Simplex method

DTU Management
Department of Technology, Management and Economics

LP models

Two examples from last week:

$$\max Z = 3x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 \quad\quad &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \end{aligned}$$

and

$$x_1 \geq 0, \; x_2 \geq 0$$

$$\min 170x_1 + 160x_2 + 175x_3 + 180x_4 + 195x_5$$

subject to

$$\begin{array}{llllll} x_1 & & & & & \geq 48 \\ x_1 & +x_2 & & & & \geq 79 \\ x_1 & +x_2 & & & & \geq 65 \\ x_1 & +x_2 & +x_3 & & & \geq 87 \\ & x_2 & +x_3 & & & \geq 64 \\ & & x_3 & +x_4 & & \geq 73 \\ & & x_3 & +x_4 & & \geq 82 \\ & & & x_4 & & \geq 43 \\ & & & x_4 & +x_5 & \geq 52 \\ & & & & x_5 & \geq 15 \\ x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0 \end{array}$$

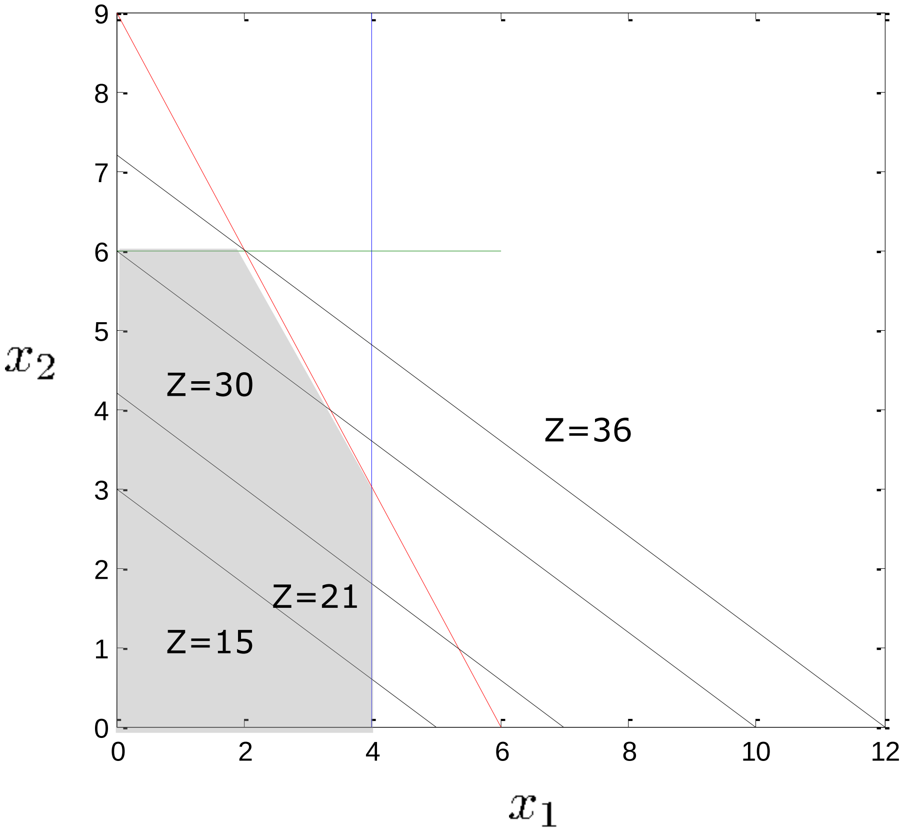
How do we solve the problem on the right?

The simplex algorithm
A method for solving any LP

- **Algorithm:**
  A precise description of the operations needed to solve a specific problem.
  Similar to a cooking recipe (but more precise).

- Some or all of you probably already knows algorithms for solving simple problems
  - Sorting
  - Searching through sorted data

Wyndor solution



$$\max Z = 3x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 \quad &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \end{aligned}$$

and

$$x_1 \geq 0,\ x_2 \geq 0$$

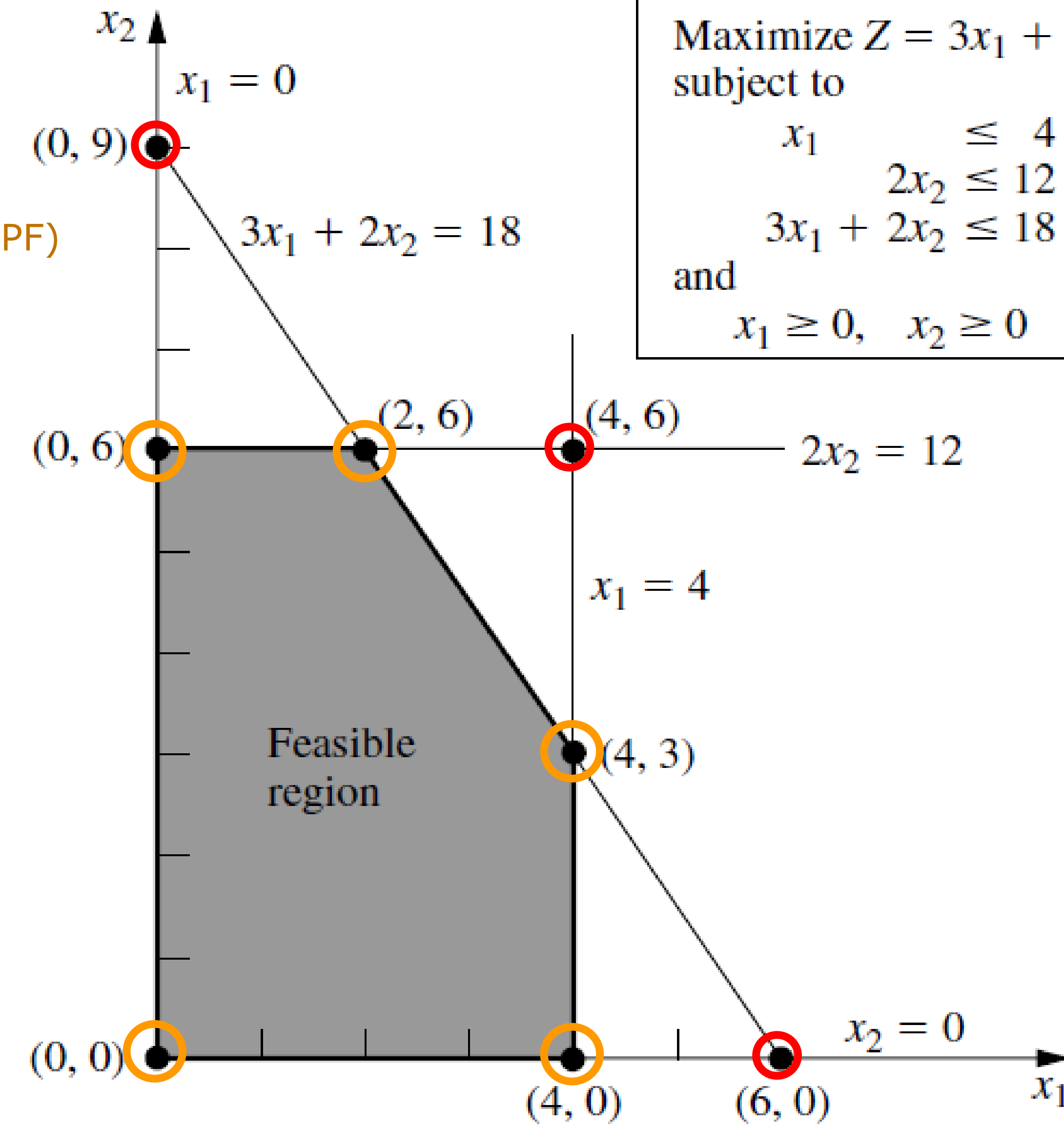
To find the optimal solution it is enough to exame all corner points (intersection between two lines).

Geometry

Corner point feasible solutions (CPF)

Corner point infeasible solutions

Two cornerpoints are neighbors if they share a line.



Maximize $Z = 3x_1 + 5x_2$,
subject to

$$
\begin{aligned}
x_1 \quad\quad\quad &\leq 4 \\
2x_2 &\leq 12 \\
3x_1 + 2x_2 &\leq 18
\end{aligned}
$$

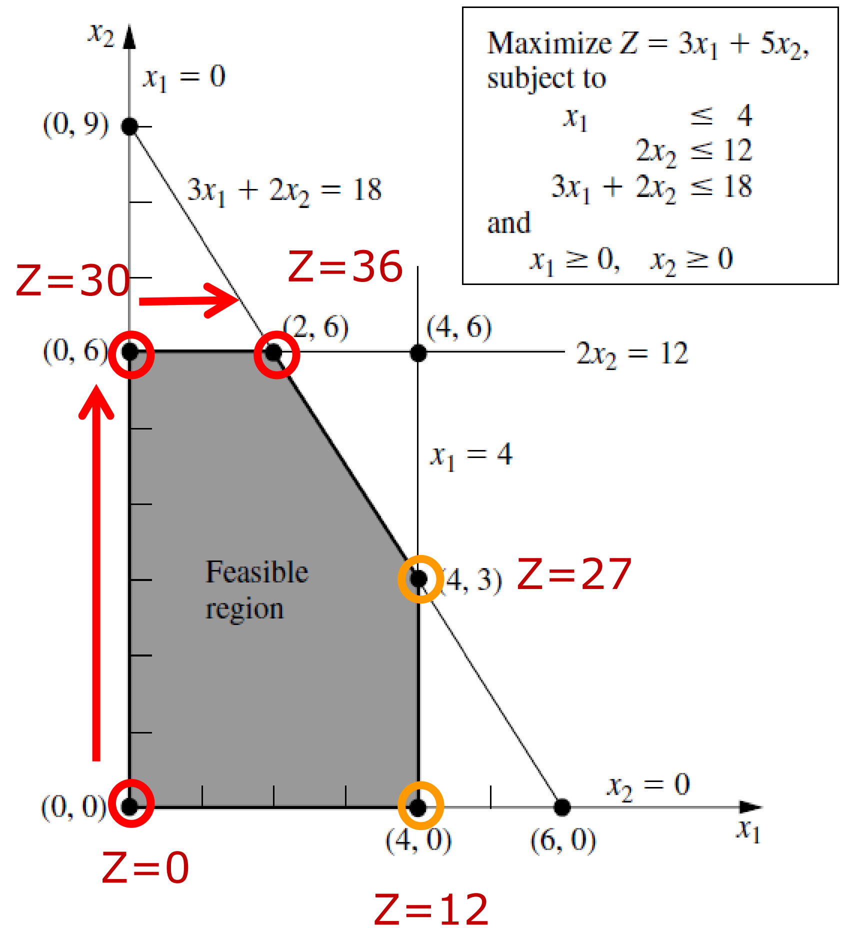
and

$$x_1 \geq 0, \quad x_2 \geq 0$$

Geometry

Idea behind Simplex algorithm:

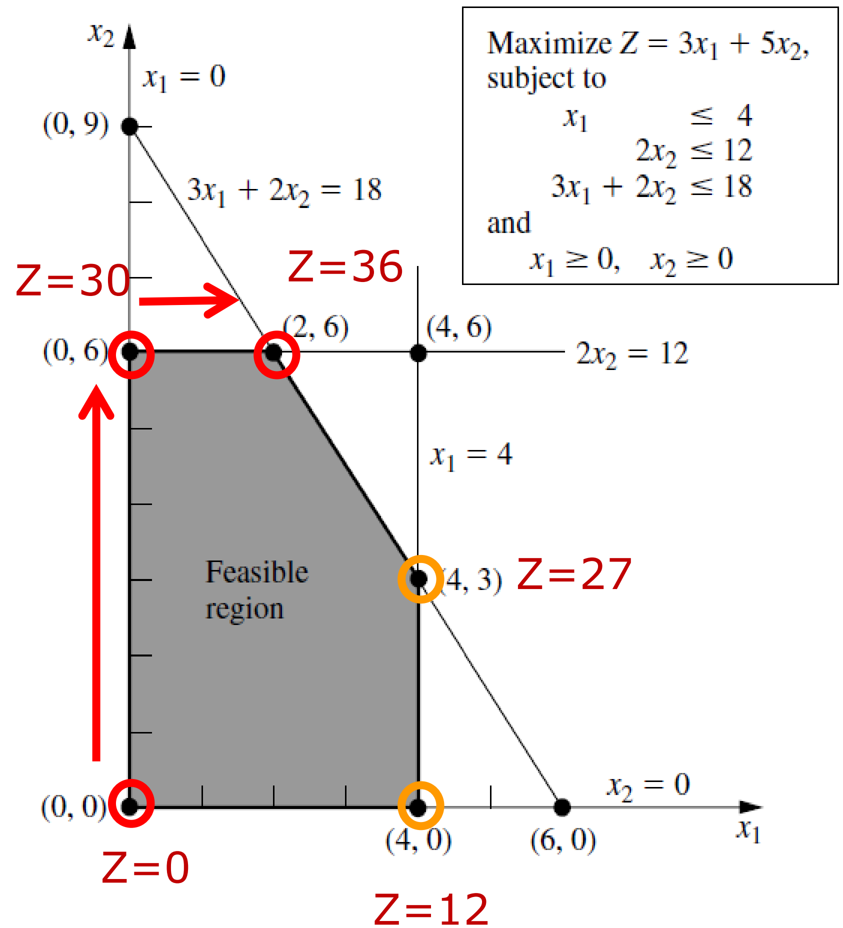
1. Start in $(x_1, x_2) = (0, 0)$.
2. If there is a neighboring feasible corner point with larger objective then jump to that corner point. Otherwise stop.
3. Jump back to step 2.



Geometry

Idea behind Simplex algorithm:

1. Start in $(x_1, x_2) = (0, 0)$.
2. If there is a neighboring feasible corner point with larger objective then jump to that corner point. Otherwise stop.
3. Jump back to step 2.



Optimality test: Conside an LP with at least one optimal solution. If a feasible corner point do not have any neighboring solutions that are better with respect to Z then the corner point is optimal.

Augmented form

Wyndor example again:

$$\max Z = 3x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 \quad\quad &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \end{aligned}$$

and

$$x_1 \ge 0, \; x_2 \ge 0$$

We wish to convert "$\le$" constraints to "$=$". We do this so we can re-use some of the machinery we learned when solving systems of linear equations.

We add "slack variables" for every constraint. For the constraint $x_1 \le 4$ we add x_3 defined by:

$$x_3 = 4 - x_1$$

We can now rewrite the equality as:

$$x_1 + x_3 = 4 \quad \text{and} \quad x_3 \ge 0$$

LP written in augmented form

Standard form

$$\max Z = 3x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 \qquad\quad &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \end{aligned}$$

and

$$x_1 \geq 0,\ x_2 \geq 0$$

Augmented form

$$\max Z = 3x_1 + 5x_2$$

subject to

$$\begin{array}{rrrrrl} x_1 & & +x_3 & & & = 4 \\ & 2x_2 & & +x_4 & & = 12 \\ 3x_1 & +2x_2 & & & +x_5 & = 18 \end{array}$$

and

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Basic solution

Standard form

$$\max Z = 3x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 \quad &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \end{aligned}$$

and

$$x_1 \ge 0,\ x_2 \ge 0$$

Augmented form

$$\max Z = 3x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 \quad + x_3 \quad &= 4 \\ 2x_2 \quad + x_4 \quad &= 12 \\ 3x_1 + 2x_2 \quad + x_5 &= 18 \end{aligned}$$

and

$$x_1, x_2, x_3, x_4, x_5 \ge 0$$

Augmented solution: original variabels + slack variabels
Eks. (0,0) -> (0,0,4,12,18)

Basic solution: augmented corner point solution (not necessarily feasible)

basic feasible solution: feasible augmented corner point solution

Basic solution

Augmented form

$$\max Z = 3x_1 + 5x_2$$

subject to

$$\begin{array}{rcl} x_1 + x_3 &=& 4 \\ 2x_2 + x_4 &=& 12 \\ 3x_1 + 2x_2 + x_5 &=& 18 \end{array}$$

and

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

5 variables, 3 equations.

In general: 5-3=2 degrees of freedom

In general we can assign any value to two of the variables and then it will be possible to find values for the remaining three variables such that the equations match.

The Simplex algorithm assigns the value 0 to the free variables.

Basic solution

Augmented form

$$\max Z = 3x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 \quad\quad + x_3 \quad\quad\quad &= 4 \\ 2x_2 \quad\quad + x_4 \quad &= 12 \\ 3x_1 + 2x_2 \quad\quad\quad + x_5 &= 18 \end{aligned}$$

and

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

5 variables, 3 equations.

In general: 5-3=2 degrees of freedom

In general we can assign any value to two of the variables and then it will be possible to find values for the remaining three variables such that the equations match.

The Simplex algorithm assigns the value 0 to the free variables.

Basic solution

1. Every variable is either denoted a basic or an non-basic variable.
2. Number of basic variables = number of constraints
3. Number of non-basic variables
   = (total number of variables) – (number of constraints)
4. Non-basic variables are always assigned the value 0.
5. The value of basic variables is found by removing non-basic variables from the system of equation and solve the remaining system.
6. If all basic variables are >=0 then the solution is feasible.

Rewriting the objective function

$$\boxed{\max Z = 3x_1 + 5x_2}$$

subject to

$$
\begin{array}{rrrrrl}
x_1 & & +x_3 & & & = 4 \\
 & 2x_2 & & +x_4 & & =12 \\
3x_1 & +2x_2 & & & +x_5 & =18
\end{array}
$$

and

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

$$\max Z$$

subject to

$$
\begin{array}{rrrrrrl}
Z & -3x_1 & -5x_2 & & & & = 0 \quad (0) \\
 & x_1 & & +x_3 & & & = 4 \quad (1) \\
 & & 2x_2 & & +x_4 & & =12 \quad (2) \\
 & 3x_1 & +2x_2 & & & +x_5 & =18 \quad (3)
\end{array}
$$

and

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Looks like a system of linear equations…

…but maximization and non-negativity is "new"

The simplex method – tableau form

TABLE 4.3 Initial system of equations for the Wyndor Glass Co. problem

(a) Algebraic Form	(b) Tabular Form								
	Basic Variable	**Eq.**	**Coefficient of:**						**Right Side**
			Z	x_1	x_2	x_3	x_4	x_5	
(0) $\boldsymbol{Z} - 3x_1 - 5x_2 = 0$	Z	(0)	1	−3	−5	0	0	0	0
(1) $x_1 + \boldsymbol{x_3} = 4$	x_3	(1)	0	1	0	1	0	0	4
(2) $2x_2 + \boldsymbol{x_4} = 12$	x_4	(2)	0	0	2	0	1	0	12
(3) $3x_1 + 2x_2 + \boldsymbol{x_5} = 18$	x_5	(3)	0	3	2	0	0	1	18

Initialization: Start by having x_1 and x_2 as non-basic variables. That makes it easy to find the value of the basic variables.

$$\begin{aligned}
x_1 \quad\quad + x_3 \quad\quad\quad &= 4 \qquad (1)\\
2x_2 \quad + x_4 \quad &= 12 \qquad (2)\\
3x_1 + 2x_2 \quad\quad\quad + x_5 &= 18 \qquad (3)
\end{aligned}$$

x_1	x_2	x_3	x_4	x_5
0	0	4	12	18

The simplex method

1. Optimality test: no negative numbers in row zero $\Rightarrow$ we are done.
2. Choose incoming basic variable (most negative in row 0).
3. Choose leaving basic variable (min ratio-test).
4. Restore the tableau to proper form with respect to the new basic variable (Gauss operations).
5. Jump back to 1.

The simplex method – 1: optimality test

TABLE 4.3 Initial system of equations for the Wyndor Glass Co. problem

(a) Algebraic Form	(b) Tabular Form								
	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
(0) $\boldsymbol{Z} - 3x_1 - 5x_2 = 0$	Z	(0)	1	−3	−5	0	0	0	0
(1) $x_1 + \boldsymbol{x_3} = 4$	x_3	(1)	0	1	0	1	0	0	4
(2) $2x_2 + \boldsymbol{x_4} = 12$	x_4	(2)	0	0	2	0	1	0	12
(3) $3x_1 + 2x_2 + \boldsymbol{x_5} = 18$	x_5	(3)	0	3	2	0	0	1	18

Optimality test:

$$Z - 3x_1 - 5x_2 = 0$$

We can obtain a better solution by increasing either x_1 or x_2 since they both have negative coefficients in row zero.

The simplex method
Step 2: choosing incoming basic variable

TABLE 4.3 Initial system of equations for the Wyndor Glass Co. problem

(a) Algebraic Form

$$
\begin{aligned}
&(0)\quad \boldsymbol{Z} - 3x_1 - 5x_2 &&= 0\\
&(1)\quad x_1 + \boldsymbol{x_3} &&= 4\\
&(2)\quad 2x_2 + \boldsymbol{x_4} &&= 12\\
&(3)\quad 3x_1 + 2x_2 + \boldsymbol{x_5} &&= 18
\end{aligned}
$$

(b) Tabular Form

Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
Z	(0)	1	−3	−5	0	0	0	0
x_3	(1)	0	1	0	1	0	0	4
x_4	(2)	0	0	2	0	1	0	12
x_5	(3)	0	3	2	0	0	1	18

choosing incoming basic variable:

- which of the two non-basic variabels should we increase?
- We chose x_2 since $5 > 3$. If we can increase x_2 by changing the values of the current basic variables then Z is going to increase by 5 per unit we increase x_2 (because all the current basic variables have coefficient 0 in row 0).
- x_2 is the entering basic variable.

Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side	Ratio
		Coefficient of:							
Z	(0)	1	−3	−5	0	0	0	0	
x_3	(1)	0	1	0	1	0	0	4	
x_4	(2)	0	0	2	0	1	0	12	$\rightarrow \frac{12}{2} = 6 \leftarrow$ minimum
x_5	(3)	0	3	2	0	0	1	18	$\rightarrow \frac{18}{2} = 9$

- We have found entering variable (pivot column). Mark column with a box.
- We keep $x_1 = 0$ and attempt to increase x_2.

$$\begin{array}{lllr} x_1 & +x_3 & & = 4 \quad (1) \\ & 2x_2 & +x_4 & =12 \quad (2) \\ 3x_1+2x_2 & & +x_5 & =18 \quad (3) \end{array}$$

$$x_3 = 4$$
$$x_4 = 12 - 2x_2$$
$$x_5 = 18 - 2x_2$$

- We can increase x_2 to 6. At that point $x_4 = 0$ and if we increase x_2 anymore then x_4 becomes negative (and all variables are required to be ≥ 0).
- x_4 leaves the basis (leaving variable).
- "Minimum ratio test"

x_1	x_2	x_3	x_4	x_5
0	6	?	0	?

TABLE 4.5 Simplex tableaux for the Wyndor Glass Co. problem after the first pivot row is divided by the first pivot number

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18

We have decided the pivot row. Put a box around this.

Element in both boxes is the *pivot element.*

TABLE 4.6 First two simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1						30
	x_3	(1)	0						4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0						6

$$r_2^{\text{new}} = \frac{1}{2} r_2^{\text{old}}$$

We use gausian elemination to create a new tableau. In this tableau

- The columns of the basic variables contains one ”1”, the rest of the elements are zero
- The ”1” in a column for a basic variable is in the row for which the variable is basic.

If we fail to do so: Optimality test and minimum ratio test in the next iteration is not going to work.

TABLE 4.6 First two simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side
0	Z	(0)	1	−3	−5	0	0	0	0
	x_3	(1)	0	1	0	1	0	0	4
	x_4	(2)	0	0	2	0	1	0	12
	x_5	(3)	0	3	2	0	0	1	18
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30
	x_3	(1)	0	1	0	1	0	0	4
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6
	x_5	(3)	0	3	0	0	−1	1	6

$$r_0^{\text{new}} = r_0^{\text{old}} + 5r_2^{\text{new}}$$
$$r_1^{\text{new}} = r_1^{\text{old}} + 0r_2^{\text{new}}$$
$$r_2^{\text{new}} = \tfrac{1}{2}r_2^{\text{old}}$$
$$r_3^{\text{new}} = r_3^{\text{old}} - 2r_2^{\text{new}}$$

We use gausian elemination to create a new tableau. In this tableau

- The columns of the basic variables contains one ”1”, the rest of the elements are zero
- The ”1” in a column for a basic variable is in the row for which the variable is basic.

If we fail to do so: Optimality test and minimum ratio test in the next iteration is not going to work.

TABLE 4.7 Steps 1 and 2 of iteration 2 for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	x_4	x_5	Right Side	Ratio
1	Z	(0)	1	-3	0	0	$\frac{5}{2}$	0	30	
	x_3	(1)	0	$\boxed{1}$	0	1	0	0	4	$\frac{4}{1} = 4$
	x_2	(2)	0	$\boxed{0}$	1	0	$\frac{1}{2}$	0	6	
	x_5	(3)	0	$\boxed{3}$	$\boxed{0}$	$\boxed{0}$	$\boxed{-1}$	$\boxed{1}$	$\boxed{6}$	$\frac{6}{3} = 2 \leftarrow$ minimum

Optimality test:

$$Z - 3x_1 + \frac{5}{2}x_4 = 30$$

We can improve Z by increasing x_1 (since the coefficient in front of x_1 is negative).
We have to continue.

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side	
0	Z	(0)	1	−3	−5	0	0	0	0	
	x_3	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	2	0	1	0	12	
	x_5	(3)	0	3	2	0	0	1	18	
1	Z	(0)	1	−3	0	0	$\frac{5}{2}$	0	30	$r_0^{\text{new}} = r_0^{\text{old}} + 5r_2^{\text{new}}$
	x_3	(1)	0	1	0	1	0	0	4	$r_1^{\text{new}} = r_1^{\text{old}} + 0r_2^{\text{new}}$
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6	$r_2^{\text{new}} = \frac{1}{2}r_2^{\text{old}}$
	x_5	(3)	0	3	0	0	−1	1	6	$r_3^{\text{new}} = r_3^{\text{old}} - 2r_2^{\text{new}}$
2	Z	(0)	1	0	0	0	$\frac{3}{2}$	1	36	$r_0^{\text{new}} = r_0^{\text{old}} + 3r_3^{\text{new}}$
	x_3	(1)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	$r_1^{\text{new}} = r_1^{\text{old}} - r_3^{\text{new}}$
	x_2	(2)	0	0	1	0	$\frac{1}{2}$	0	6	$r_2^{\text{new}} = r_2^{\text{old}} + 0r_3^{\text{new}}$
	x_1	(3)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	$r_3^{\text{new}} = \frac{1}{3}r_3^{\text{old}}$

Done!

Assumptions necessary for simplex algorithm to work:

- LP should be on standard form:

$$\max Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n &\leq b_m \\ x_1, x_2, \ldots, x_n &\geq 0 \end{aligned}$$

- **AND** Right hand side values should be non-negative, that is

$$b_i \geq 0, \forall i = 1, \ldots, m$$

Simplex algorithm

- On the DTU inside 42101 group there is a video with another example (under file sharing). The video is in Danish.

Simplex algorithm – special cases

- What if there are two equally good variables when we choose incoming variable in step 2?
- This could for example happen if our objective function was

$$\max Z = 3x_1 + 3x_2$$

- No problem: We just choose one of the variables as incoming. We are going to reach the optimal solution anyway.

Simplex algorithm – special cases

- What if no variable can leave the basis in minimum ratio test in step 3?
- In this case solution is unbounded!

TABLE 4.9 Initial simplex tableau for the Wyndor Glass Co. problem without the last two functional constraints

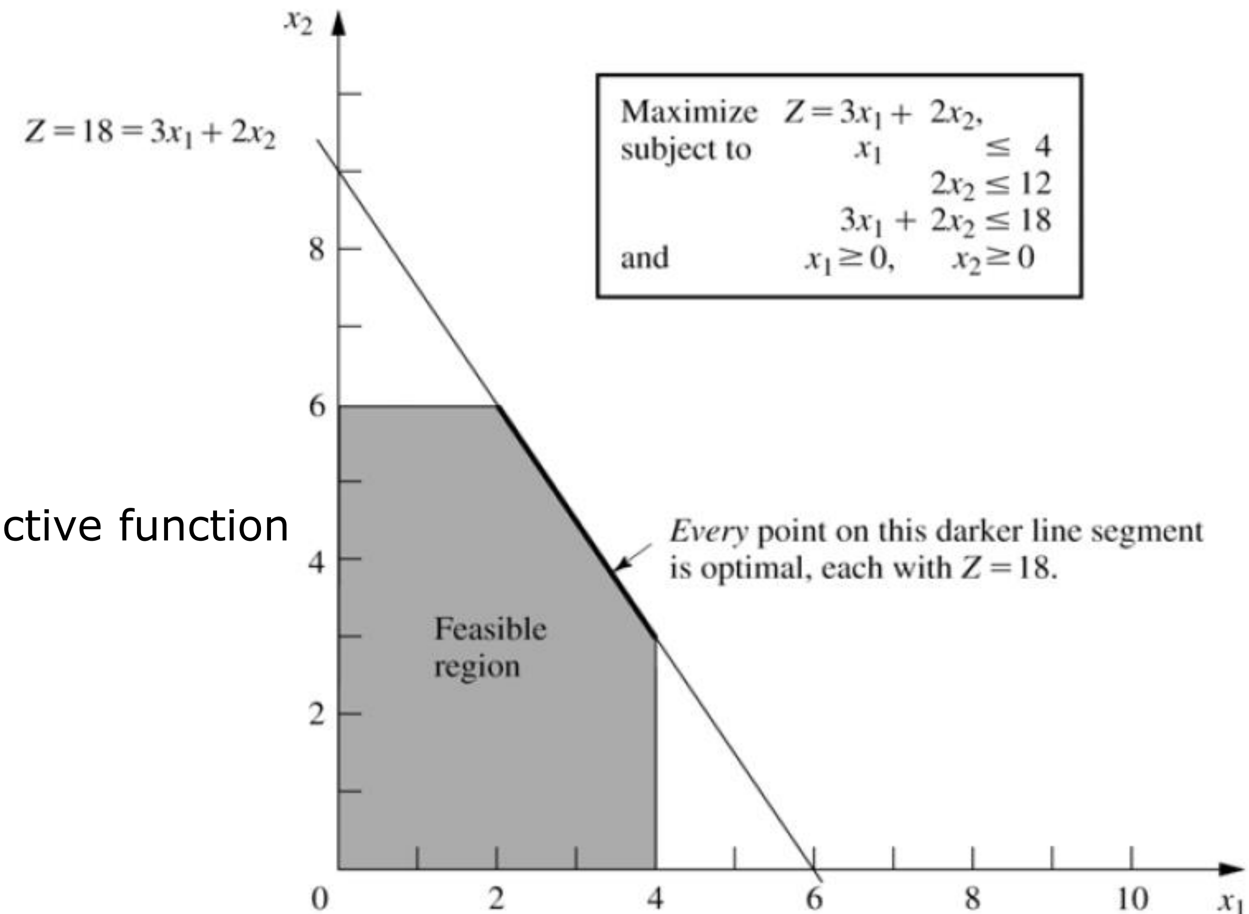
Basic Variable	Eq.	Coefficient of: Z	x_1	x_2	x_3	Right Side	Ratio
Z	(0)	1	−3	−5	0	0	
x_3	(1)	0	1	$\boxed{0}$	1	4	None

With $x_1 = 0$ and x_2 increasing,
$x_3 = 4 - 1x_1 - 0x_2 = 4 > 0$.

Simplex algorithm – special cases

- What if there are several optimal solutions?
- If we just want to know one optimal solution we can just do as we "always have done".

Wyndor with a new objective function



Simplex algorithm – special cases

- If we want to know all corner point feasible optimal solutions?
- When simplex ends: Check if any non-basis variable has coefficient 0 in row 0.
- If this is the case then there are multiple optimal CPF solutions.
- Extra pivots give the other optimal solutions.

If we want to know all corner point feasible optimal solutions?

TABLE 4.10 Complete set of simplex tableaux to obtain all optimal BF solutions for the Wyndor Glass Co. problem with $c_2 = 2$

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	Right Side	Solution Optimal?
0	Z	(0)	1	−3	−2	0	0	0	0	No
	x_3	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	2	0	1	0	12	
	x_5	(3)	0	3	2	0	0	1	18	
1	Z	(0)	1	0	−2	3	0	0	12	No
	x_1	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	2	0	1	0	12	
	x_5	(3)	0	0	2	−3	0	1	6	
2	Z	(0)	1	0	0	**0**	0	1	18	Yes
	x_1	(1)	0	1	0	1	0	0	4	
	x_4	(2)	0	0	0	3	1	−1	6	
	x_2	(3)	0	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	3	
Extra	Z	(0)	1	0	0	0	**0**	1	18	Yes
	x_1	(1)	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	
	x_3	(2)	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	
	x_2	(3)	0	0	1	0	$\frac{1}{2}$	0	6	

We are done, but x3 is not in the basis and has coefficient 0 in row 0. This means that there is at least one more optimal CPF solution.

Another non-basic variable with coefficient 0 in row 0.

However, if we pivot this variable in, we would go back to the previous tableau. We have found all optimal CPF solutions

How to handle LPs that are not on the standard form? *Minimization*

- Transform minimization to maximization:

$$\min x \Leftrightarrow \max -x$$

- or

$$\min 5x_1 + 6x_2 \Leftrightarrow \max -5x_1 - 6x_2$$

- In general:

$$\min Z = c_1x_1 + c_2x_2 \ldots + c_nx_n \Leftrightarrow \max -Z = -c_1x_1 - c_2x_2 \ldots - c_nx_n$$

How to handle LPs that are not on the standard form? *Negative variables*

- Assume we have a variable x_j that can take negative values.
- If we know a lower bound L_j for the negative variable then we can define a new variable and substitute:

$$x'_j = x_j - L_j \quad \text{og} \quad x'_j \geq 0$$

- Example. Let's assume

$$x_1 \geq -10$$

in the Wyndor eksemplet. We then introduce a new variable x'_1 and substitute:

$$x'_1 = x_1 + 10,\ x'_1 \geq 0 \quad \Leftrightarrow \quad x_1 = x'_1 - 10,\ x'_1 \geq 0$$

Let's also change the objective to

$$\max Z = -3x_1 + 5x_2$$

to make the problem more interesting

$$\boxed{\begin{aligned} &Z = -3x_1 + 5x_2 \\ &x_1 \leq 4 \\ &2x_2 \leq 12 \\ &3x_1 + 2x_2 \leq 18 \\ &x_1 \geq -10, \quad x_2 \geq 0 \end{aligned}} \rightarrow \boxed{\begin{aligned} &Z = -3(x'_1 - 10) + 5x_2 \\ &x'_1 - 10 \leq 4 \\ &2x_2 \leq 12 \\ &3(x'_1 - 10) + 2x_2 \leq 18 \\ &x'_1 - 10 \geq -10, \quad x_2 \geq 0 \end{aligned}} \rightarrow \boxed{\begin{aligned} &Z = 30 - 3x'_1 + 5x_2 \\ &x'_1 \leq 14 \\ &2x_2 \leq 12 \\ &3x'_1 + 2x_2 \leq 48 \\ &x'_1 \geq 0, \quad x_2 \geq 0 \end{aligned}}$$

How to handle LPs that are not on the standard form? *Negative variables*

b.v.	eq.	Z	x1'	x2	x3	x4	x5	RHS
Z	0	1	3.00	-5.00	0.00	0.00	0.00	30.00
x3	1	0	1.00	0.00	1.00	0.00	0.00	14.00
x4	2	0	0.00	2.00	0.00	1.00	0.00	12.00
x5	3	0	3.00	2.00	0.00	0.00	1.00	48.00

b.v.	eq.	Z	x1'	x2	x3	x4	x5	RHS
Z	0	1	3.00	0.00	0.00	2.50	0.00	60.00
x3	1	0	1.00	0.00	1.00	0.00	0.00	14.00
x2	2	0	0.00	1.00	0.00	0.50	0.00	6.00
x5	3	0	3.00	0.00	0.00	-1.00	1.00	36.00

Solution:

$$x_1' = 0, x_2 = 6, Z = 60 \quad \Leftrightarrow \quad x_1 = x_1' - 10 = -10, x_2 = 6, Z = 60$$

$$\boxed{\begin{aligned} &Z = -3x_1 + 5x_2 \\ &x_1 \leq 4 \\ &2x_2 \leq 12 \\ &3x_1 + 2x_2 \leq 18 \\ &x_1 \geq -10, \quad x_2 \geq 0 \end{aligned}} \rightarrow \boxed{\begin{aligned} &Z = -3(x_1' - 10) + 5x_2 \\ &x_1' - 10 \leq 4 \\ &2x_2 \leq 12 \\ &3(x_1' - 10) + 2x_2 \leq 18 \\ &x_1' - 10 \geq -10, \quad x_2 \geq 0 \end{aligned}} \rightarrow \boxed{\begin{aligned} &Z = 30 - 3x_1' + 5x_2 \\ &x_1' \leq 14 \\ &2x_2 \leq 12 \\ &3x_1' + 2x_2 \leq 48 \\ &x_1' \geq 0, \quad x_2 \geq 0 \end{aligned}}$$

How to handle LPs that are not on the standard form? *Negative variables*

- If there is no lower bounds on the negative variable(s) then we need to represent each negative variable by two variables in the LP:
- assume that $x_j \in \mathbb{R}$. Then we rewrite

$$x_j = x_j^+ - x_j^-, \quad x_j^+ \geq 0, \quad x_j^- \geq 0$$

- x_j^+ is the positive part of x_j. x_j^- is the negative part.
- Example:

$$\begin{array}{rrl} & \max 3x_1 + 5x_2 & \\ \text{s.t.} & x_1 & \leq 4 \\ & 2x_2 & \leq 20 \\ & 3x_1 + 2x_2 & \leq 18 \\ & x_1 \in \mathbb{R}, \quad x_2 & \geq 0 \end{array} \longrightarrow \begin{array}{rrl} & \max 3x_1^+ - 3x_1^- + 5x_2 & \\ \text{s.t.} & x_1^+ - x_1^- & \leq 4 \\ & 2x_2 & \leq 20 \\ & 3x_1^+ - 3x_1^- + 2x_2 & \leq 18 \\ & x_1^+, \quad x_1^-, \quad x_2 & \geq 0 \end{array}$$

$$\begin{array}{llll} \max & 3x_1 + 5x_2 & & \\ \text{s.t.} & x_1 & \leq & 4 \\ & 2x_2 & \leq & 20 \\ & 3x_1 + 2x_2 & \leq & 18 \\ & x_1 \in \mathbb{R}, \quad x_2 & \geq & 0 \end{array} \longrightarrow \begin{array}{llll} \max & 3x_1^+ - 3x_1^- + 5x_2 & & \\ \text{s.t.} & x_1^+ - x_1^- & \leq & 4 \\ & 2x_2 & \leq & 20 \\ & 3x_1^+ - 3x_1^- + 2x_2 & \leq & 18 \\ & x_1^+, \quad x_1^-, \quad x_2 & \geq & 0 \end{array}$$

b.v.	eq.	Z	x1+	x1-	x2	x3	x4	x5	RHS
Z	0	1	-3.00	3.00	-5.00	0.00	0.00	0.00	0.00
x3	1	0	1.00	-1.00	0.00	1.00	0.00	0.00	4.00
x4	2	0	0.00	0.00	2.00	0.00	1.00	0.00	20.00
x5	3	0	3.00	-3.00	2.00	0.00	0.00	1.00	18.00

b.v.	eq.	Z	x1+	x1-	x2	x3	x4	x5	RHS
Z	0	1	4.50	-4.50	0.00	0.00	0.00	2.50	45.00
x3	1	0	1.00	-1.00	0.00	1.00	0.00	0.00	4.00
x4	2	0	-3.00	3.00	0.00	0.00	1.00	-1.00	2.00
x2	3	0	1.50	-1.50	1.00	0.00	0.00	0.50	9.00

b.v.	eq.	Z	x1+	x1-	x2	x3	x4	x5	RHS
Z	0	1	0.00	0.00	0.00	0.00	1.50	1.00	48.00
x3	1	0	0.00	0.00	0.00	1.00	0.33	-0.33	4.67
x1-	2	0	-1.00	1.00	0.00	0.00	0.33	-0.33	0.67
x2	3	0	0.00	0.00	1.00	0.00	0.50	0.00	10.00

$$\begin{aligned} \max\ & 3x_1 + 5x_2 \\ \text{s.t.}\ & x_1 \le 4 \\ & 2x_2 \le 20 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1 \in \mathbb{R},\quad x_2 \ge 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} \max\ & 3x_1^+ - 3x_1^- + 5x_2 \\ \text{s.t.}\ & x_1^+ - x_1^- \le 4 \\ & 2x_2 \le 20 \\ & 3x_1^+ - 3x_1^- + 2x_2 \le 18 \\ & x_1^+,\quad x_1^-,\quad x_2 \ge 0 \end{aligned}$$

b.v.	eq.	Z	x1+	x1-	x2	x3	x4	x5	RHS
Z	0	1	-3.00	3.00	-5.00	0.00	0.00	0.00	0.00
x3	1	0	1.00	-1.00	0.00	1.00	0.00	0.00	4.00
x4	2	0	0.00	0.00	2.00	0.00	1.00	0.00	20.00
x5	3	0	3.00	-3.00	2.00	0.00	0.00	1.00	18.00

b.v.	eq.	Z						5	RHS
Z	0	1						0	45.00
x3	1	0						0	4.00
x4	2	0	-3.00	3.00	0.00	0.00	1.00	-1.00	2.00
x2	3	0	1.50	-1.50	1.00	0.00	0.00	0.50	9.00

$$x_1^- = \frac{2}{3}, x_2 = 10 \quad \rightarrow \quad x_1 = -\frac{2}{3}, x_2 = 10$$

b.v.	eq.	Z	x1+	x1-	x2	x3	x4	x5	RHS
Z	0	1	0.00	0.00	0.00	0.00	1.50	1.00	48.00
x3	1	0	0.00	0.00	0.00	1.00	0.33	-0.33	4.67
x1-	2	0	-1.00	1.00	0.00	0.00	0.33	-0.33	0.67
x2	3	0	0.00	0.00	1.00	0.00	0.50	0.00	10.00

How to handle LPs that are not on the standard form?

- How to handle "=" constraints?
- How to handle "$\geq$" constraints?
- How to handle negative right hand sides of constraints?

More on that next week!

Simplex algorithm

- George Dantzig (1914-2005) invented the Simplex algorithm in 1947.
- Invented several other OR techniques.

- One of the most important algorithms discovered in the last century?
  "The Best of the 20th Century: Editors Name Top 10 Algorithms" (SIAM, 2004)

- Important in many follow up OR courses
  E.g.
  - 42114 – integer programming
  - 42115 – network optimization
  - 42136 - Large Scale Optimization using Decomposition
  - 42116 - Implementing OR Solution Methods

Professor George B. Dantzig på Lundtoftesletten i går (t. v.) sammen med sin danske vært, lektor Oli G. B. Madsen fra højskolens institut for matematisk statistik og operationsanalyse.

Han laver indviklet planlægnings-værktøj

En lille, varm mand med en stor, kold hjerne.

Sådan karakteriserede en af de studerende ved Danmarks tekniske Højskole i går dagens gæsteforelæser, professor George B. Dantzig fra Stanford University, USA.

En anden erklærede efter forelæsningen i det fyldte auditorium 11/308, at Dantzig formentlig er den mest overbevisende, personlige protest, der findes, imod den udbredte, filosofiske tese om, at mennesket i virkeligheden slet ikke er indrettet til rationel forudsigelse.

Professor Dantzig er ikke nogen helt almindelig spåmand. Han har arbejdet med virkelighedsmodeller siden 1940 og bl.a. 'opfundet' den såkaldte simplex-metode til løsning af lineære programmerings-problemer.

Det er et planlægningsværktøj til bearbejdning af information på en sådan måde, at resultaterne kan trækkes ud over nu-punktet og danne grundlag for beslutninger om, hvordan fremtiden skal se ud.

Professorens budskab til sine yngre, danske kolleger inden for operationsanalyse og virksomhedsledelse m.v. er, at mennesket ikke længer kan nøjes med at tilrettelægge dagen og vejen og måske tænke lidt på morgendagen.

Det omgivende samfund er nu så kompliceret, at der må komplicerede planlægningsværktøjer til, hvis vi skal lede udviklingen netop i den retning, vi ønsker det.

Det er de færreste, der kan klare sig gennem tilværelsen alene ved hjælp af følelser og fornemmelser.

cauchi.

Politikken, 1976

Abstract and concrete models
Concrete Wyndor model

$$\max Z = 3x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 \quad\quad &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \end{aligned}$$

and

$$x_1 \geq 0, \; x_2 \geq 0$$

- 2 products (glass doors, windows)
- Requires time on 3 plants

	1	2	Capacity (hours/week)
1	1	0	4
2	0	2	12
3	3	2	18
profit	3	5	

- Profit is per unit and should be multiplied by 1000$
- We want to plan the production so that our profit is maximized

Abstract and concrete models
Abstract Wyndor model

- We may want to extend the model with more products and more plants
- Or we may want to use the model for another company
- It would be nice to write the model in a more generic way!

- 2 products (glass doors, windows)
- Requires time on 3 plants

	1	2	Capacity (hours/week)
1	1	0	4
2	0	2	12
3	3	2	18
profit	3	5	

Abstract and concrete models
Abstract Wyndor model

It would be nice to write the model in a more generic way!

- m: number of plants
- n: number of products
- b_i: hours available on plant i per week ($i = 1, \ldots, m$)
- p_j: profit per batch of product j ($j = 1, \ldots, n$)
- a_{ij}: how many hours we need on plant i when producing one batch of product j

- 2 products (glass doors, windows)
- Requires time on 3 plants

	1	2	Capacity (hours/week)
1	1	0	4
2	0	2	12
3	3	2	18
profit	3	5	

Abstract and concrete models
Abstract Wyndor model

It would be nice to write the model in a more generic way!

- m: number of plants
- n: number of products
- b_i: hours available on plant i per week ($i = 1, \ldots, m$)
- p_j: profit per batch of product j ($j = 1, \ldots, n$)
- a_{ij}: how many hours we need on plant i when producing one batch of product j
- x_j: Our decision variables. Indicates how many batches of product j we should make per week.
- The abstract model:

$$\max \sum_{j=1}^{n} p_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \forall i = 1, \ldots, m$$

$$x_j \geq 0 \quad \forall j = 1, \ldots, n$$

Abstract Wyndor model in Julia (1)

```
using JuMP, GLPK

m = 3          # number of plants
n = 2          # number of products
b = [ 4 12 18] # hours available on plant i per week
p = [ 3 5 ]      # profit per batch of product
a = [   1 0;
        0 2;
        3 2]

model = Model(with_optimizer(GLPK.Optimizer))
# x[j]: how many batches of product j should we make per week
@variable(model, x[j=1:n] >= 0 )

@objective(model, Max, sum(p[j]*x[j] for j=1:n) )
@constraint(model, [i=1:m], sum(a[i,j]*x[j] for j=1:n) <= b[i] )

print(model)

optimize!(model)

println("Objective value: ", JuMP.objective_value(model))
println("x = ", JuMP.value.(x))
```

Abstract Wyndor model in Julia (2)

```
using JuMP, GLPK

include("wyndor-data2.jl")

model = Model(with_optimizer(GLPK.Optimizer))
# x[j]: how many batches of product j should we make per week
@variable(model, x[j=1:n] >= 0 )

@objective(model, Max, sum(p[j]*x[j] for j=1:n) )
@constraint(model, [i=1:m], sum(a[i,j]*x[j] for j=1:n) <= b[i] )

print(model)

optimize!(model)

println("Objective value: ", JuMP.objective_value(model))
println("x = ", JuMP.value.(x))
```

That's it!
Now let's work on the exercises