Mathematical Analysis 2 – Call 2 – 24/02/2026

Part 1 – 9:30-11:00

Question 1. For each of the following functions, compute the gradient ∇f (i.e. $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$).

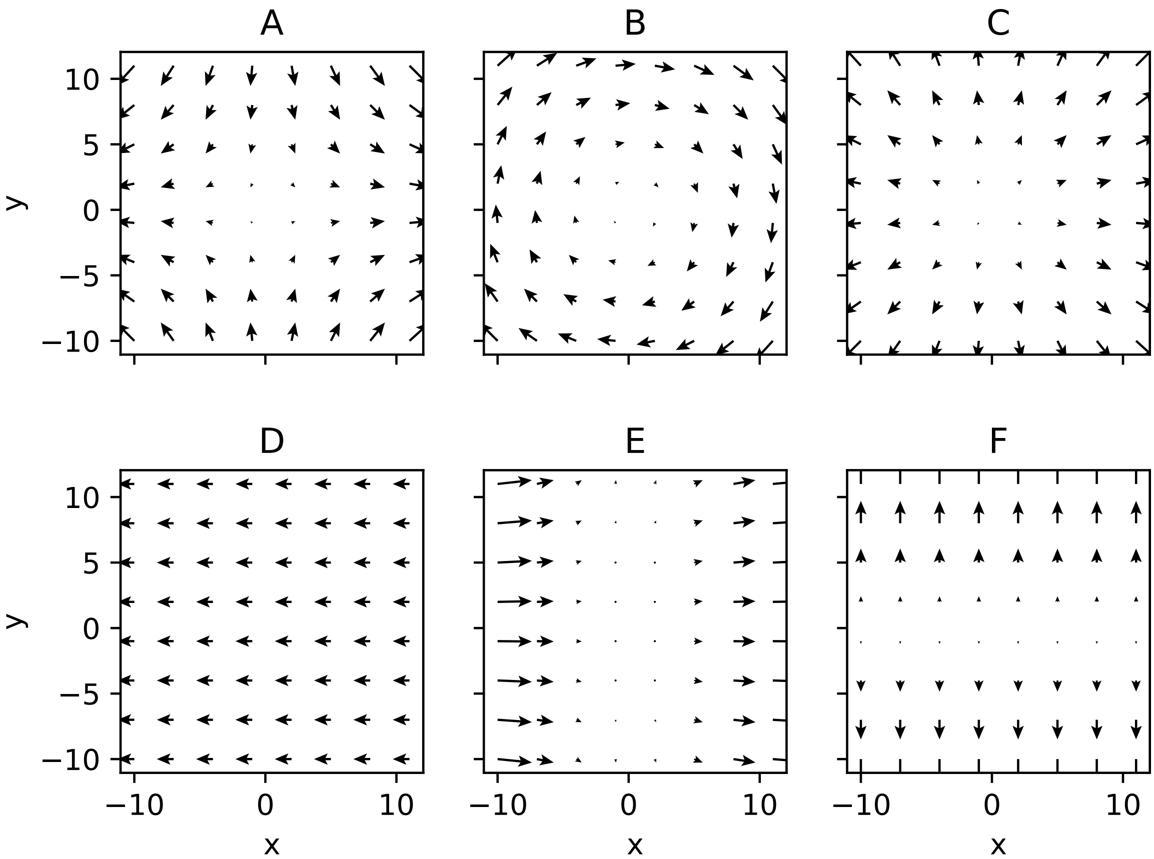
1. $f(x, y) = x^3 \ln(xy^2)$
2. $g(x, y) = e^{x^2 - y} \sin(xy)$
3. $h(x, y) = \displaystyle\int_0^{xy^2} \sin(t^2)\, dt$

Question 2. Find all stationary points of $f(x, y) = x^4 + y^4 - 16xy$ and classify each as a local maximum, local minimum, or saddle point.

Question 3. Consider the following vector fields defined on $\mathbb{R}^2$:

$$
\begin{array}{lll}
\mathbf{F}_1(x, y) = x\mathbf{i} + y\mathbf{j} & \mathbf{F}_2(x, y) = y\mathbf{i} - x\mathbf{j} & \mathbf{F}_3(x, y) = x^2\mathbf{i} + y\mathbf{j} \\
\mathbf{F}_4(x, y) = -\mathbf{i} & \mathbf{F}_5(x, y) = y\mathbf{j} & \mathbf{F}_6(x, y) = x\mathbf{i} - y\mathbf{j}
\end{array}
$$

1. Match each to one of the plots and briefly explain the logic/calculation for matching each.
2. Calculate the curl, $\nabla \times \mathbf{F}_n(x, y)$ for each of the vector fields.


A
B
C
D
E
F
y
x
10
5
0
−5
−10
−10
0
10

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Part 2 – 11:30-13:00

Question 4. Verify Green's theorem for the vector field $\mathbf{F}(x,y) = P\mathbf{i} + Q\mathbf{j} = (x^2 - y)\mathbf{i} + (2xy + 3)\mathbf{j}$ and the region D which is the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. That is, compute both sides of

$$\oint_C \mathbf{F} \cdot d\boldsymbol{\alpha} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where $\boldsymbol{\alpha}$ denotes the path counter-clockwise around the boundry of D and verify that both sides they are equal.

Hint: The boundary C consists of three line segments. Parametrize each one separately.

Question 5. Evaluate the double integral

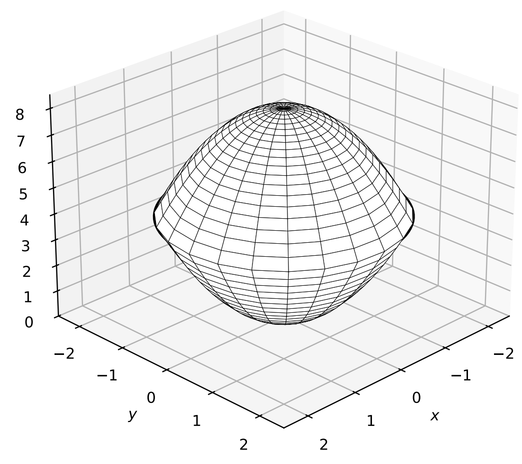
$$\iint_D xy \, dA$$

where D is the region in the first quadrant bounded by the curves $y = x$, $y = 3x$, $xy = 1$, and $xy = 4$.

Hint: Use the substitution $u = y/x$, $v = xy$.

Question 6. Compute the volume of the solid region bounded between the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$.

Method: First find where the two surfaces intersect. Then set up and evaluate the appropriate integral using cylindrical coordinates.



Solutions – Call 2 – 24/02/2026

Question 1 Solution:
Version A:

1. $f(x, y) = x^3 \ln(xy^2)$

   Note that $\ln(xy^2) = \ln x + 2 \ln y$ (for $x > 0, y > 0$). Using the product rule:

$$\frac{\partial f}{\partial x} = 3x^2 \ln(xy^2) + x^3 \cdot \frac{1}{x} = 3x^2 \ln(xy^2) + x^2$$
$$\frac{\partial f}{\partial y} = x^3 \cdot \frac{2y}{xy^2} = x^3 \cdot \frac{2}{y} = \frac{2x^3}{y}$$

$$\nabla f = \left(3x^2 \ln(xy^2) + x^2\right) \mathbf{i} + \frac{2x^3}{y} \mathbf{j}$$

2. $g(x, y) = e^{x^2-y} \sin(xy)$

   Using the product rule and chain rule:

$$\frac{\partial g}{\partial x} = 2x\, e^{x^2-y} \sin(xy) + e^{x^2-y} \cdot y \cos(xy) = e^{x^2-y}\big(2x \sin(xy) + y \cos(xy)\big)$$
$$\frac{\partial g}{\partial y} = (-1)\, e^{x^2-y} \sin(xy) + e^{x^2-y} \cdot x \cos(xy) = e^{x^2-y}\big(x \cos(xy) - \sin(xy)\big)$$

$$\nabla g = e^{x^2-y}\big(2x \sin(xy) + y \cos(xy)\big)\mathbf{i} + e^{x^2-y}\big(x \cos(xy) - \sin(xy)\big)\mathbf{j}$$

3. $h(x, y) = \displaystyle\int_0^{xy^2} \sin(t^2)\, dt$

   Using the Fundamental Theorem of Calculus with chain rule:

$$\frac{\partial h}{\partial x} = \sin\!\big((xy^2)^2\big) \cdot \frac{\partial}{\partial x}(xy^2) = y^2 \sin(x^2y^4)$$
$$\frac{\partial h}{\partial y} = \sin\!\big((xy^2)^2\big) \cdot \frac{\partial}{\partial y}(xy^2) = 2xy \sin(x^2y^4)$$

$$\nabla h = y^2 \sin(x^2y^4)\, \mathbf{i} + 2xy \sin(x^2y^4)\, \mathbf{j}$$

Version B:

1. $f(x, y) = x^3 \ln(x^2y)$

   Note that $\ln(x^2y) = 2 \ln x + \ln y$ (for $x > 0, y > 0$). Using the product rule:

$$\frac{\partial f}{\partial x} = 3x^2 \ln(x^2y) + x^3 \cdot \frac{2}{x} = 3x^2 \ln(x^2y) + 2x^2$$
$$\frac{\partial f}{\partial y} = x^3 \cdot \frac{1}{y} = \frac{x^3}{y}$$

$$\nabla f = \left(3x^2 \ln(x^2y) + 2x^2\right) \mathbf{i} + \frac{x^3}{y} \mathbf{j}$$

2. $g(x,y) = e^{x-y^2}\sin(xy)$

Using the product rule and chain rule:

$$\frac{\partial g}{\partial x} = e^{x-y^2}\sin(xy) + e^{x-y^2}\cdot y\cos(xy) = e^{x-y^2}\big(\sin(xy) + y\cos(xy)\big)$$
$$\frac{\partial g}{\partial y} = (-2y)\,e^{x-y^2}\sin(xy) + e^{x-y^2}\cdot x\cos(xy) = e^{x-y^2}\big(x\cos(xy) - 2y\sin(xy)\big)$$

$$\nabla g = e^{x-y^2}\big(\sin(xy) + y\cos(xy)\big)\mathbf{i} + e^{x-y^2}\big(x\cos(xy) - 2y\sin(xy)\big)\mathbf{j}$$

3. $h(x,y) = \displaystyle\int_0^{x^2y}\sin(t^2)\,dt$

Using the Fundamental Theorem of Calculus with chain rule:

$$\frac{\partial h}{\partial x} = \sin\big((x^2y)^2\big)\cdot\frac{\partial}{\partial x}(x^2y) = 2xy\sin(x^4y^2)$$
$$\frac{\partial h}{\partial y} = \sin\big((x^2y)^2\big)\cdot\frac{\partial}{\partial y}(x^2y) = x^2\sin(x^4y^2)$$

$$\nabla h = 2xy\sin(x^4y^2)\,\mathbf{i} + x^2\sin(x^4y^2)\,\mathbf{j}$$

Question 2 Solution:

Both versions follow the same method. We find the stationary points by solving $\nabla f = \mathbf{0}$.

Version A: $f(x,y) = x^4 + y^4 - 4xy$.

$$\frac{\partial f}{\partial x} = 4x^3 - 4y = 0 \quad\Longrightarrow\quad y = x^3$$
$$\frac{\partial f}{\partial y} = 4y^3 - 4x = 0 \quad\Longrightarrow\quad x = y^3$$

Substituting $y = x^3$ into $x = y^3$: $x = (x^3)^3 = x^9$, so $x^9 - x = 0$, i.e. $x(x^8 - 1) = 0$.

This gives $x = 0$ or $x^8 = 1$. The real solutions of $x^8 = 1$ are $x = 1$ and $x = -1$.

Critical points: $(0,0)$, $(1,1)$, and $(-1,-1)$.

The Hessian matrix is: $H = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$

At $(0,0)$**:** $\det(H) = 0 - 16 = -16 < 0 \Rightarrow$ **saddle point**.

At $(1,1)$**:** $\det(H) = 144 - 16 = 128 > 0$, $f_{xx} = 12 > 0 \Rightarrow$ **local minimum**, $f(1,1) = 1 + 1 - 4 = -2$.

At $(-1,-1)$**:** $\det(H) = 128 > 0$, $f_{xx} = 12 > 0 \Rightarrow$ **local minimum**, $f(-1,-1) = 1 + 1 - 4 = -2$.

Version B: $f(x,y) = x^4 + y^4 - 16xy$.

$$\frac{\partial f}{\partial x} = 4x^3 - 16y = 0 \quad\Longrightarrow\quad y = x^3/4$$
$$\frac{\partial f}{\partial y} = 4y^3 - 16x = 0 \quad\Longrightarrow\quad x = y^3/4$$

Substituting $y = x^3/4$ into $x = y^3/4$: $x = (x^3/4)^3/4 = x^9/256$, so $x^9 - 256x = 0$, i.e. $x(x^8 - 256) = 0$.

This gives $x = 0$ or $x^8 = 256 = 2^8$, so $x = \pm 2$.
Critical points: $(0,0)$, $(2,2)$, and $(-2,-2)$.
The Hessian matrix is: $H = \begin{pmatrix} 12x^2 & -16 \\ -16 & 12y^2 \end{pmatrix}$
At $(0,0)$**:** $\det(H) = 0 - 256 = -256 < 0 \Rightarrow$ **saddle point**.
At $(2,2)$**:** $\det(H) = 48 \cdot 48 - 256 = 2048 > 0$, $f_{xx} = 48 > 0 \Rightarrow$ **local minimum**, $f(2,2) = 16 + 16 - 64 = -32$.
At $(-2,-2)$**:** $\det(H) = 2048 > 0$, $f_{xx} = 48 > 0 \Rightarrow$ **local minimum**, $f(-2,-2) = 16 + 16 - 64 = -32$.

Question 3 Solution:

1. Matching:
   - $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$ matches **C**: At any point (x,y), the vector points radially outward from the origin with magnitude growing proportionally to the distance. This is the classic source field.
   - $\mathbf{F}_2 = y\mathbf{i} - x\mathbf{j}$ matches **B**: At $(1,0)$ the vector is $(0,-1)$ (pointing down), at $(0,1)$ it is $(1,0)$ (pointing right). This traces clockwise circles around the origin.
   - $\mathbf{F}_3 = x^2\mathbf{i} + y\mathbf{j}$ matches **E**: The horizontal component x^2 is always non-negative, so all vectors have a rightward (or zero) horizontal component regardless of the sign of x. Near $x = 0$ the vertical component y dominates.
   - $\mathbf{F}_4 = -\mathbf{i}$ matches **D**: A constant field where every vector points to the left with the same magnitude.
   - $\mathbf{F}_5 = y\mathbf{j}$ matches **F**: Purely vertical vectors, pointing upward above the x-axis, downward below, with zero on the x-axis itself.
   - $\mathbf{F}_6 = x\mathbf{i} - y\mathbf{j}$ matches **A**: Along the x-axis vectors point outward; along the y-axis vectors point toward the origin. This is a saddle pattern with expansion in the x-direction and contraction in the y-direction.
2. Curls (for $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, the curl is $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$):

$$\begin{aligned}
\nabla \times \mathbf{F}_1(x,y) &= \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) = 0 - 0 = 0 \\
\nabla \times \mathbf{F}_2(x,y) &= \frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y) = -1 - 1 = -2 \\
\nabla \times \mathbf{F}_3(x,y) &= \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x^2) = 0 - 0 = 0 \\
\nabla \times \mathbf{F}_4(x,y) &= \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(-1) = 0 - 0 = 0 \\
\nabla \times \mathbf{F}_5(x,y) &= \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(0) = 0 - 0 = 0 \\
\nabla \times \mathbf{F}_6(x,y) &= \frac{\partial}{\partial x}(-y) - \frac{\partial}{\partial y}(x) = 0 - 0 = 0
\end{aligned}$$

Question 4 Solution:

We verify Green's theorem: $\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$
The triangle D has vertices $(0,0)$, $(1,0)$, $(0,1)$, so $0 \le x \le 1$ and $0 \le y \le 1-x$.

Version A: $P = x^2 - y, Q = 2xy + 3.$
Right-hand side (double integral): $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y - (-1) = 2y + 1.$

$$\begin{aligned}\iint_D (2y+1)\,dA &= \int_0^1 \int_0^{1-x} (2y+1)\,dy\,dx = \int_0^1 \left[y^2 + y\right]_0^{1-x} dx \\ &= \int_0^1 \left((1-x)^2 + (1-x)\right) dx = \int_0^1 (2 - 3x + x^2)\,dx \\ &= \left[2x - \frac{3x^2}{2} + \frac{x^3}{3}\right]_0^1 = 2 - \frac{3}{2} + \frac{1}{3} = \boxed{\frac{5}{6}}\end{aligned}$$

Left-hand side (line integral): The boundary consists of three segments.
C_1: $(0,0) \to (1,0)$, $y = 0, dy = 0$: $\int_{C_1} = \int_0^1 x^2\,dx = \frac{1}{3}$
C_2: $(1,0) \to (0,1)$, $x = 1-t, y = t, dx = -dt, dy = dt$:

$$P = (1-t)^2 - t = 1 - 3t + t^2, \qquad Q = 2(1-t)t + 3 = 2t - 2t^2 + 3$$

$$\int_{C_2} = \int_0^1 (2 + 5t - 3t^2)\,dt = 2 + \frac{5}{2} - 1 = \frac{7}{2}$$

C_3: $(0,1) \to (0,0)$, $x = 0, dx = 0$: $\int_{C_3} = \int_1^0 3\,dy = -3$

Total: $\frac{1}{3} + \frac{7}{2} - 3 = \boxed{\frac{5}{6}}$ ✓

Version B: $P = x^2 + y, Q = 2xy - 3.$
Right-hand side (double integral): $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y - 1.$

$$\begin{aligned}\iint_D (2y-1)\,dA &= \int_0^1 \int_0^{1-x} (2y-1)\,dy\,dx = \int_0^1 \left[y^2 - y\right]_0^{1-x} dx \\ &= \int_0^1 \left((1-x)^2 - (1-x)\right) dx = \int_0^1 (-x + x^2)\,dx \\ &= \left[-\frac{x^2}{2} + \frac{x^3}{3}\right]_0^1 = -\frac{1}{2} + \frac{1}{3} = \boxed{-\frac{1}{6}}\end{aligned}$$

Left-hand side (line integral):
C_1: $(0,0) \to (1,0)$, $y = 0, dy = 0$: $\int_{C_1} = \int_0^1 x^2\,dx = \frac{1}{3}$
C_2: $(1,0) \to (0,1)$, $x = 1-t, y = t, dx = -dt, dy = dt$:

$$P = (1-t)^2 + t = 1 - t + t^2, \qquad Q = 2(1-t)t - 3 = 2t - 2t^2 - 3$$

$$\begin{aligned}\int_{C_2} &= \int_0^1 \left[-(1 - t + t^2) + (2t - 2t^2 - 3)\right] dt = \int_0^1 (-4 + 3t - 3t^2)\,dt \\ &= \left[-4t + \frac{3t^2}{2} - t^3\right]_0^1 = -4 + \frac{3}{2} - 1 = -\frac{7}{2}\end{aligned}$$

C_3: $(0,1) \to (0,0), x = 0, dx = 0$: $\displaystyle\int_{C_3} = \int_1^0 (-3)\,dy = 3$

Total: $\frac{1}{3} - \frac{7}{2} + 3 = \frac{2}{6} - \frac{21}{6} + \frac{18}{6} = \boxed{-\dfrac{1}{6}}$ ✓

Question 5 Solution:

We use the substitution $u = y/x, v = xy$. The Jacobian computation is the same for both versions.

Jacobian: We compute $\dfrac{\partial(u,v)}{\partial(x,y)}$:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -y/x^2 & 1/x \\ y & x \end{vmatrix} = \frac{-y}{x^2} \cdot x - \frac{1}{x} \cdot y = -\frac{2y}{x} = -2u$$

Therefore $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \dfrac{1}{2u}$. The integrand is $xy = v$.

Version A: $y = x, y = 3x, xy = 1, xy = 4 \;\Rightarrow\; u \in [1,3], v \in [1,4]$.

$$\iint_D xy\,dA = \int_1^3 \int_1^4 v \cdot \frac{1}{2u}\,dv\,du = \frac{1}{2}\int_1^3 \frac{du}{u} \cdot \int_1^4 v\,dv$$
$$= \frac{1}{2} \cdot \ln 3 \cdot \frac{15}{2} = \boxed{\frac{15\ln 3}{4}}$$

Version B: $y = x, y = 3x, xy = 1, xy = 3 \;\Rightarrow\; u \in [1,3], v \in [1,3]$.

$$\iint_D xy\,dA = \int_1^3 \int_1^3 v \cdot \frac{1}{2u}\,dv\,du = \frac{1}{2}\int_1^3 \frac{du}{u} \cdot \int_1^3 v\,dv$$
$$= \frac{1}{2} \cdot \ln 3 \cdot 4 = \boxed{2\ln 3}$$

Question 6 Solution:

The two paraboloids are $z = x^2 + y^2$ (opening upward) and $z = 8 - x^2 - y^2$ (opening downward).

Intersection: Setting them equal: $x^2 + y^2 = 8 - x^2 - y^2$, so $2(x^2 + y^2) = 8$, giving $x^2 + y^2 = 4$, i.e. $r = 2$ in cylindrical coordinates. The surfaces meet at $z = 4$.

Using cylindrical coordinates (r, θ, z), where z ranges from r^2 (lower paraboloid) to $8 - r^2$ (upper paraboloid):

$$V = \int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} r\,dz\,dr\,d\theta$$
$$= 2\pi \int_0^2 r\big(8 - r^2 - r^2\big)\,dr = 2\pi \int_0^2 (8r - 2r^3)\,dr$$
$$= 2\pi \left[4r^2 - \frac{r^4}{2}\right]_0^2 = 2\pi(16 - 8) = \boxed{16\pi}$$