UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

INTERVAL MAPS WITH SINGULARITIES AND CRITICAL POINTS OF INFLECTION TYPE.

TESIS PARA OPTAR AL GRADO DE
DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA

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Este trabajo ha sido parcialmente financiado por Agencia Nacional de Investigación y Desarrollo (ANID), Beca de Doctorado Nacional Chile/2016 - 21160715. Centro de Modelamiento Matemático (CMM) Proyectos ANID PIA AFB170001, BASAL ACE210010, BASAL FB210005, Proyecto Fondecyt 1201125, 11180874 y CMM ANID BASAL FB210005.

SANTIAGO DE CHILE
2022

RESUMEN DE LA TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA
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FUNCIONES DEL INTERVALO CON SINGULARIDADES Y PUNTOS CRÍTICOS DE INFLEXIÓN.

En este trabajo de tesis estudiamos la regularidad de funciones unimodales del intervalo, analizando medidas invariantes, a través de los exponentes de Lyapunov y su regularidad, exponentes de Lyapunov puntual y la recurrencia hacia el conjunto singular. Presentaremos una nueva familia de funciones unimodales que es combinatorialmente completa, presenta una singularidad de tipo Lorenz en el punto de doblez y dos puntos críticos de inflexión planos. Como esta familia es combinatorialmente completa, podemos encontrar un representante con cualquier combinatoria admisible.

De particular interes serán las funciones con combinatoria de Fibonacci, la cual estudiaremos utilizando herramientas analíticas, medibles y combinatoriales. Para este tipo de combinatoria construimos una medida ergódica e invariante cuyo exponente de Lyapuno no está definido, más aún, para casi todo punto con respecto a esta medida el exponente de Lyapunov puntual no está definido.

Finalmente, presentamos una nueva familia de funciones unimodales con dos puntos críticos no-planos que son de inflexión y la geometría del punto de doblez cambia de forma continua de ser un punto crítico a ser una singularidad de tipo Lorenz. En esta familia podemos encontrar a la familia cuadrática, y cuando el punto de doblez no es una singularidad de tipo Lorenz, la familia tiene derivada Schwarziana negativa. Proponemos un estudio sistemático de esta familia planteando algunas preguntas que surgen de la observación de diferentes fenomenos presentes en experimentos hechos para familias estudiadas anteriormente.

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INTERVAL MAPS WITH SINGULARITIES AND CRITICAL POINTS OF INFLECTION TYPE.

In this thesis work we study regularity of unimodal interval maps, focusing on the Lyapunov exponents of invariant measures, non-degeneracy of ergodic invariant measures, pointwise Lyapunov exponents, and the recurrence to the singular set. We introduce a new families of unimodal maps that are full and presents a Lorenz-like singularity at the turning point and two non-flat critical points. Since this families are full we find all possible admissible combinatorics on them.

In particular, we study the maps in these families with Fibonacci combinatorics using analytic, measure theoretical, and combinatorial tools. For these type of combinatorics we find an ergodic invariant measure whose Lyapunov exponent is not defined. Moreover, for almost every point with respect to this measures the pointwise Lyapunov exponent is not defined.

Finally, we introduce a new family of symmetric unimodal maps with two non-flat critical points of inflection type and the geometry of the turning point changing continuously from a non-flat critical point to a Lorenz-like singularity. This family contains the quadratic family, and whenever the turning point is not a Lorenz-like singularity, the Schwarzian derivative of the map is negative. We give an overview about the natural steps to follow in order to make a systematic study of this family, and some questions that arise from observation.

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Introduction

One-dimensional dynamical systems have been a subject of great interest in the last decades. One of the reasons is that one-dimensional dynamics display many features of dynamics in higher dimension. For example, the Hénon map, one of the simplest smooth chaotic dynamical systems, has been studied through its on-dimensional reduction. Another important example is the geometric model of the Lorenz attractor. For several years the chaotic behavior of the Lorenz equations was mysterious. Only after realized that some first return map is, in essence, one-dimensional, these equations became understood.

Another reason is that the behavior of one-dimensional systems is far from being trivial, and yet the good mathematical framework has made this theory remarkably well understood. For example, the order structure of the interval leads to the combinatorial theory, from which we have in the early 60's Sarkovskii's Theorem about the coexistence of multiple periodic orbits of any given period [73]. This result was rediscovered in the mid-'70s by Li and Yorke in their famous theorem "Period three implies chaos" [43]. There is also the universality found in the period-doubling bifurcation by Feigenbaum[27] and Coullet and Tresser [70]. Therefore, the existence of general results, in spite of the complexity, is remarkable.

The study of one-dimensional dynamical systems fits in four categories; combinatorial, topological, ergodic, and smooth. In the ergodic theory of interval maps, Lyapunov exponents play an important role. In particular, in the seminal work of Pesin (referred to as "Pesin Theory"), the existence and positivity of Lyapunov exponents were used to study the dynamics of non-uniformly hyperbolic systems, see for example [36, Supplement]. Using these ideas, Ledrappier [41] studied ergodic properties of absolutely continuous invariant measures for regular maps of the interval under the assumption that the Lyapunov exponent exists and is positive. Recently Dobbs [19], [20] developed the Pesin theory for noninvertible interval maps with Lorenz-like singularities and non-flat critical points. Lima [44] constructs a symbolic extension for these maps that codes the measures with positive Lyapunov exponents.

In the case of continuously differentiable interval maps, Przytycki proved that ergodic invariant measures have nonnegative Lyapunov exponent, or they are supported on a strictly attracting periodic orbit of the system. Moreover, there exists a set of full measure for which the pointwise Lyapunov exponent exists and is nonnegative, see [64], [66, Appendix A].

The goal of this work is to show that the result above cannot be extended to continuous piecewise differentiable interval maps with a finite number of non-flat critical points and Lorenz-like singularities. In particular, we construct a measure for a unimodal map with a Lorenz-like singularity and two non-flat critical points for which the Lyapunov exponent does not exist. Moreover, for this map, the pointwise Lyapunov exponent does not exist for a set of full measure. Thus, our example shows that the techniques developed by Dobbs [19], [20], and Lima, [44], cannot be extended to all maps with critical points and Lorenz-like singularities.

Maps with Lorenz-like singularities are of interest since they appear in the study of the Lorenz attractor, see [31], [46], and references therein. Apart from these motivations, these types of maps are of interest on their own since the presence of these types of singularities create expansion, and hence enforce the chaotic behavior of the system, see [2], [45], [19], and references therein.

Additionally, the unimodal map that we consider has Fibonacci recurrence of the turning point (or just Fibonacci recurrence). Maps with Fibonacci recurrence first appeared in the work of Hofbauer and Keller [34] as possible interval maps having a wild attractor. Lyubich and Milnor [49] proved that unimodal maps with a quadratic critical point and Fibonacci recurrence do not only have any Cantor attractor but also have a finite absolutely continuous invariant measure, see also [37]. Finally, Bruin, Keller, Nowicki, and van Strien [8] proved that a C^2-unimodal interval map with a critical point of order big enough and with Fibonacci recurrence has a wild Cantor attractor. On the other hand, in the work of Branner and Hubbard [5], in the case of complex cubic polynomials, and the work of Yoccoz, in the case of complex quadratic polynomials, Fibonacci recurrence appeared as the worst pattern of recurrence, see for example [35], and [55]. Maps with Fibonacci recurrence also play an important role in the renormalization theory, see for example [68], [42], [29], and references therein.

We now describe the organization of this thesis and give a brief description of each Chapter.

In Chapter 1, we review prerequisites for each of the topics covered in this thesis, interval dynamics, unimodal maps, combinatorics of unimodal maps, and ergodic theory. The objective of this chapter is to be a quick reference more than an in-depth exposition of the general theory of one-dimensional dynamical systems. We end this chapter by describing some of the most important results in the last decades.

Chapter 2, we study the combinatorial properties of a unimodal map with Fibonacci recurrence. In particular, we will be interested in the Fibonacci tent map. We make a detailed description of the set $\omega-$limit set of the turning point, and following [49], we construct a partition of it that will allow us to estimate close return times to the turning point and lower bounds for the distances of these close returns. Then we estimate how fast the orbit of the turning point return to itself in terms of the return time (see Lemma 2.3.1). This estimation is of importance since it gives us an exact estimation of the growth of the geometry near the turning point for our map f. We finish this chapter describing the unique ergodic invariant measure μ_P supported on $\omega-$limit set of the turning point for the tent map, restricted to the partition constructed previously. We will use this estimations in the rest of the thesis.

In Chapter 3, we introduce the class of unimodal maps to be studied in this thesis. We give a description of the singular set for this class, and give some important bound for the derivative in terms of the conjugacy with the Fibonacci tent map. We also present some examples and we finish the chapter by studying the recurrence of the turning point.

In Chapter 4, we study the Lyapunov exponent of the the unique ergodic invariant measure μ_P supported on $\omega-$limit set of the turning point for the class of maps defined on Chapter 3.

Finally, in Chapter 5, we will discuss some further work that we expect it can be done by using the techniques developed for this thesis, and some questions related to the properties of the class of maps introduced in Chapter 3.

The results presented in Chapter 2, Chapter 3, and Chapter 4 are adaptations of *Invariant measures for interval maps without Lyapunov exponents* [60]. Chapter 5 was written just as part of this thesis.

Chapter 1

Preliminaries

In this chapter we introduce the basics of one-dimensional dynamical systems needed as background for this work. This preliminaries are intended as a reference, so proofs are omitted.

Most of the definitions and results presented in this chapter are stated in the context of our work, although they can be stated in a more general setting. We refer the interested reader to [18], [36], [6] for a more detailed exposition.

Throughout the rest of this work, we will denote by I the closed interval $[-1,1] \subset \mathbb{R}$. We use $\mathbb{N}$ to denote the set of integers that are greater than or equal to 1 and put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

We endow I with the distance induced by the absolute value $|\cdot|$ on $\mathbb{R}$. For $x \in \mathbb{R}$ and $r > 0$, we denote by $B(x,r)$ the open ball of I with center at x and radius r. For an interval $J \subset I$, we denote by $|J|$ its length.

For real numbers a, b we put

$$[a,b] := [\min\{a,b\}, \max\{a,b\}],$$

in the same way

$$(a,b) := (\min\{a,b\}, \max\{a,b\}).$$

1.1 Interval dynamics

Given a continuous function $f \colon I \longrightarrow I$, the pair (f, I) form a dynamical systems, thus, a set I of possible states and a rule f that determine the present state in terms of the past state. We consider iterates f^n of f, defined inductively by $f^0(x) = x$, and $f^n(x) = f(f^{n-1}(x))$, for every $x \in I$, and every $n \in \mathbb{N}$. For a point $x \in I$, the *orbit of x under the map f*, is the set

$$\mathcal{O}_f(x) := \{f^n(x) \colon n \geq 0\}.$$

The aim in dynamical systems is to describe how these orbits are distributed. The simplest behavior shown by an orbit is periodicity.

Definition 1.1.1. We say that $x \in I$ is a *periodic point of period* n if $f^n(x) = x$ and $f^k(x) \neq x$ for every $1 \leq k < n$. In this case we say that $\mathcal{O}_f(x)$ is a *periodic orbit of period* n. If x is a periodic point of period 1, we call x a fixed point. If x is not periodic and there exist some $k \geq 1$ so that $f^k(x)$ is periodic, then we say that x *is preperiodic.*

If we want to study the asymptotic distribution of the orbit of a point in I, we need to look at the accumulation points of this orbit.

Definition 1.1.2. For $x \in I$ we define the ω*-limit set of* x *under the map* f as the set of accumulation points of $\mathcal{O}_f(x)$, thus

$$\omega_f(x) := \{y \in I\colon \text{ there exists } n_1 < n_2 < \dots \text{ such that } \lim{}_{k\to\infty} f^{n_k}(x) = y\}.$$

Intuitively, a point y is in $\omega_f(x)$ if there are points in the orbit of x that get arbitrarily close to y. In the case of a periodic or preperiodic point for the map f, the $\omega-$limit set is the periodic orbit.

Definition 1.1.3. We call a set $J \subset I$ *invariant* if $f(J) \subset J$. If $f(J) = J$, we say that J is *strongly invariant.*

The following theorem is a direct consequence of the definition of $\omega_f(x)$.

Theorem 1. *Let $f\colon I \longrightarrow I$ be continuous. For every $x \in I$ we have:*

1. *$\omega_f(x)$ is not empty,*
2. *$\omega_f(x)$ is closed, then compact,*
3. *$f(\omega_f(x)) = \omega_f(x)$, thus, $\omega_f(x)$ is strongly invariant.*

It follows from point 3 in the previous theorem that given a point $x \in I$, the pair $(f, \omega_f(x))$ is a dynamical system. Then, it is interesting to know what kind of sets can be $\omega-$limit sets for continuous maps from I to itself. The following theorem will give us an answer. Let

$$\mathcal{C} := \{f\colon I \to I\colon \ f \text{ is continuous}\}.$$

Theorem 2. *([4]) A set $M \subset I$ is an $\omega-$limit set for some $f \in \mathcal{C}$ if and only if either M is a nonempty nowhere dense set or M is a finite union of closed intervals.*

So, if for some x, $\omega_f(x)$ is not a finite union of closed intervals, then is nowhere dense, thus, cannot contain an open interval and hence must be totally disconnected. If $\omega_f(x)$ is finite, it must be a periodic cycle. In case is not finite it could be countable or uncountable. In the uncountable case, we cannot assure that $\omega_f(x)$ is a Cantor set. In fact, it is possible to construct examples of continuous maps $f\colon I \longrightarrow I$ such that, for some $x \in I$, the set $\omega_f(x)$ can be expressed as the union of a Cantor set K and a countable set $\omega_f(x) \setminus K$ such that $f(K) = K$ and $\omega_f(X) \setminus K$ is not closed, see [6, Section 3.2].

With more information about the dynamical systems $(f, \omega_f(x))$ we can conclude if $\omega_f(x)$ is a Cantor set.

Definition 1.1.4. Let $f \in \mathcal{C}$. We say that $x \in I$ is *recurrent*, if for every open set U containing x, there exists $n \in \mathbb{N}$ such that $f^n(x) \in U$.

From the definition we have that $x \in \omega_f(x)$ if and only if x is recurrent.

Definition 1.1.5. Let $f \in \mathcal{C}$. We say that a nonempty subset F of I is *minimal* if it is closed, invariant, and there is no proper nonempty subset with these two properties.

Lemma 1.1.6. *Let $f \in \mathcal{C}$. A nonempty subset F of I is minimal if and only if $\omega_f(x) = F$ for all $x \in F$.*

Now we can give a more detail description of $\omega-$limit sets using minimality.

Lemma 1.1.7. *[3, Chapter 5, Section 1, Lemma 4] Let $f \in \mathcal{C}$. If F is an infinite and minimal subset of I, then F is a Cantor set.*

Another important concept in dynamical systems is *conjugacy*.

Definition 1.1.8. Let $f, g \in \mathcal{C}$. We will say that the dynamical systems (f, I) and (g, I) are *topologically conjugated* if there exits a continuous bijection $h\colon I \longrightarrow I$, such that $h \circ f = g \circ h$. We say that h *is a conjugacy between* f *and* g. If h is only surjective, then we say that f and g are *semi-conjugated*, or that g is a *factor of* f, or that f is an *extension of* g.

If f and g are topologically conjugate, many dynamical features of f are preserved by the conjugacy, and thus, we will see the same features in g. As the name suggest, topological conjugacy preserves topological behaviors, as periodicity and recurrence.

1.2 Unimodal maps

Definition 1.2.1. A map $f \in \mathcal{C}$ is called *unimodal* if there is $c \in I \setminus \{-1, 1\}$ such that $f|_{[-1,c)}$ is increasing and $f|_{(c,1]}$ is decreasing. We call c the *turning point of* f.

When there is no confusion, we will denote the turning point of a unimodal map by c, and $c_i := f^i(c)$. For a unimodal map $f\colon I \longrightarrow I$, with $c_2 < c < c_1$, and $c_2 \leq c_3$, the interval $[c_2, c_1]$ is called the *core of the system* (f, I).

A prototype of polynomial symmetric unimodal map is given by

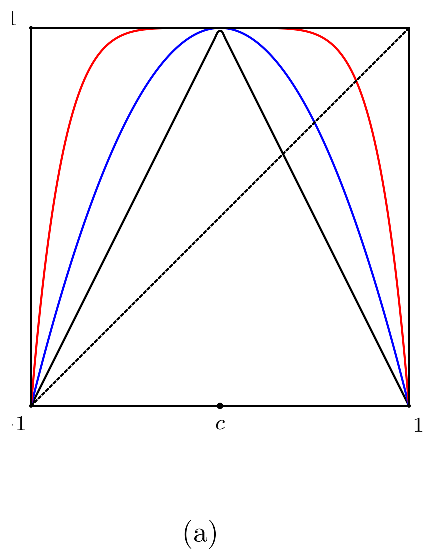
$$f_{\lambda,\alpha}(x) = \lambda(1 - |x|^\alpha) - 1, \tag{1.2.1}$$

with $\lambda \in (0, 2]$, observe that if $\lambda > 2$, then there exists $\delta > 0$ such that for every $x \in (-\delta, \delta)$ we have that $f^n(x) \longrightarrow \infty$ as $n \longrightarrow \infty$. The turning point is $c = 0$. When $\alpha > 1$, the function is differentiable at the turning point and c is a critical point, i.e. $f'(0) = 0$. We call the number α the *order of the critical point.*

When $\alpha = 1$, $f_{\lambda,1}(x)$ is a piecewise linear map with slope λ in $[-1, c)$, and slope $-\lambda$ in $(c, 1]$. We will use the notation $T_\lambda := f_{\lambda,1}$, and we will call the family $\{T_\lambda\}_{\lambda\in[0,2]}$ the *symmetric family of tent maps.* This example has been studied extensively in the last decades for its simplicity and its connection with general unimodal maps. It is not hard to see that for

$\lambda \in [0,1)$ the map T_λ has a single fixed point at $x = -1$, and the orbit of every $x \in I$ converges to -1. For $\lambda \in (1,2]$ it has two fixed points at $x = 0$, and $x = \frac{\lambda-1}{\lambda+1}$, and the behavior of the orbits can be chaotic. For the function T_1, every $x \in [-1,0]$ is a fixed point, and every $x \in (0,1]$ is preperiodic, in fact, if $x \in (0,1]$, then $T_1(x) = (1-|x|-1) = -x$. Thus, $T_1(x) \in [-1,0)$ for every $x \in (0,1]$, so $T_1(x)$ is fixed for every $x \in (0,1]$. Some examples of the tent map are shown in Figure 1.1.

Another important example, is when $\alpha = 2$. In this case we call the family $\{f_{\lambda,2}\}_{\lambda\in(0,2]}$ *the quadratic family*. Usually written as $f_c(x) = x^2 + c$ or $f_a(x) = ax(1-x)$, defined on their respective domain. The quadratic family has been recognized as an interesting and representative model of chaotic dynamics. It appears as the one-dimensional reduction of the Henon family, and as a model for population dynamics. See Figure 1.1.



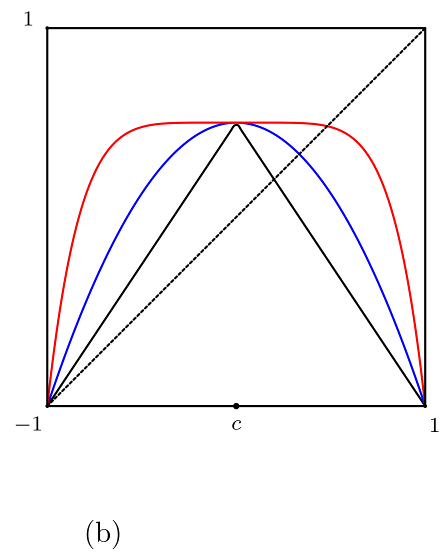


Figure 1.1: (a) Graphics of the symmetric tent map T_2 (black), the quadratic map $f_{2,2}$ (blue), and the map $f_{2,6}$ (red). (b) Graphics of the symmetric tent map $T_{1.5}$ (black), the quadratic map $f_{1.5,2}$ (blue), and the map $f_{1.5,6}$ (red).

1.3 Combinatorics of unimodal maps

In this section we will introduce the combinatorial theory developed by Milnor and Thurston [56]. Their work [56] was published in 1988, although the majority of their work was already available as a preprint since 1977. The central idea of this theory, that is the use of symbolic dynamics, was used before, see for example [62], [53]. This theory led to a topological classification of unimodal maps, under mild additional hypothesis, representation through piecewise continuous linear models, counting of fixed points, and constructing measures of maximal entropy, among others. For a detailed exposition about the combinatorics of interval maps see [56], [18, Chapter 2, Section 3].

For the rest of this section, we will consider $f\colon I \longrightarrow I$ to be a unimodal map with turning point c, and $c_2 < c < c_1$. Let $\Sigma := \{0, 1, c\}^{\mathbb{N}_0}$ be the space of sequences $\underline{x} = (x_0, x_1, x_2, \ldots)$. In Σ we consider the topology generated by the cylinders

$$[a_0 a_1 \ldots a_{n-1}]_k := \{\underline{x} \in \Sigma\colon x_{k+\mathrm{i}} = a_{\mathrm{i}} \text{ for all } \mathrm{i} = 0, 1, \ldots, n-1\}.$$

With this topology, Σ is a compact space. Let us define

$$\begin{aligned} \underline{\mathrm{i}}\colon I &\longrightarrow \Sigma \\ x &\longmapsto (\mathrm{i}_0(x), \mathrm{i}_1(x), \ldots) \end{aligned}$$

where

$$\mathrm{i}_n(x) = \begin{cases} 0 & \text{if } f^n(x) \in [-1, c) \\ 1 & \text{if } f^n(x) \in (c, 1] \\ c & \text{if } f^n(x) = c. \end{cases}$$

The sequence $\underline{\mathrm{i}}(x)$ is called *the itinerary of x under f*. Given $n \in \mathbb{N}$ and $x \in I$ there exists $\delta > 0$ such that $\mathrm{i}_n(y) \in \{0, 1\}$ and is constant for every $y \in (x, x+\delta)$. Observe that this value is not the same as $\mathrm{i}_n(x)$ if x is the turning point. It follows that

$$\underline{\mathrm{i}}(x^+) := \lim_{y \downarrow x} \underline{\mathrm{i}}(y) \qquad \text{and} \qquad \underline{\mathrm{i}}(x^-) := \lim_{y \uparrow x} \underline{\mathrm{i}}(y)$$

always exist. Notice that $\underline{\mathrm{i}}(x^-)$ and $\underline{\mathrm{i}}(x^+)$ belong to $\{0, 1\}^{\mathbb{N}_0}$. The sequence $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \ldots$ defined by $\mathrm{e}_j := \mathrm{i}_j(c_0^+)$ is called the *kneading invariant* of f. A sequence $\underline{a} \in \{0, 1\}^{\mathbb{N}}$ is *admisible* if there exists a unimodal map $f\colon I \to I$ with kneading invariant $\underline{a}$.

If $J \subset I$ is a maximal closed interval on which f^n is monotone, then $f^n\colon J \longrightarrow f^n(J)$ is called a *branch of f^n*. If c is in the boundary of J, we say that $f^n\colon J \longrightarrow f^n(J)$ is a *central branch of f*. By Definition 1.2.1, f^n has two central branches, and they have the same image, or the image of one branch contains the other, under f^n. Denotes the largest of this images by H_n. If $c \in H_n$, then n is called a *cutting time*. Denote the cutting times by $\{S(\mathrm{i})\}_{\mathrm{i} \geq 0}$. Then, $S(0) < S(1) < S(2) < \ldots$, and $S(0) = 1$ and $S(1) = 2$.

One of the important feature of the cutting time, is that they completely determine the kneading invariant (and the kneading invariant completely determine the cutting times) by the rule

$$S(k) := \min\{n \geq S(k-1)\colon \mathrm{e}_n \neq \mathrm{e}_{n-S(k-1)}\}.$$

If cutting times $S(k)$ are defined for every $k \geq 0$, then the difference between two consecutive cutting time is again a cutting time. So, there is a function $Q\colon \mathbb{N} \to \mathbb{N}_0$ such that

$$S(k) - S(k-1) = S(Q(k)).$$

The map Q is called the *kneading map* of f. A kneading map leads to an admissible kneading sequence $\{\mathrm{e}_j\}_{j \geq 1}$ by the relation

$$\mathrm{e}_{S(k-1)+1}\mathrm{e}_{S(k-1)+2} \ldots \mathrm{e}_{S(k)-1}\mathrm{e}_{S(k)} = \mathrm{e}_1\mathrm{e}_2 \ldots \mathrm{e}_{S(Q(k))-1}(1 - \mathrm{e}_{S(Q(k))}), \tag{1.3.1}$$

for $k \geq 1$. The length of each string in (1.3.1) is $S(Q(k))$, thus at the kth-step of the process we can construct $S(Q(k))$ symbols of the sequence.

The following *admissibility condition* characterizes the possibles kneading maps:

Theorem 3. *([33], [7]) A map $Q\colon \mathbb{N} \longrightarrow \mathbb{N}_0$ is the kneading map of some unimodal map if and only in*

$$(Q(j))_{k<j<\infty} \geq (Q(Q(Q(k)) + j - k))_{k<j<\infty}$$

for all k with $Q(k) > 0$ ($\geq$ is the lexicographical order). The only exception is when the critical point is attracted to an orientation reversing periodic attractor, in that case $Q(k)$ is defined only for finitely many k.

From the admissibility condition, every non-decreasing kneading map is admissible.

Example 1.3.1. For every $\mathrm{d} \in \mathbb{N}$, the map

$$\begin{aligned} Q_{\mathrm{d}}\colon \mathbb{N} &\longrightarrow \mathbb{N}_0 \\ k &\longmapsto \max\{0, k - \mathrm{d}\}, \end{aligned}$$

is non decreasing for every $\mathrm{d} \in \mathbb{N}$, then it defines an admissible kneading map.

For $\mathrm{d} = 1$, the sequence of cutting times starts like

$$S(0) = 1, S(1) = 2, S(2) = 4, S(3) = 8, S(4) = 16, S(5) = 32, S(6) = 64, \ldots,$$

and the kneading invariant starts like

$$10111010101110111011101010111010\ldots$$

Unimodal maps with associated kneading map $Q_1(k)$ are known as Feigenbaum maps. This map was described independently by Feigenbaum and Coullet and Tresser, and it represents an important example in the renormalization and bifurcation theory of unimodal maps.

For $\mathrm{d} = 2$, the sequence of cutting times starts like

$$S(0) = 1, S(1) = 2, S(2) = 3, S(3) = 5, S(4) = 8, S(5) = 13, S(6) = 21, \ldots,$$

and the kneading invariant starts like

$$100111011001010011100\ldots$$

Unimodal maps with associated kneading maps $Q_2(k)$ are known as Fibonacci maps. See Chapter chapter 2.

Unimodal maps with associated kneading map $Q_{\mathrm{d}}(k)$ for $\mathrm{d} > 2$ are known as Fibonacci-like maps.

The condition $Q(k) \longrightarrow \infty$ as $k \longrightarrow \infty$ not only ensure the admissibility of Q, the following theorem shows some of its implications.

Theorem 4. *Let f be a unimodal map with kneading map Q. If $Q(k) \longrightarrow \infty$ as $k \longrightarrow \infty$, then*

1. *$|H_n| \longrightarrow 0$, as $n \longrightarrow \infty$;*

2. *c is recurrent and $\omega_f(c)$ is a minimal Cantor set;*

3. *if in addition f is a C^2 unimodal map, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log |Df(x)| \le 0,$$

for every $x \in \omega_f(c)$.

For a Proof of statements 1 and 2 see [12, Theorem 2]. For a Proof of statement 3 see [11, Theorem 2'].

From the above theorem, if f is a unimodal map with turning point c and with associated kneading map given by $Q_{\mathrm{d}}(k) = \max\{0, k - \mathrm{d}\}$, $\omega_f(c)$ is a cantor set and f restricted to it is minimal.

We will say that a family of unimodal maps $\{f_t\}_{t\in\mathcal{I}}$ is *full*, if for each kneading map Q there is a parameter $t \in \mathcal{I}$ such that the kneading map of f_t is equal to Q. It is known that the family of symmetric tent maps is full, see [56], [18, Chapter 2].

1.4 Ergodic theory

The results and definitions of this section are stated in the particular setting of this thesis. To see a general exposition we refer the reader to [74], [18, Chapter 5], [72].

All the measure considered in this section are Borel measures on I.

For a dynamical system (f, I), the set $\omega_f(x)$ tells what parts of I are visited by x. We can give quantitative information about the frequency of visits for a set. Let $x \in I$ and $A \subset I$, put

$$F_A(f, x, n) := \sharp\{k \in [0, n-1] \cap \mathbb{Z} \colon f^k(x) \in A\}.$$

Thus, $F_A(f, x, n)$ is the number of times that the orbit of x under f visit A up to time n. Then, the limit

$$F_A(f, x) := \lim_{n\to\infty} \frac{F_A(f, x, n)}{n},$$

gives the *average visit time of x to the set A*. So we would like to know if the limit above exists, or when is positive.

The notion of invariant measure is a fundamental part in the study of the above problem.

Definition 1.4.1. Let $f \in \mathcal{C}$. We say that a measure μ in I is f-invariant, or that μ is preserved by f, if

$$\mu(f^{-1}(A)) = \mu(A),$$

for every measurable set $A \subset X$.

Then we have the following theorem

Theorem 5. *Let $f \in \mathcal{C}$ and μ a $f-$invariant probability measure. Given any measurable set $A \subset I$, the average visit time*

$$F_A(f,x) := \lim_{n\to\infty} \frac{F_A(f,x,n)}{n},$$

exists for $\mu-$a.e. $x \in I$. Moreover, $\int F_A(f,x)\mathrm{d}\mu(x) = \mu(A)$.

The previous theorem is a consequence of the more general Birkhoff Ergodic Theorem, see [72, Section 3.2], and references therein. In the particular case when $F_A(f,x) = \mu(A)$ for $\mu-$a.e. $x \in I$, we say that the measure μ is *ergodic with respect to* f. Then, an $f-$invariant measure determines the asymptotic distribution of $\mu-$a.e. point if it is ergodic.

The following theorem gives equivalents ways to formulate ergodicity.

Theorem 6. *Let $f \in \mathcal{C}$ and μ a $f-$invariant probability measure on I. The, the following are equivalents:*

1. *For every measurable set $A \subset I$ we have $F_A(f,x) = \mu(A)$, for $\mu-$a.e. $x \in I$.*

2. *For every measurable set $A \subset I$, the function $F_A(f,\cdot)$ is constant $\mu-$a.e.*

3. *For every set $E \subset I$, with $f^{-1}(E) = E$, we have $\mu(E) = 1$ or $\mu(E) = 0$.*

Definition 1.4.2. A map $f \in \mathcal{C}$ is called *uniquely ergodic* if it has only one invariant probability measure.

Another important feature of invariant measure is that they give raise to a non-trivial recurrence. More precisely, the Poincaré Recurrence Theorem implies that recurrence is a generic property of orbits of measure-preserving dynamical systems. Thus, let

$$\mathcal{R}(f) := \{x \in I : x \in \omega_f(x)\}.$$

Theorem 7. *[36, Proposition 4.1.18] Let $f \in \mathcal{C}$ and μ be an $f-$invariant probability measure. Then:*

1. *μ-a.e point is recurrent, thus* $\mathrm{supp}(\mu) \subset \mathcal{R}(f)$.

2. *If μ is ergodic then $f|_{\mathrm{supp}(\mu)}$ has a dense orbit.*

3. *If X is compact and $f|_{\mathrm{supp}(\mu)}$ is uniquely ergodic, then $(f, \mathrm{supp}(\mu))$ is a minimal set.*

The following theorem tell us that the combinatorial type of a unimodal map determine the topological structure of the space of invariant probability measures supported on $\omega_f(c)$.

Theorem 8. *Let Q be a kneading map with $Q(k) \longrightarrow \infty$ as $k \longrightarrow \infty$. If f, and g are two unimodal maps with associated kneading map Q, then the space of invariant probability measure of f supported on $\omega_f(c)$ is affine homeomorphic to that of g.*

For a proof of this theorem see [17, Proposition 4] and references therein.

Chapter 2

Fibonacci combinatorics

In this chapter we will study the Fibonacci combinatorics and the combinatorial structure of the $\omega-$limit set of the turning point.

In the first section, we will introduce the Fibonacci combinatorics and give some known facts that we will use in the rest of this thesis.

In the second section, we will study the combinatorial structure of the $\omega-$limit set associated with the Fibonacci tent map, and We introduce a tower system constructed by Milnor and Lyubich [49]. This tower system is going to be a central tool throughout the following sections.

In the Thirds section we will study the rate of the decay of the geometry near to the turning point for the Fibonacci tent map.

Finally, in the fourth section, we will study the unique ergodic invariant measure supported in the $\omega-$limit set of the turning point. In particular, we will compute the value of this measure restricted to each level in the tower system defined in the second section.

2.1 Fibonacci combinatorics

In this section we will present an important combinatorial class of unimodal maps, the so-called Fibonacci unimodal maps. This class has been well studied because of its extreme topological and measure theoretical properties, [49], [37], [8] [12].

We will say that a unimodal map f has *Fibonacci recurrence* or it is a *Fibonacci unimodal map*, if the kneading map associated to it is given by

$$Q(k) := \max\{0, k-2\}. \tag{2.1.1}$$

So the sequence $\{S(n)\}_{n\geq 0}$ of cutting times, is given by the Fibonacci numbers

$$S(0) = 1,\ S(1) = 2,\ S(2) = 3,\ S(3) = 5, \ldots.$$

For a Fibonacci unimodal map f we have that

$$|c_{S(0)} - c| > |c_{S(1)} - c| > \dots |c_{S(n)} - c| > |c_{S(n+1)} - c| > \dots, \tag{2.1.2}$$

and

$$|c_3 - c| < |c_4 - c|. \tag{2.1.3}$$

See [49, Lemma 2.1] and references therein.

Maps with Fibonacci recurrence first appeared in the work of Branner and Hubbard [5] on cubic polynomials, and in Yoccoz work on complex quadratic polynomials, as the worst pattern of recurrence. In the work of Hofbauer and Keller [34], a real quadratic map with Fibonacci recurrence was suggested as a candidate for a map having a *wild attractor* (i.e. a sets X that is the $\omega-$limit set for Lebesgue almost every orbit, but is strictly smaller than the $\omega-$limit set for a generic orbit). Lyubich and Milnor [49] proved that a unimodal map with a quadratic critical point and Fibonacci recurrence do not only have no Cantor attractor but also have a finite absolutely continuous invariant measure. More precisely,

Theorem 9 ([49]). *There is one and only one real quadratic map $f_a(x) = x^2 + a$ that has Fibonacci recurrence. For this map we have:*

1. *$\omega_{f_a}(c)$ is a Cantor set.*

2. *The map $f_a\colon \omega_{f_a}(c) \longrightarrow \omega_{f_a}(c)$ is onto, is one-to-one except at the critical point that has two preimages, is minimal, and uniquely ergodic.*

Keller and Nowicki [37] extended this result to unimodal maps with criticality different from 2. More precisely, they prove the following

Theorem 10 ([37]). *There is a $\ell_1 > 2$ such that each unimodal map with negative Schwarzian derivative Fibonacci recurrence and critical point of order $\ell \in (1, \ell_1)$ has a finite absolutely continuous invariant measure and hence no Cantor attractor.*

Despite this results, in 1992 Lyubich and Tangerman made computer estimates suggesting that wild attractors do exist for maps with Fibonacci recurrence of the form $x \mapsto x^6 + c$. Finally, in 1996, Bruin, Keller, Nowicki, and van Strien proved that for a C^2 unimodal interval map with a critical point c of order large enough and with Fibonacci recurrence, the set $\omega_f(c)$ is a wild cantor attractor for f.

From the definition, we see that the kneading map Q associated to a unimodal map f with Fibonacci recurrence satisfy $Q(k) \longrightarrow \infty$ if $k \longrightarrow \infty$. Then, by Theorem 4, the set $\omega_f(c)$ is a Cantor set and the restriction of f to this set is minimal and uniquely ergodic. The kneading invariant for a Fibonacci unimodal map starts like

$$10011101100101001110 0\dots$$

Let us consider the tent family $\{T_\lambda\}_{\lambda\in(0,2]}$ defined by $T_\lambda(x) = \lambda(1-|x|)-1$ for every $x \in I$ and every $\lambda \in (0, 2]$, see example 1.3.1. This family is *full*, thus for every kneading map Q there is a parameter $\lambda \in (0, 2]$ so that the kneading map of T_λ is Q, see [56], [18, Chapter 2]. So there exists $\lambda_F \in (0, 2]$ such that the kneading map associated to T_{λ_F} is given by

$Q(k) = \max\{0, k-2\}$. By Theorem 4, Theorem 8, and Theorem 9, we have that $\omega_{T_{\lambda_F}}(c)$ is a cantor set, and the restriction of T_{λ_F} to $\omega_{T_{\lambda_F}}(c)$ is minimal and uniquely ergodic. Denote by μ_P be the unique measure that is ergodic, invariant by T_{λ_F}, and supported on $\omega_{T_{\lambda_F}}(c)$.

From now on we use the notations $T := T_{\lambda_F}$, $\lambda := \lambda_F$, $c := 0$, and $c_i := T^i(c)$.

2.2 The combinatorics of the set $\omega_T(c)$

In this section we will give an explicit description of the set $\omega_T(c)$ following [49].

Put $S(-2) = 0$ and $S(-1) = 1$. From (1.3.1) and (2.1.1) we obtain that for every $k \geq 0$ the points $c_{S(k)}$ and $c_{S(k+2)}$ are on opposite sides of c. Since

$$c_{S(1)} = c_2 < c < c_1 = c_{S(0)},$$

we conclude that for $k \equiv 0 \pmod 4$, $c_{S(k)}$ is to the right of c and if $k \equiv 2 \pmod 4$, $c_{S(k)}$ is to the left of c. Since we also know that $c_{S(1)}$ is to the left of c, we can conclude that for $k \equiv 1 \pmod 4$, $c_{S(k)}$ is to the left of c, and for $k \equiv 3 \pmod 4$, $c_{S(k)}$ is to the right of c. From this, we can conclude that if k is even the points $c_{S(k)}$ and $c_{S(k+1)}$ are in opposite sides of c, and therefore

$$[c_{S(k+1)}, c_{S(k)}] \supseteq [c_{S(k+2)}, c_{S(k)}].$$

In the case that k is odd, $c_{S(k)}$ and $c_{S(k+1)}$ are on the same side with respect to c, and therefore

$$[c_{S(k+1)}, c_{S(k)}] \subseteq [c_{S(k+2)}, c_{S(k)}].$$

For each $k \geq 0$ let I_k be the smallest closed interval containing all of the points $c_{S(l)}$ for every $l \geq k$. For each $n \geq 0$ define $I_k^n := T^n(I_k)$. By the above discussion

$$I_k = \begin{cases} [c_{S(k)}, c_{S(k+1)}] \text{ if } k \text{ is even,} \\ [c_{S(k)}, c_{S(k+2)}] \text{ if } k \text{ is odd.} \end{cases} \tag{2.2.1}$$

Lemma 2.2.1. *For every $k \geq 1$ and every $j \in \{1, \ldots, S(k-1)-1\}$, we have that T^j is injective on $[c_1, c_{S(k)+1}]$. In particular $I_k^{j+1} = [c_{j+1}, c_{S(k)+1+j}]$.*

Proof: Since for every $0 \leq k < m$ we have that $|c - c_{S(k)}| > |c - c_{S(m)}|$, and for every $\ell \geq 1$, $|T([c_{S(k)}, c])| = \lambda |c - c_{S(k)}|$, we get that $c_{S(k)+1} < c_{S(m)+1} < c_1$, in particular $I_k^1 = [c_1, c_{S(k)+1}]$. In the case $k \geq 1$, by (1.3.1), with k replaced by $k+1$, for every $j \in \{1, \ldots, S(k-1)-1\}$ we have that $c_{S(k)+j}$ and c_j are in the same side respect to c. Thus $c \notin [c_{S(k)+j}, c_j] = T^{j-1}[c_{S(k)+1}, c_1]$, and then the map T^j is injective on $[c_1, c_{S(k)+1}]$. In particular, for $1 < j \leq S(k-1)$

$$\begin{aligned} I_k^j &= T^{j-1}([c_1, c_{S(k)+1}]) \\ &= [c_j, c_{S(k)+j}] \end{aligned} \tag{2.2.2}$$

$\square$

Note that for $k \geq 1$, by Lemma 2.2.1, with $j = S(k-1) - 1$

$$I_k^{S(k-1)} = [c_{S(k-1)}, c_{S(k)+S(k-1)}] = [c_{S(k-1)}, c_{S(k+1)}]. \tag{2.2.3}$$

Then, by (1.3.1), $c \in I_k^{S(k-1)}$ and $c \notin I_k^n$ for every $0 < n < S(k-1)$. Also, by (2.1.2) and Lemma 2.2.1, for every $k \geq 1$ we have that

$$c_{S(k-1)+1} < c_{S(k)+1} < c_1. \tag{2.2.4}$$

Lemma 2.2.2. *For all $k \geq 0$ we have that*

$$|c_{\rm i} - c| > |c_{S(k-1)} - c|,$$

for all $0 < {\rm i} < S(k)$, *with* ${\rm i} \neq S(k-1)$.

Proof: We will use induction on k. The cases $k = 0$ and 1 are vacuously true, the cases $k = 2$ and 3 are true by the definition of Fibonacci map and (2.1.3). Suppose now that it is true for k. We will prove that is true for $k+1$. We do this in four steps, in a first we prove that the result is true for $0 < {\rm i} < S(k)$, in a second step for $S(k) < {\rm i} < S(k) + S(k-2)$, in a third step for $S(k) + S(k-2) < {\rm i} < S(k+1)$, and finally we prove that the result is true for ${\rm i} = S(k) + S(k-2)$

Case 1: From (2.1.2), and the induction hypothesis, we have that

$$|c_{\rm i} - c| > |c_{S(k)} - c|,$$

for all $0 < {\rm i} < S(k)$.

Case 2: By (2.2.4) and by Lemma 2.2.1, we have that $c_{S(k)+{\rm i}} \in (c_{S(k-1)+{\rm i}}, c_{\rm i})$, for $0 < {\rm i} < S(k-2)$. By the induction hypothesis,

$$|c_{\rm i} - c| > |c_{S(k-1)} - c| > |c_{S(k)} - c|$$

and

$$|c_{S(k-1)+{\rm i}} - c| > |c_{S(k-1)} - c| > |c_{S(k)} - c|,$$

for $0 < {\rm i} < S(k-2)$. By 1.3.1 $c_{\rm i}$ and $c_{S(k-1)+{\rm i}}$ lie on the same side of c for $0 < {\rm i} < S(k-2)$. The above implies that

$$|c_{S(k)+{\rm i}} - c| > |c_{S(k)} - c|$$

for all $0 < {\rm i} < S(k-2)$.

Case 3: Since $T^{S(k-2)-1}$ is injective on $[c_{S(k-1)+1}, c_1]$ we get that $c_{S(k)+S(k-2)} \in (c_{S(k)}, c_{S(k-2)})$. Also, by (1.3.1), $c_{S(k)+S(k-2)}$ and $c_{S(k-2)}$ lie on the same side of c, and opposite to $c_{S(k)}$. Then

$$|c_{S(k)+S(k-2)} - c| < |c_{S(k-2)} - c|.$$

Hence

$$c_{S(k-2)+1} < c_{S(k)+S(k-2)+1} < c_1.$$

So by Lemma 2.2.1, $c_{S(k)+S(k-2)+{\rm i}} \in (c_{S(k-2)+{\rm i}}, c_{\rm i})$ for $0 < {\rm i} < S(k-3)$. By the induction hypothesis

$$|c_{\rm i} - c| > |c_{S(k)} - c| \quad \text{and} \quad |c_{S(k-2)+{\rm i}} - c| > |c_{S(k)} - c|,$$

for $0 < \mathrm{i} < S(k-3)$. Since, by (1.3.1), $c_{S(k-2)+\mathrm{i}}$ and c_{i} lie on the same side of c for $0 < \mathrm{i} < S(k-3)$ we get

$$|c_{S(k)+S(k-2)+\mathrm{i}} - c| > |c_{S(k)} - c|,$$

for all $0 < \mathrm{i} < S(k-3)$.
Case 4: It remains to prove that

$$|c_{S(k)+S(k-2)} - c| > |c_{S(k)} - c|.$$

Suppose by contradiction that

$$|c_{S(k)+S(k-2)} - c| < |c_{S(k)} - c|.$$

Then $c_{S(k)+1} < c_{S(k)+S(k-2)+1} < c_1$. Since $T^{S(k-3)-1}$ is injective on $[c_{S(k)+1}, c_1]$ we get that $c_{S(k+1)} \in (c_{S(k-3)}, c_{S(k)+S(k-3)})$. Noting that by (1.3.1) $c_{S(k-3)}$ and $c_{S(k)+S(k-3)}$ are in the same side with respect to c we have either

$$|c_{S(k+1)} - c| > |c_{S(k)+S(k-3)} - c| > |c_{S(k)} - c|$$

or

$$|c_{S(k+1)} - c| > |c_{S(k-3)} - c| > |c_{S(k)} - c|,$$

a contradiction. So we must have

$$|c_{S(k)+S(k-2)} - c| > |c_{S(k)} - c|,$$

and this conclude the proof. □

Let us denote

$$J_k := I_{k+1}^{S(k-1)} = [c_{S(k-1)}, c_{S(k+1)+S(k-1)}],$$

and put

$$D_k := [c, c_{S(k)}]$$

for every $k \geq 1$. For every $n \geq 0$ we use the notation

$$J_k^n := T^n(J_k) = I_{k+1}^{S(k-1)+n}. \tag{2.2.5}$$

Note that by definition $D_{k'} \subset I_k$, for every $k' \geq k \geq 1$.

Lemma 2.2.3. *For all $0 < k < k'$ we have $J_{k'} \subset D_{k'-1} \subset I_k$ and $J_k \cap J_{k'} = \emptyset$.*

Proof: First we will prove that J_{k+1} is contained in D_k and $c \notin J_{k+1}$ for every $k \geq 0$. Fix $k \geq 0$. and Lemma 2.2.1, $T^{S(k)-1}$ is injective on I_{k+1}^1. Then, $c_{S(k+2)+S(k)} \in (c_{S(k+2)}, c_{S(k)})$. By (1.3.1) with k replaced by $k+3$, we thus conclude $c_{S(k+2)+S(k)} \in (c_{S(k)}, c)$. Then

$$J_{k+1} = [c_{S(k)}, c_{S(k+2)+S(k)}] \subset [c, c_{S(k)}] = D_k \subseteq I_k$$

and $c \notin J_{k+1}$. Since, by definition, for every $k' > k$ we have $D_{k'-1} \subset I_{k'-1} \subset I_k$, we get

$$J_{k'} \subset D_{k'-1} \subset I_k.$$

Now we will prove that J_{k+1} and I_{k+1} are disjoint. If k is even then $I_{k+1} = [c_{S(k+1)}, c_{S(k+3)}]$. Since $c_{S(k)}$ and $c_{S(k+1)}$ lie on opposite sides respect to c, we have that $c_{S(k)}$, $c_{S(k+2)+S(k)}$, and

$c_{S(k+3)}$ lie on the same side of c. By Lemma (2.2.2), with k replaced by $k+4$,and since $c_{S(k+2)+S(k)} \in (c_{S(k)}, c)$, we get

$$|c_{S(k)} - c| > |c_{S(k+2)+S_k} - c| > |c_{S(k+3)} - c|.$$

So $J_{k+1} \cap I_{k+1} = \emptyset$. Now, if k is odd $I_{k+1} = [c_{S(k+1)}, c_{S(k+2)}]$ and $c_{S(k)}$, $c_{s(k+1)}$ and $c_{S(k+2)+S(k)}$ lie on the same side of c. Suppose that $|c_{S(k+1)} - c| > |c_{S(k+2)+S(k)} - c|$, then

$$[c_{S(k+1)}, c_{S(k+2)+S(k)}] \subset [c_{S(k)}, c_{S(k+2)+S(k)}].$$

Since $T^{S(k-1)}$ is injective on $[c_{S(k)}, c_{S(k+2)+S(k)}]$, then $T^{S(k-1)}$ is injective on $[c_{S(k+1)}, c_{S(k+2)+S(k)}]$. So we get

$$T^{S(k-1)}([c_{S(k+1)}, c_{S(k+2)+S(k)}]) = [c_{S(k+1)+S(k-1)}, c_{S(k+3)}].$$

Since $S(k+1) + S(k-1) < S(k+4)$, by Lemma 2.2.2, with k replaced by $k+4$, we get

$$|c_{S(k+1)+S(k-1)} - c| > |c_{S(k+3)} - c|.$$

On the other hand, $T^{S(k-1)}(J_{k+1}) = [c_{S(k+1)}, c_{S(k+3)}]$, then

$$T^{S(k-1)}(c_{S(k+1)}) = c_{S(k+1)+S(k-1)} \in (c_{S(k+1)}, c_{S(k+3)})$$

and by (1.3.1), with k replaced by $k+1$, we have that $c_{S(k+1)+S(k-1)}$ and $c_{S(k-1)}$ are in the same side of c. Since $k-1 \equiv k+3 \pmod 4$, we have that $c_{S(k-1)}$ and $c_{S(k+3)}$ are on the same side of c, so $c_{S(k+1)+S(k-1)} \in (c, c_{S(k+3)})$. Thus

$$|c_{S(k+1)+S(k-1)} - c| < |c_{S(k+3)} - c|,$$

a contradiction. So we must have $c_{S(k)+S(k+2)} \in (c_{S(k)}, c_{S(k+1)})$ and

$$J_{k+1} \ \cap \ I_{k+1} \ = \ \emptyset. \tag{2.2.6}$$

This conclude the proof of the lemma. □

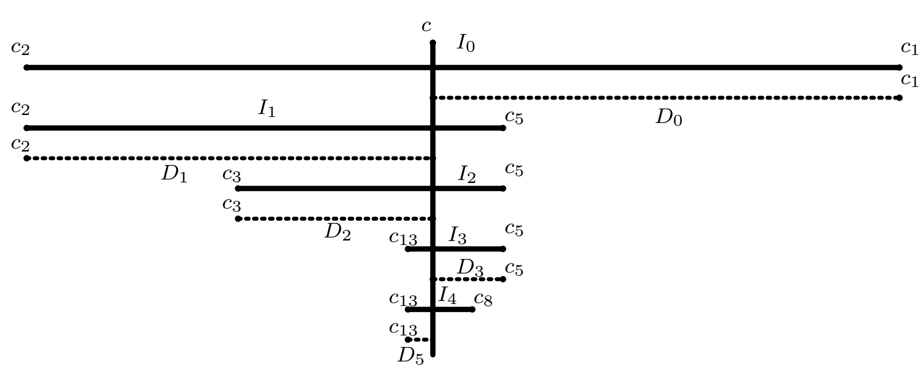


Figure 2.1: First five I_k (solid line), and D_k (dashed line) intervals.

Taking $k' = k+1$ in Lemma (2.2.3) we get $J_{k+1} \subset I_k$, and then

$$J_{k+1} \cup I_{k+1} \subset I_k \subseteq I_k^{S(k-1)}, \tag{2.2.7}$$

for every $k \geq 1$

Definition 2.2.4. For $k \geq 0$ let M_k be the $S(k)-$fold union

$$M_k = \bigcup_{0 \leq n < S(k-1)} I_k^n \cup \bigcup_{0 \leq n < S(k-2)} J_k^n.$$

Some examples of M_k,

$$\begin{aligned}
M_0 &= I_0 \\
&= [c_1, c_2], \\
M_1 &= I_1 \cup J_1 \\
&= [c_2, c_5] \cup [c_4, c_1], \\
M_2 &= I_2 \cup I_2^1 \cup J_2 \\
&= [c_3, c_5] \cup [c_4, c_1] \cup [c_2, c_7] \\
M_3 &= I_3 \cup I_3^1 \cup I_3^2 \cup J_3 \cup J_3^1 \\
&= [c_{13}, c_5] \cup [c_6, c_1] \cup [c_2, c_7] \cup [c_3, c_{11}] \cup [c_4, c_{12}] \\
M_4 &= I_4 \cup I_4^1 \cup I_4^2 \cup I_4^3 \cup I_4^4 \cup J_4 \cup J_4^1 \cup J_4^2 \\
&= [c_{13}, c_8] \cup [c_9, c_1] \cup [c_2, c_{10}] \cup [c_3, c_{11}] \cup [c_4, c_{12}] \cup [c_{14}, c_5] \cup [c_6, c_{19}] \cup [c_{20}, c_7],
\end{aligned}$$

and so on. In Figure 2.2, we see a diagram with the first five levels of M_k.

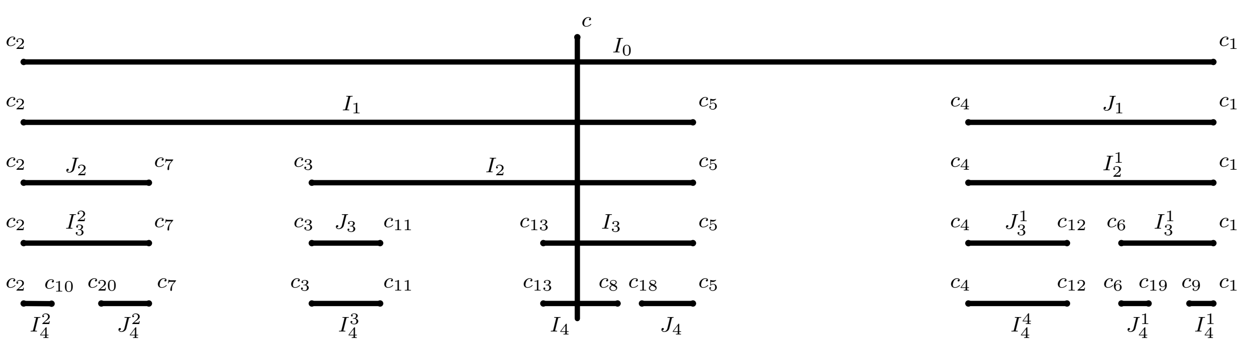


Figure 2.2: First five levels of M_k.

From the definition, for every $k \geq 0$ the $S(k)$ closed intervals

$$I_k, I_k^1, \ldots, I_k^{S(k-1)-1}, J_k, J_k^1, \ldots, J_k^{S(k-2)-1},$$

are pairwise disjoint, each M_k contains the set $\overline{\mathcal{O}_T(c)}$ and they form a nested sequence of closed sets $M_1 \supset M_2 \supset M_3 \supset \ldots$ with intersection equal to the Cantor set $\overline{\mathcal{O}_T(c)}$, for a proof of this statements see [49, Lemma 3.5]. Now by (2.2.7) we have that for every $1 \leq m < S(k-1)$,

$$I_{k+1}^m \cup J_{k+1}^m \subset I_k^m.$$

Also by (2.2.5) for every $0 \leq n < S(k-2)$,

$$I_{k+1}^{S(k-1)+n} = J_k^n.$$

Since the sets in M_k are disjoint we get that

$$\cup_{A \in M_{k+1}} (A \cap I_k) = I_{k+1} \cup J_{k+1}. \tag{2.2.8}$$

2.3 Diameter estimates

In this section we will give an estimate on how the distances $|c_{S(k)} - c|$ decrease as $k \to \infty$.

Lemma 2.3.1. *The following limit*

$$\lim_{k\to\infty} \lambda^{S(k+1)}|D_k|$$

exists and is strictly positive.

Proof: Since

$$T(D_k) = T([c, c_{S(k)}]) = I_k^1,$$

by (2.2.3) we have that

$$T^{S(k-1)}(D_k) = [c_{S(k-1)}, c_{S(k+1)}].$$

By (1.3.1), $c_{S(k-1)}$ and $c_{S(k+1)}$ are in opposite sides of c. Then $D_{k-1} \cap D_{k+1} = \{c\}$, so

$$T^{S(k-1)}(D_k) = D_{k-1} \cup D_{k+1}.$$

Since, by Lemma 2.2.1, $T^{S(k-1)}$ is injective on I_k and $D_k \subset I_k$ we get that

$$|T^{S(k-1)}(D_k)| = \lambda^{S(k-1)}|D_k| = |D_{k-1}| + |D_{k+1}|.$$

For $k \geq 0$ put $\nu_k := |D_k|/|D_{k+1}|$. By the above we get

$$\lambda^{S(k-1)} = \nu_{k-1} + \frac{1}{\nu_k}.$$

By (2.1.2), $\nu_k > 1$, so $0 < \nu_k^{-1} < 1$. Since $\lambda > 1$, we have $\lambda^{S(k-1)} \longrightarrow \infty$ as $k \longrightarrow \infty$. Then $\nu_k \longrightarrow \infty$ as $k \longrightarrow \infty$. So $\lambda^{S(k-1)} - \nu_{k-1} \longrightarrow 0$ as $k \longrightarrow \infty$. Then, if we define $C_k := \nu_k \lambda^{-S(k)}$, we have $0 < C_k < 1$ and $C_k \nearrow 1$ exponentially fast as $n \longrightarrow \infty$. By definition of ν_k, we have that

$$\frac{|D_0|}{|D_{k+1}|} = \prod_{\mathrm{i}=0}^{k} \nu_{\mathrm{i}} = \prod_{\mathrm{i}=0}^{k} \lambda^{S(\mathrm{i})} C_{\mathrm{i}} = \lambda^{S(k+2)-S(1)} \prod_{\mathrm{i}=0}^{k} C_{\mathrm{i}}.$$

Then

$$|D_{k+1}|\lambda^{S(k+2)-S(1)} = |D_0| \left[\prod_{\mathrm{i}=0}^{k} C_{\mathrm{i}}\right]^{-1}. \tag{2.3.1}$$

Since $\prod_{\mathrm{i}=0}^{k} C_{\mathrm{i}}$ converge to a strictly positive number as $k \longrightarrow \infty$, the proof is complete. $\square$

2.4 The invariant measure

Let us denote by μ_P the unique ergodic invariant measure of T restricted to $\omega_T(c)$. As in the previous, section put

$$S(-2) = 0, S(-1) = 1, S(0) = 1, S(1) = 2, \ldots$$

and put $\varphi := \frac{1+\sqrt{5}}{2}$. In this section we will estimate the value of μ_P over the elements of M_k for every $k \geq 1$.

As mentioned in Section 2.2, we know that the set $\omega_T(c)$ is contained in M_k for very $k \geq 1$ and the sets

$$I_k, I_k^1, \dots I_k^{S(k-1)-1}, J_k, J_k^1, \dots, J_k^{S(k-2)-1},$$

are disjoint. Then

$$\sum_{\mathrm{i}=0}^{S(k-1)-1} \mu_P(I_k^{\mathrm{i}}) + \sum_{j=0}^{S(k-2)-1} \mu_P(J_k^j) = 1. \tag{2.4.1}$$

Since T restricted to $\omega_T(c)$ is injective, except at the critical point that has two preimages, we have that

$$\begin{aligned} \mu_P(I_k^{\mathrm{i}}) &= \mu_P(I_k^j) && (2.4.2) \\ \mu_P(J_k^p) &= \mu_P(J_k^q), && (2.4.3) \end{aligned}$$

for every $0 \leq \mathrm{i}, j < S(k-1)$ and $0 \leq p, q < S(k-2)$. Then we can write (2.4.1) as

$$S(k-1)\mu_P(I_k) + S(k-2)\mu_P(J_k) = 1 \tag{2.4.4}$$

Since

$$I_k \sqcup J_k \subset I_{k-1},$$

and

$$\omega_T(c) \cap (I_k \sqcup J_k) = \omega_T(c) \cap I_{k-1},$$

we have that

$$\mu_P(I_k) + \mu_P(J_k) = \mu_P(I_{k-1}). \tag{2.4.5}$$

And since

$$J_{k-1} = I_k^{S(k-1)},$$

using (2.4.2) with k replaced by $k-1$, we have that

$$\mu_P(J_{k-1}) = \mu_P(I_k). \tag{2.4.6}$$

Combining (2.4.5) and (2.4.6) we can write

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mu_P(I_k) \\ \mu_P(J_k) \end{bmatrix} = \begin{bmatrix} \mu_P(I_{k-1}) \\ \mu_P(J_{k-1}) \end{bmatrix}. \tag{2.4.7}$$

Lemma 2.4.1. *For every $m \geq 1$ we have*

$$\mu_P(I_m) = \frac{1}{\varphi^m} \qquad \text{and} \qquad \mu_P(J_m) = \frac{1}{\varphi^{m+1}}.$$

Proof: We will use induction to prove the lemma. For $m = 1$, we can apply $k-2$ times the equation (2.4.7) to

$$\begin{bmatrix} \mu_P(I_{k-1}) \\ \mu_P(J_{k-1}) \end{bmatrix},$$

and we can write (2.4.7) as

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} \mu_P(I_k) \\ \mu_P(J_k) \end{bmatrix} = \begin{bmatrix} \mu_P(I_1) \\ \mu_P(J_1) \end{bmatrix}. \tag{2.4.8}$$

Using that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k-1} = \begin{bmatrix} S(k-2) & S(k-3) \\ S(k-3) & S(k-4) \end{bmatrix},$$

for $k \geq 2$. We can write

$$\begin{bmatrix} \mu_P(I_1) \\ \mu_P(J_1) \end{bmatrix} = \begin{bmatrix} S(k-2)\mu_P(I_k) + S(k-3)\mu_P(J_k) \\ S(k-3)\mu_P(I_k) + S(k-4)\mu_P(J_k) \end{bmatrix}. \tag{2.4.9}$$

Multiplying the first equation in (2.4.9) by $\frac{S(k)}{S(k-1)}$ we get

$$\frac{S(k)}{S(k-1)}\mu_P(I_1) = \frac{S(k-2)}{S(k-1)}S(k)\mu_P(I_k) + \frac{S(k-3)}{S(k-2)}\frac{S(k)}{S(k-1)}S(k-2)\mu_P(J_k). \tag{2.4.10}$$

Using (2.4.4) we can write (2.4.10) as

$$\frac{S(k)}{S(k-1)}\mu_P(I_1) = S(k)\mu_P(I_k)\left[\frac{S(k-2)}{S(k-1)} - \frac{S(k-3)}{S(k-2)}\right] + \frac{S(k-3)}{S(k-2)}\frac{S(k)}{S(k-1)}. \tag{2.4.11}$$

Since $\frac{S(k)}{S(k-1)} \longrightarrow \varphi$ as $k \longrightarrow \infty$, taking limit on (2.4.11) over k we get

$$\varphi\mu_P(I_1) = 1,$$

and then

$$\mu_P(I_1) = \frac{1}{\varphi}.$$

Using (2.4.4) with k replaced by 1 we get that

$$\mu_P(J_1) = \frac{1}{\varphi^2}.$$

So the lemma holds for $m = 1$.

Suppose now that the result is true for m. By (2.4.6) we have that

$$\mu_P(I_{m+1}) = \mu_P(J_m) = \frac{1}{\varphi^{m+1}}.$$

By (2.4.5) we have that

$$\begin{aligned} \mu_P(J_{m+1}) &= \mu_P(I_m) - \mu_P(I_{m+1}) \\ &= \frac{1}{\varphi^m} - \frac{1}{\varphi^{m+1}} \\ &= \frac{1}{\varphi^m}\left(1 - \frac{1}{\varphi}\right) \\ &= \frac{1}{\varphi^m}\frac{1}{\varphi^2} \\ &= \frac{1}{\varphi^{m+2}}, \end{aligned}$$

and we get the result. □

Chapter 3

Maps with unbounded derivative

In this chapter we will present a new class of unimodal maps that has Fibonacci combinatorics, so the $\omega-$limit set of the turning point is a uniquely ergodic minimal Cantor system. This class has two non-flat critical points and the turning point is a Lorenz-like Singularity.

3.1 Maps with singularities

In this section, we will introduce the family of unimodal maps that we will study in this thesis.

We need to recall some definitions. As in the previous chapter, $\lambda := \lambda_F$ will denote the parameter for which the tent map T_λ has Fibonacci combinatorics. We write $T := T_\lambda$. For every $A \subset [-1,1]$ and every $x \in [-1,1]$, we denote the *distance from* x *to* A by

$$\operatorname{dist}(x, A) := \inf\{|x-y| \colon y \in A\}.$$

Let $f \colon [-1,1] \longrightarrow [-1,1]$, be a continuous map. We will use f' to denote the derivative of f when it exits. We will say that the point $c \in [-1,1]$ is a *Lorenz-like singularity* if there exists ℓ^+ and ℓ^- in $(0,1)$, $L > 0$, and $\delta > 0$ such that the following holds: For every $x \in (c, c+\delta)$

$$\frac{1}{L|x-c|^{\ell^+}} \leq |f'(x)| \leq \frac{L}{|x-c|^{\ell^+}}, \tag{3.1.1}$$

and for every $x \in (c-\delta, c)$

$$\frac{1}{L|x-c|^{\ell^-}} \leq |f'(x)| \leq \frac{L}{|x-c|^{\ell^-}}. \tag{3.1.2}$$

We call ℓ^+ and ℓ^- the *right and left order of* c respectively. A point $\hat{c} \in [-1,1]$ is called a *critical point of* f if the derivative of f is defined at $\hat{c}$, and $f'(\hat{c}) = 0$. We will say that a critical point $\hat{c}$ is *non-flat* if there exist $\alpha^+ > 0$, $\alpha^- > 0$, $M > 0$, and $\delta > 0$ such that the following holds:
For every $x \in (\hat{c}, \hat{c}+\delta)$

$$\left|\log \frac{|f'(x)|}{|x-\hat{c}|^{\alpha^+}}\right| \leq M, \tag{3.1.3}$$

and for every $x \in (\hat{c} - \delta, \hat{c})$

$$\left| \log \frac{|f'(x)|}{|x - \hat{c}|^{\alpha^-}} \right| \leq M. \tag{3.1.4}$$

We call α^+ and α^- the *right and left order of* $\hat{c}$ respectively. Let us denote by $\mathrm{Crit}(f)$ the set of critical points of f. If f is a unimodal map with turning point c, we will use the notation $\mathcal{S}(f) := \mathrm{Crit}(f) \cup \{c\}$. Let us denote by C^ω the class of analytic maps. Here we will say that f is a *C^ω-unimodal map* if it is of class C^ω outside $\mathcal{S}(f)$.

For a probability measure μ on $[-1, 1]$, we define the *pushforward of μ by f* as

$$f_*\mu := \mu \circ f^{-1}.$$

Let $h\colon [-1, 1] \to [-1, 1]$ be a homeomorphism of class C^ω on $[-1, 1] \setminus \{0\}$ with a unique non-flat critical point at 0. In this thesis we will study the C^ω-unimodal map defined by

$$f := h \circ T_{\lambda_F} \circ h^{-1}.$$

In the following theorem, we will prove that the map f has a Lorenz-like singularity at $\tilde{c} := h(0)$ and two non-flat critical points, given by the preimages by f of the Lorenz-like singularity $\tilde{c}$. We will also prove a key bound on the derivative of f in terms of h^{-1}. Without loss of generality, through the rest of this work, we will assume that h preserves orientation.

Theorem 11. *Let $h\colon [-1, 1] \to [-1, 1]$ be a homeomorphism of class C^ω on $[-1, 1] \setminus \{0\}$ with a unique non-flat critical point at 0. Then, the C^ω-unimodal map $f := h \circ T_{\lambda_F} \circ h^{-1}$ has a Lorenz-like singularity at $\tilde{c}$ and two non-flat critical points at the preimages of $\tilde{c}$. Moreover there exists $\alpha^+ > 1$, $\alpha^- > 1$, $K > 0$, and $\delta > 0$ such that the following property holds: For every $x \in (\tilde{c}, h(\delta))$,*

$$K^{-1}|h^{-1}(x)|^{-\alpha^+} \leq |f'(x)| \leq K|h^{-1}(x)|^{-\alpha^+}, \tag{3.1.5}$$

and for every $x \in (h(-\delta), \tilde{c})$,

$$K^{-1}|h^{-1}(x)|^{-\alpha^-} \leq |f'(x)| \leq K|h^{-1}(x)|^{-\alpha^-}. \tag{3.1.6}$$

Remark 3.1.1. Observe that since the map f defined in Theorem 11 has Fibonacci recurrence, by Theorem 8 implies that f restricted to $\omega_f(\tilde{c})$ is uniquely ergodic.

Before presenting the proof, we make the following observation. Let $h\colon [-1, 1] \to [-1, 1]$ be as in Theorem 11 and $f = h \circ T \circ h^{-1}$. As in Theorem 11, put $\tilde{c} := h(0)$. Since h has a non-flat critical point at 0, by (3.1.3) and (3.1.4) there are $\alpha^+ > 0$, $\alpha^- > 0$, and $\delta > 0$ such that

$$\mathrm{e}^{-M}|\hat{x}|^{\alpha^+} \leq |h'(\hat{x})| \leq \mathrm{e}^{M}|\hat{x}|^{\alpha^+}, \tag{3.1.7}$$

for every $\hat{x} \in (0, \delta)$ and

$$\mathrm{e}^{-M}|\hat{x}|^{\alpha^-} \leq |h'(\hat{x})| \leq \mathrm{e}^{M}|\hat{x}|^{\alpha^-}, \tag{3.1.8}$$

for every $\hat{x} \in (-\delta, 0)$. Since $c = 0$ and for every $k \in \mathbb{N}$, $c \notin I_k^1 = [c_{c_{S(k)+1}}, c_1]$, we have that there exist positive real numbers W_1 and W_2 such that for every $x \in I_k$

$$W_1 \leq |h'(T(x))| \leq W_2. \tag{3.1.9}$$

Proof: [Proof of Theorem 11] Let $\delta > 0$ be small enough so that (3.1.7) and (3.1.8) holds, and let $k \in \mathbb{N}$ large enough so that $I_k \subset (-\delta, \delta)$. By the chain rule we have

$$f'(x) = \lambda \frac{h'(T(h^{-1}(x)))}{h'(h^{-1}(x))}, \tag{3.1.10}$$

for every $x \in (h(-\delta), h(\delta)) \setminus \{\tilde{c}\}$. Let $K := \max\{\lambda^{-1} e^M W_1^{-1}, \lambda e^M W_2\}$. Then by (3.1.9), (3.1.7), (3.1.8), and (3.1.10) we have that for every $x \in (\tilde{c}, h(\delta))$

$$\frac{1}{K|h^{-1}(x)|^{\alpha^+}} \leq |f'(x)| \leq \frac{K}{|h^{-1}(x)|^{\alpha^+}}, \tag{3.1.11}$$

and for every $x \in (h(-\delta), \tilde{c})$

$$\frac{1}{K|h^{-1}(x)|^{\alpha^-}} \leq |f'(x)| \leq \frac{K}{|h^{-1}(x)|^{\alpha^-}}. \tag{3.1.12}$$

Now, from (3.1.7) and (3.1.8) there exist $M_1 > 0$ and $M_2 > 0$ such that for every $x \in (0, \delta)$

$$M_1^{-1}|x|^{\alpha^+ + 1} \leq |h(x)| \leq M_1|x|^{\alpha^+ + 1},$$

and for every $x \in (-\delta, 0)$

$$M_2^{-1}|x|^{\alpha^- + 1} \leq |h(x)| \leq M_2|x|^{\alpha^- + 1}.$$

Since h is a homeomorphism, there exist constants $M_3 > 0$ and $M_4 > 0$ such that for every $x \in (\tilde{c}, h(\delta))$

$$M_3^{-1}|x - \tilde{c}|^{\frac{1}{\alpha^+ + 1}} \leq |h^{-1}(x)| \leq M_3|x - \tilde{c}|^{\frac{1}{\alpha^+ + 1}}, \tag{3.1.13}$$

and for every $x \in (h(-\delta), \tilde{c})$

$$M_4^{-1}|x - \tilde{c}|^{\frac{1}{\alpha^- + 1}} \leq |h^{-1}(x)| \leq M_4|x - \tilde{c}|^{\frac{1}{\alpha^- + 1}}. \tag{3.1.14}$$

Then, by (3.1.11), (3.1.12), (3.1.13), and (3.1.14) we have that for every $x \in (\tilde{c}, h(\delta))$

$$\frac{1}{M_3|x - \tilde{c}|^{\frac{\alpha^+}{\alpha^+ + 1}}} \leq |f'(x)| \frac{M_3}{|x - \tilde{c}|^{\frac{\alpha^+}{\alpha^+ + 1}}},$$

and for every $x \in (\tilde{c}, h(-\delta))$

$$\frac{1}{M_4|x - \tilde{c}|^{\frac{\alpha^-}{\alpha^- + 1}}} \leq |f'(x)| \leq \frac{M_4}{|x - \tilde{c}|^{\frac{\alpha^-}{\alpha^- + 1}}}.$$

Thus, f has a Lorenz-like singularity at $\tilde{c}$ with left order $\ell^- = \frac{\alpha^-}{\alpha^- + 1}$, and right order $\ell^+ = \frac{\alpha^+}{\alpha^+ + 1}$. $\square$

3.2 Example

Now we will provide an example of a map f as in Theorem 11. Fix ℓ^+ and ℓ^- in $(0,1)$. Put $\alpha^+ := \frac{1}{1-\ell^+}$ and $\alpha^- := \frac{1}{1-\ell^-}$. Define

$$h_{\alpha^+,\alpha^-} : [-1,1] \longrightarrow [-1,1]$$

as

$$h_{\alpha^+,\alpha^-}(x) = \begin{cases} |x|^{\alpha^+} & \text{if } x \geq 0 \\ -|x|^{\alpha^-} & \text{if } x < 0. \end{cases} \tag{3.2.1}$$

So

$$h^{-1}_{\alpha^+,\alpha^-}(x) = \begin{cases} |x|^{1/\alpha^+} & \text{if } x \geq 0 \\ -|x|^{1/\alpha^-} & \text{if } x < 0. \end{cases} \tag{3.2.2}$$

Put $h = h_{\alpha^+,\alpha^-}$. Then by the definition of the tent map, (3.2.1),(3.2.2), and the chain rule, we have

$$f'(x) = \lambda_F \frac{h'(T_(h^{-1}(x)))}{h'(h^{-1}(x))}$$

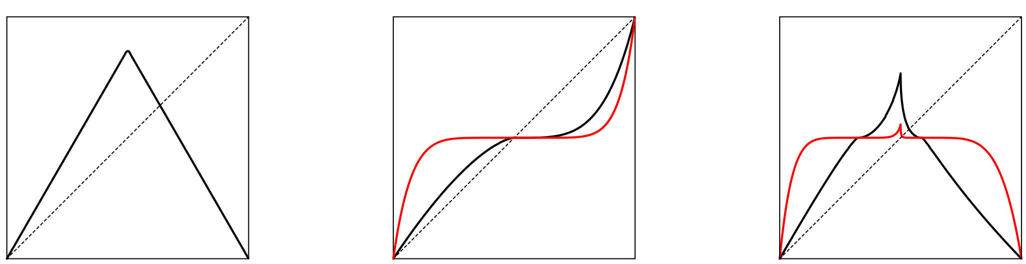
for every $x \in [-1,1] \setminus \{h(0)\}$. The function $h'(T(h^{-1}(x)))$ is bounded for x close enough to 0. Then, by Theorem 11, there exists $L > 0$ such that for every $x \in (h(0), h(\delta))$,

$$\frac{1}{L|x|^{\ell^+}} \leq |f'(x)| \leq \frac{L}{|x|^{\ell^+}},$$

and for every $x \in (h(0), h(-\delta))$,

$$\frac{1}{L|x|^{\ell^-}} \leq |f'(x)| \leq \frac{L}{|x|^{\ell^-}}.$$

Thus, $h(0)$ is a Lorenz-like singularity of f, see Figure 3.1. Also, by (3.2.1) and (3.2.2), if δ is small enough so that $T^{-1}_{\lambda_F}(0) \cap (-\delta, \delta) = \emptyset$, the two critical points of f are non-flat. The one to the left of $h(0)$ has right order α^+ and left order α^-, and the one to the right of $h(0)$ has right order α^- and left order α^+.



(a) (b) (c)

Figure 3.1: Graphics of the functions $T_{\lambda_F}(x)$ (Figure (A)), $h_\alpha(x)$ for $\alpha^+ = 2$ and $\alpha^- = 1.2$ (Figure (B)) and $f(x)$ (Figure (C)).

3.3 Recurrence of the turning point

In this section, we will study the recurrence of the Lorenz-like singularity of the map f constructed in the previous section. Since the critical points of f are the preimages of $\tilde{c}$, the recurrence of the Lorenz-like singularity also give us information about the recurrence of the critica lset. In other words, the recurrence of the singular set $\mathcal{S}(f)$ can be understand by just studying the recurrence of the Lorenz-like singularity.

For interval dynamics, the recurrence of the critical points has played an important role in the study of topological and metric properties. This item represents a crucial difference between smooth interval maps, and the case of interval maps with critical points and Lorenz-like singularities. In the case of interval maps with critical points and singularities, control over the recurrence of the singular set is needed for the existence of absolutely continuous invariant probability measures, see [46], [2], [45]. In the smooth case, certain conditions on the growth of the derivative restrict the recurrence to the critical set, leading to conditions on the existences and properties of absolutely continuous invariant probability measures. This growth conditions were first introduced by Collet and Eckmann [16]. More precisely, they proved that a unimodal map with negative Schwarzian derivative f satisfying the following condition

$$\liminf_{n\to\infty} \frac{\log|(f^n)'(f(c))|}{n} > 0. \tag{CE}$$

admits a finite absolutely continuous invariant measure. This is known as the *Collet-Eckmann condition*. Further work has been done in this direction, getting less restrictive conditions on the growth of the derivative of the critical orbit, see [59], [13], [9].

In our setting, the existence of critical points and a Lorenz-like singularity, and their interaction, can give rise to asymptotic growth in the derivative due to the recurrence to the region of unbounded derivative.

Since f is topologically conjugated to the Fibonacci tent map we know that $\tilde{c}$ is recurrent, and that the recurrence times are given by the Fibonacci numbers. Then, to have an estimate on the recurrence of the turning point it is enough to estimate the decay of the distances $|f^{S(k)}(\tilde{c}) - \tilde{c}|$, where

$$S(0) = 1,\ S(1) = 2,\ S(2) = 3,\ S(3) = 5, S(4) = 8, S(6) = 13\ldots,$$

are the Fibonacci numbers. More precisely, we will prove the following

Theorem 12. *The map f, given in Theorem 11 has* exponential recurrence *of the Lorenz-like singularity orbit, thus,*

$$\limsup_{n\to\infty} \frac{-\log|f^n(\tilde{c}) - \tilde{c}|}{n} \in (0, +\infty).$$

Theorem 12 is a direct consequence of the following proposition.

Proposición 3.1 *Let f be as in Theorem 11. There exist Θ, α', α'' positive numbers, such that*

$$\lambda^{-S(k)\alpha''}\Theta^{-1} \le |f^{S(k)}(\tilde{c}) - \tilde{c}| \le \lambda^{-S(k)\alpha'}\Theta, \tag{3.3.1}$$

for every $k \geq 1$.

For every $k \geq 1$, put

$$D_k^+ := (|c_{S(k)}|, c) \qquad \text{and} \qquad D_k^- := (-|c_{S(k)}|, c)$$

$$A_k^+ := D_k^+ \setminus \overline{D_{k+1}^+} \qquad \text{and} \qquad A_k^- := D_k^- \setminus \overline{D_{k+1}^-}.$$

Observe that

$$|A_k^+| = |D_k| - |D_{k+1}| = |A_k^-|. \tag{3.3.2}$$

Lemma 3.3.1. *There exist* α''_+, α''_-, α'_+, α'_-, K, *and* Q *positive real numbers such that*

$$\lambda^{-S(k)\alpha''_+}Q^{-1} \leq |h(A_k^+)| \leq \lambda^{-S(k)\alpha'_+}Q, \tag{3.3.3}$$

and

$$\lambda^{-S(k)\alpha''_-}Q^{-1} \leq |h(A_k^-)| \leq \lambda^{-S(k)\alpha'_-}Q, \tag{3.3.4}$$

for every $k \geq K$.

Proof: By Lemma 2.3.1, there exists $\beta > 0$ such that

$$\lim_{n\to\infty} \lambda^{S(k+1)}|D_k| = \beta.$$

Let $\varepsilon > 0$ be small enough so that $(\beta - \varepsilon)/(\beta + \varepsilon) \geq 1/2$. Let $M > 0$ be as in (3.1.7) and (3.1.8). Fix $K > 0$ big enough so that, for every $k \geq K$, (3.1.7), (3.1.8) holds on A_k, and the following holds:

$$\lambda^{-S(k+1)}(\beta - \varepsilon) \leq |D_k| \leq \lambda^{-S(k+1)}(\beta + \varepsilon), \tag{3.3.5}$$

$$\lambda^{-S(k)} \leq \frac{1}{4}, \tag{3.3.6}$$

and

$$\frac{S(k+1)}{S(k)} < \varphi + \varepsilon. \tag{3.3.7}$$

By (3.3.5), with k replaced by $k+1$, we get

$$\lambda^{S(k+2)}\frac{1}{\beta + \varepsilon} \leq \frac{1}{|D_{k+1}|} \leq \lambda^{S(k+2)}\frac{1}{\beta - \varepsilon}. \tag{3.3.8}$$

Combining (3.3.5) and (3.3.8), we get

$$\lambda^{S(k)}\frac{\beta - \varepsilon}{\beta + \varepsilon} \leq \frac{|D_k|}{|D_{k+1}|} \leq \lambda^{S(k)}\frac{\beta + \varepsilon}{\beta - \varepsilon}. \tag{3.3.9}$$

For $k \geq K$, using the mean value theorem on the function $h\colon A_k^+ \longrightarrow h(A_k^+)$, there exists $\gamma^+ \in A_k^+$ such that

$$\frac{|h(A_k^+)|}{|A_k^+|} = |h'(\gamma^+)|. \tag{3.3.10}$$

Let α^+, be the right order of 0 as a critical point of h. By (3.1.7), we have

$$\mathrm{e}^{-M}|\gamma^+|^{\alpha^+} \leq |h'(\gamma^+)| \leq \mathrm{e}^{M}|\gamma^+|^{\alpha^+}. \tag{3.3.11}$$

Since $\gamma^+ \in A_k^+$, we have that

$$|D_{k+1}| \leq |\gamma^+| \leq |D_k|.$$

Then, by (3.3.5), (3.3.6), (3.3.10) and (3.3.11) we have that

$$\mathrm{e}^{-M}|D_{k+1}|^{\alpha^+ +1}\left(\frac{|D_k|}{|D_{k+1}|}-1\right) \leq |h(A_k^+)| \leq \mathrm{e}^{M}|D_k|^{\alpha^+ +1}\left(1-\frac{|D_{k+1}|}{|D_k|}\right). \tag{3.3.12}$$

Using (3.3.9) in (3.3.12), we obtain

$$\mathrm{e}^{-M}(\beta-\varepsilon)^{\alpha^+ +1}\lambda^{-S(k+2)(\alpha^+ +1)}\left(\lambda^{S(k)}\frac{\beta-\varepsilon}{\beta+\varepsilon}-1\right) \leq |h(A_k^+)| \leq$$
$$\mathrm{e}^{M}(\beta+\varepsilon)^{\alpha^+ +1}\lambda^{-S(k+1)(\alpha^+ +1)}\left(1-\lambda^{-S(k)}\frac{\beta-\varepsilon}{\beta+\varepsilon}\right). \tag{3.3.13}$$

Put

$$Q_1 := \mathrm{e}^{-M}\left(\frac{\beta}{3}\right)^{\alpha^+ +1}\frac{1}{4} \qquad \text{and} \qquad Q_2 := \mathrm{e}^{M}2\beta.$$

By (3.3.6) and since $(\beta-\varepsilon)/(\beta+\varepsilon) \geq 1/2$, we have that

$$Q_1 \leq \mathrm{e}^{-M}(\beta-\varepsilon)^{\alpha^+ +1}\left(\frac{\beta-\varepsilon}{\beta+\varepsilon}-\lambda^{-S(k)}\right),$$

and

$$\mathrm{e}^{M}(\beta+\varepsilon)^{\alpha^+ +1}\left(1-\lambda^{-S(k)}\frac{\beta-\varepsilon}{\beta+\varepsilon}\right) \leq Q_2,$$

for every $k \geq K$. Then

$$\lambda^{-S(k+2)(\alpha^+ +1)}\lambda^{S(k)}Q_1 \leq |h(A_k^+)| \leq \lambda^{-S(k+1)(\alpha^+ +1)}Q_2. \tag{3.3.14}$$

Finally, put

$$\alpha'_+ := \alpha^+ + 1 \quad \text{and} \quad \alpha''_+ := (\varphi+\varepsilon)^2(\alpha^+ +1)-1.$$

Since $S(k) = S(k+2) - S(k-1)$, by (3.3.7) we have

$$\begin{aligned} -S(k+2)(\alpha^+ +1)+S(k) &= -S(k)\left(\frac{S(k+2)}{S(k)}(\alpha^+ +1)-1\right) \\ &\geq -S(k)\alpha''_+. \end{aligned}$$

Then, taking $Q := \max\{Q_1^{-1}, Q_2\}$ we have

$$\lambda^{-S(k)\alpha''_+}Q^{-1} \leq |h(A_k^+)| \leq \lambda^{S(k)\alpha'_+}Q.$$

In the same way we can prove (3.3.4). □

Proof: [Proof of Proposition 3.1] Let α''_+, α''_-, α'_+, α'_-, K, and Q, be as in Lemma 3.3.1. By (3.3.2), we have that

$$\sum_{m=0}^{n} |h(A^+_{k+m})| = |h(D^+_k)| - |h(D^+_{k+n+1})|, \tag{3.3.15}$$

for every $n \geq 0$. Then by (3.3.3) and (3.3.15) we get

$$Q^{-1} \sum_{m=0}^{n} \lambda^{-S(k+m)\alpha''_+} \leq |h(D^+_k)| - |h(D^+_{k+n+1})| \leq Q \sum_{m=0}^{n} \lambda^{-S(k+m)\alpha'_+}, \tag{3.3.16}$$

for every $n \geq 0$. Now, for $m \geq 0$ we have that

$$S(k+m) = S(k) + \sum_{j=0}^{m-1} S(k+j-1). \tag{3.3.17}$$

Put

$$F_{k+m} := \sum_{j=0}^{m-1} S(k+j-1), \quad \text{and} \quad \sigma'(k) := 1 + \sum_{\mathrm{i}=0}^{\infty} \lambda^{-\alpha'_+ F_{k+\mathrm{i}}}.$$

Then, combining (3.3.16) and (3.3.17) we obtain

$$\lambda^{-S(k)\alpha''_+} Q^{-1} \leq |h(D^+_k)| - |h(D^+_{k+n+1})| \leq \lambda^{-S(k)\alpha'_+} Q\sigma'(k). \tag{3.3.18}$$

If we put

$$\Theta := Q\left(1 + \sum_{\mathrm{i}=0}^{\infty} \lambda^{-\alpha''_+ S(\mathrm{i})}\right),$$

then for every $k \geq K$ and every $m \geq 0$ we have

$$Q\sigma'(k) \leq \Theta, \quad \text{and} \quad \Theta^{-1} \leq Q^{-1}.$$

Then

$$\lambda^{-S(k)\alpha''_+}\Theta^{-1} \leq |h(D^+_k)| - |h(D^+_{k+n+1})| \leq \lambda^{-S(k)\alpha'_+}\Theta. \tag{3.3.19}$$

Since $|D_{k+n+1}| \longrightarrow 0$ as $n \longrightarrow \infty$, and h is continuous, taking the limit in (3.3.19) as $n \longrightarrow \infty$ we obtain

$$\lambda^{-S(k)\alpha''_+}\Theta^{-1} \leq |h(D^+_k)| \leq \lambda^{-S(k)\alpha'_+}\Theta. \tag{3.3.20}$$

In the same way we can prove that

$$\lambda^{-S(k)\alpha''_-}\Theta^{-1} \leq |h(D^-_k)| \leq \lambda^{-S(k)\alpha'_-}\Theta. \tag{3.3.21}$$

Finally, put $\alpha'' := \max\{\alpha''_-, \alpha''_+\}$, and $\alpha' := \min\{\alpha'_-, \alpha'_+\}$. For any $k \geq K$ we have that

$$|f^{S(k)}(\tilde{c}) - \tilde{c}| = |h(D^+_k)| \qquad \text{or} \qquad |f^{S(k)}(\tilde{c}) - \tilde{c}| = |h(D^-_k)|.$$

In any case, by (3.3.20) and (3.3.21) the result follows.

$\square$

Chapter 4

Regular measures and Lyapunov exponents

In this chapter we will study the unique ergodic invariant measure for the map f given by Theorem 11, see Section 3.1, Remark 3.1.1.

In the first section we will study the Lyapunov exponent of this measure, and the pointwise Lyapunov exponent on the $\omega-$limit set of the turning point of f.

In the second section, we will study a regularity condition for the invariant measure that has been widely used in the study of interval maps.

4.1 Lyapunov exponents

From now on we will denote by μ_P the unique ergodic measure for T supported on $\omega_T(c)$, and by $\widetilde{\mu}_P$ the pushforward of μ_T by h. Thus, $\widetilde{\mu}_P = h_* \mu_P$.

Given a function $f \in \mathcal{C}$ and μ a $f-$invariant probability measure, denote by

$$\chi_\mu(f) := \int \log |f'| \mathrm{d}\mu,$$

its *Lyapunov exponent*, if the integral exists. Similarly, for every $x \in [-1,1]$, such that $\mathcal{O}_f(x) \cap \mathcal{S}(f) = \emptyset$, denote by

$$\chi_f(x) := \lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|,$$

the *pointwise Lyapunov exponent of f at x*, if the limit exists.

Lyapunov exponents play an important role in the study of ergodic properties of dynamical systems. In particular, in the seminal work of Pesin (referred to as "Pesin Theory"), the existence and positivity of Lyapunov exponents were used to study the dynamics of non-uniformly hyperbolic systems, see for example [36, Supplement]. Using these ideas, Ledrappier [41] studied ergodic properties of absolutely continuous invariant measures for regular maps of the interval under the assumption that the Lyapunov exponent exists and is positive. Recently Dobbs [19], [20] developed the Pesin theory for noninvertible interval maps with Lorenz-like singularities and non-flat critical points. Lima [44] constructs a symbolic extension for these maps that code the measures with positive Lyapunov exponents.

In the case of continuously differentiable interval maps, Przytycki proved that ergodic invariant measures have nonnegative Lyapunov exponent, or they are supported on a strictly attracting periodic orbit of the system. Moreover, there exists a set of full measure for which the pointwise Lyapunov exponent exists and is nonnegative, see [64], [66, Appendix A].

The following theorem tell us that Przytycki result do not extend to maps with Lorenz-like singularities and non-flat critical points.

Theorem 13. *Let h and f be as in Theorem 11. Then*

1. $\chi_{\tilde{\mu}_P}(f)$ *is not defined.*

2. *For $x \in \omega_f(\tilde{c})$, the pointwise Lyapunov exponent of f at x does not exist if $\mathcal{O}_f(x) \cap \mathcal{S}(f) = \emptyset$, and it is not defined if $\mathcal{O}_f(x) \cap \mathcal{S}(f) \neq \emptyset$.*

Dobbs constructed an example of a unimodal map with a flat critical point and singularities at the boundary, for which the Lyapunov exponent of an invariant measure does not exist, see [19, Proposition 43]. For interval maps with infinite Lyapunov exponent see [63, Theorem A], and references therein.

The proof of the theorem above is consequence of the following propositions. Define

$$\log^+ |f'| := \max\{0, \log |f'|\} \qquad \text{and} \qquad \log^- |f'| := \max\{0, -\log |f'|\},$$

on $I \setminus \{\tilde{c}\}$.

Proposición 4.1 *Let h and f be as in Theorem 11. Then*

(i) $\int \log^+ |f'| \mathrm{d}\tilde{\mu}_P = +\infty$, *and*

(ii) $\int \log^- |f'| \mathrm{d}\tilde{\mu}_P = +\infty$.

Recall that for $x \in \omega_f(\tilde{c})$ such that $\tilde{c} \in \mathcal{O}_f(x)$ we have that the pointwise Lyapunov exponent is not defined, since for n large enough $\log |(f^n)'(x)|$ is not defined.

Proposición 4.2 *Let h and f be as in Theorem 11. Then there exists a positive number α such that, for every $x \in \omega_f(\tilde{c})$ with $\tilde{c} \notin \mathcal{O}_f(x)$, we have that*

$$\liminf_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| \leq \left(1 - \frac{\alpha}{\varphi}\right) \log \lambda < \log \lambda \leq \limsup_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|, \tag{4.1.1}$$

where $\varphi := \frac{1+\sqrt{5}}{2}$.

The following lemma is going to be an essential part on the proof of part (i) in Proposition 4.1 and the next section.

Lemma 4.1.1. *Let f and h be as in Theorem 11. There exist $k^* \geq 2$ and $\Lambda > 0$ such that for every $n \geq k^*$ we have*

$$\int_{h(J_n)} \log^+ |f'| \mathrm{d}\widetilde{\mu}_P \geq \Lambda > 0.$$

Proof: Let α^+ and α^-, K, and δ be given be Theorem 11. First, take $\alpha := \max\{\alpha^+, \alpha^-\}$. From (3.1.5) and (3.1.6), we get that for every $x \in (h(-\delta), h(\delta)) \setminus \{\tilde{c}\}$

$$\frac{1}{K|h^{-1}(x)|^\alpha} \leq |f'(x)|. \tag{4.1.2}$$

By Lemma 2.3.1, there is $k \geq 2$ so that (4.1.2) holds on $h(I_k) \setminus \{\tilde{c}\}$ and such that $|f'| > 1$ on $h(I_k) \setminus \{\tilde{c}\}$. For $n > k$ put $L_n := \lambda^{S(n+1)}|D_n|$. Recall the definition

$$\log^+ |f'| := \max\{0, \log |f'|\} \quad \text{and} \quad \log^- |f'| := \max\{0, -\log |f'|\},$$

on $I \setminus \{\tilde{c}\}$.

Then for each $n > k$ and $x \in J_n$, we have by Lemma (2.2.3) and (4.1.2)

$$|f'(h(x))| \geq K^{-1} \frac{1}{|D_{n-1}|^\alpha} = K^{-1} \lambda^{\alpha S(n)} L_{n-1}^{-\alpha}. \tag{4.1.3}$$

By the above together with Lemma 2.4.1 and the fact that $S(n) \geq \frac{1}{3}\varphi^{n+2}$

$$\begin{aligned}
\int_{h(J_n)} \log |f'| \mathrm{d}\tilde{\mu}_P &\geq \tilde{\mu}_P(h(J_n)) \log \left|K^{-1}\lambda^{\alpha S(n)} L_{n-1}^{-\alpha}\right| \\
&\geq \left(\frac{1}{\varphi}\right)^{n+1} \left[\alpha S(n) \log(\lambda) + \alpha \log(K^{1/\alpha} L_{n-1})^{-1}\right] \\
&\geq \frac{\varphi\alpha}{3} \log(\lambda) + \alpha \left(\frac{1}{\varphi}\right)^{n+1} \log(K^{1/\alpha} L_{n-1})^{-1}.
\end{aligned} \tag{4.1.4}$$

By Lemma 2.3.1, $\left(\frac{1}{\varphi}\right)^{n+1} \log(K^{1/\alpha} L_{n-1})^{-1} \longrightarrow 0$ as $n \longrightarrow \infty$. Let $k' \geq 2$ be so that

$$\alpha \left(\frac{1}{\varphi}\right)^{n+1} \log(K^{1/\alpha} L_{n-1})^{-1} < \frac{\varphi\alpha}{6} \log(\lambda).$$

Taking $k^* = \max\{k, k'\}$, and $\Lambda = \frac{\varphi\alpha}{6} \log(\lambda)$, the result follows. $\square$

Proof: [Proof of Proposition 4.1]

First we prove (i). Let k^* and Λ as in Lemma 4.1.1. By Lemma 2.3.1, there is $k \geq k^*$ such that $|f'| > 1$ on $h(I_k) \setminus \{\tilde{c}\}$. By Lemma 2.2.3, for every $n > k$, we have $J_n \subset I_k$ and for every $k < n < n'$, we have $J_n \cap J_{n'} = \emptyset$. So, since $\tilde{\mu}_P(\{\tilde{c}\}) = 0$

$$\int \log^+ |f'| \mathrm{d}\tilde{\mu}_P \geq \int_{h(I_k)} \log |f'| \mathrm{d}\tilde{\mu}_P \geq \sum_{n>k} \int_{h(J_n)} \log |f'| \mathrm{d}\tilde{\mu}_P.$$

Since $\int_{h(J_n)} \log |f'| \mathrm{d}\tilde{\mu}_P \geq \Lambda$, for every $n > k$, we get that

$$\int \log^+ |f'| \mathrm{d}\tilde{\mu}_P = +\infty. \tag{4.1.5}$$

Now we prove (ii). Suppose by contradiction that

$$\int \log^- |f'| \mathrm{d}\tilde{\mu}_P < +\infty. \tag{4.1.6}$$

By the chain rule

$$\log |f'(x)| = \log(\lambda) + \log |h'(T(h^{-1}(x)))| - \log |h'(h^{-1}(x))|, \tag{4.1.7}$$

on $I \setminus \{\tilde{c}\}$. Since h has a unique critical point, $\log |h'|$ is bounded away from the critical point. In particular, is bounded from above in all I. Then $-\log |h'|$ is bounded from below on I. In particular, since $\tilde{\mu}_P(\{\tilde{c}\}) = 0$, the integral

$$\int \log |h' \circ h^{-1}| \mathrm{d}\tilde{\mu}_P,$$

is defined. Since the only critical points of f are the points in $f^{-1}(\{\tilde{c}\})$, we have that $\log |f'|$ is bounded away from $\{\tilde{c}\} \cup f^{-1}\{\tilde{c}\}$. Let $\widetilde{V} \subset I \setminus \{\tilde{c}\}$ be a neighborhood of $f^{-1}\{\tilde{c}\}$ such that $\log |f'(x)| < 0$ for $x \in \widetilde{V}$, then by (4.1.6) and (4.1.7)

$$-\infty < \int_{\tilde{V}} \log |f'| \mathrm{d}\tilde{\mu}_P = \log(\lambda) \mathrm{d}\tilde{\mu}_P(\tilde{V}) + \int_{\tilde{V}} \left(\log |h' \circ T \circ h^{-1}| - \log |h' \circ h^{-1}|\right) \mathrm{d}\tilde{\mu}_P.$$

Since $h^{-1}(\widetilde{V})$ is a neighborhood of $T^{-1}(c)$, the function $-\log |h' \circ h^{-1}|$ is bounded on $\widetilde{V}$. On the other hand, since

$$h^{-1} \circ f(x) = T \circ h^{-1}(x) = c,$$

we have $h' \circ T \circ h^{-1}(x) = 0$ if $x \in f^{-1}(\tilde{c})$. Thus $h' \circ T \circ h^{-1}(x) \neq 0$ for $x \in I \setminus \widetilde{V}$. Then $\log |h' \circ T \circ h^{-1}|$ is bounded in $I \setminus \widetilde{V}$. So

$$\int_{I \setminus \widetilde{V}} \log |h' \circ T \circ h^{-1}| \mathrm{d}\tilde{\mu}_P > -\infty. \tag{4.1.8}$$

Now,

$$\begin{aligned} -\infty &< \int_{\tilde{V}} \log |f'| \mathrm{d}\tilde{\mu}_P \\ &\leq \left(\log(\lambda) + \max_{x \in \tilde{V}} \{-\log |h' \circ h^{-1}(x)|\}\right) \tilde{\mu}_P(\tilde{V}) + \int_{\tilde{V}} \left(\log |h' \circ T \circ h^{-1}|\right) \mathrm{d}\tilde{\mu}_P. \end{aligned}$$

So

$$\int_{\tilde{V}} \log|h' \circ T \circ h^{-1}| \mathrm{d}\tilde{\mu}_P > -\infty. \tag{4.1.9}$$

Together with (4.1.8) this implies that

$$\int \log|h' \circ T \circ h^{-1}| \mathrm{d}\tilde{\mu}_P,$$

is finite. Since the integral

$$\int -\log|h' \circ h^{-1}| \mathrm{d}\tilde{\mu}_P,$$

is defined, we have

$$\begin{aligned}
\int \log|f'| \mathrm{d}\tilde{\mu}_P &= \log(\lambda) + \int \log|h' \circ T \circ h^{-1}| \mathrm{d}\tilde{\mu}_P + \int -\log|h' \circ h^{-1}| \mathrm{d}\tilde{\mu}_P \\
&= \log(\lambda) + \int \log|h' \circ h^{-1} \circ f| \mathrm{d}\tilde{\mu}_P + \int -\log|h' \circ h^{-1}| \mathrm{d}\tilde{\mu}_P,
\end{aligned}$$

and since $\tilde{\mu}_P$ is f invariant we get

$$\int \log|f'| \mathrm{d}\tilde{\mu}_P = \log(\lambda),$$

contradicting (4.1.5). This contradiction completes the proof of part (ii). □

For the proof of Proposition 4.2, we use the following lemma

Lemma 4.1.2. *For every $\hat{x} \in I_k \cap \omega_T(c)$ there exists an increasing sequence of positive integers $\{n_{\mathrm{i}}\}_{\mathrm{i} \geq 1}$ such that*

$$T^{n_{\mathrm{i}}}(\hat{x}) \in I_{k+\mathrm{i}} \qquad \textit{and} \qquad T^m(\hat{x}) \notin I_{k+\mathrm{i}+1},$$

for all $\mathrm{i} \geq 1$ *and all* $n_{\mathrm{i}} + 1 \leq m < n_{\mathrm{i}+1}$. *Moreover,*

$$S(k+\mathrm{i}) - S(k) \leq n_{\mathrm{i}} \leq S(k+\mathrm{i}+2) - S(k+2), \tag{4.1.10}$$

for all $\mathrm{i} > 1$.

Proof: We will prove the lemma by induction. Let α^+ and α^- be the right and left critical orders of 0 as the critical point of h, and let $\alpha := \max\{\alpha^+, \alpha^-\}$. Let $M > 0$ be as in (3.1.7) and (3.1.8). Fix $k > 2$ big enough so that (4.1.2) holds on $h(I_k)$, and (3.1.7), and (3.1.8) holds on I_k. For any $x \in h(I_k)$ we put $\hat{x} := h^{-1}(x) \in I_k$.

Let $\hat{x} \in I_k \cap \omega_T(c)$. Recall that for any integer $k' \geq 1$, we have that

$$I_{k'}, I_{k'}^1, \ldots I_{k'}^{S(k'-1)-1}, J_{k'}, J_{k'}^1, \ldots, J_{k'}^{S(k'-2)-1},$$

are pairwise disjoint. Now, by (2.2.8)

$$\hat{x} \in I_{k+1} \qquad \text{or} \qquad \hat{x} \in J_{k+1}.$$

If $\hat{x} \in J_{k+1}$, for every $1 \leq m < S(k-1)$

$$T^m(\hat{x}) \in J_{k+1}^m,$$

thus

$$T^m(\hat{x}) \notin I_{k+1}$$

and

$$T^{S(k-1)}(\hat{x}) \in I_{k+1}.$$

In this case $n_1 = S(k-1)$ satisfies the desired properties. If $\hat{x} \in I_{k+1}$, for every $1 \leq m < S(k)$

$$T^m(\hat{x}) \in I_{k+1}^m,$$

thus

$$T^m(\hat{x}) \notin I_{k+1},$$

and

$$T^{S(k)}(\hat{x}) \in I_{k+1} \qquad \text{or} \qquad T^{S(k)}(\hat{x}) \in J_{k+1}.$$

In the former case $n_1 = S(k)$ satisfies the desired properties. In the later case we have that for $1 \leq m < S(k) + S(k-1) = S(k+1)$

$$T^m(\hat{x}) \notin I_{k+1}$$

and

$$T^{S(k+1)}(\hat{x}) \in I_{k+1}.$$

So $n_1 = S(k+1)$ satisfies the desired properties. So we have

$$S(k-1) \leq n_1 \leq S(k+1).$$

Now suppose that for some $\mathrm{i} \geq 1$ there is n_i satisfying the conclusions of the lemma. Thus

$$T^{n_\mathrm{i}}(\hat{x}) \in I_{k+\mathrm{i}}$$

and

$$S(k+\mathrm{i}) - S(k) \leq n_\mathrm{i} \leq S(k+\mathrm{i}+2) - S(k+2).$$

By (2.2.8)

$$T^{n_\mathrm{i}}(\hat{x}) \in I_{k+\mathrm{i}+1} \qquad \text{or} \qquad T^{n_\mathrm{i}}(\hat{x}) \in J_{k+\mathrm{i}+1}.$$

If $T^{n_\mathrm{i}}(\hat{x}) \in J_{k+\mathrm{i}+1}$, for every $1 \leq m < S(k+\mathrm{i}-1)$

$$T^{m+n_\mathrm{i}}(\hat{x}) \in J_{k+\mathrm{i}+1}^m,$$

thus

$$T^{m+n_\mathrm{i}}(\hat{x}) \notin I_{k+\mathrm{i}+1}$$

and

$$T^{S(k+\mathrm{i}-1)+n_\mathrm{i}}(\hat{x}) \in I_{k+\mathrm{i}+1}.$$

In this case $n_{\mathrm{i}+1} = S(k+\mathrm{i}-1) + n_\mathrm{i}$ satisfies the desired properties. If $T^{n_\mathrm{i}}(\hat{x}) \in I_{k+\mathrm{i}+1}$, for every $1 \leq m < S(k+\mathrm{i})$

$$T^{m+n_\mathrm{i}}(\hat{x}) \in I_{k+\mathrm{i}+1}^m,$$

thus

$$T^{m+n_{\mathrm{i}}}(\hat{x}) \notin I_{k+\mathrm{i}+1},$$

and

$$T^{S(k+\mathrm{i})+n_{\mathrm{i}}}(\hat{x}) \in I_{k+\mathrm{i}+1} \qquad \text{or} \qquad T^{S(k+\mathrm{i})+n_{\mathrm{i}}}(\hat{x}) \in J_{k+\mathrm{i}+1}.$$

In the former case $n_{\mathrm{i}+1} = S(k+\mathrm{i}) + n_{\mathrm{i}}$ satisfies the desired properties. In the later case we have that for $1 \leq m < S(k+\mathrm{i}) + S(k+\mathrm{i}-1) = S(k+\mathrm{i}+1)$

$$T^{m+n_{\mathrm{i}}}(\hat{x}) \notin I_{k+\mathrm{i}+1}$$

and

$$T^{S(k+\mathrm{i}+1)+n_{\mathrm{i}}}(\hat{x}) \in I_{k+\mathrm{i}+1}.$$

So $n_{\mathrm{i}+1} = S(k+\mathrm{i}+1) + n_{\mathrm{i}}$ satisfies the desired properties. So we have

$$S(k+\mathrm{i}-1) \leq n_{n_{\mathrm{i}}+1} \leq S(k+\mathrm{i}+1) + n_{\mathrm{i}}.$$

Since n_{i} satisfies (4.1.10)

$$S(k+\mathrm{i}+1) - S(k) \leq n_{\mathrm{i}+1} \leq S(k+\mathrm{i}+3) - S(k+2).$$

This conclude the proof of the lemma. □

For every $x \in I$ such that $\tilde{c} \notin \mathcal{O}_f(x)$ we put

$$\chi_f^+(x) := \limsup_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|, \tag{4.1.11}$$

and

$$\chi_f^-(x) := \liminf_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|. \tag{4.1.12}$$

Proof: [Proof of Proposition 4.2] Let α^+ and α^- be the right and left critical orders of 0 as the critical point of h, and let $\alpha := \max\{\alpha^+, \alpha^-\}$. Let $M > 0$ be as in (3.1.7) and (3.1.8). Fix $k > 2$ big enough so that (4.1.2) holds on $h(I_k)$, and (3.1.7), and (3.1.8) holds on I_k. For any $x \in h(I_k)$ we put $\hat{x} := h^{-1}(x) \in I_k$. Let $x \in \omega_f(\tilde{c})$, with $\tilde{c} \notin \mathcal{O}_f(x)$. Then $\hat{x} = h^{-1}(x) \in \omega_T(c)$, and $c \notin \mathcal{O}_T(\hat{x})$. By the chain rule, we have that for very $n \geq 1$

$$(f^n)'(x) = \prod_{\mathrm{i}=0}^{n-1} \lambda \frac{h'(T(T^{\mathrm{i}}(\hat{x})))}{h'(T^{\mathrm{i}}(\hat{x}))} = \lambda^n \frac{h'(T^n(\hat{x}))}{h'(\hat{x})}. \tag{4.1.13}$$

Then,

$$\frac{1}{n} \log |(f^n)'(x)| = \log \lambda + \frac{1}{n} \log |h'(T^n(\hat{x}))| - \frac{1}{n} \log |h'(\hat{x})|, \tag{4.1.14}$$

for every $n \geq 1$. So, using (4.1.11) and (4.1.12), we get

$$\chi_f^+(x) = \log \lambda + \limsup_{n\to\infty} \frac{1}{n} \log |h'(T^n(\hat{x}))|, \tag{4.1.15}$$

and

$$\chi_f^-(x) = \log \lambda + \liminf_{n\to\infty} \frac{1}{n} \log |h'(T^n(\hat{x}))|. \tag{4.1.16}$$

Now, since $\hat{x} \in \omega_T(c)$, we have that $\hat{x}$ belongs to one of the following sets

$$I_k, I_k^1, \dots, I_k^{S(k-1)-1}, J_k, \dots, J_k^{S(k-2)-1}.$$

So there exists $0 \leq l_k(x) < S(k)$ such that $T^{l_k(x)}(\hat{x}) \in I_k$. Let $\{n_\mathrm{i}\}_{\mathrm{i}\geq 1}$ be as in Lemma 4.1.2, for $T^{l_k(x)}(x)$. Note that for every $\mathrm{i} \geq 1$

$$T^{n_\mathrm{i}+l_k(x)+1}(\hat{x}) \in I_{k+\mathrm{i}}^1 \subset [c_{S(k)+1}, c_1].$$

Then by (3.1.9) and (4.1.15) we have that

$$\log \lambda \leq \chi_f^+(x).$$

Now by (4.1.10) we have that

$$\frac{1}{S(k+\mathrm{i}+2)} \leq \frac{1}{n_\mathrm{i}}. \tag{4.1.17}$$

Also, since for every $\mathrm{i} \geq 1$

$$T^{n_\mathrm{i}+l_k(x)}(\hat{x}) \in I_{k+\mathrm{i}},$$

by (2.1.2) and (2.2.1), we get

$$|T^{n_\mathrm{i}+l_k(x)}(\hat{x})| \leq |c_{S(k+\mathrm{i})}| = |D_{k+\mathrm{i}}|. \tag{4.1.18}$$

By (4.1.2) we have

$$\frac{1}{\lambda K |T^{n_\mathrm{i}+l_k(x)}(\hat{x})|^\alpha} \leq \frac{|h'(T^{n_\mathrm{i}+l_k(x)+1}(\hat{x}))|}{|h'(T^{n_\mathrm{i}+l_k(x)}(\hat{x}))|}. \tag{4.1.19}$$

Combining (3.1.9), (4.1.18) and (4.1.19) we get

$$|h'(T^{n_\mathrm{i}+l_k(x)}(\hat{x}))| \leq \lambda K W_2 |D_{k+\mathrm{i}}|^\alpha. \tag{4.1.20}$$

Since $|D_{k+\mathrm{i}}| \longrightarrow 0$ as $\mathrm{i} \longrightarrow \infty$, there exists $\mathrm{i}' \geq 1$ such that for every $\mathrm{i} \geq \mathrm{i}'$

$$|D_{k+\mathrm{i}}| < \left(\frac{1}{\lambda K W_2}\right)^{1/\alpha}.$$

Then for every $\mathrm{i} \geq \mathrm{i}'$ from (4.1.20) we get

$$\log |h'(T^{n_\mathrm{i}+l_k(x)}(\hat{x}))| \leq \log\left(\lambda K W_2 |D_{k+\mathrm{i}}|^\alpha\right) < 0.$$

By the above and (4.1.17)

$$\frac{1}{n_\mathrm{i}} \log |h'(T^{n_\mathrm{i}+l_k(x)}(\hat{x}))| \leq \frac{1}{S(k+\mathrm{i}+2)} \log(\lambda K W_2 |D_{k+\mathrm{i}}|^\alpha).$$

Taking limit as $\mathrm{i} \longrightarrow \infty$,

$$\lim_{\mathrm{i}\to\infty} \frac{1}{n_\mathrm{i}} \log |h'(T^{n_\mathrm{i}+l_k(x)}(\hat{x}))| \leq \lim_{\mathrm{i}\to\infty} \frac{1}{S(k+\mathrm{i}+2)} \log |D_{k+\mathrm{i}}|^\alpha.$$

Using Lemma 2.3.1

$$\begin{aligned}
\lim_{\mathrm{i}\to\infty} \frac{1}{S(k+\mathrm{i}+2)} \log |D_{k+\mathrm{i}}|^{\alpha} &= \lim_{\mathrm{i}\to\infty} \frac{1}{S(k+\mathrm{i}+2)} \log \lambda^{-\alpha S(k+\mathrm{i}+1)} \\
&= -\alpha \log \lambda \lim_{\mathrm{i}\to\infty} \frac{S(k+\mathrm{i}+1)}{S(k+\mathrm{i}+2)} \\
&= -\frac{\alpha}{\varphi} \log \lambda.
\end{aligned}$$

Then by (4.1.16)

$$\chi_f^-(x) \le (1 - \frac{\alpha}{\varphi}) \log \lambda < \log \lambda.$$

This conclude the proof of the proposition.

□

4.2 Adapted measure

In this section we will prove the following

Theorem 14. *Let h and f be as in Theorem 11. Then*

$$\log(\mathrm{dist}(\cdot, \mathcal{S}(f))) \notin L^1(\tilde{\mu}_P).$$

The negation of the theorem above is considered in several works as a regularity condition to study ergodic invariant measures. In [44], Lima studied measures satisfying this condition for interval maps with critical points and discontinuities, he called measures satisfying this condition $f-$*adapted*. By the Birkhoff ergodic theorem, if $\log(\mathrm{dist}(\cdot, \mathcal{S}(f))) \in L^1(\mu)$, then for an ergodic invariant measure μ, we have

$$\lim_{n\to\infty} \frac{1}{n} \log(\mathrm{dist}(f^n(x), \mathcal{S}(f))) = 0,$$

$\mu-$a.e. Ledrappier called measures satisfying this last condition *non-degenerated*, for interval maps with a finite number of critical points, see [41]. The measure $\tilde{\mu}_P$ does not satisfy the non-degenerated condition. For more results related to this condition see [44] and references therein.

For continuously differentiable interval maps with a finite number of critical points, every ergodic invariant measure that is not supported on an attracting periodic point satisfies $\lim_{n\to\infty} \frac{1}{n} \log(\mathrm{dist}(f^n(x), \mathcal{S}(f))) = 0$, a.e., see [64] and [66, Appendix]. Item (3) in Theorem 11 tells us that we cannot extend this to piecewise differentiable maps with a finite number of critical points and Lorenz-like singularities.

Theorem 14 is a direct consequence of the following proposition.

Proposición 4.3 *Let h and f be as in Theorem 11. Then*

$$\int |\log(\mathrm{dist}(\cdot, \mathcal{S}(f)))| \, \tilde{\mu}_P = +\infty.$$

Proof: By Theorem 11, f has a Lorenz-like singularity at $\tilde{c}$, then there exist $\delta > 0$, $\ell^+ > 0$, $\ell^- > 0$ and $L > 0$ such that (3.1.1) holds for every $x \in (\tilde{c}, \tilde{c} + \delta)$ and (3.1.2) holds for every $x \in (\tilde{c} - \delta, \tilde{c})$. Let $\ell := \max\{\ell^+, \ell^-\}$, and choose $0 < \hat{\delta} \leq \delta$, so that $\log|f'(x)| > 0$ and $\mathrm{dist}(x, \mathcal{S}(f)) = |x - \tilde{c}|$ for every $x \in (\tilde{c} - \hat{\delta}, \tilde{c} + \hat{\delta}) \setminus \{\tilde{c}\}$. Let $k^* \geq 2$ and Λ be as in Lemma 4.1.1, and let $m \geq k^*$ be so that $I_m \subset (\tilde{c} - \hat{\delta}, \tilde{c} + \hat{\delta})$. Then, for every $x \in I_m \setminus \{\tilde{c}\}$ we have

$$\log|f'(x)| \leq \log(L) - \ell \log(\mathrm{dist}(x, \mathcal{S}(f))).$$

So, for every $n \geq m$

$$\int_{h(J_n)} \log|f'| \mathrm{d}\tilde{\mu}_P \leq \log(L)\tilde{\mu}_P(J_n) + \ell \int_{h(J_n)} |\log|x - \tilde{c}|| \mathrm{d}\tilde{\mu}_P(x).$$

By Lemma 4.1.1 and Lemma 2.3.1 we get that

$$+\infty = \int_{h(I_m)} |\log(\mathrm{dist}(x, \mathcal{S}(f)))| \mathrm{d}\tilde{\mu}_P(x),$$

so $\log(\mathrm{dist}(x, \mathcal{S}(f))) \notin L^1(\tilde{\mu}_P)$. This conclude the proof of the proposition. $\square$

Chapter 5

Future work

In this chapter chapter we discuss further problems arising from the study developed in this thesis.

5.1 Families of Lorenz-like maps

In Chapter 3 and Chapter 4 we saw that the presence of critical points of inflection type and singularities can lead to some new chaotic phenomenon without loosing much regularity. In particular, from a topological point of view, the map introduced in Chapter 3 share the same topological behavior as the Fibonacci tent map (see Chapter 2). Motivated by this phonomenon, we propose a systematic study for families of unimodal maps with a Lorenz-like singularity at the turning point and non-flat critical points of inflection type. For β positive real number, λ nonnegative real number and $c \in \mathbb{R}$, then we define the continuous map

$$f_{\lambda,\beta,c}(x) := \left| |x|^{\frac{2}{\beta}} - \lambda^{\frac{2}{\beta}} \right| \left(|x|^{\frac{\beta+1}{\beta}} - \lambda^{\frac{\beta+1}{\beta}} \right) + c. \tag{5.1.1}$$

We can write it as

$$f_{\lambda,\beta,c}(x) = \begin{cases} \left(x^{\frac{2}{\beta}} - \lambda^{\frac{2}{\lambda}}\right)\left(x^{\frac{1+\beta}{\beta}} - \lambda^{\frac{1+\beta}{\beta}}\right) + c & \text{if } x \in [\lambda, \infty) \\ \left(\lambda^{\frac{2}{\lambda}} - x^{\frac{2}{\beta}}\right)\left(x^{\frac{1+\beta}{\beta}} - \lambda^{\frac{1+\beta}{\beta}}\right) + c & \text{if } x \in [0, \lambda) \\ \left(\lambda^{\frac{2}{\lambda}} - (-x)^{\frac{2}{\beta}}\right)\left((-x)^{\frac{1+\beta}{\beta}} - \lambda^{\frac{1+\beta}{\beta}}\right) + c & \text{if } x \in [-\lambda, 0) \\ \left((-x)^{\frac{2}{\beta}} - \lambda^{\frac{2}{\lambda}}\right)\left((-x)^{\frac{1+\beta}{\beta}} - \lambda^{\frac{1+\beta}{\beta}}\right) + c & \text{if } x \in (-\infty, -\lambda). \end{cases} \tag{5.1.2}$$

Then, we have that

For $x \geq \lambda$

$$f'_{\lambda,\beta,c}(x) = \frac{2}{\beta} x^{\frac{2-\beta}{\beta}} \left(x^{\frac{1+\beta}{\beta}} - \lambda^{\frac{1+\beta}{\beta}} \right) + \frac{1+\beta}{\beta} x^{\frac{1}{\beta}} \left(x^{\frac{2}{\beta}} - \lambda^{\frac{2}{\beta}} \right). \tag{5.1.3}$$

For $0 \leq x < \lambda$

$$f'_{\lambda,\beta,c}(x) = -\frac{2}{\beta} x^{\frac{2-\beta}{\beta}} \left(x^{\frac{1+\beta}{\beta}} - \lambda^{\frac{1+\beta}{\beta}} \right) + \frac{1+\beta}{\beta} x^{\frac{1}{\beta}} \left(\lambda^{\frac{2}{\beta}} - x^{\frac{2}{\beta}} \right). \tag{5.1.4}$$

For $-\lambda \leq x < 0$

$$f'_{\lambda,\beta,c}(x) = \frac{2}{\beta}(-x)^{\frac{2-\beta}{\beta}} \left((-x)^{\frac{1+\beta}{\beta}} - \lambda^{\frac{1+\beta}{\beta}} \right) - \frac{1+\beta}{\beta}(-x)^{\frac{1}{\beta}} \left(\lambda^{\frac{2}{\beta}} - (-x)^{\frac{2}{\beta}} \right). \tag{5.1.5}$$

For $x \leq -\lambda$

$$f'_{\lambda,\beta,c}(x) = -\frac{2}{\beta}(-x)^{\frac{2-\beta}{\beta}} \left((-x)^{\frac{1+\beta}{\beta}} - \lambda^{\frac{1+\beta}{\beta}} \right) - \frac{1+\beta}{\beta}(-x)^{\frac{1}{\beta}} \left((-x)^{\frac{2}{\beta}} - \lambda^{\frac{2}{\beta}} \right). \tag{5.1.6}$$

From (5.1.3) and (5.1.4), we see that $f'_{\lambda,\beta,c}(x) > 0$ for $x > 0$. On the other had, from (5.1.5) and (5.1.6), we see that $f'_{\lambda,\beta,c}(x) < 0$ for $x < 0$. Then, by continuity, we have that for every $\lambda \geq 0$, $\beta > 0$, and $c \in \mathbb{R}$, the map $f_{\lambda,\beta,c}$ is a unimodal map with turning point at $x = 0$.

From (5.1.3), (5.1.4), (5.1.5), and (5.1.6), we can see that the geometry of the family $f_{\lambda,\beta,c}$ depends on the parameters λ and β. Motivated by this observation we study the parameter space given by λ and β in terms of the critical points and Lorenz-like singularity.

Case $\lambda = 0$. In this case we have that the map (5.1.1) take the form

$$f_{0,\beta,c}(x) = |x|^{\frac{3+\beta}{\beta}} + c,$$

and

$$f'_{0,\beta,c}(x) = \begin{cases} \frac{3+\beta}{\beta} x^{\frac{3}{\beta}} & \text{if } x \geq 0 \\ -\frac{3+\beta}{\beta}(-x)^{\frac{3}{\beta}} & \text{if } x \leq 0. \end{cases}$$

Then $f_{0,\beta,c}$ has a unique non-flat critical point at the turning point $x = 0$, of order $\frac{3}{\beta}$. In particular, for $\lambda = 0$ and $\beta = 3$ we recover the quadratic family $f_{0,3,c}(x) = x^2 + c$. Also, as β growth to infinite, the order of the critical point approaches to 1.

Case $\lambda > 0$. From (5.1.3) and (5.1.5), we see that if $\lambda > 0$, the map $f_{\lambda,\beta,c}$ has two critical points at $x = \pm\lambda$ of order $\frac{3}{\beta}$. Since the map is unimodal, these critical points are of inflection type.

From (5.1.4) and (5.1.5), we can see that the geometry around the turning point $x = 0$ depends on the value of β. More precisely

1. For $\beta > 2$ we have that $\frac{2-\beta}{\beta} < 0$. Then, by (5.1.4), and (5.1.5),

$$\lim_{x\to 0} |f'_{\lambda,\beta,c}(x)| = \infty.$$

Thus, the turning point of $f_{\lambda,\beta,c}$ is a Lorenz-like singularity of order $\left|\frac{2-\beta}{\beta}\right|$.

2. For $\beta = 2$ we have that

$$f'_{\lambda,2,c}(x) = \begin{cases} -\left(x^{\frac{3}{2}} - \lambda^{\frac{3}{2}}\right) + \frac{3}{2}x^{\frac{1}{2}}(\lambda - x) & \text{if } 0 \leq x < \lambda \\ \left(x^{\frac{3}{2}} - \lambda^{\frac{3}{2}}\right) - \frac{3}{2}(-x)^{\frac{1}{2}}(\lambda + x) & \text{if } -\lambda \leq x < 0. \end{cases}$$

Then,

$$\lim_{x\to 0^+} f'_{\lambda,2,c}(x) = \lambda^{\frac{3}{2}} \qquad \text{and} \qquad \lim_{x\to 0^-} f'_{\lambda,2,c}(x) = -\lambda^{\frac{3}{2}}.$$

Thus, the map $f_{\lambda,2,c}$ is not differentiable at the turning point.

3. For $1 \geq \beta < 2$, from (5.1.4), and (5.1.5), we can see that the turning point $x = 0$ is a non-flat critical point of order $\frac{1}{\beta}$.

4. For $0 < \beta < 1$, from (5.1.4), and (5.1.5), we can see that the turning point $x = 0$ is a non-flat critical point of order $\frac{2-\beta}{\beta}$.

From the continuous map $f_{\lambda,\beta,c}$ defined in (5.1.1) we can extract a one-parameter families of unimodal maps, fixing $\lambda \geq 0$ and $\beta > 0$ we get the family $\{f_{\lambda,\beta,c}\}_{c\in\mathbb{R}}$.

The first natural question that we could ask is the following.

Question 1. For what values of λ and β the family $\{f_{\lambda,\beta,c}\}_{c\in\mathbb{R}}$ is a full family?

A positive answer to this question will imply that this family contain all possible combinatorial type. In particular, given any quadratic interval map g, there would be $c \in \mathbb{R}$ such that g and f_c have the same combinatorics.

To give an answer to this question it seems reasonable to follow the ideas of Milnor and Thurston [56]. The proof relies on some type of intermediate value theorem in the space of kneading sequences (see Chapter 1, Section 1.3), by analyzing the discontinuities of the map $c \mapsto \underline{\mathrm{i}}(f_{\lambda,\beta,c}(0)^+)$, that assignee to each parameter $c \in \mathbb{R}$ the kneading invariant of the map $f_{\lambda,\beta,c}$, and using the following observations:

1. The map $c \mapsto \underline{\mathrm{i}}(f_{\lambda,\beta,c}(0)^+)$ is continuous at parameters c_* for which the turning points is non-periodic.

2. If for c_* the turning point of f_{λ,β,c_*} is periodic, then for c close enough to c_*, the map f_{λ,β,c_*} still has a periodic attractor. If this holds, then we can show that for each c, c' close enough to c_* the kneading sequences of $f_{\lambda,\beta,c}$ and $f_{\lambda,\beta,c'}$ are "almost" the same. This will implies that each admissible kneading sequence can be obtained as the kneading invariant of some map $f_{\lambda,\beta,c}$.

5.1.1 Saddle-node bifurcation

Although one-parameter families of interval dynamical systems are simple to represent, they have a complex dependence on the parameters. When the qualitative properties of the map changes as the parameter varies we call it a *bifurcation*, and the parameter values at which they occur are called *bifurcation points*. The *saddle-node bifurcation* is the mechanism by which fixed points are "created" or "destroy" in interval dynamics.

For $\lambda > 0$ and $\beta > 0$ large enough, this family show a saddle-node bifurcation phenomenon more complicated than in the smooth unimodal case. We can find $c_2 < c_1 < c_0$, so that the following holds. For $c > c_0$ f_c has no fixed points, so the orbit of every point escape to infinity, see Figure 5.1a. At c_0, the map f_{λ,β,c_0} has a unique fixed point (f_{λ,β,c_0} is tangent to the identity line), see figure 5.1b. For $c_1 < c < c_0$, the map $f_{\lambda,\beta,ce}$ has two fixed points, see Figure 5.1c. At c_1, the map f_{λ,β,c_1} has three fixed points, one of them being the turning point, see Figure 5.1d. For $c_2 < c < c_1$, the map $f_{\lambda,\beta,c}$ has four fixed points, see Figure 5.1e. As c approaches c_2, two of the fixed points get closer to finally collapse in a tangent fixed point when $c = c_2$, so f_{λ,β,c_2} has 3 fixed points, see figure 5.1f. Finally, for $c < c_2$, then map $f_{\lambda,\beta,c}$ has 2 fixed points, see Figure 5.1g. This behavior is suggesting that this family could be chaotic in a different way than the quadratic family.

In the mid 1980s Douady and Hubbard [21], [22], [23], and Milnor and Thurston [56] showed that for the quadratic family $g_a(x) = ax(1-x)$ with $a \in [0, 4]$, periodic orbits do not disappear when a increases, this means that as a increases, new periodic orbits are created by saddle-node bifurcations and by period-doubling bifurcations (see Section 5.2). That this is true, follows essentially from *Thurston rigidity* that state that combinatorially equivalent critically finite unimodal maps are unique.

Question 2. Can we expect Thurston ridigity to hold for the family $f_{\lambda,\beta,c}$ with λ and β fixed?

The discussion above, about the saddle-node bifurcation, tell us that for the family $\{f_{\lambda,\beta,c}\}_{c\in[c_2,c_0]}$ periodic orbits (fixed points) can disappear as c decreases (since the turning point of the family $f_{\lambda,\beta,c}$ is a minimum, we look at the growth of periodic orbits as c decreases).

Question 3. Will periodic orbits of period larger or equal than two disappear for the family $f_{\lambda,\beta,c}$?

Question 4. Can we find intervals $(a, b) \subset [c_2, c_0]$ such that new periodic orbits are created as $c \in (a, b)$ decreases?

5.1.2 Real Bounds and distortion estimates

One of the main tool in the study of interval dynnamical systems is the control of the distortion of iterates of a map on some interval under some disjointness assumptions on the iterates of this interval. The distortion of a differentiable map on an interval $J \subset \mathbb{R}$ is the maximal ratio of the absolute values of the derivative in two different points of J. This number measures the non-linearity of the map on J. Another way to measure the non-linearity is to consider pairs of adjacent interval L and R intersecting at a common boundary point and

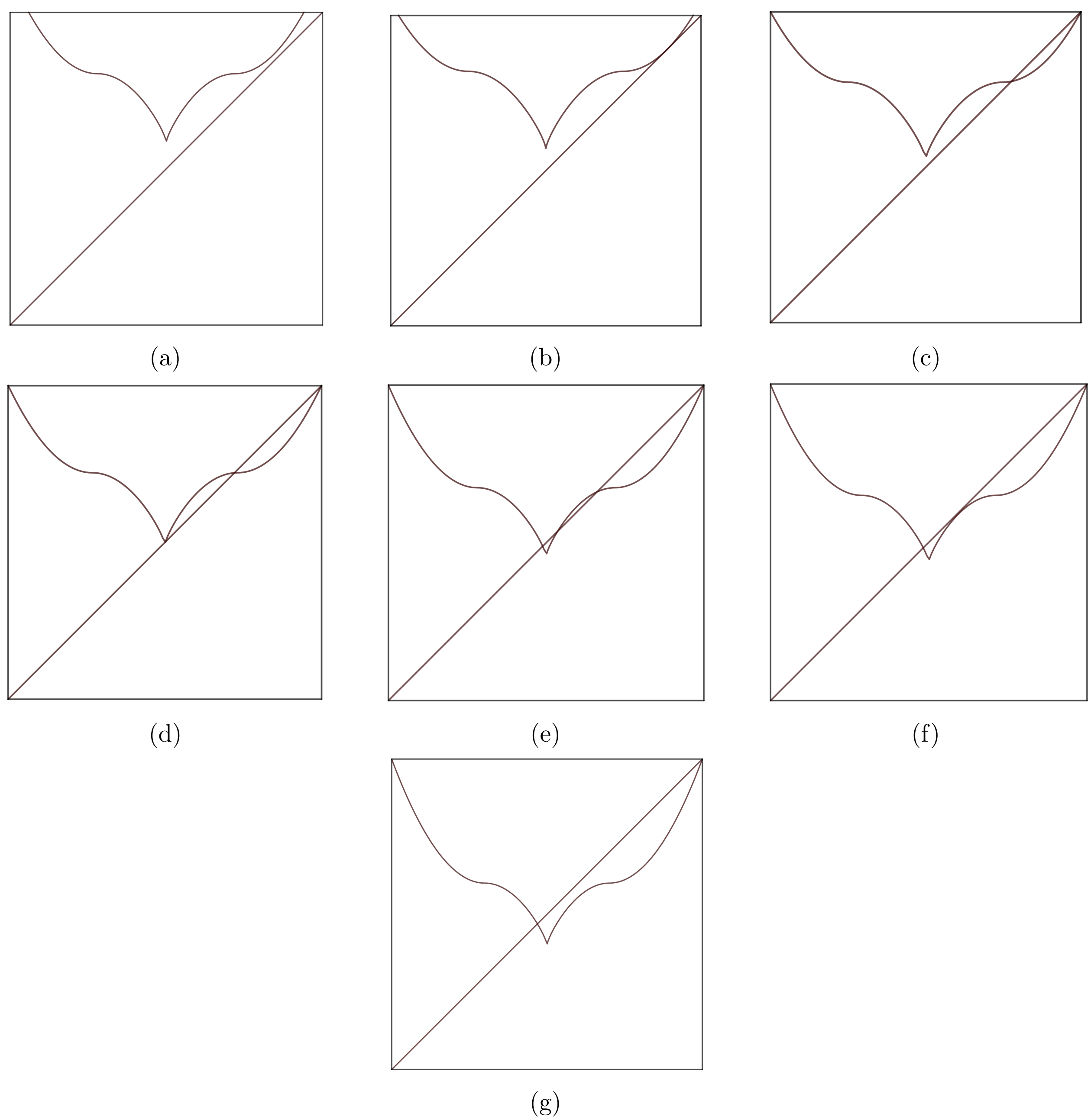


Figure 5.1: Graphics of all the saddle-node bifurcation in the family $\{f_c\}$ for $\lambda = 1$, and $\beta = 3$.

look at the distortion by the map f of the ratio $\mathcal{R}(L,R) := \frac{|L|}{|R|}$, thus we evaluate the number $\mathcal{R}(f,L,R) := \frac{\mathcal{R}(f(L),f(R)}{\mathcal{R}(L,R)}$. It is not hard to see that the distortion of a differentiable map f in the interval $J \subset \mathbb{R}$ is bounded if and only if there is a bound for $\mathcal{R}(L,R)$ for any pair of adjacent intervals $L, R \subset J$. If the map f has critical points or Lorenz-like singularities we cannot hope to get a bound of its non-linearity (the distortion is infinity, and we cannot consider the ration of the length of a pair of adjacent intervals).

One way of being able to deal with this situation is to assume that the *Schwarzian derivative*

$$S(f) := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2, \tag{5.1.7}$$

of f is negative. The reason why this assumption is very helpful is that if $I \subset \mathbb{R}$ is a compact interval and $f\colon I \longrightarrow I$ has negative negative Schwarzian derivative, then

1. For every $n \geq 1$ we have $S(f^n) < 0$;

2. The map $x \mapsto |f'(x)|$ has no strictly positive local minimum:

3. (*Koebe Principle*) On each interval $T \subset I$ such that $f|_T$ is a diffeomorphism the following holds. For each $\xi > 0$ there exists a $K > 0$ independent of f, such that for every interval $J \subset T$ for which $f(J)$ is $\xi-$well inside $f(T)$ (this means that each connected component of $f(T) \setminus f(J)$ have at least length $\xi|f(J)|$), one has

$$\frac{1}{K} \leq \frac{|f'(x)|}{|f'(y)|} \leq K,$$

for all x, y in J.

This condition was first introduced in the context of interval maps by Singer [67], who proved 1 and 2 above. Moreover, he proved the following

Theorem 15 (Singer [67])**.** *If $f\colon I \longrightarrow I$ is a C^3 map with negative Shwarzian derivative, then*

1. *the immediate basin of any attracting periodic orbit contains either a critical point of f or a boundary point of the interval I;*

2. *each neutral periodic point is attracting;*

3. *there exists no interval of periodic points.*

In particular, the number of non-repelling periodic orbits is bounded if the number of critical points of f is finite.

Later, Guckenheimer [30], Misiurewicz [58], and van Strien [71] showed how useful is the Schwarzian derivative in the study of several dynamical properties.

For the family given by (5.1.1) we have that the Schwarzian derivative is independent of the parameter c, and in terms of λ and β we have the following.

Define the sets $\mathcal{S}^- := \{0\} \times \mathbb{R}^+ \cup (0, 2] \times \mathbb{R}^+$, and $\mathcal{S}^+ := \mathbb{R}_{\geq 0} \times \mathbb{R}^+ \setminus \mathcal{S}^+$. Then

1. If $(\lambda, \beta) \in \mathcal{S}^-$, then we have that $S(f_{\lambda,\beta,c}) < 0$.
2. If $(\lambda, \beta) \in \mathcal{S}^+$, then we have that there is $x_+ \in (0, \lambda)$ such that $S(f_{\lambda,\beta,c}(x_+)) = 0$, and

$$\frac{S(f_{\lambda,\beta,c}(x))}{|S(f_{\lambda,\beta,c}(x))|} = \begin{cases} +1 & \text{if } |x| < x_+ \\ -1 & \text{if } |x| > x_+. \end{cases}$$

Now we prove the observation above. Let $(\lambda, \beta, c) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^+ \times \mathbb{R}$. If $x \geq \lambda$, from (5.1.3), we have that

$$\begin{aligned} f'_{\lambda,\beta,c}(x) &= \frac{1}{\beta} x^{\frac{3}{\beta}} \left((3+\beta) - (1+\beta) \left(\frac{\lambda}{x}\right)^{\frac{2}{\beta}} - 2\left(\frac{\lambda}{x}\right)^{\frac{1+\beta}{\beta}} \right); && (5.1.8) \\ f''_{\lambda,\beta,c}(x) &= \frac{1}{\beta^2} x^{\frac{3-\beta}{\beta}} \left(3(3+\beta) - (1+\beta) \left(\frac{\lambda}{x}\right)^{\frac{2}{\beta}} - 2(2-\beta)\left(\frac{\lambda}{x}\right)^{\frac{1+\beta}{\beta}} \right); && (5.1.9) \\ f'''_{\lambda,\beta,c}(x) &= \frac{1}{\beta^3} x^{\frac{3-2\beta}{\beta}} \left(3(9-\beta^2) - (1+\beta^2) \left(\frac{\lambda}{x}\right)^{\frac{2}{\beta}} - 4(2-\beta)(1-\beta)\left(\frac{\lambda}{x}\right)^{\frac{1+\beta}{\beta}} \right). && (5.1.10) \end{aligned}$$

If we write

$$\begin{aligned} A(x) &= (3+\beta) - (1+\beta) \left(\frac{\lambda}{x}\right)^{\frac{2}{\beta}} - 2\left(\frac{\lambda}{x}\right)^{\frac{1+\beta}{\beta}}; && (5.1.11) \\ B(x) &= 3(3+\beta) - (1+\beta) \left(\frac{\lambda}{x}\right)^{\frac{2}{\beta}} - 2(2-\beta)\left(\frac{\lambda}{x}\right)^{\frac{1+\beta}{\beta}}; && (5.1.12) \\ C(x) &= 3(9-\beta^2) - (1+\beta^2) \left(\frac{\lambda}{x}\right)^{\frac{2}{\beta}} - 4(2-\beta)(1-\beta)\left(\frac{\lambda}{x}\right)^{\frac{1+\beta}{\beta}}. && (5.1.13) \end{aligned}$$

We have that

$$S(f_{\lambda,\beta,c})(x) = \frac{1}{2A(x)^2\beta^2} x^{-2}(2A(x)C(x) - 3B^2(x)).$$

Since the term $(2A(x)^2\beta^2)^{-1}x^{-2}$ is positive for any $x \neq 0$, we need to study the sign of $2A(x)C(x) - 3B^2(x)$. Making the product

$$\begin{aligned} A(x)C(x) = {} & 3(9-\beta^2)(3+\beta) - 2(1+\beta^2)(3+\beta)(5-\beta)\left(\frac{\lambda}{x}\right)^{\frac{2}{\beta}} - 2(3+\beta)(13-9\beta+2\beta^2)\left(\frac{\lambda}{x}\right)^{\frac{1+\beta}{\beta}} \\ & + 2(1-\beta^2)(5-\beta)\left(\frac{\lambda}{x}\right)^{\frac{3+\beta}{\beta}} + (1-\beta^2)(1+\beta)\left(\frac{\lambda}{x}\right)^{\frac{4}{\beta}} + 8(2-\beta)(1-\beta^2)\left(\frac{\lambda}{x}\right)^{\frac{2+2\beta}{\beta}}, \end{aligned} \quad (5.1.14)$$

and

$$B^2(x) = 9(3+\beta)^2 - 6(1+\beta)(3+\beta)\left(\frac{\lambda}{x}\right)^{\frac{2}{\beta}} - 12(3+\beta)(2-\beta)\left(\frac{\lambda}{x}\right)^{\frac{1+\beta}{\beta}}$$
$$+ 4(1+\beta^2)(2-\beta)\left(\frac{\lambda}{x}\right)^{\frac{3+\beta}{\beta}} + (1+\beta)^2\left(\frac{\lambda}{x}\right)^{\frac{4}{\beta}} + 4(2-\beta)^2\left(\frac{\lambda}{x}\right)^{\frac{2+2\beta}{\beta}}. \quad (5.1.15)$$

So we get

$$2A(x)C(x) - 3B^2(x) = -3(3+\beta)^2(3+2\beta) - 2(1+\beta)(3+\beta)(1-2\beta)\left(\frac{\lambda}{x}\right)^{\frac{2}{\beta}}$$
$$-4(3+\beta)(2\beta^2-5)\left(\frac{\lambda}{x}\right)^{\frac{1+\beta}{\beta}} - 4(1+\beta)(1+4\beta-2\beta^2)\left(\frac{\lambda}{x}\right)^{\frac{3+\beta}{\beta}}$$
$$-(1+\beta)^2(1+2\beta)\left(\frac{\lambda}{x}\right)^{\frac{4}{\beta}} - 4(2-\beta)(2+\beta)\left(\frac{\lambda}{x}\right)^{\frac{2+2\beta}{\beta}}. \quad (5.1.16)$$

Put

$$\begin{aligned}
a_0 = a_0(\beta) &:= -3(3+\beta)^2(3+2\beta), & (5.1.17)\\
a_1 = a_1(\beta) &:= -2(1+\beta)(3+\beta)(1-2\beta), & (5.1.18)\\
a_2 = a_2(\beta) &:= -4(3+\beta)(2\beta^2-5), & (5.1.19)\\
a_3 = a_3(\beta) &:= -4(1+\beta)(1+4\beta-2\beta^2), & (5.1.20)\\
a_4 = a_4(\beta) &:= -(1+\beta)^2(1+2\beta), & (5.1.21)\\
a_5 = a_5(\beta) &:= -4(2-\beta)(2+\beta). & (5.1.22)
\end{aligned}$$

we can write

$$S(f_{\lambda,\beta,c})(x) = a_0 + a_1\left(\frac{\lambda}{x}\right)^{\frac{2}{\beta}} + a_2\left(\frac{\lambda}{x}\right)^{\frac{1+\beta}{\beta}} + a_3\left(\frac{\lambda}{x}\right)^{\frac{3+\beta}{\beta}} + a_4\left(\frac{\lambda}{x}\right)^{\frac{4}{\beta}} + a_5\left(\frac{\lambda}{x}\right)^{\frac{2+2\beta}{\beta}}.$$

We call these polynomials the *coefficients* of $S(f_{\lambda,\beta,c})$. In Figure 5.2 we see how the sign of the coefficients of $S(f_{\lambda,\beta,c})$ changes for $\beta \in (0,\infty)$. Since $x \geq \lambda$, we have that $\frac{\lambda}{x} \leq 1$. Using this, we have the following:

Case $0 < \beta \leq 1/2$. The coefficient $a_2(\beta)$ is positive and the other coefficients are nonpositve. Then

$$\begin{aligned}
2A(x)C(x) - 3B^2(x) &< a_0(\beta) + a_2(\beta)\\
&= -3(3+\beta)^2(3+2\beta) - 4(3+\beta)(2\beta^2-5)\\
&= -(3+\beta)(14\beta^2+27\beta+7)\\
&< 0.
\end{aligned}$$

		$\frac{1}{2}$		$\sqrt{\frac{5}{3}}$		2		$1+\sqrt{\frac{3}{2}}$	
$a_0(\beta)$	$-$		$-$		$-$		$-$		$-$
$a_1(\beta)$	$-$	•	$+$		$+$		$+$		$+$
$a_2(\beta)$	$+$		$+$	•	$-$		$-$		$-$
$a_3(\beta)$	$-$		$-$		$-$		$-$	•	$+$
$a_4(\beta)$	$-$		$-$		$-$		$-$		$-$
$a_5(\beta)$	$-$		$-$		$-$	•	$+$		$+$

Figure 5.2: Sign of the coefficients $a_i(\beta)$ in terms of β. The dot at each row represents the value at which $a_i(\beta)$ vanish.

Case $1/2 < \beta \leq \sqrt{\frac{5}{3}}$. The coefficients $a_1(\beta)$ and $a_2(\beta)$ are nonnegative and the other coefficients are negative. Then

$$\begin{aligned}
2A(x)C(x) - 3B^2(x) &< a_0(\beta) + a_1(\beta) + a_2(\beta) \\
&= -3(3+\beta)^2(3+2\beta) - 2(1+\beta)(3+\beta)(1-2\beta) - 4(3+\beta)(2\beta^2-5) \\
&= -(3+\beta)(10\beta^2 + 25\beta + 9) \\
&< 0.
\end{aligned}$$

Case $\sqrt{\frac{5}{3}} < \beta \leq 2$. The coefficient $a_1(\beta)$ is positive and the other coefficients are nonpositive. Then

$$\begin{aligned}
2A(x)C(x) - 3B^2(x) &< a_0(\beta) + a_1(\beta) \\
&= -3(3+\beta)^2(3+2\beta) - 2(1+\beta)(3+\beta)(1-2\beta) \\
&= -(3+\beta)(2\beta^2 + 25\beta + 29) \\
&< 0.
\end{aligned}$$

Case $\sqrt{\frac{5}{3}} < \beta \leq 2$. The coefficient $a_1(\beta)$ is positive and the other coefficients are nonpositive. Then

$$\begin{aligned}
2A(x)C(x) - 3B^2(x) &< a_0(\beta) + a_1(\beta) \\
&= -3(3+\beta)^2(3+2\beta) - 2(1+\beta)(3+\beta)(1-2\beta) \\
&= -(3+\beta)(2\beta^2 + 25\beta + 29) \\
&< 0.
\end{aligned}$$

Case $2 < \beta \leq 1+\sqrt{\frac{3}{2}}$. The coefficients $a_1(\beta)$ and $a_5(\beta)$ are positive and the other coefficients are nonpositive. Then

$$
\begin{aligned}
2A(x)C(x) - 3B^2(x) &< a_0(\beta) + a_1(\beta) + a_5(\beta) \\
&= -3(3+\beta)^2(3+2\beta) - 2(1+\beta)(3+\beta)(1-2\beta) - 4(2-\beta)(2+\beta) \\
&= -(2\beta^3 + 35\beta^2 + 104\beta + 71) \\
&< 0.
\end{aligned}
$$

Case $1 + \sqrt{\frac{3}{2}} < \beta$**.** The coefficients $a_1(\beta)$, $a_3(\beta)$, and $a_5(\beta)$ are positive and the other coefficients are negative. Then

$$
\begin{aligned}
2A(x)C(x) - 3B^2(x) &< a_0(\beta) + a_1(\beta) + a_3(\beta) + a_5(\beta) \\
&= -3(3+\beta)^2(3+2\beta) - 2(1+\beta)(3+\beta)(1-2\beta) \\
&\quad -4(1+\beta)(1+4\beta-2\beta^2) - 4(2-\beta)(2+\beta) \\
&= 6\beta^3 - 43\beta^2 - 124\beta - 75.
\end{aligned}
\tag{5.1.23}
$$

So, for $x \geq \lambda$ we have that $S(f_{\lambda,\beta,c}) < 0$, for every $\beta \in (0, \frac{1}{12}(49 + \sqrt{4201}))$, where $\frac{1}{12}(49 + \sqrt{4201}$ is the unique positive root of the polynomial $6\beta^3 - 43\beta^2 - 124\beta - 75$, this polynomial is negative in the interval $(0, \frac{1}{12}(49 + \sqrt{4201}))$.

Now we study the Shwarzian derivative for $0 < x < \lambda$. In this case $\frac{\lambda}{x} > 1$, and $\frac{\lambda}{x} \longrightarrow +\infty$ as $x \to 0^+$.

By (5.1.3) and (5.1.4), we can see that the formulas for the first, second, and third derivative of $f_{\lambda,\beta,c}$ are given by (5.1.8), (5.1.9), and (5.1.10) with opposite sing, so we can write

$$f'_{\lambda,\beta,c}(x) = -\frac{1}{\beta} x^{\frac{3}{\beta}} A(x); \tag{5.1.24}$$

$$f''_{\lambda,\beta,c}(x) = -\frac{1}{\beta^2} x^{\frac{3-\beta}{\beta}} B(x); \tag{5.1.25}$$

$$f'''_{\lambda,\beta,c}(x) = -\frac{1}{\beta^3} x^{\frac{3-2\beta}{\beta}} C(x), \tag{5.1.26}$$

where the functions $A(x)$, $B(x)$, and $C(x)$ are given by (5.1.11), (5.1.12), and (5.1.13) respectively. Then we have the same formula for the Schwarzian derivative of $f_{\lambda,\beta,c}$ for $0 < x < \lambda$ and $\lambda \geq x$. So we need to study the sign of $2A(x)B(x) - 3B^2(x)$ for $0 < x < \lambda$ in terms of β. First we write

$$a_0(\beta) = -6\beta^3 - 45\beta^2 - 108\beta - 81, \tag{5.1.27}$$

$$a_1(\beta) = 4\beta^3 + 14\beta^2 + 4\beta - 6, \tag{5.1.28}$$

$$a_2(\beta) = -8\beta^3 - 24\beta^2 + 20\beta + 60, \tag{5.1.29}$$

$$a_3(\beta) = 8\beta^3 - 8\beta^2 - 20\beta - 4, \tag{5.1.30}$$

$$a_4(\beta) = -2\beta^3 - 5\beta^2 - 4\beta - 1, \tag{5.1.31}$$

$$a_5(\beta) = 4\beta^2 - 16. \tag{5.1.32}$$

Case $0 < \beta \leq 1/2$**.** The coefficient $a_2(\beta)$ is positive and the other coefficients are nonpositve. Put

$$p_1(x) := a_0 + a_2 \left(\frac{\lambda}{x}\right)^{\frac{1+\beta}{\beta}} + a_5 \left(\frac{\lambda}{x}\right)^{\frac{2+2\beta}{\beta}},$$

and $p_2(x) := 2A(x)C(x) - 3B^2(x) - p_1(X)$. We will use the following three observations. First, by definition we have that $2A(x)C(x) - 3B^2(x) \leq p_1(x)$. Also,

$$\begin{aligned} p_1(\lambda) &< a_0 + a_2 \\ &= -(3+\beta)(2\beta^2 + 25\beta + 29) \\ &< 0. \end{aligned} \tag{5.1.33}$$

Second, by continuity of p_1, there is $\varepsilon > 0$ such that for every $x \in (\lambda - \varepsilon, \lambda)$ we have that $p_1(x) < 0$. Finally, since $a_5 < 0$, and $\frac{2+2\beta}{\beta} > \frac{1+\beta}{\beta}$ there is $\delta > 0$ such that for every $x \in (\delta, 0)$ we have that $p_1(x) < 0$.

The first observation is telling us that in order to prove that $S(f_{\lambda,\beta,c}) < 0$, is enough to prove that $p_1 < 0$.

Now, if $p_1(x)$ is positive for some $x \in (0, \lambda)$,then by continuity it must has at least one critical point that is a local maximum. Since

$$p_1'(x) = \frac{-\lambda}{x^2}\left(\frac{1+\beta}{\beta}\right) a_2 \left(\frac{\lambda}{x}\right)^{\frac{1}{\beta}} + \frac{-\lambda}{x^2}\left(\frac{2+2\beta}{\beta}\right) a_5 \left(\frac{\lambda}{x}\right)^{\frac{2+\beta}{\beta}},$$

Dirichlet's Theorem for generalized polynomials tell us that p_1' has at most one root in $(0, \lambda)$. Writing

$$p_1'(x) = \frac{-\lambda}{x^2}\frac{1+\beta}{\beta}\left(\frac{\lambda}{x}\right)^{\frac{1}{\beta}}\left(a_2 + 2a_5\left(\frac{\lambda}{x}\right)^{\frac{1+\beta}{\beta}}\right),$$

we see that p_1 has a critical point at

$$x_1 = \lambda \left(\frac{-a_2}{2a_5}\right)^{-\frac{\beta}{3-\beta}}.$$

If w_1 is a critical point of inflexion type or a local minimum, by the second and third observations above, we have that $p_1 < 0$. Also,

$$p_1(x_1) = a_0 < 0.$$

Then, if the critical point is a local maximum we get that $p_1 < 0$. Now, since p_2 is increasing on $(0, \lambda)$, we have that for every $x \in (0, \lambda)$

$$2A(x)C(x) - 3B^2(x) \leq p_1(w_1) + p_2(\lambda) < 0.$$

Case $1/2 < \beta \leq \sqrt{\frac{3}{2}}$**.** The coefficients $a_1(\beta)$ and $a_2(\beta)$ are positive and the other coefficients are nonpositve. Put

$$a_{01}(\beta) := -30\beta - 81 \qquad \text{and } a_{02}(\beta) := -6\beta^3 - 45\beta^2 - 78\beta.$$

We see that $a_{01}+a_{02}=a_0$. Now, let

$$p_1(x) := a_{01} + a_2\left(\frac{\lambda}{x}\right)^{\frac{1+\beta}{\beta}} + a_5\left(\frac{\lambda}{x}\right)^{\frac{2+2\beta}{\beta}},$$

$$p_2(x) := a_{02} + a_1\left(\frac{\lambda}{x}\right)^{\frac{2}{\beta}} + a_4\left(\frac{\lambda}{x}\right)^{\frac{4}{\beta}},$$

and $p_3(x) := 2A(x)C(x) - 3B^2(x) - p_1(X) - p_2(x)$. We will use the following three observations. First, by definition we have that $2A(x)C(x) - 3B^2(x) \leq p_1(x) + p_2(x)$. Also,

$$\begin{aligned} p_1(\lambda) &< a_{01} + a_2 \\ &= -8\beta^3 - 24\beta^2 - 10\beta - 19 \\ &< 0, \end{aligned} \tag{5.1.34}$$

and

$$\begin{aligned} p_2(\lambda) &< a_{02} + a_1 \\ &= -2\beta^3 - 31\beta^2 - 58\beta - 6 \\ &< 0. \end{aligned} \tag{5.1.35}$$

Second, by continuity of p_1 and p_2, there is $\varepsilon > 0$ such that for every $x \in (\lambda-\varepsilon, \lambda)$ we have that $p_1(x) < 0$ and $p_2(x) < 0$. Finally, since $a_5 < 0$and $a_4 < 0$, and $\frac{2+2\beta}{\beta} > \frac{1+\beta}{\beta}$ and $\frac{4}{\beta} > \frac{2}{\beta}$ there is $\delta > 0$ such that for every $x \in (\delta, 0)$ we have that $p_1(x) < 0$ and $p_2(x) < 0$.

The first observation is telling us that in order to prove that $S(f_{\lambda,\beta,c}) < 0$, is enough to prove that $p_1 < 0$ and $p_2 < 0$. By what we prove in the previous case, we see that $p_1(x)$ has a a critical point at

$$x_1 = \lambda\left(\frac{-a_2}{2a_5}\right)^{-\frac{\beta}{1+\beta}},$$

and at this critical point $p_1(x_1) < 0$.

Now, if $p_2(x)$ is positive for some $x \in (0, \lambda)$,then by continuity it must has at least one critical point that is a local maximum. Since

$$p_2'(x) = \frac{-\lambda}{x^2}\left(\frac{2}{\beta}\right)a_1\left(\frac{\lambda}{x}\right)^{\frac{2-\beta}{\beta}} + \frac{-\lambda}{x^2}\left(\frac{4}{\beta}\right)a_4\left(\frac{\lambda}{x}\right)^{\frac{4-\beta}{\beta}},$$

Dirichlet's Theorem for generalized polynomials tell us that p_2' has at most one root in $(0, \lambda)$. Writing

$$p_2'(x) = \frac{-\lambda}{x^2}\frac{2}{\beta}\left(\frac{\lambda}{x}\right)^{\frac{2-\beta}{\beta}}\left(a_1 + 2a_4\left(\frac{\lambda}{x}\right)^{\frac{2}{\beta}}\right),$$

we see that p_2 has a critical point at

$$x_2 = \lambda\left(\frac{-a_1}{2a_4}\right)^{-\frac{\beta}{2}}.$$

If x_2 is a critical point of inflexion type or a local minimum, by the second and third observations above, we have that $p_2 < 0$. Also,

$$p_2(x_2) = a_{02} < 0.$$

Then, if the critical point is a local maximum we get that $p_2 < 0$. Now, since p_3 is increasing on $(0, \lambda)$, we have that for every $x \in (0, \lambda)$

$$2A(x)C(x) - 3B^2(x) \leq p_1(x_1) + p_2(x_2) + p_3(\lambda) < 0.$$

Case $\sqrt{\frac{3}{2}} < \beta \leq 2$. The coefficient $a_1(\beta)$ is positive and the other coefficients are nonpositve. Put

$$p_1(x) := a_0 + a_1 \left(\frac{\lambda}{x}\right)^{\frac{2}{\beta}} + a_4 \left(\frac{\lambda}{x}\right)^{\frac{4}{\beta}},$$

and $p_2(x) := 2A(x)C(x) - 3B^2(x) - p_1(X)$. The we can proceed as in the previous case to show that $p_1 < 0$.

Case $\sqrt{2} < \beta$. The coefficient $a_1(\beta)$ and $a_5(\beta)$ are positive and the other coefficients are nonpositve. Since a_5 is the coefficient of the dominant term $\left(\frac{\lambda}{x}\right)^{\frac{2+2\beta}{\beta}}$, then there is a $x_0 \in (0, \lambda)$ such that $S(f_{\lambda,\beta,c})(x) \geq 0$ for $x \in (0, x_0)$.

By symmetry we have the same behavior for the case $x \in [-\lambda, 0)$ and for $x \in (-\infty, -\lambda)$.

So we have that Theorem 15 holds if $\beta \in (0, 2]$. In particular, the number of attracting cycles for the map $f_{\lambda,\beta,c}$ is at most two for $0 < \beta \leq 2$ (since both critical points of inflection type have the same image).

Question 5. Can we find parameter $\lambda \geq 0$, $\beta > 2$, and $c \in [c_2, c_0]$ so that the map $f_{\lambda,\beta,c}$ has a attracting cycle that doesn't attract a critical point?

5.2 Feigenbaum-Coullet-Tresser universality

In the 1970's Feigenbaum [27], [28], and independently Coullet and Tresser [70] discovered a universal scaling law of transition from regular to chaotic dynamics through computer experiments. The computer experiments goes in the following way: first we choose a family of quadratic unimodal maps depending on one parameter, for instance we can think on the family $f_{0,3/2,c} \colon \mathbb{R} \longrightarrow \mathbb{R}$, with $c \in [-2, 1/4]$ defined in (5.1.1), whose turning point ($x = 0$) is a critical point. Now we can iterate the critical point $x = 0$ and look for the values that these iterations accumulates for differents values of c. We can make a graph of this point by considering the different values of c in the horizontal axis, and for each value of c we draw (vertically) the values accumulated by the orbit of $x = 0$ (see Figure 5.7a, here the horizontal axis is in reversed order, thus in the horizontal axis the leftmost value is $c = 1/4$ and the rightmost value is $c = -2$), this graph is called the *bifurcation diagram* of the family $f_{0,3/2,c}$. From this bifurcation diagram in Figure 5.7a, we can see a curve coming from left that splits into two curves, then these two curves splits into four curves, and so on. Now let a_1 be the parameter value where the curve coming from left to right splits into two, a_2 where the two curves split into four, a_3 where the four curves split into eight, etc (thus, the parameter values a_i represent where a bifurcation happens). Form the distances $\mathrm{d}_1 = a_2 - a_1$, $\mathrm{d}_2 = a_3 - a_3$, ... The sequence of ratios $\mathrm{d}_1/\mathrm{d}_2$, $\mathrm{d}_2/\mathrm{d}_3$, ... converges to the number $\delta \approx 4.6692\ldots$. The number δ is known as the *Feigenbaum constant*. This phenomenon can be observe if we replace the family $f_{0,3/2,c}$ for any other one-parameter family that "looks like the quadratic family". This is called the *Feigenbaum-Coullet-Tresser universality*.

In order to explain this universality phenomenon, Feigenbaum and Coullet and Tresser introduced a *dynamical renormalization* theory.

Considere the space $\mathcal{U}^r$ of C^r unimodal maps whose turning point is a non-flat with right and left orders equal to one.

Definition 5.2.1. Let $f \in \mathcal{U}^r$. An interval J containing the turning point of f in its interior is called *restrictive*, or *periodic*, of period $n > 1$ if $f^n(J) \subset J$, and for $1 \leq k < n$ we have $f^k(J) \not\subset J$.

Observe that if J is a restrictive interval of period $n > 1$ for f, then $J, f(J), \ldots, f^{n-1}(J)$ are disjoints (see [18, Chapter 2, Section 5]).

Definition 5.2.2. We say that a unimodal map $f \in \mathcal{U}^r$, with turning point c is *renormalizable* if there exists a restrictive interval J of period $n > 1$ such that $f^n|_J$ is again a unimodal map. We denote by $\mathcal{D}^r(\mathcal{R})$ the set of renormalizable maps in $\mathcal{U}^r$. The *renormalization operator* is the map $\mathcal{R}\colon \mathcal{D}^r(\mathcal{R}) \longrightarrow \mathcal{U}^r$ defined by $\mathcal{R}(f) = H^{-1} \circ f^n \circ H$, where $n > 1$ is as above, and H is a suitable affine homeomorphism.

Remark 5.2.3.

1. We can assume that for every $f \in \mathcal{U}^r$ with turning point c we have that $f^2(c) < c < f(c)$, and $f^2(c) < f^3(c) < f(c)$. This assumption, just ruled out some trivial dynamical situations, see [18, Chapter 2], [6, Chapter 3].

2. Assuming the previous point, the interval $[f^2(c), f(c)]$, which is the core of the system (see Chapter 1), is the smallest invariant interval that contains the turning point. Conjugating by an affine transformation we can assume that the core of the system is the interval $[0, 1]$, thus $f(c) = 1$, and $f^2(c) = 0$.

Using remarks 1 and 2 above, we can describe $\mathcal{U}^r$ as the space of all maps $f\colon [0,1] \longrightarrow [0,1]$ of the form $f = \phi \circ Q \circ \psi$ where $\psi\colon [0,1] \longrightarrow [\psi(0), 1]$ is an orientation reversing diffeomorphism of class C^r with $\psi(0) \in (-1, 0)$, $Q(x) = x^2$, and $\phi\colon [0,1] \longrightarrow [0,1]$ is an orientation reversing diffeomorphism of class C^r.

Let $\mathcal{U}_0^r \subset \mathcal{U}^r$ be the set of maps f whose critical point c satisfy the order condition $0 = f^2(c) < c < f^4(c) < f^3(c) < f^5(c) < f(c) = 1$. We have that the renormalization operator restricted to $\mathcal{U}_0^r$ acts as a doubling operator, thus $\mathcal{R}(f) = H^{-1} \circ f^2 \circ H$ for every $f \in \mathcal{U}_0^r$, where $H\colon [0,1] \longrightarrow [f^2(c), f^4(c)]$ is an orientation reversing affine map. If $f \in \mathcal{U}^r$ can be renormalized infinitely many times and all its images under the renormalization operator belong to $\mathcal{U}_0^r$, we say that f is a *Feigenbaum* map. Fiegenbaum and Coullet and Tresser introduce the renormalization in the context described above (the doubling operator), and in order to explain the numerical discovery in the bifurcation structure of one-parameter families of unimodal maps they made the following conjectures:

1. There exists a Banach space of analytic functions $\mathcal{B}$ such that the restriction of the renormalization operator to $\mathcal{B} \cap \mathcal{U}_0^2$ is a bounded C^2 operator which has a fixed point f_* (often refered as the Feigenbaum fixed point);

2. The derivative of $\mathcal{R}$ at f_* has discrete spectrum with a unique eigenvalue outside the unit disk equal to δ, and all other eigenvalues in the interior of the unit disk;

3. Let $\Sigma_n \subset \mathcal{B} \cap \mathcal{U}_0^2$ be the set of maps having zero topological entropy and for which the turning point is periodic of period 2^n. Then Σ_n intersects the local unstable manifold of $\mathcal{R}$ at f_* transversally for n sufficiently large (for this point, observe that Σ_n is characterized by the critical value, thus it is a one-dimensional condition, so Σ_n is a codimension one submanifold).

So, using the above conjecture, the universality phenomenon goes as follows. Since conjugation preserve periodic points, we have that $f \in \mathcal{U}_0^r$ has a periodic point of period $2n$ if and only if $\mathcal{R}(f)$ has a periodic point of period n. Hence, $\mathcal{R}(\Sigma_{n+1}) \subset \mathcal{R}(\Sigma_n)$, then the $\lambda-$lemma (see [61, Chapter 2, Section 7]) will imply that the submanifolds Σ_n converges toward the stable manifold of $\mathcal{R}$ at f_* and that the rate of convergence is determined by δ. A one-parameter family of unimodal maps in $\mathcal{B}$ is a curve, so we can look for the parameter a_n at which the curve crosses Σ_n. This values converge (if the family intersect Σ_n transversally for every n large enough) and the rate of convergence, given by the ratios $(a_{n+1} - a_n)/(a_n - a_{n-1})$, is δ^{-1}. This only depends on the on f_* and not on the family under consideration.

This conjecture was firstly proved using computer estimates by Landford [39], [40], and Eckmann and Wittwer [25].

Later on, many ingredients of the conjecture were proven without computers. The existence of the fixed point was proved by Campanino and Epstein [14], Campanino, Epstein, and Ruelle [15], and Epstein [26]. Existence of an unstable eigenvalue at this fixed point was proved by Eckmann and Epstein [24]. The stable manifold was constructed by Sullivan [69] and McMullen [51]. Finally, Lyubich [48] proved that the renormalization fixed point f_* is hyperbolic with one-dimensional unstable manifold. This completed the proof of the Renormalization Conjecture for quadratic-like maps.

In the last decades, great effort has been done in order to extend the dynamical renormalization theory to different classes of dynamical systems, see [50], [75], for the Lorenz maps. In the case of asymmetrical unimodal maps see [52], [38]. In the case of the Henon family see [47], [32].

Bifurcations diagrams for the family $f_{\lambda,\beta,c}$ for fixed values of $\lambda > 0$ and $\beta > 2$ show that the behavior of this family changes almost "continuously" starting from the smooth unimodal case for $\lambda = 0$ (see Figure 5.7a), to a very irregular diagram when λ is large enough (see Figure 5.4d). In this case, since $\beta > 2$, the map $f_{\lambda,\beta,c}$ has positive Schwarzian derivative in a neighborhood of the turning point, then we should not expect that the orbit of the turning point keeps track of the attracting cycles.

Question 6. For what values of $\lambda > 0$ and $\beta > 2$ can we find a period-doubling, or other orders, bifurcations in the family $f_{\lambda,\beta,c}$?

Question 7. Could one expect universality if the family contains period-doubling, or other orders, bifurcation?

On the other hand, if $\beta \in (0, 2]$, we can keep track of the attracting cycles by looking at the orbit of the critical points. The bifurcation diagram for $0 \leq \lambda$ and $0 < \beta \leq 2$ can be highly different, compare Figure 5.5 and Figure 5.6, but in all of them it is possible to find parts that are similar to the bifurcation diagram of the quadratic family, see 5.7a.

Question 8. For what values of $\lambda > 0$ and $0 < \beta \leq 2$ can we find a period-doubling cascade, or other orders cascade, in the family $f_{\lambda,\beta,c}$?

Question 9. Could one expect universality if the family contains cascades of period-doubling, or other orders, bifurcation?

5.3 Monotonicity of entropy

The topological entropy was first introduce by Adler, Konheim, and MacAndrew [1] for continuous map on compact metric spaces and is a measure of the dynamical complexity of the map. It was inspired by the Kolmogorov-Sinai or metric entropy, and it measures the asymptotic growth rate of the number of different orbits of length n if we use a precision ε to distinguish two orbits. For unimodal maps the topological entropy coincides with the growth of the number of points in the backward orbits of the turning point. More precisely, let $I \subset \mathbb{R}$ be a compact interval and $f\colon I \longrightarrow I$ a continuous map. Suppose we can find a finite partition $\{a_0 < a_1 < \ldots < a_k\}$ of I such that $f|_{(a_i,a_{i+1})}$ is monotone for every $i = 0, 1 \ldots k-1$. The smallest cardinality among these partitions is called *the lap number* of f, and we denote it by $\ell(f)$. For example, if f is a unimodal map, then $\ell(f) = 2$. We can see that for a unimodal map $f\colon I \longrightarrow I$ the lap number $\ell(f^n)$ is equal to the number of points in the backward orbits of the turning point. Then

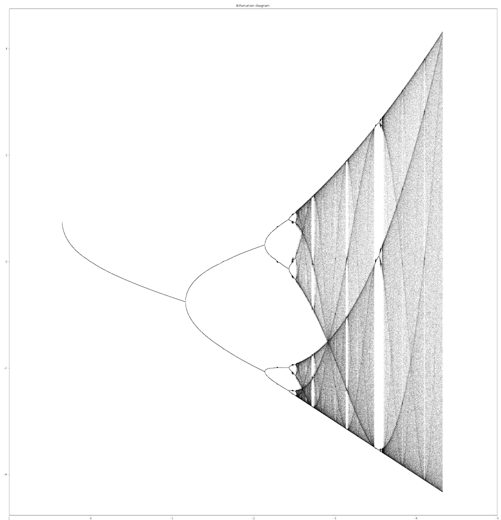
$$h_{\text{top}}(f) = \lim_{n\to\infty} \frac{1}{n} \log(\ell(f^n)).$$

This result was proved, for piecewise monotone maps of the interval, by Misiurewicz and Szlenk [57], and Young [76]. Topological entropy can be used to classify unimodal maps, up to semi-conjugacy. Conventionally, a map is called *chaotic* if it has positive topological entropy.

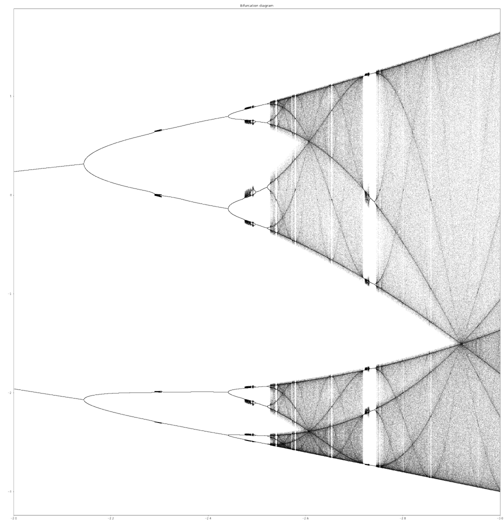
The questions whether $h_{\text{top}}(f)$ is "monotone" in terms of f goes back to the 1970's (see [54]), and in our setting we have a simple way of asking this question. Fix $\lambda \geq 0$, and $\beta > 0$. Let c_0, and c_2 as in Section 5.1.1.

Question 10. Does the topological entropy of $f_{\lambda,\beta,c}$ increases when $c \in [c_2, c_0]$ decrease?

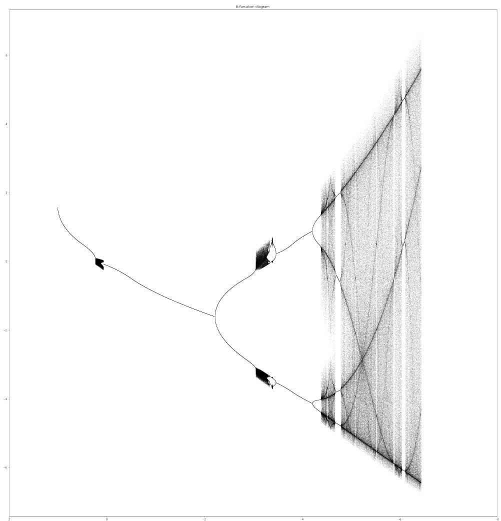
In the unimodal case, it is known that the entropy cannot be strictly increasing if the family presents parameter intervals with a period-doubling bifurcation (in these parameter intervals the entropy is constant). It has been conjectured that if a C^3 unimodal convex map



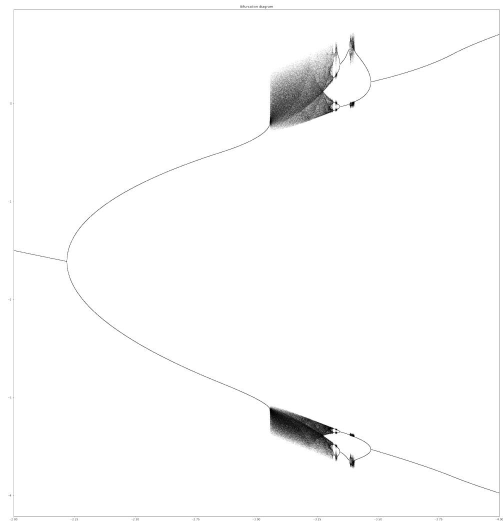
(a) $\lambda = 0.05$, $\beta = 3$, $c \in [1, -5]$



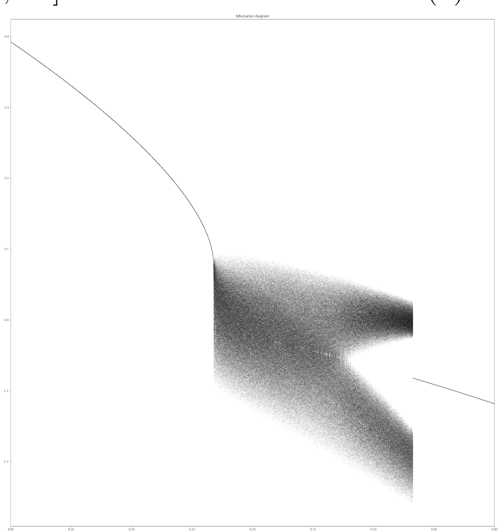
(b) $\lambda = 0.05$, $\beta = 3$, $c \in [-2, -3]$



(c) $\lambda = 0.5$, $\beta = 3$, $c \in [2, -8]$

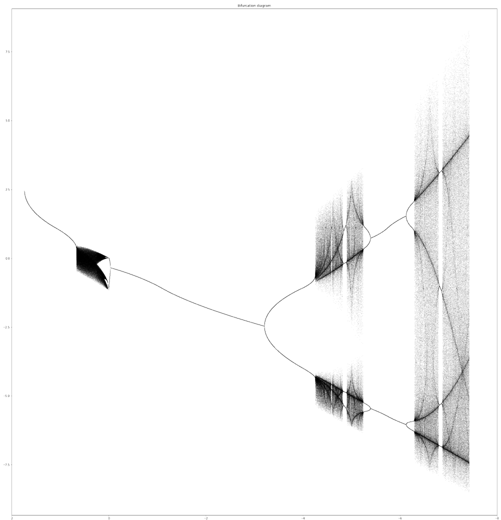


(d) $\lambda = 0.5$, $\beta = 3$, $c \in [-2, -4]$

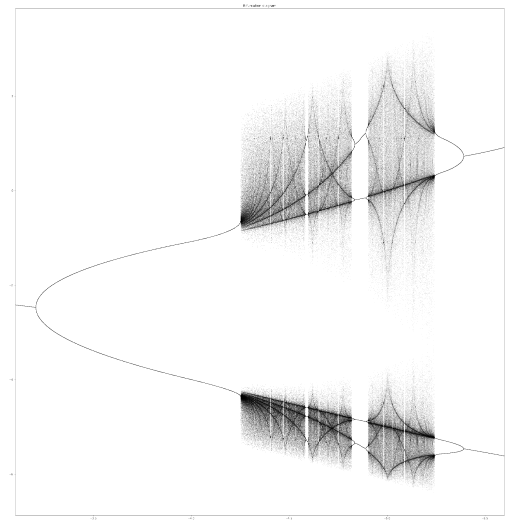


(e) $\lambda = 0.5$, $\beta = 3$, $c \in [0.4, 0]$

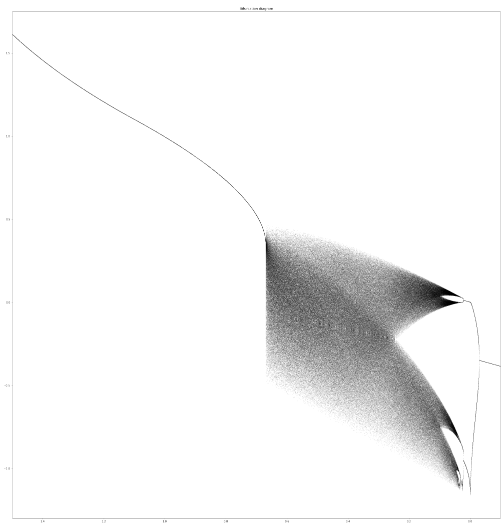
Figure 5.3: Graphics corresponding to the bifurcation diagram for the family $f_{\lambda,\beta,c}$ starting at $x = 0$, for the corresponding value of λ, β, and c in the indicated range.



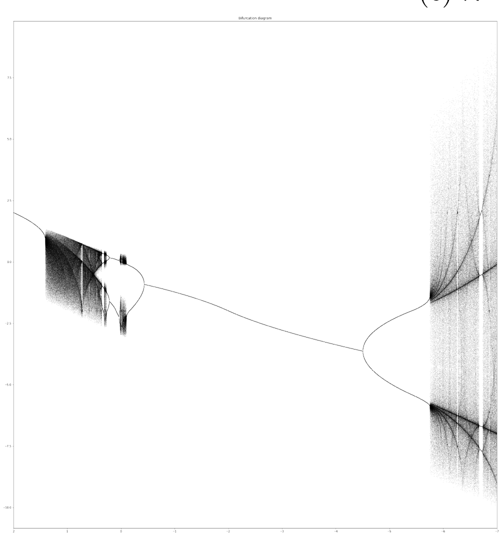
(a) $\lambda = 1.1$, $\beta = 3$, $c \in [2, -8]$



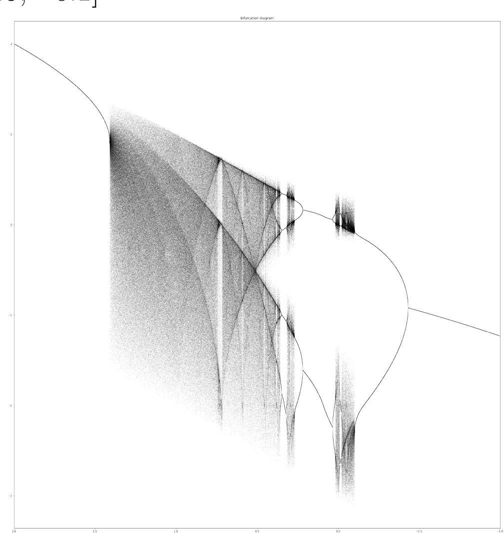
(b) $\lambda = 1.1$, $\beta = 3$, $c \in [-3.1, -5.6]$



(c) $\lambda = 1.1$, $\beta = 3$, $c \in [1.5, -0.1]$

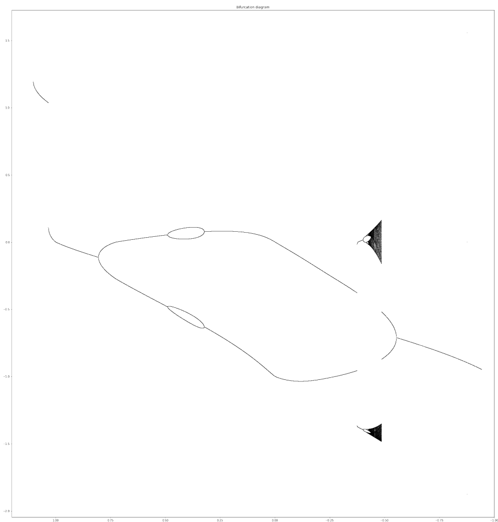


(d) $\lambda = 2$, $\beta = 3$, $c \in [2, -7]$

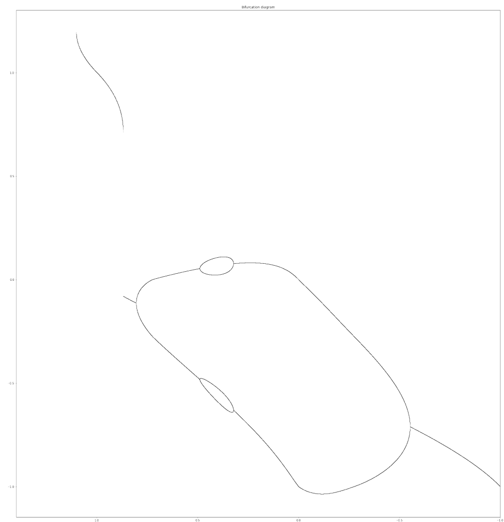


(e) $\lambda = 2$, $\beta = 3$, $c \in [2, -1]$

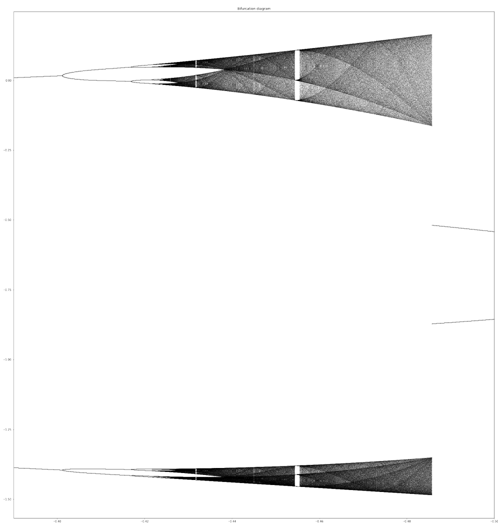
Figure 5.4: Graphics corresponding to the bifurcation diagram for the family $f_{\lambda,beta,c}$ starting at $x = 0$ for the corresponding value of λ, β, c in the indicated range.



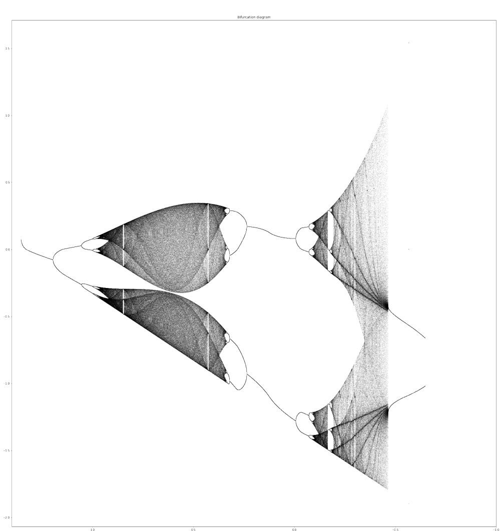
(a) $\lambda = 1$, $\beta = 1.5$, $c \in [1.5, -1]$, $x = 0$



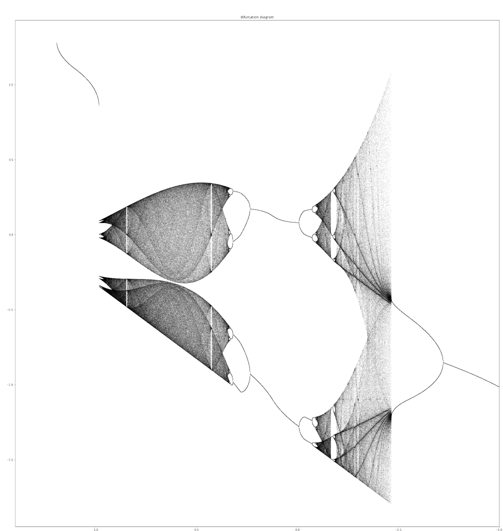
(b) $\lambda = 1$, $\beta = 1.5$, $c \in [1.5, -1]$, $x = 1$



(c) $\lambda = 1$, $\beta = 1.5$, $c \in [-0.39, -0.5]$, $x = 0$

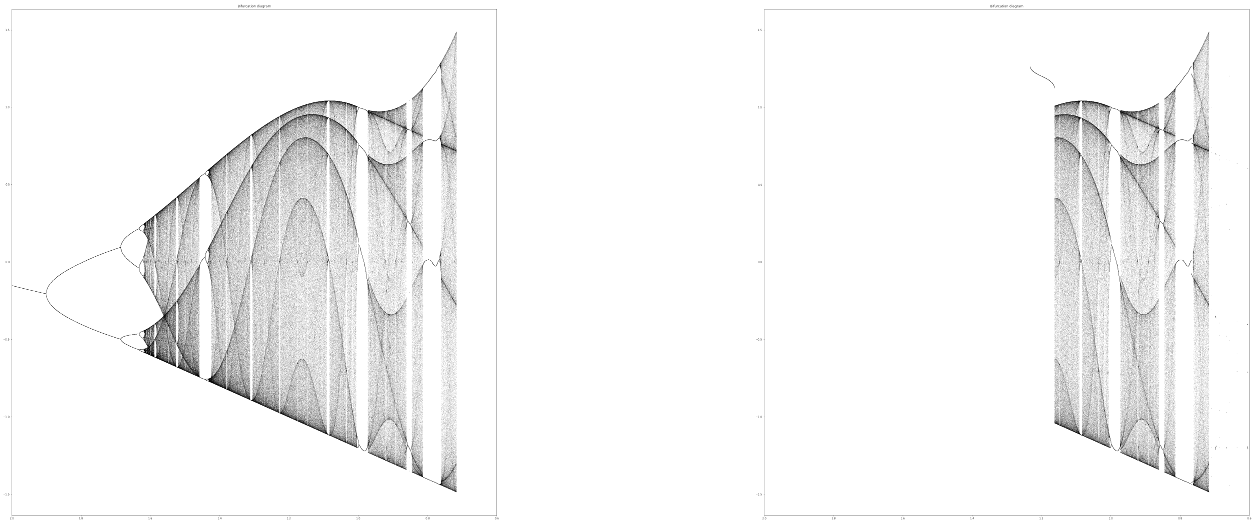


(d) $\lambda = 1.1$, $\beta = 1.5$, $c \in [2, -7]$, $x = 0$



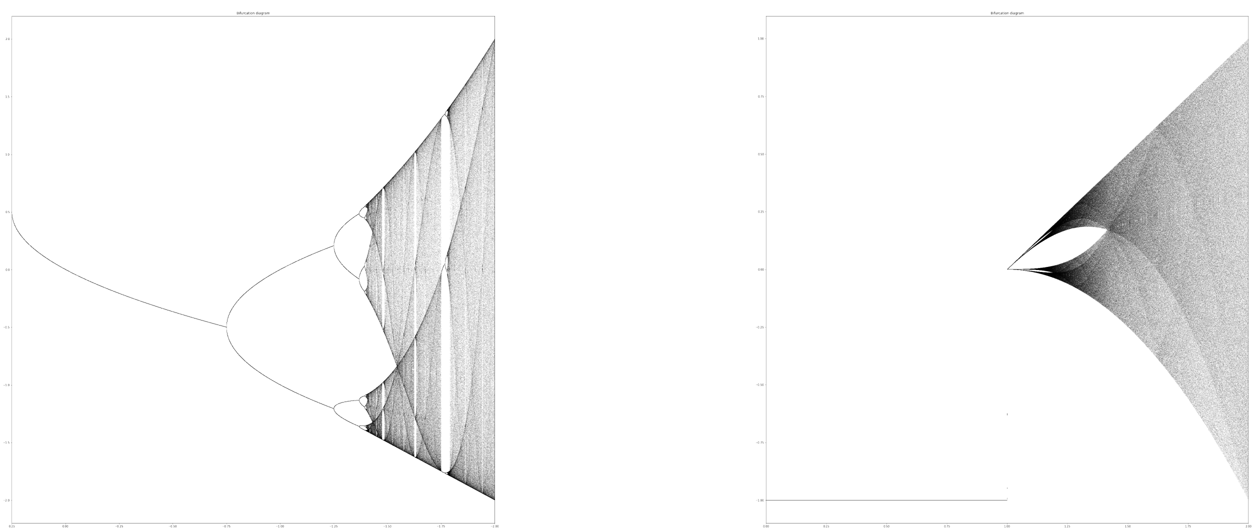
(e) $\lambda = 1.1$, $\beta = 1.5$, $c \in [2, -1]$, $x = 1.1$

Figure 5.5: Graphics corresponding to the bifurcation diagram for the family $f_{\lambda,\beta,c}$ for the corresponding value of λ, β, c in the indicated range, and x the point.



(a) $\lambda = 1.2$, $\beta = 0.9$, $c \in [2, 0.6]$, $x = 0$ (b) $\lambda = 1.2$, $\beta = 0.9$, $c \in [2, 0.6]$, $x = 1.2$

Figure 5.6: Graphics corresponding to the bifurcation diagram for the family $f_{\lambda,\beta,c}$ for the corresponding value of λ, β, c in the indicated range, and x the corresponding point.



(a) (b)

Figure 5.7: (a) Bifurcation diagram for the quadratic family. (b) Bifurcation diagram for the family of symmetric tent maps.

$f\colon [0,1] \longrightarrow [0,1]$ with $f(0) = 0 = f(1)$, $f(1/2) = 1$, negative Schwarzian derivative, and symmetric around the turning point, then the topological entropy of $f_a(x) = af(x)$ increases with $a \in [0,1]$. Recall that the Schwarzian derivative of a map $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is defined as

$$S(f)(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

The more general questions if $f, g\colon [0,1] \longrightarrow [0,1]$ are two unimodal maps with $g \leq f$ implies $h_{\text{top}}(g) \leq h_{\text{top}}(f)$ is not true in general. In [10] Bruin proved that there is a "large" set of unimodal maps for which $g \leq f$ and $h_{\text{top}}(g) > h_{\text{top}}(f)$.

In the 1980's Douady and Hubbard [21, 22, 23], and Milnor and Thurston [56] showed that for the quadratic family $f_a(x) = 4ax(1-x)$ the entropy $h_{\text{top}}(f_a)$ depends monotonically on $a \in [0,1]$. The proof of this use that the quadratic family can be extended to the complex plane and require tools from complex analysis. Recently Rempe-Gillen and vans Strien [65] proved that for each unimodal map $f\colon [0,1] \longrightarrow [0,1]$ with $f(0) = 0 = f(1)$ which extends to an entire transcendental map on the complex plane, with a finite number of singular values and satisfying the so-called sector condition, the topological entropy of $f_a(x) = af(x)$ depends monotonically on $a \in [0,1]$.

In the case of the Lorenz-like family $f_{\lambda,\beta,c}$ an interesting phenomenon can be observe in the bifurcation diagrams, for $\beta > 2$, as $\lambda \geq 0$ increases. A region of "chaotic behavior" appeared at the beginning of the diagram (see Figure 5.3c, Figure 5.3e, Figure 5.4a, Figure 5.4c, Figure 5.4d, and Figure 5.4e), it seems that for λ big enough the dynamics of $f_{\lambda,\beta,c}$ changes from having an attracting fixed point to jump into chaotic behavior, that can be seeing toward a small interval of parameters c, for later collapse into a single attracting fixed points. For $\lambda = 0.5$ we see that region is similar to the bifurcation diagram for the symmetric family of tent maps (see Figure 5.3e and Figure 5.7b), and when the value of λ increases we start to see bifurcations inside this region (see Figure 5.4c).

The above observations suggest that the answer for Questions10 is negative when the turning point is a Lorenz-like singularity.

Bibliography

[1] R. L. Adler, A. G. Konheim, and M. H. McAndrew. Topological entropy. *Trans. Amer. Math. Soc.*, 114:309–319, 1965.

[2] Vítor Araújo, Stefano Luzzatto, and Marcelo Viana. Invariant measures for interval maps with critical points and singularities. *Adv. Math.*, 221(5):1428–1444, 2009.

[3] L. S. Block and W. A. Coppel. *Dynamics in one dimension*, volume 1513 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1992.

[4] Alexander Blokh, A. M. Bruckner, P. D. Humke, and J. Smítal. The space of ω-limit sets of a continuous map of the interval. *Trans. Amer. Math. Soc.*, 348(4):1357–1372, 1996.

[5] Bodil Branner and John H. Hubbard. The iteration of cubic polynomials. II. Patterns and parapatterns. *Acta Math.*, 169(3-4):229–325, 1992.

[6] Karen M. Brucks and Henk Bruin. *Topics from one-dimensional dynamics*, volume 62 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004.

[7] H. Bruin. Combinatorics of the kneading map. In *Proceedings of the Conference "Thirty Years after Sharkovskiĭ's Theorem: New Perspectives" (Murcia, 1994)*, volume 5, pages 1339–1349, 1995.

[8] H. Bruin, G. Keller, T. Nowicki, and S. van Strien. Wild Cantor attractors exist. *Ann. of Math. (2)*, 143(1):97–130, 1996.

[9] H. Bruin, J. Rivera-Letelier, W. Shen, and S. van Strien. Large derivatives, backward contraction and invariant densities for interval maps. *Invent. Math.*, 172(3):509–533, 2008.

[10] Henk Bruin. Non-monotonicity of entropy of interval maps. *Phys. Lett. A*, 202(5-6):359–362, 1995.

[11] Henk Bruin. Quasi-symmetry of conjugacies between interval maps. *Nonlinearity*, 9(5):1191–1207, 1996.

[12] Henk Bruin. Combinatorics of Fibonacci-like unimodal maps. In *École de Théorie*

Ergodique, volume 20 of *Sémin. Congr.*, pages 17–37. Soc. Math. France, Paris, 2010.

[13] Henk Bruin, Weixiao Shen, and Sebastian van Strien. Invariant measures exist without a growth condition. *Comm. Math. Phys.*, 241(2-3):287–306, 2003.

[14] M. Campanino and H. Epstein. On the existence of Feigenbaum's fixed point. *Comm. Math. Phys.*, 79(2):261–302, 1981.

[15] M. Campanino, H. Epstein, and D. Ruelle. On Feigenbaum's functional equation $g \circ g(\lambda x) + \lambda g(x) = 0$. *Topology*, 21(2):125–129, 1982.

[16] P. Collet and J.-P. Eckmann. Positive Liapunov exponents and absolute continuity for maps of the interval. *Ergodic Theory Dynam. Systems*, 3(1):13–46, 1983.

[17] María Isabel Cortez and Juan Rivera-Letelier. Invariant measures of minimal post-critical sets of logistic maps. *Israel J. Math.*, 176:157–193, 2010.

[18] Welington de Melo and Sebastian van Strien. *One-dimensional dynamics*, volume 25 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1993.

[19] Neil Dobbs. On cusps and flat tops. *Ann. Inst. Fourier (Grenoble)*, 64(2):571–605, 2014.

[20] Neil Dobbs. Pesin theory and equilibrium measures on the interval. *Fund. Math.*, 231(1):1–17, 2015.

[21] A. Douady and J. H. Hubbard. *Étude dynamique des polynômes complexes. Partie I*, volume 84 of *Publications Mathématiques d'Orsay [Mathematical Publications of Orsay]*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1984.

[22] A. Douady and J. H. Hubbard. *Étude dynamique des polynômes complexes. Partie II*, volume 85 of *Publications Mathématiques d'Orsay [Mathematical Publications of Orsay]*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1985. With the collaboration of P. Lavaurs, Tan Lei and P. Sentenac.

[23] Adrien Douady. Topological entropy of unimodal maps: monotonicity for quadratic polynomials. In *Real and complex dynamical systems (Hillerød, 1993)*, volume 464 of *NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci.*, pages 65–87. Kluwer Acad. Publ., Dordrecht, 1995.

[24] J.-P. Eckmann and H. Epstein. Bounds on the unstable eigenvalue for period doubling. *Comm. Math. Phys.*, 128(2):427–435, 1990.

[25] Jean-Pierre Eckmann and Peter Wittwer. A complete proof of the Feigenbaum conjectures. *J. Statist. Phys.*, 46(3-4):455–475, 1987.

[26] H. Epstein. New proofs of the existence of the Feigenbaum functions. *Comm. Math. Phys.*, 106(3):395–426, 1986.

[27] Mitchell J. Feigenbaum. Quantitative universality for a class of nonlinear transformations. *J. Statist. Phys.*, 19(1):25–52, 1978.

[28] Mitchell J. Feigenbaum. The universal metric properties of nonlinear transformations. *J. Statist. Phys.*, 21(6):669–706, 1979.

[29] Rui Gao and Wei Xiao Shen. Decay of correlations for Fibonacci unimodal interval maps. *Acta Math. Sin. (Engl. Ser.)*, 34(1):114–138, 2018.

[30] John Guckenheimer. Sensitive dependence to initial conditions for one-dimensional maps. *Comm. Math. Phys.*, 70(2):133–160, 1979.

[31] John Guckenheimer and R. F. Williams. Structural stability of Lorenz attractors. *Inst. Hautes Études Sci. Publ. Math.*, (50):59–72, 1979.

[32] P. E. Hazard, M. Lyubich, and M. Martens. Renormalizable Hénon-like maps and unbounded geometry. *Nonlinearity*, 25(2):397–420, 2012.

[33] Franz Hofbauer. The topological entropy of the transformation $x \mapsto ax(1-x)$. *Monatsh. Math.*, 90(2):117–141, 1980.

[34] Franz Hofbauer and Gerhard Keller. Some remarks on recent results about S-unimodal maps. volume 53, pages 413–425. 1990. Hyperbolic behaviour of dynamical systems (Paris, 1990).

[35] J. H. Hubbard. Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz. In *Topological methods in modern mathematics (Stony Brook, NY, 1991)*, pages 467–511. Publish or Perish, Houston, TX, 1993.

[36] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[37] Gerhard Keller and Tomasz Nowicki. Fibonacci maps re(al)visited. *Ergodic Theory Dynam. Systems*, 15(1):99–120, 1995.

[38] Oleg Kozlovski and Sebastian van Strien. Asymmetric unimodal maps with non-universal period-doubling scaling laws. *Comm. Math. Phys.*, 379(1):103–143, 2020.

[39] Oscar E. Lanford, III. A computer-assisted proof of the Feigenbaum conjectures. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):427–434, 1982.

[40] Oscar E. Lanford, III. A shorter proof of the existence of the Feigenbaum fixed point. *Comm. Math. Phys.*, 96(4):521–538, 1984.

[41] François Ledrappier. Some properties of absolutely continuous invariant measures on an interval. *Ergodic Theory Dynam. Systems*, 1(1):77–93, 1981.

[42] Genadi Levin and Grzegorz Świątek. Common limits of Fibonacci circle maps. *Comm. Math. Phys.*, 312(3):695–734, 2012.

[43] T. Y. Li and James A. Yorke. Period three implies chaos. *Amer. Math. Monthly*, 82(10):985–992, 1975.

[44] Yuri Lima. Symbolic dynamics for one dimensional maps with nonuniform expansion. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 37(3):727–755, 2020.

[45] Stefano Luzzatto and Ian Melbourne. Statistical properties and decay of correlations for interval maps with critical points and singularities. *Comm. Math. Phys.*, 320(1):21–35, 2013.

[46] Stefano Luzzatto and Warwick Tucker. Non-uniformly expanding dynamics in maps with singularities and criticalities. *Inst. Hautes Études Sci. Publ. Math.*, (89):179–226 (2000), 1999.

[47] M. Lyubich and M. Martens. Renormalization of Hénon maps. In *Dynamics, games and science. I*, volume 1 of *Springer Proc. Math.*, pages 597–618. Springer, Heidelberg, 2011.

[48] Mikhail Lyubich. Feigenbaum-Coullet-Tresser universality and Milnor's hairiness conjecture. *Ann. of Math. (2)*, 149(2):319–420, 1999.

[49] Mikhail Lyubich and John Milnor. The Fibonacci unimodal map. *J. Amer. Math. Soc.*, 6(2):425–457, 1993.

[50] Marco Martens and Björn Winckler. Instability of renormalization, 2016.

[51] Curtis T. McMullen. *Renormalization and 3-manifolds which fiber over the circle*, volume 142 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1996.

[52] B. D. Mestel and A. H. Osbaldestin. Feigenbaum theory for unimodal maps with asymmetric critical point: rigorous results. *Comm. Math. Phys.*, 197(1):211–228, 1998.

[53] N. Metropolis, M. L. Stein, and P. R. Stein. On finite limit sets for transformations on the unit interval. *J. Combinatorial Theory Ser. A*, 15:25–44, 1973.

[54] N. Metropolis, M. L. Stein, and P. R. Stein. On finite limit sets for transformations on the unit interval. *J. Combinatorial Theory Ser. A*, 15:25–44, 1973.

[55] John Milnor. Local connectivity of Julia sets: expository lectures. In *The Mandelbrot set, theme and variations*, volume 274 of *London Math. Soc. Lecture Note Ser.*, pages 67–116. Cambridge Univ. Press, Cambridge, 2000.

[56] John Milnor and William Thurston. On iterated maps of the interval. In *Dynamical systems (College Park, MD, 1986–87)*, volume 1342 of *Lecture Notes in Math.*, pages 465–563. Springer, Berlin, 1988.

[57] M. Misiurewicz and W. Szlenk. Entropy of piecewise monotone mappings. *Studia Math.*,

67(1):45–63, 1980.

[58] MichałMisiurewicz. Structure of mappings of an interval with zero entropy. *Inst. Hautes Études Sci. Publ. Math.*, (53):5–16, 1981.

[59] Tomasz Nowicki and Sebastian van Strien. Invariant measures exist under a summability condition for unimodal maps. *Invent. Math.*, 105(1):123–136, 1991.

[60] Jorge Olivares-Vinales. Invariant measures for interval maps without lyapunov exponents. *Ergodic Theory and Dynamical Systems*, page 1–29, 2021.

[61] Jacob Palis, Jr. and Welington de Melo. *Geometric theory of dynamical systems.* Springer-Verlag, New York-Berlin, 1982. An introduction, Translated from the Portuguese by A. K. Manning.

[62] William Parry. Symbolic dynamics and transformations of the unit interval. *Trans. Amer. Math. Soc.*, 122:368–378, 1966.

[63] Fabiola Pedreira. *On the behaviour of the singular values of expanding Lorenz maps.* PhD thesis, Universidade Federal da Bahia, 2020.

[64] Feliks Przytycki. Lyapunov characteristic exponents are nonnegative. *Proc. Amer. Math. Soc.*, 119(1):309–317, 1993.

[65] Lasse Rempe-Gillen and Sebastian van Strien. Density of hyperbolicity for classes of real transcendental entire functions and circle maps. *Duke Math. J.*, 164(6):1079–1137, 2015.

[66] Juan Rivera-Letelier. Asymptotic expansion of smooth interval maps. *Astérisque*, (416, Quelques aspects de la théorie des systèmes dynamiques: un hommage à Jean-Christophe Yoccoz.II):33–63, 2020.

[67] David Singer. Stable orbits and bifurcation of maps of the interval. *SIAM J. Appl. Math.*, 35(2):260–267, 1978.

[68] Daniel Smania. Puzzle geometry and rigidity: the Fibonacci cycle is hyperbolic. *J. Amer. Math. Soc.*, 20(3):629–673, 2007.

[69] Dennis Sullivan. Bounds, quadratic differentials, and renormalization conjectures. In *American Mathematical Society centennial publications, Vol. II (Providence, RI, 1988)*, pages 417–466. Amer. Math. Soc., Providence, RI, 1992.

[70] Charles Tresser and Pierre Coullet. Itérations d'endomorphismes et groupe de renormalisation. *C. R. Acad. Sci. Paris Sér. A-B*, 287(7):A577–A580, 1978.

[71] Sebastian J. van Strien. On the bifurcations creating horseshoes. In *Dynamical systems and turbulence, Warwick 1980 (Coventry, 1979/1980)*, volume 898 of *Lecture Notes in Math.*, pages 316–351. Springer, Berlin-New York, 1981.

[72] Marcelo Viana and Krerley Oliveira. *Foundations of ergodic theory*, volume 151 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.

[73] O. M. Šarkovskiĭ. Co-existence of cycles of a continuous mapping of the line into itself. *Ukrain. Mat. Ž.*, 16:61–71, 1964.

[74] Peter Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[75] Björn Winckler. The lorenz renormalization conjecture, 2018.

[76] Lai Sang Young. On the prevalence of horseshoes. *Trans. Amer. Math. Soc.*, 263(1):75–88, 1981.