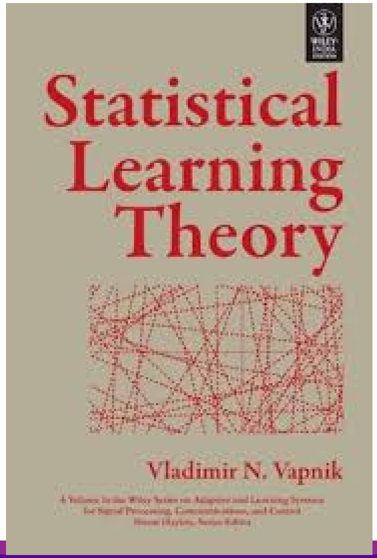
Support Vector Machine

- Support Vector Machine (SVM)
  - Vladimir Vapnik, 1963
  - But became widely-used with kernel trick, 1993
  - More on this later

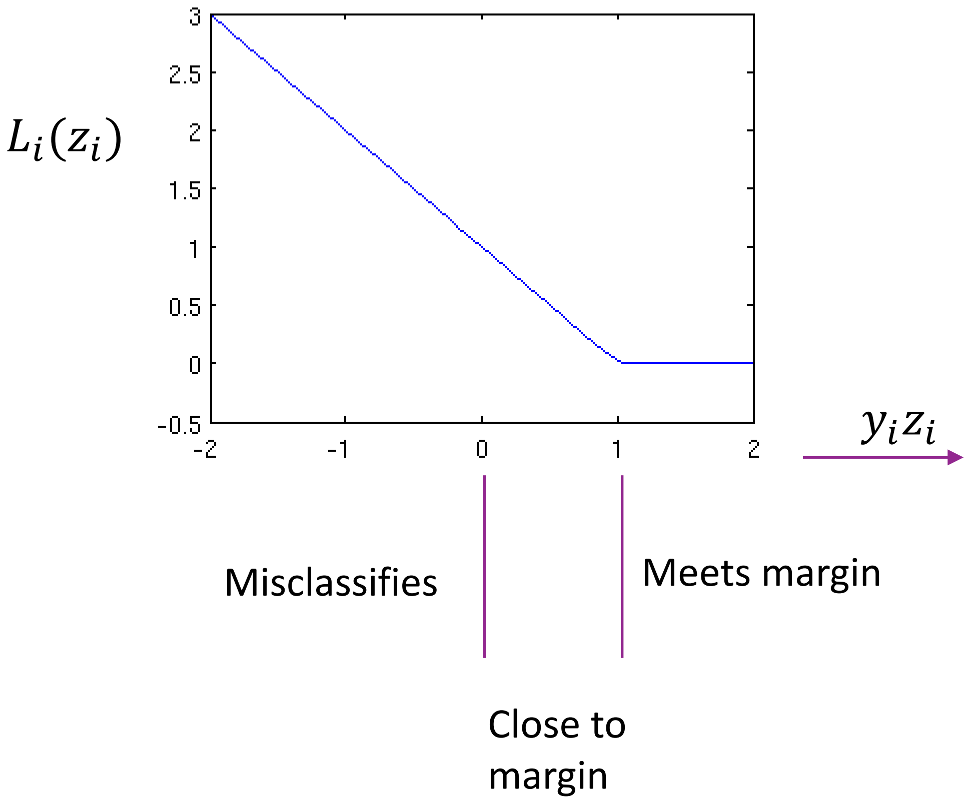
- Got best results on character recognition

- Key idea: Allow "slack" in the classification
  - Support vector classifier (SVC): Directly use raw features. Good when the original feature space is roughly linearly separable
  - Support vector machine (SVM): Map the raw features to some other domain through a kernel function



Hinge Loss

- Fix $\gamma = 1$
- Want ideally: $y_i(\boldsymbol{w}^T\boldsymbol{x} + b) \geq 1$ for all samples i
  - Equivalently, $y_i z_i \geq 1, \quad z_i = b + \boldsymbol{w}^T\boldsymbol{x}$
- But perfect separation may not be possible
- Define hinge loss or soft margin:
  - $L_i(\boldsymbol{w}, b) = \max(0, 1 - y_i z_i)$
- Starts to increase as sample is misclassified:
  - $y_i z_i \geq 1 \;\Rightarrow$ Sample meets margin target, $L_i(w) = 0$
  - $y_i z_i \in [0,1) \;\Rightarrow$ Sample margin too small, small loss
  - $y_i z_i \leq 0 \;\Rightarrow$ Sample misclassified, large loss



SVM Optimization

- Given data $(\boldsymbol{x}_i, y_i)$
- Optimization $\min_{w,b} J(\boldsymbol{w}, b)$

$$J(\boldsymbol{w}, b) = C \sum_{i=1}^{N} \max(0, 1 - y_i(\boldsymbol{w}^T \boldsymbol{x}_i + b)) + \frac{1}{2}\|\boldsymbol{w}\|^2$$

C controls final margin

Hinge loss term
Attempts to reduce
Misclassifications

margin=$1/\|\boldsymbol{w}\|$

- Constant $C > 0$ will be discussed below
- Note: ISL book uses different naming conventions.
  - We have followed convention in sklearn

Alternate Form of SVM Optimization

- Equivalent optimization:

$$\min J_1(\boldsymbol{w}, b, \boldsymbol{\epsilon}), \qquad J_1(\boldsymbol{w}, b, \boldsymbol{\epsilon}) = C \sum_{i=1}^{N} \epsilon_i + \frac{1}{2} \|\boldsymbol{w}\|^2$$

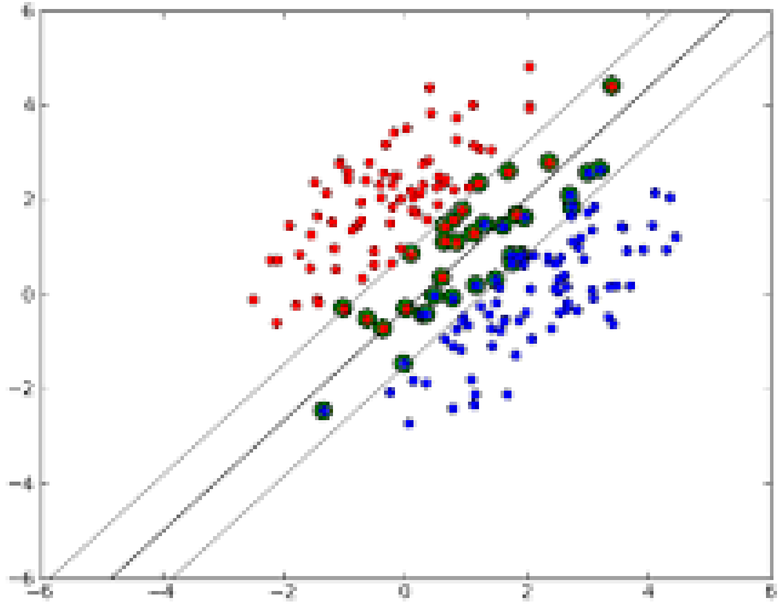
- Subject to constraints:

$$y_i(\boldsymbol{w}^T \boldsymbol{x}_i + b) \geq 1 - \epsilon_i \ \text{ for all } i = 1, \dots, N$$

  - ϵ_i = amount sample i misses margin target
- Sometimes write as $J_1(\boldsymbol{w}, b, \boldsymbol{\epsilon}) = C\|\boldsymbol{\epsilon}\|_1 + \frac{1}{2}\|\boldsymbol{w}\|^2$
  - $\|\boldsymbol{\epsilon}\|_1 = \sum_{i=1}^{N} \epsilon_i$ called the “one-norm”
  - Generally one-norm would have absolute sign over ϵ_i.
  - But in this case, when the constraint is met, ϵ_i>=0.

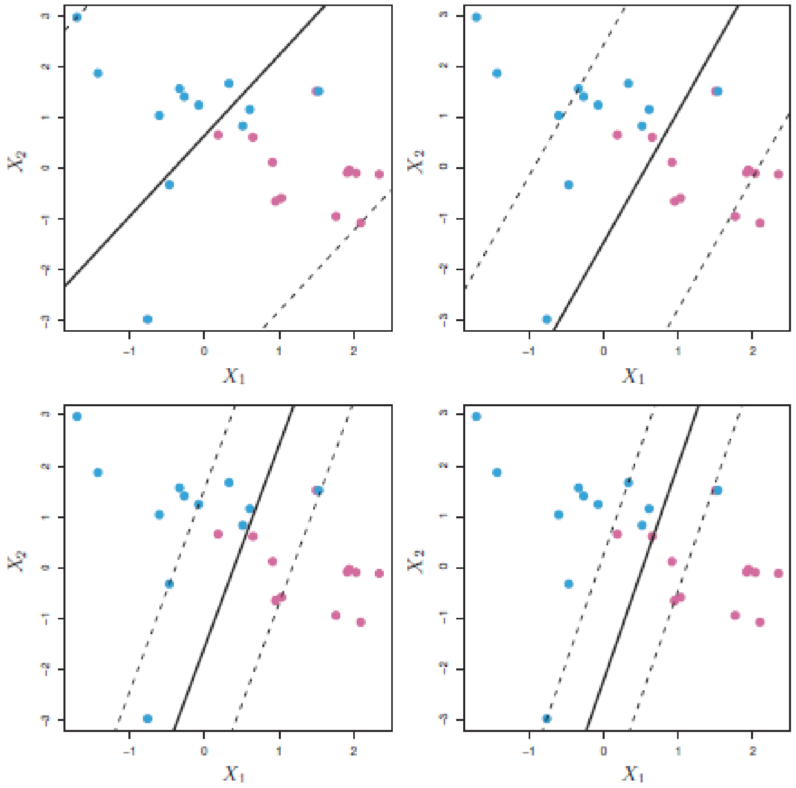
Support Vectors

- **Support vectors:** Samples that either:
  - Are exactly on margin: $y_{\mathrm{i}}(\boldsymbol{w}^T\boldsymbol{x}_i + b) = 1$
  - Or, on wrong side of margin: $y_{\mathrm{i}}(\boldsymbol{w}^T\boldsymbol{x}_i + b) \leq 1$
- Changing samples that are not SVs
  - Does not change solution
  - Provides robustness



Illustrating Effect of C

Low $C = C_1$ C_2



C_3 High C_4

- Fig. 9.7 of ISL
  - Note: C has opposite meaning in ISL than python
  - Here, we use python meaning
- Low C:
  - Leads to large margin
  - But allow many violations of margin.
  - Many more SVs
  - Reduces variance by using more samples
- Large C:
  - Leads to small margin
  - Reduce number of violations, and fewer SVs.
  - Highly fit to data. Low bias, higher variance
  - More chance to overfit

Relation to Logistic Regression

- Logistic regression also minimizes a loss function:

$$J(\boldsymbol{w}, b) = \sum_{i=1}^{N} L_i(w, b)\,, \qquad L_i(w, b) = \ln P(y_i|\boldsymbol{x}_i) = -\ln(1 + e^{-y_i z_i})$$