Plate Bending

Siva Srinivas Kolukula
Structural Mechanics Laboratory
IGCAR, Kalpakkam
INDIA
Email-allwayzitzme@gmail.com

1. Introduction

A plate is a flat body whose thickness is much smaller than its other dimensions. Plate carries a lateral load by bending. A plate develops bending moments in two directions and a twisting moment. Two plate theories are well known depending on whether transverse shear deformation is considered or not considered.

1.1 Thin plate theory/ Kirchhoff plate theory

The basic assumption for the classical Kirchhoff plate bending theory is that a straight line normal to the midplane of the plate before deformation remains normal even after deformation. In this theory transverse shear deformation is neglected.

1.2 Mindlin theory/ Mindlin—Reissner theory

The plane normal to the midplane before deformation will not remain normal any longer after the deformation. In this theory transverse shear deformation is accounted.

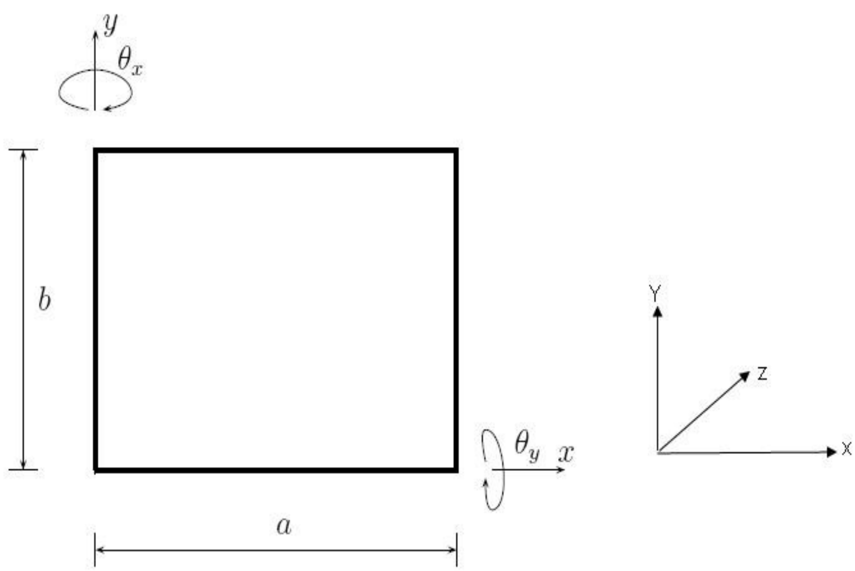

Figure 1: Plate, geometry and degrees of freedom

Figure 1 shows the plate, its geometry and degrees of freedom. Plate is placed in xy plane. θ_x and θ_y represent rotations along Y-axes and X-axes respectively. A plate has a thickness t and has a midsurface at a distance t/2 from each lateral surface. We locate plate midsurface along the xy plane, z = 0 is the plate mid surface. Length of the plate is 'a' and its breadth 'b'.

2. Kirchhoff plate theory

In this theory the transverse deformation is neglected. The inplane displacement u, v are expresses as

$$u = -z\frac{\partial w}{\partial x}$$
$$v = -z\frac{\partial w}{\partial y} \tag{2.1}$$

$$w = w(x, y)$$

Where x and y are the inplane axes located at the midplane of the plate, and z is along the thickness of the plate as shown in Figure 1. Thus u, v are the displacements along the x-axes and y-axes respectively, and w is the transverse displacement or deflection along the z-axes.

The inplane strains in terms of the displacements are written as

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2} \qquad \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2} \qquad \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x\,\partial y} \tag{2.2}$$

As the transverse deformation is not considered, we have

$$\gamma_{yz} = \gamma_{zx} = 0 \tag{2.3}$$

For a thin plate assuming the plane stress condition, we have

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \end{Bmatrix} = -z\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}\begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \end{Bmatrix} \qquad \tau_{xy} = -2zG\frac{\partial^2 w}{\partial x\,\partial y} \tag{2.4}$$

The above stresses give rise to the following moments

$$M_x = \int_{-t/2}^{t/2} \sigma_x\, z\, dz \qquad M_y = \int_{-t/2}^{t/2} \sigma_y\, z\, dz \qquad M_{xy} = \int_{-t/2}^{t/2} \tau_{xy}\, z\, dz \tag{2.5}$$

Thus the moment-curvature relations for a homogenous and isotropic Kirchhoff plate are

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \cdot \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\cdot\frac{\partial^2 w}{\partial x \partial x} \end{Bmatrix} \tag{2.6}$$

Where D is flexural rigidity, given by $D = \frac{E \cdot h^3}{12 \cdot (1 - \nu^2)}$

3. Mindlin plate theory

It assumes that transverse deformation occurs in the plate. The deformations and strains are given by

$$u = z\theta_y \qquad \varepsilon_x = z\frac{\partial \theta_y}{\partial x} \qquad \gamma_{xy} = z\left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}\right)$$

$$v = -z\theta_x \qquad \varepsilon_y = -z\frac{\partial \theta_x}{\partial y} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} - \theta_x \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \theta_y \tag{3.1}$$

The moment curvature relations will be for bending and shear deformations given as follows

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \cdot \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \cdot \frac{\partial^2 w}{\partial x \partial x} \end{Bmatrix} \tag{3.2}$$

For bending and

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} Gt & 0 \\ 0 & Gt \end{bmatrix} \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} \tag{3.3}$$

For the transverse shear deformation. Where Q_x and Q_y are transverse shear force.

Stress on cross section of the plate is shown in Figure 2.

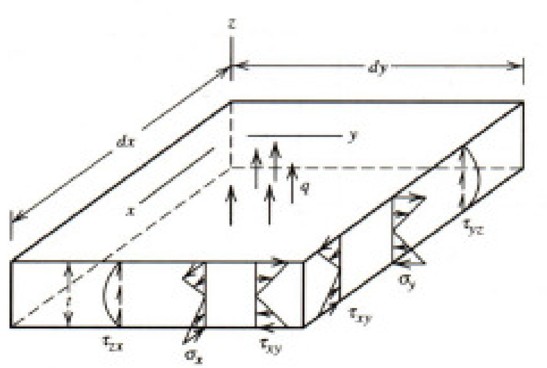

Figure 2: Stresses and distributed lateral forces on plate

4. Finite Element Method for Plate

Mindlin plate theory is considered in finite elements of plates. The displacement interpolation for plates is given by

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} = \mathbf{Nd} \tag{4.1}$$

Where N_i is the shape function associated with the plate. Most common shape functions used for the plate are Q4 and Q8.

5. Example Problem

A simple supported and a clamped square plate under uniform transverse pressure is considered. The following is the data used in the code.

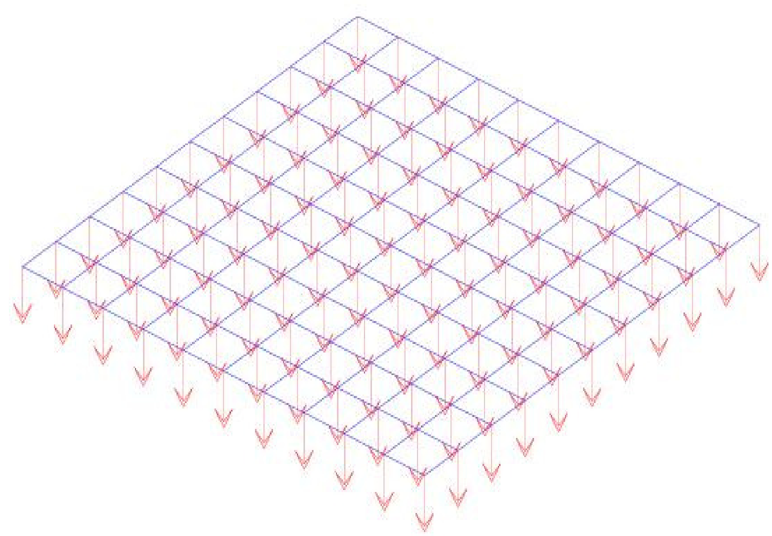

Figure 3: Plate under Uniform transverse pressure

Young's Modulus, E = 10920 Nm/Kg^2
Plate thickness, t = 0.1 m
Poisson's ratio, υ = 0.3
Length of the plate, a = 1 m
Breadth of the plate, b = 1m

Pressure = 1 unit

Four node Isoparametric elements are used. Plate is meshed with 200 elements. Meshing of the plate and the transverse load applied on the plate are shown in the Figure 3. Static analysis is done and the values obtained with present code are compared with standard FEM software. To show the accuracy of the code, values of the present code are compared with standard FEM software. Table 1 shows the transverse displacement 'w' obtained using present code and FEM software. The transverse displacement is written in the non-dimensional from as

$$\bar{w} = w\frac{D}{Pa^4} \tag{5.1}$$

Table 1 : Transverse displacement of the plate

Boundary Condition	MATLAB	FEM software
Simple supported	0.0042703 m	0.0042699 m
Clamped	0.0015113 m	0.0015102 m

Figure 4 shows the deformation of the simply supported plate under transverse pressure.

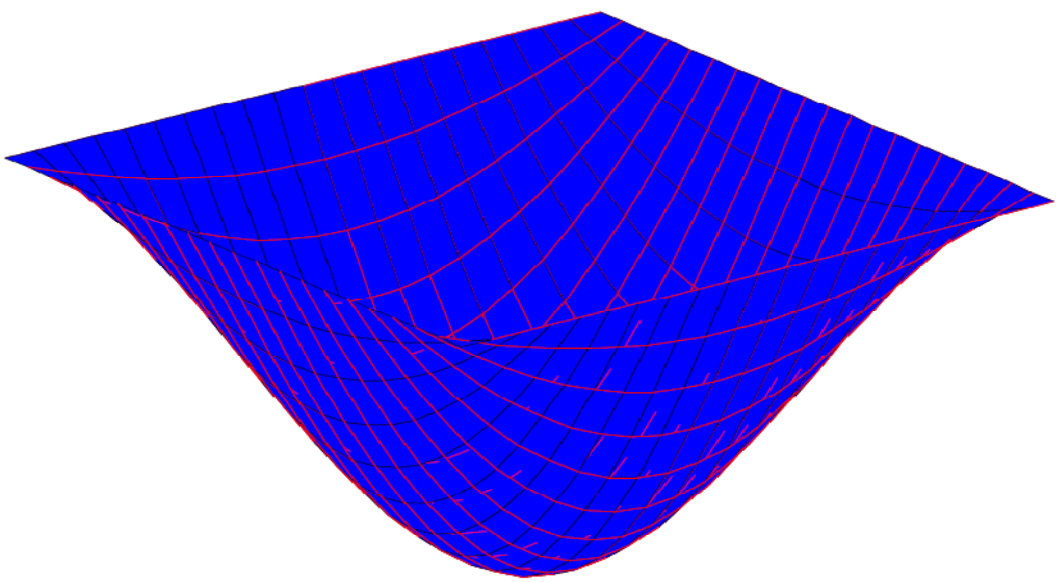

Figure 4: Plate deformation under transverse pressure

Figure 5 shows the contour plot of transverse displacement of the simply supported plate from FEM software and Figure 6 shows the contour plot from the present code. Both the plots are in good agreement.

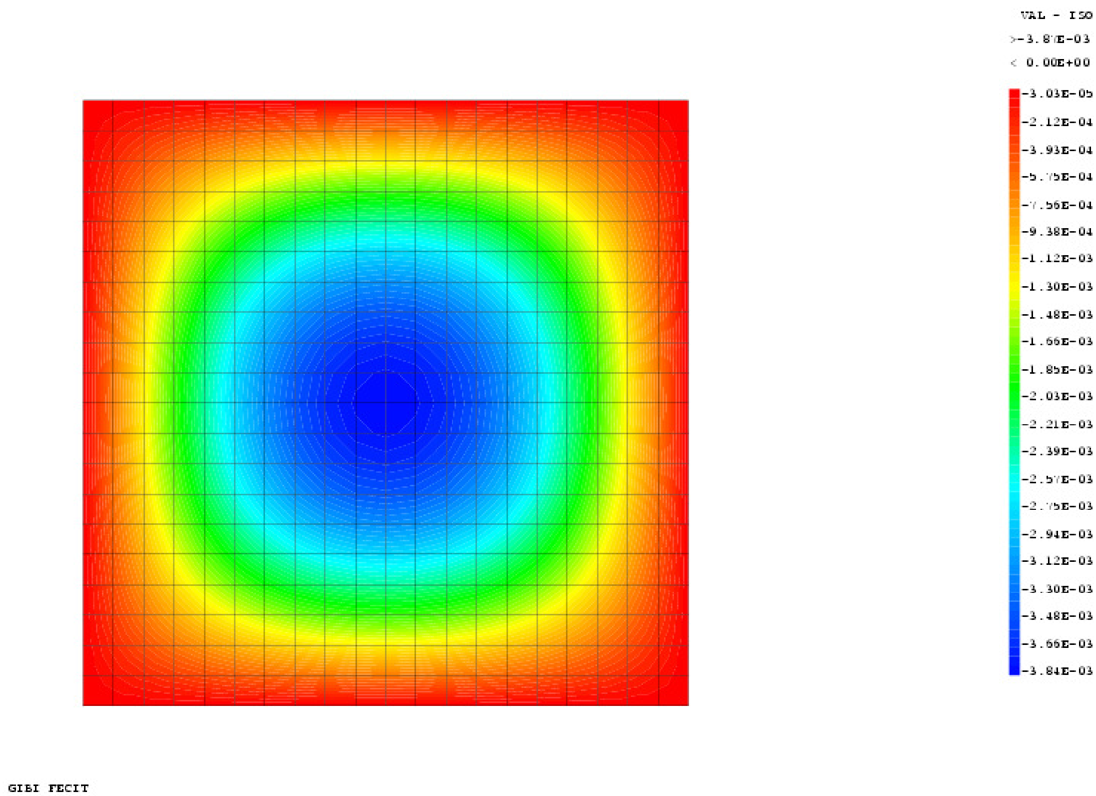

Figure 5: Contour plot of w from FEM software

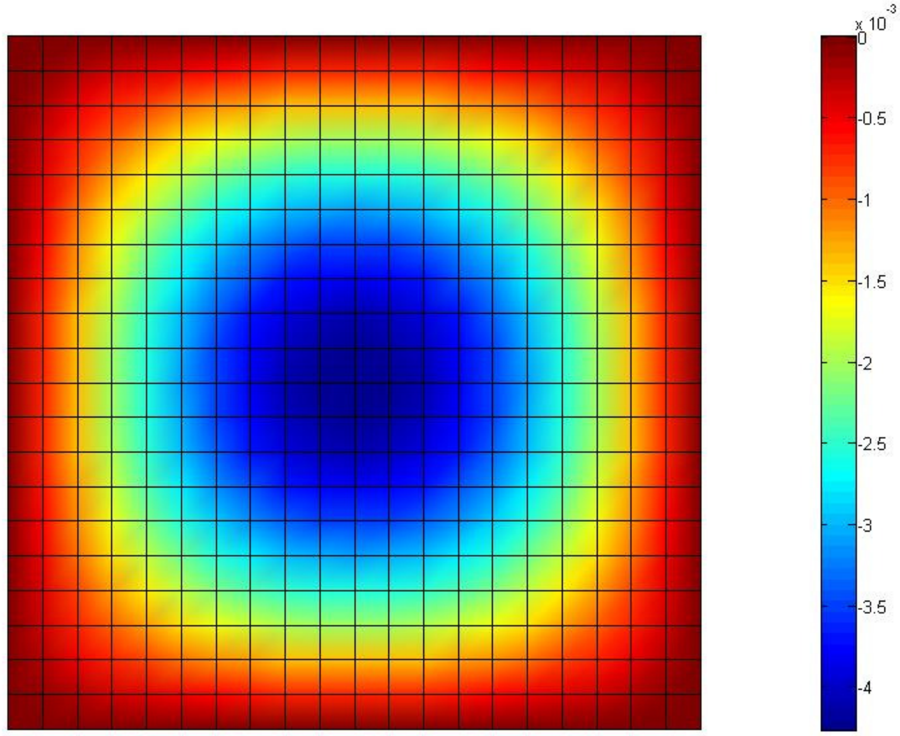

Figure 6: Contour plot of w from present code

References:

1. Concepts and Applications of Finite Element Analysis – Robert D. Cook

2. The Finite Element Method Edition 3 - Zienkiewicz