42101 Introduction to Operations research
Linear programming
Project 1, Fall 2019

October 7, 2019

Formalities

The project starts on September the 23th of September and a report has to be handed in as a PDF file on Sunday the 13th of October using DTU inside. The report should be made in groups of two to three participants. In the evaluation of the report we put emphasis on the solution method so remember to explain your calculations. You obtain no points for a result without calculations. Both project 1 and project 2 should be passed in order to attend the exam (overall assessment of the two reports). There are no "second attempts" for passing projects. Remember to write the study number and name for all group members on the front page of the report. You can write the report by hand and scan the pages but remember to write clearly. You can also write using the computer. Consider using latex (https://www.latex-project.org/) or lyx (https://www.lyx.org/).

You are allowed to talk to other groups about the assignment but the same text, figures, models, tables and so on must not be present in more than one hand-in. Likewise, you are NOT allowed to copy text, figures, models, tables , etc. from hand-ins from earlier years, solutions, documents from the internet etc.

1 Exercise 1

A DTU student has got her hands on 7 liters of pure Ethanol, 21 liters of apple juice and 20 liters of Coca-Cola. It is getting near the end of the month and the student is low on cash, so she decides to try to mix the ingredients and sell it to her fellow students. She tries different combinations and finds that the following products taste okay:

- Product A: 2 deciliter (dl) apple juice, 3 dl Coca-Cola and 1 dl Ethanol.
- Product B: 2 dl apple juice, 1 dl Coca-Cola and 2 dl Ethanol.
- Product C: 3 dl apple juice, 1 dl Coca-Cola and 1 dl Ethanol.

She learns that she can sell product A for 60 kr per liter, product B for 70 kr per liter and product C for 30 kr per liter.

- Formulate an LP that decides how the student should mix the three products in order to maximize her profit. Explain the model and the constraints
- In the optimal solution she only makes product A and product B and she does not use up all the apple juice. Can you based on this information say which variables are basic in the optimal solution? Explain how you reach your conclusion.
- Now that you know the optimal basic solution it is possible to write up the final (optimal) tableau using the fundamental insight (lecture 4). Please do so.

2 Exercise 2

Consider the following LP:

$$\max Z = x_1 + 3x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &= 18 \\ -2x_1 + x_2 &\leq 3 \\ -x_1 + 2x_2 &\geq -2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

- Write the LP in augmented form.
- Solve the LP, step by step, using the two phase method. Remember to explain your calculations.
- Draw the solution space to the LP and show the solutions that the two-phase method visited. Explain why these solutions were visited. Figure 4.6 from the book can be used as inspiration.

3 Exercise 3

Consider the following LP (notice that the variables can be negative in the LP)

$$\max Z = -2x_1 - x_2$$

subject to

$$\begin{aligned} -2x_1 + x_2 &\leq 3 \\ x_1 &\leq 5 \\ x_1 + 3x_2 &\leq 8 \\ x_1 &\in \mathbb{R} \\ x_2 &\geq -2 \end{aligned}$$

- Transform the problem to one that is on standard form (no negative variables). Hint: Look at slides from lecture 2
- Solve the resulting problem using the matrix version of the Simplex algorithm. Remember to explain your calculations.
- What is the solution to the original problem?

4 Exercise 4

4.1 Background

The company *DTU tours* rents out buses to schools, tourist agencies and companies that need to transport a large number of people. On a particular day DTU tours have to provide buses for the following 5 trips

	From	To	Departure
1	A	B	8:00
2	B	A	16:00
3	C	D	10:00
4	C	E	9:00
5	E	B	10:00

The travel time (in minutes) between locations are given in the table below. The buses should start and end their day at location A (that's the bus depot).

	A	B	C	D	E
A	0	42	50	57	35
B	42	0	24	45	39
C	50	24	0	22	27
D	57	45	22	0	23
E	35	39	27	23	0

The bus should arrive to the origin of the trip at least 15 minutes before departure and it will be ready to start another trip 20 minutes after arriving at the destination (to let people off the bus and to have some buffer in case of delays). As an example consider the first trip.

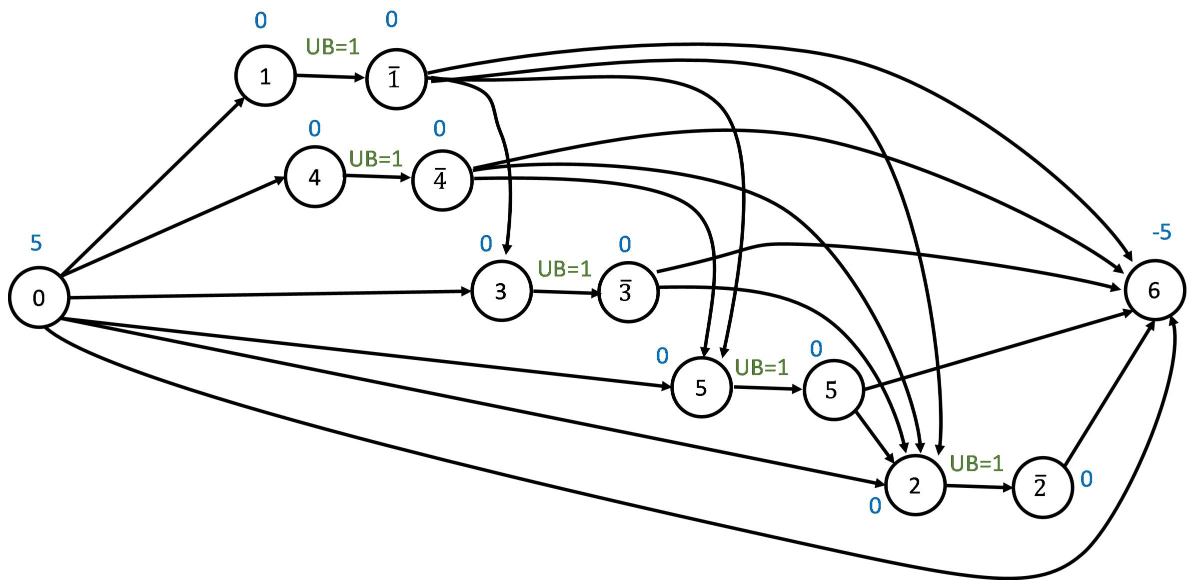
- The bus should arrive at A at 7:45 at the latest.
- At 8:00 the bus will depart for site B. According to the travel time matrix the bus will arrive at B at time 8:42
- At time 9:02 the bus is ready for a new trip. It cannot do trip 4 as it would arrive too late to the origin of this trip, but it could do trip 3 (for example). In that case the bus would drive from site B to site C which take 24 minutes and the bus would be ready at site C at time 9:26 which is early enough to start trip 3 at time 10:00.

When planning the bus schedule for the day DTU tours wishes to

- Carry out all trips
- Minimize the driving cost which are equal to 10 kr per minute driven plus 1000 kr per bus used (that is, in general the company wants to use few buses)

The owner of DTU tours have studied Introduction to Operations Research and realizes that the problem can be solved as a minimum cost flow problem. An important property of the minimum cost flow problem is that the x variables always take integer values as long as the surplus and demands are integer (more on that in lecture 10).

The minimum cost flow graph for the scheduling problem is shown below. On the graph node 0 corresponds to the start of the day (leaving the bus depot) and node 6 is the end of the day (returning to the bus depot). Each trip is represented by two nodes. For example is trip 1 represented by node 1 (departure from A) and $\bar{1}$ (arrival at B). From node $\bar{1}$ there are arcs to the start nodes of the trips that can follow after trip 1. That is why there is no arc from node $\bar{1}$ to the start node for trip 4, but there are arcs to node 3,5,2 and 6.

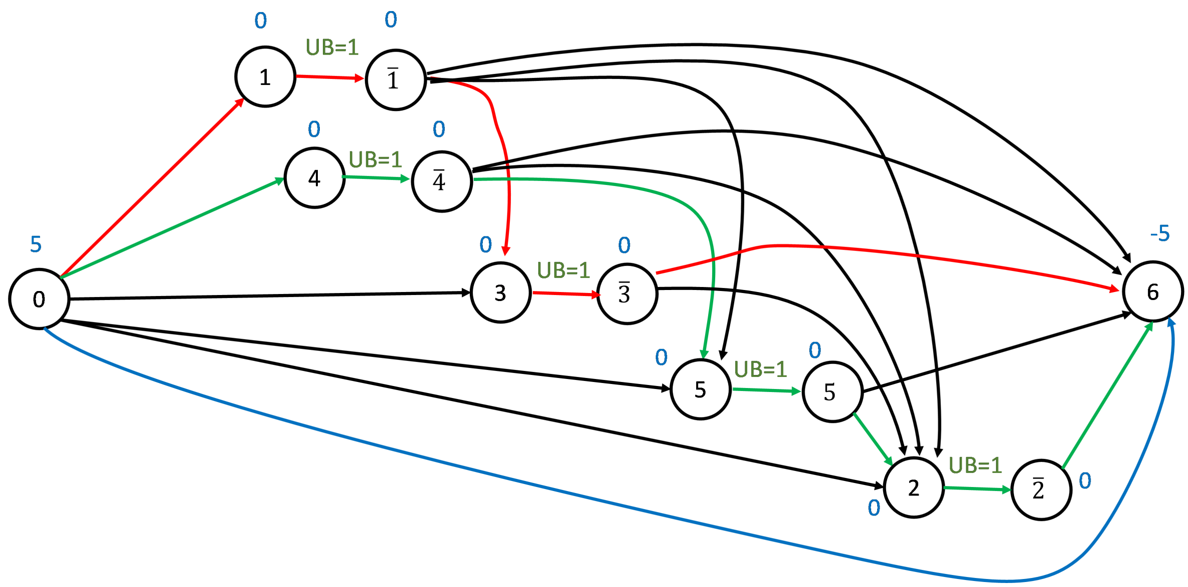


Node 0 has a surplus of 5 buses and node 6 needs these 5 buses. All other nodes have surplus/demand equal to 0. Sending flow from node 0 to node 6 using arc (0,6) corresponds to leaving buses unused. Arcs between start and end nodes of a trip can at most carry one unit of flow (to prevent the trip from being done more than once). All other arcs have no upper bounds. The cost of the arcs are given by the table below. The costs can be explained as follows:

- Arcs leaving node 0: If we go from node 0 to one of the trip nodes this corresponds to starting a new bus tour. This therefore cost 1000 plus 10 times the travel time from the depot (location A) to the start of the trip. I.e. the cost of arc (0,2) is 1000+420=1420 kr. Arc (0,6) is free since it corresponds to leaving a bus unused.
- Arcs from the start of a trip to the end of a trip (i.e.$(1,\bar{1})$) has a big negative cost to force the arc to be used in the solution (to ensure that the trip is made). Arc $(1,\bar{1})$ has, for example, cost -10000+420=-9580 kr. The 420 since the trip goes from A to B and the travel time between these two location are 42 minutes.
- Arcs from the end of a trip to the start of a compatible trip has a cost that is equal to the travel time times 10.
- Arcs from the end of a trip to the end depot has also a cost that is equal to the travel time times 10.

	0	1	$\bar{1}$	2	$\bar{2}$	3	$\bar{3}$	4	$\bar{4}$	5	$\bar{5}$	6
0		1000		1420		1500		1500		1350		0
1			-10000+420									
$\bar{1}$				0		240				390		420
2					-10000+420							
$\bar{2}$												0
3							-10000+220					
$\bar{3}$				450								570
4									-10000+270			
$\bar{4}$				390						0		350
5											-10000+390	
$\bar{5}$				0								420
6												

The solution to this instance of the min cost flow problem is shown below



One unit is sent from 0 to 6 using the red path, one unit is sent using the green path and three units are sent using the blue path. This solution is interpreted as follows:

- One bus first does trip 1 and then trip 3.
- Another bus does trip 4 , then trip 5 and then trip 2.
- The three other buses are not used.

The cost of the solution is -44970. Since every arc connecting the start and end node of a trip has a bonus of -10000 the real cost of the solution is -44970+50000 = 5030 kr. This makes sense since the cost for the first bus is 1000+0+420+240+220+570 =2450kr and the cost for the second bus is 1000+500+270+0+390+0+420+0=2580.

4.2 Assignment

Now consider the following trips:

	From	To	Departure
1	E	C	14:00
2	F	A	10:50
3	A	B	12:10
4	A	E	10:00
5	D	B	10:20
6	E	A	11:20

The travel time (in minutes) between locations are given in the table below. The buses should start and end their day at location A (that's the bus depot).

	A	B	C	D	E	F
A	0	18	61	49	30	33
B	18	0	43	46	12	14
C	61	43	0	59	32	29
D	49	46	59	0	49	47
E	30	12	32	49	0	4
F	33	14	29	47	4	0

- Show the min cost flow graph that results from this instance (including costs, demands and upper bounds). Explain how you got to this result.

- Solve the min cost flow problem using Julia. You can use the program from lecture 3. When constructing the graph in Julia, it would be smart to number the nodes from 1 to $2n+2$ (where n is the number of trips).
- Show the solution and explain how to interpret it.
- Consider if there was a certain payment associated with each trip and that DTU tours could decide if they wanted to do each trip (in order to maximize profit). Could you modify the a min-cost flow graph such that the model decided which trips to perform and computed the corresponding bus trips? It is enough to explain how this could be done.
- On DTU inside there are data for two large instances with 40 trips in each. In one instance the bus cost is 1000 and in the other it is zero, apart from that the two instances are identical. Try to solve both. Does the number of buses used change when decreasing the bus cost?