering the bank will be given by

$$\frac{\lambda/(1 + \lambda\mu_G)}{\lambda} = \frac{1}{1 + \lambda\mu_G}$$

In particular if $\lambda = 2$ and $\mu_G = 2$, then only one customer out of five will actually enter the system. ■

A somewhat unusual application of Proposition 7.1 is provided by our next example.

Example 7.7 A sequence of independent trials, each of which results in outcome number i with probability P_i, $i = 1, \ldots, n$, $\sum_{i=1}^{n} P_i = 1$, is observed until the same outcome occurs k times in a row; this outcome then is declared to be the winner of the game. For instance, if $k = 2$ and the sequence of outcomes is $1, 2, 4, 3, 5, 2, 1, 3, 3$, then we stop after nine trials and declare outcome number 3 the winner. What is the probability that i wins, $i = 1, \ldots, n$, and what is the expected number of trials?

Solution: We begin by computing the expected number of coin tosses, call it $E[T]$, until a run of k successive heads occurs when the tosses are independent and each lands on heads with probability p. By conditioning on the time of the first nonhead, we obtain

$$E[T] = \sum_{j=1}^{k} (1 - p)p^{j-1}(j + E[T]) + kp^k$$

Solving this for $E[T]$ yields

$$E[T] = k + \frac{(1 - p)}{p^k} \sum_{j=1}^{k} jp^{j-1}$$

Upon simplifying, we obtain

$$E[T] = \frac{1 + p + \cdots + p^{k-1}}{p^k} = \frac{1 - p^k}{p^k(1 - p)} \tag{7.7}$$

Now, let us return to our example, and let us suppose that as soon as the winner of a game has been determined we immediately begin playing another game. For each i let us determine the rate at which outcome i wins. Now, every time i wins, everything starts over again and thus wins by i constitute renewals. Hence, from Proposition 7.1, the

$$\text{rate at which } i \text{ wins} = \frac{1}{E[N_i]}$$

where N_i denotes the number of trials played between successive wins of outcome i. Hence, from Equation (7.7) we see that

$$\text{rate at which } i \text{ wins} = \frac{P_i^k(1 - P_i)}{1 - P_i^k} \tag{7.8}$$

Hence, the long-run proportion of games that are won by number i is given by

$$\text{proportion of games } i \text{ wins} = \frac{\text{rate at which } i \text{ wins}}{\sum_{j=1}^{n} \text{rate at which } j \text{ wins}} = \frac{P_i^k(1 - P_i)/(1 - P_i^k)}{\sum_{j=1}^{n}(P_j^k(1 - P_j)/(1 - P_j^k))}$$

However, it follows from the strong law of large numbers that the long-run proportion of games that i wins will, with probability 1, be equal to the probability that i wins any given game. Hence,

$$P\{i \text{ wins}\} = \frac{P_i^k(1 - P_i)/(1 - P_i^k)}{\sum_{j=1}^{n}(P_j^k(1 - P_j)/(1 - P_j^k))}$$

To compute the expected time of a game, we first note that the

$$\text{rate at which games end} = \sum_{i=1}^{n} \text{rate at which } i \text{ wins} = \sum_{i=1}^{n} \frac{P_i^k(1 - P_i)}{1 - P_i^k} \quad (\text{from Equation (7.8)})$$

Now, as everything starts over when a game ends, it follows by Proposition 7.1 that the rate at which games end is equal to the reciprocal of the mean time of a game. Hence,

$$E[\text{time of a game}\} = \frac{1}{\text{rate at which games end}}$$
$$= \frac{1}{\sum_{i=1}^{n}(P_i^k(1-P_i)/(1-P_i^k))} \qquad \blacksquare$$

Proposition 7.1 says that the average renewal rate up to time t will, with probability 1, converge to $1/\mu$ as $t \to \infty$. What about the expected average renewal rate? Is it true that $m(t)/t$ also converges to $1/\mu$? This result is known as the *elementary renewal theorem.*

Theorem 7.1 Elementary Renewal Theorem

$$\frac{m(t)}{t} \to \frac{1}{\mu} \quad \text{as } t \to \infty$$

As before, $1/\mu$ is interpreted as 0 when $\mu = \infty$.

Remark At first glance it might seem that the elementary renewal theorem should be a simple consequence of Proposition 7.1. That is, since the average renewal rate will, with probability 1, converge to $1/\mu$, should this not imply that the expected average renewal rate also converges to $1/\mu$? We must, however, be careful; consider the next example.

Example 7.8 Let U be a random variable which is uniformly distributed on (0, 1); and define the random variables Y_n, $n \geqslant 1$, by

$$Y_n = \begin{cases} 0, & \text{if } U > 1/n \\ n, & \text{if } U \leqslant 1/n \end{cases}$$

Now, since, with probability 1, U will be greater than 0, it follows that Y_n will equal 0 for all sufficiently large n. That is, Y_n will equal 0 for all n large enough so that $1/n < U$. Hence, with probability 1,

$$Y_n \to 0 \quad \text{as } n \to \infty$$

However,

$$E[Y_n] = nP\left\{U \leqslant \frac{1}{n}\right\} = n\frac{1}{n} = 1$$

Therefore, even though the sequence of random variables Y_n converges to 0, the expected values of the Y_n are all identically 1. ■

A key element in the proof of the elementary renewal theorem, which is also of independent interest, is the establishment of a relationship between $m(t)$, the mean number of renewals by time t, and $E[S_{N(t)+1}]$, the expected time of the first renewal after t. Letting

$$g(t) = E[S_{N(t)+1}]$$

we will derive an integral equation, similar to the renewal equation, for $g(t)$ by conditioning on the time of the first renewal. This yields

$$g(t) = \int_0^\infty E[S_{N(t)+1}|X_1 = x]f(x)\,dx$$

where we have supposed that the interarrival times are continuous with density f. Now, if the first renewal occurs at time x and $x > t$, then clearly the time of the first renewal after t is x. On the other hand, if the first renewal occurs at a time $x < t$, then by regarding x as the new origin, it follows that the expected time, from this origin, of the first renewal occurring after a time $t - x$ from this origin is $g(t - x)$. That is, we see that

$$E[S_{N(t)+1}|X_1 = x] = \begin{cases} g(t-x) + x, & \text{if } x < t \\ x, & \text{if } x > t \end{cases}$$

Substituting this into the preceding equation gives

$$\begin{aligned} g(t) &= \int_0^t (g(t-x) + x)\,f(x)\,dx + \int_t^\infty xf(x)\,dx \\ &= \int_0^t g(t-x)\,f(x)\,dx + \int_0^\infty xf(x)\,dx \end{aligned}$$

or

$$g(t) = \mu + \int_0^t g(t-x)\,f(x)\,dx$$

which is quite similar to the renewal equation

$$m(t) = F(t) + \int_0^t m(t-x)f(x)\,ds$$

Indeed, if we let

$$g_1(t) = \frac{g(t)}{\mu} - 1$$

we see that

$$g_1(t) + 1 = 1 + \int_0^t [g_1(t-x) + 1] f(x)\, dx$$

or

$$g_1(t) = F(t) + \int_0^t g_1(t-x) f(x)\, dx$$

That is, $g_1(t) = E[S_{N(t)+1}]/\mu - 1$ satisfies the renewal equation and thus, by uniqueness, must be equal to $m(t)$. We have thus proven the following.

Proposition 7.2

$$E[S_{N(t)+1}] = \mu[m(t) + 1]$$

A second derivation of Proposition 7.2 is given in Exercises 13 and 14. To see how Proposition 7.2 can be used to establish the elementary renewal theorem, let $Y(t)$ denote the time from t until the next renewal. $Y(t)$ is called the *excess*, or *residual life*, at t. As the first renewal after t will occur at time $t + Y(t)$, we see that

$$S_{N(t)+1} = t + Y(t)$$

Taking expectations and utilizing Proposition 7.2 yields

$$\mu[m(t) + 1] = t + E[Y(t)] \tag{7.9}$$

which implies that

$$\frac{m(t)}{t} = \frac{1}{\mu} - \frac{1}{t} + \frac{E[Y(t)]}{t\mu}$$

The elementary renewal theorem can now be proven by showing that

$$\lim_{t\to\infty} \frac{E[Y(t)]}{t} = 0$$

(see Exercise 14).

Relation (7.9) shows that if we can determine $E[Y(t)]$, the mean excess at t, then we can compute $m(t)$ and vice versa.

Example 7.9 Consider the renewal process whose interarrival distribution is the convolution of two exponentials; that is,

$$F = F_1 * F_2, \quad \text{where } F_i(t) = 1 - e^{-\mu_i t}, \quad i = 1, 2$$

We will determine the renewal function by first determining $E[Y(t)]$. To obtain the mean excess at t, imagine that each renewal corresponds to a new machine being put in use, and suppose that each machine has two components—initially component 1 is employed and this lasts an exponential time with rate μ_1, and then component 2, which functions for an exponential time with rate μ_2, is employed. When component 2 fails, a new machine is put in use (that is, a renewal occurs). Now consider the process $\{X(t),\ t \geqslant 0\}$ where $X(t)$ is i if a type i component is in use at time t. It is easy to see that $\{X(t),\ t \geqslant 0\}$ is a two-state continuous-time Markov chain, and so, using the results of Example 6.11, its transition probabilities are

$$P_{11}(t) = \frac{\mu_1}{\mu_1 + \mu_2} e^{-(\mu_1+\mu_2)t} + \frac{\mu_2}{\mu_1 + \mu_2}$$

To compute the expected remaining life of the machine in use at time t, we condition on whether it is using its first or second component: for if it is still using its first component, then its remaining life is $1/\mu_1 + 1/\mu_2$, whereas if it is already using its second component, then its remaining life is $1/\mu_2$. Hence, letting $p(t)$ denote the probability that the machine in use at time t is using its first component, we have

$$\begin{aligned} E[Y(t)] &= \left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) p(t) + \frac{1 - p(t)}{\mu_2} \\ &= \frac{1}{\mu_2} + \frac{p(t)}{\mu_1} \end{aligned}$$

But, since at time 0 the first machine is utilizing its first component, it follows that $p(t) = P_{11}(t)$, and so, upon using the preceding expression of $P_{11}(t)$, we obtain

$$E[Y(t)] = \frac{1}{\mu_2} + \frac{1}{\mu_1 + \mu_2} e^{-(\mu_1+\mu_2)t} + \frac{\mu_2}{\mu_1(\mu_1 + \mu_2)} \tag{7.10}$$

Now it follows from Equation (7.9) that

$$m(t) + 1 = \frac{t}{u} + \frac{E[Y(t)]}{\mu} \tag{7.11}$$

where μ, the mean interarrival time, is given in this case by

$$\mu = \frac{1}{\mu_1} + \frac{1}{\mu_2} = \frac{\mu_1 + \mu_2}{\mu_1 \mu_2}$$

Substituting Equation (7.10) and the preceding equation into (7.11) yields, after simplifying,

$$m(t) = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} t - \frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)^2} [1 - e^{-(\mu_1+\mu_2)t}] \qquad \blacksquare$$

Remark Using the relationship of Equation (7.11) and results from the two-state continuous-time Markov chain, the renewal function can also be obtained in the same manner as in Example 7.9 for the interarrival distributions

$$F(t) = pF_1(t) + (1-p)F_2(t)$$

and

$$F(t) = pF_1(t) + (1-p)(F_1 * F_2)(t)$$

when $F_i(t) = 1 - e^{-\mu_i t}$, $t > 0$, $i = 1, 2$. ■

Suppose the interarrival times of a renewal process are all positive integer valued. Let

$$I_i = \begin{cases} 1, & \text{if there is a renewal at time } i \\ 0, & \text{otherwise} \end{cases}$$

and note that $N(n)$, the number of renewals by time n, can be expressed as

$$N(n) = \sum_{i=1}^{n} I_i$$

Taking expectations of both sides of the preceding shows that

$$m(n) = E\left[N(n)\right] = \sum_{i=1}^{n} P(\text{renewal at time } i)$$

Hence, the elementary renewal theorem yields

$$\frac{\sum_{i=1}^{n} P(\text{renewal at time } i)}{n} \to \frac{1}{E[\text{time between renewals}]}$$

Now, for a sequence of numbers $a_1, a_2, \ldots$ it can be shown that

$$\lim_{n\to\infty} a_n = a \quad \Rightarrow \quad \lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i}{n} = a$$

Hence, if $\lim_{n\to\infty} P(\text{renewal at time } n)$ exists then that limit must equal $\frac{1}{E[\text{time between renewals}]}$.

Example 7.10 Let $X_i, i \geqslant 1$ be independent and identically distributed random variables, and set

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, n > 0$$

The process $\{S_n, n \geqslant 0\}$ is called a *random walk process*. Suppose that $E[X_i] < 0$. The strong law of large numbers yields

$$\lim_{n\to\infty} \frac{S_n}{n} \to E[X_i]$$

But if S_n divided by n is converging to a negative number, then S_n must be going to minus infinity. Let α be the probability that the random walk is always negative after the initial movement. That is,

$$\alpha = P(S_n < 0 \text{ for all } n \geqslant 1)$$

To determine α, define a counting process by saying that an event occurs at time n if $S(n) < \min(0, S_1, \ldots, S_{n-1})$. That is, an event occurs each time the random walk process reaches a new low. Now, if an event occurs at time n, then the next event will occur k time units later if

$$X_{n+1} \geqslant 0, X_{n+1} + X_{n+2} \geqslant 0, \ldots, X_{n+1} + \cdots + X_{n+k-1} \geqslant 0,$$
$$X_{n+1} + \cdots + X_{n+k} < 0$$

Because $X_i, i \geqslant 1$ are independent and identically distributed the preceding event is independent of the values of $X_1, \ldots, X_n$, and its probability of occurrence does not depend on n. Consequently, the times between successive events are independent and identically distributed, showing that the counting process is a renewal process. Now,

$$\begin{aligned} P(\text{renewal at } n) &= P\big(S_n < 0, S_n < S_1, S_n < S_2, \ldots, S_n < S_{n-1}\big) \\ &= P(X_1 + \cdots + X_n < 0, X_2 + \cdots + X_n < 0, \\ &\qquad X_3 + \cdots + X_n < 0, \ldots, X_n < 0) \end{aligned}$$

Because $X_n, X_{n-1}, \ldots, X_1$ has the same joint distribution as does $X_1, X_2, \ldots, X_n$ it follows that the value of the preceding probability would be unchanged if X_1 were replaced by X_n; X_2 were replaced by X_{n-1}; X_3 were replaced by X_{n-2}; and so on. Consequently,

$$\begin{aligned} P(\text{renewal at } n) &= P\big(X_n + \cdots + X_1 < 0, X_{n-1} + \cdots + X_1 < 0, \\ &\qquad X_{n-2} + \cdots + X_1 < 0, X_1 < 0\big) \\ &= P\big(S_n < 0, S_{n-1} < 0, S_{n-2} < 0, \ldots, S_1 < 0\big) \end{aligned}$$

Hence,

$$\lim_{n\to\infty} P(\text{renewal at } n) = P\big(S_n < 0 \text{ for all } n \geqslant 1\big) = \alpha$$

But, by the elementary renewal theorem, this implies that

$$\alpha = 1/E[T]$$

where T is the mean time between renewals. That is,

$$T = \min\{n : S_n < 0\}$$

For instance, in the case of left skip free random walks (which are ones for which $\sum_{j=-1}^{\infty} P(X_i = j) = 1$) we showed in Section 3.6.6 that $E[T] = -1/E[X_i]$ when $E[X_i] < 0$, showing that for skip free random walks having a negative mean,

$$P(S_n < 0 \text{ for all } n) = -E[X_i]$$

which verifies a result previously obtained in Section 3.6.6. ■

An important limit theorem is the central limit theorem for renewal processes. This states that, for large t, $N(t)$ is approximately normally distributed with mean t/μ and variance $t\sigma^2/\mu^3$, where μ and σ^2 are, respectively, the mean and variance of the interarrival distribution. That is, we have the following theorem which we state without proof.

Theorem 7.2 Central Limit Theorem for Renewal Processes

$$\lim_{t\to\infty} P\left\{\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2}\, dx$$

In addition, as might be expected from the central limit theorem for renewal processes, it can be shown that $\mathrm{Var}(N(t))/t$ converges to σ^2/μ^3. That is, it can be shown that

$$\lim_{t\to\infty} \frac{\mathrm{Var}(N(t))}{t} = \sigma^2/\mu^3 \tag{7.12}$$

Example 7.11 Two machines continually process an unending number of jobs. The time that it takes to process a job on machine 1 is a gamma random variable with parameters $n = 4$, $\lambda = 2$, whereas the time that it takes to process a job on machine 2 is uniformly distributed between 0 and 4. Approximate the probability that together the two machines can process at least 90 jobs by time $t = 100$.

Solution: If we let $N_i(t)$ denote the number of jobs that machine i can process by time t, then $\{N_1(t), t \geqslant 0\}$ and $\{N_2(t), t \geqslant 0\}$ are independent renewal processes. The interarrival distribution of the first renewal process is gamma with parameters $n = 4$, $\lambda = 2$, and thus has mean 2 and variance 1.

Correspondingly, the interarrival distribution of the second renewal process is uniform between 0 and 4, and thus has mean 2 and variance 16/12.

Therefore, $N_1(100)$ is approximately normal with mean 50 and variance 100/8; and $N_2(100)$ is approximately normal with mean 50 and variance 100/6. Hence, $N_1(100) + N_2(100)$ is approximately normal with mean 100 and variance 175/6. Thus, with Φ denoting the standard normal distribution function, we have

$$\begin{aligned}
P\{N_1(100)+N_2(100)>89.5\} &= P\left\{\frac{N_1(100)+N_2(100)-100}{\sqrt{175/6}} > \frac{89.5-100}{\sqrt{175/6}}\right\} \\
&\approx 1-\Phi\left(\frac{-10.5}{\sqrt{175/6}}\right) \\
&\approx \Phi\left(\frac{10.5}{\sqrt{175/6}}\right) \\
&\approx \Phi(1.944) \\
&\approx 0.9741
\end{aligned}$$

■

7.4 Renewal Reward Processes

A large number of probability models are special cases of the following model. Consider a renewal process $\{N(t),\ t \geqslant 0\}$ having interarrival times $X_n,\ n \geqslant 1$, and suppose that each time a renewal occurs we receive a reward. We denote by R_n the reward earned at the time of the nth renewal. We shall assume that the $R_n,\ n \geqslant 1$, are independent and identically distributed. However, we do allow for the possibility that R_n may (and usually will) depend on X_n, the length of the nth renewal interval. If we let

$$R(t) = \sum_{n=1}^{N(t)} R_n$$

then $R(t)$ represents the total reward earned by time t. Let

$$E[R] = E[R_n], \quad E[X] = E[X_n]$$

Proposition 7.3 If $E[R] < \infty$ and $E[X] < \infty$, then

(a) with probability 1, $$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E[R]}{E[X]}$$

(b) $$\lim_{t\to\infty} \frac{E[R(t)]}{t} = \frac{E[R]}{E[X]}$$

Proof. We give the proof for (a) only. To prove this, write

$$\frac{R(t)}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{t} = \left(\frac{\sum_{n=1}^{N(t)} R_n}{N(t)}\right)\left(\frac{N(t)}{t}\right)$$

By the strong law of large numbers we obtain

$$\frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \rightarrow E[R] \quad \text{as } t \rightarrow \infty$$

and by Proposition 7.1

$$\frac{N(t)}{t} \rightarrow \frac{1}{E[X]} \quad \text{as } t \rightarrow \infty$$

The result thus follows. ■

Remark

(i) If we say that a *cycle* is completed every time a renewal occurs, then Proposition 7.3 states that the long-run average reward per unit time is equal to the expected reward earned during a cycle divided by the expected length of a cycle. For instance, in Example 7.6 if we suppose that the amounts that the successive customers deposit in the bank are independent random variables having a common distribution H, then the rate at which deposits accumulate—that is, $\lim_{t\to\infty}$ (total deposits by the time $t)/t$—is given by

$$\frac{E[\text{deposits during a cycle}]}{E[\text{time of cycle}]} = \frac{\mu_H}{\mu_G + 1/\lambda}$$

where $\mu_G + 1/\lambda$ is the mean time of a cycle, and μ_H is the mean of the distribution H.

(ii) Although we have supposed that the reward is earned at the time of a renewal, the result remains valid when the reward is earned gradually throughout the renewal cycle.

Example 7.12 (A Car Buying Model) The lifetime of a car is a continuous random variable having a distribution H and probability density h. Mr. Brown has a policy that he buys a new car as soon as his old one either breaks down or reaches the age of T years. Suppose that a new car costs C_1 dollars and also that an additional cost of C_2 dollars is incurred whenever Mr. Brown's car breaks down. Under the assumption that a used car has no resale value, what is Mr. Brown's long-run average cost?

If we say that a cycle is complete every time Mr. Brown gets a new car, then it follows from Proposition 7.3 (with costs replacing rewards) that his long-run average cost equals

$$\frac{E[\text{cost incurred during a cycle}]}{E[\text{length of a cycle}]}$$

Now letting X be the lifetime of Mr. Brown's car during an arbitrary cycle, then the cost incurred during that cycle will be given by

$$\begin{aligned} &C_1, && \text{if } X > T \\ &C_1 + C_2, && \text{if } X \leqslant T \end{aligned}$$

so the expected cost incurred over a cycle is

$$C_1 P\{X > T\} + (C_1 + C_2) P\{X \leqslant T\} = C_1 + C_2 H(T)$$

Also, the length of the cycle is

$$\begin{aligned} &X, && \text{if } X \leqslant T \\ &T, && \text{if } X > T \end{aligned}$$

and so the expected length of a cycle is

$$\int_0^T x h(x)\, dx + \int_T^\infty T h(x)\, dx = \int_0^T x h(x)\, dx + T[1 - H(T)]$$

Therefore, Mr. Brown's long-run average cost will be

$$\frac{C_1 + C_2 H(T)}{\int_0^T x h(x)\, dx + T[1 - H(T)]} \tag{7.13}$$

Now, suppose that the lifetime of a car (in years) is uniformly distributed over (0, 10), and suppose that C_1 is 3 (thousand) dollars and C_2 is $\frac{1}{2}$ (thousand) dollars. What value of T minimizes Mr. Brown's long-run average cost?

If Mr. Brown uses the value T, $T \leqslant 10$, then from Equation (7.13) his long-run average cost equals

$$\begin{aligned} \frac{3 + \frac{1}{2}(T/10)}{\int_0^T (x/10)\, dx + T(1 - T/10)} &= \frac{3 + T/20}{T^2/20 + (10T - T^2)/10} \\ &= \frac{60 + T}{20T - T^2} \end{aligned}$$

We can now minimize this by using the calculus. Toward this end, let

$$g(T) = \frac{60 + T}{20T - T^2}$$

then

$$g'(T) = \frac{(20T - T^2) - (60 + T)(20 - 2T)}{(20T - T^2)^2}$$

Equating to 0 yields

$$20T - T^2 = (60 + T)(20 - 2T)$$

or, equivalently,

$$T^2 + 120\,T - 1200 = 0$$

which yields the solutions

$$T \approx 9.25 \quad \text{and} \quad T \approx -129.25$$

Since $T \leqslant 10$, it follows that the optimal policy for Mr. Brown would be to purchase a new car whenever his old car reaches the age of 9.25 years. ■

Example 7.13 (Dispatching a Train) Suppose that customers arrive at a train depot in accordance with a renewal process having a mean interarrival time μ. Whenever there are N customers waiting in the depot, a train leaves. If the depot incurs a cost at the rate of nc dollars per unit time whenever there are n customers waiting, what is the average cost incurred by the depot?

If we say that a cycle is completed whenever a train leaves, then the preceding is a renewal reward process. The expected length of a cycle is the expected time required for N customers to arrive and, since the mean interarrival time is μ, this equals

$$E[\text{length of cycle}] = N\mu$$

If we let T_n denote the time between the nth and $(n+1)$st arrival in a cycle, then the expected cost of a cycle may be expressed as

$$E[\text{cost of a cycle}] = E[c\,T_1 + 2c\,T_2 + \cdots + (N-1)\,c\,T_{N-1}]$$

which, since $E[T_n] = \mu$, equals

$$c\mu\,\frac{N}{2}(N-1)$$

Hence, the average cost incurred by the depot is

$$\frac{c\mu\,N(N-1)}{2N\mu} = \frac{c(N-1)}{2}$$

Suppose now that each time a train leaves, the depot incurs a cost of six units. What value of N minimizes the depot's long-run average cost when $c = 2$, $\mu = 1$?

In this case, we have that the average cost per unit time N is

$$\frac{6 + c\mu N(N-1)/2}{N\mu} = N - 1 + \frac{6}{N}$$

By treating this as a continuous function of N and using the calculus, we obtain that the minimal value of N is

$$N = \sqrt{6} \approx 2.45$$

Hence, the optimal integral value of N is either 2 which yields a value 4, or 3 which also yields the value 4. Hence, either $N = 2$ or $N = 3$ minimizes the depot's average cost. ■

Example 7.14 Suppose that customers arrive at a single-server system in accordance with a Poisson process with rate λ. Upon arriving a customer must pass through a door that leads to the server. However, each time someone passes through, the door becomes locked for the next t units of time. An arrival finding a locked door is lost, and a cost c is incurred by the system. An arrival finding the door unlocked passes through to the server. If the server is free, the customer enters service; if the server is busy, the customer departs without service and a cost K is incurred. If the service time of a customer is exponential with rate μ, find the average cost per unit time incurred by the system.

Solution: The preceding can be considered to be a renewal reward process, with a new cycle beginning each time a customer arrives to find the door unlocked. This is so because whether or not the arrival finds the server free, the door will become locked for the next t time units and the server will be busy for a time X that is exponentially distributed with rate μ. (If the server is free, X is the service time of the entering customer; if the server is busy, X is the remaining service time of the customer in service.) Since the next cycle will begin at the first arrival epoch after a time t has passed, it follows that

$$E[\text{time of a cycle}] = t + 1/\lambda$$

Let C_1 denote the cost incurred during a cycle due to arrivals finding the door locked. Then, since each arrival in the first t time units of a cycle will result in a cost c, we have

$$E[C_1] = \lambda t c$$

Also, let C_2 denote the cost incurred during a cycle due to an arrival finding the door unlocked but the server busy. Then because a cost K is incurred if the server is still busy a time t after the cycle began and, in addition, the next arrival after that time occurs before the service completion, we see that

$$E[C_2] = Ke^{-\mu t} \frac{\lambda}{\lambda + \mu}$$

Consequently,

$$\text{average cost per unit time} = \frac{\lambda tc + \lambda K e^{-\mu t}/(\lambda + \mu)}{t + 1/\lambda} \qquad \blacksquare$$

Example 7.15 Consider a manufacturing process that sequentially produces items, each of which is either defective or acceptable. The following type of sampling scheme is often employed in an attempt to detect and eliminate most of the defective items. Initially, each item is inspected and this continues until there are k consecutive items that are acceptable. At this point 100% inspection ends and each successive item is independently inspected with probability α. This partial inspection continues until a defective item is encountered, at which time 100% inspection is reinstituted, and the process begins anew. If each item is, independently, defective with probability q,

(a) what proportion of items are inspected?
(b) if defective items are removed when detected, what proportion of the remaining items are defective?

Remark Before starting our analysis, note that the preceding inspection scheme was devised for situations in which the probability of producing a defective item changed over time. It was hoped that 100% inspection would correlate with times at which the defect probability was large and partial inspection when it was small. However, it is still important to see how such a scheme would work in the extreme case where the defect probability remains constant throughout.

Solution: We begin our analysis by noting that we can treat the preceding as a renewal reward process with a new cycle starting each time 100% inspection is instituted. We then have

$$\text{proportion of items inspected} = \frac{E[\text{number inspected in a cycle}]}{E[\text{number produced in a cycle}]}$$

Let N_k denote the number of items inspected until there are k consecutive acceptable items. Once partial inspection begins—that is, after N_k items have been produced—since each inspected item will be defective with probability q, it follows that the expected number that will have to be inspected to find a defective item is $1/q$. Hence,

$$E[\text{number inspected in a cycle}] = E[N_k] + \frac{1}{q}$$

In addition, since at partial inspection each item produced will, independently, be inspected and found to be defective with probability αq, it follows that the number of items produced until one is inspected and found to be defective is $1/\alpha q$, and so

$$E[\text{number produced in a cycle}] = E[N_k] + \frac{1}{\alpha q}$$

Also, as $E[N_k]$ is the expected number of trials needed to obtain k acceptable items in a row when each item is acceptable with probability $p = 1 - q$, it follows from Example 3.14 that

$$E[N_k] = \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^k} = \frac{(1/p)^k - 1}{q}$$

Hence, we obtain

$$P_I \equiv \text{proportion of items that are inspected} = \frac{(1/p)^k}{(1/p)^k - 1 + 1/\alpha}$$

To answer (b), note first that since each item produced is defective with probability q it follows that the proportion of items that are both inspected and found to be defective is qP_{I}. Hence, for N large, out of the first N items produced there will be (approximately) NqP_{I} that are discovered to be defective and thus removed. As the first N items will contain (approximately) Nq defective items, it follows that there will be $Nq - NqP_{\mathrm{I}}$ defective items not discovered. Hence,

$$\text{proportion of the nonremoved items that are defective} \approx \frac{Nq(1 - P_{\mathrm{I}})}{N(1 - qP_{\mathrm{I}})}$$

As the approximation becomes exact as $N \to \infty$, we see that

$$\text{proportion of the nonremoved items that are defective} = \frac{q(1 - P_{\mathrm{I}})}{(1 - qP_{\mathrm{I}})} \qquad \blacksquare$$

Example 7.16 (The Average Age of a Renewal Process) Consider a renewal process having interarrival distribution F and define $A(t)$ to be the time at t since the last renewal. If renewals represent old items failing and being replaced by new ones, then $A(t)$ represents the age of the item in use at time t. Since $S_{N(t)}$ represents the time of the last event prior to or at time t, we have

$$A(t) = t - S_{N(t)}$$

We are interested in the average value of the age—that is, in

$$\lim_{s\to\infty} \frac{\int_0^s A(t)\,dt}{s}$$

To determine this quantity, we use renewal reward theory in the following way: Let us assume that at any time we are being paid money at a rate equal to the age of the renewal process at that time. That is, at time t, we are being paid at rate $A(t)$, and so $\int_0^s A(t)\,dt$ represents our total earnings by time s. As everything starts over again when a renewal occurs, it follows that

$$\frac{\int_0^s A(t)\,dt}{s} \to \frac{E[\text{reward during a renewal cycle}]}{E[\text{time of a renewal cycle}]}$$

Now, since the age of the renewal process a time t into a renewal cycle is just t, we have

$$\begin{aligned}\text{reward during a renewal cycle} &= \int_0^X t\,dt\\ &= \frac{X^2}{2}\end{aligned}$$

where X is the time of the renewal cycle. Hence, we have that

$$\begin{aligned}\text{average value of age} &\equiv \lim_{s\to\infty} \frac{\int_0^s A(t)\,dt}{s}\\ &= \frac{E[X^2]}{2E[X]}\end{aligned} \tag{7.14}$$

where X is an interarrival time having distribution function F. ■

Example 7.17 (The Average Excess of a Renewal Process) Another quantity associated with a renewal process is $Y(t)$, the excess or residual time at time t. $Y(t)$ is defined to equal the time from t until the next renewal and, as such, represents the remaining (or residual) life of the item in use at time t. The average value of the excess, namely,

$$\lim_{s\to\infty} \frac{\int_0^s Y(t)\,dt}{s}$$

also can be easily obtained by renewal reward theory. To do so, suppose that we are paid at time t at a rate equal to $Y(t)$. Then our average reward per unit time will, by renewal reward theory, be given by

$$
\begin{aligned}
\text{average value of excess} &\equiv \lim_{s\to\infty} \frac{\int_0^s Y(t)\,dt}{s} \\
&= \frac{E[\text{reward during a cycle}]}{E[\text{length of a cycle}]}
\end{aligned}
$$

Now, letting X denote the length of a renewal cycle, we have

$$
\begin{aligned}
\text{reward during a cycle} &= \int_0^X (X-t)\,dt \\
&= \frac{X^2}{2}
\end{aligned}
$$

and thus the average value of the excess is

$$
\text{average value of excess} = \frac{E[X^2]}{2E[X]}
$$

which was the same result obtained for the average value of the age of a renewal process. ■

7.5 Regenerative Processes

Consider a stochastic process $\{X(t),\ t \geqslant 0\}$ with state space $0, 1, 2, \ldots$, having the property that there exist time points at which the process (probabilistically) restarts itself. That is, suppose that with probability 1, there exists a time T_1, such that the continuation of the process beyond T_1 is a probabilistic replica of the whole process starting at 0. Note that this property implies the existence of further times $T_2, T_3, \ldots$, having the same property as T_1. Such a stochastic process is known as a *regenerative process*.

From the preceding, it follows that $T_1, T_2, \ldots$, constitute the arrival times of a renewal process, and we shall say that a cycle is completed every time a renewal occurs.

Examples

(1) A renewal process is regenerative, and T_1 represents the time of the first renewal.
(2) A recurrent Markov chain is regenerative, and T_1 represents the time of the first transition into the initial state.

We are interested in determining the long-run proportion of time that a regenerative process spends in state j. To obtain this quantity, let us imagine that we earn a reward at a rate 1 per unit time when the process is in state j and at rate 0 otherwise. That is, if $I(s)$ represents the rate at which we earn at time s, then

$$I(s) = \begin{cases} 1, & \text{if } X(s) = j \\ 0, & \text{if } X(s) \neq j \end{cases}$$

and

$$\text{total reward earned by } t = \int_0^t I(s)\, ds$$

As the preceding is clearly a renewal reward process that starts over again at the cycle time T_1, we see from Proposition 7.3 that

$$\text{average reward per unit time} = \frac{E[\text{reward by time } T_1]}{E[T_1]}$$

However, the average reward per unit is just equal to the proportion of time that the process is in state j. That is, we have the following.

Proposition 7.4 For a regenerative process, the long-run

$$\text{proportion of time in state } j = \frac{E[\text{amount of time in } j \text{ during a cycle}]}{E[\text{time of a cycle}]}$$

Remark If the cycle time T_1 is a continuous random variable, then it can be shown by using an advanced theorem called the "key renewal theorem" that the preceding is equal also to the limiting probability that the system is in state j at time t. That is, if T_1 is continuous, then

$$\lim_{t\to\infty} P\{X(t) = j\} = \frac{E[\text{amount of time in } j \text{ during a cycle}]}{E[\text{time of a cycle}]}$$

Example 7.18 Consider a positive recurrent continuous-time Markov chain that is initially in state i. By the Markovian property, each time the process reenters state i it starts over again. Thus returns to state i are renewals and constitute the beginnings of new cycles. By Proposition 7.4, it follows that the long-run

$$\text{proportion of time in state } j = \frac{E[\text{amount of time in } j \text{ during an } i\text{–}i \text{ cycle}]}{\mu_{ii}}$$

where μ_{ii} represents the mean time to return to state i. If we take j to equal i, then we obtain

$$\text{proportion of time in state } i = \frac{1/v_i}{\mu_{ii}}$$ ■

Example 7.19 (A Queueing System with Renewal Arrivals) Consider a waiting time system in which customers arrive in accordance with an arbitrary renewal process and are served one at time by a single server having an arbitrary service distribution. If we suppose that at time 0 the initial customer has just arrived, then $\{X(t),\ t \geqslant 0\}$ is a regenerative process, where $X(t)$ denotes the number of customers in the system at time t. The process regenerates each time a customer arrives and finds the server free. ■

Example 7.20 Although a system needs only a single machine to function, it maintains an additional machine as a backup. A machine in use functions for a random time with density function f and then fails. If a machine fails while the other one is in working condition, then the latter is put in use and, simultaneously, repair begins on the one that just failed. If a machine fails while the other machine is in repair, then the newly failed machine waits until the repair is completed; at that time the repaired machine is put in use and, simultaneously, repair begins on the recently failed one. All repair times have density function g. Find P_0, P_1, P_2, where P_i is the long-run proportion of time that exactly i of the machines are in working condition.

Solution: Let us say that the system is in state i whenever i machines are in working condition $i = 0, 1, 2$. It is then easy to see that every time the system enters state 1 it probabilistically starts over. That is, the system restarts every time that a machine is put in use while, simultaneously, repair begins on the other one. Say that a cycle begins each time the system enters state 1. If we let X denote the working time of the machine put in use at the beginning of a cycle, and let R be the repair time of the other machine, then the length of the cycle, call it T_c, can be expressed as

$$T_c = \max(X, R)$$

The preceding follows when $X \leqslant R$, because, in this case, the machine in use fails before the other one has been repaired, and so a new cycle begins when that repair is completed. Similarly, it follows when $R < X$, because then the repair occurs first, and so a new cycle begins when the machine in use fails. Also, let T_i, $i = 0, 1, 2$, be the amount of time that the system is in state i during a cycle. Then, because the amount of time during a cycle that neither machine is working is $R - X$ provided that this quantity is positive or 0 otherwise, we have

$$T_0 = (R - X)^+$$

Similarly, because the amount of time during the cycle that a single machine is working is $\min(X, R)$, we have

$$T_1 = \min(X, R)$$

Finally, because the amount of time during the cycle that both machines are working is $X - R$ if this quantity is positive or 0 otherwise, we have

$$T_2 = (X - R)^+$$

Hence, we obtain

$$P_0 = \frac{E[(R - X)^+]}{E[\max(X, R)]}$$

$$P_1 = \frac{E[\min(X, R)]}{E[\max(X, R)]}$$

$$P_2 = \frac{E[(X - R)^+]}{E[\max(X, R)]}$$

That $P_0 + P_1 + P_2 = 1$ follows from the easily checked identity

$$\max(x, r) = \min(x, r) + (x - r)^+ + (r - x)^+$$

The preceding expectations can be computed as follows:

$$\begin{aligned}
E[\max(X, R)] &= \int_0^\infty \int_0^\infty \max(x, r) f(x)\, g(r)\, dx\, dr \\
&= \int_0^\infty \int_0^r r f(x)\, g(r)\, dx\, dr + \int_0^\infty \int_r^\infty x f(x)\, g(r)\, dx\, dr \\
E[(R - X)^+] &= \int_0^\infty \int_0^x (r - x)^+ f(x)\, g(r)\, dx\, dr \\
&= \int_0^\infty \int_0^r (r - x) f(x)\, g(r)\, dx\, dr \\
E[\min(X, R)] &= \int_0^\infty \int_0^\infty \min(x, r)\, f(x)\, g(r)\, dx\, dr \\
&= \int_0^\infty \int_0^r x f(x)\, g(r)\, dx\, dr + \int_0^\infty \int_r^\infty r f(x)\, g(r)\, dx\, dr \\
E[(X - R)^+] &= \int_0^\infty \int_0^x (x - r)\, f(x)\, g(r)\, dr\, dx
\end{aligned}$$

■

7.5.1 Alternating Renewal Processes

Another example of a regenerative process is provided by what is known as an *alternating renewal process*, which considers a system that can be in one of two states: on or off. Initially it is on, and it remains on for a time Z_1; it then goes off and remains off for a time Y_1. It then goes on for a time Z_2; then off for a time Y_2; then on, and so on.

We suppose that the random vectors (Z_n, Y_n), $n \geqslant 1$ are independent and identically distributed. That is, both the sequence of random variables $\{Z_n\}$ and the sequence $\{Y_n\}$ are independent and identically distributed; but we allow Z_n and Y_n to be dependent. In other words, each time the process goes on, everything starts over again, but when it then goes off, we allow the length of the off time to depend on the previous on time.

Let $E[Z] = E[Z_n]$ and $E[Y] = E[Y_n]$ denote, respectively, the mean lengths of an on and off period.

We are concerned with P_{on}, the long-run proportion of time that the system is on. If we let

$$X_n = Y_n + Z_n, \quad n \geqslant 1$$

then at time X_1 the process starts over again. That is, the process starts over again after a complete cycle consisting of an on and an off interval. In other words, a renewal occurs whenever a cycle is completed. Therefore, we obtain from Proposition 7.4 that

$$\begin{aligned} P_{\text{on}} &= \frac{E[Z]}{E[Y] + E[Z]} \\ &= \frac{E[\text{on}]}{E[\text{on}] + E[\text{off}]} \end{aligned} \tag{7.15}$$

Also, if we let P_{off} denote the long-run proportion of time that the system is off, then

$$\begin{aligned} P_{\text{off}} &= 1 - P_{\text{on}} \\ &= \frac{E[\text{off}]}{E[\text{on}] + E[\text{off}]} \end{aligned} \tag{7.16}$$

Example 7.21 (A Production Process) One example of an alternating renewal process is a production process (or a machine) that works for a time Z_1, then breaks down and has to be repaired (which takes a time Y_1), then works for a time Z_2, then is down for a time Y_2, and so on. If we suppose that the process is as good as new after each repair, then this constitutes an alternating renewal process. It is worthwhile to note that in this example it makes sense to suppose that the repair time will depend on the amount of time the process had been working before breaking down. ■

Example 7.22 The rate a certain insurance company charges its policyholders alternates between r_1 and r_0. A new policyholder is initially charged at a rate of

r_1 per unit time. When a policyholder paying at rate r_1 has made no claims for the most recent s time units, then the rate charged becomes r_0 per unit time. The rate charged remains at r_0 until a claim is made, at which time it reverts to r_1. Suppose that a given policyholder lives forever and makes claims at times chosen according to a Poisson process with rate λ, and find

(a) P_i, the proportion of time that the policyholder pays at rate r_i, $i = 0, 1$;
(b) the long-run average amount paid per unit time.

Solution: If we say that the system is "on" when the policyholder pays at rate r_1 and "off" when she pays at rate r_0, then this on–off system is an alternating renewal process with a new cycle starting each time a claim is made. If X is the time between successive claims, then the on time in the cycle is the smaller of s and X. (Note that if $X < s$, then the off time in the cycle is 0.) Since X is exponential with rate λ, the preceding yields

$$\begin{aligned} E[\text{on time in cycle}] &= E[\min(X, s)] \\ &= \int_0^s x\lambda e^{-\lambda x}\,dx + se^{-\lambda s} \\ &= \frac{1}{\lambda}(1 - e^{-\lambda s}) \end{aligned}$$

Since $E[X] = 1/\lambda$, we see that

$$P_1 = \frac{E[\text{on time in cycle}]}{E[X]} = 1 - e^{-\lambda s}$$

and

$$P_0 = 1 - P_1 = e^{-\lambda s}$$

The long-run average amount paid per unit time is

$$r_0P_0 + r_1P_1 = r_1 - (r_1 - r_0)e^{-\lambda s}$$ ■

Example 7.23 (The Age of a Renewal Process) Suppose we are interested in determining the proportion of time that the age of a renewal process is less than some constant c. To do so, let a cycle correspond to a renewal, and say that the system is "on" at time t if the age at t is less than or equal to c, and say it is "off" if the age at t is greater than c. In other words, the system is "on" the first c time units of a renewal interval, and "off" the remaining time. Hence, letting X denote a renewal interval, we have, from Equation (7.15),

$$\begin{aligned} \text{proportion of time age is less than } c &= \frac{E[\min(X, c)]}{E[X]} \\ &= \frac{\int_0^\infty P\{\min(X, c) > x\}\,dx}{E[X]} \end{aligned}$$

$$= \frac{\int_0^c P\{X > x\}\,dx}{E[X]}$$

$$= \frac{\int_0^c (1 - F(x))\,dx}{E[X]} \tag{7.17}$$

where F is the distribution function of X and where we have used the identity that for a nonnegative random variable Y

$$E[Y] = \int_0^\infty P\{Y > x\}\,dx \qquad \blacksquare$$

Example 7.24 (The Excess of a Renewal Process) Let us now consider the long-run proportion of time that the excess of a renewal process is less than c. To determine this quantity, let a cycle correspond to a renewal interval and say that the system is on whenever the excess of the renewal process is greater than or equal to c and that it is off otherwise. In other words, whenever a renewal occurs the process goes on and stays on until the last c time units of the renewal interval when it goes off. Clearly this is an alternating renewal process, and so we obtain Equation (7.16) that

$$\text{long-run proportion of time the excess is less than } c = \frac{E[\text{off time in cycle}]}{E[\text{cycle time}]}$$

If X is the length of a renewal interval, then since the system is off the last c time units of this interval, it follows that the off time in the cycle will equal $\min(X, c)$. Thus,

$$\text{long-run proportion of time the excess is less than } c = \frac{E[\min(X, c)]}{E[X]}$$

$$= \frac{\int_0^c (1 - F(x))\,dx}{E[X]}$$

where the final equality follows from Equation (7.17). Thus, we see from the result of Example 7.23 that the long-run proportion of time that the excess is less than c and the long-run proportion of time that the age is less than c are equal. One way to understand this equivalence is to consider a renewal process that has been in operation for a long time and then observe it going backwards in time. In doing so, we observe a counting process where the times between successive events are independent random variables having distribution F. That is, when we observe a renewal process going backwards in time we again observe a renewal process having the same probability structure as the original. Since the excess (age) at any time for the backwards process corresponds to the age (excess)

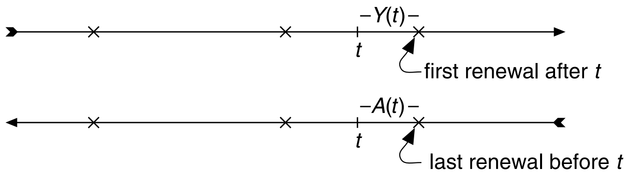

Figure 7.3 Arrowheads indicate direction of time.

at that time for the original renewal process (see Figure 7.3), it follows that all long-run properties of the age and the excess must be equal. ■

Example 7.25 (The Busy Period of the $M/G/\infty$ Queue) The infinite server queueing system in which customers arrive according to a Poisson process with rate λ, and have a general service distribution G, was analyzed in Section 5.3, where it was shown that the number of customers in the system at time t is Poisson distributed with mean $\lambda \int_0^t \bar{G}(y)dy$. If we say that the system is busy when there is at least one customer in the system and is idle when the system is empty, find $E[B]$, the expected length of a busy period.

Solution: If we say that the system is on when there is at least one customer in the system, and off when the system is empty, then we have an alternating renewal process. Because $\int_0^\infty \bar{G}(t)dt = E[S]$, where $E[S]$ is the mean of the service distribution G, it follows from the result of Section 5.3 that

$$\lim_{t\to\infty} P\{\text{system off at } t\} = e^{-\lambda E[S]}$$

Consequently, from alternating renewal process theory we obtain

$$e^{-\lambda E[S]} = \frac{E[\text{off time in cycle}]}{E[\text{cycle time}]}$$

But when the system goes off, it remains off only up to the time of the next arrival, giving that

$$E[\text{off time in cycle}] = 1/\lambda$$

Because

$$E[\text{on time in cycle}] = E[B]$$

we obtain

$$e^{-\lambda E[S]} = \frac{1/\lambda}{1/\lambda + E[B]}$$

or

$$E[B] = \frac{1}{\lambda}\left(e^{\lambda E[S]} - 1\right)$$ ■

If μ is the mean interarrival time, then the distribution function F_e, defined by

$$F_e(x) = \int_0^x \frac{1 - F(y)}{\mu}\, dy$$

is called the *equilibrium distribution* of F. From the preceding, it follows that $F_e(x)$ represents the long-run proportion of time that the age, and the excess, of the renewal process is less than or equal to x.

Example 7.26 (An Inventory Example) Suppose that customers arrive at a specified store in accordance with a renewal process having interarrival distribution F. Suppose that the store stocks a single type of item and that each arriving customer desires a random amount of this commodity, with the amounts desired by the different customers being independent random variables having the common distribution G. The store uses the following (s, S) ordering policy: If its inventory level falls below s then it orders enough to bring its inventory up to S. That is, if the inventory after serving a customer is x, then the amount ordered is

$$\begin{array}{ll} S - x, & \text{if } x < s \\ 0, & \text{if } x \geqslant s \end{array}$$

The order is assumed to be instantaneously filled.

For a fixed value y, $s \leqslant y \leqslant S$, suppose that we are interested in determining the long-run proportion of time that the inventory on hand is at least as large as y. To determine this quantity, let us say that the system is "on" whenever the inventory level is at least y and is "off" otherwise. With these definitions, the system will go on each time that a customer's demand causes the store to place an order that results in its inventory level returning to S. Since whenever this occurs a customer must have just arrived it follows that the times until succeeding customers arrive will constitute a renewal process with interarrival distribution F; that is, the process will start over each time the system goes back on. Thus, the on and off periods so defined constitute an alternating renewal process, and from Equation (7.15) we have that

$$\text{long-run proportion of time inventory} \geqslant y = \frac{E[\text{on time in a cycle}]}{E[\text{cycle time}]} \tag{7.18}$$

Now, if we let $D_1, D_2, \ldots$ denote the successive customer demands, and let

$$N_x = \min(n : D_1 + \cdots + D_n > S - x) \tag{7.19}$$

then it is the N_y customer in the cycle that causes the inventory level to fall below y, and it is the N_s customer that ends the cycle. As a result, if we let X_i, $i \geqslant 1$, denote the interarrival times of customers, then

$$\text{on time in a cycle} = \sum_{i=1}^{N_y} X_i \tag{7.20}$$

$$\text{cycle time} = \sum_{i=1}^{N_s} X_i \tag{7.21}$$

Assuming that the interarrival times are independent of the successive demands, we have that

$$\begin{aligned} E\left[\sum_{i=1}^{N_y} X_i\right] &= E\left[E\left[\sum_{i=1}^{Ny} X_i | N_y\right]\right] \\ &= E[N_y E[X]] \\ &= E[X]E[N_y] \end{aligned}$$

Similarly,

$$E\left[\sum_{i=1}^{N_s} X_i\right] = E[X]E[N_s]$$

Therefore, from Equations (7.18), (7.20), and (7.21) we see that

$$\text{long-run proportion of time inventory} \geqslant y = \frac{E[N_y]}{E[N_s]} \tag{7.22}$$

However, as the D_i, $i \geqslant 1$, are independent and identically distributed nonnegative random variables with distribution G, it follows from Equation (7.19) that N_x has the same distribution as the index of the first event to occur after time $S-x$ of a renewal process having interarrival distribution G. That is, $N_x - 1$ would be the number of renewals by time $S - x$ of this process. Hence, we see that

$$\begin{aligned} E[N_y] &= m(S - y) + 1, \\ E[N_s] &= m(S - s) + 1 \end{aligned}$$

where

$$m(t) = \sum_{n=1}^{\infty} G_n(t)$$

From Equation (7.22), we arrive at

$$\text{long-run proportion of time inventory} \geqslant y = \frac{m(S-y)+1}{m(S-s)+1}, \quad s \leqslant y \leqslant S$$

For instance, if the customer demands are exponentially distributed with mean $1/\mu$, then

$$\text{long-run proportion of time inventory} \geqslant y = \frac{\mu(S-y)+1}{\mu(S-s)+1}, \quad s \leqslant y \leqslant S$$

■

7.6 Semi-Markov Processes

Consider a process that can be in state 1 or state 2 or state 3. It is initially in state 1 where it remains for a random amount of time having mean μ_1, then it goes to state 2 where it remains for a random amount of time having mean μ_2, then it goes to state 3 where it remains for a mean time μ_3, then back to state 1, and so on. What proportion of time is the process in state i, $i = 1, 2, 3$?

If we say that a cycle is completed each time the process returns to state 1, and if we let the reward be the amount of time we spend in state i during that cycle, then the preceding is a renewal reward process. Hence, from Proposition 7.3 we obtain that P_i, the proportion of time that the process is in state i, is given by

$$P_i = \frac{\mu_i}{\mu_1 + \mu_2 + \mu_3}, \quad i = 1, 2, 3$$

Similarly, if we had a process that could be in any of N states $1, 2, \ldots, N$ and that moved from state $1 \to 2 \to 3 \to \cdots \to N-1 \to N \to 1$, then the long-run proportion of time that the process spends in state i is

$$P_i = \frac{\mu_i}{\mu_1 + \mu_2 + \cdots + \mu_N}, \quad i = 1, 2, \ldots, N$$

where μ_i is the expected amount of time the process spends in state i during each visit.

Let us now generalize the preceding to the following situation. Suppose that a process can be in any one of N states $1, 2, \ldots, N$, and that each time it enters state i it remains there for a random amount of time having mean μ_i and then makes a transition into state j with probability P_{ij}. Such a process is called a *semi-Markov process*. Note that if the amount of time that the process spends in each state before making a transition is identically 1, then the semi-Markov process is just a Markov chain.

Let us calculate P_i for a semi-Markov process. To do so, we first consider π_i, the proportion of transitions that take the process into state i. Now, if we let X_n denote the state of the process after the nth transition, then $\{X_n, n \geqslant 0\}$ is a Markov chain with transition probabilities P_{ij}, $i, j = 1, 2, \ldots, N$. Hence, π_i will just be the limiting (or stationary) probabilities for this Markov chain (Section 4.4). That is, π_i will be the unique nonnegative solution* of

$$\sum_{i=1}^{N} \pi_i = 1,$$

$$\pi_i = \sum_{j=1}^{N} \pi_j P_{ji}, \quad i = 1, 2, \ldots, N \tag{7.23}$$

Now, since the process spends an expected time μ_i in state i whenever it visits that state, it seems intuitive that P_i should be a weighted average of the π_i where π_i is weighted proportionately to μ_i. That is,

$$P_i = \frac{\pi_i \mu_i}{\sum_{j=1}^{N} \pi_j \mu_j}, \quad i = 1, 2, \ldots, N \tag{7.24}$$

where the π_i are given as the solution to Equation (7.23).

Example 7.27 Consider a machine that can be in one of three states: *good condition, fair condition*, or *broken down*. Suppose that a machine in good condition will remain this way for a mean time μ_1 and then will go to either the fair condition or the broken condition with respective probabilities $\frac{3}{4}$ and $\frac{1}{4}$. A machine in fair condition will remain that way for a mean time μ_2 and then will break down. A broken machine will be repaired, which takes a mean time μ_3, and when repaired will be in good condition with probability $\frac{2}{3}$ and fair condition with probability $\frac{1}{3}$. What proportion of time is the machine in each state?

Solution: Letting the states be 1, 2, 3, we have by Equation (7.23) that the π_i satisfy

$$\pi_1 + \pi_2 + \pi_3 = 1,$$
$$\pi_1 = \frac{2}{3}\pi_3,$$
$$\pi_2 = \frac{3}{4}\pi_1 + \frac{1}{3}\pi_3,$$
$$\pi_3 = \frac{1}{4}\pi_1 + \pi_2$$

* We shall assume that there exists a solution of Equation (7.23). That is, we assume that all of the states in the Markov chain communicate.

The solution is

$$\pi_1 = \frac{4}{15}, \quad \pi_2 = \frac{1}{3}, \quad \pi_3 = \frac{2}{5}$$

Hence, from Equation (7.24) we obtain that P_i, the proportion of time the machine is in state i, is given by

$$P_1 = \frac{4\mu_1}{4\mu_1 + 5\mu_2 + 6\mu_3},$$

$$P_2 = \frac{5\mu_2}{4\mu_1 + 5\mu_2 + 6\mu_3},$$

$$P_3 = \frac{6\mu_3}{4\mu_1 + 5\mu_2 + 6\mu_3}$$

For instance, if $\mu_1 = 5$, $\mu_2 = 2$, $\mu_3 = 1$, then the machine will be in good condition $\frac{5}{9}$ of the time, in fair condition $\frac{5}{18}$ of the time, in broken condition $\frac{1}{6}$ of the time. ■

Remark When the distributions of the amount of time spent in each state during a visit are continuous, then P_i also represents the limiting (as $t \to \infty$) probability that the process will be in state i at time t.

Example 7.28 Consider a renewal process in which the interarrival distribution is discrete and is such that

$$P\{X = i\} = p_i, \quad i \geqslant 1$$

where X represents an interarrival random variable. Let $L(t)$ denote the length of the renewal interval that contains the point t (that is, if $N(t)$ is the number of renewals by time t and X_n the nth interarrival time, then $L(t) = X_{N(t)+1}$). If we think of each renewal as corresponding to the failure of a lightbulb (which is then replaced at the beginning of the next period by a new bulb), then $L(t)$ will equal i if the bulb in use at time t dies in its ith period of use.

It is easy to see that $L(t)$ is a semi-Markov process. To determine the proportion of time that $L(t) = j$, note that each time a transition occurs—that is, each time a renewal occurs—the next state will be j with probability p_j. That is, the transition probabilities of the embedded Markov chain are $P_{ij} = p_j$. Hence, the limiting probabilities of this embedded chain are given by

$$\pi_j = p_j$$

and, since the mean time the semi-Markov process spends in state j before a transition occurs is j, it follows that the long-run proportion of time the state is j is

$$P_j = \frac{jp_j}{\sum_i ip_i} \qquad \blacksquare$$

7.7 The Inspection Paradox

Suppose that a piece of equipment, say, a battery, is installed and serves until it breaks down. Upon failure it is instantly replaced by a like battery, and this process continues without interruption. Letting $N(t)$ denote the number of batteries that have failed by time t, we have that $\{N(t), t \geqslant 0\}$ is a renewal process.

Suppose further that the distribution F of the lifetime of a battery is not known and is to be estimated by the following sampling inspection scheme. We fix some time t and observe the total lifetime of the battery that is in use at time t. Since F is the distribution of the lifetime for all batteries, it seems reasonable that it should be the distribution for this battery. However, this is the *inspection paradox* for it turns out that the *battery in use at time t tends to have a larger lifetime than an ordinary battery*.

To understand the preceding so-called paradox, we reason as follows. In renewal theoretic terms what we are interested in is the length of the renewal interval containing the point t. That is, we are interested in $X_{N(t)+1} = S_{N(t)+1} - S_{N(t)}$ (see Figure 7.2). To calculate the distribution of $X_{N(t)+1}$ we condition on the time of the last renewal prior to (or at) time t. That is,

$$P\{X_{N(t)+1} > x\} = E[P\{X_{N(t)+1} > x | S_{N(t)} = t - s\}]$$

where we recall (Figure 7.2) that $S_{N(t)}$ is the time of the last renewal prior to (or at) t. Since there are no renewals between $t-s$ and t, it follows that $X_{N(t)+1}$ must be larger than x if $s > x$. That is,

$$P\{X_{N(t)+1} > x | S_{N(t)} = t - s\} = 1 \quad \text{if } s > x \tag{7.25}$$

On the other hand, suppose that $s \leqslant x$. As before, we know that a renewal occurred at time $t - s$ and no additional renewals occurred between $t - s$ and t, and we ask for the probability that no renewals occur for an additional time $x - s$. That is, we are asking for the probability that an interarrival time will be greater than x given that it is greater than s. Therefore, for $s \leqslant x$,

$$\begin{aligned}
&P\{X_{N(t)+1} > x | S_{N(t)} = t - s\} \\
&\quad = P\{\text{interarrival time} > x | \text{interarrival time} > s\} \\
&\quad = P\{\text{interarrival time} > x\}/P\{\text{interarrival time} > s\}
\end{aligned}$$

$$= \frac{1 - F(x)}{1 - F(s)}$$
$$\geqslant 1 - F(x) \tag{7.26}$$

Hence, from Equations (7.25) and (7.26) we see that, for all s,

$$P\{X_{N(t)+1} > x | S_{N(t)} = t - s\} \geqslant 1 - F(x)$$

Taking expectations on both sides yields

$$P\{X_{N(t)+1} > x\} \geqslant 1 - F(x) \tag{7.27}$$

However, $1 - F(x)$ is the probability that an ordinary renewal interval is larger than x, that is, $1 - F(x) = P\{X_n > x\}$, and thus Equation (7.27) is a statement of the inspection paradox that the length of the renewal interval containing the point t tends to be larger than an ordinary renewal interval.

Remark To obtain an intuitive feel for the so-called inspection paradox, reason as follows. We think of the whole line being covered by renewal intervals, one of which covers the point t. Is it not more likely that a larger interval, as opposed to a shorter interval, covers the point t?

We can explicitly calculate the distribution of $X_{N(t)+1}$ when the renewal process is a Poisson process. (Note that, in the general case, we did not need to calculate explicitly $P\{X_{N(t)+1} > x\}$ to show that it was at least as large as $1 - F(x)$.) To do so we write

$$X_{N(t)+1} = A(t) + Y(t)$$

where $A(t)$ denotes the time from t since the last renewal, and $Y(t)$ denotes the time from t until the next renewal (see Figure 7.4). $A(t)$ is the *age* of the process at time t (in our example it would be the age at time t of the battery in use at time t), and $Y(t)$ is the *excess life* of the process at time t (it is the additional time from t until the battery fails). Of course, it is true that $A(t) = t - S_{N(t)}$, and $Y(t) = S_{N(t)+1} - t$.

To calculate the distribution of $X_{N(t)+1}$ we first note the important fact that, for a Poisson process, $A(t)$ and $Y(t)$ are independent. This follows since by the memoryless property of the Poisson process, the time from t until the next renewal will be exponentially distributed and will be independent of all that has previously

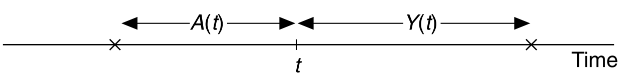

Figure 7.4

occurred (including, in particular, $A(t)$). In fact, this shows that if $\{N(t),\ t \geqslant 0\}$ is a Poisson process with rate λ, then

$$P\{Y(t) \leqslant x\} = 1 - e^{-\lambda x} \tag{7.28}$$

The distribution of $A(t)$ may be obtained as follows

$$\begin{aligned} P\{A(t) > x\} &= \begin{cases} P\{0 \text{ renewals in } [t-x,\ t]\}, & \text{if } x \leqslant t \\ 0, & \text{if } x > t \end{cases} \\ &= \begin{cases} e^{-\lambda x}, & \text{if } x \leqslant t \\ 0, & \text{if } x > t \end{cases} \end{aligned}$$

or, equivalently,

$$P\{A(t) \leqslant x\} = \begin{cases} 1 - e^{-\lambda x}, & x \leqslant t \\ 1, & x > t \end{cases} \tag{7.29}$$

Hence, by the independence of $Y(t)$ and $A(t)$ the distribution of $X_{N(t)+1}$ is just the convolution of the exponential distribution seen in Equation (7.28) and the distribution of Equation (7.29). It is interesting to note that for t large, $A(t)$ approximately has an exponential distribution. Thus, for t large, $X_{N(t)+1}$ has the distribution of the convolution of two identically distributed exponential random variables, which by Section 5.2.3 is the gamma distribution with parameters $(2, \lambda)$. In particular, for t large, the expected length of the renewal interval containing the point t is approximately *twice* the expected length of an ordinary renewal interval.

Using the results obtained in Examples 7.16 and 7.17 concerning the average values of the age and of the excess, it follows from the identity

$$X_{N(t)+1} = A(t) + Y(t)$$

that the average length of the renewal interval containing a specified point is

$$\lim_{s\to\infty} \frac{\int_0^s X_{N(t)+1}\,dt}{s} = \frac{E[X^2]}{E[X]}$$

where X has the interarrival distribution. Because, except for when X is a constant, $E[X^2] > (E[X])^2$, this average value is, as expected from the inspection paradox, greater than the expected value of an ordinary renewal interval.

We can use an alternating renewal process argument to determine the long-run proportion of time that $X_{N(t)+1}$ is greater than c. To do so, let a cycle correspond to a renewal interval, and say that the system is on at time t if the renewal interval containing t is of length greater than c (that is, if $X_{N(t)+1} > c$), and say that the system is off at time t otherwise. In other words, the system is always on during a cycle if the cycle time exceeds c or is always off during the cycle if the cycle time is less than c. Thus, if X is the cycle time, we have

$$\text{on time in cycle} = \begin{cases} X, & \text{if } X > c \\ 0, & \text{if } X \leqslant c \end{cases}$$

Therefore, we obtain from alternating renewal process theory that

$$\begin{aligned}\text{long-run proportion of time } X_{N(t)+1} > c &= \frac{E[\text{on time in cycle}]}{E[\text{cycle time}]} \\ &= \frac{\int_c^\infty xf(x)\,dx}{\mu}\end{aligned}$$

where f is the density function of an interarrival.

7.8 Computing the Renewal Function

The difficulty with attempting to use the identity

$$m(t) = \sum_{n=1}^{\infty} F_n(t)$$

to compute the renewal function is that the determination of $F_n(t) = P\{X_1 + \cdots + X_n \leqslant t\}$ requires the computation of an n-dimensional integral. Following, we present an effective algorithm that requires as inputs only one-dimensional integrals.

Let Y be an exponential random variable having rate λ, and suppose that Y is independent of the renewal process $\{N(t), t \geqslant 0\}$. We start by determining $E[N(Y)]$, the expected number of renewals by the random time Y. To do so, we first condition on X_1, the time of the first renewal. This yields

$$E[N(Y)] = \int_0^\infty E[N(Y)|X_1 = x]f(x)\,dx \tag{7.30}$$

where f is the interarrival density. To determine $E[N(Y)|X_1 = x]$, we now condition on whether or not Y exceeds x. Now, if $Y < x$, then as the first renewal occurs at time x, it follows that the number of renewals by time Y is equal to 0. On the other hand, if we are given that $x < Y$, then the number of renewals by time Y will equal 1 (the one at x) plus the number of additional renewals between x and Y. But by the memoryless property of exponential random variables, it follows that, given that $Y > x$, the amount by which it exceeds x is also exponential with rate λ, and so given that $Y > x$ the number of renewals between x and Y will have the same distribution as $N(Y)$. Hence,

$$E[N(Y)|X_1 = x,\ Y < x] = 0,$$
$$E[N(Y)|X_1 = x,\ Y > x] = 1 + E[N(Y)]$$

and so,

$$\begin{aligned} E[N(Y)|X_1 = x] &= E[N(Y)|X_1 = x,\ Y < x]P\{Y < x|X_1 = x\} \\ &\quad + E[N(Y)|X_1 = x,\ Y > x]P\{Y > x|X_1 = x\} \\ &= E[N(Y)|X_1 = x,\ Y > x]P\{Y > x\} \\ &\qquad \text{since } Y \text{ and } X_1 \text{ are independent} \\ &= (1 + E[N(Y)])e^{-\lambda x} \end{aligned}$$

Substituting this into Equation (7.30) gives

$$E[N(Y)] = (1 + E[N(Y)]) \int_0^\infty e^{-\lambda x} f(x)\, dx$$

or

$$E[N(Y)] = \frac{E[e^{-\lambda X}]}{1 - E[e^{-\lambda X}]} \tag{7.31}$$

where X has the renewal interarrival distribution.

If we let $\lambda = 1/t$, then Equation (7.31) presents an expression for the expected number of renewals (not by time t, but) by a random exponentially distributed time with mean t. However, as such a random variable need not be close to its mean (its variance is t^2), Equation (7.31) need not be particularly close to $m(t)$. To obtain an accurate approximation suppose that $Y_1, Y_2, \ldots, Y_n$ are independent exponentials with rate λ and suppose they are also independent of the renewal process. Let, for $r = 1, \ldots, n$,

$$m_r = E[N(Y_1 + \cdots + Y_r)]$$

To compute an expression for m_r, we again start by conditioning on X_1, the time of the first renewal:

$$m_r = \int_0^\infty E[N(Y_1 + \cdots + Y_r)|X_1 = x] f(x)\, dx \tag{7.32}$$

To determine the foregoing conditional expectation, we now condition on the number of partial sums $\sum_{i=1}^{j} Y_i$, $j = 1, \ldots, r$, that are less than x. Now, if all r

partial sums are less than x—that is, if $\sum_{i=1}^{r} Y_i < x$—then clearly the number of renewals by time $\sum_{i=1}^{r} Y_i$ is 0. On the other hand, given that $k, k < r$, of these partial sums are less than x, it follows from the lack of memory property of the exponential that the number of renewals by time $\sum_{i=1}^{r} Y_i$ will have the same distribution as 1 plus $N(Y_{k+1} + \cdots + Y_r)$. Hence,

$$E\left[N(Y_1 + \cdots + Y_r)\Big|X_1 = x,\ k \text{ of the sums } \sum_{i=1}^{j} Y_i \text{ are less than } x\right]$$
$$= \begin{cases} 0, & \text{if } k = r \\ 1 + m_{r-k}, & \text{if } k < r \end{cases} \tag{7.33}$$

To determine the distribution of the number of the partial sums that are less than x, note that the successive values of these partial sums $\sum_{i=1}^{j} Y_i, j = 1, \ldots, r$, have the same distribution as the first r event times of a Poisson process with rate λ (since each successive partial sum is the previous sum plus an independent exponential with rate λ). Hence, it follows that, for $k < r$,

$$P\left\{k \text{ of the partial sums } \sum_{i=1}^{j} Y_i \text{ are less than } x\Big|X_1 = x\right\}$$
$$= \frac{e^{-\lambda x}(\lambda x)^k}{k!} \tag{7.34}$$

Upon substitution of Equations (7.33) and (7.34) into Equation (7.32), we obtain

$$m_r = \int_0^{\infty} \sum_{k=0}^{r-1} (1 + m_{r-k}) \frac{e^{-\lambda x}(\lambda x)^k}{k!} f(x)\, dx$$

or, equivalently,

$$m_r = \frac{\sum_{k=1}^{r-1}(1 + m_{r-k}) E\left[X^k e^{-\lambda X}\right](\lambda^k/k!) + E\left[e^{-\lambda X}\right]}{1 - E\left[e^{-\lambda X}\right]} \tag{7.35}$$

If we set $\lambda = n/t$, then starting with m_1 given by Equation (7.31), we can use Equation (7.35) to recursively compute $m_2, \ldots, m_n$. The approximation of $m(t) = E[N(t)]$ is given by $m_n = E[N(Y_1 + \cdots + Y_n)]$. Since $Y_1 + \cdots + Y_n$ is the sum of n independent exponential random variables each with mean t/n, it follows that it is (gamma) distributed with mean t and variance $nt^2/n^2 = t^2/n$. Hence, by choosing n large, $\sum_{i=1}^{n} Y_i$ will be a random variable having most of its probability concentrated about t, and so $E\left[N\left(\sum_{i=1}^{n} Y_i\right)\right]$ should be quite close to $E[N(t)]$. (Indeed, if $m(t)$ is continuous at t, it can be shown that these approximations converge to $m(t)$ as n goes to infinity.)

Table 7.1 Approximating $m(t)$

F_i		Exact	Approximation				
i	t	$m(t)$	$n = 1$	$n = 3$	$n = 10$	$n = 25$	$n = 50$
1	1	0.2838	0.3333	0.3040	0.2903	0.2865	0.2852
1	2	0.7546	0.8000	0.7697	0.7586	0.7561	0.7553
1	5	2.250	2.273	2.253	2.250	2.250	2.250
1	10	4.75	4.762	4.751	4.750	4.750	4.750
2	0.1	0.1733	0.1681	0.1687	0.1689	0.1690	—
2	0.3	0.5111	0.4964	0.4997	0.5010	0.5014	—
2	0.5	0.8404	0.8182	0.8245	0.8273	0.8281	0.8283
2	1	1.6400	1.6087	1.6205	1.6261	1.6277	1.6283
2	3	4.7389	4.7143	4.7294	4.7350	4.7363	4.7367
2	10	15.5089	15.5000	15.5081	15.5089	15.5089	15.5089
3	0.1	0.2819	0.2692	0.2772	0.2804	0.2813	—
3	0.3	0.7638	0.7105	0.7421	0.7567	0.7609	—
3	1	2.0890	2.0000	2.0556	2.0789	2.0850	2.0870
3	3	5.4444	5.4000	5.4375	5.4437	5.4442	5.4443

Example 7.29 Table 7.1 compares the approximation with the exact value for the distributions F_i with densities f_i, $i = 1, 2, 3$, which are given by

$$\begin{aligned} f_1(x) &= xe^{-x}, \\ 1 - F_2(x) &= 0.3e^{-x} + 0.7e^{-2x}, \\ 1 - F_3(x) &= 0.5e^{-x} + 0.5e^{-5x} \end{aligned}$$

■

7.9 Applications to Patterns

A counting process with independent interarrival times $X_1, X_2, \ldots$ is said to be a *delayed* or *general* renewal process if X_1 has a different distribution from the identically distributed random variables $X_2, X_3, \ldots$. That is, a delayed renewal process is a renewal process in which the first interarrival time has a different distribution than the others. Delayed renewal processes often arise in practice and it is important to note that all of the limiting theorems about $N(t)$, the number of events by time t, remain valid. For instance, it remains true that

$$\frac{E[N(t)]}{t} \to \frac{1}{\mu} \quad \text{and} \quad \frac{\text{Var}(N(t))}{t} \to \sigma^2/\mu^3 \quad \text{as } t \to \infty$$

where μ and σ^2 are the expected value and variance of the interarrivals X_i, $i > 1$.

7.9.1 Patterns of Discrete Random Variables

Let $X_1, X_2, \ldots$ be independent with $P\{X_i = j\} = p(j)$, $j \geqslant 0$, and let T denote the first time the pattern $x_1, \ldots, x_r$ occurs. If we say that a renewal occurs at time n, $n \geqslant r$, if $(X_{n-r+1}, \ldots, X_n) = (x_1, \ldots, x_r)$, then $N(n)$, $n \geqslant 1$, is a delayed renewal process, where $N(n)$ denotes the number of renewals by time n. It follows that

$$\frac{E[N(n)]}{n} \to \frac{1}{\mu} \quad \text{as } n \to \infty \tag{7.36}$$

$$\frac{\operatorname{Var}(N(n))}{n} \to \frac{\sigma^2}{\mu^3} \quad \text{as } n \to \infty \tag{7.37}$$

where μ and σ are, respectively, the mean and standard deviation of the time between successive renewals. Whereas, in Section 3.6.4, we showed how to compute the expected value of T, we will now show how to use renewal theory results to compute both its mean and its variance.

To begin, let $I(i)$ equal 1 if there is a renewal at time i and let it be 0 otherwise, $i \geqslant r$. Also, let $p = \prod_{i=1}^{r} p(x_i)$. Since,

$$P\{I(i) = 1\} = P\{X_{i-r+1} = i_1, \ldots, X_i = i_r\} = p$$

it follows that $I(i)$, $i \geqslant r$, are Bernoulli random variables with parameter p. Now,

$$N(n) = \sum_{i=r}^{n} I(i)$$

so

$$E[N(n)] = \sum_{i=r}^{n} E[I(i)] = (n - r + 1)p$$

Dividing by n and then letting $n \to \infty$ gives, from Equation (7.36),

$$\mu = 1/p \tag{7.38}$$

That is, the mean time between successive occurrences of the pattern is equal to $1/p$. Also,

$$\begin{aligned}
\frac{\operatorname{Var}(N(n))}{n} &= \frac{1}{n} \sum_{i=r}^{n} \operatorname{Var}(I(i)) + \frac{2}{n} \sum_{i=r}^{n-1} \sum_{n \geqslant j > i} \operatorname{Cov}(I(i), I(j)) \\
&= \frac{n - r + 1}{n} p(1 - p) + \frac{2}{n} \sum_{i=r}^{n-1} \sum_{i < j \leqslant \min(i+r-1, n)} \operatorname{Cov}(I(i), I(j))
\end{aligned}$$

where the final equality used the fact that $I(i)$ and $I(j)$ are independent, and thus have zero covariance, when $|i-j| \geqslant r$. Letting $n \to \infty$, and using the fact that $\mathrm{Cov}(I(i), I(j))$ depends on i and j only through $|j-i|$, gives

$$\frac{\mathrm{Var}(N(n))}{n} \quad \to \quad p(1-p) + 2\sum_{j=1}^{r-1} \mathrm{Cov}(I(r), I(r+j))$$

Therefore, using Equations (7.37) and (7.38), we see that

$$\sigma^2 = p^{-2}(1-p) + 2p^{-3}\sum_{j=1}^{r-1} \mathrm{Cov}(I(r), I(r+j)) \tag{7.39}$$

Let us now consider the amount of "overlap" in the pattern. The overlap, equal to the number of values at the end of one pattern that can be used as the beginning part of the next pattern, is said to be of size k, $k > 0$, if

$$k = \max\{j < r : (i_{r-j+1}, \ldots, i_r) = (i_1, \ldots, i_j)\}$$

and is of size 0 if for all $k = 1, \ldots, r-1$, $(i_{r-k+1}, \ldots, i_r) \neq (i_1, \ldots, i_k)$. Thus, for instance, the pattern 0, 0, 1, 1 has overlap 0, whereas 0, 0, 1, 0, 0 has overlap 2. We consider two cases.

Case 1: The Pattern Has Overlap 0

In this case, $N(n)$, $n \geqslant 1$, is an ordinary renewal process and T is distributed as an interarrival time with mean μ and variance σ^2. Hence, we have the following from Equation (7.38):

$$E[T] = \mu = \frac{1}{p} \tag{7.40}$$

Also, since two patterns cannot occur within a distance less than r of each other, it follows that $I(r)I(r+j) = 0$ when $1 \leqslant j \leqslant r-1$. Hence,

$$\mathrm{Cov}(I(r), I(r+j)) = -E[I(r)]E[I(r+j)] = -p^2, \quad \text{if } 1 \leqslant j \leqslant r-1$$

Hence, from Equation (7.39) we obtain

$$\mathrm{Var}(T) = \sigma^2 = p^{-2}(1-p) - 2p^{-3}(r-1)p^2 = p^{-2} - (2r-1)p^{-1} \tag{7.41}$$

Remark In cases of "rare" patterns, if the pattern hasn't yet occurred by some time n, then it would seem that we would have no reason to believe that the remaining time would be much less than if we were just beginning from scratch. That is, it would seem that the distribution is approximately memoryless and would thus be approximately exponentially distributed. Thus, since the variance of an exponential is equal to its mean squared, we would expect when μ is large that $\mathrm{Var}(T) \approx E^2[T]$, and this is borne out by the preceding, which states that $\mathrm{Var}(T) = E^2[T] - (2r-1)E[T]$.

Example 7.30 Suppose we are interested in the number of times that a fair coin needs to be flipped before the pattern h, h, t, h, t occurs. For this pattern, $r = 5$, $p = \frac{1}{32}$, and the overlap is 0. Hence, from Equations (7.40) and (7.41)

$$E[T] = 32, \quad \mathrm{Var}(T) = 32^2 - 9 \times 32 = 736,$$

and

$$\mathrm{Var}(T)/E^2[T] = 0.71875$$

On the other hand, if $p(i) = i/10$, $i = 1, 2, 3, 4$ and the pattern is 1, 2, 1, 4, 1, 3, 2 then $r = 7$, $p = 3/625{,}000$, and the overlap is 0. Thus, again from Equations (7.40) and (7.41), we see that in this case

$$E[T] = 208{,}333.33, \quad \mathrm{Var}(T) = 4.34 \times 10^{10},$$
$$\mathrm{Var}(T)/E^2[T] = 0.99994$$ ■

Case 2: The Overlap Is of Size k

In this case,

$$T = T_{i_1,\ldots,i_k} + T^*$$

where $T_{i_1,\ldots,i_k}$ is the time until the pattern $i_1, \ldots, i_k$ appears and T^*, distributed as an interarrival time of the renewal process, is the additional time that it takes, starting with $i_1, \ldots, i_k$, to obtain the pattern $i_1, \ldots, i_r$. Because these random variables are independent, we have

$$E[T] = E[T_{i_1,\ldots,i_k}] + E[T^*] \tag{7.42}$$

$$\mathrm{Var}(T) = \mathrm{Var}(T_{i_1,\ldots,i_k}) + \mathrm{Var}(T^*) \tag{7.43}$$

Now, from Equation (7.38)

$$E[T^*] = \mu = p^{-1} \tag{7.44}$$

Also, since no two renewals can occur within a distance $r-k-1$ of each other, it follows that $I(r)I(r+j)=0$ if $1 \leqslant j \leqslant r-k-1$. Therefore, from Equation (7.39) we see that

$$\begin{aligned}\operatorname{Var}(T^*) = \sigma^2 &= p^{-2}(1-p) + 2p^{-3}\left(\sum_{j=r-k}^{r-1} E[I(r)I(r+j)] - (r-1)p^2\right)\\ &= p^{-2} - (2r-1)p^{-1} + 2p^{-3}\sum_{j=r-k}^{r-1} E[I(r)I(r+j)] \end{aligned} \tag{7.45}$$

The quantities $E[I(r)I(r+j)]$ in Equation (7.45) can be calculated by considering the particular pattern. To complete the calculation of the first two moments of T, we then compute the mean and variance of $T_{i_1,\ldots,i_k}$ by repeating the same method.

Example 7.31 Suppose that we want to determine the number of flips of a fair coin until the pattern h, h, t, h, h occurs. For this pattern, $r = 5$, $p = \frac{1}{32}$, and the overlap parameter is $k = 2$. Because

$$E[I(5)I(8)] = P\{h,h,t,h,h,t,h,h\} = \tfrac{1}{256},$$

$$E[I(5)I(9)] = P\{h,h,t,h,h,h,t,h,h\} = \tfrac{1}{512}$$

we see from Equations (7.44) and (7.45) that

$$E[T^*] = 32,$$

$$\operatorname{Var}(T^*) = (32)^2 - 9(32) + 2(32)^3\left(\tfrac{1}{256} + \tfrac{1}{512}\right) = 1120$$

Hence, from Equations (7.42) and (7.43) we obtain

$$E[T] = E[T_{h,h}] + 32, \quad \operatorname{Var}(T) = \operatorname{Var}(T_{h,h}) + 1120$$

Now, consider the pattern h, h. It has $r = 2$, $p = \frac{1}{4}$, and overlap parameter 1. Since, for this pattern, $E[I(2)I(3)] = \frac{1}{8}$, we obtain, as in the preceding, that

$$E[T_{h,h}] = E[T_h] + 4,$$

$$\operatorname{Var}(T_{h,h}) = \operatorname{Var}(T_h) + 16 - 3(4) + 2\left(\tfrac{64}{8}\right) = \operatorname{Var}(T_h) + 20$$

Finally, for the pattern h, which has $r = 1, p = \frac{1}{2}$, we see from Equations (7.40) and (7.41) that

$$E[T_h] = 2, \quad \operatorname{Var}(T_h) = 2$$

Putting it all together gives

$$E[T] = 38, \quad \mathrm{Var}(T) = 1142, \quad \mathrm{Var}(T)/E^2[T] = 0.79086 \qquad \blacksquare$$

Example 7.32 Suppose that $P\{X_n = i\} = p_i$, and consider the pattern 0, 1, 2, 0, 1, 3, 0, 1. Then $p = p_0^3 p_1^3 p_2 p_3$, $r = 8$, and the overlap parameter is $k = 2$. Since

$$E[I(8)I(14)] = p_0^5 p_1^5 p_2^2 p_3^2,$$
$$E[I(8)I(15)] = 0$$

we see from Equations (7.42) and (7.44) that

$$E[T] = E[T_{0,1}] + p^{-1}$$

and from Equations (7.43) and (7.45) that

$$\mathrm{Var}(T) = \mathrm{Var}(T_{0,1}) + p^{-2} - 15p^{-1} + 2p^{-1}(p_0 p_1)^{-1}$$

Now, the r and p values of the pattern 0, 1 are $r(0,1) = 2$, $p(0,1) = p_0 p_1$, and this pattern has overlap 0. Hence, from Equations (7.40) and (7.41),

$$E[T_{0,1}] = (p_0 p_1)^{-1}, \quad \mathrm{Var}(T_{0,1}) = (p_0 p_1)^{-2} - 3(p_0 p_1)^{-1}$$

For instance, if $p_i = 0.2$, $i = 0, 1, 2, 3$ then

$$E[T] = 25 + 5^8 = 390{,}650$$
$$\mathrm{Var}(T) = 625 - 75 + 5^{16} + 35 \times 5^8 = 1.526 \times 10^{11}$$
$$\mathrm{Var}(T)/E^2[T] = 0.99996 \qquad \blacksquare$$

Remark It can be shown that T is a type of discrete random variable called *new better than used* (NBU), which loosely means that if the pattern has not yet occurred by some time n then the additional time until it occurs tends to be less than the time it would take the pattern to occur if one started all over at that point. Such a random variable is known to satisfy (see Proposition 9.6.1 of Ref. [4])

$$\mathrm{Var}(T) \leqslant E^2[T] - E[T] \leqslant E^2[T] \qquad \blacksquare$$

Now, suppose that there are s patterns, $A(1), \ldots, A(s)$ and that we are interested in the mean time until one of these patterns occurs, as well as the probability mass function of the one that occurs first. Let us assume, without any loss of generality, that none of the patterns is contained in any of the others. (That is, we rule out such trivial cases as $A(1) = h, h$ and $A(2) = h, h, t$.) To determine the quantities of interest, let $T(i)$ denote the time until pattern $A(i)$ occurs, $i = 1, \ldots, s$, and let $T(i, j)$ denote the additional time, starting with the occurrence of pattern $A(i)$, until pattern $A(j)$ occurs, $i \neq j$. Start by computing the expected values of these random variables. We have already shown how to compute $E[T(i)]$, $i = 1, \ldots, s$. To compute $E[T(i, j)]$, use the same approach, taking into account any "overlap" between the latter part of $A(i)$ and the beginning part of $A(j)$. For instance, suppose $A(1) = 0, 0, 1, 2, 0, 3$, and $A(2) = 2, 0, 3, 2, 0$. Then

$$T(2) = T_{2,0,3} + T(1, 2)$$

where $T_{2,0,3}$ is the time to obtain the pattern 2, 0, 3. Hence,

$$\begin{aligned} E[T(1, 2)] &= E[T(2)] - E[T_{2,0,3}] \\ &= \left(p_2^2 p_0^2 p_3\right)^{-1} + (p_0 p_2)^{-1} - (p_2 p_0 p_3)^{-1} \end{aligned}$$

So, suppose now that all of the quantities $E[T(i)]$ and $E[T(i, j)]$ have been computed. Let

$$M = \min_i T(i)$$

and let

$$P(i) = P\{M = T(i)\}, \quad i = 1, \ldots, s$$

That is, $P(i)$ is the probability that pattern $A(i)$ is the first pattern to occur. Now, for each j we will derive an equation that $E[T(j)]$ satisfies as follows:

$$\begin{aligned} E[T(j)] &= E[M] + E[T(j) - M] \\ &= E[M] + \sum_{i: i \neq j} E[T(i, j)] P(i), \quad j = 1, \ldots, s \end{aligned} \tag{7.46}$$

where the final equality is obtained by conditioning on which pattern occurs first. But Equations (7.46) along with the equation

$$\sum_{i=1}^{s} P(i) = 1$$

constitute a set of $s + 1$ equations in the $s + 1$ unknowns $E[M]$, $P(i)$, $i = 1, \ldots, s$. Solving them yields the desired quantities.

Example 7.33 Suppose that we continually flip a fair coin. With $A(1) = h, t, t, h, h$ and $A(2) = h, h, t, h, t$, we have

$$\begin{aligned}
E[T(1)] &= 32 + E[T_h] = 34,\\
E[T(2)] &= 32,\\
E[T(1,2)] &= E[T(2)] - E[T_{h,h}] = 32 - (4 + E[T_h]) = 26,\\
E[T(2,1)] &= E[T(1)] - E[T_{h,t}] = 34 - 4 = 30
\end{aligned}$$

Hence, we need, solve the equations

$$\begin{aligned}
34 &= E[M] + 30P(2),\\
32 &= E[M] + 26P(1),\\
1 &= P(1) + P(2)
\end{aligned}$$

These equations are easily solved, and yield the values

$$P(1) = P(2) = \tfrac{1}{2}, \quad E[M] = 19$$

Note that although the mean time for pattern $A(2)$ is less than that for $A(1)$, each has the same chance of occurring first. ■

Equations (7.46) are easy to solve when there are no overlaps in any of the patterns. In this case, for all $i \neq j$

$$E[T(i,j)] = E[T(j)]$$

so Equations (7.46) reduce to

$$E[T(j)] = E[M] + (1 - P(j))E[T(j)]$$

or

$$P(j) = E[M]/E[T(j)]$$

Summing the preceding over all j yields

$$E[M] = \frac{1}{\sum_{j=1}^{s} 1/E[T(j)]}, \tag{7.47}$$

$$P(j) = \frac{1/E[T(j)]}{\sum_{j=1}^{s} 1/E[T(j)]} \tag{7.48}$$

In our next example we use the preceding to reanalyze the model of Example 7.7.

Example 7.34 Suppose that each play of a game is, independently of the outcomes of previous plays, won by player i with probability p_i, $i = 1, \ldots, s$. Suppose further that there are specified numbers $n(1), \ldots, n(s)$ such that the first player i to win $n(i)$ consecutive plays is declared the winner of the match. Find the expected number of plays until there is a winner, and also the probability that the winner is i, $i = 1, \ldots, s$.

Solution: Letting $A(i)$, for $i = 1, \ldots, s$, denote the pattern of n_i consecutive values of i, this problem asks for $P(i)$, the probability that pattern $A(i)$ occurs first, and for $E[M]$. Because

$$E[T(i)] = (1/p_i)^{n(i)} + (1/p_i)^{n(i)-1} + \cdots + 1/p_i = \frac{1 - p_i^{n(i)}}{p_i^{n(i)}(1 - p_i)}$$

we obtain, from Equations (7.47) and (7.48), that

$$E[M] = \frac{1}{\sum_{j=1}^{s} \left[p_j^{n(j)}(1 - p_j)/\left(1 - p_j^{n(j)}\right)\right]},$$

$$P(i) = \frac{p_i^{n(i)}(1 - p_i)/\left(1 - p_i^{n(i)}\right)}{\sum_{j=1}^{s} \left[p_j^{n(j)}(1 - p_j)/\left(1 - p_j^{n(j)}\right)\right]}$$ ■

7.9.2 The Expected Time to a Maximal Run of Distinct Values

Let X_i, $i \geqslant 1$, be independent and identically distributed random variables that are equally likely to take on any of the values $1, 2, \ldots, m$. Suppose that these random variables are observed sequentially, and let T denote the first time that a run of m consecutive values includes all the values $1, \ldots, m$. That is,

$$T = \min\{n : X_{n-m+1}, \ldots, X_n \text{ are all distinct}\}$$

To compute $E[T]$, define a renewal process by letting the first renewal occur at time T. At this point start over and, without using any of the data values up to T, let the next renewal occur the next time a run of m consecutive values are all distinct, and so on. For instance, if $m = 3$ and the data are

$$1, 3, 3, 2, 1, 2, 3, 2, 1, 3, \ldots, \tag{7.49}$$

then there are two renewals by time 10, with the renewals occurring at times 5 and 9. We call the sequence of m distinct values that constitutes a renewal a *renewal run*.

Let us now transform the renewal process into a delayed renewal reward process by supposing that a reward of 1 is earned at time n, $n \geqslant m$, if the values

$X_{n-m+1}, \ldots, X_n$ are all distinct. That is, a reward is earned each time the previous m data values are all distinct. For instance, if $m = 3$ and the data values are as in (7.49) then unit rewards are earned at times 5, 7, 9, and 10. If we let R_i denote the reward earned at time i, then by Proposition 7.3,

$$\lim_n \frac{E\left[\sum_{i=1}^n R_i\right]}{n} = \frac{E[R]}{E[T]} \tag{7.50}$$

where R is the reward earned between renewal epochs. Now, with A_i equal to the set of the first i data values of a renewal run, and B_i to the set of the first i values following this renewal run, we have the following:

$$\begin{aligned}
E[R] &= 1 + \sum_{i=1}^{m-1} E[\text{reward earned a time } i \text{ after a renewal}] \\
&= 1 + \sum_{i=1}^{m-1} P\{A_i = B_i\} \\
&= 1 + \sum_{i=1}^{m-1} \frac{i!}{m^i} \\
&= \sum_{i=0}^{m-1} \frac{i!}{m^i}
\end{aligned} \tag{7.51}$$

Hence, since for $i \geqslant m$

$$E[R_i] = P\{X_{i-m+1}, \ldots, X_i \text{ are all distinct}\} = \frac{m!}{m^m}$$

it follows from Equation (7.50) that

$$\frac{m!}{m^m} = \frac{E[R]}{E[T]}$$

Thus, from Equation (7.51) we obtain

$$E[T] = \frac{m^m}{m!} \sum_{i=0}^{m-1} i!/m^i$$

The preceding delayed renewal reward process approach also gives us another way of computing the expected time until a specified pattern appears. We illustrate by the following example.

Example 7.35 Compute $E[T]$, the expected time until the pattern h, h, h, t, h, h, h appears, when a coin that comes up heads with probability p and tails with probability $q = 1 - p$ is continually flipped.

Solution: Define a renewal process by letting the first renewal occur when the pattern first appears, and then start over. Also, say that a reward of 1 is earned whenever the pattern appears. If R is the reward earned between renewal epochs, we have

$$\begin{aligned} E[R] &= 1 + \sum_{i=1}^{6} E[\text{reward earned } i \text{ units after a renewal}] \\ &= 1 + 0 + 0 + 0 + p^3 q + p^3 qp + p^3 qp^2 \end{aligned}$$

Hence, since the expected reward earned at time i is $E[R_i] = p^6 q$, we obtain the following from the renewal reward theorem:

$$\frac{1 + qp^3 + qp^4 + qp^5}{E[T]} = qp^6$$

or

$$E[T] = q^{-1} p^{-6} + p^{-3} + p^{-2} + p^{-1} \qquad \blacksquare$$

7.9.3 Increasing Runs of Continuous Random Variables

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed continuous random variables, and let T denote the first time that there is a string of r consecutive increasing values. That is,

$$T = \min\{n \geqslant r : X_{n-r+1} < X_{n-r+2} < \cdots < X_n\}$$

To compute $E[T]$, define a renewal process as follows. Let the first renewal occur at T. Then, using only the data values after T, say that the next renewal occurs when there is again a string of r consecutive increasing values, and continue in this fashion. For instance, if $r = 3$ and the first 15 data values are

$$12, 20, 22, 28, 43, 18, 24, 33, 60, 4, 16, 8, 12, 15, 18$$

then 3 renewals would have occurred by time 15, namely, at times 3, 8, and 14. If we let $N(n)$ denote the number of renewals by time n, then by the elementary renewal theorem

$$\frac{E[N(n)]}{n} \to \frac{1}{E[T]}$$

To compute $E[N(n)]$, define a stochastic process whose state at time k, call it S_k, is equal to the number of consecutive increasing values at time k. That is, for $1 \leqslant j \leqslant k$

$$S_k = j \quad \text{if } X_{k-j} > X_{k-j+1} < \cdots < X_{k-1} < X_k$$

where $X_0 = \infty$. Note that a renewal will occur at time k if and only if $S_k = ir$ for some $i \geqslant 1$. For instance, if $r = 3$ and

$$X_5 > X_6 < X_7 < X_8 < X_9 < X_{10} < X_{11}$$

then

$$S_6 = 1, \quad S_7 = 2, \quad S_8 = 3, \quad S_9 = 4, \quad S_{10} = 5, \quad S_{11} = 6$$

and renewals occur at times 8 and 11. Now, for $k > j$

$$\begin{aligned} P\{S_k = j\} &= P\{X_{k-j} > X_{k-j+1} < \cdots < X_{k-1} < X_k\} \\ &= P\{X_{k-j+1} < \cdots < X_{k-1} < X_k\} \\ &\quad - P\{X_{k-j} < X_{k-j+1} < \cdots < X_{k-1} < X_k\} \\ &= \frac{1}{j!} - \frac{1}{(j+1)!} \\ &= \frac{j}{(j+1)!} \end{aligned}$$

where the next to last equality follows since all possible orderings of the random variables are equally likely.

From the preceding, we see that

$$\lim_{k\to\infty} P\{\text{a renewal occurs at time } k\} = \lim_{k\to\infty} \sum_{i=1}^{\infty} P\{S_k = ir\} = \sum_{i=1}^{\infty} \frac{ir}{(ir+1)!}$$

However,

$$E[N(n)] = \sum_{k=1}^{n} P\{\text{a renewal occurs at time } k\}$$

Because we can show that for any numbers a_k, $k \geqslant 1$, for which $\lim_{k\to\infty} a_k$ exists that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k}{n} = \lim_{k\to\infty} a_k$$

we obtain from the preceding, upon using the elementary renewal theorem,

$$E[T] = \frac{1}{\sum_{i=1}^{\infty} ir/(ir+1)!}$$

7.10 The Insurance Ruin Problem

Suppose that claims are made to an insurance firm according to a Poisson process with rate λ, and that the successive claim amounts $Y_1, Y_2, \ldots$ are independent random variables having a common distribution function F with density $f(x)$. Suppose also that the claim amounts are independent of the claim arrival times. Thus, if we let $M(t)$ be the number of claims made by time t, then $\sum_{i=1}^{M(t)} Y_i$ is the total amount paid out in claims by time t. Supposing that the firm starts with an initial capital x and receives income at a constant rate c per unit time, we are interested in the probability that the firm's net capital ever becomes negative; that is, we are interested in

$$R(x) = P\left\{\sum_{i=1}^{M(t)} Y_i > x + ct \text{ for some } t \geqslant 0\right\}$$

If the firm's capital ever becomes negative, we say that the firm is ruined; thus $R(x)$ is the probability of ruin given that the firm begins with an initial capital x.

Let $\mu = E[Y_i]$ be the mean claim amount, and let $\rho = \lambda\mu/c$. Because claims occur at rate λ, the long-run rate at which money is paid out is $\lambda\mu$. (A formal argument uses renewal reward processes. A new cycle begins when a claim occurs; the cost for the cycle is the claim amount, and so the long-run average cost is μ, the expected cost incurred in a cycle, divided by $1/\lambda$, the mean cycle time.) Because the rate at which money is received is c, it is clear that $R(x) = 1$ when $\rho > 1$. As $R(x)$ can be shown to also equal 1 when $\rho = 1$ (think of the recurrence of the symmetric random walk), we will suppose that $\rho < 1$.

To determine $R(x)$, we start by deriving a differential equation. To begin, consider what can happen in the first h time units, where h is small. With probability $1 - \lambda h + o(h)$ there will be no claims and the firm's capital at time h will be $x + ch$; with probability $\lambda h + o(h)$ there will be exactly one claim and the firm's capital at time h will be $x + ch - Y_1$; with probability $o(h)$ there will be two or more claims. Therefore, conditioning on what happens during the first h time units yields

$$R(x) = (1 - \lambda h)R(x + ch) + \lambda h\, E[R(x + ch - Y_1)] + o(h)$$

Equivalently,

$$R(x + ch) - R(x) = \lambda h\, R(x + ch) - \lambda h\, E[R(x + ch - Y_1)] + o(h)$$

Dividing through by ch gives

$$\frac{R(x+ch)-R(x)}{ch} = \frac{\lambda}{c}\,R(x+ch) - \frac{\lambda}{c}\,E[R(x+ch-Y_1)] + \frac{1}{c}\,\frac{o(h)}{h}$$

Letting h go to 0 yields the differential equation

$$R'(x) = \frac{\lambda}{c}\,R(x) - \frac{\lambda}{c}\,E[R(x-Y_1)]$$

Because $R(u)=1$ when $u<0$, the preceding can be written as

$$R'(x) = \frac{\lambda}{c}R(x) - \frac{\lambda}{c}\int_0^x R(x-y)f(y)\,dy - \frac{\lambda}{c}\int_x^\infty f(y)\,dy$$

or, equivalently,

$$R'(x) = \frac{\lambda}{c}\,R(x) - \frac{\lambda}{c}\int_0^x R(x-y)f(y)\,dy - \frac{\lambda}{c}\,\bar{F}(x) \tag{7.52}$$

where $\bar{F}(x) = 1 - F(x)$.

We will now use the preceding equation to show that $R(x)$ also satisfies the equation

$$R(x) = R(0) + \frac{\lambda}{c}\int_0^x R(x-y)\bar{F}(y)\,dy - \frac{\lambda}{c}\int_0^x \bar{F}(y)\,dy, \quad x \geqslant 0 \tag{7.53}$$

To verify Equation (7.53), we will show that differentiating both sides of it results in Equation (7.52). (It can be shown that both (7.52) and (7.53) have unique solutions.) To do so, we will need the following lemma, whose proof is given at the end of this section.

Lemma 7.5 For a function k, and a differentiable function t,

$$\frac{d}{dx}\int_0^x t(x-y)k(y)\,dy = t(0)k(x) + \int_0^x t'(x-y)k(y)\,dy$$

Differentiating both sides of Equation (7.53) gives, upon using the preceding lemma,

$$R'(x) = \frac{\lambda}{c}\left[R(0)\bar{F}(x) + \int_0^x R'(x-y)\bar{F}(y)\,dy - \bar{F}(x)\right] \tag{7.54}$$

Differentiation by parts $[u = \bar{F}(y),\, dv = R'(x-y)\,dy]$ shows that

$$\int_0^x R'(x-y)\bar{F}(y)\,dy = -\bar{F}(y)R(x-y)|_0^x - \int_0^x R(x-y)f(y)\,dy$$
$$= -\bar{F}(x)R(0) + R(x) - \int_0^x R(x-y)f(y)\,dy$$

Substituting this result back in Equation (7.54) gives Equation (7.52). Thus, we have established Equation (7.53).

To obtain a more usable expression for $R(x)$, consider a renewal process whose interarrival times $X_1, X_2, \ldots$ are distributed according to the equilibrium distribution of F. That is, the density function of the X_i is

$$f_e(x) = F_e'(x) = \frac{\bar{F}(x)}{\mu}$$

Let $N(t)$ denote the number of renewals by time t, and let us derive an expression for

$$q(x) = E\left[\rho^{N(x)+1}\right]$$

Conditioning on X_1 gives

$$q(x) = \int_0^\infty E\left[\rho^{N(x)+1}|X_1 = y\right]\frac{\bar{F}(y)}{\mu}\,dy$$

Because, given that $X_1 = y$, the number of renewals by time x is distributed as $1 + N(x-y)$ when $y \leqslant x$, or is identically 0 when $y > x$, we see that

$$E[\rho^{N(x)+1}|X_1 = y] = \begin{cases} \rho\, E\left[\rho^{N(x-y)+1}\right], & \text{if } y \leqslant x \\ \rho, & \text{if } y > x \end{cases}$$

Therefore, $q(x)$ satisfies

$$q(x) = \int_0^x \rho q(x-y)\frac{\bar{F}(y)}{\mu}\,dy + \rho\int_x^\infty \frac{\bar{F}(y)}{\mu}\,dy$$
$$= \frac{\lambda}{c}\int_0^x q(x-y)\,\bar{F}(y)\,dy + \frac{\lambda}{c}\left[\int_0^\infty \bar{F}(y)\,dy - \int_0^x \bar{F}(y)\,dy\right]$$
$$= \frac{\lambda}{c}\int_0^x q(x-y)\,\bar{F}(y)\,dy + \rho - \frac{\lambda}{c}\int_0^x \bar{F}(y)\,dy$$

Because $q(0) = \rho$, this is exactly the same equation that is satisfied by $R(x)$, namely Equation (7.53). Therefore, because the solution to (7.53) is unique, we obtain the following.

Proposition 7.6

$$R(x) = q(x) = E\big[\rho^{N(x)+1}\big]$$

Example 7.36 Suppose that the firm does not start with any initial capital. Then, because $N(0) = 0$, we see that the firm's probability of ruin is $R(0) = \rho$. ■

Example 7.37 If the claim distribution F is exponential with mean μ, then so is F_e. Hence, $N(x)$ is Poisson with mean x/μ, giving the result

$$\begin{aligned}
R(x) = E\big[\rho^{N(x)+1}\big] &= \sum_{n=0}^{\infty} \rho^{n+1} e^{-x/\mu}(x/\mu)^n/n! \\
&= \rho\, e^{-x/\mu} \sum_{n=0}^{\infty} (\rho\, x/\mu)^n/n! \\
&= \rho e^{-x(1-\rho)/\mu}
\end{aligned}$$

■

To obtain some intuition about the ruin probability, let T be independent of the interarrival times X_i of the renewal process having interarrival distribution F_e, and let T have probability mass function

$$P\{T = n\} = \rho^n(1-\rho), \quad n = 0, 1, \ldots$$

Now consider $P\Big\{\sum_{i=1}^{T} X_i > x\Big\}$, the probability that the sum of the first T of the X_i exceeds x. Because $N(x) + 1$ is the first renewal that occurs after time x, we have

$$N(x) + 1 = \min\left\{n : \sum_{i=1}^{n} X_i > x\right\}$$

Therefore, conditioning on the number of renewals by time x gives

$$\begin{aligned}
P\left\{\sum_{i=1}^{T} X_i > x\right\} &= \sum_{j=0}^{\infty} P\left\{\sum_{i=1}^{T} X_i > x \,\middle|\, N(x) = j\right\} P\{N(x) = j\} \\
&= \sum_{j=0}^{\infty} P\{T \geqslant j+1 | N(x) = j\} P\{N(x) = j\}
\end{aligned}$$

$$= \sum_{j=0}^{\infty} P\{T \geqslant j+1\} P\{N(x) = j\}$$

$$= \sum_{j=0}^{\infty} \rho^{j+1} P\{N(x) = j\}$$

$$= E\big[\rho^{N(x)+1}\big]$$

Consequently, $P\big\{\sum_{i=1}^{T} X_i > x\big\}$ is equal to the ruin probability. Now, as noted in Example 7.36, the ruin probability of a firm starting with 0 initial capital is ρ. Suppose that the firm starts with an initial capital x, and suppose for the moment that it is allowed to remain in business even if its capital becomes negative. Because the probability that the firm's capital ever falls below its initial starting amount x is the same as the probability that its capital ever becomes negative when it starts with 0, this probability is also ρ. Thus, if we say that a low occurs whenever the firm's capital becomes lower than it has ever previously been, then the probability that a low ever occurs is ρ. Now, if a low does occur, then the probability that there will be another low is the probability that the firm's capital will ever fall below its previous low, and clearly this is also ρ. Therefore, each new low is the final one with probability $1-\rho$. Consequently, the total number of lows that ever occur has the same distribution as T. In addition, if we let W_i be the amount by which the ith low is less than the low preceding it, it is easy to see that $W_1, W_2, \ldots$ are independent and identically distributed, and are also independent of the number of lows. Because the minimal value over all time of the firm's capital (when it is allowed to remain in business even when its capital becomes negative) is $x - \sum_{i=1}^{T} W_i$, it follows that the ruin probability of a firm that starts with an initial capital x is

$$R(x) = P\left\{\sum_{i=1}^{T} W_i > x\right\}$$

Because

$$R(x) = E\big[\rho^{N(x)+1}\big] = P\left\{\sum_{i=1}^{T} X_i > x\right\}$$

we can identify W_i with X_i. That is, we can conclude that each new low is lower than its predecessor by a random amount whose distribution is the equilibrium distribution of a claim amount.

Remark Because the times between successive customer claims are independent exponential random variables with mean $1/\lambda$ while money is being paid to the insurance firm at a constant rate c, it follows that the amounts of money paid in to the insurance company between consecutive claims are independent exponential random variables with mean c/λ. Thus, because ruin can only occur when a claim arises, it follows that the expression given in Proposition 7.6 for the ruin probability $R(x)$ is valid for any model in which the amounts of money paid to the insurance firm between claims are independent exponential random variables with mean c/λ and the amounts of the successive claims are independent random variables having distribution function F, with these two processes being independent.

Now imagine an insurance model in which customers buy policies at arbitrary times, each customer pays the insurance company a fixed rate c per unit time, the time until a customer makes a claim is exponential with rate λ, and each claim amount has distribution F. Consider the amount of money the insurance firm takes in between claims. Specifically, suppose a claim has just occurred and let X be the amount the insurance company takes in before another claim arises. Note that this amount increases continuously in time until a claim occurs, and suppose that at the present time the amount t has been taken in since the last claim. Let us compute the probability that a claim will be made before the amount taken in increases by an additional amount h, when h is small. To determine this probability, suppose that at the present time the firm has k customers. Because each of these k customers is paying the insurance firm at rate c, it follows that the additional amount taken in by the firm before the next claim occurs will be less than h if and only if a claim is made within the next $\frac{h}{kc}$ time units. Because each of the k customers will register a claim at an exponential rate λ, the time until one of them makes a claim is an exponential random variable with rate $k\lambda$. Calling this random variable $E_{k\lambda}$, it follows that the probability that the additional amount taken in is less than h is

$$\begin{aligned}
P(\text{additional amount} < h|k \text{ customers}) &= P\left(E_{k\lambda} < \frac{h}{kc}\right) \\
&= 1 - e^{-\lambda h/c} \\
&= \frac{\lambda}{c}h + o(h)
\end{aligned}$$

Thus,

$$P(X < t + h|X > t) = \frac{\lambda}{c}h + o(h)$$

showing that the failure rate function of X is identically $\frac{\lambda}{c}$. But this means that the amounts taken in between claims are exponential random variables with mean $\frac{c}{\lambda}$. Because the amounts of each claim have distribution function F, we can thus conclude that the firm's failure probability in this insurance model is exactly the same as in the previously analyzed classical model. ■

Let us now give the proof of Lemma 7.5.

Proof of Lemma 7.5. Let $G(x) = \int_0^x t(x-y)k(y)\,dy$. Then

$$\begin{aligned} G(x+h) - G(x) &= G(x+h) - \int_0^x t(x+h-y)k(y)\,dy \\ &\quad + \int_0^x t(x+h-y)k(y)\,dy - G(x) \\ &= \int_x^{x+h} t(x+h-y)k(y)\,dy \\ &\quad + \int_0^x [t(x+h-y) - t(x-y)]k(y)\,dy \end{aligned}$$

Dividing through by h gives

$$\begin{aligned} \frac{G(x+h) - G(x)}{h} &= \frac{1}{h}\int_x^{x+h} t(x+h-y)k(y)\,dy \\ &\quad + \int_0^x \frac{t(x+h-y) - t(x-y)}{h}\,k(y)\,dy \end{aligned}$$

Letting $h \to 0$ gives the result

$$G'(x) = t(0)\,k(x) + \int_0^x t'(x-y)\,k(y)\,dy \qquad \blacksquare$$

Exercises

1. Is it true that
   (a) $N(t) < n$ if and only if $S_n > t$?
   (b) $N(t) \leqslant n$ if and only if $S_n \geqslant t$?
   (c) $N(t) > n$ if and only if $S_n < t$?
2. Suppose that the interarrival distribution for a renewal process is Poisson distributed with mean μ. That is, suppose

   $$P\{X_n = k\} = e^{-\mu}\frac{\mu^k}{k!}, \quad k = 0, 1, \ldots$$

   (a) Find the distribution of S_n.
   (b) Calculate $P\{N(t) = n\}$.
*3. If the mean-value function of the renewal process $\{N(t), t \geqslant 0\}$ is given by $m(t) = t/2$, $t \geqslant 0$, what is $P\{N(5) = 0\}$?

4. Let $\{N_1(t), t \geqslant 0\}$ and $\{N_2(t), t \geqslant 0\}$ be independent renewal processes. Let $N(t) = N_1(t) + N_2(t)$.
   (a) Are the interarrival times of $\{N(t), t \geqslant 0\}$ independent?
   (b) Are they identically distributed?
   (c) Is $\{N(t), t \geqslant 0\}$ a renewal process?
5. Let $U_1, U_2, \ldots$ be independent uniform (0, 1) random variables, and define N by

$$N = \min\{n : U_1 + U_2 + \cdots + U_n > 1\}$$

   What is $E[N]$?
*6. Consider a renewal process $\{N(t), t \geqslant 0\}$ having a gamma (r, λ) interarrival distribution. That is, the interarrival density is

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{r-1}}{(r-1)!}, \quad x > 0$$

   (a) Show that

$$P\{N(t) \geqslant n\} = \sum_{i=nr}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

   (b) Show that

$$m(t) = \sum_{i=r}^{\infty} \left[\frac{i}{r}\right] \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

   where $[i/r]$ is the largest integer less than or equal to i/r.

   Hint: Use the relationship between the gamma (r, λ) distribution and the sum of r independent exponentials with rate λ to define $N(t)$ in terms of a Poisson process with rate λ.
7. Mr. Smith works on a temporary basis. The mean length of each job he gets is three months. If the amount of time he spends between jobs is exponentially distributed with mean 2, then at what rate does Mr. Smith get new jobs?
*8. A machine in use is replaced by a new machine either when it fails or when it reaches the age of T years. If the lifetimes of successive machines are independent with a common distribution F having density f, show that
   (a) the long-run rate at which machines are replaced equals

$$\left[\int_0^T x f(x)\, dx + T(1 - F(T))\right]^{-1}$$

   (b) the long-run rate at which machines in use fail equals

$$\frac{F(T)}{\int_0^T x f(x)\, dx + T[1 - F(T)]}$$

9. A worker sequentially works on jobs. Each time a job is completed, a new one is begun. Each job, independently, takes a random amount of time having distribution F to complete. However, independently of this, shocks occur according to a Poisson process with rate λ. Whenever a shock occurs, the worker discontinues working on the present job and starts a new one. In the long run, at what rate are jobs completed?

10. Consider a renewal process with mean interarrival time μ. Suppose that each event of this process is independently "counted" with probability p. Let $N_C(t)$ denote the number of counted events by time t, $t > 0$.
    (a) Is $N_C(t)$, $t \geqslant 0$ a renewal process?
    (b) What is $\lim_{t\to\infty} N_C(t)/t$?

11. A renewal process for which the time until the initial renewal has a different distribution than the remaining interarrival times is called a *delayed* (or a *general*) *renewal process*. Prove that Proposition 7.1 remains valid for a delayed renewal process. (In general, it can be shown that all of the limit theorems for a renewal process remain valid for a delayed renewal process provided that the time until the first renewal has a finite mean.)

12. Events occur according to a Poisson process with rate λ. Any event that occurs within a time d of the event that immediately preceded it is called a d-event. For instance, if $d = 1$ and events occur at times $2, 2.8, 4, 6, 6.6, \ldots$, then the events at times 2.8 and 6.6 would be d-events.
    (a) At what rate do d-events occur?
    (b) What proportion of all events are d-events?

13. Let $X_1, X_2, \ldots$ be a sequence of independent random variables. The nonnegative integer valued random variable N is said to be a *stopping time* for the sequence if the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$. The idea being that the X_i are observed one at a time—first X_1, then X_2, and so on—and N represents the number observed when we stop. Hence, the event $\{N = n\}$ corresponds to stopping after having observed $X_1, \ldots, X_n$ and thus must be independent of the values of random variables yet to come, namely, $X_{n+1}, X_{n+2}, \ldots$.
    (a) Let $X_1, X_2, \ldots$ be independent with

$$P\{X_i = 1\} = p = 1 - P\{X_i = 0\}, \quad i \geqslant 1$$

Define

$$N_1 = \min\{n : X_1 + \cdots + X_n = 5\}$$

$$N_2 = \begin{cases} 3, & \text{if } X_1 = 0 \\ 5, & \text{if } X_1 = 1 \end{cases}$$

$$N_3 = \begin{cases} 3, & \text{if } X_4 = 0 \\ 2, & \text{if } X_4 = 1 \end{cases}$$

Which of the N_i are stopping times for the sequence $X_1, \ldots$? An important result, known as *Wald's equation* states that if $X_1, X_2, \ldots$ are independent and identically distributed and have a finite mean $E(X)$, and if N is a stopping time for this sequence having a finite mean, then

$$E\left[\sum_{i=1}^{N} X_i\right] = E[N]E[X]$$

To prove Wald's equation, let us define the indicator variables $I_i, i \geqslant 1$ by

$$I_i = \begin{cases} 1, & \text{if } i \leqslant N \\ 0, & \text{if } i > N \end{cases}$$

(b) Show that

$$\sum_{i=1}^{N} X_i = \sum_{i=1}^{\infty} X_i I_i$$

From part (b) we see that

$$E\left[\sum_{i=1}^{N} X_i\right] = E\left[\sum_{i=1}^{\infty} X_i I_i\right]$$
$$= \sum_{i=1}^{\infty} E[X_i I_i]$$

where the last equality assumes that the expectation can be brought inside the summation (as indeed can be rigorously proven in this case).

(c) Argue that X_i and I_i are independent.

Hint: I_i equals 0 or 1 depending on whether or not we have yet stopped after observing which random variables?

(d) From part (c) we have

$$E\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{\infty} E[X]E[I_i]$$

Complete the proof of Wald's equation.

(e) What does Wald's equation tell us about the stopping times in part (a)?

14. Wald's equation can be used as the basis of a proof of the elementary renewal theorem. Let $X_1, X_2, \ldots$ denote the interarrival times of a renewal process and let $N(t)$ be the number of renewals by time t.

(a) Show that whereas $N(t)$ is *not* a stopping time, $N(t) + 1$ is.

Hint: Note that

$$N(t) = n \quad \Leftrightarrow \quad X_1 + \cdots + X_n \leqslant t \quad \text{and} \quad X_1 + \cdots + X_{n+1} > t$$

(b) Argue that

$$E\left[\sum_{i=1}^{N(t)+1} X_i\right] = \mu[m(t) + 1]$$

(c) Suppose that the X_i are bounded random variables. That is, suppose there is a constant M such that $P\{X_i < M\} = 1$. Argue that

$$t < \sum_{i=1}^{N(t)+1} X_i < t + M$$

(d) Use the previous parts to prove the elementary renewal theorem when the interarrival times are bounded.

15. Consider a miner trapped in a room that contains three doors. Door 1 leads him to freedom after two days of travel; door 2 returns him to his room after a four-day journey; and door 3 returns him to his room after a six-day journey. Suppose at all times he is equally likely to choose any of the three doors, and let T denote the time it takes the miner to become free.

(a) Define a sequence of independent and identically distributed random variables $X_1, X_2 \ldots$ and a stopping time N such that

$$T = \sum_{i=1}^{N} X_i$$

Note: You may have to imagine that the miner continues to randomly choose doors even after he reaches safety.

(b) Use Wald's equation to find $E[T]$.

(c) Compute $E\left[\sum_{i=1}^{N} X_i | N = n\right]$ and note that it is not equal to $E[\sum_{i=1}^{n} X_i]$.

(d) Use part (c) for a second derivation of $E[T]$.

16. A deck of 52 playing cards is shuffled and the cards are then turned face up one at a time. Let X_i equal 1 if the ith card turned over is an ace, and let it be 0 otherwise, $i = 1, \ldots, 52$. Also, let N denote the number of cards that need be turned over until all four aces appear. That is, the final ace appears on the Nth card to be turned over. Is the equation

$$E\left[\sum_{i=1}^{N} X_i\right] = E[N]E[X_i]$$

valid? If not, why is Wald's equation not applicable?

17. In Example 7.6, suppose that potential customers arrive in accordance with a renewal process having interarrival distribution F. Would the number of events by time t constitute a (possibly delayed) renewal process if an event corresponds to a customer

(a) entering the bank?

(b) leaving the bank?

What if F were exponential?

*18. Compute the renewal function when the interarrival distribution F is such that

$$1 - F(t) = pe^{-\mu_1 t} + (1 - p)e^{-\mu_2 t}$$

19. For the renewal process whose interarrival times are uniformly distributed over $(0, 1)$, determine the expected time from $t = 1$ until the next renewal.

20. For a renewal reward process consider

$$W_n = \frac{R_1 + R_2 + \cdots + R_n}{X_1 + X_2 + \cdots + X_n}$$

where W_n represents the average reward earned during the first n cycles. Show that $W_n \to E[R]/E[X]$ as $n \to \infty$.

21. Consider a single-server bank for which customers arrive in accordance with a Poisson process with rate λ. If a customer will enter the bank only if the server is free when he arrives, and if the service time of a customer has the distribution G, then what proportion of time is the server busy?

*22. The lifetime of a car has a distribution H and probability density h. Ms. Jones buys a new car as soon as her old car either breaks down or reaches the age of T years. A new car costs C_1 dollars and an additional cost of C_2 dollars is incurred whenever a car breaks down. Assuming that a T-year-old car in working order has an expected resale value $R(T)$, what is Ms. Jones' long-run average cost?

23. Consider the gambler's ruin problem where on each bet the gambler either wins 1 with probability p or loses 1 with probability $1 - p$. The gambler will continue to play until his winnings are either $N - i$ or $-i$. (That is, starting with i the gambler will quit when his fortune reaches either N or 0.) Let T denote the number of bets made before the gambler stops. Use Wald's equation, along with the known probability that the gambler's final winnings are $N - i$, to find $E[T]$.

Hint: Let X_j be the gambler's winnings on bet $j, j \geqslant 1$. What are the possible values of $\sum_{j=1}^{T} X_j$? What is $E\left[\sum_{j=1}^{T} X_j\right]$?

24. Wald's equation can also be proved by using renewal reward processes. Let N be a stopping time for the sequence of independent and identically distributed random variables X_i, $i \geqslant 1$.
    (a) Let $N_1 = N$. Argue that the sequence of random variables $X_{N_1+1}, X_{N_1+2}, \ldots$ is independent of $X_1, \ldots, X_N$ and has the same distribution as the original sequence $X_i, i \geqslant 1$.

    Now treat $X_{N_1+1}, X_{N_1+2}, \ldots$ as a new sequence, and define a stopping time N_2 for this sequence that is defined exactly as N_1 is on the original sequence. (For instance, if $N_1 = \min\{n: X_n > 0\}$, then $N_2 = \min\{n: X_{N_1+n} > 0\}$.) Similarly, define a stopping time N_3 on the sequence $X_{N_1+N_2+1}, X_{N_1+N_2+2}, \ldots$ that is identically defined on this sequence as N_1 is on the original sequence, and so on.
    (b) Is the reward process in which X_i is the reward earned during period i a renewal reward process? If so, what is the length of the successive cycles?
    (c) Derive an expression for the average reward per unit time.
    (d) Use the strong law of large numbers to derive a second expression for the average reward per unit time.
    (e) Conclude Wald's equation.

25. Suppose in Example 7.13 that the arrival process is a Poisson process and suppose that the policy employed is to dispatch the train every t time units.

(a) Determine the average cost per unit time.
(b) Show that the minimal average cost per unit time for such a policy is approximately $c/2$ plus the average cost per unit time for the best policy of the type considered in that example.

26. Consider a train station to which customers arrive in accordance with a Poisson process having rate λ. A train is summoned whenever there are N customers waiting in the station, but it takes K units of time for the train to arrive at the station. When it arrives, it picks up all waiting customers. Assuming that the train station incurs a cost at a rate of nc per unit time whenever there are n customers present, find the long-run average cost.

27. A machine consists of two independent components, the ith of which functions for an exponential time with rate λ_i. The machine functions as long as at least one of these components function. (That is, it fails when both components have failed.) When a machine fails, a new machine having both its components working is put into use. A cost K is incurred whenever a machine failure occurs; operating costs at rate c_i per unit time are incurred whenever the machine in use has i working components, $i = 1, 2$. Find the long-run average cost per unit time.

28. In Example 7.15, what proportion of the defective items produced is discovered?

29. Consider a single-server queueing system in which customers arrive in accordance with a renewal process. Each customer brings in a random amount of work, chosen independently according to the distribution G. The server serves one customer at a time. However, the server processes work at rate i per unit time whenever there are i customers in the system. For instance, if a customer with workload 8 enters service when there are three other customers waiting in line, then if no one else arrives that customer will spend 2 units of time in service. If another customer arrives after 1 unit of time, then our customer will spend a total of 1.8 units of time in service provided no one else arrives.

    Let W_i denote the amount of time customer i spends in the system. Also, define $E[W]$ by

$$E[W] = \lim_{n\to\infty} (W_1 + \cdots + W_n)/n$$

and so $E[W]$ is the average amount of time a customer spends in the system.

    Let N denote the number of customers that arrive in a busy period.

(a) Argue that

$$E[W] = E[W_1 + \cdots + W_N]/E[N]$$

Let L_i denote the amount of work customer i brings into the system; and so the L_i, $i \geqslant 1$, are independent random variables having distribution G.

(b) Argue that at any time t, the sum of the times spent in the system by all arrivals prior to t is equal to the total amount of work processed by time t.

Hint: Consider the rate at which the server processes work.

(c) Argue that

$$\sum_{i=1}^{N} W_i = \sum_{i=1}^{N} L_i$$

(d) Use Wald's equation (see Exercise 13) to conclude that

$$E[W] = \mu$$

where μ is the mean of the distribution G. That is, the average time that customers spend in the system is equal to the average work they bring to the system.

*30. For a renewal process, let $A(t)$ be the age at time t. Prove that if $\mu < \infty$, then with probability 1

$$\frac{A(t)}{t} \to 0 \quad \text{as } t \to \infty$$

31. If $A(t)$ and $Y(t)$ are, respectively, the age and the excess at time t of a renewal process having an interarrival distribution F, calculate

$$P\{Y(t) > x | A(t) = s\}$$

32. Determine the long-run proportion of time that $X_{N(t)+1} < c$.

33. In Example 7.14, find the long-run proportion of time that the server is busy.

34. An $M/G/\infty$ queueing system is cleaned at the fixed times $T, 2T, 3T, \ldots$. All customers in service when a cleaning begins are forced to leave early and a cost C_1 is incurred for each customer. Suppose that a cleaning takes time $T/4$, and that all customers who arrive while the system is being cleaned are lost, and a cost C_2 is incurred for each one.
(a) Find the long-run average cost per unit time.
(b) Find the long-run proportion of time the system is being cleaned.

*35. Satellites are launched according to a Poisson process with rate λ. Each satellite will, independently, orbit the earth for a random time having distribution F. Let $X(t)$ denote the number of satellites orbiting at time t.
(a) Determine $P\{X(t) = k\}$.

Hint: Relate this to the $M/G/\infty$ queue.

(b) If at least one satellite is orbiting, then messages can be transmitted and we say that the system is functional. If the first satellite is orbited at time $t = 0$, determine the expected time that the system remains functional.

Hint: Make use of part (a) when $k = 0$.

36. Each of n skiers continually, and independently, climbs up and then skis down a particular slope. The time it takes skier i to climb up has distribution F_i, and it is independent of her time to ski down, which has distribution H_i, $i = 1, \ldots, n$. Let $N(t)$ denote the total number of times members of this group have skied down the slope by time t. Also, let $U(t)$ denote the number of skiers climbing up the hill at time t.
(a) What is $\lim_{t\to\infty} N(t)/t$?
(b) Find $\lim_{t\to\infty} E[U(t)]$.
(c) If all F_i are exponential with rate λ and all G_i are exponential with rate μ, what is $P\{U(t) = k\}$?

37. There are three machines, all of which are needed for a system to work. Machine i functions for an exponential time with rate λ_i before it fails, $i = 1, 2, 3$. When a machine fails, the system is shut down and repair begins on the failed machine. The time to fix machine 1 is exponential with rate 5; the time to fix machine 2 is uniform on (0, 4); and the time to fix machine 3 is a gamma random variable with parameters $n = 3$ and $\lambda = 2$. Once a failed machine is repaired, it is as good as new and all machines are restarted.
    (a) What proportion of time is the system working?
    (b) What proportion of time is machine 1 being repaired?
    (c) What proportion of time is machine 2 in a state of suspended animation (that is, neither working nor being repaired)?
38. A truck driver regularly drives round trips from A to B and then back to A. Each time he drives from A to B, he drives at a fixed speed that (in miles per hour) is uniformly distributed between 40 and 60; each time he drives from B to A, he drives at a fixed speed that is equally likely to be either 40 or 60.
    (a) In the long run, what proportion of his driving time is spent going to B?
    (b) In the long run, for what proportion of his driving time is he driving at a speed of 40 miles per hour?
39. A system consists of two independent machines that each function for an exponential time with rate λ. There is a single repairperson. If the repairperson is idle when a machine fails, then repair immediately begins on that machine; if the repairperson is busy when a machine fails, then that machine must wait until the other machine has been repaired. All repair times are independent with distribution function G and, once repaired, a machine is as good as new. What proportion of time is the repairperson idle?
40. Three marksmen take turns shooting at a target. Marksman 1 shoots until he misses, then marksman 2 begins shooting until he misses, then marksman 3 until he misses, and then back to marksman 1, and so on. Each time marksman i fires he hits the target, independently of the past, with probability P_i, $i = 1, 2, 3$. Determine the proportion of time, in the long run, that each marksman shoots.
41. Each time a certain machine breaks down it is replaced by a new one of the same type. In the long run, what percentage of time is the machine in use less than one year old if the life distribution of a machine is
    (a) uniformly distributed over (0, 2)?
    (b) exponentially distributed with mean 1?

*42. For an interarrival distribution F having mean μ, we defined the equilibrium distribution of F, denoted F_e, by

$$F_e(x) = \frac{1}{\mu} \int_0^x [1 - F(y)]\, dy$$

    (a) Show that if F is an exponential distribution, then $F = F_e$.
    (b) If for some constant c,

$$F(x) = \begin{cases} 0, & x < c \\ 1, & x \geqslant c \end{cases}$$

show that F_e is the uniform distribution on $(0, c)$. That is, if interarrival times are identically equal to c, then the equilibrium distribution is the uniform distribution on the interval $(0, c)$.

(c) The city of Berkeley, California, allows for two hours parking at all non-metered locations within one mile of the University of California. Parking officials regularly tour around, passing the same point every two hours. When an official encounters a car he or she marks it with chalk. If the same car is there on the official's return two hours later, then a parking ticket is written. If you park your car in Berkeley and return after three hours, what is the probability you will have received a ticket?

43. Consider a renewal process having interarrival distribution F such that

$$\bar{F}(x) = \tfrac{1}{2}e^{-x} + \tfrac{1}{2}e^{-x/2}, \quad x > 0$$

That is, interarrivals are equally likely to be exponential with mean 1 or exponential with mean 2.

(a) Without any calculations, guess the equilibrium distribution F_e.
(b) Verify your guess in part (a).

44. An airport shuttle bus picks up all passengers waiting at a bus stop and drops them off at the airport terminal; it then returns to the stop and repeats the process. The times between returns to the stop are independent random variables with distribution F, mean μ, and variance σ^2. Passengers arrive at the bus stop in accordance with a Poisson process with rate λ. Suppose the bus has just left the stop, and let X denote the number of passengers it picks up when it returns.

(a) Find $E[X]$.
(b) Find $\mathrm{Var}(X)$.
(c) At what rate does the shuttle bus arrive at the terminal without any passengers?

Suppose that each passenger that has to wait at the bus stop more than c time units writes an angry letter to the shuttle bus manager.

(d) What proportion of passengers write angry letters?
(e) How does your answer in part (d) relate to $F_e(x)$?

45. Consider a system that can be in either state 1 or 2 or 3. Each time the system enters state i it remains there for a random amount of time having mean μ_i and then makes a transition into state j with probability P_{ij}. Suppose

$$P_{12} = 1, \quad P_{21} = P_{23} = \tfrac{1}{2}, \quad P_{31} = 1$$

(a) What proportion of transitions takes the system into state 1?
(b) If $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 3$, then what proportion of time does the system spend in each state?

46. Consider a semi-Markov process in which the amount of time that the process spends in each state before making a transition into a different state is exponentially distributed. What kind of process is this?

47. In a semi-Markov process, let t_{ij} denote the conditional expected time that the process spends in state i given that the next state is j.

(a) Present an equation relating μ_i to the t_{ij}.
(b) Show that the proportion of time the process is in i and will next enter j is equal to $P_i P_{ij} t_{ij}/\mu_i$.

Hint: Say that a cycle begins each time state i is entered. Imagine that you receive a reward at a rate of 1 per unit time whenever the process is in i and heading for j. What is the average reward per unit time?

48. A taxi alternates between three different locations. Whenever it reaches location i, it stops and spends a random time having mean t_i before obtaining another passenger, $i = 1, 2, 3$. A passenger entering the cab at location i will want to go to location j with probability P_{ij}. The time to travel from i to j is a random variable with mean m_{ij}. Suppose that $t_1 = 1$, $t_2 = 2$, $t_3 = 4$, $P_{12} = 1$, $P_{23} = 1$, $P_{31} = \frac{2}{3} = 1 - P_{32}$, $m_{12} = 10$, $m_{23} = 20$, $m_{31} = 15$, $m_{32} = 25$. Define an appropriate semi-Markov process and determine
    (a) the proportion of time the taxi is waiting at location i, and
    (b) the proportion of time the taxi is on the road from i to j, $i, j = 1, 2, 3$.

*49. Consider a renewal process having the gamma (n, λ) interarrival distribution, and let $Y(t)$ denote the time from t until the next renewal. Use the theory of semi-Markov processes to show that

$$\lim_{t\to\infty} P\{Y(t) < x\} = \frac{1}{n}\sum_{i=1}^{n} G_{i,\lambda}(x)$$

where $G_{i,\lambda}(x)$ is the gamma (i, λ) distribution function.

50. To prove Equation (7.24), define the following notation:

$$X_i^j \equiv \text{time spent in state } i \text{ on the } j\text{th visit to this state;}$$
$$N_i(m) \equiv \text{number of visits to state } i \text{ in the first } m \text{ transitions}$$

In terms of this notation, write expressions for
    (a) the amount of time during the first m transitions that the process is in state i;
    (b) the proportion of time during the first m transitions that the process is in state i.

Argue that, with probability 1,
    (c) $\displaystyle\sum_{j=1}^{N_i(m)} \frac{X_i^j}{N_i(m)} \to \mu_i \qquad \text{as } m \to \infty$
    (d) $N_i(m)/m \to \pi_i \qquad$ as $m \to \infty$.
    (e) Combine parts (a), (b), (c), and (d) to prove Equation (7.24).

51. In 1984 the country of Morocco in an attempt to determine the average amount of time that tourists spend in that country on a visit tried two different sampling procedures. In one, they questioned randomly chosen tourists as they were leaving the country; in the other, they questioned randomly chosen guests at hotels. (Each tourist stayed at a hotel.) The average visiting time of the 3000 tourists chosen from hotels was 17.8, whereas the average visiting time of the 12,321 tourists questioned at departure was 9.0. Can you explain this discrepancy? Does it necessarily imply a mistake?

52. Let X_i, $i = 1, 2, \ldots$, be the interarrival times of the renewal process $\{N(t)\}$, and let Y, independent of the X_i, be exponential with rate λ.

(a) Use the lack of memory property of the exponential to argue that

$$P\{X_1 + \cdots + X_n < Y\} = (P\{X < Y\})^n$$

(b) Use part (a) to show that

$$E[N(Y)] = \frac{E[e^{-\lambda X}]}{1 - E[e^{-\lambda X}]}$$

where X has the interarrival distribution.

53. Write a program to approximate $m(t)$ for the interarrival distribution $F * G$, where F is exponential with mean 1 and G is exponential with mean 3.

54. Let X_i, $i \geqslant 1$, be independent random variables with $p_j = P\{X = j\}$, $j \geqslant 1$. If $p_j = j/10$, $j = 1, 2, 3, 4$, find the expected time and the variance of the number of variables that need be observed until the pattern 1, 2, 3, 1, 2 occurs.

55. A coin that comes up heads with probability 0.6 is continually flipped. Find the expected number of flips until either the sequence *thht* or the sequence *ttt* occurs, and find the probability that *ttt* occurs first.

56. Random digits, each of which is equally likely to be any of the digits 0 through 9, are observed in sequence.
(a) Find the expected time until a run of 10 distinct values occurs.
(b) Find the expected time until a run of 5 distinct values occurs.

57. Let $h(x) = P\{\sum_{i=1}^{T} X_i > x\}$ where $X_1, X_2, \ldots$ are independent random variables having distribution function F_e and T is independent of the X_i and has probability mass function $P\{T = n\} = \rho^n(1 - \rho)$, $n \geqslant 0$. Show that $h(x)$ satisfies Equation (7.53).

Hint: Start by conditioning on whether $T = 0$ or $T > 0$.

References

The results in Section 7.9.1 concerning the computation of the variance of the time until a specified pattern appears are new, as are the results of Section 7.9.2. The results of Section 7.9.3 are from Ref. [3].

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