$\boxed{Q1}$ let $f(y) = \sqrt{y}$

$f(1) = 1$

$f'(1) = 1/2$

$f''(1) = -\frac{1}{2}\cdot\frac{1}{2}$

$f'''(1) = +\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}$

$f^{iv}(1) = -\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}$

$f'(y) = \frac{1}{2}y^{-1/2}$

$f''(y) = \frac{1}{2}(-\frac{1}{2})y^{-3/2}$

$f'''(y) = \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})y^{-5/2}$

$f^{iv}(y) = \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})y^{-7/2}$

$$\sqrt{y} \approx 1 + \frac{1}{2}(\overset{y-1}{\cancel{1-y}}) - \frac{1}{2!}\cdot\frac{1}{2}(\overset{y-1}{\cancel{1-y}})^2 + \frac{1}{3!}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}(\overset{y-1}{\cancel{1-y}})^3$$

$$- \frac{1}{4!}\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}(\overset{y-1}{\cancel{1-y}})^4 + \cdots$$

$$\sqrt{1+x^3} \approx 1 + \frac{1}{2}x^3 - \frac{1}{2!}\cdot\frac{1}{2}x^6 + \frac{1}{3!}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}x^9$$

$$- \frac{1}{4!}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}x^{12} + \cdots$$

$y = 1+x^3$

$\Leftrightarrow$

$\cancel{1-y = -x^3}$

$\boxed{y-1 = x^3}$

Integrating termwise:

$$\int\sqrt{1+x^3}\,dx = C + x + \frac{1}{2}\cdot\frac{1}{4}x^4 - \frac{1}{2!}\cdot\frac{1}{2}\cdot\frac{1}{7}x^7 + \frac{1}{3!}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{10}x^{10}$$

$$- \frac{1}{4!}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{13}x^{13} + \cdots$$

$$= C + x + \frac{x^4}{8} - \frac{x^7}{28} + \frac{x^{10}}{160} - \frac{5}{1664}x^{13} + \cdots$$

$$\left(\begin{array}{r} 128 \\ \times\ 13 \\ \hline 384 \\ 1280 \end{array} \Big/ 1664\right)$$

$\boxed{Q2}$ $f(x,y) = 9x^2 + y^2 + 3.$

let $g(x,y) = \frac{4}{9}x^2 + y^2$

$\nabla f(x,y) = \begin{pmatrix} 18x \\ 2y \end{pmatrix}, \quad \nabla g(x,y) = \begin{pmatrix} 8/9\, x \\ 2y \end{pmatrix}$

$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 9 \end{cases}$

$\Leftrightarrow \begin{cases} 18x = \lambda \, 8/9\, x & (1) \Rightarrow x = 0 \text{ or } \lambda = 18 \cdot \frac{9}{8} = \frac{9^2}{4} \\ 2y = \lambda 2y & (2) \Rightarrow y = 0 \text{ or } \lambda = 1 \\ \frac{4}{9}x^2 + y^2 = 9 & (3) \end{cases}$

Case $x = 0$

$(3) \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$

Case $y = 0$

$(3) \Rightarrow \frac{4}{9}x^2 = 9 \Rightarrow x^2 = \left(\frac{9}{2}\right)^2 \Rightarrow x = \pm\frac{9}{2}$

4 solutions: $(0,-3), (0,3), (-\frac{9}{2},0), (\frac{9}{2},0).$

$f(0,-3) = 9(0^2) + (-3)^2 + 3 = 9 + 3 = 12$

$f(0,3) = 12$

$f(-\frac{9}{2},0) = 9\left(-\frac{9}{2}\right)^2 + 0^2 + 3 = 9 \cdot \frac{81}{2} = 729$

$f(\frac{9}{2},0) = 729$

$\begin{array}{r} 81 \\ \times \quad 9 \\ \hline 729 \end{array}$

$\boxed{Q3}$ $\nabla \cdot F_1(x,y) = 0$ $\qquad$ $\nabla \cdot F_4(x,y) = 0$

$\nabla \cdot F_2(x,y) = 0$ $\qquad$ $\nabla \cdot F_5(x,y) = 2-2 = 0$

$\nabla \cdot F_3(x,y) = 0$ $\qquad$ $\nabla \cdot F_6(x,y) = \frac{1}{2}\cos\left(\frac{x}{2}\right) + \frac{1}{2}\cos\left(\frac{y}{2}\right)$

$F_1 \longrightarrow D$ (y component always positive)

$F_2 \longrightarrow B$ ~~plot~~($F_2(1,1) = -\underline{i} + \underline{j}$

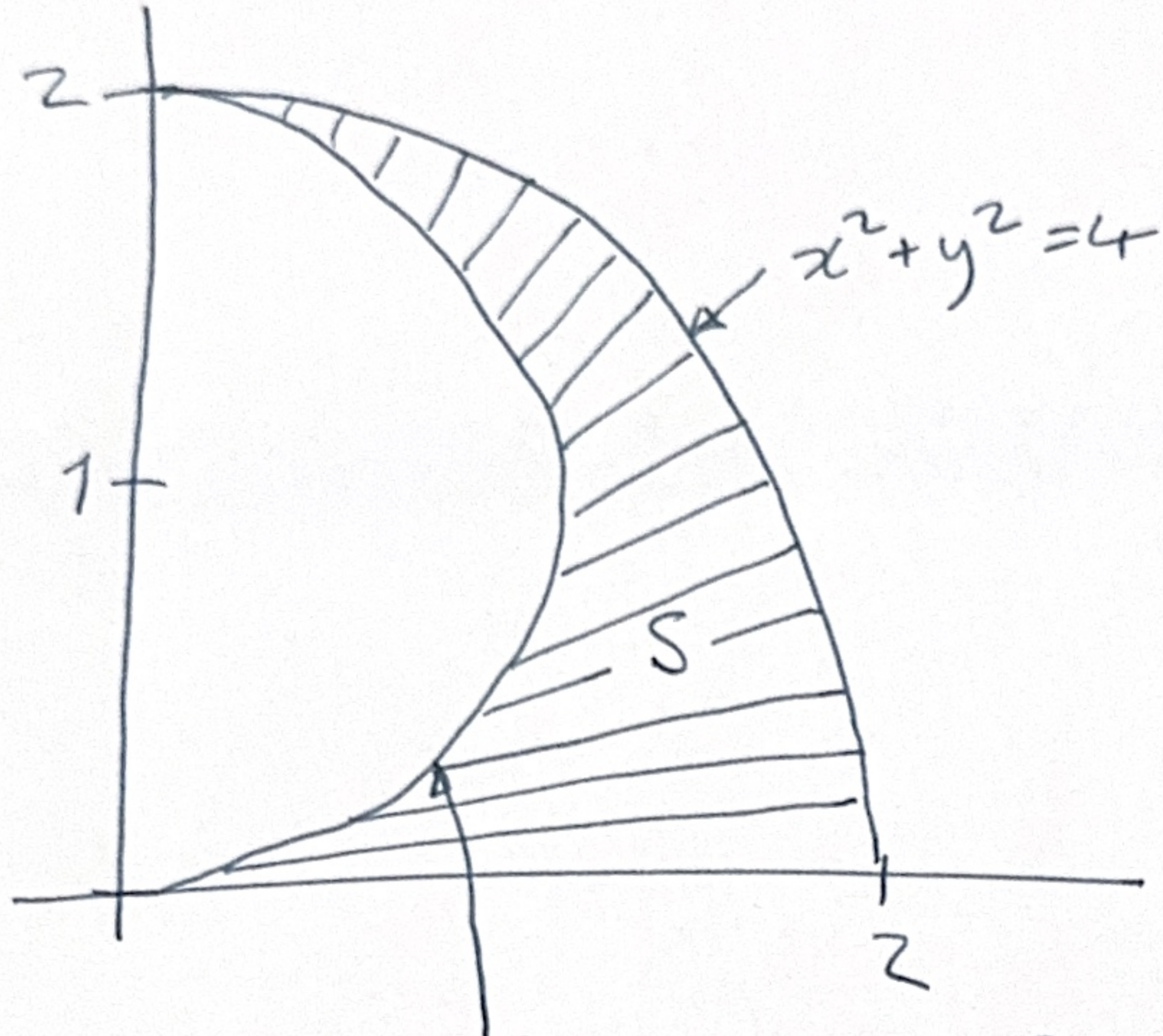
$F_3 \longrightarrow C$ $\left(F_3(1,1) = \underline{i} + \underline{j}\right)$

$F_4 \longrightarrow F$ $\left(F_4(1,1) = 2\underline{i} - 2\underline{j}\right)$

$F_5 \longrightarrow E$ (x +ve $\Rightarrow$ x comp. +ve,
x -ve $\Rightarrow$ x comp. -ve)

$F_6 \longrightarrow A$ (periodic in x,y)

Q4



$$x^2+y^2-2y=0 \iff x^2+(y-1)^2=1$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Calculating new region:

$$\boxed{0 \le \theta \le \pi/2}$$

$$x^2+y^2=4 \Rightarrow \boxed{r \le 2}$$

$$x^2+y^2-2y=0 \Rightarrow r^2-2r\sin\theta=0$$

$$\iff r(r-2\sin\theta)=0 \Rightarrow r=0 \text{ or } \boxed{r=2\sin\theta}$$

Let $T = \{(r,\theta): 0 \le \theta \le \pi/2,\ 2\sin\theta \le r \le 2\}$

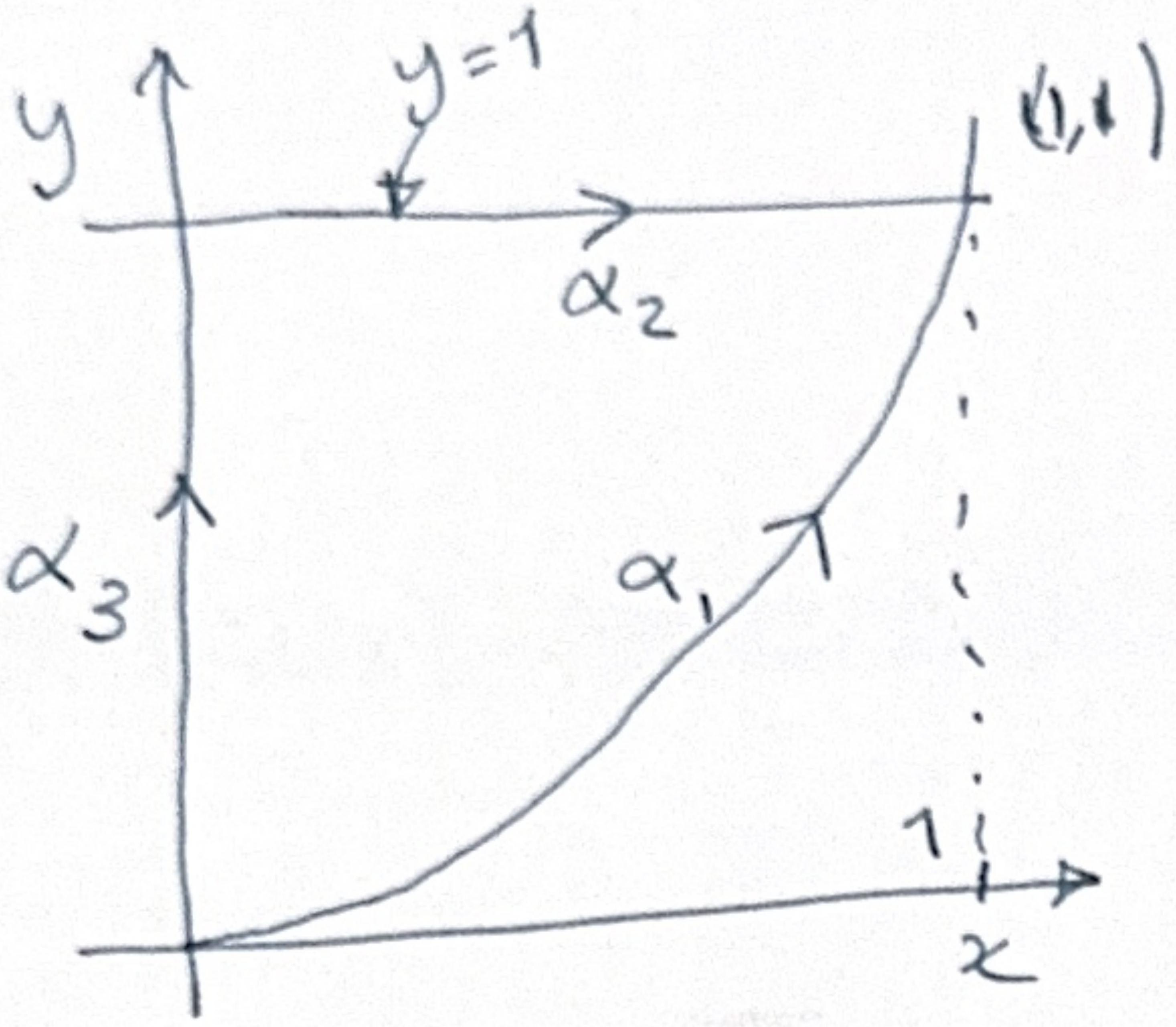
$$\iint_S x\,dx\,dy = \iint_T (r\cos\theta)(r)\,dr\,d\theta \quad \text{(Jacobian determinant)}$$

$$= \int_0^{\pi/2}\left(\int_{2\sin\theta}^{2} r^2\,dr\right)\cos\theta\,d\theta = \int_0^{\pi/2}\left(\left[\frac{r^3}{3}\right]_{2\sin\theta}^{2}\right)\cos d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2}(1-\sin^3\theta)\cos\theta\,d\theta$$

$$= \frac{8}{3}\left(\left[\sin\theta\right]_0^{\pi/2} - \left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2}\right) = \frac{8}{3}\left(1-\frac{1}{4}\right) = \frac{8}{3}\cdot\frac{3}{4} = 2$$

$\boxed{Q5}$ $F(x,y) = (ax^2y - \sin x)\mathbf{i} + (x^3 + y\ln y)\mathbf{j}$



$\alpha_1(t) = (t, t^2),\ t \in [0,1]$

$\alpha_2(t) = (t, 1),\ t \in [0,1]$

$\alpha_3(t) = (0, t),\ t \in [0,1]$

$\alpha_1'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix},\ \alpha_2'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$

$\alpha_3'(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\underline{\alpha_1:}$ $F(\alpha_1(t)) = \begin{pmatrix} at^2t^2 - \sin t \\ t^3 + t^2\ln(t^2) \end{pmatrix}$

$$\underbrace{F(\alpha_1(t))\cdot\alpha_1'(t)}_{\swarrow} = at^4 - \sin t + 2t^4 + t^3\ln(t^2)$$

$$\int_0^1 \swarrow\ dt = \left[\frac{at^5}{5} + \cos t + \frac{2}{5}t^5 + \frac{1}{4}t^4\ln(t^2) - \frac{t^4}{4}\right]_0^1$$

$$= \frac{a}{5} + \cos 1 + \frac{2}{5} - \frac{1}{4} - 1$$

$\underline{\alpha_2}$ $F(\alpha_2(t)) = \begin{pmatrix} at^2 - \sin t \\ \_\_\ ? \end{pmatrix}$

$$\underbrace{F(\alpha_2(t))\cdot\alpha_2'(t)}_{\swarrow} = at^2 - \sin t$$

$$\int_0^1 \swarrow\ dt = \left[\frac{at^3}{3} + \cos t\right]_0^1 = \frac{a}{3} + \cos 1 - 1$$

$\underline{\alpha_3}$ $F(\alpha_3(t)) = \begin{pmatrix} 0 \\ t\ln t \end{pmatrix}$

$$\underbrace{F(\alpha_3(t))\cdot\alpha_3'(t)}_{\swarrow} = t\ln t$$

$$\int_0^1 \swarrow\ dt = \left[\frac{t^2}{2}\ln t - \frac{t^2}{4}\right]_0^1 = -\frac{1}{4}$$

$$\int F\cdot d\alpha = \int F\cdot d\alpha_1 - \int F\cdot d\alpha_2 - \int F\cdot d\alpha_3$$

$$= \frac{a}{5} + \cancel{\cos 1} + \frac{2}{5} - \cancel{\frac{1}{4}} - \cancel{1} - \left(\frac{a}{3} + \cancel{\cos 1} - \cancel{1}\right) - \left(\cancel{-\frac{1}{4}}\right)$$

$$= -\frac{2}{15}a + \frac{2}{5}$$ [Or use Green's theorem for a quicker calculation]

$\boxed{Q6}$ $G(u,v) = (u, v^2 - u, u+v)$, $T = [0,2] \times [0,1]$

F.dP :

$$N(u,v) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2v \\ 1 \end{pmatrix} = \begin{pmatrix} -1-2v \\ -1 \\ 2v \end{pmatrix}$$

$$F(G(u,v)) = \begin{pmatrix} u - v^2 \\ u \\ 0 \end{pmatrix}$$

$$\begin{aligned} F(G(u,v)) \cdot N(u,v) &= (u - v^2)(-1-2v) + (u)(-1) \\ &= -u - 2uv + v^2 + 2v^3 - u \\ &= -2u - 2uv + v^2 + 2v^3 \end{aligned}$$

$$\begin{aligned} &\int_0^1 \int_0^2 F(G(u,v)) \cdot N(u,v) \, du \, dv \\ &= \int_0^1 \int_0^2 -2(1-v)u + v^2 + 2v^3 \, du \, dv \\ &= \int_0^1 -2(1-v)(2) + (v^2 + 2v^3)(2) \, dv \\ &= \left[-4v - 2v^2 + \frac{2}{3}v^3 + v^4 \right]_0^1 \\ &= -4 - 2 + \frac{2}{3} + 1 \\ &= \frac{2}{3} - 5 \\ &= -\frac{13}{3} \end{aligned}$$