The Spectral Theorem

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February 14, 2025

0.1 The Spectral Theorem

Functional Analysis by Walter Rudin 1991, extract from Chapter 12

The principal assertion of the spectral theorem is that every bounded normal operator T on a Hilbert space induces (in a canonical way) a resolution E of the identity on the Borel subsets of its spectrum $\sigma(T)$ and that T can be reconstructed from E by an integral of the type discussed in Theorem 12.21. A large part of the theory of normal operators depends on this fact.

It should perhaps be stated explicitly that the spectrum $\sigma(T)$ of an operator $T \in \mathcal{B}(H)$ will always refer to the full algebra $\mathcal{B}(H)$. In other words, $\lambda \in \sigma(T)$ if and only if $T - \lambda I$ has no inverse in $\mathcal{B}(H)$. Sometimes we shall also be concerned with closed subalgebras A of $\mathcal{B}(H)$ which have the additional property that $I \in A$ and $T^* \in A$ whenever $T \in A$. (Such algebras are sometimes called $*$-algebras.)

Let A be such an algebra, and suppose that $T \in A$ and $T^{-1} \in \mathcal{B}(H)$. Since TT^* is self-adjoint, $\sigma(TT^*)$ is a compact subset of the real line (Theorem 12.15), hence does not separate $\mathbb{C}$, and therefore $\sigma_A(TT^*) = \sigma(TT^*)$, by the corollary to Theorem 10.18. Since TT^* is invertible in $\mathcal{B}(H)$, this equality shows that $(TT^*)^{-1} \in A$, and therefore $T^{-1} = T^(TT^*)^{-1}$ is also in A.

Thus T has the same spectrum relative to all closed *-algebras in $\mathcal{B}(H)$ that contain T.

Theorem 12.23 will be obtained as a special case of the following result, which deals with normal algebras of operators rather than with individual ones.

Theorem 1 (12.22)**.** *If A is a closed normal subalgebra of $\mathcal{B}(H)$ which contains the identity operator I and if Δ is the maximal ideal space of A, then the following assertions are true:*

1. *There exists a unique resolution E of the identity on the Borel subsets of Δ which satisfies*
$$T = \int_\Delta \widehat{T}\, dE \tag{1}$$
for every $T \in A$, where $\widehat{T}$ is the Gelfand transform of T.

2. *The inverse of the Gelfand transform (i.e., the map that takes $\widehat{T}$ back to T) extends to an isometric *-isomorphism of the algebra $L^\infty(E)$ onto a closed subalgebra B of $\mathcal{B}(H)$, $B \supset A$, given by*
$$\Phi f = \int_\Delta f\, dE \quad (f \in L^\infty(E)). \tag{2}$$
Explicitly, Φ is linear and multiplicative and satisfies
$$\Phi(\bar{f}) = (\Phi f)^*, \|\Phi f\| = \|f\|_\infty \quad (f \in L^\infty(E)).$$

3. *B is the closure [in the norm topology of $\mathcal{B}(H)$] of the set of all finite linear combinations of the projections $E(\omega)$.*

4. *If $\omega \subset \Delta$ is open and nonempty, then $E(\omega) \neq 0$.*

5. *An operator $S \in \mathcal{B}(H)$ commutes with every $T \in A$ if and only if S com mutes with every projection $E(\omega)$.*

Proof. Recall that (1) is an abbreviation for
$$(Tx, y) = \int_\Delta \widehat{T}\, dE_{x,y} \quad (x, y \in H, T \in A). \tag{3}$$

Since $\mathcal{B}(H)$ is a B^*-algebra (Section 12.9), our given algebra A is a commutative B^*-algebra. The Gelfand-Naimark theorem 11.18 asserts therefore that $T \to \widehat{T}$ is an isometric *-isomorphism of A onto $C(\Delta)$.

This leads to an easy proof of the uniqueness of E. Suppose E satisfies (3). Since $\widehat{T}$ ranges over all of $C(\Delta)$, the assumed regularity of the complex Borel measures $E_{x,y}$ shows that each $E_{x,y}$ is uniquely determined by (3); this follows from the uniqueness assertion that is part of the Riesz representation theorem ([23], Th. 6.19). Since, by definition, $(E(\omega)x, y) = E_{x,y}(\omega)$, each projection $E(\omega))$ is also uniquely determined by (3).

This uniqueness proof motivates the following proof of the existence of E. If $x \in H$ and $y \in H$, Theorem 11.18 shows that $\widehat{T} \mapsto (Tx, y)$ is a bounded linear functional on $C(\Delta)$, of norm $\le \|x\|\|y\|$, since $\|\widehat{T}\|_\infty = \|T\|$. The Riesz representation theorem supplies us therefore with unique regular complex Borel measures $\mu_{x,y}$ on Δ such that

$$(Tx, y) = \int_\Delta \widehat{T}\, d\mu_{x,y} \quad (x, y \in H, T \in A). \tag{4}$$

For fixed T, the left side of (4) is a bounded sesquilinear functional on H, hence so is the right side, and it remains so if the continuous function $\widehat{T}$ is replaced by an arbitrary bounded Borel function f. To each such f corresponds therefore an operator $\Phi f \in \mathcal{B}(H)$ (see Theorem 12.8) such that

$$((\Phi f)x, y) = \int_\Delta f\, d\mu_{x,y} \quad (x, y \in H). \tag{5}$$

Comparison of (4) and (5) shows that $\Phi\hat{T} = T$. Thus Φ is an extension of the inverse of the Gelfand transform.

It is clear that Φ is linear.

Part of the Gelfand-Naimark theorem states that T is self-adjoint if and only if $\hat{T}$ is real-valued. For such T,

$$\int_\Delta \widehat{T}\, d\mu_{x,y} = (Tx, y) = (x, Ty) = \overline{(Ty, x)} = \overline{\int_\Delta \hat{T} d\mu_{y,x}},$$

and this implies that $\mu_{y,x} = \overline{\mu_{x,y}}$. Hence,

$$((\Phi\bar{f})x, y) = \int_\Delta \bar{f}\, d\mu_{x,y} = \overline{\int_\Delta f\, d\mu_{y,x}} = \overline{((\Phi f)y, x)} = (x, (\Phi f)y)$$

for all $x, y \in H$, so that

$$\Phi\bar{f} = (\Phi f)^*. \tag{6}$$

Our next objective is the equality

$$\Phi(fg) = (\Phi f)(\Phi g) \tag{7}$$

for bounded Borel functions f, g on Δ. If $S \in A$ and $T \in A$, then $(ST)^\wedge = \widehat{S}\widehat{T}$; hence

$$\int_\Delta \hat{S}\hat{T}\, d\mu_{x,y} = (STx, y) = \int_\Delta \widehat{S}\, d\mu_{Tx,y}.$$

This holds for every $\widehat{S} \in C(\Delta)$; hence the two integrals are equal if $\widehat{S}$ is replaced by any bounded Borel function f. Thus

$$\int_\Delta f\widehat{T}d\mu_{x,y} = \int_\Delta f\ d\mu_{Tx,y} = ((\Phi f)Tx, y) = (Tx, z) = \int_\Delta \hat{T}d\mu_{x,z},$$

where we put $z = (\Phi f)^* y$. Again, the first and last integrals remain equal if $\widehat{T}$ is replaced by g. This gives

$$\begin{aligned}(\Phi(fg)x, y) &= \int_\Delta fg\ d\mu_{x,y} = \int_\Delta g\ d\mu_{x,z} \\ &= ((\Phi g)x, z) = ((\Phi g)x, (\Phi f)^* y) = (\Phi(f)\Phi(g)x, y),\end{aligned}$$

and (7) is proved.

We are finally ready to define E: If ω is a Borel subset of Δ, let χ_ω be its characteristic function, and put

$$E(\omega) = \Phi(\chi_\omega).$$

By (7), $E(\omega \cap \omega') = E(\omega)E(\omega')$. With $\omega' = \omega$, this shows that each $E(\omega)$ is a projection. Since Φf is self-adjoint when f is real, by (6), each $E(\omega)$ is self-adjoint. It is clear that $E(\emptyset) = \Phi(0) = 0$. That $E(\Delta) = I$ follows from (4) and (5). The finite additivity of E is a consequence of (5), and, for all $x, y \in H$,

$$E_{x,y}(\omega) = (E(\omega)x, y) = \int_\Delta \chi_\omega\ d\mu_{x,y} = \mu_{x,y}(\omega).$$

Thus (5) becomes (2). That $\|\Phi f\| = \|f\|_\infty$ follows now from Theorem 12.21.

This completes the proof of (1) and (2).

Part (3) is now clear because every $f \in L^\infty(E)$ is a uniform limit of simple functions (i.e., of functions with only finitely many values).

Suppose next that ω is open and $E(\omega) = 0$. If $T \in A$ and $\widehat{T}$ has its support in ω, (1) implies that $T = 0$; hence $\widehat{T} = 0$. Since $\widehat{A} = C(\Delta)$, Urysohn's lemma implies now that $\omega = \emptyset$. This proves (4).

To prove (5), choose $S \in \mathcal{B}(H)$, $x \in H$, $y \in H$, and put $z = S^* y$. For any $T \in A$ and any Borel set $\omega \subset \Delta$ we then have

$$(STx, y) = (Tx, z) = \int_\Delta \widehat{T}\ dE_{x,z}, \tag{8}$$

$$(TSx, y) = \int_\Delta \hat{T}\ dE_{Sx,y}, \tag{9}$$

$$(SE(\omega)x, y) = (E(\omega)x, z) = E_{x,z}(\omega),$$

$$(E(\omega)Sx, y) = E_{Sx,y}(\omega).$$

If $ST = TS$ for every $T \in A$, the measures in (8) and (9) are equal, so that $SE(\omega) = E(\omega)S$. The same argument establishes the converse. This completes the proof. $\square$