Stronger version of Birkhoff's Ergodic Thm. If $f \in L^1(M, \mathcal{A}, \mu)$

(a) $\exists f^* = \mathbb{E}(f \mid \mathcal{I})$, where $\mathcal{I} = \{A \in \mathcal{A} \mid T^{-1}A = A \mod \mu\}$
(invariant σ-algebra)

HW: Prove that $\mathcal{I}$ is a σ-alg.

(b) $\int f^* d\mu = \int f d\mu$

Obs. (b) is obvious by (a): $\int \mathbb{E}(f \mid \mathcal{I}) d\mu = \int f d\mu$

Pf Given $\varphi \in L^1(M, \mathcal{A}, \mu)$, define

all these are evaluated in x

$$\Phi_n := \max_{1 \le k \le n} \sum_{i=0}^{k-1} \varphi \circ T^i = \max\{\varphi, \varphi + \varphi \circ T, \ldots, \varphi + \varphi \circ T + \cdots + \varphi \circ T^{n-1}\}$$

$\boxed{\text{Rk}: \Phi_n(x) \nearrow}$

Since $\Phi_n(x) \ge \sum_{i=0}^{n-1} \varphi \circ T^i(x)$

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ T^k \le \limsup_{n \to \infty} \frac{\Phi_n}{n} \quad (\star)$$

Set $A := \{x \in M \mid \Phi_n \xrightarrow[n \to \infty]{} +\infty\}$

So $(\star) \le 0$ on A^c

Claim 1: (again pointwise) $\Phi_{n+1} - \Phi_n \circ T = \varphi - \min\{0, \Phi_n \circ T\}$

Pf of claim 1:

$$\Phi_{n+1} = \max\{\varphi, \varphi + \varphi \circ T, \varphi + \varphi \circ T + \varphi \circ T^2, \ldots, \varphi + \varphi \circ T + \cdots + \varphi \circ T^n\}$$

$$\Phi_n \circ T = \max\{\varphi \circ T, \varphi \circ T + \varphi \circ T^2, \ldots, \varphi \circ T + \cdots + \varphi \circ T^n\}$$

Now there are two cases for $\Phi_{n+1}(x) = \max\{\varphi(x), \dots\}$

(1) the max is achieved by $\varphi(x)$ (1^{st} element of list). In this case $\sum_{i=1}^{k} \varphi\circ T^i(x) \leq 0 \quad \forall k\in\{1,2,\dots,n\} \iff \Phi_n\circ T(x) \leq 0$

$\iff \min\{0, \Phi_n\circ T(x)\} = \Phi_n\circ T(x)$

$$\Rightarrow \Phi_{n+1}(x) - \Phi_n\circ T(x) = \varphi(x) - \Phi_n\circ T(x) = \varphi(x) - \min\{0, \Phi_n\circ T(x)\}$$

(2) the max is achieved by $\sum_{i=0}^{k} \varphi\circ T^i(x)$, for some $k\in\{1,\dots,n\}$

Then clearly $\Phi_{n+1}(x) - \Phi_n\circ T(x) = \varphi(x)$

On the other hand, since we are not in case (1) $\Phi_n\circ T(x) > 0$,

$$\Phi_{n+1}(x) - \Phi_n\circ T(x) = \varphi(x) - 0 = \varphi(x) - \min\{0, \Phi_n\circ T(x)\}$$

QED Claim 1

<u>Claim 2</u> $A \in \mathcal{I}$, i.e. $T^{-1}A = A \mod \mu$

Clearly $A = \{x\in M \mid \Phi_{n+1}(x) \to +\infty\}$

$T^{-1}A = \{x\in M \mid \Phi_n\circ T(x) \to +\infty\}$

- If $\Phi_n(x) \to +\infty$, clearly, for n large $\Phi_n(x) > \varphi(x)$, so we are in case (2) above and $\Phi_n\circ T(x) = \Phi_{n+1}(x) - \varphi(x) \to \infty$
- If $\Phi_n\circ T(x) \to \infty$ clearly for n large $\Phi_n\circ T(x) > 0$, so $\Phi_n\circ T(x) + \varphi(x) > \varphi(x)$, so we are in case (2) above and once again

$$\Phi_{n+1}(x) = \Phi_n\circ T(x) + \varphi(x) \to +\infty$$

QED Claim 2

Claims 1&2 says that $\boxed{\Phi_{n+1} - \Phi_n \circ T \underset{n\to\infty}{\searrow} \varphi}$ in A

(in fact by Claim 2 $x \in A \iff T(x) \in A$ and so $\Phi_n \circ T(x) \nearrow +\infty$, and by claim 1 we get the framed formula

On the other hand $\Phi_{n+1} \geq \Phi_n$ (pointwise, like everything else). So

$$0 \leq \int_A (\Phi_{n+1} - \Phi_n)\, d\mu = \int_A \Phi_{n+1}\, d\mu - \int_A \Phi_n\, d\mu = \cancel{\int_A} \quad \Big| \text{ by invariance of } A \text{ and } \mu$$

$$= \int_A \Phi_{n+1}\, d\mu - \int_A (\Phi_n \circ T)\, d\mu = \int_A (\Phi_{n+1} - \Phi_n \circ T) \searrow \int_A \varphi\, d\mu$$

by monotonic convergence

$$= \int_A E(\varphi|\mathcal{I})\, d\mu$$

general property of $E(\cdot|\cdot)$

For short denote $\varphi_{\mathcal{I}} := E(\varphi|\mathcal{I})$.

If we knew that $\varphi_{\mathcal{I}} < 0$ by the above we'd get $\mu(A) = 0$

So now set $\varphi := f - f_{\mathcal{I}} - \varepsilon \Rightarrow \varphi_{\mathcal{I}} = \cancel{f_{\mathcal{I}}} - \cancel{f_{\mathcal{I}}} - \varepsilon < 0$

So for this φ we have $\mu(A) = 0 \iff \mu(A^c) = 1$

By def of A and $(\star)$

$\underline{\mu\text{-a.e.}}$ $0 \geq \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ T^k(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) - f_{\mathcal{I}} - \varepsilon$

$\iff \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) \leq f_{\mathcal{I}} + \varepsilon$

$f_{\mathcal{I}} \circ T^k = f_{\mathcal{I}}$ b/c $f_{\mathcal{I}}$ is T-invariant

same here

Repeating the same reasoning to $\varphi := -f + f_{\mathcal{J}} - \varepsilon$, we get

$$\liminf_{n\to\infty} \left(\_\_\_ \right) \ge f_{\mathcal{J}} - \varepsilon$$

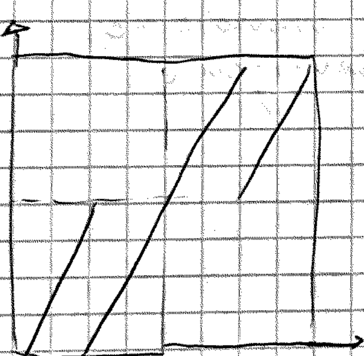
So (ε arbitrary)

$$\exists \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k = f_{\mathcal{J}} \quad \mu\text{-a. everywhere}$$

QED (a)

About the previous remark that it is not true that the statement: "Every cont. invariant fn is constant the DS is ergodic" we give a conterexample

$T : [0,2] \to [0,2]$, preserving $\mu = \frac{\underline{Leb|_{[0,2]}}}{2}$



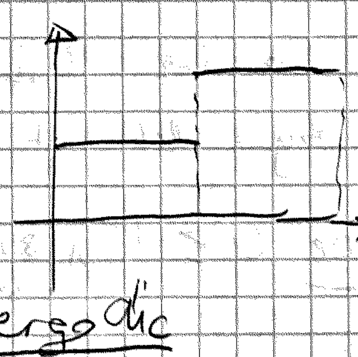
$T|_{[0,1]} : [0,1] \circlearrowleft$ cat map

$T|_{(1,2]} : (1,2] \circlearrowleft$ cat map

Clearly not ergodic: $T^{-1}[0,1] = [0,1]$

Easy to see: $\mathcal{J} = \{\emptyset, [0,1), [1,2], [0,2]\}$ and leb

Take an invariant fn f. Then $f^* = f$. But by the strong version of BET f^* is $\mathcal{J}$-measurable so f^* must be constant on $[0,1)$ and $(1,2]$



The only cont. fns of this kind are the constan fns & the above premise (every cont. inv fn is constant) is true but the system is <u>not ergodic</u>

On the other hand, if it weren't so we would have proved that under suitable topological assumptions