FEM MATLAB Code for Linear and Nonlinear Bending Analysis of Plates

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References:

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`http://www.mathworks.com/matlabcentral/fileexchange/32029-plate-bending`
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4] A. Pica, R.D. Wood, and E. Hinton, Finite Element Analysis of Geometrically Nonlinear Plate Behaviour using a Mindlin Formulation, Computers and Structures, 11, 203–215, 1980.
5] O.C. Zienkiewicz, R.L. Taylor and D.D. Fox, The Finite Element Method for Solid and Structural Mechanics, Elsvier Publications, 2014.
6] MATLAB R2012b, MathWorks Inc., 2015
7] COMSOL Multiphysics 5, COMSOL AB, 2015

Contribution

1] The linear MATLAB code is taken from first two references, our contribution is limited to few modifications.
2] The nonlinear FEM code is written by us.
3] The linear FEM theory is based on reference [2] and [5].
4] The nonlinear FEM theory is based on references [3], [4] and [5].
5] The verifications of results is done with help of FEM software, COMSOL Multiphysics.

Notes

1] This open-source code is for FEM and MATLAB beginners. So if you find any mistake, please write us an email.
2] Only brief introduction of theory is given here; for detail theory please read books on Mechanics of Solids, Finite Element Methods.
3] While writing theory it is assumed that student have some basic knowledge of symbols, notations, nomenclatures and definitions etc.

URL for downloading FEM MATLAB code files

Download all MATLAB and COMSOL Multiphysics files from the URL.
FEM MATLAB Code for Linear and Nonlinear Bending Analysis of Plates
http://www.mathworks.com/matlabcentral/fileexchange/54226-fem-matlab-code-for-linear-and-nonlinear-bending-analysis-of-plates

Part 1: Linear Bending Analysis of Plates

1. Theory and assumptions

1. Only linear static analysis is considered. Material is considered to homogeneous, isotropic and linear elastic (Material linearity). Dynamical effects are neglected.
2. Deformations are small enough to use an infinitesimal deformation theory (Geometric linearity).
3. The Mindlin plate theory is used, Mindlin plate theory includes transverse shear deformations.

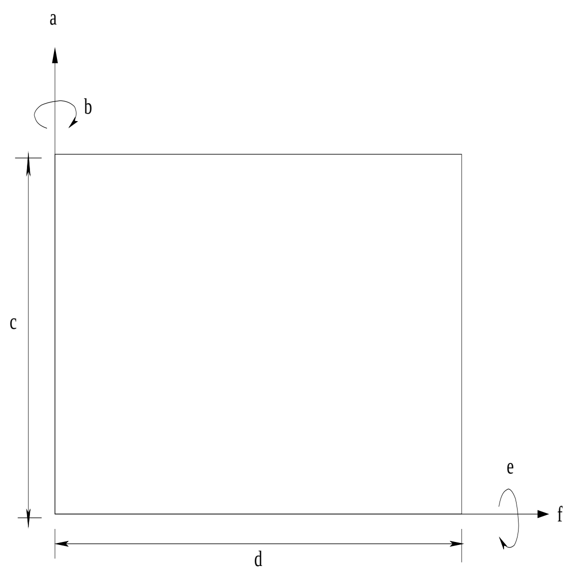


Figure 1: Deformation of Mindlin Plate

In linear Mindlin plate theory, the displacements $\{u, v, w\}$ at (x, y, z) are expressed as function of mid-plane deformations $\{w, \theta_x, \theta_y\}$, where w is transverse deformation, θ_x and θ_y are rotations of normal. These normals are remains straight after deformation but need not remain perpendicular to mid-plane, this allows shear deformation in the plates (thick plates particularly).
The displacement field of plate is

$$a = \{w \quad \theta_x \quad \theta_y\}^T \tag{1}$$

The total potential energy U of the plate can be written as

$$U = \frac{1}{2}\int\limits_V \sigma_b \epsilon_b dV + \frac{\alpha}{2}\int\limits_V \sigma_s \epsilon_s dV \tag{2}$$

where subscript b is related to bending and subscript s is related to shear part. Here, α is a shear correction factor and equal to 5/6. The bending stress and strain are written as

$$
\begin{aligned}
\{\sigma_b\} &= \{\sigma_x \quad \sigma_y \quad \tau_{xy}\}^T \\
\{\epsilon_b\} &= \{\epsilon_x \quad \epsilon_y \quad \gamma_{xy}\}^T
\end{aligned} \tag{3}
$$

The transverse shear stress and strain are given by

$$
\begin{aligned}
\{\sigma_s\} &= \{\tau_{xz} \quad \tau_{yz}\}^T \\
\{\epsilon_s\} &= \{\gamma_{xz} \quad \gamma_{yz}\}^T
\end{aligned} \tag{4}
$$

Remember that γ_{xy} is engineering component of shear strain, the tensor component of shear strain is ϵ_{xy} and $\epsilon_{xy} = \gamma_{xy}/2$.
The assumed displacement field for Mindlin plate is u (along x axis), v (along y axis) and w (along z axis) written as,

$$
\begin{aligned}
u &= -z\theta_x \\
v &= -z\theta_y \\
w &= w
\end{aligned} \tag{5}
$$

The kinematic equations are written as (Linear strain-displacement relation)

$$
\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} = -z\frac{\partial \theta_x}{\partial x} \\
\epsilon_y &= \frac{\partial v}{\partial y} = -z\frac{\partial \theta_y}{\partial y} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) \\
\gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} - \theta_x \\
\gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y} - \theta_y
\end{aligned} \tag{6}
$$

The linear stress-strain relationship is

$$
\sigma_b = D_b\epsilon_b, \tag{7}
$$

$$
\text{where } D_b = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}
$$

$$
\sigma_s = D_s\epsilon_s, \tag{8}
$$

$$
\text{where } D_s = \frac{E}{2(1+\nu)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

2. Finite element discretization

Quadrilateral (Four node Q4) isoparametric element is used for discretization.The displacement within element domain is given by

$$w = \sum_{i=1}^{4} N_i(\xi,\eta) w_i$$
$$\theta_x = \sum_{i=1}^{4} N_i(\xi,\eta)(\theta_x)_i \tag{9}$$
$$\theta_y = \sum_{i=1}^{4} N_i(\xi,\eta)(\theta_y)_i$$

The coordinates x and y within element can be obtained by the same shape functions

$$x = \sum_{i=1}^{4} N_i(\xi,\eta) x_i$$
$$y = \sum_{i=1}^{4} N_i(\xi,\eta) y_i \tag{10}$$

where $N_i(\xi,\eta)$ are shape functions for four noded Q4 element and ξ and η are natural coordinates.

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

3. Stiffness and force matrix formulation for each element

The displacement vector a for each element is given by

$$a^e = \{w_1 \ (\theta_x)_1 \ (\theta_y)_1 \ w_2 \ (\theta_x)_2 \ (\theta_y)_2 \ w_3 \ (\theta_x)_3 \ (\theta_y)_3 \ w_4 \ (\theta_x)_4 \ (\theta_y)_4\}^T \tag{11}$$

The potential energy can be written as

$$U^e = \frac{1}{2}\int_{-h/2}^{h/2} z^2 \int_A (a^e)^T (B_b^e)^T D_b^e B_b^e a^e dAdz + \frac{\alpha}{2}\int_{-h/2}^{h/2}\int_A (a^e)^T (B_s^e)^T D_s^e B_s^e a^e dAdz \tag{12}$$

The element stiffness matrix is written as

$$K^e = \frac{h^3}{12}\int\limits_A (B_b^e)^T D_b^e B_b^e dA + \alpha h \int\limits_A (B_s^e)^T D_s^e B_s^e dA \tag{13}$$

converting above equations into natural coordinates,

$$K^e = \frac{h^3}{12}\int\limits_{-1}^{1}\int\limits_{-1}^{1} (B_b^e)^T D_b^e B_b^e \left|J\right| d\xi d\eta + \alpha h \int\limits_{-1}^{1}\int\limits_{-1}^{1} (B_s^e)^T D_s^e B_s^e \left|J\right| d\xi d\eta \tag{14}$$

where J is Jacobian and written as

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{bmatrix}$$

The kinematic matrix B_b^e and B_s^e for bending and shear can be written as

$$B_b^e = \begin{bmatrix} 0 & -\frac{\partial N_1}{\partial x} & 0 & 0 & -\frac{\partial N_2}{\partial x} & 0 & 0 & -\frac{\partial N_3}{\partial x} & 0 & 0 & -\frac{\partial N_4}{\partial x} & 0 \\ 0 & 0 & -\frac{\partial N_1}{\partial y} & 0 & 0 & -\frac{\partial N_2}{\partial y} & 0 & 0 & -\frac{\partial N_3}{\partial y} & 0 & 0 & -\frac{\partial N_4}{\partial y} \\ 0 & -\frac{\partial N_1}{\partial y} & -\frac{\partial N_1}{\partial x} & 0 & -\frac{\partial N_2}{\partial y} & -\frac{\partial N_2}{\partial x} & 0 & -\frac{\partial N_3}{\partial y} & -\frac{\partial N_3}{\partial x} & 0 & -\frac{\partial N_4}{\partial x} & -\frac{\partial N_4}{\partial x} \end{bmatrix}$$

$$B_s^e = \begin{bmatrix} \frac{\partial N_1}{\partial x} & -N_1 & 0 & \frac{\partial N_2}{\partial x} & -N_2 & 0 & \frac{\partial N_3}{\partial x} & -N_3 & 0 & \frac{\partial N_4}{\partial x} & -N_4 & 0 \\ \frac{\partial N_1}{\partial y} & 0 & -N_1 & \frac{\partial N_2}{\partial y} & 0 & -N_2 & \frac{\partial N_3}{\partial y} & 0 & -N_3 & \frac{\partial N_4}{\partial y} & 0 & -N_4 \end{bmatrix}$$

The element stiffness matrix found by numerical integration of above shown equations. The Guass-quadrature integration rule is used for numerical integration. For bending part in stiffness matrix, 2×2 Guass-quadrature integration used and for shear part, 1×1 Guass-quadrature integration is used (This method is called as lower order integration for shear part to avoid shear locking). The Guass-quadrature integration is summarized bellow

$$\int\limits_{-1}^{1}\int\limits_{-1}^{1} f(\xi, \eta) d\xi d\eta = \sum_i \sum_j f(\xi_i, \eta_j) W_{ij} \tag{15}$$

where $f(\xi_i, \eta_j)$ function values at sample points and W_{ij} are weightage for sample points. The pressure work is written as

$$W^e = \int\limits_A P w^e dA \tag{16}$$

The force vector is written as

$$F^e = \int\limits_{-1}^{1}\int\limits_{-1}^{1} P N \left|J\right| d\xi d\eta \tag{17}$$

where N is vector of shape functions.

4. Assemblage of element stiffness matrix and force vector into global level

Now each element stiffness and force matrix is added to the appropriate location (based on element position, nodal connectivity) to the global stiffness and force matrix. This process is called as assembly.

$$\sum_e K^e a^e = \sum_e F^e \tag{18}$$

$$K\ a = F \tag{19}$$

5. Numerical solution

The DOF related to boundary conditions are identified and equation $K\ a = F$ is solved by Newton-Raphson method for active DOF only. Remember that $K\ a = F$ is algebraic equation. The basic of FEM is to convert weak form of equilibrium equations of system into set of algebraic equations, and solve it by Newton-Raphson or Modified-Newton methods.

6. Results and verification

Here we will show results of simply supported and clamped plate, using a 20×20 Q4 mesh. The following non-dimensional parameter are chosen for the study. For square plate; length $a = 1$, breadth $b = 1$, thickness $h = 0.1$, Young Modulus $E = 10920$, Poisson's ratio $\nu = 0.3$. For rectangular plate; only length is taken as $a = 1.5$ by keeping all other parameters same. For verifications of the results, COMSOL Multiphysics software used.

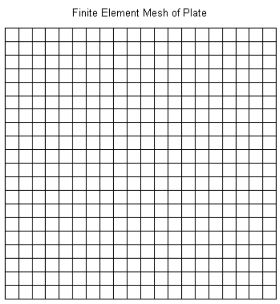
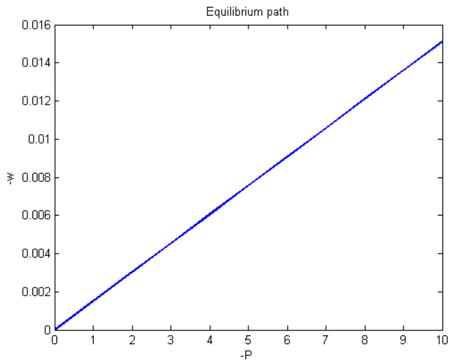
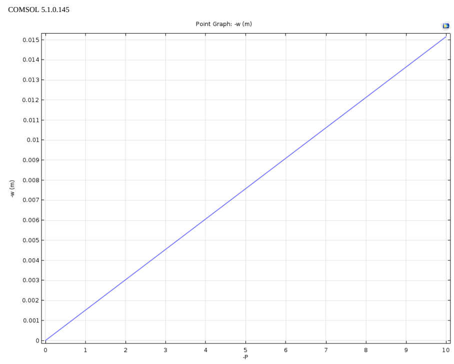


Figure 2: 20×20 meshing of square plate

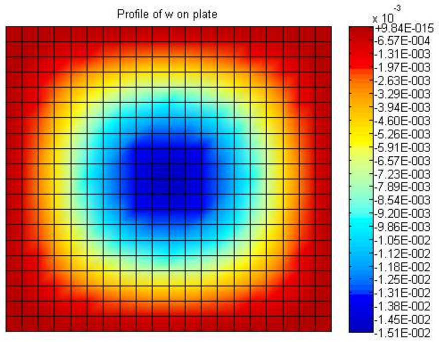


(a) MATLAB Results

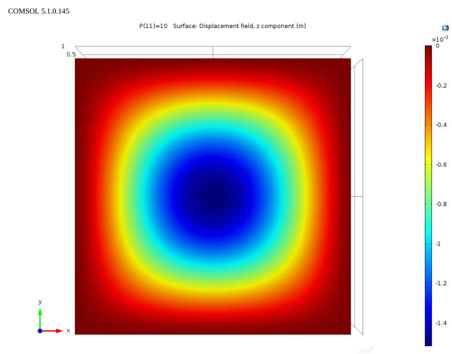


(b) COMSOL Results

Figure 3: Variation of maximum transverse displacement $-w$ with pressure $-P$ for clammed square plate

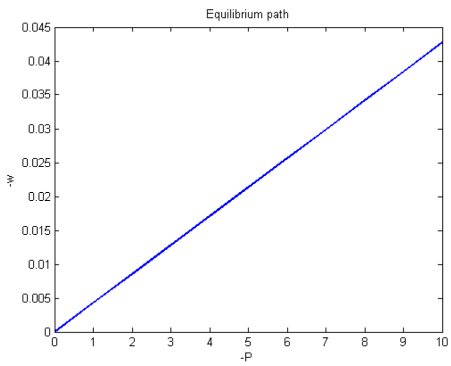


(a) MATLAB Results

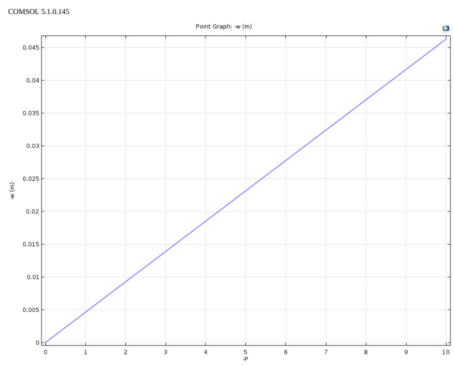


(b) COMSOL Results

Figure 4: Counter plot of w at pressure $P = -10$ for clamped square plate

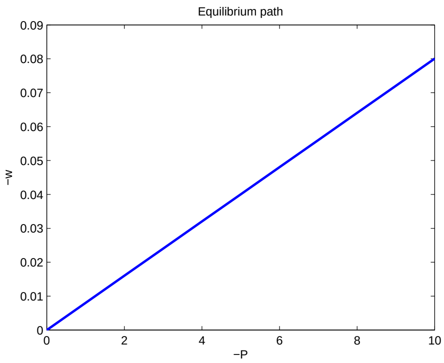


(a) MATLAB Results

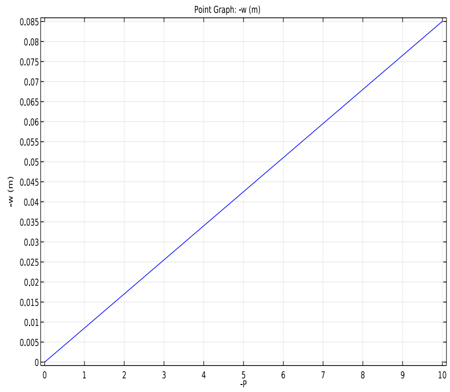


(b) COMSOL Results

Figure 5: Variation of maximum transverse displacement $-w$ with pressure $-P$ for simply supported square plate



(a) MATLAB Results



(b) COMSOL Results

Figure 6: Variation of maximum transverse displacement $-w$ with pressure $-P$ for simply supported rectangular plate

Part 2: Non-linear Bending Analysis of Plates

7. Theory and assumptions

1. The non-linear static analysis is considered. Only geometric non-linearity is considered. Material is considered to homogeneous, isotropic and linear elastic. Dynamical effects are neglected.
2. Deformations are large to use finite deformation theory (Geometric non-linearity).
3. The Mindlin plate theory is used, Mindlin plate theory includes transverse shear deformations.

In nonlinear Mindlin plate theory, the displacements $\{\bar{u}, \bar{v}, \bar{w}\}$ at (x, y, z) are expressed as function of mid-plane translations u, v, w and mid-plane normal rotations θ_x, θ_y.So,

$$\begin{aligned} \bar{u} &= u - z\theta_x \\ \bar{v} &= v - z\theta_y \\ \bar{w} &= w \end{aligned} \tag{20}$$

where u and v are inplane displacement, w is out of plane displacement, θ_x and θ_y are rotation of transverse normals about Z axis. The displacement field of plate is taken as

$$a = \{u \quad v \quad w \quad \theta_x \quad \theta_y\}^T \tag{21}$$

For nonlinear problems, the virtual work principal is written as

$$\int_A d\epsilon^T \hat{\sigma} dA = \int_A da^T P dA \tag{22}$$

where ϵ, $\hat{\sigma}$ a and P are strains, stress resultants, displacements and external force vectors respectively.
For a Mindlin plate, the strain vector is written as

$$\epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \frac{\partial \bar{u}}{\partial x} + \frac{1}{2}\left(\frac{\partial \bar{w}}{\partial x}\right)^2 \\ \frac{\partial \bar{v}}{\partial y} + \frac{1}{2}\left(\frac{\partial \bar{w}}{\partial y}\right)^2 \\ \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{w}}{\partial x}\frac{\partial \bar{w}}{\partial y} \\ \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x} \\ \frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial y} \end{bmatrix} \tag{23}$$

$$\epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \epsilon_i^0 \\ 0 \end{bmatrix} + \begin{bmatrix} z\epsilon_b^0 \\ \epsilon_s^0 \end{bmatrix} + \begin{bmatrix} \epsilon_i^L \\ 0 \end{bmatrix} \tag{24}$$

where the linear inplane strains are

$$\epsilon_i^0 = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} \tag{25}$$

the linear bending strains are

$$\epsilon_b^0 = \begin{bmatrix} -\frac{\partial \theta_x}{\partial x} \\ -\frac{\partial \theta_y}{\partial y} \\ -\frac{\partial \theta_x}{\partial y} - \frac{\partial \theta_y}{\partial x} \end{bmatrix} \tag{26}$$

the shear strains are

$$\epsilon_s^0 = \begin{bmatrix} \frac{\partial w}{\partial x} - \theta_x \\ \frac{\partial w}{\partial y} - \theta_y \end{bmatrix} \tag{27}$$

and the nonlinear inplane strains are

$$\epsilon_i^L = \begin{bmatrix} \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \left(\frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y}\right) \end{bmatrix} \tag{28}$$

For a Mindlin plate the stress resultant vector is written as

$$\hat{\sigma} = \begin{bmatrix} \hat{\sigma}_i \\ \hat{\sigma}_b \\ \hat{\sigma}_s \end{bmatrix} \tag{29}$$

which contains following components like, inplane stress resultants are

$$\hat{\sigma}_i = [N_x, N_y, N_{xy}]^T = \left[\int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy}) dz \right]^T \tag{30}$$

bending stress resultants are

$$\hat{\sigma}_b = [M_x, M_y, M_{xy}]^T = \left[\int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy}) z dz \right]^T \tag{31}$$

shear stress resultants

$$\hat{\sigma}_s = [Q_x, Q_y]^T = \left[\int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) dz \right]^T \tag{32}$$

The virtual work is re-written as

$$
\begin{aligned}
\int_A d\epsilon^T \hat{\sigma} dA &= \int_A da^T P dA \\
\int_A d\epsilon_{ib}^T \hat{\sigma}_{ib} dA + \int_A d\epsilon_s^T \hat{\sigma}_s dA &= \int_A da^T P dA
\end{aligned}
\tag{33}
$$

where ϵ_{ib} and ϵ_s are inplane-bending and shear strains , σ_{ib} and σ_s are inplane-bending and shear stress resultants. The stress (stress resultants)-strain relationship for combined inplane and bending action is

$$
\hat{\sigma}_{ib} = D_{ib}\epsilon_{ib},
$$

$$
\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} \frac{Eh}{(1-\nu^2)} & \frac{Eh\nu}{(1-\nu^2)} & 0 & 0 & 0 & 0 \\ \frac{Eh\nu}{(1-\nu^2)} & \frac{Eh}{(1-\nu^2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{Eh}{2(1+\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{Eh^3}{12(1-\nu^2)} & \frac{Eh^3\nu}{12(1-\nu^2)} & 0 \\ 0 & 0 & 0 & \frac{Eh^3\nu}{12(1-\nu^2)} & \frac{Eh^3}{12(1-\nu^2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{Eh^3}{24(1+\nu)} \end{bmatrix} \begin{bmatrix} \epsilon_{xi} \\ \epsilon_{yi} \\ \gamma_{xyi} \\ \epsilon_{xb} \\ \epsilon_{yb} \\ \gamma_{xyb} \end{bmatrix} \tag{34}
$$

The stress-strain relationship for shear action is

$$
\hat{\sigma_s} = D_s \epsilon_s,
$$

$$
\begin{bmatrix} Q_{xz} \\ Q_{yz} \end{bmatrix} = \begin{bmatrix} \frac{E\alpha}{2(1+\nu)} & 0 \\ 0 & \frac{E\alpha}{2(1+\nu)} \end{bmatrix} \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \tag{35}
$$

where α is shear correction factor,$\alpha = 5/6$.

8. Finite element discretization

Quadrilateral (Four node Q4) isoparametric element is used for discretization.The displacement within element domain is given by

$$
\begin{aligned}
u &= \sum_{i=1}^{4} N_i(\xi,\eta)u_i \\
v &= \sum_{i=1}^{4} N_i(\xi,\eta)v_i \\
w &= \sum_{i=1}^{4} N_i(\xi,\eta)w_i \\
\theta_x &= \sum_{i=1}^{4} N_i(\xi,\eta)(\theta_x)_i \\
\theta_y &= \sum_{i=1}^{4} N_i(\xi,\eta)(\theta_y)_i
\end{aligned}
\tag{36}
$$

The coordinates x and y within element can be obtained by the same shape functions

$$
\begin{aligned}
x &= \sum_{i=1}^{4} N_i(\xi,\eta)x_i \\
y &= \sum_{i=1}^{4} N_i(\xi,\eta)y_i
\end{aligned}
\tag{37}
$$

where $N_i(\xi,\eta)$ are shape functions for four noded Q4 element and ξ and η are natural coordinates.

$$
\begin{aligned}
N_1 &= \frac{1}{4}(1-\xi)(1-\eta) \\
N_2 &= \frac{1}{4}(1+\xi)(1-\eta) \\
N_3 &= \frac{1}{4}(1+\xi)(1+\eta) \\
N_4 &= \frac{1}{4}(1-\xi)(1+\eta)
\end{aligned}
$$

9. Stiffness and force matrix formulation for each element

The displacement vector a for each element is given by

$$
a^e = \{u_1 \;\; v_1 \;\; w_1 \;\; (\theta_x)_1 \;\; (\theta_y)_1 \;\; u_2 \;\; v_2 \;\; w_2 \;\; (\theta_x)_2 \;\; (\theta_y)_2 \;\; u_3 \;\; v_3 \;\; w_3 \;\; (\theta_x)_3 \;\; (\theta_y)_3 \;\; u_4 \;\; v_4 \;\; w_4 \;\; (\theta_x)_4 \;\; (\theta_y)_4\}^T \tag{38}
$$

henceforth to avoid confusions of subscripts and superscripts, we drop superscript e from all terms derived for a single element. It is assumed that all terms written in this particular section is only for a single element.
The first variation of potential energy is written as

$$\begin{aligned} \delta\Pi &= \delta U - \delta W = 0 \\ \delta U &= \delta W \end{aligned} \tag{39}$$

The variation of strain energy of the plate is written as summation of inplane-bending strain energy and transverse shear strain energy

$$\delta U = \int_A d\epsilon_{ib}^T \hat{\sigma}_{ib} dA + \int_A d\epsilon_s^T \hat{\sigma}_s dA \tag{40}$$

where $\epsilon_{ib} = B_{ib} a^e$ and $\epsilon_s = B_s^0 a^e$.

$$\epsilon_{ib} = (B_{ib}^0 + \frac{1}{2} B_i^L) a, \qquad \hat{\sigma}_{ib} = D_{ib} \epsilon_{ib} \tag{41}$$

$$d\epsilon_{ib} = (B_{ib}^0 + B_i^L) da, \qquad d\epsilon_{ib} = B_{ib} da \tag{42}$$

$$\epsilon_s = B_s^0 a, \qquad \hat{\sigma}_s = D_s \epsilon_s \tag{43}$$

$$d\epsilon_s = B_s^0 da \tag{44}$$

where B_{ib}^0 is linear inplane-bending matrix, same as in infinitesimal theory, B_i^L is a non-linear matrix depends on displacement a. The B_s^0 is a linear matrix for shear action and same as infinitesimal theory.
For non-linear system, the element equilibrium equation is

$$R = \int_A B_{ib}^T \hat{\sigma}_{ib} dA + \int_A B_s^T \hat{\sigma}_s dA - F = 0 \tag{45}$$

where R is residual and F is a generalized forces comes from variation of external work done.
The element stiffness matrix K_S is written as

$$K_S = \int_A ((B_{ib}^0)^T D_{ib} B_{ib}^0 + 1/2 (B_{ib}^0)^T D_{ib} B_i^L + (B_i^L)^T D_{ib} B_{ib}^0 + 1/2 (B_i^L)^T D_{ib} B_i^L) dA + \int_A (B_s^0)^T D_s B_s^0 dA \tag{46}$$

The solution algorithm for the assembled nonlinear equilibrium equations (45) is based on Newton-Raphson method which need linearization of equations at equilibrium point. The Taylor series expansion of residual $R(a_{i+1})$ in the neighbourhood of a_i is

$$R(a_{i+1}) \approx R(a_i) + K_T \Delta a = 0 \tag{47}$$

where K_T is assembled tangent stiffness matrix and given by

$$K_T = \frac{\partial R}{\partial a} \tag{48}$$

$$K_T = \int_A (B_{ib})^T d\hat{\sigma_{ib}} + (dB_{ib})^T \hat{\sigma}_{ib} dA + \int_A (B_s^0)^T d\hat{\sigma_s} + (dB_s^0)^T \hat{\sigma}_s dA \tag{49}$$

here $d\hat{\sigma_{ib}} = D_{ib} d\epsilon_{ib} = D_{ib}(B_{ib}^0 + B_i^L)da$ and $dB_s^0 = 0$. So tangent stiffness matrix K_T is written as

$$K_T = \int_A ((B_{ib}^0)^T D_{ib} B_{ib}^0 + (B_{ib}^0)^T D_{ib} B_i^L + (B_i^L)^T D_{ib} B_{ib}^0 + (B_i^L)^T D_{ib} B_i^L) dA + \int_A (dB_{ib})^T \hat{\sigma}_{ib} dA \tag{50}$$

$$+ \int_A (B_s^0)^T D_s B_s^0 dA$$

$$K_T = K_0 + K_L + K_\sigma + K_s \tag{51}$$

The linear inplane-bending stiffness matrix is given by

$$K_0 = \int_A (B_{ib}^0)^T D_{ib} B_{ib}^0 dA \tag{52}$$

The nonlinear inplane-bending stiffness matrix can be written as

$$K_L = \int_A ((B_{ib}^0)^T D_{ib} B_i^L + (B_i^L)^T D_{ib} B_{ib}^0 + (B_i^L)^T D_{ib} B_i^L) dA \tag{53}$$

The linear shear stiffness matrix is given by

$$K_s = \int_A (B_s^0)^T D_s B_s^0 dA \tag{54}$$

The initial stress matrix or geometric matrix is written as

$$\begin{aligned} K_\sigma &= \int_A (dB_{ib})^T \hat{\sigma}_{ib} dA \\ &= \int_A (dB_i^L)^T \hat{\sigma}_{ib} dA \\ &= \int_A G^T dC^T \hat{\sigma}_{ib} dA \\ &= \int_A G^T S G dA \end{aligned} \tag{55}$$

The mathematical property of matrix C insures that $dC^T \hat{\sigma}_{ib} = SG$.

$$S = \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix}_{2\times 2}$$

$$G = \begin{bmatrix} 0 & 0 & \frac{\partial N_1}{\partial x} & 0 & 0 & - - - - - - - - - - - - - - - - - - 0 & 0 & \frac{\partial N_4}{\partial x} & 0 & 0 \\ 0 & 0 & \frac{\partial N_1}{\partial y} & 0 & 0 & - - - - - - - - - - - - - - - - - - 0 & 0 & \frac{\partial N_4}{\partial y} & 0 & 0 \end{bmatrix}_{2\times 20}$$

The inplane and bending strain can be written as

$$\{\epsilon\} = \{\epsilon_{ib}^0\} + \{\epsilon_i^L\}$$

$$\{\epsilon\} = \begin{bmatrix} \frac{\partial u}{\partial x} + \frac{\partial \theta_x}{\partial x} \\ \frac{\partial v}{\partial y} + \frac{\partial \theta_y}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \left(\frac{\partial w}{\partial x}\right)^2 \\ \left(\frac{\partial w}{\partial y}\right)^2 \\ 2\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \end{bmatrix}$$

$$d\epsilon = (B_{ib}^0 + B_i^L)da$$

$$\epsilon_i^L = \frac{1}{2} \begin{bmatrix} \frac{\partial w}{\partial x} & 0 \\ 0 & \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix}$$

$$\epsilon_i^L = \frac{1}{2} C\theta$$

$$d\epsilon_i^L = \frac{1}{2}(dC\theta + Cd\theta)$$

$$d\epsilon_i^L = CGda$$

$$B_i^L = CG$$

here G is derivative of shape functions.
The kinematic stiffness matrix B_{ib}^0 can be written as

$$B_{ib}^0 = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & 0 & 0 & 0 & - - - - - - - - - - - - & \frac{\partial N_4}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & 0 & 0 & - - - - - - - - - - - - & 0 & \frac{\partial N_4}{\partial y} & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 & 0 & 0 & - - - - - - - - - - - - & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\partial N_1}{\partial x} & 0 & - - - - - - - - - - - - - - & 0 & 0 & 0 & -\frac{\partial N_4}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & -\frac{\partial N_1}{\partial y} & - - - - - - - - - - - - - - & 0 & 0 & 0 & 0 & -\frac{\partial N_4}{\partial y} \\ 0 & 0 & 0 & -\frac{\partial N_1}{\partial y} & -\frac{\partial N_1}{\partial x} & - - - - - - - - - - - - & 0 & 0 & 0 & -\frac{\partial N_4}{\partial y} & -\frac{\partial N_4}{\partial x} \end{bmatrix}_{6\times 20}$$

The kinematic stiffness matrix B_i^L can be written as

$$B_i^L = \begin{bmatrix} 0 & 0 & \frac{\partial w}{\partial x}\frac{\partial N_1}{\partial x} & 0 & 0 & - - - - - - - - - - - - - - 0 & 0 & \frac{\partial w}{\partial x}\frac{\partial N_4}{\partial x} & 0 & 0 \\ 0 & 0 & \frac{\partial w}{\partial y}\frac{\partial N_1}{\partial y} & 0 & 0 & - - - - - - - - - - - - - - 0 & 0 & \frac{\partial w}{\partial x}\frac{\partial N_4}{\partial y} & 0 & 0 \\ 0 & 0 & \frac{\partial w}{\partial y}\frac{\partial N_1}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial N_1}{\partial y} & 0 & 0 & - - - - - - - - - - - - - - 0 & 0 & \frac{\partial w}{\partial y}\frac{\partial N_4}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial N_4}{\partial y} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & - - - - - - - - - - - - - - 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & - - - - - - - - - - - - - - 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & - - - - - - - - - - - - - - 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 20}$$

The kinematic stiffness matrix B_s^0 can be written as

$$B_s^0 = \begin{bmatrix} 0 & 0 & \frac{\partial N_1}{\partial x} & -N_1 & 0 & ----------------- 0 & 0 & \frac{\partial N_4}{\partial x} & -N_4 & 0 \\ 0 & 0 & \frac{\partial N_1}{\partial y} & 0 & -N_1 & ----------------- 0 & 0 & \frac{\partial N_4}{\partial y} & 0 & -N_4 \end{bmatrix}_{2\times 20}$$

converting above all equations into natural coordinates as,

$$\int_A dA = \int_{-1}^{1}\int_{-1}^{1} |J|\, d\xi d\eta \tag{56}$$

where J is Jacobian and written as

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{bmatrix}$$

The pressure work is written as

$$W = \int_A PwdA \tag{57}$$

The force vector is written as

$$F = \int_{-1}^{1}\int_{-1}^{1} PN\, |J|\, d\xi d\eta \tag{58}$$

where N is vector of shape functions.

$$N = \begin{bmatrix} 0 & 0 & N_1 & 0 & 0 & ----------------- 0 & 0 & N_4 & 0 & 0 \end{bmatrix}_{1\times 20}$$

The Guass-quadrature integration rule is used for numerical integration. For linear/nonlinear inplane-bending stiffness matrix, initial stress stiffness matrix and force vector 2×2 Guass-quadrature integration used and for shear part 1×1 Guass-quadrature integration is used (This method is called as lower order integration for shear part to avoid shear locking).

10. Assemblage of element stiffness matrix and force vector into global level

Now each element stiffness and force matrix is added to the appropriate location (based on element position, nodal connectivity) of the overall, or global stiffness and force matrix. This process is called assembly.

11. Newton-Raphson method

The DOF related to boundary conditions are identified and equations are solved by Newton-Raphson method for active DOF only. For linear problems we solve algebraic equations of form $Ka = F$; but for non-linear systems equations need to be linearized around equilibrium point and solution must be sought by iterative procedure. For non-linear system either force or displacement can be a controlling parameter. To circumvent limit points in non-linear equilibrium paths, displacement control is used, but here chances of encountering limit point are slim, so we are using force as controlling parameter. Here linearized equations are of the form

$$K_T da^i_{n+1} = R^i_{n+1}$$

where R is residual , and i denotes iteration number. The residual can be written as

$$R = K_S\ a - F_{n+1}$$

where F and $K_S\ a$ are external and internal forces respectively. Note that system stiffness matrix K_S and tangent stiffness matrix K_T are not same. The increment in displacement is written as

$$\begin{aligned} R(a^{i+1}_{n+1}) &\approx R(a^i_{n+1}) + \frac{\partial R^i}{\partial a}_{n+1} da^i_{n+1} = 0 \\ K_T &= \frac{\partial R}{\partial a} \\ K^i_T da^i_{n+1} &= -R^i_{n+1} \\ da^i_{n+1} &= -(K^i_T)^{-1} R^i_{n+1} \end{aligned}$$

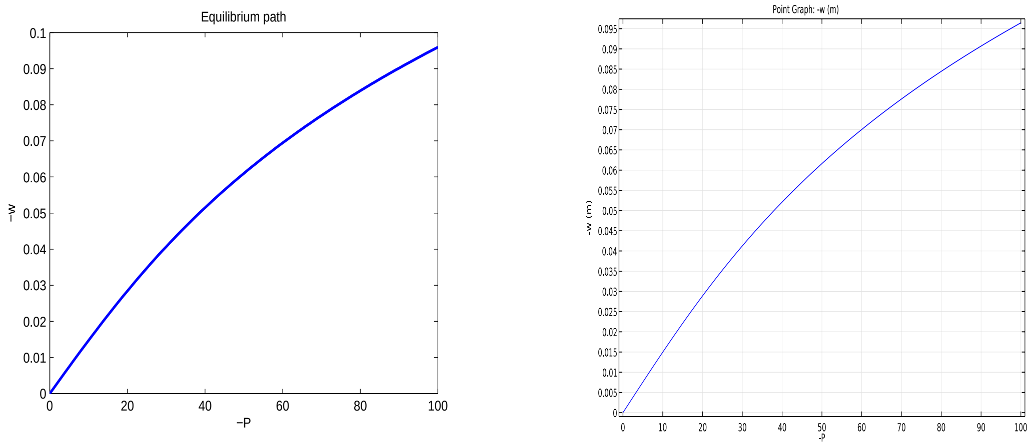
The series of successive approximation gives

$$a^{i+1}_{n+1} = a^i_{n+1} + da^i_{n+1}$$

where $n + 1$ is a equilibrium solution number and i is a Newton iteration number at specific equilibrium point. The Newton iterations at each force level is continued until residual becomes less than tolerance level. (approximately zero,say 10^{-4}).

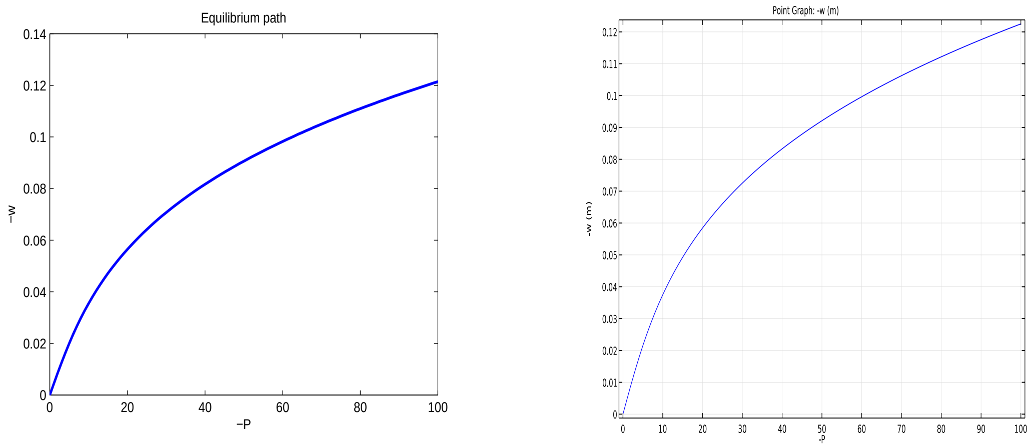
12. Results and verification

Here we will show results of simply supported and clamped plate, using a 15×15 Q4 mesh. The following non-dimensional parameter are chosen for the study. For square plate; length $a = 1$, breadth $b = 1$, thickness $h = 0.1$ (thick plate) and $h = 0.01$ (thin plate), Young Modulus $E = 10920$, Poisson's ratio $\nu = 0.3$. For rectangular plate; only length is taken as $a = 1.5$ by keeping all other parameters same. For verifications of the results COMSOL Multiphysics software is used.



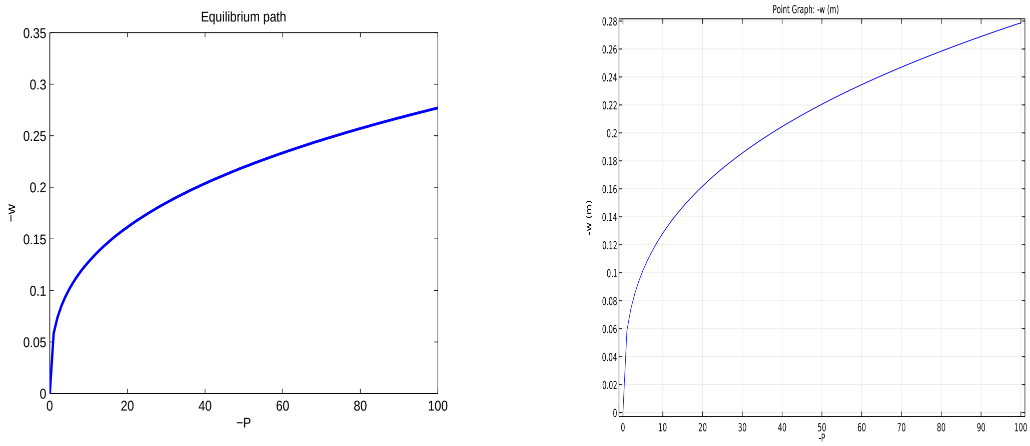
(a) MATLAB Results (b) COMSOL Results

Figure 7: Variation of maximum transverse displacement $-w$ with pressure $-P$ for clammed square plate with thickness $h = 0.1$



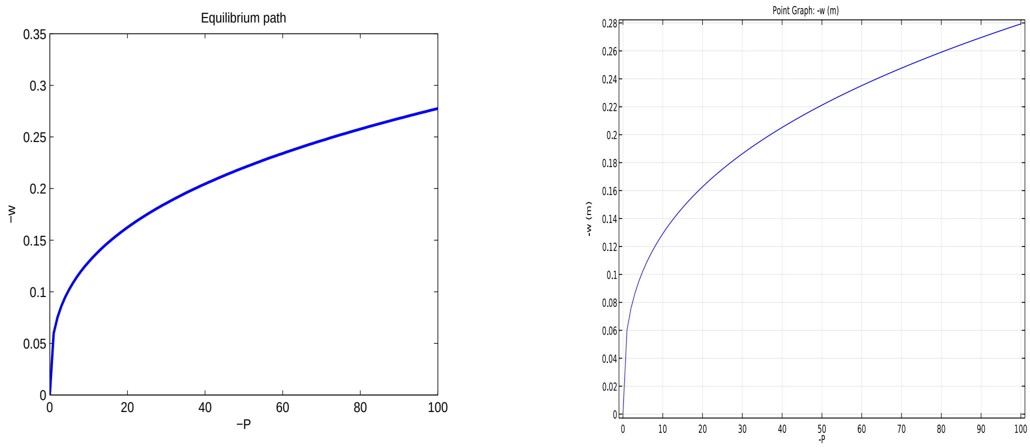
(a) MATLAB Results (b) COMSOL Results

Figure 8: Variation of maximum transverse displacement $-w$ with pressure $-P$ for simply supported square plate with thickness $h = 0.1$



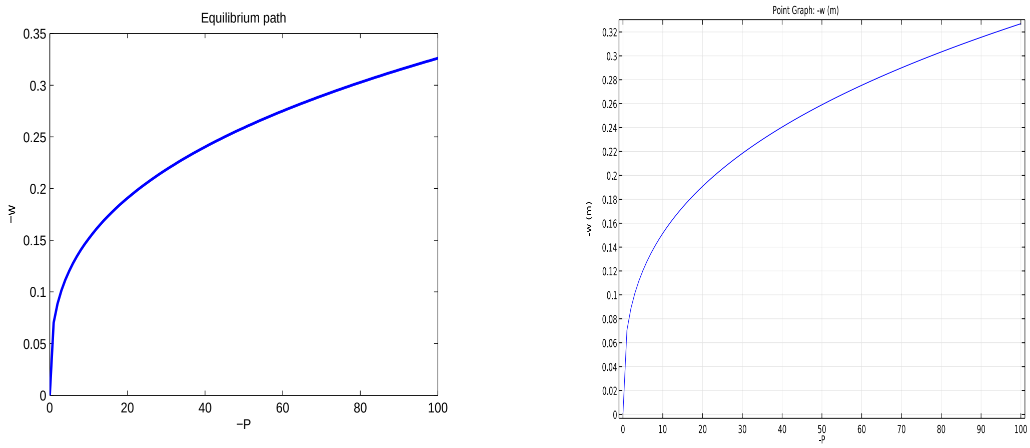
(a) MATLAB Results (b) COMSOL Results

Figure 9: Variation of maximum transverse displacement $-w$ with pressure $-P$ for clammed square plate with thickness $h = 0.01$



(a) MATLAB Results (b) COMSOL Results

Figure 10: Variation of maximum transverse displacement $-w$ with pressure $-P$ for simply supported square plate with thickness $h = 0.01$



(a) MATLAB Results

(b) COMSOL Results

Figure 11: Variation of maximum transverse displacement $-w$ with pressure $-P$ for simply supported rectangular plate with thickness $h = 0.01$

13. Membrane and shear locking

Membrane locking: Membrane locking occur in curved shell elements. The term describes a stiffening effect that occurs from pure bending deformations which are accompanied by membrane stresses. As membrane locking is associated with the curvature of a structure it occurs if the elements are actually curved. The inplane deformations energy is proportional to thickness but bending energy is proportional to cube of thickness (h^3) and so when thickness becomes very small, the bending energy contributes less compare to inplane energy, which is not true and lead towards wrong results.

Shear locking: Shear locking can occur in beam, plate, and shell elements. From a mathematical point of view, there is no ill-conditioning in corresponding equilibrium equations. The important parameter in the case of shear locking is the thickness of the element. This can be understood most easily with the help of following example. If you consider a plate under combined shear and bending deformations, the portion of the internal energy associated with the bending deformation is proportional to cube of thickness (h^3), but the portion related to shear deformations is proportional with thickness. Now if we reduce the thickness , the value of h^3 approaches zero much faster than h and as a result, all the strain energy of the plate will come from shear deformation. This is not correct because for a thin plate, the bending deformations provides most part of the energy. So the shear deformation beam can lead to wrong results for very thin plate. This is called shear locking.