Graph Directed Constructions of Iterated Function Systems, Characterization and Special Cases

Oliver Adams

14^{th} December 2018

Abstract

The study of Iterated Function Systems has been an attractive and popular research field for decades. Iterated function systems exhibit an intriguing form of self similarity with profound geometric properties. Understanding the underlying geometry of abstract systems can give rise to many insights on visualizing DNA sequences, defining Laplacians on fractal spaces, and analyzing chaotic and nonlinear time series. In this paper we formulate and introduce graph directed constructions of iterated function systems, using a metric theoretic and graph theoretic approach.

// Fractals; Scissor-Resistant!

If you were to cut the following paper up with scissors; to dissect each line from the page, you would just be left with a pile of disconnected sentences that are individually distinct from the paper you started with. The dissected pieces are visually dissimilar from the whole. No matter how you choose to cut apart this paper, you will never end up with a smaller copy of this paper! Sure, a lot of objects we deal with every day are like this - but not all! The wood from which this paper came, for example, is a different story; if you were to go outside and cut off a small branch of tree (please don't), the resulting object you are left with resembles the whole:

Roughly speaking: *fractals* are objects which can be decomposed, or cut, into smaller parts resemble looking like what you started with (same shape, just a different size). Some objects with fractal-like properties are: the surface of the ocean, mountains, roadway/highway systems, white noise / TV static, internet traffic and neural networks [1] [7]. Understanding the underlying geometry of abstract systems can give rise to many insights on visualizing DNA sequences, defining Laplacians on fractal spaces, and analyzing chaotic and nonlinear time series [3] [6]. The objective of this paper is to introduce the formalities and various theorems related to the construction of *graph generated* fractals. We now jump into the mathematical background of this topic.

// Iterated Function Systems:

An iterated function system is a set of functions, which when composed infinitely many times in all possible ways gives rise to a fractal like structure. These functions can be informally thought of as the actions which "grow" a fractal. In most scenarios, the type of functions used in an iterated function system are *contraction* mappings.

Definition 1. *Let δ represent the Euclidean metric in $\mathbb{R}^d$. A mapping $f : \mathbb{R}^d \to \mathbb{R}^d$ is called a* ***contraction*** *if for some $r \in (0, 1)$ and for all $x, y \in \mathbb{R}^d$:*

$$\delta(f(x), f(y)) \leq r \cdot \delta(x, y)$$

We will call f a *strict contraction* if $\delta(f(x), f(y))$ is strictly equal to $r \cdot \delta(x, y)$, for some *scaling factor* r. Some examples of contraction mappings are: linear affine transformations with determinant less than 1, dilations and or rotations around a point, etc. Note that f satisfies the Lipschitz condition of continuity, and so f must also be continuous. There is an important classical theorem regarding the nature of contraction mappings in $\mathbb{R}^d$:

Theorem 1. *Let X be a metric space equipped with metric δ, then for any contraction mapping f, there is a unique $k \in X$ such that $k = f(k)$. We call such k the fixed point of f.*

Proof. Assume that $k, k_0 \in X$ are both fixed points of f. Then, by the fact that f is a contraction, we have

$$\delta(k_0, k) = \delta(f(k_0), f(k)) \leq r\delta(k_0, k)$$

However, since $r < 1$, then it follows that $\delta(k_0, k) = 0$, so $k = k_0$. Hence there is at most one fixed point. Now to show that k exists; we begin by choosing any $x_0 \in X$. Define $x_1 = f(x_0)$, and more generally let $x_n = f(x_{n-1})$ hold for $n > 1$. Let $a = \delta(x_1, x_0) \geq 0$, then $\delta(x_2, x_1) = \delta(f(x_1), f(x_2) \leq r\delta(x_1, x_0) = ra$. Observe that $\delta(x_{n+1}, x_n) \leq ar^n$ holds by induction. We will to show that the sequence of points $\{x_n\} \subset X$ is Cauchy, and converges to a fixed point of f. Fix $\epsilon > 0$, and choose an integer N that is large enough that $\frac{ar^N}{1-r} < \epsilon$. Then for integers m, n with $n > m \geq N$ we have

$$\delta(x_m, x_n) \leq \sum_{j=m}^{n-1} \delta(x_{j+1}, x_j) \leq \sum_{j=m}^{n-1} ar^j = \frac{a(r^m - r^n)}{1-r} = \frac{ar^m(1-r^{n-m})}{1-r} \leq \frac{ar^m}{1-r} < \epsilon.$$

Since X is complete and $\{x_n\}$ is a Cauchy sequence, $x_k \to k$ for some $k \in X$. Since f is continuous, $f(x_n) \to f(k)$. But since $f(x_{n-1}) = x_n$, we have that $f(x_{n-1}) \to k$. Thus $f(k) = k$, as needed. ☐

Note that if $k \in \mathbb{R}^d$ is a fixed point of a given contraction f, then k is also a fixed point of map $f^n(x) = f \circ f \circ \cdots \circ f(x)$, where n is any nonnegative integer.

Theorem 2. *Let f_1 and f_2 be contraction mappings from complete metric space X to X, with scaling factors r_1 and r_2 respectively. Then the composition $f_1 \circ f_2$ is a contraction mapping with scaling factor $r_1 r_2$.*

Proof. Let $x, y \in \mathbb{R}^d$, and r_1 and r_2 be the scaling factors for f_1 and f_2 respectively. Then

$$\delta(f_1 \circ f_2(y), f_1 \circ f_2(y)) \leq r_1\delta(f_2(y), f_2(x)) \leq r_1 r_2 \delta(x, y).$$

Thus the composition of two contraction mappings is also a contraction mapping with scaling ratio $r_1 r_2$. ☐

Remark 1. *By induction we have that any finite composition of contraction mappings is also a contraction mapping.*

Now if we have two different contraction mappings, f_1 and f_2, with distinct fixed points of their own: k_1 and k_2 respectively, the nature of the fixed points of the finite compositions: $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}$, for $i_j \in \{1, 2\}$, is not easy to pin down. When one begins considering the set of the fixed points of all the possible finite compositions of a set of contraction mappings, that is when fractal-like behavior begins to arise. Let's now refer to a set of contraction mappings as a family, which we will typically name as F, where:

$$F = \{f_i : \mathbb{R}^d \to \mathbb{R}^d | f_i \text{ is a strict contraction with scaling factor } r_i < 1, i \in \{0, 1, 2, \ldots, n-1\}\}$$

Theorem 3. *(Hutchinson) Let $F = \{f_0, f_1, \ldots, f_{n-1}\}$ be a family of contraction maps on complete metric space X. There then exists a unique closed and bounded set K such that $K = \cup_{i=1}^{n-1} f_i(K)$. Furthermore, K is the closure of the set of fixed points $k_{i_1 i_2 \ldots i_p}$ of finite compositions $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_p}$ for $i_j \in \{0, 1, \ldots, n-1\}$.*

We will leave the formal proof of Hutchinson's theorem to Hutchinson, though some explanation is warranted. The basic idea is this; by defining a distance between two compact sets, in this case the Hausdorff metric, one can realize the set H of all compact subsets of $\mathbb{R}^d$ as a complete metric space. Then with $F = \{f_0, f_1, \ldots f_{n-1}\}$ a family of contraction mappings on $\mathbb{R}^d$, we define the function $\mathcal{F} : H \to H$ by $\mathcal{F}(A) \equiv f_0(A) \cup f_1(A) \cup \cdots \cup f_{n-1}(A)$ for $A \in H$. It can be shown that $\mathcal{F}$ is a contraction mapping on H. Then invoking the result of Theorem 1, it can then we can then show that the contraction mapping $\mathcal{F} : H \to H$ has a unique fixed set $K \in H$.

Typically, Hutchinson's set K is called the *attractor* for a given family F. Though each family F defines a unique attractor, it is not always true that a given compact set can be represented as the attractor of a certain family of maps – though much work has been done on this "inverse problem" [7]. In Hutchinson's construction of the attractor K, all possible compositions of members of F are considered – though this need not be the case in general. By "forbidding" certain compositions of maps $f_i \circ f_j$ for some $i, j \in \{1, \ldots, n-1\}$ for example, an attractor still exists, though it may have a different form (Mauldin Williams). Let's now work towards understanding the fractal behavior of the attractor of an IFS with forbidden compositions.

In Hutchinson's construction of the attractor K for a given family F, we take K to be the closure of the fixed points of the finite compositions of functions in F. However Hutchinson's theorem does not put any restriction on *which* finite compositions we choose to consider, only that K will be the closure of the fixed points of these compositions. That is, if we disallow certain compositions of the form $f_i \circ f_j$ with $f_i, f_j \in F$, then the set of fixed points will change; but a unique attractor will still exist and it will still be the closure of all *possible* fixed points. To keep track of, or to illustrate, the compositions of functions which *are* allowed, we use graphs and sets of strings as tools.

Let $\mathcal{E} = \{0, 1, 2, \ldots, n-1\}$ be an **alphabet** of n symbols. We consider strings I made up of symbols from $\mathcal{E}$, where I can be identified as $a_1 a_2 a_3 \ldots a_k$ with $a_i \in \mathcal{E}$. For $I = a_1 a_2 a_3 \ldots a_k$, we say that the **length** of I is k. We denote by $\mathcal{E}^{(m)}$ the set of all strings of length m. Note that $\mathcal{E}^{(1)} = \mathcal{E}$. A **substring** of I of length m is a string $a_i a_{i+1} a_{i+2} \ldots a_{i+m}$ with $i + m \leq k$. For example, if $n = 2$, then $\mathcal{E}^{(3)} = \{000, 001, 010, \ldots, 101, 110, 111\}$and the string 00 is a length 2 substring of some but not all strings in $\mathcal{E}^{(3)}$. We identify the infinite set of finite strings, $\mathcal{E}^{(\omega)}$, as the union:

$$\mathcal{E}^{(\omega)} = \mathcal{E}^{(0)} \cup \mathcal{E}^{(1)} \cup \mathcal{E}^{(2)} \cup \mathcal{E}^{(3)} \cup \ldots$$

Definition 2. *The concatenation of strings $I, J \in \mathcal{E}^{(\omega)}$ is defined as*

$$I|J = a_1 \ldots a_n | b_1 \ldots b_m = a_1 \ldots a_n b_1 \ldots b_m$$

The string $I|J$ has length $n + m$, and is not in general equal to $J|I$.

Now connecting back to functions and other things; we identify the finite composition $f_{a_1} \circ f_{a_2} \circ \ldots f_{a_{k-1}} \circ f_{a_k}$ as the string $a_1 a_2 \ldots a_{k-1} a_k = I \in \mathcal{E}^{(\omega)}$. That is, we can use strings $I \in \mathcal{E}^{(\omega)}$to represent different compositions of functions from F, with $|F| = |\mathcal{E}|$. In parallel with our definition of $\mathcal{E}^{(m)}$, we can define $F^{(m)}$ as the set of all possible compositions of functions from F:

$$F^{(m)} = \{f_{a_1} \circ f_{a_2} \circ \dots f_{a_m} : \mathbb{R}^d \to \mathbb{R}^d |\ a_j \in \mathcal{E}, f_i \in F \text{ and } |\mathcal{E}| = |F|\}$$

We define $F^{(\omega)}$ in the same way that we define $\mathcal{E}^{(\omega)}$; as the infinite union: $F^{(0)} \cup F^{(1)} \cup F^{(3)} \cup \dots$. With this, we can consider the attractor K as being the closure of the set of fixed points of the contraction mappings in $F^{(\omega)}$. With a string $I = a_1 \dots a_n \in \mathcal{E}^{(\omega)}$, we identify composition $f_{a_1} \circ \dots \circ f_{a_n}$ as f_I. With F a family of functions, we fix a **construction set** $S \subseteq \mathcal{E}^{(2)}$ where $a_1 a_2 \in S$ indicates that the composition $f_{a_2} \circ f_{a_1} \in F^{(2)}$, is allowed. Similar to the case for the edge set E, we say that S is **complete** if $S = \mathcal{E}^{(2)}$.

Definition 3. *The k^{th} **construction band**, call it $X_S(k)$, for a given construction set S, is the set of all strings I of length $k \in \mathbb{N}, k \geq 2$, such that all substrings of length 2 of I belong to S:*

$$X_S(k) = \{a_1 a_2 a_3 \dots a_k \in \mathcal{E}^{(k)} | a_i a_{i+1} \in S, \forall i = 1, 2, \dots, k-1\}$$

It is worth mentioning that if $S = \mathcal{E}^{(2)}$, with $|\mathcal{E}| = n$, then $|X_S(k)| = |\mathcal{E}^{(k)}| = n^k$. When S is a proper subset of $\mathcal{E}^{(2)}$, the structure and size of $X_S(k)$ as k varies is not straightforward to compute. In fact, $X_S(k)$ itself has self-similar characteristics; note that each string $I \in X_S(k-1)$ is a substring of some $I' \in X_S(k)$. Later on, there will be examples that illustrate the nature of $X_S(k)$ for different families F and construction sets S.

Definition 4. *Let K' be the attractor for a family F with a complete construction set S', and fix S as some arbitrary construction set for F. With the construction band $X_S(k)$, the k^{th} **approximant** of the attractor K' for F is the set $\mathcal{K}^{(k)}$ with:*

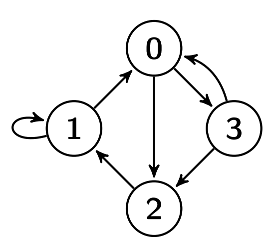
$$\mathcal{K}^{(k)} = \bigcup_{I \in X_S(k)} f_I(K')$$

With this, we define the attractor $\mathcal{K}$ for F with construction set S as:

$$\mathcal{K} = \bigcap_{k \geq 2}^{\infty} \left[\bigcup_{I \in X_S(k)} f_I(K') \right] = \bigcap_{k \geq 2}^{\infty} \mathcal{K}^{(k)}$$

So in general we have that $\mathcal{K} \subseteq K'$. Closely related to the alphabet $\mathcal{E}$ and construction string S, is the concept of directed graphs and edge sets;

Let S be a construction set for some alphabet $\mathcal{E}$ with n symbols and family $F = \{f_0, f_1, \dots, f_{n-1}\}$. Let G be a directed graph on n vertices labeled by elements of $\mathcal{E}$, with the directed edge set $E = \{e_{(i,j)} | ij \in S\}$, where $e_{(i,j)}$ is a directed edge from vertex i to vertex j. For example If G has four vertices, with $E = \{e_{(0,3)}, e_{(1,1)}, e_{(1,0)}, e_{(2,1)}, e_{(3,0)}, e_{(3,2)}, e_{(0,2)}\}$, then we may present G graphically as:



Definition 5. *The edge set E for a directed graph G with n vertices is said to be **complete** if its edge set E contains all possible directed edges, so that $|E| = n^2$.*

A natural way of representing the set of edges in G is by using an $n \times n$ matrix A with 1 in the i^{th} row and j^{th} column indicating that $e_{(i,j)}$ is a directed edge in G. Here, we call A the **edge-matrix** for G. In abstract graph theory, this set is traditionally called the adjacency matrix. For the example above, the corresponding A would be:

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

In short, we may write A for a given G as $A = [e_{(i,j)}]_{i,j \leq n}$, where $e_{(i,j)}$ is understood to be a zero or one depending on $e_{(i,j)} \in E$ or not. If E is complete, then A is the $n \times n$ matrix made up of only 1's.

Definition 6. *The **construction matrix** for a given family F of n functions is the $n \times n$ matrix $A_r = [r_i \cdot e_{(i,j)}]_{i,j \leq n}$, with r_i being the scaling factor for $f_i \in F$ and $e_{(i,j)} \in E$ of G.*

Let M be an $n \times n$ matrix with eigenvalues $\{\lambda_{1,2}, \ldots, \lambda_k\}$, and let λ^* be the largest eigenvalue of M. The **spectral radius** of a M, denoted $\Phi(M)$, is defined as the magnitude of λ^*. There's a really powerful theorem connecting the spectral radius of a general construction matrix to the attractor of a given F with edge set E:

Theorem 4. *(Mauldin Williams) Let F be a family of n contraction mappings with scaling factors r_i, and edge set E. With $A_r^\alpha = [r_i^\alpha \cdot e_{(i,j)}]_{i,j \leq n}$ and $e_{(i,j)} \in E$, the value of α for which the spectral radius $\Phi(A_r^\alpha)$ equals 1 is the dimension of the attractor $\mathcal{K}$ of F.*

Remark 2. *We will use $\mathcal{K}$ as the attractor for a family F with an incomplete edge set, and K as the attractor for F with a complete edge set.*

A proof of the previous theorem can be found, and accredited, to Mauldin and Williams [2]. We will now begin to show how this specific notion of dimension is an abstraction of the more familiar definitions for fractal dimension. There's a complex hierarchy of different ways to compute the "dimension", and while most fit with and result in what we would colloquially expect as the dimension of a space, it is usually in an unexpected or mathematically obscured manner. A more familiar and tangible definition of dimension is the *similarity value*:

Definition 7. *The **similarity value** for a family F of contraction mappings with scaling ratios $\{r_i\}$, is the value d_s for which the following holds:*

$$1 = \sum_{i=0}^{n-1} r_i^{d_s}$$

Theorem 5. *The similarity value d_s for a given family F always exists, and is unique.*

Proof. Let $\phi(d) = \sum_{i=0}^{n-1} r_i^d$. Observe that $\phi(d)$ is continuous, and that $\phi(d) \to 0$ as $d \to \infty$ since $r_i \in (0,1)$ for all $i \in \{0, 1, \ldots, n-1\}$. Now by the intermediate value theorem, we know there must exist some x such that $\phi(x) = 1$. Since $\ln(r_i) < 0$ for all r_i, then the derivative of ϕ with respect to d, namely $\sum_{i=0}^{n-1} \ln(r_i) r_i^d$, must be less than 0 for all d. So $\phi(d)$ is a strictly decreasing function for $d \in (0, \infty)$. Hence the value x for which $\phi(x) = 1$ is unique. $\square$

Theorem 6. *For a family F of n functions, and a complete edge set E, the similarity value d_s is equal to the Hausdorff dimension α of the attractor K.*

Proof. If E is complete, then the edge matrix for G is filled with only 1's. Hence the construction matrix is then:

$$A_r^d = \begin{bmatrix} r_0^d & r_1^d & \dots & r_{n-1}^d \\ r_0^d & r_1^d & \dots & r_{n-1}^d \\ \vdots & \vdots & \ddots & \vdots \\ r_0^d & r_1^d & \dots & r_{n-1}^d \end{bmatrix}$$

Row reduction on A_r^d shows that it has rank 1, and so there is only one nonzero eigenvector for A_r^d with a 1-D eigenspace. Let's now show that $v = [1, 1, \dots, 1]^\intercal$ that an eigenvector for A_r^d with eigenvalue $\phi(d)$ from before:

$$A_r^d \cdot v = \begin{bmatrix} r_0^d & r_1^d & \dots & r_{n-1}^d \\ r_0^d & r_1^d & \dots & r_{n-1}^d \\ \vdots & \vdots & \ddots & \vdots \\ r_0^d & r_1^d & \dots & r_{n-1}^d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} r_0^d + r_1^d + \dots + r_{n-1}^d \\ r_0^d + r_1^d + \dots + r_{n-1}^d \\ \vdots \\ r_0^d + r_1^d + \dots + r_{n-1}^d \end{bmatrix} = \sum_{i=0}^{n-1} r_i^d \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$
$$= \phi(d)v.$$

Since v is the only nonzero eigenvector for A_r^d, then $\phi(d)$ is the largest eigenvalue of A_r^d. Hence the spectral radius of A_r^d is $\phi(d) = \Phi(A_r^d)$; so by Theorem 4 the value d for which $\Phi(A_r^d) = 1$ is exactly the Hausdorff dimension α. □

The result of Theorem 6 means that if we have a family F of contraction mappings and we consider *all* possible compositions of functions of F, then we will get an attractor with dimension d_s. This is especially nice since calculating the similarity value for a set of ratios is generally much easier than computing Hausdorff dimension of a family F with edge set E. Up to now though, the way that we have defined dimension may not be so satisfactory, so let's bring this definition of dimension down another level.

Let's consider the case of a family F has contraction mappings f_i which all have the same scaling factor. That is we have $r_i = r$ for all $i \in \{0, 1, \dots, n-1\}$ and r_i the scaling factor for $f_i \in F$. Also, let the edge set E for F be complete. If this is the case, then $\phi(d) = nr^d$, and the similarity value for such an F is be the number d such that $nr^d = 1$.

Definition 8. *Let $F = \{f_0, f_1, \dots, f_{n-1}\}$ be a family of contraction mappings with scaling ratios r_i all equal to some $r \in (0, 1)$, and let the edge set E for F be complete. The **box counting dimension** for F is the real number d_B such that $nr^{d_B} = 1$, or rather:*

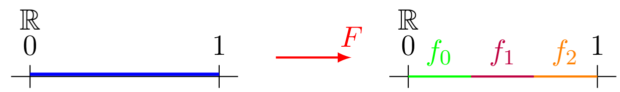
$$d_B = \frac{\log(n)}{\log(1/r)}$$

The box counting dimension is a great because we can compute it easily, provided the right conditions are met. Later on we will discuss how the box counting dimension for a given compact set $K \subset \mathbb{R}^d$, with some slight modifications can be calculated without any knowledge of F at all. Before we get there though, let's go over some concrete examples:

IFS on the Real Line, Example 1: To begin to understand the behavior and use of such systems, we will first consider the construction of an elementary example; The Cantor Set. Introduced in 1883 by Georg Cantor, this set has remarkably deep properties arising from various methods of construction. We will get to the construction of the Cantor set soon, but before then we have a little more ground to cover. In our case, we will consider the construction of this set using translations and dilations from the family $F = \{f_0, f_1, f_2\} : \mathbb{R} \to \mathbb{R}$ with functions defined as follows:

$$f_0(x) = \frac{x}{3}$$
$$f_1(x) = \frac{x+1}{3}$$
$$f_2(x) = \frac{x+2}{3}$$

Figure 1: The images of F



Some simple algebra shows that the fixed points for $f_0(x)$, $f_1(x)$, and $f_2(x)$ are $x = 0$, $x = \frac{1}{2}$ and $x = \frac{1}{2}$ respectively. Let's now compute the scaling ratios r_0, r_1 and r_2 for the functions f_0, f_1, and f_2 respectively. With $x, y \in [0, 1]$ and δ being the usual Euclidean metric, we consider the distance between the images of x and y under the contraction mappings above:

$$\delta(f_0(x), f_0(y)) = \delta\left(\frac{x}{3}, \frac{y}{3}\right) = \frac{1}{3}|x - y| = \frac{1}{3}\delta(x, y) \qquad \implies r_0 = \frac{1}{3}$$
$$\delta(f_1(x), f_1(y)) = \delta\left(\frac{x+1}{3}, \frac{y+1}{3}\right) = \frac{1}{3}|x + 1 - y - 1| = \frac{1}{3}\delta(x, y) \qquad \implies r_1 = \frac{1}{3}$$
$$\delta(f_2(x), f_2(y)) = \delta\left(\frac{x+2}{3}, \frac{y+2}{3}\right) = \frac{1}{3}|x + 2 - y - 2| = \frac{1}{3}\delta(x, y) \qquad \implies r_2 = \frac{1}{3}$$

Note that the image of the compact interval $[0, 1]$ under f_0, f_1 and f_2 is $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$ respectively. So we have that $[0, 1] = f_0([0, 1]) \cup f_1([0, 1]) \cup f_2([0, 1])$. Then by Theorem 3 we have that $[0, 1]$ is the unique attractor K for the family F.

Though working with the functions in this way is completely valid, keeping track of the output of different compositions functions can be somewhat clunky for our purposes. There is, however, a more clean way of representing the images of these functions. By representing numbers $x \in [0, 1]$ in base 3 notation, rather than base 10, we can more easily exhibit the behavior of compositions of functions from F on x.

Representation of numbers in base 10 is of course familiar; each digit of a given number corresponds to the multiple of certain power of 10. We put a parenthesis with a subscript n around a number or set of numbers to denote they are being represented in base n. For example:

$$(0.1539)_{10} = 1 \times 10^{-1} + 5 \times 10^{-2} + 3 \times 10^{-3} + 9 \times 10^{-4}$$

Base 3 representation is similar, where each digit of a given number corresponds to the multiple of a certain power of 3. In base 3, each digit is chosen from the alphabet $\mathcal{E} = \{0, 1, 2\}$. For example:

$$(0.2101)_3 = 2 \times 3^{-1} + 1 \times 3^{-2} + 0 \times 3^{-3} + 1 \times 3^{-4}$$
and
$$(0.\overline{3})_{10} = (0.1\overline{0})_3 = (0.0\overline{2})_3$$

Now if we start to represent the images of F using base 3 we arrive to equalities such as $f_0([0,1]_{10}) = [0, .1]_3$, $f_2([0,1]_{10}) = [.2, 1]_3$, and so on (we are using interval notation with an added subscript).

Theorem 7. *The action of the map $f_i \in F$ on some rational $x \in [0,1]$ corresponds to concatenating an $i \in \{0,1,2\}$ to the beginning of the base 3 representation of x, that is $f_i(x) = i|(x)_3$*

Proof. We will show this is true for the case of f_0, as the same method of proof can be applied to the other two functions of F. Let's begin by representing the action of f_0 in base 3 representation:

$$f_0((x)_{10}) = \frac{(x)_{10}}{(3)_{10}} = \frac{(x)_3}{(10)_3} = (x)_3 \cdot 3^{-1}$$

With $(x)_3 = 0.a_1a_2\dots a_k = a_1 \times 3^{-1} + a_2 \times 3^{-2} + \dots + a_k \times 3^{-k}$ and $a_i \in \{0,1,2\}$, we multiply:

$$\begin{aligned} f_0((x)_{10}) = (x)_3 \cdot (1 \times 3^{-1}) &= (a_1 \times 3^{-1} + a_2 \times 3^{-2} + \dots + a_k \times 3^{-k})(1 \times 3^{-1}) \\ &= (a_1 \times 3^{-2} + a_2 \times 3^{-3} + \dots + a_k \times 3^{-k-1}) \\ &= 0.0a_1a_2a_3\dots a_k \\ &= 0|(x)_3 \end{aligned}$$

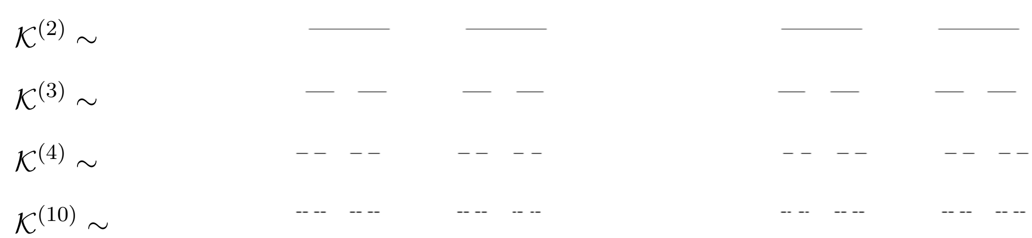
So we have that the action of f_0 on $x \in [0,1]$ is the same as concatenating a 0 to the beginning of the base 3 decimal expansion of x. Likewise, we can see that $f_1(x) = 1|(x)_3$ and $f_2(x) = 2|(x)_3$. $\square$

Corollary 1. *The action of the composition $f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_k} \in F^{(k)}$ on $x \in [0,1]$ can be represented as concatenating the string $i_1i_2\dots i_k$ to the beginning of the base 3 representation of x, i.e.: $f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_k}(x) = i_1i_2\dots i_k|(x)_3$.*

Let's now construct the Cantor set using the following machinery; With construction set $S = \{00, 02, 20, 22\}$, we have :

$$X_S(3) = \{000, 002, 020, 022, 200, 202, 220, 222\}.$$

In general $X_S(k)$ is the set of all strings in $\mathcal{E}^{(k)}$ which do not contain the symbol 1. Let's now depict graphically the approximants of F for a few values of k:



Now since S does not contain any strings which contain a 1, then we want all compositions of functions which involve f_1 to be disallowed. Hence the construction matrix, A_r^d, has 0's along the entries in the 1^{st} row and column (zero indexed):

$$A_r^d = \left(\frac{1}{3}\right)^d \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

One can easily show that A_r^d has eigenvalues 0 and $2(\frac{1}{3})^d$, hence the spectral radius of A_r^d is $\Phi(A_r^d) = 2(\frac{1}{3})^d$, and computing the d for which $\Phi(A_r^d) = 1$ reveals that $d = \frac{\log(2)}{\log(3)}$.

Another way of showing that the dimension of the Cantor set is $\frac{\log(2)}{\log(3)}$ is by noting that the Cantor set is the attractor for the family $F' = \{f_0, f_2\}$ with a complete edge set E. Now since $f_1, f_2 \in F$ both have the same scaling factor of $\frac{1}{3}$, we can use the box counting dimension to immediately conclude that $d_b = \frac{\log(n)}{\log(1/r)} = \frac{\log(2)}{\log(3)}$. Also note that by the same reasoning, we have that the attractor for $S = \{00, 01, 10, 11\}$ has the same dimension as the Cantor set. We begin to wrap up this example with a small gallery of various edge matrices their adjacent 10^{th} approximant, $F(X_S(10))$:

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longmapsto$$

$$A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \longmapsto$$

Above, we represent the construction sets for the above attractors using edge matrices, as this is typically a more compact way of expressing the elements of S. Concerning the first attractor for the edge matrix A_1; Observe that $e_{(0,2)}$ and $e_{(2,0)}$ are both zero, indicating that the compositions $f_0 \circ f_2$ and $f_2 \circ f_0$ are both neglected. By neglected, this means that the images of $[0,1]$ by $f_0 \circ f_2$ and $f_2 \circ f_0$, namely $[\frac{2}{9}, \frac{1}{3}]$ and $[\frac{2}{3}, \frac{7}{9}]$ are removed from the attractor $[0,1]$. Similarly, the images of any composition $f_{a_1} \circ \cdots \circ f_{a_k}$ which contains a $f_0 \circ f_2$ or $f_2 \circ f_0$ are removed.

Similarly, the attractor for the edge matrix A_2, the intervals $[0, \frac{1}{9}]$, $[\frac{4}{9}]$, and $[\frac{7}{9}, 1]$ are missing from the attractor since the compositions $f_0 \circ f_0$, $f_1 \circ f_1$ and $f_2 \circ f_2$ are neglected. These attractors for this family F are beautiful and interesting in their own right, but as we consider families with higher dimensions, things get more interesting.

IFS on $\mathbb{R}^2$, Example 2: Let's now consider another slightly more complicated: Let $F = \{f_0, f_1, f_2, f_3\}$ be a family of four contraction mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. Wherein the individual functions of F are defined as follows:

$$f_0(\vec{v}) = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\vec{v}$$

$$f_1(\vec{v}) = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\vec{v} + \begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}$$

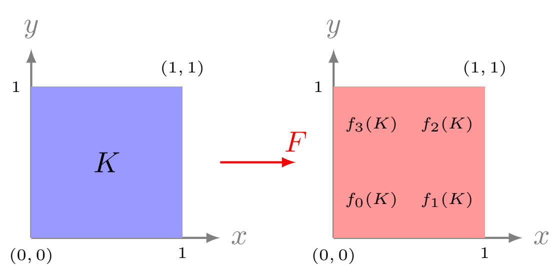
$$f_2(\vec{v}) = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\vec{v} + \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$$

$$f_3(\vec{v}) = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\vec{v} + \begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}$$

Function Name	Δx	Δy	s_x	s_y
f_0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
f_1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
f_2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
f_3	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The two adjacent definitions for the functions of F are saying the same thing using different notation; for the definition on the upper left, we imply that $\vec{v} \in \mathbb{R}^2$ is some vector which is being scaled by $\frac{1}{2}$, and then translated by some amount depending on the function. While on the upper right, the Δx and s_x columns indicate the amount by which a given contraction mapping translates and scales the plane in the x direction – similarly for the Δy and s_y columns. So by definition we have that the scaling factors for each function in F are all equal to $\frac{1}{2}$. The definition of the family F on the right is traditionally called the *Iterated Function System Rules* [1]. A more clean way of visually representing the family F is found by considering the images of the unit square under F.

Figure 2: The Attractor for F



From above, the set K is the unit square, $[0,1] \times [0,1]$, and we have drawn the various images of K by functions in F. From here it is clear to see that $K = f_0(K) \cup f_1(K) \cup f_2(K) \cup f_3(K)$, which implies by Theorem 3 that K is the attractor for F with a complete construction set S. Let us now use strings from the alphabet $\mathcal{E} = \{0, 1, 2, 3\}$ as a more compact way of labeling the images of various compositions of function from F:

Figure 3: 'Embedding' $\mathcal{E}^{(n)}$ into K

3	2
0	1

33	23	32	22
03	13	02	12
30	20	31	21
00	10	01	11

333	233	323	223	332	232	322	222
033	133	023	123	032	132	022	122
303	203	313	213	302	202	312	212
003	103	013	113	002	102	012	112
330	230	320	220	331	231	321	221
030	130	020	120	031	131	021	121
300	200	310	210	301	201	311	211
000	100	010	110	001	101	011	111

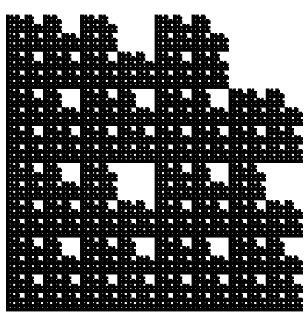
From above, the square on the left with a 0 inside denotes the set $f_0(K)$, likewise for 1, 2, and 3. As we look to the center square, we have the strings $ij \in \mathcal{E}^{(2)}$ labeling squares which denote the set $f_i(f_j(K))$. In this representation, when we consider a certain construction set S, the attractor will be a subset of the union of all squares above labeled by strings in $X_S(k)$ for all $k \in \mathbb{N}$.

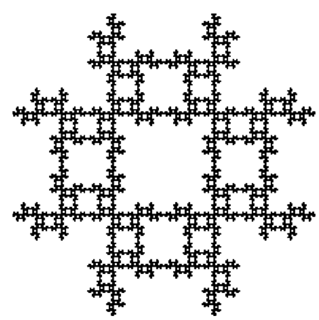
Some interesting things to note that in the limit, the squares denoted by the strings $a = 000\cdots0002$ and $b = 222\cdots2220$ become arbitrarily close, likewise for $1\cdots13$ and $3\cdots31$. For future explorations, it may be interesting to consider the application different string metrics on the above square; to search for the metric δ^* through which we see that $\delta^*(a,b) = 0$, and investigate consequences.

Let's now consider the attractors $\mathcal{K}$ for various edge matrices (whereby edge matrices are just another representation of a construction set), with edge matrix A_i corresponding to the attractor $\mathcal{K}_i$;

$$A_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = A_1$$

$\mathcal{K}_0 \rightarrow$



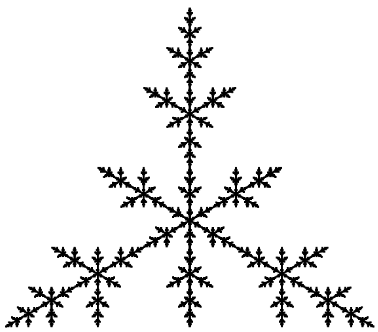
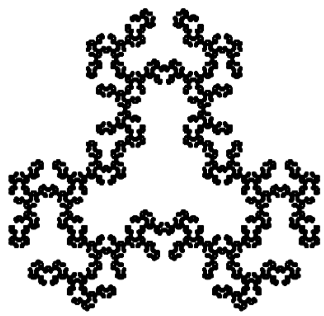
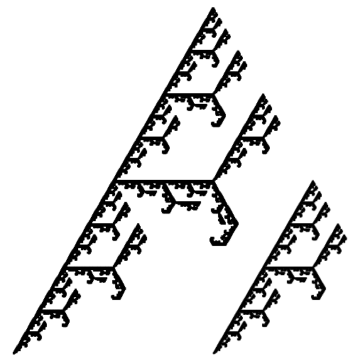


$\leftarrow \mathcal{K}_1$

Informally, we may think of the attractor for A_0 being an accumulation of squares with no top right corner; we disallow 22. The attractor for A_1 is similar, though all corners have been removed: we disallow $00, 11, 22, 33$. Many familiar fractals can be constructed using this specific family of functions, like the Sierpinski Gasket and members of its variation [8]. Loosely speaking, the construction of these images is just the tip of the iceberg. We now end with a gallery of various families and interesting attractors, to illustrate this point.

Gallery: Examples from a family of four contraction mappings on $\mathbb{R}^2$. With the attractor being the equilateral triangle with side length one:

$$\begin{bmatrix}1&1&1&1\\1&1&0&0\\1&0&1&0\\1&0&0&1\end{bmatrix} \begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix} \begin{bmatrix}1&1&1&1\\0&0&0&1\\1&1&1&1\\0&1&1&1\end{bmatrix}$$

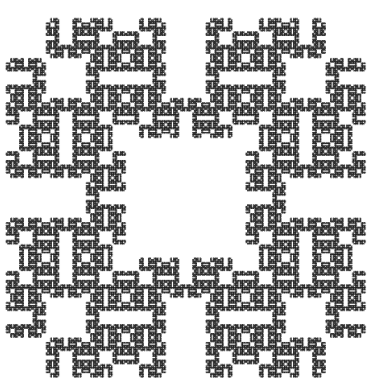
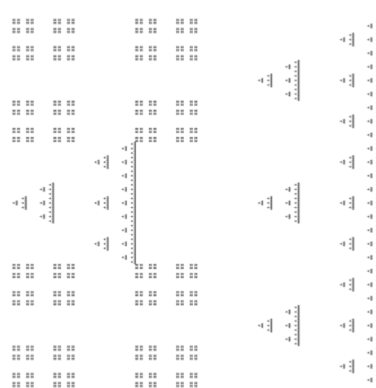
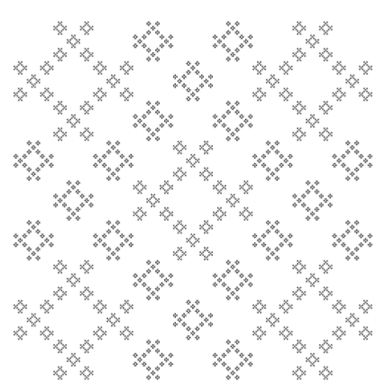




Examples from a family of nine contraction mappings on $\mathbb{R}^2$, $F = \{f_0, f_1, \ldots, f_8\}$. With the attractor being the unit square with labeling

8	7	6
5	4	3
0	1	2

:

$$\begin{bmatrix}0&1&1&1&0&1&1&1&1\\1&0&1&1&0&1&1&1&1\\1&1&0&1&0&1&1&1&1\\1&1&1&0&0&1&1&1&1\\0&0&0&0&0&0&0&0&0\\1&1&1&1&0&0&1&1&1\\1&1&1&1&0&1&0&1&1\\1&1&1&1&0&1&1&0&1\\1&1&1&1&0&1&1&1&0\end{bmatrix} \begin{bmatrix}1&1&0&0&0&0&0&1&1\\1&1&0&0&0&0&0&1&1\\0&0&1&1&0&1&1&0&0\\0&0&1&1&0&1&1&0&0\\0&0&0&0&0&0&0&0&0\\0&0&1&1&0&1&1&1&1\\0&0&1&1&0&1&1&0&0\\1&1&0&0&0&0&0&1&1\\1&1&0&0&0&0&0&1&1\end{bmatrix} \begin{bmatrix}1&0&1&0&1&0&1&0&1\\0&1&0&1&0&1&0&1&0\\1&0&1&0&1&0&1&0&1\\0&1&0&1&0&1&0&1&0\\1&0&1&0&1&0&1&0&0\\0&1&0&1&0&1&0&1&0\\1&0&1&0&1&0&1&0&1\\0&1&0&1&0&1&0&1&0\\1&0&1&0&1&0&1&1&1\end{bmatrix}$$

References

[1] Bedient, Richard & Frame, Michael & Gross, Keith & Lanski, Jennifer & Sullivan, Brendan. (2010). *Higher block IFS 1: Memory reduction and dimension computations*. S0218348X10004804 Fractals. 729. 218-348. 10.1142/S0218348X10004804.

[2] Mauldin, R., & Williams, S. (1988). *Hausdorff Dimension in Graph Directed Constructions*. Transactions of the American Mathematical Society, 309(2), 811-829. doi:10.2307/2000940

[3] Metz, V. Math. Ann. (2004).*Laplacians on finitely ramified, graph directed fractals* 330: 809. https://doi.org/10.1007/s00208-004-0571-9

[4] Hutchinson, J. (1981). *Fractals and Self Similarity*. Indiana University Mathematics Journal, 30(5), 713-747. Retrieved from http://www.jstor.org/stable/24893080

[5] G. Edgar and R. Mauldin, *Multifractal decompositions of digraph recursive fractals*, Proc. Lond. Math. Soc. 65 (1992) 604-628

[6] H. Jeffrey,*Chaos game visualization of sequences*, Comput. Graph. 16 (1992) 25-33.

[7] Edgar, G. A. (2011). , Topology, and Fractal Geometry. New York: Springer. doi:https://doi.org/10.1007/978-0-387-74749-1

[8] Taylor, T.D. "Connectivity Properties of Sierpinski Relatives," Fractals, Vol. 19, No. 4 (2011), 481-506.