Mathematical Analysis 2

Midterm 3 – 18/12/2023 – 14:00-15:30

<u>Do **not** turn this sheet over until instructed to do so.</u>

Name: , Seat:

This test consists of 5 questions, you have 1.5 hours to attempt to solve them.

Respond to the questions with **fully and clearly argued solutions**. If there is something that you are uncertain about, do the best you can and demonstrate your knowledge of the relevant concepts.

Include your seat number () on each sheet of paper on which you write your solutions.

- Students are permitted to bring only the following items to their desk in the test room: pens / pencils, drinking bottle. Calculators and any other electronic items are not permitted.
- Paper for rough calculations will be provided in the test room. After the test the paper used during the test remains in the test room.
- Under penalty of exclusion, during the test it is forbidden to communicate, using any means, with anyone except the test invigilators. If you want to communicate, raise your hand and an invigilator will come to you.
- It is forbidden to have anything that can be used for communication (e.g., telephone, smartwatch, computer, headphones, earbuds, messenger pigeon) on your person or close to you during the test.
- You may choose to leave the test early but only after receiving confirmation from an invigilator.
- In case of any inconsistencies with the written exam, an interview may be requested to confirm the grade.

I affirm that I will not give or receive any unauthorized help on this test. If I am offered unauthorized help I will notify an invigilator.

Name: , Signature:

Question 1. Evaluate the integral $I = \iint_R f(x,y)\,dxdy$ by iterated integration, where:

(a) $R = [2,3] \times [0,1]$, $f(x,y) = 4y^3$

(b) $R = [0,1] \times [0,2]$, $f(x,y) = 4x(x+y)$

(c) $R = [0,\pi] \times [0,1]$, $f(x,y) = x \sin xy$

Question 2. Let

$$\mathbf{f}(x,y,z) = \begin{pmatrix} e^{y-1} \\ x^2+y^2 \\ z^2 \end{pmatrix}, \quad \mathbf{g}(x,y,z) = \begin{pmatrix} 0 \\ x - \sin z \\ x + y\cos z \end{pmatrix}.$$

Evaluate $(\nabla \cdot \mathbf{f})(2,1,3)$, $(\nabla \times \mathbf{f})(2,1,3)$, $(\nabla \cdot \mathbf{g})(2,7,\pi/2)$ and $(\nabla \times \mathbf{g})(2,7,\pi/2)$.

Question 3. Use polar coordinates to evaluate the integral

$$I = \int_0^2 \int_0^{\sqrt{3}x} \frac{2x^2+y^2}{x^2+y^2}\,dy\,dx.$$

Hints: $\int \sec^2 x\,dx = \tan x + C$. *The answer has the form* $\frac{a\pi}{b} + a\sqrt{b}$ *with integers a, b.*

Question 4. (a) Which of the following choices of $\mathbf{r}(u,v)$ is a valid parameterization of the surface $z = x^2 - y^2$:

$$[\alpha]\ (u\cos v, u\sin v, u^2), \quad [\beta]\ (u+v, u-v, 4uv), \quad [\gamma]\ (u+v, u-v, uv)?$$

(b) Find the fundamental vector product $\mathbf{N}$ for that parameterization.

(c) Use the results of the previous parts to evauate the surface integral $\iint_S \mathbf{f}\cdot\mathbf{n}\,dS$, where S is the part of the surface $z = x^2 - y^2$ with (x,y) in the square with corners $(2,0)$, $(0,2)$, $(-2,0)$, $(0,-2)$ and $\mathbf{n}$ is the normal vector in the upwards direction, and $\mathbf{f}(x,y,z) = (y, x, z^2/16)$.

Question 5. Let V be the truncated cone $\left\{(x,y,z) : \sqrt{x^2+y^2} \le 2 - z,\ 0 \le z \le 1\right\}$, and let S be the curved part of the boundary of V, with outward normal vector $\mathbf{n}$.

Find the integral $\iint_S \mathbf{f}\cdot\mathbf{n}\,dS$, where $\mathbf{f}(x,y,z) = (2xz + \tan y, e^{x^2} - yz, z^2+1)$.

Hint: use Gauss' theorem to rewrite this in terms of three other integrals.