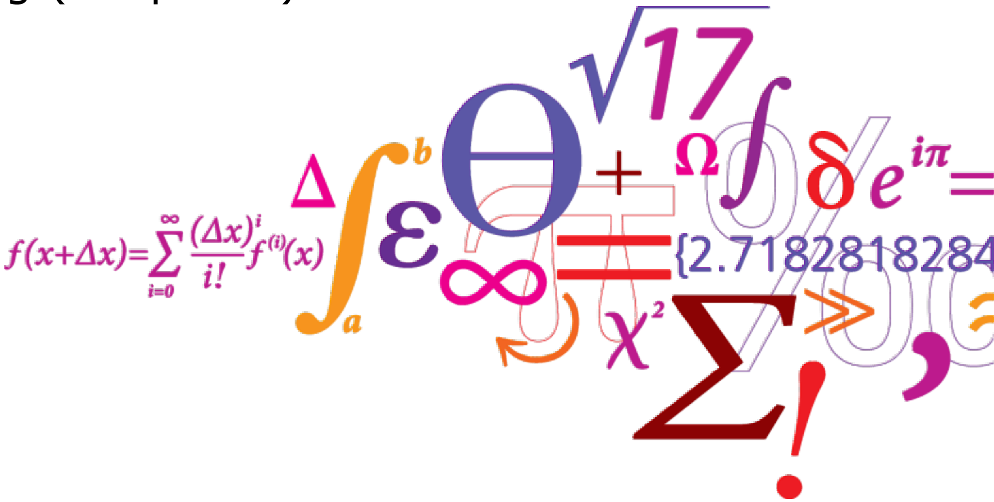
42101 Introduction to Operations Research

- What is Operations Research?
- Overview of the course
- Introduction to Linear Programming (Chapter 3)

Stefan Røpke
Richard Lusby



DTU Management Engineering
Department of Management Engineering

What is Operations Research (OR)?

- Very short definition:
  Application of mathematical techniques to decision making

- Examples of applications
  - Planning of production: which products should be produced at what time?
  - Work planning: Which employee should be on duty at what time?

- … it becomes clear as we go on!

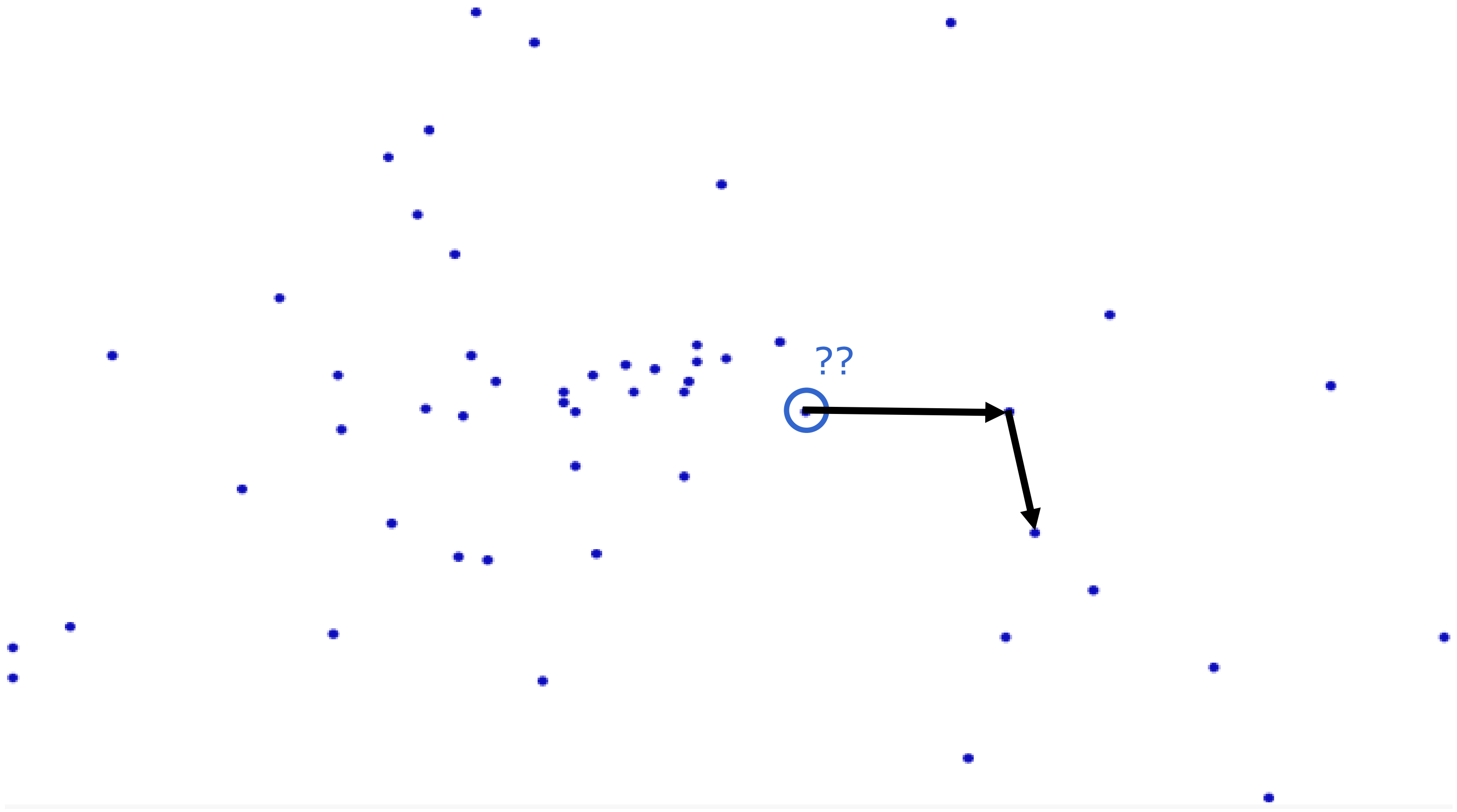
What is Operations Research?

- Useful skills when studying Operations Research
  - Mathematics
  - Computer Science
  - Machine learning/Artificial intellingece
  - Ability to understand new application areas
  - Common sense

Very short history of operations research

- Debatable when and where operations research started.
- However, a mile stone is the use of operations research in UK during World War 2 in order to improve war efforts.
- After the second world war the techniques were used for peaceful purposes and accelerated due to the arrival of computers
- Now OR techniques are used in many places. Everyday we are in contact with products and serviced that in one way or the other have been influenced by OR.

The traveling salesman problem (TSP)



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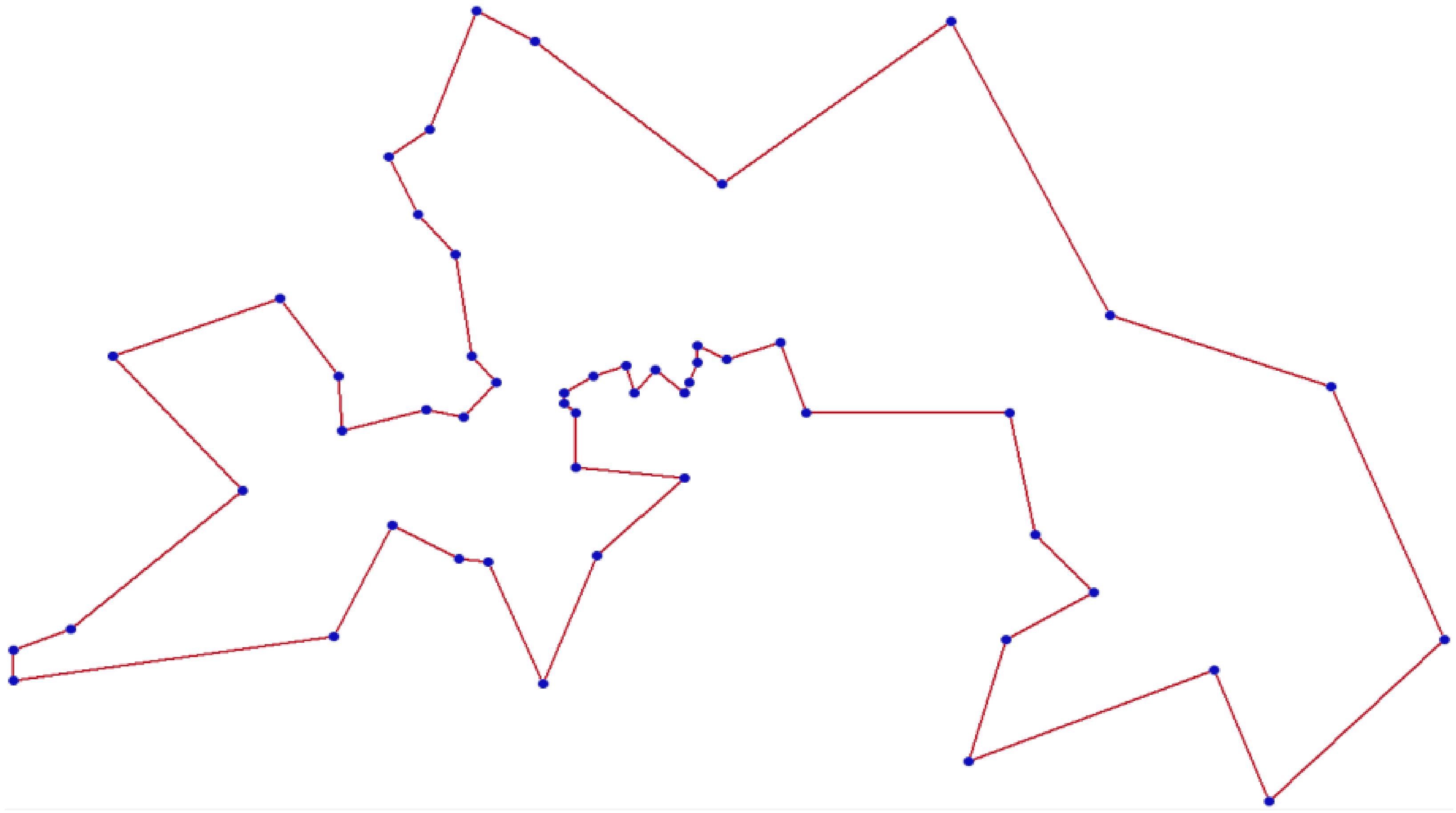
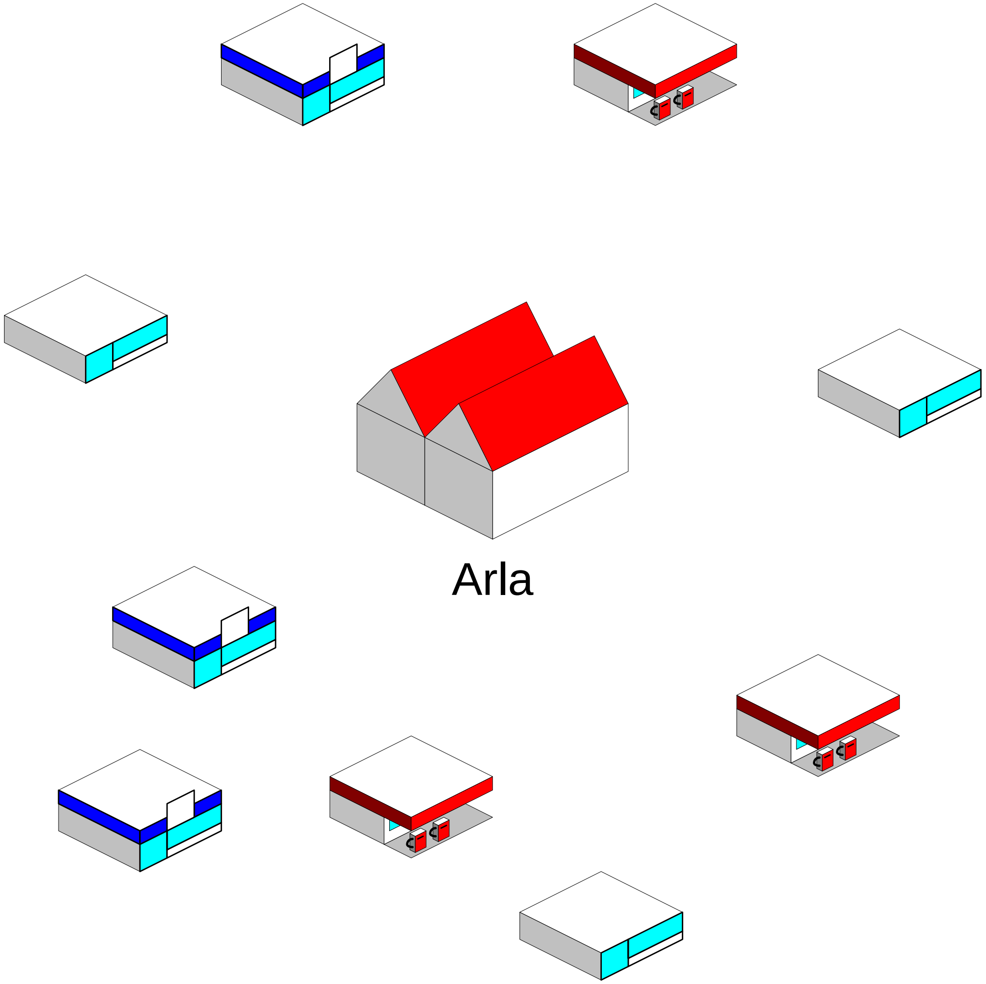


Distribution planning

Our task: supply super markets, gas stations, corner stores etc. with dairy (milk) products.
Each truck can at most serve 4 shops.



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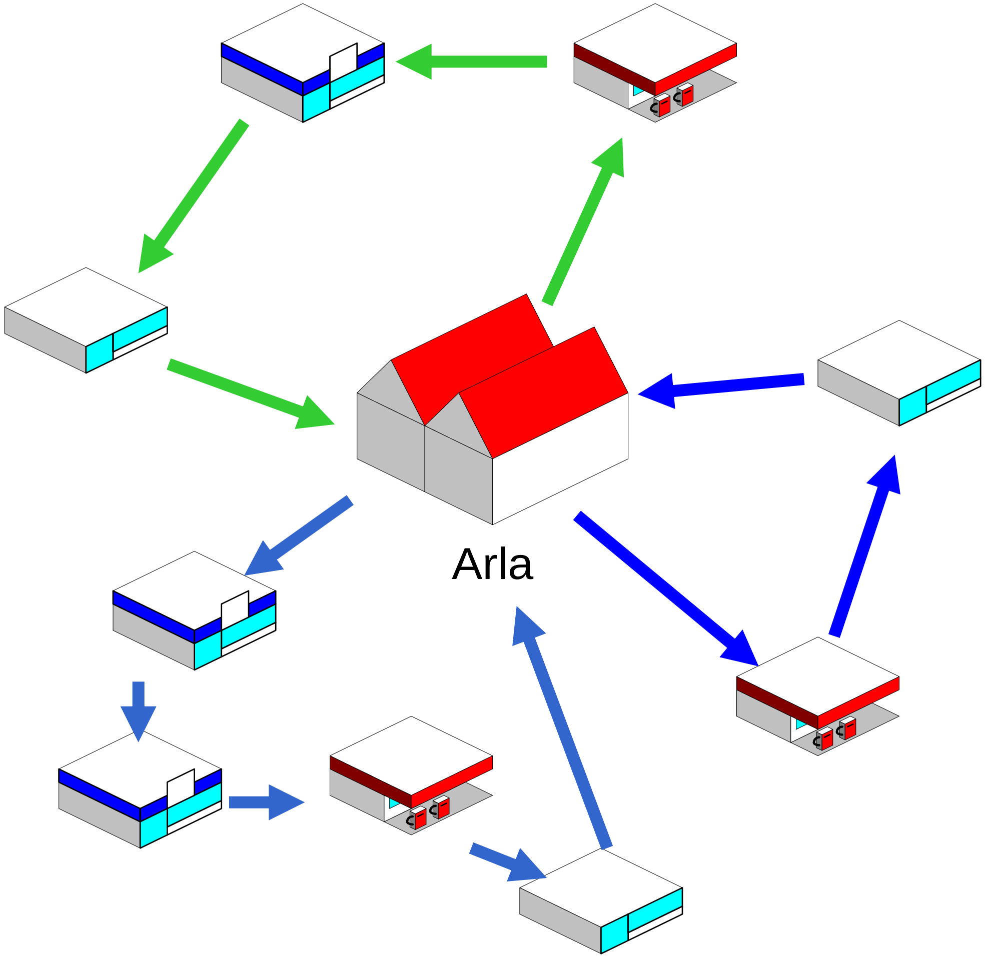


Operations research is everywhere

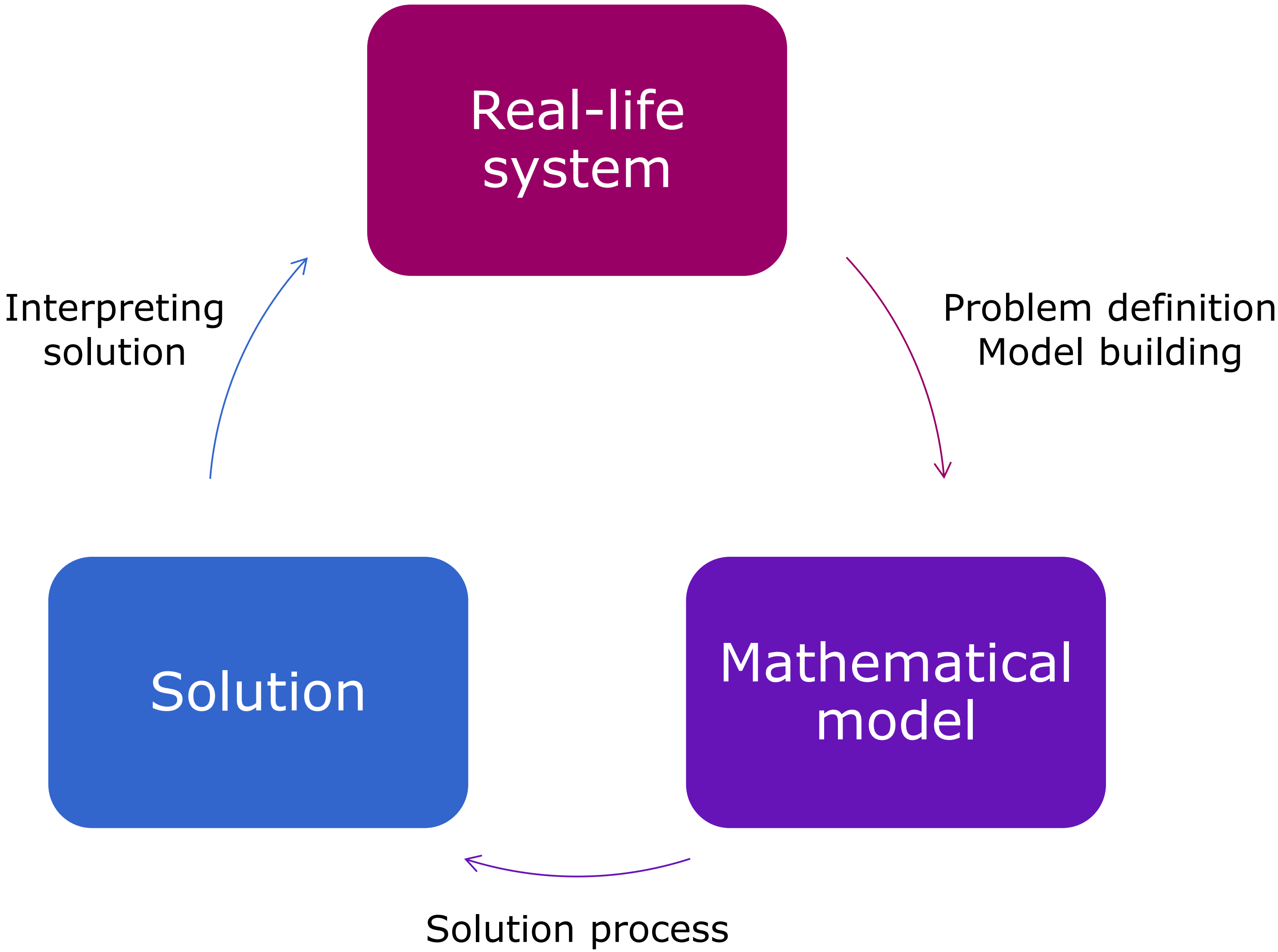
- Products are often brought to the end customer using trucks, ships or planes. OR may have been involved in planning the transport.

- The production of products you buy may have been planned using OR

- When you ride on the bus, metro or the S-train OR may have been used in the planning processes

- OR methods may have been used to design the electronics in your smartphone, computer, etc. Used to optimize signal path, for example.

Work cycle in OR



Course overview

#	Date	Topic	Reading material	Teacher	Project
1	02-sep	Introduction to Operations Research and Linear Programming	3.1-3.4. (1.1-1.4) (2.1-2.7)	SR	
2	09-sep	Modeling with linear programming The Simplex algorithm	3.1-3.4. 4.1-4.4. (Appendix 6)	SR	
3	16-sep	Basic graph theory. Handling non-standard LPs	4.5-4.7. 10.1-10.2.	SR	
4	23-sep	Simplex using matrix computations	5.1-5.5. (Appendix 4)	SR	Project 1 start. Hand in: 13th of October
5	30-sep	Duality and sensitivity analysis	6.1-6.6 7.1-7.2	SR	
6	07-okt	Modeling with integer variables	12.1-12.4.	SR	
	14-okt	Autumn holiday			

Course overview

7	21-okt	Modeling with integer variables	12.1-12.4.	RL	
8	28-okt	Modeling with integer variables. Greedy algorithm	12.1-12.4.	RL	Project 2 start (preliminary)
9	04-nov	Modeling with integer variables. Solving a model iteratively	12.1-12.4.	RL	
10	11-nov	Total unimodularity: Assignment problem Min cost flow	9.3,9.4, 10.6 Lecture notes on the assignment problem.	RL	
11	18-nov	Linear relaxation, Branch and bound	12.5-12.7	RL	
12	25-nov	Dual simplex transport problem	8.1, 9.1 Lecture notes on the network simplex algorithm	RL	
13	02-dec	Solve old exam		RL / SR	
	???	Question hour. Date is to be decided.		RL / SR	
	11-dec	Exam. See `www.eksamensplan.dtu.dk`			

Practical info

- **Main book**
  - Hillier, Lieberman: Introduction to Operations Research, 10th edition. [earlier version are similar, but page and chapter references in class may be “off”]
  - Material on DTU inside.
- **Prerequisites** (Simultaneous Linear Equations, Matrix operations)
  - Read appendix 4 and 6
  - Appendix 6 you can get from the book web-page or from DTU inside.
- **DTU inside**
  - Course overview, slides, projects, additional reading material
  - “Final” version of slides ready around noon Sunday.
- **Language:** English
  - You can hand in projects in Danish or English
  - You can answer exam in Danish or English

Practical information

- **Lectures**
  - Monday from 13 to 15 building 341, room 21
- **Exercises**
  - Monday from 15 to 17
  - Building 324, room 040, 050 and 060.
- **Preparation:**
  - Read the text and look at slides before lecture.
  - If you do not finish exercises, then complete them before next lecture.
- **Teachers**
  - Stefan Røpke, ropke@dtu.dk (first part)
  - Richard Lusby, rmlu@dtu.dk (second part)
- **Teaching assistants**

Kristine Børsting	Morten Bondorf Gerdes	**Elisabeth Marie Heegaard**
		

Demanding course?

- You have to make an effort to meet the learning objectives!
    - Be active from the start – we keep building on material from past lectures.
    - Read the book. It’s not enough to be present at the lectures.
    - Solve all exercises
    - Exercises from 15 to 17 are more important compared to the lectures.

Two projects

- Show what you can do with OR.
- Go a little further with some topics.
- Preparation for the exam.

- Projects start on the 23th of September and 28th of October (Preliminary plan)

- Content is the material covered until (and including) the project start day

- Do projects in groups of two to three

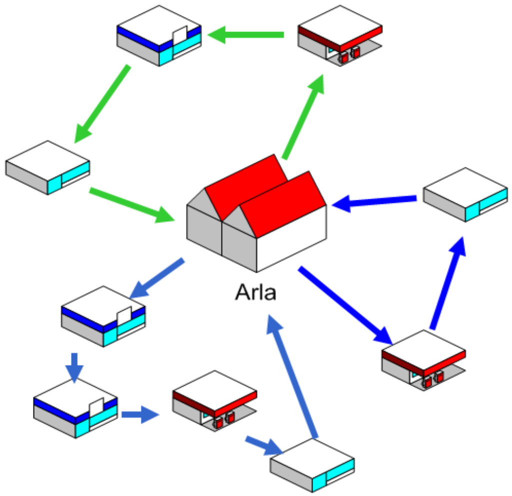
- You need to pass both projects in order to attend the exam. We make an overall assessment of the two projects.
  – Perfect project: 100 point
  – Passed: 50-100 point

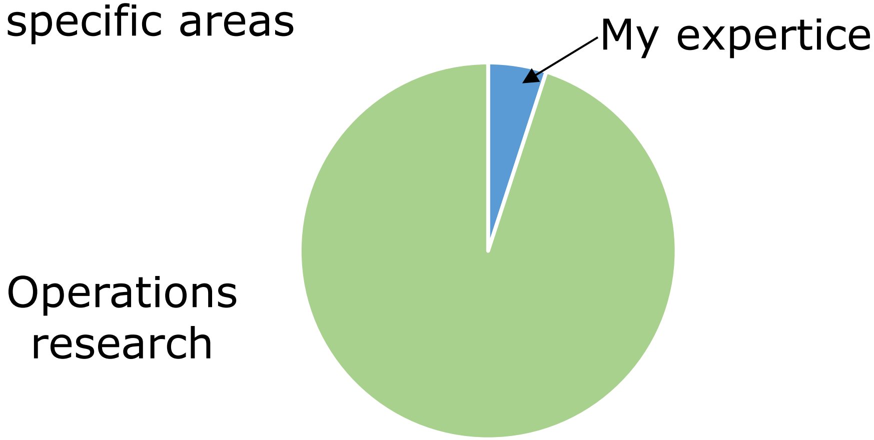
Exam

- 4 hours written exam
- “All aids allowed”.
- Combination of multiple choice and text questions. Mainly multiple-choice (could be multiple-choice only).

About myself

- Contact:
  - Email: ropke@dtu.dk
  - Office 224, building 424.
- Background
  - Been employed at DTU since 2008
  - Professor since 2012.
  - Education in Computer Science (University of Copenhagen)
  - I do research in operations research.
  - Is especially interested in applications within tranport and solution methods.
  - World class research in a few specific areas





Introduction to Linear programming (LP)

Programming = make a program = planning

Content of chapter 3

- 3. Introduction to LP
- 3.1 Prototype example: Wyndor
- 3.2 LP-model
- 3.3 Assumptions for LP
- 3.4 More examples

3.1 Wyndor: problem

- 2 products (glass doors, windows)
- Requires time on 3 plants

	1	2	Capacity (hours/week)
1	1	0	4
2	0	2	12
3	3	2	18
profit	3	5	

- Profit is per unit and should be multiplied by 1000$
- We want to plan the production so that our profit is maximized

3.1 Wyndor model

Decision variables:
x_1 = production of product 1 (units/week)
x_2 = production af product 2 (units/week)

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3.1 Wyndor model

Decision variables:
x_1 = production of product 1 (units/week)
x_2 = production af product 2 (units/week)

Symbolic model (LP)

$$\max Z = 3x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 \quad\quad &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \end{aligned}$$

and

$$x_1 \ge 0,\ x_2 \ge 0$$

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Components of the model:

- Decision variables (what we can decide)
- Objective function (what we wish to optimize)
- Constraints (that we have to respect)

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	1	2	Capacity (Hours/week)
1	1	0	4
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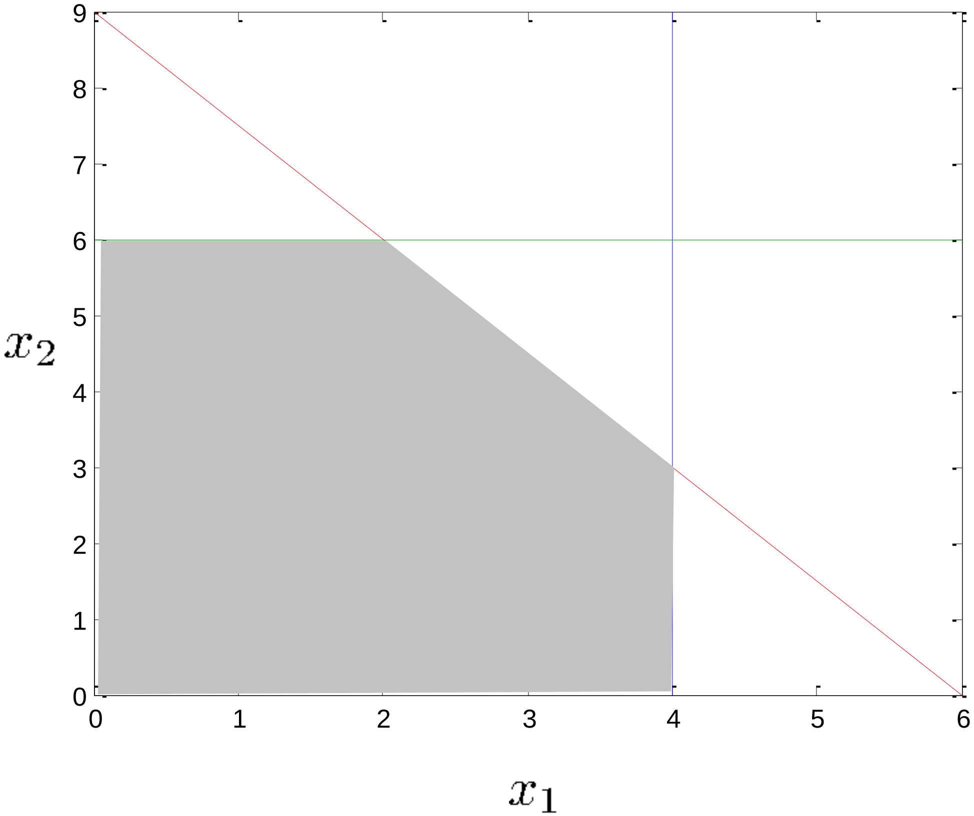
3	3	2	18
profit	3	5	

Is it ok if decision variable is non-integer?
E.g. $x_1 = 2.76$**?**

Wyndor solution



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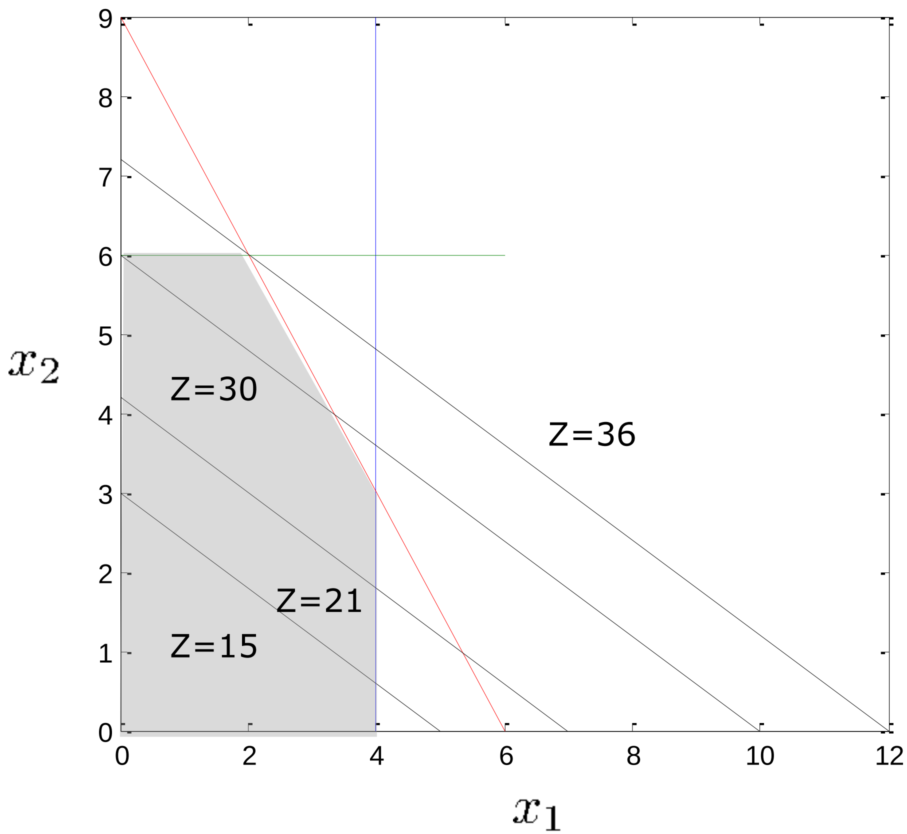

$$\max Z = 3x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \end{aligned}$$

and

$$x_1 \geq 0, \;\; x_2 \geq 0$$

$$\begin{aligned} x_1^* &= 2 \text{ units per week} \\ x_2^* &= 6 \text{ units per week} \\ Z^* &= 36\text{k\$ per week} \\ &= 36{,}000\text{\$ per week} \end{aligned}$$

The graphical method

1. Draw the line for the objective function for a fixed Z value. Make sure that the line intersects the feasible area.
2. While keeping the slope fixed, "move" this line toward increasing values for Z.
3. The point (or the line) that the line touch just before leaving the feasible area is the optimal solution(s).

3.2 LP-model in general

- Allocating m resources to n activities
- Decision variables: x_j: level of activity j
- Parameters (input)
  - c_j: How useful is activity j
  - b_i: How many units of resource i are available
  - a_{ij}: units of resource i consumed by each unit of activity j
- Z: measure of performance

$$\max Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n &\leq b_m \\ x_1, x_2, \ldots, x_n &\geq 0 \end{aligned}$$

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- Z: measure of performance

Not always useful or possible to interpret an LP model this way

"Our" standard form

$$\max Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n &\leq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n &\leq b_m \\
x_1, x_2, \ldots, x_n &\geq 0
\end{aligned}$$

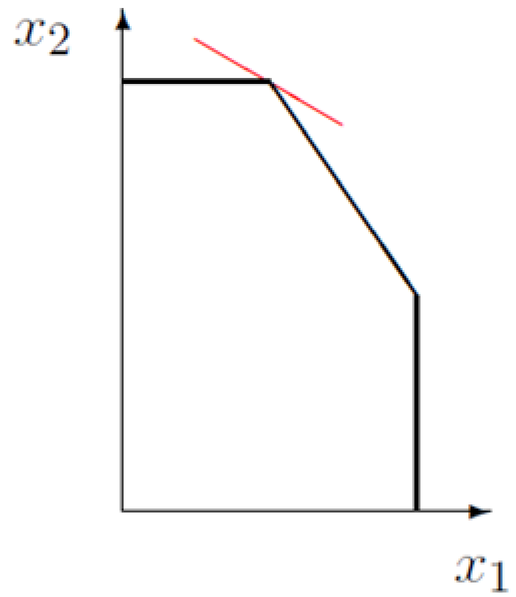
LP's as above are said to be on the *standard form.*
Deviation from the standard form:

- Minimization
- $\geq$ constraints
- $=$ constraints
- Decision variables that can take negative values.

Terminology

Terminology

- Feasible solution
- Infeasible solution
- Feasible region
- Optimal solution
- Objective function
- Objective value
- Unbounded solution
- Corner-point feasible solution – CPF



Assumptions for linear programming

$$\max Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n &\leq b_m \\ x_1, x_2, \ldots, x_n &\geq 0 \end{aligned}$$

- Linear objective function and constraints.
- c_j and a_{ij} are constants, only x_j are variables. Objective function and constraints are **always** on this form.
- Variables are real numbers. We cannot force them to be integers, for example.

3.4 Examples

- p. 45 Design of Radiation Therapy
- p. 47 Regional Planning
- p. 51 Controlling Air Pollution
- p. 53 Reclaiming Solid Wastes
- p. 57 Personnel Scheduling
- p. 60 Distributing Goods Through a Distribution Network

p. 45 Design of Radiation Therapy

- A tumor is treated with radiation therapy.
- Doctors are considering two different entry points for the radiation beams

	Beam 1	Beam 2	Restrictions
Healthy anatomy	0.4	0.5	
Critical tissues	0.3	0.1	≤ 2.7
Tumor region	0.5	0.5	$= 6$
Center of tumor	0.6	0.4	≥ 6

- All numbers in the table are in kilorads
- Numbers for beam 1 and 2 are for a normal dose. We can give higher and lower doses as necessary.
- How to model this? Think of
  - Decision variables
  - Constraints
  - Objective function

p. 45 Radiation Therapy: model

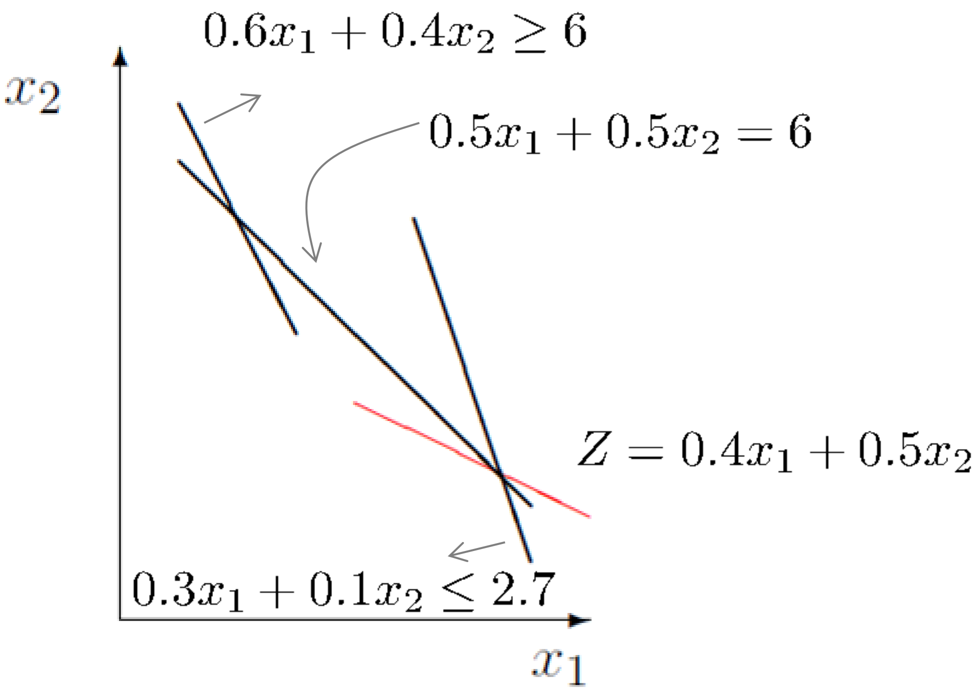
- x_j: intensity of beam j ($j = 1, 2$)
- Z: radiation dose to healthy anatomy (kilorads)

$$\min Z = 0.4x_1 + 0.5x_2$$

subject to

$$\begin{aligned} 0.3x_1 + 0.1x_2 &\le 2.7 \\ 0.5x_1 + 0.5x_2 &= 6 \\ 0.6x_1 + 0.4x_2 &\ge 6 \\ x_1, x_2 &\ge 0 \end{aligned}$$

p. 45 Radiation Therapy: solution



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subject to

$$\begin{aligned} 0.3x_1 + 0.1x_2 &\leq 2.7 \\ 0.5x_1 + 0.5x_2 &= 6 \\ 0.6x_1 + 0.4x_2 &\geq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$x_1^* = 7.5$ times standard dose
$x_2^* = 4.5$ times standard dose
$Z^* = 5.25$ kilorads

Modelbuilding – how to?

- First of all: Understand what the model has to decide.
- Figure out what decision variables are needed
- How do you interpret the variables? Write down the interpretation of the variables and consider the unit attached to the variable.
- Examples
  - Wyndor: x_1 : Production of doors (number of door batches per week)
  - Radiation therapy: x_1 : Intensity of beam 1 (relative to a standard beam)
- Write up objective function and constraints

Personnel Scheduling

Time	Shift 1	2	3	4	5	Agents needed
6-8	☑					48
8-10	☑	☑				79
10-12	☑	☑				65
12-14	☑	☑	☑			87
14-16		☑	☑			64
16-18			☑	☑		73
18-20			☑	☑		82
20-22				☑		43
22-24				☑	☑	52
24-6					☑	15
Cost ($ per day per agent)	170	160	175	180	195	

- How to model this? Think of
  - Decision variables
  - Constraints
  - Objective function

Personnel Scheduling: model

x_j : number of agents used on shift j $(j = 1, 2, \ldots, 5)$

$$\min 170x_1 + 160x_2 + 175x_3 + 180x_4 + 195x_5$$

subject to

$$\begin{array}{llllll}
x_1 & & & & & \geq 48 \\
x_1 & +x_2 & & & & \geq 79 \\
x_1 & +x_2 & & & & \geq 65 \\
x_1 & +x_2 & +x_3 & & & \geq 87 \\
 & x_2 & +x_3 & & & \geq 64 \\
 & & x_3 & +x_4 & & \geq 73 \\
 & & x_3 & +x_4 & & \geq 82 \\
 & & & x_4 & & \geq 43 \\
 & & & x_4 & +x_5 & \geq 52 \\
 & & & & x_5 & \geq 15 \\
x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0 \text{ and integer}
\end{array}$$

Personnel Scheduling - solution

Time	Shift 1	2	3	4	5	Agents needed
6-8	☑					48
8-10	☑	☑				79
10-12	☑	☑				65
12-14	☑	☑	☑			87
14-16		☑	☑			64
16-18			☑	☑		73
18-20			☑	☑		82
20-22				☑		43
22-24				☑	☑	52
24-6					☑	15
Cost ($ per day per agent)	170	160	175	180	195	

$Z^* = 30610\$$ per day

$x_1^* = 48, x_2^* = 31, x_3^* = 39, x_4^* = 43, x_5^* = 15$

Very short history of Linear Programming

- First used in general form: Kantorovich (1939)

- Dantzig proposed the simplex algorithm (1947)
  – We learn about this method next week

- Numerous application since then

Solve LP models using software

- See Julia slides

Exercises

- See tasks on DTU inside

- Let’s move to the exercise rooms:
- Building 324, room 040, 050 and 060