Differential calculus in higher dimension

In this part of the course we work on the following skills:

- Become comfortable working with coordinates in arbitrary dimension.
- Develop an intuition for working with vector fields.
- Understand the subtleties of derivatives in dimension greater than 1, evaluate and manipulate partial derivatives, directional derivatives, Jacobian.

See also the exercises associated to this part of the course.

Here we start to consider higher dimensional space. That is, instead of $\mathbb{R}$ we consider $\mathbb{R}^n$ for $n \in \mathbb{N}$. We will particularly focus on 2D and 3D but everything also holds in any dimension. Going beyond $\mathbb{R}$ we have more options for functions and correspondingly more options for derivatives. Various different notation is commonly used. Here we will primarily use $(x, y) \in \mathbb{R}^2$, $(x, y, z) \in \mathbb{R}^3$ or, more generally, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ where $x_1 \in \mathbb{R}, \dots, x_n \in \mathbb{R}$. For example, $\mathbb{R}^2$ is the plane, $\mathbb{R}^3$ is 3D space.

Definition (inner product)

$$\mathbf{x} \cdot \mathbf{y} = \sum_{k=1}^{n} x_k y_k \in \mathbb{R}$$

We recall that the inner product being zero has a geometric meaning, it means that the two vectors are orthogonal. We also recall that the "length" of a vector is given by the norm, defined as follows.

Definition (norm)

$$\|\mathbf{x}\| = \sqrt{\mathbf{x}\cdot\mathbf{x}} = \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}}.$$

For example, in $\mathbb{R}^2$ then $\|(x,y)\| = \sqrt{x^2+y^2}$. There are various convenient properties for working with norms and inner products, in particular, the Cauchy-Schwarz inequality $|x\cdot y| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$ and the triangle inequality $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

The primary higher-dimensional functions we consider in this course are:

- Scalar fields: $f : \mathbb{R}^n \to \mathbb{R}$
- Vector fields: $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$
- Paths: $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^n$
- Change of coordinates: $\mathbf{x} : \mathbb{R}^n \to \mathbb{R}^n$

These possibilities all fit into the general pattern of $f : \mathbb{R}^n \to \mathbb{R}^m$ for $n, m \in \mathbb{N}$ but tradition and use of the function gives us different terminology and symbols. Such functions are useful for representing various practical things, for example: gravitational force; temperature in a region; wind velocity; fluid flow; electric field; etc.

Open sets, closed sets, boundary, continuity

Let $\mathbf{a} \in \mathbb{R}^n$, $r > 0$. The open n-ball of radius r and centre $\mathbf{a}$ is written as

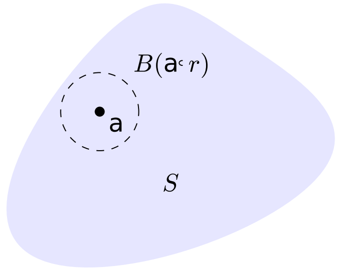
$$B(\mathbf{a}, r) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}-\mathbf{a}\| < r\}.$$

Definition (interior point)

Let $S \subset \mathbb{R}^n$. A point $\mathbf{a} \in S$ is said to be an *interior point* if there is $r > 0$ such that $B(\mathbf{a}, r) \subset S$. The set of all interior points of S is denoted $\operatorname{int} S$.

Definition (open set)

A set $S \subset \mathbb{R}^n$ is said to be *open* if all of its points are interior points, i.e., if $\operatorname{int} S = S$.



Interior points are the centre of a ball contained within the set

For example, open intervals, open disks, open balls, unions of open intervals, etc., are all open sets.

Lemma

Let $r > 0$, $\mathbf{a} \in \mathbb{R}^n$. The set $B(\mathbf{a}, r) \subset \mathbb{R}^n$ is open.

Proof

Let $\mathbf{b} \in B(\mathbf{a}, r)$. It suffices to show that $\mathbf{b}$ is an interior point. (1) Let $r_1 = \|\mathbf{b} - \mathbf{a}\| < r$. (2) Let $r_2 = (r - r_1)/2$. (3) We claim that $B(\mathbf{b}, r_2) \subset B(\mathbf{a}, r)$: In order to see this take any $\mathbf{c} \in B(\mathbf{b}, r_2)$ and observe that

$$\|\mathbf{c} - \mathbf{a}\| \leq \|\mathbf{c} - \mathbf{b}\| + \|\mathbf{b} - \mathbf{a}\| \leq r_2 + r_1 = \frac{r + r_1}{2} < r.$$

Observe that the radius of the ball will be small for points close to the boundary.

Definition (Cartesian product)

If $A_1 \subset \mathbb{R}$, $A_2 \subset \mathbb{R}$ then the *Cartesian product* is defined as

$$A_1 \times A_2 := \{(x, y) : x \in A_1, y \in A_2\} \subset \mathbb{R}^2.$$

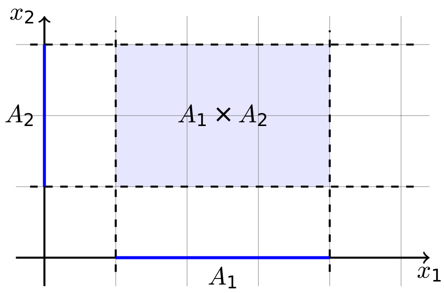
Analogously the Cartesian product can be defined in higher dimensions: If $A_1 \subset \mathbb{R}^m$, $A_2 \subset \mathbb{R}^n$ then the *Cartesian product* $A_1 \times A_2$ is defined as the set of all points $(x_1, \dots, x_m, y_1, \dots, y_n) \in \mathbb{R}^{m+n}$ such that $(x_1, \dots, x_m) \in A_1$ and $(y_1, \dots, y_n) \in A_2$.

Lemma

If A_1, A_2 are open subsets of $\mathbb{R}$ then $A_1 \times A_2$ is an open subset of $\mathbb{R}^2$.

Proof

Let $\mathbf{a} = (a_1, a_2) \in A_1 \times A_2 \subset \mathbb{R}^2$. Since A_1 is open there exists $r_1 > 0$ such that $B(a_1, r_1) \subset A_1$. Similarly for A_2. Let $r = \min\{r_1, r_2\}$. This all means that $B(\mathbf{a}, r) \subset B(a_1, r_1) \times B(a_2, r_2) \subset A_1 \times A_2$.



If A_1, A_2 are intervals then $A_1 \times A_2$ is a rectangle

Discussing the "interior" of the set naturally suggests the topic of the "boundary" of the set. In the following definitions we develop this idea.

Definition (exterior points)

Let $S \subset \mathbb{R}^n$. A point $\mathbf{a} \notin S$ is said to be an *exterior point* if there exists $r > 0$ such that $B(\mathbf{a}, r) \cap S = \emptyset$. The set of all exterior points of S is denoted $\mathbf{ext}\, S$.

Observe that $\mathbf{ext}\, S$ is an open set. We use the notation $S^c = \mathbb{R}^n \setminus S$ and we say that C^c is the *complement* of the set S.

Definition (boundary)

The set $\mathbb{R}^n \setminus (\mathbf{int}\, S \cup \mathbf{ext}\, S)$ is called the boundary of $S \subset \mathbb{R}^n$ and is denoted ∂S.

Definition (closed)

A set $S \subset \mathbb{R}^n$ is said to be *closed* if $\partial S \subset S$.

Lemma

S is open $\iff S^c$ is closed.

Proof

Observe that $\mathbb{R}^n = \textbf{int}\, S \cup \partial S \cup \textbf{ext}\, S$ (disjointly). If $\mathbf{x} \in \partial S$ then, for every $r > 0$, $B(\mathbf{x}, r) \cap S \neq \emptyset$ and so $\mathbf{x} \in \partial(S^c)$. Similarly with S and S^c swapped and so $\partial S = \partial(S^c)$. If S is open then $\textbf{int}\, S = S$ and $S^c = \textbf{ext}\, S \cup \partial S = \textbf{ext}\, S \cup \partial(S^c)$ and so S^c is closed. If S is not open then there exists $\mathbf{a} \in \partial S \cap S$. Additionally $\mathbf{a} \in \partial(S^c) \cap S$ hence S^c is not closed.

Limits and continuity

Let $S \subset \mathbb{R}^n$ and $\mathbf{f} : S \to \mathbb{R}^m$. If $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$ we write $\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ to mean that $\|\mathbf{f}(\mathbf{x}) - \mathbf{b}\| \to 0$ as $\|\mathbf{x} - \mathbf{a}\| \to 0$. Observe how, if $n = m = 1$, this is the familiar notion of continuity for functions on $\mathbb{R}$.

Definition (Continuous)

A function $\mathbf{f}$ is said to be *continuous* at $\mathbf{a}$ if $\mathbf{f}$ is defined at $\mathbf{a}$ and $\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$. We say $\mathbf{f}$ is continuous on S if $\mathbf{f}$ is continuous at each point of S.

Even functions which look "nice" can fail to be continuous as we can see in the following example.

Example (continuity in higher dimensions)

Let f be defined, for $(x, y) \neq (0, 0)$, as

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

and $f(0, 0) = 0$. What is the behaviour of f when approaching $(0, 0)$ along the following lines?

line	value
$\{x = 0\}$	$f(0, t) = 0$

line	value
$\{y=0\}$	$f(t,0)=0$
$\{x=y\}$	$f(t,t)=\frac{1}{2}$
$\{x=-y\}$	$f(t,t)=-\frac{1}{2}$

Theorem

Suppose that $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ and $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}) = \mathbf{c}$. Then

1. $\lim_{\mathbf{x}\to\mathbf{a}}(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})) = \mathbf{b} + \mathbf{c}$,
2. $\lim_{\mathbf{x}\to\mathbf{a}} \lambda\mathbf{f}(\mathbf{x}) = \lambda\mathbf{b}$ for every $\lambda \in \mathbb{R}$,
3. $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) = \mathbf{b} \cdot \mathbf{c}$,
4. $\lim_{\mathbf{x}\to\mathbf{a}} \|\mathbf{f}(\mathbf{x})\| = \|\mathbf{b}\|$.

We prove a couple of the parts of the above theorem here, the other parts are left as exercises.

Proof of part 3.

Observe that $\mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) - \mathbf{b} \cdot \mathbf{c} = (\mathbf{f}(\mathbf{x}) - \mathbf{b}) \cdot (\mathbf{g}(\mathbf{x}) - \mathbf{c}) + \mathbf{b} \cdot (\mathbf{g}(\mathbf{x}) - \mathbf{c}) + \mathbf{c} \cdot (\mathbf{f}(\mathbf{x}) - \mathbf{b})$. By the triangle inequality and Cauchy-Schwarz,

$$\begin{aligned}\|\mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) - \mathbf{b} \cdot \mathbf{c}\| \le{}& \|\mathbf{f}(\mathbf{x}) - \mathbf{b}\|\,\|\mathbf{g}(\mathbf{x}) - \mathbf{c}\| \\ &+ \|\mathbf{b}\|\,\|\mathbf{g}(\mathbf{x}) - \mathbf{c}\| \\ &+ \|\mathbf{c}\|\,\|\mathbf{f}(\mathbf{x}) - \mathbf{b}\|.\end{aligned}$$

Since we already know that $\|\mathbf{f}(\mathbf{x}) - \mathbf{b}\| \to 0$ and $\|\mathbf{g}(\mathbf{x}) - \mathbf{c}\| \to 0$ as $\mathbf{x} \to \mathbf{a}$, this implies that $\|\mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) - \mathbf{b} \cdot \mathbf{c}\| \to 0$.

Proof of part 4.

Take $\mathbf{f} = \mathbf{g}$ in part (c) implies that $\lim_{\mathbf{x}\to\mathbf{a}} \|\mathbf{f}(\mathbf{x})\|^2 = \|\mathbf{b}\|^2$.

When writing a vector field (or similar functions) it is often convenient to divide the higher-dimensional function into smaller parts. We call these parts the *components of a vector field*. For example $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}))$ in 2D, $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), F_3(\mathbf{x}))$ in 3D, etc.

Theorem

Let $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}))$. Then $\mathbf{F}$ is continuous if and only if F_1 and F_2 are continuous.

Proof

We will independently prove the two implications.

- ($\Rightarrow$) Let $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$ and observe that $F_k(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \cdot \mathbf{e}_k$. We have already shown that the continuity of two vector fields implies the continuity of the inner product.
- ($\Leftarrow$) By definition of the norm

$$\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{a})\|^2 = \sum_{k=1}^{2} (F_k(\mathbf{x}) - F_k(\mathbf{a}))^2$$

  and we know $\|F_k(\mathbf{x}) - F_k(\mathbf{a})\| \to 0$ as $\|\mathbf{x} - \mathbf{a}\| \to 0$.

In higher dimensions the analogous statement is true for the vector field $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), \dots, F_m(\mathbf{x}))$ with exactly the same proof. I.e., $\mathbf{F}$ is continuous if and only if each f_k is continuous.

Example (polynomials)

A *polynomial* in n variables is a scalar field on $\mathbb{R}^n$ of the form

$$f(x_1, \dots, x_n) = \sum_{k_1=0}^{j} \cdots \sum_{k_n=0}^{j} c_{k_1,\dots,k_n} x_1^{k_1} \cdots x_n^{k_n}.$$

E.g., $f(x, y) := x + 2xy - x^2$ is a polynomial in 2 variables. Polynomials are continuous everywhere in $\mathbb{R}^n$. This is because they are the finite sum of products of continuous scalar fields.

Example (rational functions)

A *rational function* is a scalar field

$$f(\mathbf{x}) = \frac{p(\mathbf{x})}{q(\mathbf{x})}$$

where $p(\mathbf{x})$ and $q(\mathbf{x})$ are polynomials. A rational function is continuous at every point $\mathbf{x}$ such that $q(\mathbf{x}) \neq 0$.

As described in the following result, the continuity of functions continues to hold, in an intuitive way, under composition of functions.

Theorem

Suppose $S \subset \mathbb{R}^l, T \subset \mathbb{R}^m, \mathbf{f} : S \to \mathbb{R}^m, \mathbf{g} : T \to \mathbb{R}^n$ and that $\mathbf{f}(S) \subset T$ so that

$$(\mathbf{g} \circ \mathbf{f})(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$$

makes sense. If $\mathbf{f}$ is continuous at $\mathbf{a} \in S$ and $\mathbf{g}$ is continuous at $\mathbf{f}(\mathbf{a})$ then $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{a}$.

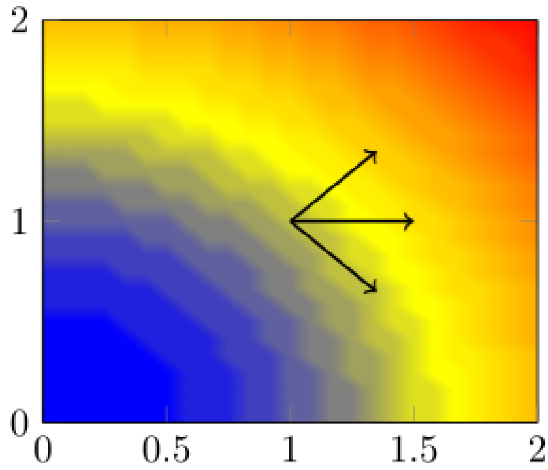
Proof

$$\lim \mathbf{x} \to \mathbf{a} \, \|\mathbf{f}(\mathbf{g}(\mathbf{x})) - \mathbf{f}(\mathbf{g}(\mathbf{a}))\| = \lim \mathbf{y} \to \mathbf{g}(\mathbf{a}) \, \|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{g}(\mathbf{a}))\| = 0$$

Example

We can consider the scalar field $f(x, y) = \sin(x^2 + y) + xy$ as the composition of functions.

Derivatives of scalar fields



Plot where colour represents the value of $f(x, y) = x^2 + y^2$. The change in f depends on direction

We can imagine, for example in the figure, that in higher dimensions, the derivative of a scalar field depends on the direction. This motivates the following.

Definition (directional derivative)

Let $S \subset \mathbb{R}^n$ and $f : S \to \mathbb{R}$. For any $\mathbf{a} \in \text{int}\, S$ and $\mathbf{v} \in \mathbb{R}^n$, $\|v\| = 1$ the directional derivative of f with respect to $\mathbf{v}$ is defined as

$$D_{\mathbf{v}} f(\mathbf{a}) = \lim_{h \to 0} \frac{1}{h}(f(\mathbf{a} + h\mathbf{v}) - f(\mathbf{a})).$$

When h is small we can guarantee that $\mathbf{a} + h\mathbf{v} \in S$ because $\mathbf{a} \in \text{int}\, S$ so this definition makes sense.

Theorem

Suppose $S \subset \mathbb{R}^n$, $f : S \to \mathbb{R}$, $\mathbf{a} \in \text{int}\, S$. Let $g(t) := f(\mathbf{a} + t\mathbf{v})$. If one of the derivatives $g'(t)$ or $D_{\mathbf{v}} f(\mathbf{a})$ exists then the other also exists and

$$g'(t) = D_{\mathbf{v}} f(\mathbf{a} + t\mathbf{v}).$$

In particular $g'(0) = D_{\mathbf{v}} f(\mathbf{a})$.

Proof

By definition $\frac{1}{h}(g(t+h) - g(h)) = \frac{1}{h}(f(\mathbf{a} + h\mathbf{v}) - f(\mathbf{a}))$.

The following result is useful for proving later results.

Theorem (mean value)

Assume that $D_{\mathbf{v}}(\mathbf{a}+t\mathbf{v})$ exists for each $t \in [0,1]$. Then for some $\theta \in (0,1)$,

$$f(\mathbf{a}+\mathbf{v}) - f(\mathbf{a}) = D_{\mathbf{v}} f(\mathbf{z}), \quad \text{where } z = \mathbf{a} + \theta \mathbf{v}.$$

Proof

Apply mean value theorem to $g(t) = f(\mathbf{a}+t\mathbf{v})$.

The following notation is convenient. For any $k \in \{1, 2, \ldots, n\}$, let $\mathbf{e}_k$ be the n-dimensional unit vector where all entries are zero except the kth position which is equal to 1. I.e., $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, $\mathbf{e}_n = (0, \ldots, 0, 1)$.

Definition (partial derivatives)

We define the *partial derivative* in x_k of $f(x_1, \ldots, x_n)$ at $\mathbf{a}$ as

$$\frac{\partial f}{\partial x_k}(\mathbf{a}) = D_{\mathbf{e}_k} f(\mathbf{a}).$$

Remark

Various symbols used for partial derivatives: $\frac{\partial f}{\partial x_k}(\mathbf{a}) = D_k f(\mathbf{a}) = \partial_k f(\mathbf{a})$. If a function is written $f(x, y)$ we write $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ for the partial derivatives. Similarly for higher dimension.

In practice, to compute the partial derivative $\frac{\partial f}{\partial x_k}$, one should consider all other x_j for $j \neq k$ as constants and take the derivative with respect to x_k. In a moment we see this rigorously.

If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then we know that, when x is close to a,

$$f(x) \approx f(a) + (x-a) f'(a).$$

More precisely, we know that $f(x) = f(a) + (x-a)f'(a) + \epsilon(x-a)$ where $|\epsilon(x-a)| = o(|x-a|)$. (This is little-o notation and here means that

$|f(x)-f(a)-(x-a)f'(a)|/|x-a| \to 0$ as $|x-a| \to 0$.) This way of seeing differentiability is convenient for the higher dimensional definition of differentiability.

Definition (differentiable)

Let $S \subset \mathbb{R}^n$ be open, $f : S \to \mathbb{R}$. We say that f is *differentiable* at $\mathbf{a} \in S$ if there exists a linear transformation $df_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}$ such that, for $\mathbf{x} \in B(\mathbf{a}, r)$,

$$f(\mathbf{x}) = f(\mathbf{a}) + df_{\mathbf{a}}(\mathbf{x}-\mathbf{a}) + \epsilon(\mathbf{x}-\mathbf{a})$$

where $|\epsilon(\mathbf{x}-\mathbf{a})| = o(\|\mathbf{x}-\mathbf{a}\|)$.

For future convenience we introduce the following notation.

Definition (gradient)

The *gradient* of the scalar field $f(x, y, z)$ at the point $\mathbf{a}$ is

$$\nabla f(\mathbf{a}) = \begin{pmatrix} \frac{\partial f}{\partial x}(\mathbf{a}) \\ \frac{\partial f}{\partial y}(\mathbf{a}) \\ \frac{\partial f}{\partial z}(\mathbf{a}) \end{pmatrix}.$$

In general, when working in $\mathbb{R}^n$ for some $n \in \mathbb{N}$, the *gradient* of the scalar field $f(x_1, \ldots, x_n)$ at the point $\mathbf{a}$ is

$$\nabla f(\mathbf{a}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{a}) \\ \frac{\partial f}{\partial x_2}(\mathbf{a}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{a}) \end{pmatrix}.$$

Theorem

If f is differentiable at $\mathbf{a}$ then $df_{\mathbf{a}}(\mathbf{v}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$. This means that, for $\mathbf{x} \in B(\mathbf{a}, r)$,

$$f(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x}-\mathbf{a}) + \epsilon(\mathbf{x}-\mathbf{a})$$

where $|\epsilon(\mathbf{x}-\mathbf{a})| = o(\|\mathbf{x}-\mathbf{a}\|)$. Moreover, for any vector $\mathbf{v}$, $\|v\| = 1$,

$$D_{\mathbf{v}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}.$$

Proof

Since f is differentiable there exists a linear transformation $df_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}$ such that $f(\mathbf{a} + h\mathbf{v}) = f(\mathbf{a}) + hdf_{\mathbf{a}}(\mathbf{v}) + \epsilon(h\mathbf{v})$ and hence

$$\begin{aligned} D_{\mathbf{v}} f(\mathbf{a}) &= \lim_{h\to 0} \frac{1}{h}(f(\mathbf{a}+h\mathbf{v}) - f(\mathbf{a})) \\ &= \lim_{h\to 0} \frac{1}{h}(h\, df_{\mathbf{a}}(\mathbf{v}) + \epsilon(h\mathbf{v})) = df_{\mathbf{a}}(\mathbf{v}). \end{aligned}$$

In particular $df_{\mathbf{a}}(\mathbf{e}_k) = D_{\mathbf{e}_k} f(\mathbf{a})$.

Theorem

If f is differentiable at $\mathbf{a}$, then it is continuous at $\mathbf{a}$.

Proof

Observe that $|f(\mathbf{a}+\mathbf{v}) - f(\mathbf{a})| = |df_{\mathbf{a}}(\mathbf{v}) + \epsilon(\mathbf{v})|$. This means that

$$|f(\mathbf{a}+\mathbf{v}) - f(\mathbf{a})| \leq \|df_{\mathbf{a}}\| \ || \mathbf{v} || \ + |\epsilon(\mathbf{v})|$$

and so this tends to 0 as $|| \mathbf{v} || \to 0$.

Theorem

Suppose that $f(x_1, \dots, x_n)$ is a scalar field. If the partial derivatives $\partial_1 f(\mathbf{x}), \dots, \partial_n f(\mathbf{x})$ exist for all $\mathbf{x} \in B(\mathbf{a}, r)$ and are continuous at $\mathbf{a}$ then f is differentiable at $\mathbf{a}$.

Proof

For convenience define the vectors

$$\begin{aligned} \mathbf{v} &= (v_1, v_2, \dots, v_n), \\ \mathbf{u}_k &= (v_1, v_2, \dots, v_k, 0, \dots, 0). \end{aligned}$$

Observe that

$$\mathbf{u}_k - \mathbf{u}_{k-1} = v_k \mathbf{e}_k, \quad \mathbf{u}_0 = (0, 0, \dots, 0), \quad \mathbf{u}_n = \mathbf{v}.$$

Using the mean value theorem we know that there exists $\mathbf{z}_k = \mathbf{u}_{k-1} + \theta_k \mathbf{e}_k$ such that $f(\mathbf{a} + \mathbf{u}_k) - f(\mathbf{a} + \mathbf{u}_{k-1}) = v_k D_{\mathbf{e}\_k} f(\mathbf{a} + \mathbf{z}\_k)$. Consequently

$$
\begin{aligned}
f(\mathbf{a}+\mathbf{v})-f(\mathbf{a}) &= \sum_{k=1}^{n} f(\mathbf{a}+\mathbf{u}_k)-f(\mathbf{a}+\mathbf{u}_{k-1}) \\
&= \sum_{k=1}^{n} v_k D_{\mathbf{e}_k} f(\mathbf{a}+\mathbf{z}\_k) \\
&= \sum_{k=1}^{n} v_k D_{\mathbf{e}_k} f(\mathbf{a}+\mathbf{u}_{k-1}) \\
&\quad + \sum_{k=1}^{n} v_k \left(D_{\mathbf{e}_k} f(\mathbf{a}+\mathbf{z}\_k) - D_{\mathbf{e}_k} f(\mathbf{a}+\mathbf{u}_{k-1})\right)
\end{aligned}
$$

To conclude, observe that the second sum vanishes as $||\mathbf{v}|| \to 0$ and that the first sum, $\sum_{k=1}^{n} v_k D_{\mathbf{e}_k} f(\mathbf{a}+\mathbf{u}_{k-1})$, converges to $\mathbf{v}\cdot\nabla f(\mathbf{a})$.

Chain rule

When we are working in $\mathbb{R}$ we know that, if g and h are differentiable, then $f(t) = g \circ h(t)$ is also differentiable and also $f'(t) = g'(h(t))\, h'(t)$. This is called the *chain rule* and is frequently very useful in calculating derivatives. We now investigate how this extends to higher dimension?

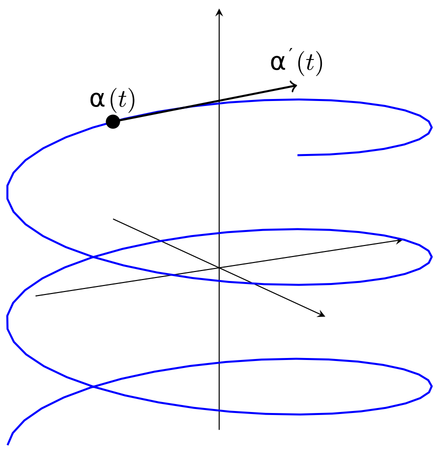
Example

Suppose that $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^3$ describes the position $\boldsymbol{\alpha}(t)$ at time t and that $f : \mathbb{R}^3 \to \mathbb{R}$ describes the temperature $f(\boldsymbol{\alpha})$ at a point $\boldsymbol{\alpha}$ The temperature at time t is equal to $g(t) = f(\boldsymbol{\alpha}(t))$. We want to calculate $g'(t)$ because this is the change in temperature with respect to time.

In situations like the above example it is convenient to consider the derivative of a path $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^n$. Let $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^n$ and suppose it has the form $\boldsymbol{\alpha}(t) = (\alpha_1(t), \dots, \alpha_n(t))$. We define the derivative as

$$
\boldsymbol{\alpha}'(t) := \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}.
$$

Here $\boldsymbol{\alpha}'$ is a vector-valued function which represents the "direction of movement".



$\boldsymbol{\alpha}(t) = (\cos t, \sin t, t)$, $t \in \mathbb{R}$

Theorem

Let $S \subset \mathbb{R}^n$ be open and $I \subset \mathbb{R}$ an interval. Let $\mathbf{x} : I \to S$ and $f : S \to \mathbb{R}$ and define, for $t \in I$,

$$g(t) = f(\mathbf{x}(t)).$$

Suppose that $t \in I$ is such that $\mathbf{x}'(t)$ exists and f is differentiable at $\mathbf{x}(t)$. Then $g'(t)$ exists and

$$g'(t) = \nabla f\,(\mathbf{x}(t)) \cdot \mathbf{x}'(t).$$

Proof

Since f is differentiable, $f(\mathbf{y}) - f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) + \epsilon(\mathbf{x}, \mathbf{y} - \mathbf{x})$ where $|\epsilon(\mathbf{x}, \mathbf{y} - \mathbf{x})| = o(\mathbf{y} - \mathbf{x})$. Let $h > 0$ be small.

$$\begin{aligned} \tfrac{1}{h}[g(t+h) - g(t)] &= \tfrac{1}{h}[f(\mathbf{x}(t+h)) - f(\mathbf{x}(t))] \\ &= \tfrac{1}{h}\nabla f(\mathbf{x}(t)) \cdot (\mathbf{x}(t+h) - \mathbf{x}(t)) \\ &\quad + \tfrac{1}{h}\epsilon(\mathbf{x}(t), \mathbf{x}(t+h) - \mathbf{x}(t)). \end{aligned}$$

Observe that $\frac{1}{h}(\mathbf{x}(t+h) - \mathbf{x}(t)) \to \mathbf{x}'(t)$ as $h \to 0$.

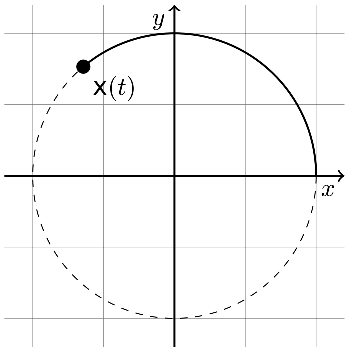
Example

A particle moves in a circle and its position at time $t \in [0, 2\pi]$ is given by

$$\mathbf{x}(t) = (\cos t, \sin t).$$

The temperature at a point $\mathbf{y} = (y_1, y_2)$ is given by the function $f(\mathbf{y}) := y_1 + y_2$. The temperature the particle experiences at time t is given by $g(t) = f(\mathbf{x}(t))$.
Temperature change:

$$g'(t) = \nabla f\left(\mathbf{x}(t)\right) \cdot \mathbf{x}'(t) = \binom{1}{1} \cdot \binom{-\sin t}{\cos t} = \cos t - \sin t.$$



$\mathbf{x}(t)$ *is the position of a particle.*

Level sets & tangent planes

Let $S \subset \mathbb{R}^2$, $f : S \to \mathbb{R}$. Suppose $c \in \mathbb{R}$ and let

$$L(c) = \{\mathbf{x} \in S : f(\mathbf{x}) = c\}.$$

The set $L(c)$ is called the *level set*. In general this set can be empty or it can be all of S. However the set $L(c)$ is often a curve and this is the case of interest. This is the same notion as that of contour lines on a map. I.e., $\mathbf{x}(t_a) = \mathbf{a}$ for some $t_a \in I$ and

$$f(\mathbf{x}(t)) = c$$

for all $t \in I$. Then

- $\nabla f(\mathbf{a})$ is normal to the curve at $\mathbf{a}$
- Tangent line at $\mathbf{a}$ is $\left\{\mathbf{x} \in \mathbb{R}^2 : \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0\right\}$

This is because the chain rule implies that $\nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = 0$.

Example

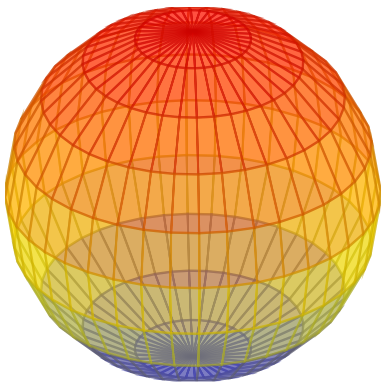
Let $f(x_1, x_2, x_3) := x_1^2 + x_2^2 + x_3^2$.

- If $c > 0$ then $L(c)$ is a sphere,
- $L(0)$ is a single point $(0, 0, 0)$,
- If $c < 0$ then $L(c)$ is empty.

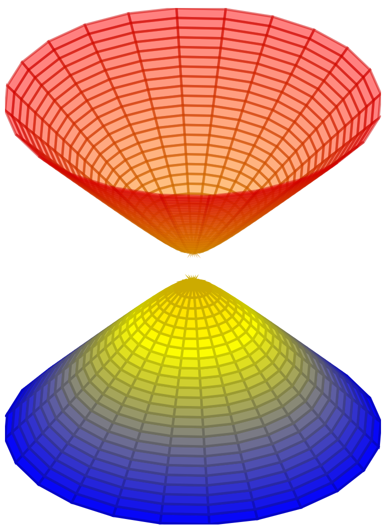
Example

Let $f(x_1, x_2, x_3) := x_1^2 + x_2^2 - x_3^2$. See figure.

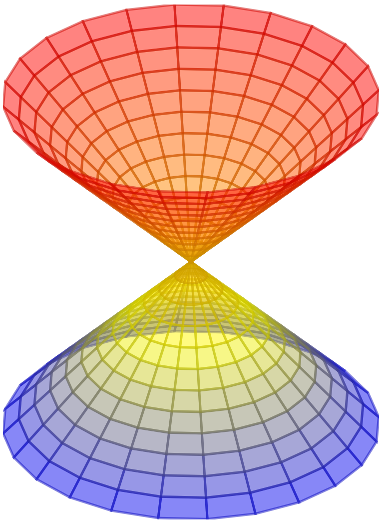
- If $c > 0$ then $L(c)$ is a one-sheeted hyperboloid,
- $L(0)$ is an infinite cone,
- If $c < 0$ then $L(c)$ is a two-sheeted hyperboloid.



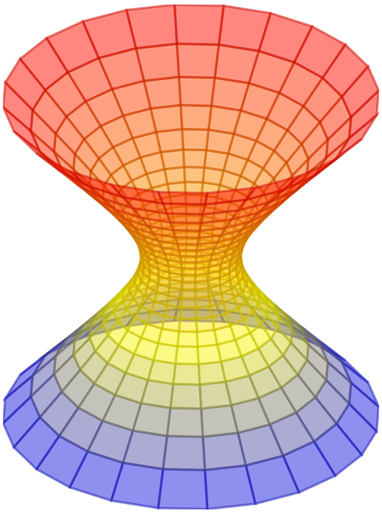
Sphere



2-sheet hyperboloid



Infinite cone

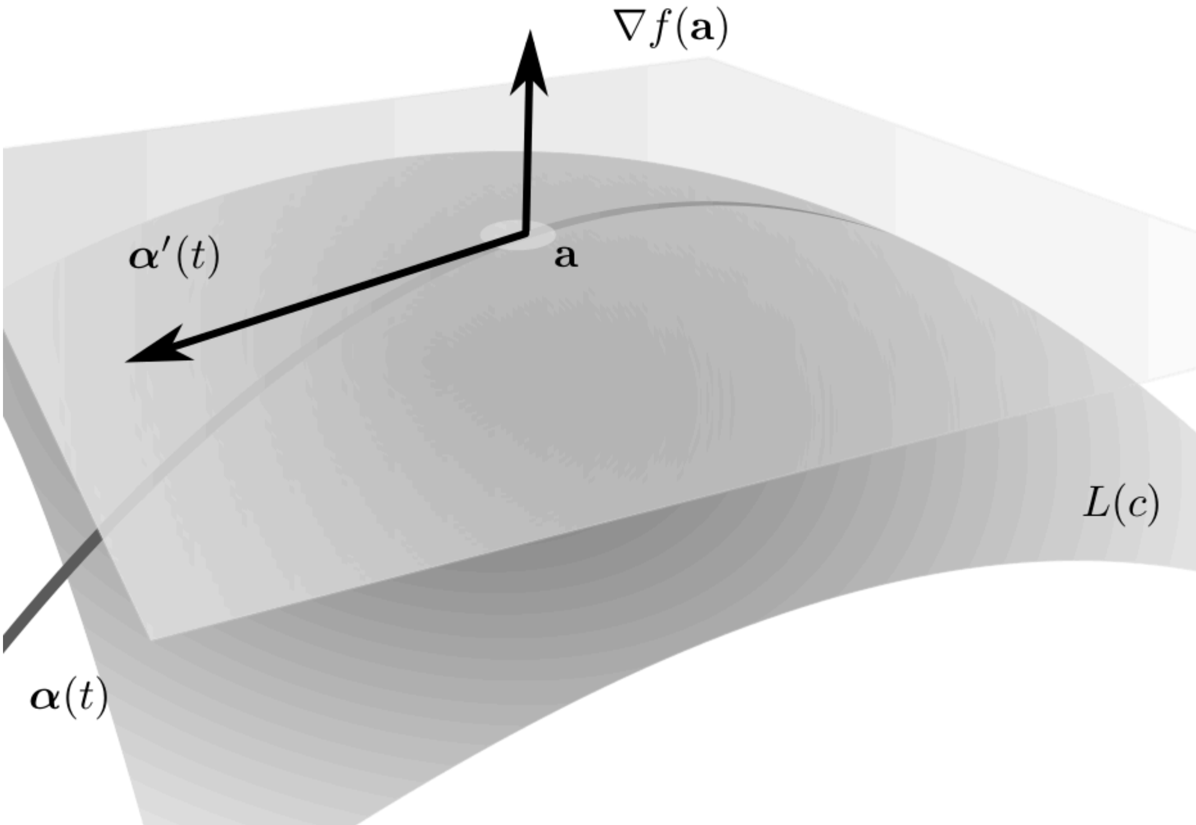


1-sheet hyperboloid

Let f be a differentiable scalar field on $S \subset \mathbb{R}^3$ and suppose that the level set $L(c) = \{\mathbf{x} \in S : f(\mathbf{x}) = c\}$ defines a surface.

- The gradient $\nabla f(\mathbf{a})$ is normal to every curve $\boldsymbol{\alpha}(t)$ in the surface which passes through $\mathbf{a}$,
- The tangent plane at $\mathbf{a}$ is $\left\{\mathbf{x} \in \mathbb{R}^3 : \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0\right\}$.

Same argument as in $\mathbb{R}^2$ works in $\mathbb{R}^n$.



Tangent plane and normal vector

Derivatives of vector fields

Essentially everything discussed above for scalar fields extends to vector fields in a predictable way. This is because of the linearity and that we can consider each *component* of the vector field independently.

Definition (directional derivative)

Let $S \subset \mathbb{R}^n$ and $\mathbf{F} : S \to \mathbb{R}^m$. For any $\mathbf{a} \in \mathbf{int}\, S$ and $\mathbf{v} \in \mathbb{R}^n$ the derivative of the vector field $\mathbf{F}$ with respect to $\mathbf{v}$ is defined as

$$D_{\mathbf{v}}\mathbf{F}(\mathbf{a}) := \lim_{h \to 0} \frac{1}{h}(\mathbf{F}(\mathbf{a} + h\mathbf{v}) - \mathbf{F}(\mathbf{a})).$$

Remark

If we use the notation $\mathbf{F} = (F_1, \dots, F_m)$, i.e., we write the function using the ``components'' where each F_k is a scalar field, then $D_{\mathbf{v}}\mathbf{F} = (D_{\mathbf{v}}F_1, \dots, D_{\mathbf{v}}F_m)$.

Definition (differentiable)

We say that $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable* at $\mathbf{a}$ if there exists a linear transformation $df_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}^m$ such that, for $\mathbf{x} \in B(\mathbf{a}, r)$,

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{a}) + df_{\mathbf{a}}(\mathbf{x} - \mathbf{a}) + \epsilon(\mathbf{x} - \mathbf{a}),$$

$\|\epsilon(\mathbf{x} - \mathbf{a})\| = o(\,\|\,\mathbf{x} - \mathbf{a}\,\|\,)$.

Theorem

If $\mathbf{F}$ is differentiable at $\mathbf{a}$ then $\mathbf{F}$ is continuous at $\mathbf{a}$ and $df_{\mathbf{a}}(\mathbf{v}) = D_{\mathbf{v}}\mathbf{F}(\mathbf{a})$.

Proof

Same as for the case of scalar fields when $f : \mathbb{R}^n \to \mathbb{R}$.

Jacobian matrix & the chain rule

The relevant differential for higher-dimensional functions is the Jacobian matrix.

Definition (Jacobian matrix)

Suppose that $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ and use the notation $\mathbf{F}(x, y) = (F_1(x, y), F_2(x, y))$. The *Jacobian matrix* of $\mathbf{F}$ at $\mathbf{a}$ is defined as

$$D\mathbf{F}(\mathbf{a}) = \begin{pmatrix} \frac{\partial F_1}{\partial x}(\mathbf{a}) & \frac{\partial F_1}{\partial y}(\mathbf{a}) \\ \frac{\partial F_2}{\partial x}(\mathbf{a}) & \frac{\partial F_2}{\partial y}(\mathbf{a}) \end{pmatrix}.$$

The *Jacobian matrix* is defined analogously in any dimension. I.e., if $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ the the Jacobian at $\mathbf{a}$ is

$$DF(\mathbf{a}) = \begin{pmatrix} \partial_1 F_1(\mathbf{a}) & \partial_2 F_1(\mathbf{a}) & \cdots & \partial_n F_1(\mathbf{a}) \\ \partial_1 F_2(\mathbf{a}) & \partial_2 F_2(\mathbf{a}) & \cdots & \partial_n F_2(\mathbf{a}) \\ \vdots & \vdots & & \vdots \\ \partial_1 F_m(\mathbf{a}) & \partial_2 F_m(\mathbf{a}) & \cdots & \partial_n F_m(\mathbf{a}) \end{pmatrix}$$

If we choose a basis then any linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ can be written as a $m \times n$ matrix. We find that $df_{\mathbf{a}}(\mathbf{v}) = D\mathbf{F}(\mathbf{a})\mathbf{v}$.

Let $S \subset \mathbb{R}^n$ and $\mathbf{F} : S \to \mathbb{R}^m$. If f is differentiable at $\mathbf{a} \in S$ then, for all $\mathbf{x} \in B(\mathbf{a}, r) \subset S$,

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{a}) + D\mathbf{F}(\mathbf{a})(\mathbf{x} - \mathbf{a}) + \epsilon(\mathbf{x} - \mathbf{a})$$

where $|\epsilon(\mathbf{x} - \mathbf{a})| = o(\,\|\mathbf{x} - \mathbf{a}\|\,)$. This is like a Taylor expansion in higher dimensions.

Here we see that in higher dimensions we have a matrix form of the chain rule.

Theorem

Let $S \subset \mathbb{R}^l, T \subset \mathbb{R}^m$ be open. Let $\mathbf{f} : S \to T$ and $\mathbf{g} : T \to \mathbb{R}^n$ and define

$$\mathbf{h} = \mathbf{g} \circ \mathbf{f} : S \to \mathbb{R}^n.$$

Let $\mathbf{a} \in S$. Suppose that $\mathbf{f}$ is differentiable at $\mathbf{a}$ and $\mathbf{g}$ is differentiable at $\mathbf{f}(\mathbf{a})$. Then $\mathbf{h}$ is differentiable at $\mathbf{a}$ and

$$D\mathbf{h}(\mathbf{a}) = D\mathbf{g}(\mathbf{f}(\mathbf{a}))\; D\mathbf{f}(\mathbf{a}).$$

Proof

Let $\mathbf{u} = \mathbf{f}(\mathbf{a} + \mathbf{v}) - \mathbf{f}(\mathbf{a})$. Since $\mathbf{f}$ and $\mathbf{g}$ are differentiable,

$$\begin{aligned} \mathbf{h}(\mathbf{a} + \mathbf{v}) - \mathbf{h}(\mathbf{a}) &= \mathbf{g}(\mathbf{f}(\mathbf{a} + \mathbf{v})) - \mathbf{g}(\mathbf{f}(\mathbf{a})) \\ &= D\mathbf{g}(\mathbf{f}(\mathbf{a}))(\mathbf{f}(\mathbf{a} + \mathbf{v}) - \mathbf{f}(\mathbf{a})) + \epsilon_{\mathbf{g}}(\mathbf{u}) \\ &= D\mathbf{g}(\mathbf{f}(\mathbf{a}))D\mathbf{f}(\mathbf{a})\mathbf{v} + D\mathbf{g}(\mathbf{f}(\mathbf{a}))\epsilon_{\mathbf{f}}(\mathbf{v}) + \epsilon_{\mathbf{g}}(\mathbf{u}). \end{aligned}$$

Example (polar coordinates)

Here we consider *polar coordinates* and calculate the Jacobian of this transformation. We can write the change of coordinates

$$(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$$

as the function $\mathbf{f}(r,\theta) = (x(r,\theta), y(r,\theta))$ where $\mathbf{f} : (0,\infty) \times [0, 2\pi) \to \mathbb{R}^2$. We calculate the Jacobian matrix of this transformation

$$D\mathbf{f}(r,\theta) = \begin{pmatrix} \frac{\partial x}{\partial r}(r,\theta) & \frac{\partial x}{\partial \theta}(r,\theta) \\ \frac{\partial y}{\partial r}(r,\theta) & \frac{\partial y}{\partial \theta}(r,\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

In particular we see that $\det D\mathbf{f}(r,\theta) = r$, the familiar value used in change of variables with polar coordinated.

Suppose now that we wish to calculate derivatives of $h := g \circ \mathbf{f}$ for some $g : \mathbb{R}^2 \to \mathbb{R}$. Here we take advantage of the theorem concerning multiplication of Jacobians.

$$Dh(r,\theta) = Dg(\mathbf{f}(r,\theta))\, D\mathbf{f}(r,\theta)$$
$$\begin{pmatrix} \frac{\partial h}{\partial r}(r,\theta) & \frac{\partial h}{\partial \theta}(r,\theta) \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial x}(\mathbf{f}(r,\theta)) & \frac{\partial g}{\partial y}(\mathbf{f}(r,\theta)) \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

In other words, we have shown that

$$\frac{\partial h}{\partial r}(r,\theta) = \frac{\partial g}{\partial x}(r\cos\theta, r\sin\theta)\cos\theta + \frac{\partial g}{\partial y}(r\cos\theta, r\sin\theta)\sin\theta$$
$$\frac{\partial h}{\partial \theta}(r,\theta) = -r\frac{\partial g}{\partial x}(r\cos\theta, r\sin\theta)\sin\theta + r\frac{\partial g}{\partial y}(r\cos\theta, r\sin\theta)\cos\theta.$$

Implicit functions & partial derivatives

Just like with derivatives, we can take higher order partial derivatives. For convenience when we want to write $\frac{\partial}{\partial y}\frac{\partial}{\partial x}f(x,y)$, i.e., differentiate first with respect to x and then with respect to y, we write instead $\frac{\partial^2 f}{\partial y \partial x}(x,y)$. The analogous notation is used for higher derivatives and any other choice of coordinates.

We first consider the question of when

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) \stackrel{?}{=} \frac{\partial^2 f}{\partial x \partial y}(x,y).$$

Example (partial derivative problem)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(0,0) = 0$ and, for $(x,y) \neq (0,0)$,

$$f(x,y) := \frac{xy(x^2-y^2)}{x^2+y^2}.$$

We calculate that $\frac{\partial^2 f}{\partial y \partial x}(0,0) = -1$ but $\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$.

Theorem

Let $f : S \to \mathbb{R}$ be a scalar field such that the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist on an open set $S \subset \mathbb{R}^2$ containing $\mathbf{x}$. Further assume that $\frac{\partial^2 f}{\partial y \partial x}$ is continuous on S. Then the derivative $\frac{\partial^2 f}{\partial x \partial y}(\mathbf{x})$ exists and

$$\frac{\partial^2 f}{\partial x \partial y}(\mathbf{x}) = \frac{\partial^2 f}{\partial y \partial x}(\mathbf{x}).$$

In many cases we can choose to write a given curve/function either in *implicit* or *explicit* form.

Implicit	Explicit
$x^2 - y = 0$	$y(x) = x^2$
$x^2 + y^2 = 1$	$y(x) = \pm\sqrt{1-x^2}$, $\lvert x \rvert \leq 1$
$x^2 - y^2 - 1 = 0$	$y(x) = \pm\sqrt{x^2-1}$, $\lvert x \rvert \geq 1$
$x^2 + y^2 - e^y - 4 = 0$	A mess?
$x^2 y^4 - 3 = \sin(xy)$	A huge mess?

Given the above observation, the following method of calculating derivatives is sometimes useful. Suppose that some $f : \mathbb{R}^2 \to \mathbb{R}$ is given and we suppose there exists some $y : \mathbb{R} \to \mathbb{R}$ such that

$$f(x, y(x)) = 0 \quad \text{for all } x.$$

Let $h(x) := f(x, y(x))$ and note that $h'(x) = 0$. Here we are using the idea that $h = f \circ g$ where $g(x) = (x, y(x))$. By the chain rule $h'(x)$ is equal to

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x, y(x)) & \frac{\partial f}{\partial y}(x, y(x)) \end{pmatrix} \begin{pmatrix} 1 \\ y'(x) \end{pmatrix} = 0.$$

Consequently

$$y'(x) = -\frac{\frac{\partial f}{\partial x}(x, y(x))}{\frac{\partial f}{\partial y}(x, y(x))}.$$

Edit this page on GitHub

Updated at: 13/10/2025, 12:08

Previous page
Lecture diary

Next page
2. Extrema

Extrema

In this part of the course we work on the following skills:

- Locating and classifying the extrema of scalar fields.
- Applying Lagrange's multipliers method to optimize quantities with respect to constraints.

See also the exercises associated to this part of the course.

In the previous chapter we introduced various notions of differentials for higher dimensional functions (scalar fields, vector fields, paths, etc.). This part of the course is devoted to searching for extrema (minima / maxima) in various different scenarios. This extends what we already know for functions in $\mathbb{R}$ and we will find that in higher dimensions more possibilities and subtleties exist.

Extrema (minima / maxima / saddle)

Let $S \subset \mathbb{R}^n$ be open, $f : S \to \mathbb{R}$ be a scalar field and $\mathbf{a} \in S$.

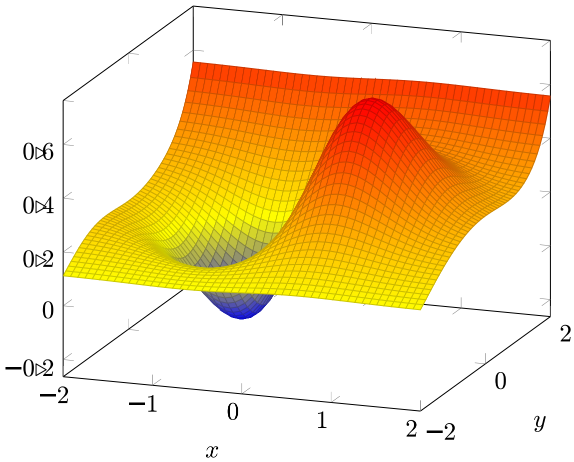
Definition

If $f(\mathbf{a}) \leq f(\mathbf{x})$ (resp. $f(\mathbf{a}) \geq f(\mathbf{x})$) for all $\mathbf{x} \in S$, then $f(\mathbf{a})$ is said to be the *absolute* minimum (resp. maximum) of f.

Definition

If $f(\mathbf{a}) \leq f(\mathbf{x})$ (resp. $f(\mathbf{a}) \geq f(\mathbf{x})$) for all $\mathbf{x} \in B(\mathbf{a}, r)$ for some $r > 0$, then $f(\mathbf{a})$ is said to be a *relative* minimum (resp. maximum) of f.

Collectively we call the these points the *extrema* of the scalar field. In the case of a scalar field defined on $\mathbb{R}^2$ we can visualize the scalar field as a 3D plot like the figure. Here we see the extrema as the "flat" places. We sometimes use *global* as a synonym of *absolute* and *local* as a synonym of *relative*.



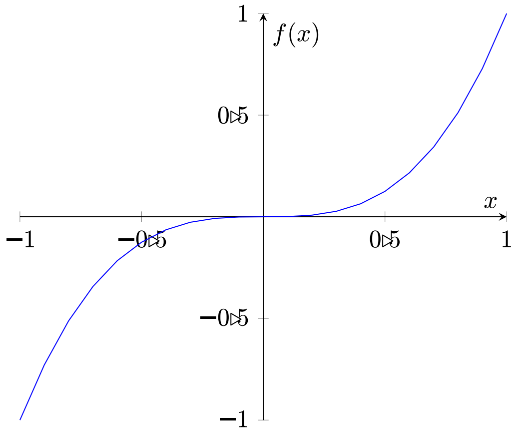
$$f(x,y) = xe^{-(x^2y^2)} + \frac{1}{4}e^{y^{\frac{3}{10}}}$$

To proceed it is convenient to connect the extrema with the behaviour of the gradient of the scalar field.

Theorem

If $f : S \to \mathbb{R}$ is differentiable and has a relative minimum or maximum at $\mathbf{a}$, then $\nabla f(\mathbf{a}) = \mathbf{0}$.

▶ **Proof**

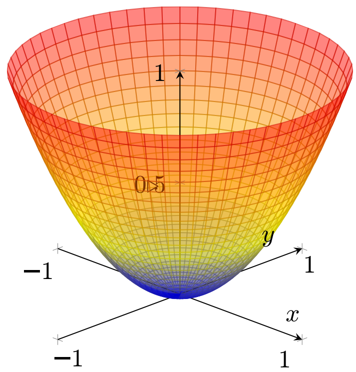


$\nabla f(\mathbf{a}) = \mathbf{0}$ *doesn't imply a minimum or maximum at* $\mathbf{a}$*, even in* $\mathbb{R}$*, as seen with the function* $f(x) = x^3$*. In higher dimensions even more is possible.*

Observe that, here and in the subsequent text, we can always consider the case of $f : \mathbb{R} \to \mathbb{R}$, i.e., the case of $\mathbb{R}^n$ where $n = 1$. Everything still holds and reduces to the arguments and formulae previously developed for functions of one variable.

Definition (stationary point)

If $\nabla f(\mathbf{a}) = \mathbf{0}$ then $\mathbf{a}$ is called a *stationary point*.



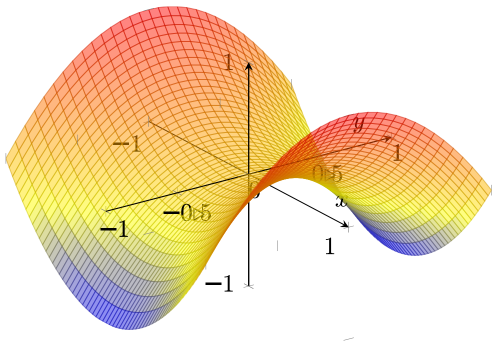
If $f(x, y) = x^2 + y^2$ *then* $\nabla f(x, y) = \binom{2x}{2y}$ *and* $\nabla f(0, 0) = \binom{0}{0}$*. The point* $(0, 0)$ *is an absolute minimum for* f*.*

As we see in the inflection example, the converse of the [above theorem] fails in the sense that a stationary point might not be a minimum or a maximum. The motivates the following.

Definition (saddle point)

If $\nabla f(\mathbf{a}) = \mathbf{0}$ and $\mathbf{a}$ is neither a minimum nor a maximum then $\mathbf{a}$ is said to be a *saddle point*.

The quintessential saddle has the shape seen in the graph. However it might be similar to an inflection in 1D or more complicated using the possibilities available in higher dimension.



If $f(x, y) = x^2 - y^2$ *then* $\nabla f(x, y) = \binom{2x}{-2y}$ *and* $\nabla f(0, 0) = \mathbf{0}$. *The point* $(0, 0)$ *is a saddle point for* f.

Hessian matrix

To proceed it is useful to develop the idea of a second order Taylor expansion in this higher dimensional setting. In particular this will allow us to identify the local behaviour close to stationary points. The main object for doing this is the *Hessian matrix*. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be twice differentiable and use the notation $f(x, y)$. The *Hessian matrix* at $\mathbf{a} \in \mathbb{R}^2$ is defined as

$$\mathbf{H}f(\mathbf{a}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(\mathbf{a}) & \frac{\partial^2 f}{\partial x\, \partial y}(\mathbf{a}) \\ \frac{\partial^2 f}{\partial y\, \partial x}(\mathbf{a}) & \frac{\partial^2 f}{\partial y^2}(\mathbf{a}) \end{pmatrix}.$$

Observe that the Hessian matrix $\mathbf{H}f(\mathbf{a})$ is a symmetric matrix since we know that

$$\frac{\partial^2 f}{\partial x\, \partial y}(\mathbf{a}) = \frac{\partial^2 f}{\partial y\, \partial x}(\mathbf{a})$$

for twice differentiable functions.

▶ **The Hessian matrix is defined analogously in any dimension.**

Observe that the Hessian matrix is a real symmetric matrix in any dimension. If $f : \mathbb{R} \to \mathbb{R}$ then $\mathbf{H}f(a)$ is a 1×1 matrix and coincides with the second derivative of f. In this sense what we know about extrema in $\mathbb{R}$ is just a special case of everything we do here.

As an example, let $f(x, y) = x^2 - y^2$ (figure). The gradient and the Hessian are respectively

$$\nabla f(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x, y) \\ \frac{\partial f}{\partial y}(x, y) \end{pmatrix} = \begin{pmatrix} 2x \\ -2y \end{pmatrix},$$

$$\mathbf{H}f(x, y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x, y) & \frac{\partial^2 f}{\partial x\, \partial y}(x, y) \\ \frac{\partial^2 f}{\partial y\, \partial x}(x, y) & \frac{\partial^2 f}{\partial y^2}(x, y) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

The point $(0, 0)$ is a stationary point since $\nabla f(0, 0) = \binom{0}{0}$. In this example $\mathbf{H}f$ does not depend on (x, y) but in general we can expect dependence and so it gives a different matrix at different points (x, y).

Theorem

If $\mathbf{v} = (v_1, \ldots, v_n)$ then,

$$\mathbf{v}\, \mathbf{H}f(\mathbf{a})\, \mathbf{v}^{\mathbf{T}} = \sum_{j,k=0}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}(\mathbf{a}) v_j v_k \in \mathbb{R}.$$

▶ **Proof**

Second order Taylor formula for scalar fields

First let's recall the first order Taylor approximation we saw before. If f is differentiable at $\mathbf{a}$ then

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}).$$

If $\mathbf{a}$ is a stationary point then this only tells us that $f(\mathbf{x}) \approx f(\mathbf{a})$ so a natural next question is to search for slightly more detailed information.

Theorem (second order Taylor for scalar fields)

Let f be a scalar field twice differentiable on $B(\mathbf{a}, r)$. Then, for $\mathbf{x}$ close to $\mathbf{a}$,

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})\, \mathbf{H}f(\mathbf{a})\, (\mathbf{x} - \mathbf{a})^{\mathbf{T}}$$

in the sense that the error is $o(\|(\mathbf{x} - \mathbf{a})\|^2)$.

▶ **Proof**

Classifying stationary points

In order to classify the stationary points we will take advantage of the Hessian matrix and therefore we need to first understand the follow fact about real symmetric matrices.

Theorem

Let A be a real symmetric matrix and let $Q(\mathbf{v}) = \mathbf{v}^{\mathbf{T}} A \mathbf{v}$. Then,

- $Q(\mathbf{v}) > 0$ for all $\mathbf{v} \neq \mathbf{0}$ $\iff$ all eigenvalues of A are positive,
- $Q(\mathbf{v}) < 0$ for all $\mathbf{v} \neq \mathbf{0}$ $\iff$ all eigenvalues of A are negative.

▶ **Proof**

Theorem (classifying stationary points)

Let f be a scalar field twice differentiable on $B(\mathbf{a}, r)$. Suppose $\nabla f(\mathbf{a}) = \mathbf{0}$ and consider the eigenvalues of $\mathbf{H}f(\mathbf{a})$. Then,

- All eigenvalues are positive $\implies$ relative minimum at $\mathbf{a}$,
- All eigenvalues are negative $\implies$ relative maximum at $\mathbf{a}$,
- Some positive, some negative $\implies$ $\mathbf{a}$ is a saddle point.

Proof

Let $Q(\mathbf{v}) = \mathbf{v}^{\mathbf{T}}\mathbf{H}f(\mathbf{a})\mathbf{v}$, $\mathbf{w} = B\mathbf{v}$ and let $\Lambda := \min_j \lambda_j$. Observe that $\|\mathbf{w}\| = \|\mathbf{v}\|$ and that $Q(\mathbf{v}) = \sum_j \lambda_j w_j^2 \geq \Lambda \sum_j w_j^2 = \Lambda \|\mathbf{v}\|^2$. We have them 2nd-order Taylor

$$
\begin{aligned}
f(\mathbf{a}+\mathbf{v}) - f(\mathbf{a}) &= \frac{1}{2}\mathbf{v}^{\mathbf{T}}\,\mathbf{H}f(\mathbf{a})\,\mathbf{v} + \epsilon(\mathbf{v}) \\
&\geq \left(\tfrac{\Lambda}{2} - \tfrac{\epsilon(\mathbf{v})}{\|\mathbf{v}\|^2}\right)\|\mathbf{v}\|^2.
\end{aligned}
$$

Since $\frac{|\epsilon(\mathbf{v})|}{\|\mathbf{v}\|^2} \to 0$ as $\|\mathbf{v}\| \to 0$, $\frac{|\epsilon(\mathbf{v})|}{\|\mathbf{v}\|^2} < \frac{\Lambda}{2}$ when $\|\mathbf{v}\|$ is small. The argument is analogous for the second part. For final part consider $\mathbf{v}_j$ which is the eigenvector for λ_j and apply the argument of the first or second part.

Attaining extreme values

Here we explore the extreme value theorem for continuous scalar fields. The argument will be in two parts: Firstly we show that continuity implies boundedness; Secondly we show that boundedness implies that the maximum and minimum are attained. We use the following notation for *interval* / *rectangle* / *cuboid* / *tesseract*, etc. If $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ then we consider the n-dimensional closed Cartesian product

$$[\mathbf{a}, \mathbf{b}] = [a_1, b_1] \times \cdots \times [a_n, b_n].$$

We call this set a *rectangle* (independent of the dimension). As a first step it is convenient to know that all sequences in our setting have convergent subsequences.

Theorem

If $\{\mathbf{x}_n\}_n$ is a sequence in $[\mathbf{a}, \mathbf{b}]$ there exists a convergent subsequence $\{\mathbf{x}_{n_j}\}_j$.

▶ **Proof**

Theorem

Suppose that f is a scalar field continuous at every point in the closed rectangle $[\mathbf{a}, \mathbf{b}]$. Then f is bounded on $[\mathbf{a}, \mathbf{b}]$ in the sense that there exists $C > 0$ such that $|f(\mathbf{x})| \leq C$ for all $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$.

▶ **Proof**

We can now use the above result on the boundedness in order to show that the extreme values are actually obtained.

Theorem

Suppose that f is a scalar field continuous at every point in the closed rectangle $[\mathbf{a}, \mathbf{b}]$. Then there exist points $\mathbf{x}, \mathbf{y} \in [\mathbf{a}, \mathbf{b}]$ such that

$$f(\mathbf{x}) = \inf f \quad \text{and} \quad f(\mathbf{y}) = \sup f.$$

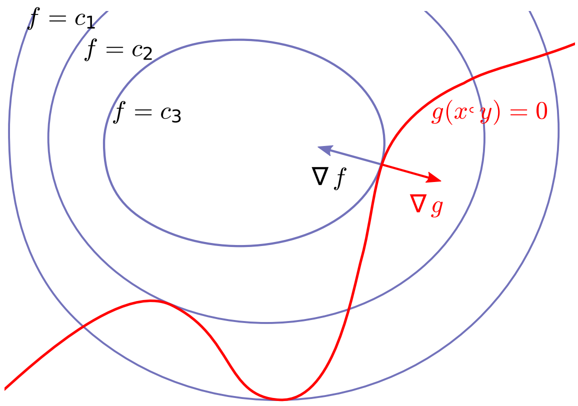
▶ **Proof**

Extrema with constraints (Lagrange multipliers)

We now consider a slightly different problem to the one earlier in this chapter. There we wished to find the extrema of a given scalar field. Here the general problem is to minimise or maximise a given scalar field $f(x, y)$ under the constraint $g(x, y) = 0$. Subsequently we will also consider the same problem but in higher dimensions. For this graphic representation we draw the constraint and also various level sets of the function that we want to find the extrema of. The graphical representation suggests to us that at the "touching point" the gradient vectors are parallel. In other words, $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$. The implementation of this idea is the [method of Lagrange multipliers].

Suppose that a differentiable scalar field $f(x, y)$ has a relative minimum or maximum when it is subject to the constraint $g(x, y) = 0$. Then there exists a scalar λ such that, at the extrema point,

$$\nabla f = \lambda \nabla g.$$



Searching extrema of f under constraint $g = 0$

In three dimensions a similar result holds. Suppose that a differentiable scalar field $f(x, y, z)$ has a relative minimum or maximum when it is subject to the constraints

$$g_1(x, y, z) = 0, \quad g_2(x, y, z) = 0$$

and the ∇g_k are linearly independent. Then there exist scalars λ_1, λ_2 such that, at the extrema point,

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2.$$

In higher dimensions and possibly with additional constraints we have the following general theorem.

Theorem (Lagrange multipliers)

Suppose that a differentiable scalar field $f(x_1, \ldots, x_n)$ has an relative extrema when it is subject to m constraints

$$g_1(x_1, \ldots, x_n) = 0, \ldots, g_m(x_1, \ldots, x_n) = 0,$$

where $m < n$, and the ∇g_k are all linearly independent. Then there exist m scalars $\lambda_1, \dots, \lambda_m$ such that, at each extrema point,

$$\nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_m \nabla g_m.$$

▶ **The Lagrange multiplier method is often stated and far less often proved.**

Idea of proof

Let us consider a particular case of the method when $n = 3$ and $m = 2$. More precisely we consider the following problem: Find the maxima and minima of $f(x, y, z)$ along the curve C defined as

$$g_1(x, y, z) = 0, \quad g_2(x, y, z) = 0$$

where g_1, g_2 are differentiable functions. In this particular case we will prove the Lagrange multiplier method. Suppose that $\mathbf{a}$ is some point in the curve. Let $\boldsymbol{\alpha}(t)$ denote a path which lies in the curve C in the sense that $\boldsymbol{\alpha}(t) \in C$ for all $t \in (-1, 1)$, $\boldsymbol{\alpha}'(t) \neq \mathbf{0}$ and $\boldsymbol{\alpha}(0) = \mathbf{a}$. If $\mathbf{a}$ is a local minimum for f restricted to C it means that $f(\boldsymbol{\alpha}(t)) \geq f(\boldsymbol{\alpha}(0))$ for all $t \in (-\delta, \delta)$ for some $\delta > 0$. In words, moving away from $\mathbf{a}$ along the curve C doesn't cause $f(\mathbf{x})$ to decrease. Let $h(t) = f(\boldsymbol{\alpha}(t))$ and observe that $h : \mathbb{R} \to \mathbb{R}$ so we know how to find the extrema. In particular we know that $h'(0) = 0$. By the chain rule $h'(t) = \nabla f(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)$ and so

$$\nabla f(\mathbf{a}) \cdot \boldsymbol{\alpha}'(0) = 0.$$

Since we know that $g_1(\boldsymbol{\alpha}(t)) = 0$ and $g_2(\boldsymbol{\alpha}(t)) = 0$, again by the chain rule,

$$\nabla g_1(\mathbf{a}) \cdot \boldsymbol{\alpha}'(0) = 0, \quad \nabla g_2(\mathbf{a}) \cdot \boldsymbol{\alpha}'(0) = 0.$$

To proceed it is convenient to isolate the following result of linear algebra.

> Consider $w, u_1, u_2 \in \mathbb{R}^3$ and let $V = \{v : u_k \cdot v = 0, k = 1, 2\}$. If $w \cdot v = 0$ for all $v \in V$ then $w = \lambda_1 u_1 + \lambda_2 u_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

In order to prove this we write $w = \lambda_1 u_1 + \lambda_2 u_2 + v_0$ where $v_0 \in V$ because u_1, u_2 together with V must span $\mathbb{R}^3$. Since $v_0 \in V$ and, by assumption, $w \cdot v_0 = 0$,

$$0 = w \cdot v_0 = (\lambda_1 u_1 + \lambda_2 u_2 + v_0) \cdot v_0 = v_0 \cdot v_0 = \|v_0\|^2.$$

This means that $v_0 = 0$ and so $w = \lambda_1 u_1 + \lambda_2 u_2$.

The above statement holds in any dimension with any number of vectors with the analogous proof. Applying this lemma to the vectors $\nabla f(\mathbf{a})$, $\nabla g_1(\mathbf{a})$ and $\nabla g_2(\mathbf{a})$ recovers exactly the Lagrange multiplier method in this setting.

Curves & line integrals

WARNING

The information in this section is being updated.

In this part of the course we work on the following skills:

- Work with parametric paths
- Evaluate and work with scalar line integrals
- Evaluate and work with vector line integrals
- Work with potentials and conservative vector fields

See also the graded exercises and additional exercises associated to this part of the course.

Curves have played a part in earlier parts of the course and now we turn our attention to precisely what we mean by this notion. Up until now we relied more on an intuition, an idea of some type of 1D subset of higher dimensional space. We will also define how we can integrate scalar and vector fields along these curves. These types of integrals have a natural and important physical relevance. We will then study some of the properties of these integrals. To start let's recall a random selection of curves we have already seen:

- Circle: $x^2 + y^2 = 4$
- Semi-circle: $x^2 + y^2 = 4, \ \ x \geq 0$
- Ellipse: $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 4$
- Line: $y = 5x + 2$

- Line (in 3D): $x + 2y + 3z = 0, \;\; x = 4y$
- Parabola (in 3D): $y = x^2, \;\; z = x$

In the above list the curves are written in a way where we are describing a set of points using certain constraint or constraints. In some cases in *implicit* form, in some cases in *explicit* form. For example, for the circle we formally mean the set $\{(x, y) : x^2 + y^2 = 4\}$. We have the idea that the curves should be sets which are single connected pieces and we vaguely have an idea that we need curves that are sufficiently smooth. To proceed we need a precise definition of the 1D objects we can work with. As part of the definition we force a structure which really allows us to work with these objects in a useful way.

Curves, paths & line integrals

Let $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^n$ be continuous. For convenience, in components we write $\boldsymbol{\alpha}(t) = (\alpha_1(t), \dots, \alpha_n(t))$. We say that $\boldsymbol{\alpha}(t)$ is *differentiable* if each component $\alpha_k(t)$ is differentiable on $[a, b]$ and $\alpha_k'(t)$ is continuous

Definition

We say that $\boldsymbol{\alpha}(t)$ is *piecewise differentiable* if $[a, b] = [a, c_1] \cup [c_1, c_2] \cup \cdots \cup [c_l, b]$ and $\boldsymbol{\alpha}(t)$ is *differentiable* on each of these intervals.

Definition

If $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^n$ is continuous and piecewise differentiable then we call it a *path*.

Note that different functions can trace out the *same* curve in different ways. Also note that a path has an inherent direction. We say that this is a *parametric representation* of a given curve. We already saw examples of paths in spiral and circular motion. A few examples of paths are as follows.

- $\boldsymbol{\alpha}(t) = (t, t)$, $t \in [0, 1]$
- $\boldsymbol{\alpha}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$
- $\boldsymbol{\alpha}(t) = (\cos t, \sin t)$, $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

- $\boldsymbol{\alpha}(t) = (\cos t, -\sin t)$, $t \in [0, 2\pi]$
- $\boldsymbol{\alpha}(t) = (t, t, t)$, $t \in [0, 1]$
- $\boldsymbol{\alpha}(t) = (\cos t, \sin t, t)$, $t \in [-10, 10]$

Observe how some of these paths represent the same curve, perhaps traversed in a different direction.

Let $\boldsymbol{\alpha}(t)$ be a (piecewise differentiable) path on $[a, b]$ and let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field. Recall that we consider $\boldsymbol{\alpha}'(t)$ and $\mathbf{f}(\mathbf{x})$ as n-vectors. I.e., in the case $n = 2$, then

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} \alpha_1'(t) \\ \alpha_2'(t) \end{pmatrix}, \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{pmatrix}.$$

Definition (line integral of vector field)

Let $\boldsymbol{\alpha}(t)$ be a (piecewise differentiable) path on $[a, b]$ and let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field. The *line integral* of the vector field $\mathbf{f}$ along the path $\boldsymbol{\alpha}$ is defined as

$$\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \int_a^b \mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) \, dt.$$

Sometimes the same integral is written as $\int_C \mathbf{f} \cdot d\boldsymbol{\alpha}$ to emphasize that the integral is along the curve C. Alternatively the integral is sometimes written as $\int f_1 \, d\alpha_1 + \cdots + f_n \, d\alpha_n$ or $\int f_1 \, dx_1 + \cdots + f_n \, dx_n$. Each of these different notations are in common usage in different contexts but the underlying quantity is always the same.

Example

Consider the vector field $\mathbf{f}(x, y) = (\sqrt{y}, x^3 + y)$ and the path $\boldsymbol{\alpha}(t) = (t^2, t^3)$ for $t \in (0, 1)$. Evaluate $\int \mathbf{f} \cdot d\boldsymbol{\alpha}$.

Solution

We start by calculating

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t)) = \begin{pmatrix} t^{\frac{3}{2}} \\ t^6 + t^3 \end{pmatrix}.$$

This means that $\mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t) = 2t^{\frac{5}{2}} + 3t^8 + 3t^5$ and so

$$\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \int_0^1 (2t^{\frac{5}{2}} + 3t^8 + 3t^5)\, dt = \frac{59}{42}.$$

Now we consider the question of defining the line integral for scalar fields. Such a line integral allows us also to define the *length of a curve* in a meaningful way. Again let $\boldsymbol{\alpha}(t)$, $t \in [a, b]$ be a path in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous scalar field.

Definition (line integral of scalar field)

Let $\boldsymbol{\alpha}(t)$, $t \in [a, b]$ be a (piecewise differentiable) path in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous scalar field. The *line integral* of the scalar field f along the path $\boldsymbol{\alpha}$ is defined as

$$\int f\, d\alpha = \int_a^b f(\alpha(t))\ \|\boldsymbol{\alpha}'(t)\|\ dt.$$

Subsequently we will primarily work with the line integral of a vector field. However the analogous results hold also for this integral and the proofs are essentially the same. Namely it is linear and also respects how a path can be decomposed or joined with other paths which changing the value of the integral. Moreover, the value of the integral along a given path is independent of the choice of parametrization of the curve. In this case, even if the curve is parametrized in the opposite direction then the integral takes the same value. Consequently it makes sense to define the length of the curve as the line integral of the unit scalar field, i.e., the length of a curve parametrized by the path $\boldsymbol{\alpha}$ is $\int_a^b \|\boldsymbol{\alpha}'(t)\|\ dt$.

Basic properties of the line integral

Having defined the line integral, the next step is to clarify its behaviour, in particular the following key properties.

Theorem

Linearity: Suppose $\mathbf{f}$, $\mathbf{g}$ are vector fields and $\boldsymbol{\alpha}(t)$ is a path. For any $c, d \in \mathbb{R}$, then

$$\int (c\mathbf{f} + d\mathbf{g}) \cdot d\boldsymbol{\alpha} = c \int \mathbf{f} \cdot d\boldsymbol{\alpha} + d \int \mathbf{g} \cdot d\boldsymbol{\alpha}.$$

Joining / splitting paths: Suppose $\mathbf{f}$ is a vector field and that

$$\boldsymbol{\alpha}(t) = \begin{cases} \boldsymbol{\alpha}_1(t) & t \in [a, c] \\ \boldsymbol{\alpha}_2(t) & t \in [c, b] \end{cases}$$

is a path. Then

$$\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \int \mathbf{f} \cdot d\boldsymbol{\alpha}_1 + \int \mathbf{f} \cdot d\boldsymbol{\alpha}_2.$$

Alternatively, if we write C, C_1, C_2 for the corresponding curves, then

$$\int_C \mathbf{f} \cdot d\boldsymbol{\alpha} = \int_{C_1} \mathbf{f} \cdot d\boldsymbol{\alpha} + \int_{C_2} \mathbf{f} \cdot d\boldsymbol{\alpha}.$$

As already mentioned, for a given curve there are many different choices of parametrization. For example, consider the curve $C = \{(x, y) : x^2 + y^2 = 1, y \geq 0\}$. This is a semi-circle and two possible parametrizations are $\boldsymbol{\alpha}(t) = (-t, \sqrt{1 - t^2})$, $t \in [-1, 1]$ and $\boldsymbol{\beta}(t) = (\cos t, \sin t)$, $t \in [0, \pi]$. These are just two possibilities among many possible choices. For a given curve, to what extent does the line integral depend on the choice of parametrization?

Definition (equivalent paths)

We say that two paths $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ are *equivalent* if there exists a differentiable function $u : [c, d] \to [a, b]$ such that $\boldsymbol{\alpha}(u(t)) = \boldsymbol{\beta}(t)$.

Furthermore, we say that $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ are

- *in the same direction* if $u(c) = a$ and $u(d) = b$,
- *in the opposite direction* if $u(c) = b$ and $u(d) = a$.

With this terminology we can precisely describe the dependence of the integral on the choice of parametrization.

Theorem

Let $\mathbf{f}$ be a continuous vector field and let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ be equivalent paths. Then

$$\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \begin{cases} \int \mathbf{f} \cdot d\boldsymbol{\beta} & \text{if the paths in the same direction,} \\ -\int \mathbf{f} \cdot d\boldsymbol{\beta} & \text{if the paths in the opposite direction.} \end{cases}$$

Proof

Suppose that the paths are continuously differentiable path, decomposing if required. Since $\boldsymbol{\alpha}(u(t)) = \boldsymbol{\beta}(t)$ the chain rule implies that $\boldsymbol{\beta}'(t) = \boldsymbol{\alpha}'(u(t))\, u'(t)$. In particular

$$\int \mathbf{f} \cdot d\boldsymbol{\beta} = \int_c^d \mathbf{f}(\boldsymbol{\beta}(t)) \cdot \boldsymbol{\beta}'(t)\, dt = \int_c^d \mathbf{f}(\boldsymbol{\alpha}(u(t))) \cdot \boldsymbol{\alpha}'(u(t))\, u'(t)\, dt.$$

Changing variables, adding a minus sign if path is opposite direction because we need to swap the limits of integration, completes the proof.

Gradients & work

Let $h(x, y)$ be a scalar field in $\mathbb{R}^2$ and recall that the gradient $\nabla h(x, y)$ is a vector field. Let $\boldsymbol{\alpha}(t)$, $t \in [0, 1]$ be a path. Now let $g(t) = h(\boldsymbol{\alpha}(t))$, consider the derivative $g'(t) = \nabla h(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)$ and evaluate the line integral

$$\begin{aligned} \int \nabla h \cdot d\boldsymbol{\alpha} &= \int_0^1 \nabla h(\boldsymbol{\alpha}(t)) \cdot \alpha'(t)\, dt \\ &= \int_0^1 g(t)\, dt = g(1) - g(0) = h(\boldsymbol{\alpha}(1)) - h(\boldsymbol{\alpha}(0)). \end{aligned}$$

This equality has the following intuitive interpretation if we suppose for a moment that h denotes altitude. In this case the line integral is the sum of all the infinitesimal altitude changes and equals the total change in altitude.

As a first example of work in physics let's consider gravity. The gravitational field on earth is $\mathbf{f}(x, y, z) = \left(\begin{smallmatrix} 0 \\ 0 \\ mg \end{smallmatrix}\right)$. If we move a particle from $\mathbf{a} = (a_1, a_2, a_3)$ to $\mathbf{b} = (b_1, b_2, b_3)$ along the path $\boldsymbol{\alpha}(t)$, $t \in [0, 1]$ then the work done is defined as $\int \mathbf{f} \cdot d\boldsymbol{\alpha}$. We calculate that

$$\begin{aligned} \int \mathbf{f} \cdot d\boldsymbol{\alpha} = \int_0^1 \mathbf{f}(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)\, dt &= \int_0^1 mg\, \alpha_3'(t)\, dt \\ = mg\, [\alpha_3(t)]_0^1 = mg(b_3 - a_3). \end{aligned}$$

This coincides we what we know, work done depends only on the change in height.

As a second example of work in physics let's consider a particle moving in a force field. Let $\mathbf{f}$ be the force field and let $\mathbf{x}(t)$ be the position at time t of a particle moving in the field. Let $\mathbf{v}(t) = \mathbf{x}'(t)$ be the velocity at time t of the particle and define kinetic energy as $\frac{m}{2}\|\mathbf{v}(t)\|^2$. According to Newton's law $\mathbf{f}(\mathbf{x}(t)) = m\mathbf{x}''(t) = m\mathbf{v}'(t)$ and so the work done is

$$\begin{aligned}\int \mathbf{f} \cdot d\mathbf{x} &= \int_0^1 \mathbf{f}(\mathbf{x}(t)) \cdot \mathbf{v}(t)\, dt = \int_0^1 m\mathbf{v}'(t) \cdot \mathbf{v}(t)\, dt \\ &= \int_0^1 \tfrac{d}{dt}\left(\tfrac{m}{2}\|\mathbf{v}(t)\|^2\right) = \left(\tfrac{m}{2}\|\mathbf{v}(1)\|^2 - \tfrac{m}{2}\|\mathbf{v}(0)\|^2\right)\end{aligned}$$

In this case we see, as expected, the work done on the particle moving in the force field is equal to the change in kinetic energy.

The second fundamental theorem

Recall that, if $\varphi : \mathbb{R} \to \mathbb{R}$ is differentiable then $\int_a^b \varphi'(t)\, dt = \varphi(b) - \varphi(a)$. This is called the second fundamental theorem of calculus and is one of the ways in which we see that differentiation and integration are opposites. The analog for line integrals is the following.

Theorem (second fundamental theorem for line integrals)

Suppose that φ is a continuously differentiable scalar field on $S \subset \mathbb{R}^n$ and suppose that $\boldsymbol{\alpha}(t)$, $t \in [a, b]$ is a path in S. Let $\mathbf{a} = \boldsymbol{\alpha}(a)$, $\mathbf{b} = \boldsymbol{\alpha}(b)$. Then

$$\int \nabla\varphi \cdot d\boldsymbol{\alpha} = \varphi(\mathbf{b}) - \varphi(\mathbf{a}).$$

Proof

Suppose that $\boldsymbol{\alpha}(t)$ is differentiable. By the chain rule $\frac{d}{dt}\varphi(\boldsymbol{\alpha}(t)) = \nabla\varphi(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)$. Consequently

$$\int \nabla\varphi \cdot d\boldsymbol{\alpha} = \int_0^1 \nabla\varphi(\boldsymbol{\alpha}(t)) \cdot \boldsymbol{\alpha}'(t)\, dt = \int_0^1 \tfrac{d}{dt}\varphi(\boldsymbol{\alpha}(t))\, dt.$$

By the 2nd fundamental theorem in $\mathbb{R}$ we know that $\int_0^1 \frac{d}{dt}\varphi(\boldsymbol{\alpha}(t))\,dt = \varphi(\boldsymbol{\alpha}(b)) - \varphi(\boldsymbol{\alpha}(a))$.

Our earth has mass M with centre at $(0,0,0)$. Suppose that there is a small particle close to earth which has mass m. The force field of gravitation and potential energy are, respectively,

$$\mathbf{f}(\mathbf{x}) = \frac{-GmM}{\|\mathbf{x}\|^3}\mathbf{x}, \quad \varphi(\mathbf{x}) = \frac{GmM}{\|\mathbf{x}\|}.$$

We can calculate $\nabla\varphi(\mathbf{x})$ and see that it is equal to $\mathbf{f}(\mathbf{x})$.

The first fundamental theorem

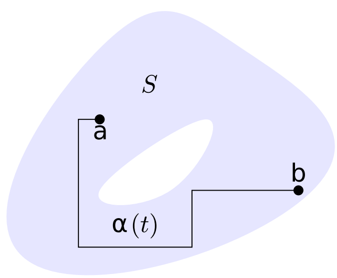
First we need to consider a basic topological property of sets. In particular we want to avoid the possibility of the set being several disconnected pieces, in other words we want to guarantee that we can get from one point to another in the set in a way without every leaving the set (see figure).

Definition

The set $S \subset \mathbb{R}^n$ is said to be *connected* if, for every pair of points $\mathbf{a}, \mathbf{b} \in S$, there exists a path $\boldsymbol{\alpha}(t), t \in [a,b]$ such that

- $\boldsymbol{\alpha}(t) \in S$ for every $t \in [a,b]$,
- $\boldsymbol{\alpha}(a) = \mathbf{a}$ and $\boldsymbol{\alpha}(b) = \mathbf{b}$.

Sometimes this property is called "path connected" to distinguish between different notions.



Recall that, if $f : \mathbb{R} \to \mathbb{R}$ is continuous and we let $\varphi(x) = \int_a^x f(t)\,dt$ then $\varphi'(x) = f(x)$. This is called the first fundamental theorem of calculus and is the other way in which we see that differentiation and integration are opposites. Again we have an analog for the line integral but here it becomes a little more subtle since there are many different paths along which we can integrate between any two points.

Theorem (first fundamental theorem for line integrals)

Let $\mathbf{f}$ be a continuous vector field on a connected set $S \subset \mathbb{R}^n$. Suppose that, for $\mathbf{x}, \mathbf{a} \in S$, the line integral $\int \mathbf{f} \cdot d\boldsymbol{\alpha}$ is equal for every path $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha}(a) = \mathbf{a}$, $\boldsymbol{\alpha}(b) = \mathbf{x}$. Fix $\mathbf{a} \in S$ and define $\varphi(\mathbf{x}) = \int \mathbf{f} \cdot d\boldsymbol{\alpha}$. Then φ is continuously differentiable and $\nabla\varphi = \mathbf{f}$.

Proof

As before let $\mathbf{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\mathbf{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$. Observe that, if we define the paths $\boldsymbol{\beta}_k(t) = \mathbf{x} + t\mathbf{e}_k$, $t \in [0, h]$, then

$$\varphi(\mathbf{x} + h\mathbf{e}_k) - \varphi(\mathbf{x}) = \int \mathbf{f} \cdot d\boldsymbol{\beta}_k.$$

Moreover $\boldsymbol{\beta}_k'(t) = \mathbf{e}_k$. Consequently

$$\begin{aligned}\frac{\partial\varphi}{\partial x_k}(\mathbf{x}) &= \lim_{h\to 0} \frac{1}{h}(\varphi(\mathbf{x} + h\mathbf{e}_k) - \varphi(\mathbf{x})) \\ &= \lim_{h\to 0} \frac{1}{h} \int_0^h \mathbf{f}(\boldsymbol{\beta}_k(t)) \cdot \mathbf{e}_k\,dt = f_k(\mathbf{x}).\end{aligned}$$

In other words, we have shown that $\nabla\varphi(\mathbf{x}) = \mathbf{f}(\mathbf{x})$.

We say a path $\boldsymbol{\alpha}(t)$, $t \in [a, b]$ is *closed* if $\boldsymbol{\alpha}(a) = \boldsymbol{\alpha}(b)$.

Observe that, if $\boldsymbol{\alpha}(t)$, $t \in [a, b]$ is a closed path then we can divided it into two paths: Let $c \in [a, b]$ and consider the two paths $\boldsymbol{\alpha}(t)$, $t \in [a, c]$ and $\boldsymbol{\alpha}(t)$, $t \in [c, b]$. On the other hand, suppose $\boldsymbol{\alpha}(t)$, $t \in [a, b]$ and $\boldsymbol{\beta}(t)$, $t \in [c, d]$ are two path starting at $\mathbf{a}$ and finishing at $\mathbf{b}$. The these can be combined to define a closed path (by following one backward).

Definition (conservative vector field)

A vector field $\mathbf{f}$, continuous on $S \subset \mathbb{R}^n$ is *conservative* if there exists a scalar field φ such that, on S,

$$\mathbf{f} = \nabla\varphi.$$

Note that some authors call such a vector field a *gradient* (i.e., the vector field is the gradient of some scalar). If $\mathbf{f} = \nabla\varphi$ then the scalar field φ is called the *potential* (associated to $\mathbf{f}$). Observe that that the potential is not unique, $\nabla\varphi = \nabla(\varphi + C)$ for any constant $C \in \mathbb{R}$.

Theorem (conservative fields)

Let $S \subset \mathbb{R}^n$ and and consider the vector field $\mathbf{f} : S \to \mathbb{R}^n$. The following are equivalent:

1. $\mathbf{f}$ is conservative, i.e., $\mathbf{f} = \nabla\varphi$ on S for some φ,
2. $\int \mathbf{f} \cdot d\boldsymbol{\alpha}$ does not depend on $\boldsymbol{\alpha}$, as long as $\boldsymbol{\alpha}(a) = \mathbf{a}$, $\boldsymbol{\alpha}(b) = \mathbf{b}$,
3. $\int \mathbf{f} \cdot d\boldsymbol{\alpha} = 0$ for any closed path $\boldsymbol{\alpha}$ contained in S.

Proof

In the previous theorems (the two fundamental theorems) we proved that (i) is equivalent to (ii).

Now we prove that (ii) implies (iii): Let $\boldsymbol{\alpha}(t)$ be a closed path and let $\boldsymbol{\beta}(t)$ be the same path in the opposite direction. Observe that $\int \mathbf{f} \cdot d\boldsymbol{\alpha} = -\int \mathbf{f} \cdot d\boldsymbol{\beta}$ but that $\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \int \mathbf{f} \cdot d\boldsymbol{\beta}$ and so $\int \mathbf{f} \cdot d\boldsymbol{\alpha} = 0$.

It remains to prove that (iii) implies (ii): The two paths between $\mathbf{a}$ and $\mathbf{b}$ can be combined (with a minus sign) to give a closed path.

Theorem (mixed partial derivatives)

Suppose that $S \subset \mathbb{R}^2$ and that $\mathbf{f} : S \to \mathbb{R}^2$ is a differentiable vector field and write $\mathbf{f} = \left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right)$.

If $\mathbf{f}$ is conservative then, on S,

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}.$$

The above result is a special case of the following general statement which holds in any dimension.

Theorem (mixed partial derivatives)

Suppose that $\mathbf{f}$ is a differentiable vector field[^1] on $S \subset \mathbb{R}^n$. If $\mathbf{f}$ is conservative then, for each l, k,

$$\frac{\partial f_l}{\partial x_k} = \frac{\partial f_k}{\partial x_l}.$$

Proof

By assumption the second order partial derivatives exist and so

$$\frac{\partial f_l}{\partial x_k} = \frac{\partial^2 \varphi}{\partial x_k \partial x_l} = \frac{\partial^2 \varphi}{\partial x_l \partial x_k} = \frac{\partial f_k}{\partial x_l}.$$

Example

Consider the vector field

$$\mathbf{f}(x, y) = \begin{pmatrix} -y(x^2+y^2)^{-1} \\ x(x^2+y^2)^{-1} \end{pmatrix}$$

on $S = \mathbb{R}^2 \setminus (0, 0)$. Calculating we verify that $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$ on S. We now evaluate the line integral $\int \mathbf{f} \cdot d\boldsymbol{\alpha}$ where $\boldsymbol{\alpha}(t) = (a \cos t, a \sin t)$, $t \in [0, 2\pi]$. We calculate that

$$\boldsymbol{\alpha}'(t) = \begin{pmatrix} -a \sin t \\ a \cos t \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\alpha}(t)) = \frac{1}{a^2} \begin{pmatrix} -a \sin t \\ a \cos t \end{pmatrix}.$$

This means that

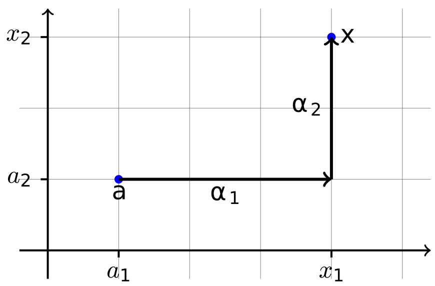
$$\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, dt = 2\pi.$$

Observe that in the above example S is somehow not a "nice" set because of the "hole" in the middle. Moreover, observe that the line integral is the same for any circle, independent of the radius.

The mixed partials theorem isn't useful in showing that a vector field is conservative because it is possible for the mixed partial derivatives to all be equal but still the field fail to be conservative. On the other hand, if a pair of mixed derivatives is not equal then $\mathbf{f}$ is *not* conservative and so it is useful for proving the negative. Later in this chapter we will return to this topic.

Potentials & conservative vector fields

We now turn our attention to the following question: Suppose we are given a vector field $\mathbf{f}$ and we know that $\mathbf{f} = \nabla\varphi$ for some φ. How can we find φ? For this we consider two methods in the following paragraphs.



The paths $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$.

First we describe the method which we call *constructing a potential by line integral*. Suppose that $\mathbf{f}$ is a conservative vector field on the rectangle $[a_1, b_1] \times [a_2, b_2]$. We define $\varphi(\mathbf{x})$ as the line integral $\int \mathbf{f} \cdot d\boldsymbol{\alpha}$ where $\boldsymbol{\alpha}$ is a path between $\mathbf{a} = (a_1, a_2)$ and $\mathbf{x}$. For any $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ consider the two paths:

- $\boldsymbol{\alpha}_1(t) = (t, a_2)$, $t \in [a_1, x_1]$,
- $\boldsymbol{\alpha}_2(t) = (x_1, t)$, $t \in [a_2, x_2]$.

Let $\boldsymbol{\alpha}(t)$ denote the concatenation of the two paths. We calculate that

$$\int \mathbf{f} \cdot d\boldsymbol{\alpha} = \int_{a_1}^{x_1} \mathbf{f}(\boldsymbol{\alpha}_1(t)) \cdot \boldsymbol{\alpha}_1'(t)\, dt + \int_{a_2}^{x_2} \mathbf{f}(\boldsymbol{\alpha}_2(t)) \cdot \boldsymbol{\alpha}_2'(t)\, dt.$$

This means that $\varphi(\mathbf{x}) = \int_{a_1}^{x_1} f_1(t, a_2)\, dt + \int_{a_2}^{x_2} f_2(x_1, t)\, dt$.

Now we describe a different method which we describe as *constructing a potential by indefinite integrals*. Again suppose that $\mathbf{f} = \nabla\varphi$ for some scalar field $\varphi(x,y)$ which we wish to find. Observe that $\frac{\partial\varphi}{\partial x} = f_1$ and $\frac{\partial\varphi}{\partial y} = f_2$. This means that

$$\int_a^x f_1(t,y)\,dt + A(y) = \varphi(x,y) = \int_b^y f_2(x,t)\,dt + B(x)$$

where $A(y)$, $B(x)$ are constants of integration. Calculating and comparing we can then obtain a formula for $\varphi(x,y)$.

Find a potential for $\mathbf{f}(x,y) = \begin{pmatrix} e^x y^2+1 \\ 2e^x y \end{pmatrix}$ on $\mathbb{R}^2$.

We calculate that

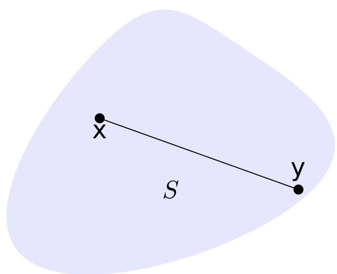
$$\begin{aligned} \int_a^x f_1(t,y)\,dt + A(y) &= e^x y^2 + x + A(y) = \varphi(x,y), \\ \int_b^y f_2(x,t)\,dt + B(x) &= e^x y^2 + B(x) = \varphi(x,y). \end{aligned}$$

From this we see that we can choose $A(y) = 0$ and $B(x) = x$ to obtain equality of the above quantities. Consequently we obtain the potential $\varphi(x,y) = e^x y^2 + x$.

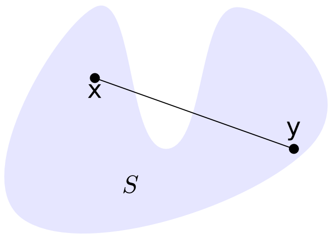
The mixed partials theorem concerning conservative fields and the mixed partial derivatives was somewhat less than satisfactory since the converse wasn't possible. In order to get a more satisfactory result we need to look at another topological details of the domain of the vector field. This concept is somewhat suggested by the methods of constructing potentials which were described above.

Definition (convex)

A set $S \subset \mathbb{R}^n$ is said to be *convex* if for any $\mathbf{x}, \mathbf{y} \in S$ the segment $\{t\mathbf{x} + (1-t)\mathbf{y}, t \in [0,1]\}$ is contained in S.



A convex set



A set which is not convex.

This extra property permits the following sufficient condition for a vector field to be conservative.

Theorem

Let $\mathbf{f}$ be a differentiable vector field on a convex region $S \subset \mathbb{R}^n$. Then $\mathbf{f}$ is conservative if and only if

$$\frac{\partial f_l}{\partial x_k} = \frac{\partial f_k}{\partial x_l}, \qquad \text{for each } l, k.$$

Proof

We have already proved that $\mathbf{f}$ being conservative implies the equality of partial derivatives ([mixed partials theorem]()) and therefore we need only assume that $\partial_g f_l = \partial_l f_k$ and construct a potential. Let $\varphi(\mathbf{x}) = \int \mathbf{f} \cdot d\boldsymbol{\alpha}$ where $\boldsymbol{\alpha}(t) = t\mathbf{x}$, $t \in [0, 1]$. Since $\boldsymbol{\alpha}'(t) = \mathbf{x}$, $\varphi(\mathbf{x}) = \int_0^1 \mathbf{f}(t\mathbf{x}) \cdot \mathbf{x}\, dt$. Also (needs proving)

$$\frac{\partial \varphi}{\partial x_k}(t\mathbf{x}) = \int_0^1 \left(t\partial_k \mathbf{f}(t\mathbf{x}) \cdot \mathbf{x} + f_k(t\mathbf{x})\right) dt.$$

This is equal to $\int_0^1 \left(t\nabla f_k(t\mathbf{x}) \cdot \mathbf{x} + f_k(t\mathbf{x})\right) dt$ because $\partial_g f_l = \partial_l f_k$; By the chain rule applied to $g(t) = t\nabla f_k(t\mathbf{x})$ this is equal to $f_k(\mathbf{x})$ as required.

The above gives us a useful tool to check if a given vector field is conservative. Using the idea of "gluing together" several convex regions this result can be manually extended to some more general settings. Later, we will take advantage of some further ideas in order to significantly extend this result.

Multiple integrals

WARNING

The information in this section is being updated.

See also the graded exercises and additional exercises associated to this part of the course. If you want more, Chapter 5 (plus sections 6.3 and 6.4) of OpenStax Calculus Volume 3 is a good option.

The extension to higher dimension of differentiation was established in the previous chapters. We then defined line integrals which are, in a sense, one dimensional integrals which exist in a high dimensional setting. We now take the next step and define higher dimensional integrals in the sense of how to integrate a scalar field defined on a subset of $\mathbb{R}^n$. The first step will be to rigorously define which scalar fields are integrable and to define the integral. Then we need to find reasonable ways to evaluate such integrals. Among other applications we will use this multiple integrals to calculate volumes and moment of inertia. In Green's Theorem we find a connection between multiple integrals and line integrals. We also develop the important topic of change of variables which takes advantage of the Jacobian determinant and is often invaluable for actually working with a given problem.

Definition of the integral

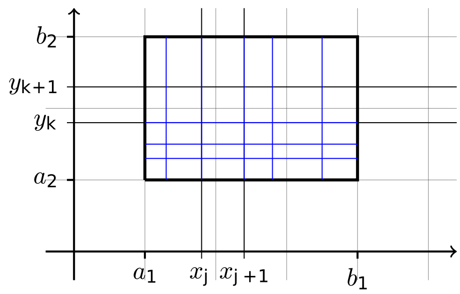
First we need to find a definition of integrability and the integral. Then we will proceed to study the properties of this higher dimensional integral. Recall that, in the one-dimensional case integration was defined using the following steps:

1. Define the integral for step functions,
2. Define integral for "integrable functions",
3. Show that continuous functions are integrable.

For higher dimensions we follow the same logic. We will then show that we can evaluate higher dimensional integrals by repeated one-dimensional integration.

Definition (partition)

Let $R = [a_1, b_1] \times [a_2, b_2]$ be a rectangle. Suppose that $P_1 = \{x_0, \ldots, x_m\}$ and $P_2 = \{y_0, \ldots, y_n\}$ such that $a_1 = x_0 < x_1 < \cdots < x_m = b_1$ and $a_2 = y_0 < y_1 < \cdots < y_n = b_2$. $P = P_1 \times P_2$ is said to be a *partition* of R.

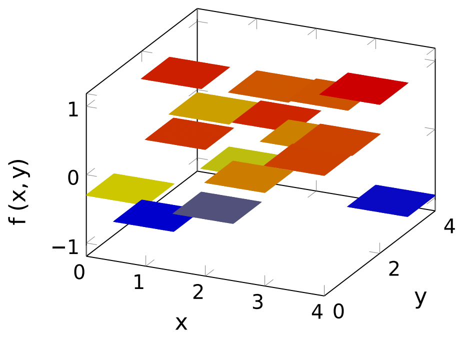


A partition of a rectangle R

Observe that a partition divides R into nm sub-rectangles. If $P \subseteq Q$ then we say that Q is a finer partition than P. Partitions are constructed in higher dimension, for $\mathbb{R}^n$, in an analogous way. Before defining integration for general functions it is convenient to make the definition for a special class of functions called step functions.

Definition (step function)

A function $f : R \to \mathbb{R}$ is said to be a *step function* if there is a partition P of R such that f is constant on each sub-rectangle of the partition.



Graph of a step function

If f and g are step functions and $c, d \in \mathbb{R}$, then $cf + dg$ is also a step function. Also note that the area of the sub-rectangle $Q_{jk} := [x_j, x_{j+1}] \times [y_k, y_{k+1}]$ is equal to $(x_{j+1} - x_j)(y_{k+1} - y_k)$.

We can now define the integral of a step function in a reasonable way. The definition here is for 2D but the analogous definition holds for any dimension.

Suppose that f is a step function with value c_{jk} on the sub-rectangle $(x_j, x_{j+1}) \times (y_k, y_{k+1})$. Then we define the integral as

$$\iint_R f \, dxdy = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} c_{jk}(x_{j+1} - x_j)(y_{k+1} - y_k).$$

This should remind you of Riemann sums from Analysis I. Observe that the value of the integral is independent of the partition, as long as the function is constant on each sub-rectangle. In this sense the integral is well-defined (not dependent on the choice of partition used to calculate it).

Theorem

Let f, g be step functions. Then

$$\begin{aligned} \iint_R (af + bg) \, dxdy &= a \iint_R f \, dxdy + b \iint_R g \, dxdy \quad \text{for all } a, b \in \mathbb{R}, \\ \iint_R f \, dxdy &= \iint_{R_1} f \, dxdy + \iint_{R_2} f \, dxdy \quad \text{if } R \text{ is divided into } R_1 \text{ and } R_2, \\ \iint_R f \, dxdy &\le \iint_R g \, dxdy \quad \text{if } f(x,y) \le g(x,y). \end{aligned}$$

Proof

All properties follow from the definition by basic calculations.

We are now in the position to define the set of integrable functions. In order to define integrability we take advantage of "upper" and "lower" integrals which "sandwich" the function we really want to integrate.

Definition (integrability on a rectangle)

Let R be a rectangle and let $f : R \to \mathbb{R}$ be a bounded function. We call f an _integrable* function if there is one and only one number $I \in \mathbb{R}$ such that

$$\iint_R g(x,y)\,dxdy \leq I \leq \iint_R h(x,y)\,dxdy$$

for every pair of step functions g, h such that, for all $(x,y) \in R$,

$$g(x,y) \leq f(x,y) \leq h(x,y).$$

This number I is called the integral of f on R and is denoted $\iint_R f(x,y)\,dxdy$.

All the basic properties of the integral of step functions, as stated in the above Theorem, also hold for the integral of any integrable functions. This can be shown by considering the limiting procedure of the upper and lower integral of step functions which are part of the definition of integrability.

The most important words in the definition are "only one number": that's what we need to check to verify that a function is integrable. That still isn't immediately easy to check and so it is convenient to now investigate the integrability of continuous functions.

Theorem

Suppose that f is a continuous function defined on the rectangle R. Then f is integrable.

Proof

Continuity implies boundedness and so upper and lower integrals exist. Let $\epsilon > 0$. Exists $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{y})| \leq \epsilon$ whenever $\|\mathbf{x} - \mathbf{y}\| \leq \delta$. We can choose a partition such that $\|\mathbf{x} - \mathbf{y}\| \leq \delta$ whenever $\mathbf{x}, \mathbf{y}$ are in the same sub-rectangle Q_{jk}. We then define the step functions g, h s.t. $g(\mathbf{x}) = \inf_{Q{jk}} f$, $h(\mathbf{x}) = \sup_{Q{jk}} f$ when $\mathbf{x} \in Q_{jk}$. To finish the proof we observe that $\left|\inf_{Qjk} f - \sup_{Qjk} f\right| \leq \epsilon$ and $\epsilon > 0$ can be made arbitrarily small, so we can make the upper and lower integrals as close as we want.

Evaluation of multiple integrals

Now we have a definition, so we know what a multidimensional integral is, and we also know that some interesting ones exist, but it is essential to also have a way to practically evaluate any given integral. It turns out we can do that by integrating in one variable at a time:

Theorem (Fubini)

Let f be an integrable function on the rectangle $R = [a_1, b_1] \times [a_2, b_2]$. Then

$$\iint_R f(x,y)\,dxdy = \int_{a_2}^{b_2} \left[\int_{a_1}^{b_1} f(x,y)\,dx\right] dy = \int_{a_1}^{b_1} \left[\int_{a_2}^{b_2} f(x,y)\,dy\right] dx.$$

Proof

To see this, think about any pair of step functions g, h such that $g \leq f \leq h$. Since these are step functions,

$$\iint_R g(x,y)\,dxdy = \int_{a_2}^{b_2} \left[\int_{a_1}^{b_1} g(x,y)\,dx\right] dy = \int_{a_1}^{b_1} \left[\int_{a_2}^{b_2} g(x,y)\,dy\right] dx$$

since these are all just different names for the same sum, and the same is true for h. Using this,

$$\iint_R g(x,y)\,dxdy \leq \int_{a_2}^{b_2} \left[\int_{a_1}^{b_1} f(x,y)\,dx\right] dy \leq \iint_R h(x,y)\,dxdy;$$

in other words the iterated integral in the middle is bounded from above and below by the same upper and lower integrals as the integral of f, which leaves only one possible value

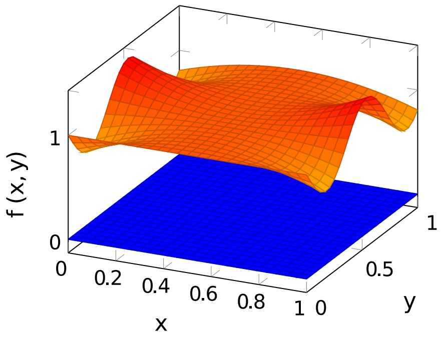
$$\int_{a_2}^{b_2} \left[\int_{a_1}^{b_1} f(x,y)\,dx\right] dy = \iint_R f(x,y)\,dxdy$$

and the other equality holds for the same reason.

This integral naturally allows us to calculate the volume of a solid. Let $f(x,y) \leq z \leq g(x,y)$ be defined on the rectangle $R \subset \mathbb{R}^2$ and consider the 3D set defined as

$$V = \{(x,y,z) : (x,y) \in R, f(x,y) \leq z \leq g(x,y)\}.$$

The volume of the set V is equal to $\mathbf{Vol}(V) = \iint_R [g(x,y) - f(x,y)]\, dxdy$.



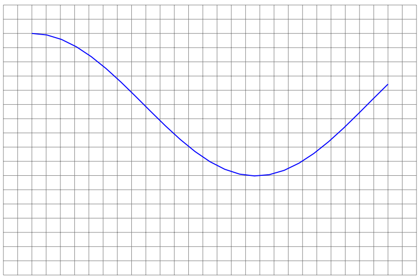
Set enclosed by xy-plane and $f(x,y)$

Up until now we have considered step function and continuous functions. As with one-dimensional integrals we can permit some discontinuities and we introduce the following concept to be able to control the functions with discontinuities sufficiently to guarantee that the integrals are well-defined.

Definition (Content zero sets)

A bounded subset $A \subset \mathbb{R}^2$ is said to have content zero if, for every $\epsilon > 0$, there exists a finite set of rectangles whose union includes A and the sum of the areas of the rectangles is not greater than ϵ.

Examples of content zero sets include: finite sets of points; bounded line segments; continuous paths.



The graph of a continuous function has content zero

Theorem

Suppose that φ is a continuous function on $[a, b]$. Then the graph $\{(x, y) : x \in [a, b], y = \varphi(x)\}$ has zero content.

Proof

By continuity, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\varphi(x) - \varphi(y)| \leq \epsilon$ whenever $|x - y| \leq \delta$. We then take partition of $[a, b]$ into subintervals of length less than δ. Using this partition we generate a cover of the graph which has area not greater than $2\epsilon\,|b - a|$.

Theorem

Let f be a bounded function on R and suppose that the set of discontinuities $A \subset R$ has content zero. Then the double integral $\iint_R f(x, y)\;dxdy$ exists.

Proof

Take a cover of A by rectangles with total area not greater than $\delta > 0$. Let P be a partition of R which is finer than the cover of A. We may assume that $\left|\inf_{Qjk} f - \sup_{Qjk} f\right| \leq \epsilon$ on each sub-rectangle of the partition which doesn't contain a discontinuity of f. The contribution to the integral of bounding step functions from the cover of A is bounded by $\delta \sup |f|$.

Regions bounded by functions

A major limitation is that we have only integrated over rectangles whereas we would like to integrate over much more general different shaped regions. This we develop now.

Suppose $S \subset R$ and f is a bounded function on S. We extend f to R by defining

$$f_R(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in S \\ 0 & \text{otherwise.} \end{cases}$$

We use this notation in the following definition.

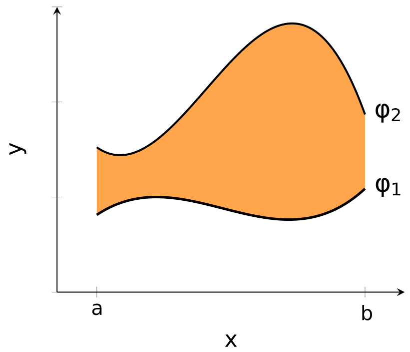
Definition

We say that $f : S \to \mathbb{R}$ is integrable if f_R is integrable and define

$$\iint_S f(x,y)\,dxdy = \iint_R f_R(x,y)\,dxdy.$$

Suppose that there are continuous functions φ_1, φ_2 on $\mathbb{R}$ and consider the set

$$S = \{(x,y) : a \leq x \leq b, \varphi_1(x) \leq y \leq \varphi_2(x)\} \subset \mathbb{R}^2.$$



A region defined by two continuous functions. The projection of the region onto the x-axis is the interval $[a,b]$

Not all sets can be written in this way but many can and such a way of describing a subset of $\mathbb{R}^2$ is convenient for evaluating integrals. We will call sets that can be written this way Type 1 sets in this section.

Let $S = \{(x,y) : x \in [a,b], \varphi_1(x) \leq y \leq \varphi_2(x)\}$ where φ_1, φ_2 are continuous and let f be a bounded continuous function of S. Then f is integrable on S and

$$\iint_S f(x,y)\,dxdy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right] dx :$$

the set of discontinuity of f_R is the boundary of S in $R = [a,b] \times [\tilde{a},\tilde{b}]$ which consists of the graphs of φ_1, φ_2. These graphs have zero content as we proved before. For each x, $f(x,y)$ is integrable since it has only two discontinuity points. Additionally $\int_{\tilde{a}}^{\tilde{b}} f_R(x,y)\,dy = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$. Observe that it doesn't make a different to the integral if we use $<$ or $\leq$ in the definition of S since the difference would be a content zero set.

We could also consider the following set

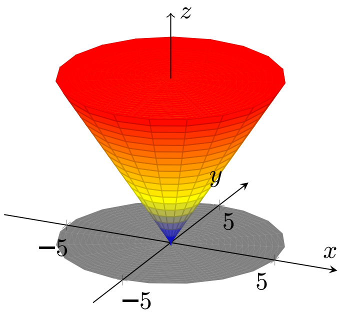
$$R = \{(x,y) : a \leq y \leq b, \varphi_1(y) \leq x \leq \varphi_2(y)\},$$

which we will call a Type 2 set. This is just the same situation as above with the roles of x and y switched, that is

$$\iint_R f(x,y)\, dxdy = \int_a^b \left[\int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)\, dx \right] dy.$$

In the first case we could describe the representation as projecting along the y-coordinate whereas in the second case we are projecting along the x-coordinate. Many interesting sets are both Type 1 and Type 2, although very often one form is more obvious or more useful than the other.

For higher dimensions we need to also have an understanding of how to represent subsets of $\mathbb{R}^n$. Take for example a 3D solid; then we would hope to be able to "project" along one of the coordinate axis and so describe it using the 2D "shadow" and a pair of continuous functions. For example, consider the upside-down cone in this figure which has base of radius 5 lying in the plane $\{z = 5\}$ and has its tip at the origin.



Upside-down cone of height 5 with tip at the origin. The solid is bounded by the surfaces $z = \sqrt{x^2+y^2}$ and $z = 5$. This solid can be "projected" onto the xy-plane.

In order to describe this set it is convenient to imagine how it projects down onto the xy-axis. We then describe it as

$$V=\{(x,y,z):(x,y)\in S,\gamma_1(x,y)\le z\le \gamma_2(x,y)\}$$

where $S\subset\mathbb{R}^2$ is the "shadow" and the functions represent the control we need in the vertical direction. In this case we must choose $S=\{(x,y):x^2+y^2\le 5^2\}$ since the base of the cone, at the top of the picture, it the largest part in terms of the shadow. We also must choose $\gamma_1(x,y)=\sqrt{x^2+y^2}$ and $\gamma_2(x,y)=5$ to correspond to the sloped lower surface and the horizontal upper surface.

Applications of multiple integrals

Multiple integrals can be used to calculate the area or volume of a given set. Suppose that

$$S=\{(x,y):x\in[a,b],\varphi_1(x)\le y\le\varphi_2(x)\}\subset\mathbb{R}^2$$

where φ_1, φ_2 are continuous functions. The the area of S is

$$\iint_S dxdy=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}dy\right]dx=\int_a^b[\varphi_2(x)-\varphi_1(x)]\,dx.$$

This corresponds to the usual notion of the integral of a function on $\mathbb{R}$ determining the area under the curve. The same idea extends to arbitrary dimension. Suppose that $\gamma_1(x,y)\le\gamma_2(x,y)$ are continuous functions on S and let

$$V=\{(x,y,z):x\in[a,b],\varphi_1(x)\le y\le\varphi_2(x),\gamma_1(x,y)\le z\le\gamma_2(x,y)\}\subset\mathbb{R}^3.$$

The volume of V is

$$\begin{aligned}\iiint_V dxdydz&=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}\left[\int_{\gamma_1(x,y)}^{\gamma_2(x,y)}dz\right]dy\right]dx\\&=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}[\gamma_2(x,y)-\gamma_1(x,y)]dy\right]dx.\end{aligned}$$

Multiple integrals also allow us to calculate the mass and centre of mass of solids. Suppose we have several particles each with mass m_k and located at point (x_k,y_k). The total mass would then be $M=\sum_k m_k$ and the centre of mass is the point (p,q) such that

$$pM = \sum_k m_k x_k \quad \text{and} \quad qM = \sum_k m_k y_k.$$

Suppose an object has the shape of a region S and the density of the material is $f(x,y)$ at point (x,y). Then, similar to the discrete case above, the total mass is $M = \iint_S f(x,y)\, dxdy$ and the centre of mass is the point (p,q) such that

$$pM = \iint_S x\, f(x,y)\, dxdy \quad \text{and} \quad qM = \iint_S y\, f(x,y)\, dxdy.$$

By tradition, if the density is constant, then the centre of mass is called the *centroid*.

Green's theorem

We can now establish a connection between multiple integrals and the line integrals of the previous chapter.

Theorem (Green's)

Let $C \subset \mathbb{R}^2$ be a piecewise-smooth simple (no intersections) curve and $\boldsymbol{\alpha}$ a path that parametrizes C in the counter-clockwise direction. Let S be the region enclosed by C. Suppose that $\mathbf{f}(x,y) = \binom{P(x,y)}{Q(x,y)}$ is a vector field continuously differentiable on an open set containing S. Then

$$\iint_S \left(\tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}\right) dxdy = \int_C \mathbf{f} \cdot d\boldsymbol{\alpha}.$$

Proof

To start we assume that S is a type 1 region and that $Q = 0$, Since $S = \{(x,y) : x \in [a,b], \varphi_1(x) \le y \le \varphi_2(x)\}$,

$$\begin{aligned}\iint_S \left(\tfrac{\partial Q}{\partial x} - \tfrac{\partial P}{\partial y}\right) dxdy &= \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} (-\tfrac{\partial P}{\partial y})\, dy\right] dx \\ &= \int_a^b [P(x,\varphi_1(x)) - P(x,\varphi_2(x))] dx,\end{aligned}$$

It is then natural to choose four paths $\boldsymbol{\alpha}_1(t) = (t, \varphi_1(t))$, $\boldsymbol{\alpha}_2(t) = (a,t)$, $\boldsymbol{\alpha}_3(t) = (t, \varphi_2(t))$, $\boldsymbol{\alpha}_4(t) = (b,t)$. We can calculate that

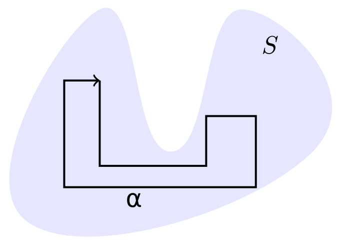
$$\int_C \mathbf{f} \cdot d\boldsymbol{\alpha} = \int \mathbf{f} \cdot d\boldsymbol{\alpha}_1 - \int \mathbf{f} \cdot d\boldsymbol{\alpha}_3 = \int_a^b P(t, \varphi_1(t))\, dt - \int_a^b P(t, \varphi_2(t))\, dt.$$

If S is also type 2 then this works for $P = 0$ and linearity means it works for $\mathbf{f} = \binom{P}{0} + \binom{0}{Q}$, More general regions can be formed by "glueing" together simpler regions of the above type to complete the argument.

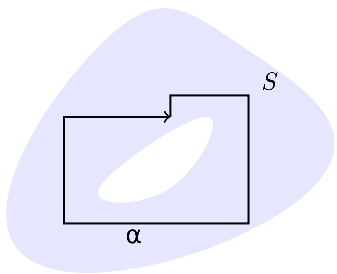
The quantity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is reminiscent of something we saw with conservative vector fields and we take advantage of this with the following application. We previously introduced the concept of *connected sets* but now we need a slight refinement of the idea.

Definition (simply connected)

A connected set $S \subset \mathbb{R}^n$ is said to be *simply-connected* if any closed path $\boldsymbol{\alpha}$, contained within S, can be contracted to a point. (This is in the sense that there exists a continuous map $F : D^2 \to S$, where $D^2 \subset \mathbb{R}^2$ denotes the unit disk, such that F restricted to the unit circle is $\boldsymbol{\alpha}$.)



Simply connected



Not simply connected

The following result extends [the theorem about conservative vector fields from Part 4](#) which was limited to convex sets.

Theorem

Let S be a simply connected region and suppose that $\mathbf{f} = \binom{P}{Q}$ is a vector field, continuously differentiable on S. Then $\mathbf{f}$ is conservative if and only if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

Proof

In the mixed partials theorem we already proved that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ whenever $\mathbf{f}$ is conservative so we need only prove the other direction of the statement. Suppose that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ and consider any closed path $\boldsymbol{\alpha}$ in S. Then Green's theorem tells us that

$$\int_C \mathbf{f} \cdot d\boldsymbol{\alpha} = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0.$$

This implies that $\mathbf{f}$ is conservative because the fact that the line integral around every closed curve is zero (using a theorem from Part 4).

A crucially important consequence of the above result is that it implies the invariance of a line integral under deformation of a path when the vector field is conservative. Observe that the result can be extended to multiply connected regions by adding additional "cuts" and keeping track of the additional line integrals.

Change of variables

When we want to identify a point in space it is common, particularly if we are pirates recording the position of treasure, that there are many alternative ways we can describe this point. For example we could write the number of steps north and the number of steps east from the central palm tree. Alternatively we can specify that we stand at the palm tree looking in a specific direction and then walk a particular number of steps. Often is is really convenient to swap from one coordinate to another and in this section we show how multiple integrals behave under change of coordinates.

To start, we recall the 1D case. If $g : [a, b] \to [g(a), g(b)]$ is onto with continuous derivative and f is continuous then

$$\int_{g(a)}^{g(b)} f(x)\, dx = \int_a^b f(g(u))\, g'(u)\, du.$$

In higher dimension we obtain a similar result but g' must be replaced by a type of derivative which works in higher dimension.

For the 2D case we have the following result.

Theorem (change of variables)

Suppose that $(u, v) \mapsto (X(u, v), Y(u, v))$ maps T to S one-to-one and X, Y are continuously differentiable. Then

$$\iint_S f(x, y)\, dxdy = \iint_T f(X(u, v), Y(u, v))\ |J(u, v)|\, dudv.$$

Here $J(u, v) = \begin{pmatrix} \partial_u X & \partial_u Y \\ \partial_v X & \partial_v Y \end{pmatrix}$ is the Jacobian matrix as used previously. Note that the Jacobian represents the scaling of volume in the sense that $\iint_S\ dxdy = \iint_T |J(u, v)|\ dudv.$

Polar coordinates

Polar coordinates correspond to the coordinate mapping

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases}$$

In this case the Jacobian determinant is

$$|J(r, \theta)| = \left| \begin{pmatrix} \partial_u X & \partial_u Y \\ \partial_v X & \partial_v Y \end{pmatrix} \right| = \left| \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \right| = r(\cos^2\theta + \sin^2\theta) = r.$$

Consequently, the change of variable in the integral gives that

$$\iint_S f(x, y)\, dxdy = \iint_T r\ f(r\cos\theta, r\sin\theta)\, drd\theta.$$

Linear transformations

In this case the coordinate mapping is

$$\begin{cases} x = Au + Bv \\ y = Cu + Dv \end{cases}$$

where $A, B, C, D \in \mathbb{R}$ are chosen fixed. The Jacobian determinant is equal to

$$|J(u,v)| = \left|\begin{pmatrix} \partial_u X & \partial_u Y \\ \partial_v X & \partial_v Y \end{pmatrix}\right| = \left|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right| = |AD - BC|.$$

Consequently the change of coordinates for the integral is

$$\iint_S f(x,y)\, dxdy = |AD - BC| \iint_T f(Au + Bv, Cu + Dv)\, dudv.$$

Extension to higher dimensions

The exact analog of [the 2D change of variables formula] holds in any dimension. In particular, in 3D, if we consider the change of variables $(u,v,w) \mapsto (X(u,v,w), Y(u,v,w), Z(u,v,w))$, then $\iiint_S f(x,y,z)\, dxdydz$ is equal to

$$\iiint_T f(X(u,v,w), Y(u,v,w), Z(u,v,w))\ |J(u,v,w)|\, dudvdw$$

where $J(u,v)$ is now the Jacobian matrix of dimension (3×3).

Cylindrical coordinates

[Cylindrical coordinates] correspond to the mapping (require $r > 0$, $0 \leq \theta \leq 2\pi$)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

and, in this case, the Jacobian determinant is

$$|J(r,\theta,z)| = \left|\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| = \left|r(\cos^2\theta + \sin^2\theta)\right| = r$$

and so the change of variables in the integral gives

$$\iiint_S f(x,y,z)\, dxdydz = \iiint_T r\, F(r,\theta,z)\, drd\theta dz.$$

where $F(r,\theta,z) = f(r\cos\theta, r\sin\theta, z)$. Note that cylindrical coordinates are closely related to polar coordinates in the sense that we don't touch the z coordinate and use polar coordinates for x and y.

Spherical coordinates

Spherical coordinates correspond to how we use latitude, longitude and altitude to specify a position on earth. It is the coordinate mapping (require $\rho > 0$, $0 \leq \theta \leq 2\pi$, $0 \leq \varphi < \pi$)

$$\begin{cases} x = \rho \cos\theta \sin\varphi \\ y = \rho \sin\theta \sin\varphi \\ z = \rho \cos\varphi. \end{cases}$$

In this case the Jacobian determinant is

$$|J(\rho,\theta,\varphi)| = \left| \begin{pmatrix} \cos\theta\sin\varphi & \sin\theta\sin\varphi & \cos\varphi \\ -\rho\sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & 0 \\ \rho\cos\theta\cos\varphi & \rho\sin\theta\cos\varphi & -\rho\sin\varphi \end{pmatrix} \right| = \left|-\rho^2 \sin\varphi\right| = \rho^2 \sin\varphi.$$

Consequently the change of variables in the integral gives that

$$\iiint_S f(x,y,z)\, dxdydz = \iiint_T F(\rho,\theta,\varphi)\rho^2 \sin\varphi \, d\rho d\theta d\varphi.$$

where $F(\rho,\theta,\varphi) = f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)$.

Surface integrals

WARNING

The information in this section is being updated.

See also the graded exercises and additional exercises associated to this part of the course. If you want more, Sections 6.4 to 6.8 of OpenStax Calculus Volume 3 are a good option.

In this section we consider surfaces and how to define integral of vector fields over these surfaces. This is similar in many ways to line integrals but a higher dimensional version. Curves (for line integrals) are 1D subsets of higher dimensional space whereas surfaces are 2D subsets of higher dimensional space. Identically to line integrals, the first step is to understand a practical way to represent the surfaces, just like with curves we used paths as the parametric representation of the curve. Once we have clarified the parametric representation of surface we can define the surface integral (of a vector field) and show that it satisfies various properties which we would expect, including that the integral is independent of the choice of parametrization. Similar to how we were able to use a line integral (of a scalar) to calculate the length of a curve we can use a surface integral (of a scalar) to calculate the area of a surface.

We then introduce two important operators that act on vector fields, namely *curl* and *divergence*. Using these operators and the surface integral we introduce two theorems, Gauss' Theorem and Stokes' Theorem. These theorems connect line integrals with surface integrals and with volume integrals.

Representation of a surface

Before developing parametric representations of surfaces let's recall an example of parametric representation of a curve (path). For example, the half circle $C = \{(x, y) : x^2 + y^2 = 1, y \geq 1\}$ can be parametrized in many ways, including the following two paths.

$$\begin{aligned} \boldsymbol{\alpha}(x) &= (x, \sqrt{1-x^2}), \quad x \in [-1, 1], \\ \boldsymbol{\alpha}(t) &= (\cos t, \sin t), \quad t \in [0, \pi]. \end{aligned}$$

In a similar way, now in 2D we can have a parametric representation of a hemisphere.

The hemisphere $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$ can be represented parametrically in many ways, including

$$\begin{aligned} \mathbf{r}(x, y) &= (x, y, \sqrt{1-x^2-y^2}), \quad (x, y) \in \{x^2 + y^2 \leq 1\}, \\ \mathbf{r}(u, v) &= (\cos u \cos v, \sin u \cos v, \sin v), \quad (u, v) \in [0, 2\pi] \times [0, \pi/2]. \end{aligned}$$

Observe that the second form above can be deduced from spherical coordinates (fixed distance from the origin).

The cone $S = \{(x, y, z) : z^2 = x^2 + y^2, z \in [0, 1]\}$ can be represented parametrically in many ways, including

$$\begin{aligned} \mathbf{r}(x, y) &= (x, y, \sqrt{x^2+y^2}), \quad (x, y) \in \{x^2 + y^2 \leq 1\}, \\ \mathbf{r}(u, v) &= (v \cos u, v \sin u, v), \quad (u, v) \in [0, 2\pi] \times [0, 1]. \end{aligned}$$

Observe that the second form can be deduced from spherical coordinates (fixed angle from z-axis) or from cylindrical coordinates (fixed ratio between vertical coordinate and distance from the the z-axis).

Fundamental vector product

A key notion for parametric surfaces and natural geometric object is the *fundamental vector product*. Consider the parametric surface, denoted $\mathbf{r}(T)$, and suppose it has the form

$$\mathbf{r}(u,v) = (X(u,v), Y(u,v), Z(u,v)), \quad (u,v) \in T.$$

The vector-valued function defined as

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} \partial_u X \\ \partial_u Y \\ \partial_u Z \end{pmatrix} \times \begin{pmatrix} \partial_v X \\ \partial_v Y \\ \partial_v Z \end{pmatrix}$$

is called the *fundamental vector product* of the representation $\mathbf{r}$.

By definition, the vector-valued functions $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are tangent to the surface. As such, assuming that they are linearly independent, the fundamental vector product $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ is normal to the surface (orthogonal to every curve which passes through the surface). Moreover the norm of the vector represents the local scaling of area (small parallelograms).

As always we need to take some care about smoothness of the objects we work with. If (u,v) is a point in T at which $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are continuous and the fundamental vector product is non-zero then $\mathbf{r}(u,v)$ is said to be a *regular point* for that representation.

A surface $\mathbf{r}(T)$ is said to be smooth if all its points are regular points.

Just like we saw with paths to represent curves, there are many different ways we can find the parametric representation of a given surface. If the surface S has the form $z = f(x,y)$ (the surface in written in explicit form) then we can use x, y as the parameters and have the representation

$$\mathbf{r}(x,y) = (x, y, f(x,y)), \quad (x,y) \in T.$$

The region T is the projection of S onto the xy-plane. For such a surface we compute

$$\frac{\partial \mathbf{r}}{\partial x} = \begin{pmatrix} 1 \\ 0 \\ \partial_x f \end{pmatrix}, \quad \frac{\partial \mathbf{r}}{\partial y} = \begin{pmatrix} 0 \\ 1 \\ \partial_y f \end{pmatrix},$$

and consequently

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \begin{pmatrix} 1 \\ 0 \\ \partial_x f \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \partial_y f \end{pmatrix} = \begin{pmatrix} -\partial_x f \\ -\partial_y f \\ 1 \end{pmatrix}.$$

An example of such a representation is as follows for the hemisphere. Let $T = \{x^2 + y^2 \leq 1\}$, and let

$$\mathbf{r}(x, y) = (x, y, \sqrt{1 - x^2 - y^2}).$$

The surface $\mathbf{r}(T)$ is the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$. The fundamental vector product of this representation is

$$\tfrac{\partial \mathbf{r}}{\partial x} \times \tfrac{\partial \mathbf{r}}{\partial y}(x, y) = \begin{pmatrix} x(1-x^2-y^2)^{-1/2} \\ y(1-x^2-y^2)^{-1/2} \\ 1 \end{pmatrix} = z^{-1}\,\mathbf{r}(x, y).$$

In this case, all points are regular except the equator $\{(x, y, 0) : x^2 + y^2 = 1\}$.

Let $T = [0, 2\pi] \times [0, \pi/2]$ and let

$$\mathbf{r}(u, v) = (\cos u \cos v, \sin u \cos v, \sin v).$$

The surface $\mathbf{r}(T)$ is the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. This representation is connected to spherical coordinates. We calculate that

$$\tfrac{\partial \mathbf{r}}{\partial u}(u, v) = \begin{pmatrix} -\sin u \cos v \\ \cos u \cos v \\ 0 \end{pmatrix}, \quad \tfrac{\partial \mathbf{r}}{\partial v}(u, v) = \begin{pmatrix} -\cos u \sin v \\ -\sin u \sin v \\ \cos v \end{pmatrix},$$

and so the fundamental vector product of this representation is

$$\tfrac{\partial \mathbf{r}}{\partial u} \times \tfrac{\partial \mathbf{r}}{\partial v}(u, v) = \cos v\,\mathbf{r}(u, v).$$

In this case many points map to the north pole $(0, 0, 1)$ and so the north pole is not a regular point. Additionally there are two points which map to each point on the line between equator and north pole $\{(x, y, z) \in \mathbf{r}(T) : x \geq 0, y = 0\}$.

Surface integral of scalar field

Mirroring the process for line integrals we will define surface integrals both for scalar fields and for vector fields. The surface integral of a scalar field is closely related to the area of a parametric surface, just like the length of a curve is closely related to the line integral of a scalar field.

Definition

The area of the parametric surface $S = \mathbf{r}(T)$ is defined as the double integral

$$\operatorname{Area}(S) = \iint_T \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \, du dv.$$

Observe that the definition is in terms of a multiple integral over the region T, and the quantity being integrated is the norm of the fundamental vector product.

Later we will show that $\operatorname{Area}(S)$ is *independent* of the choice of representation as we require for such a definition, it would be unreasonable if the area of a surface depended on the choice of representation.

We will check that this definition corresponds to a fact that we already know by computing the surface area of a hemisphere. Let, as before, $T = [0, 2\pi] \times [0, \pi/2]$ and let $\mathbf{r}(u, v) = (\cos u \cos v, \sin u \cos v, \sin v)$. The norm of the fundamental vector product (which we computed earlier) is

$$\left\| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}(u, v) \right\| = \cos v \ \| \mathbf{r}(u, v) \| = \cos v.$$

Taking the definition of area and evaluating the multiple integral, this means that

$$\operatorname{Area}(S) = \iint_T \cos v \, du dv = \int_0^{2\pi} \left[\int_0^{\pi/2} \cos v \, dv \right] du = 2\pi.$$

The surface integral of a scalar field is defined in a way similar to the area of a surface.

Definition (scalar surface integral)

Let $S = \mathbf{r}(T)$ be a parametric surface and let f be a scalar field defined on S. The surface integral of f over S is defined as

$$\iint_{\mathbf{r}(T)} f \, dS = \iint_T f(\mathbf{r}(u, v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}(u, v) \right\| \, du dv$$

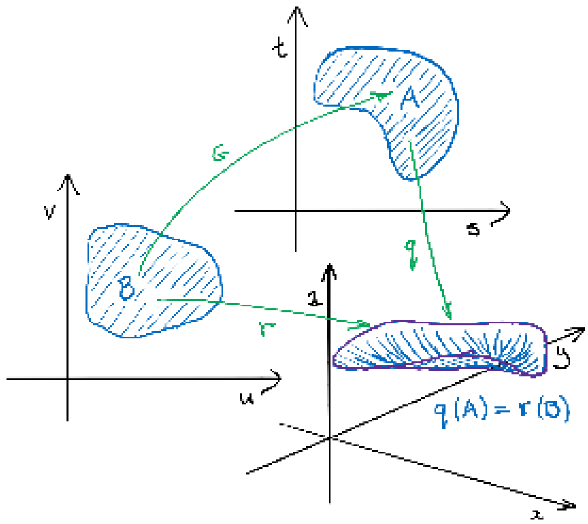
whenever the double integral on the right exists.

Observe that, if we choose $f \equiv 1$, that is we choose the scalar field identically equal to 1, then we obtain the formula for the area of the surface). This is just the same as the line integral of a scalar and the length of the corresponding curve.

From the point of view of applications, we could take f as the density of thin material which has the shape of the surface S and then $\iint_S f\, dS$ is the total mass of this piece of material. Extending this idea we could also calculate the centre of mass of this piece of material.

Change of surface parametrization

In order to validate the definition of a surface integral and consequently that of the area of a surface, we will now show that the the value of the evaluated integral doesn't depend on the choice of representation for any given surface.



Two different representations for a given surface

Suppose that $\mathbf{q}(A)$ and $\mathbf{r}(B)$ are both representations of the same surface, and that $\mathbf{r} = \mathbf{q} \circ G$ for some differentiable $G : B \to A$. Then

$$\iint_A f \circ \mathbf{q} \left\| \frac{\partial \mathbf{q}}{\partial s} \times \frac{\partial \mathbf{q}}{\partial t} \right\| dsdt = \iint_B f \circ \mathbf{r} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dudv.$$

Since $\mathbf{r}(u,v) = \mathbf{q}(S(u,v), T(u,v))$ we calculate (chain rule and vector product) that

$$\left[\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right](u,v) = \left[\left(\frac{\partial \mathbf{q}}{\partial s} \times \frac{\partial \mathbf{q}}{\partial t}\right)\left(\frac{\partial S}{\partial u}\frac{\partial T}{\partial v} - \frac{\partial S}{\partial v}\frac{\partial T}{\partial u}\right)\right](S(u,v), T(u,v)).$$

Observe that $\frac{\partial S}{\partial u}\frac{\partial T}{\partial v} - \frac{\partial S}{\partial v}\frac{\partial T}{\partial u}$ is the Jacobian determinant associated to change of variables $(u,v) \mapsto (S(u,v), T(u,v))$. Consequently, by the change of variables theorem,

$$\iint_A f \circ \mathbf{q} \left\| \frac{\partial \mathbf{q}}{\partial s} \times \frac{\partial \mathbf{q}}{\partial t} \right\| dsdt = \iint_B f \circ \mathbf{r} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dudv$$

so the definition does make sense.

Surface integral of a vector field

In preparation for defining the surface integral of a vector field we need the notion of the *normal* vector of a surface. This is a natural geometric notion, for each point in the surface it is the unit vector field which is orthogonal to the surface.

Definition (unit normal)

Let $S = \mathbf{r}(T)$ be a parametric surface. At each regular point the two unit normals are

$$\mathbf{n}_1 = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\|} \quad \text{and} \quad \mathbf{n}_2 = -\mathbf{n}_1.$$

This definition makes $\|\mathbf{n}_1\| = \|\mathbf{n}_2\| = 1$. That there are two normal vectors is expected because there are two sides to the surface at each point, one is just the opposite direction to the other. When we have two parameterizations of the same surface, they always have the same *pair* of normals at any regular point, but which one is $\mathbf{n}_1$ and which one is $\mathbf{n}_2$ can be different.

If $\mathbf{f}$ is a vector field then $\mathbf{f} \cdot \mathbf{n}$ is the component of the flow in direction of $\mathbf{n}$.

Definition (vector surface integral)

Let $S = \mathbf{r}(T)$ be a parametric surface and $\mathbf{f}$ a vector field. The integral

$$\iint_S \mathbf{f} \cdot \mathbf{n} \, dS$$

is said to be the *surface integral of* $\mathbf{f}$ *with respect to the normal* $\mathbf{n}$.

For convenience let $\mathbf{N} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ and $\mathbf{n} = \mathbf{N} / \|\mathbf{N}\|$. Observe that

$$\iint_S \mathbf{f} \cdot \mathbf{n} \, dS = \iint_T (\mathbf{f} \circ \mathbf{r}) \cdot \mathbf{n} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dudv = \iint_T (\mathbf{f} \circ \mathbf{r}) \cdot \mathbf{N} \, dudv$$

and so for evaluating the surface integral of a vector field there is typically no need to evaluate the norm of the fundamental vector product. Also note that $\iint_S \mathbf{f} \cdot \mathbf{n}_1 \, dS = -\iint_S \mathbf{f} \cdot \mathbf{n}_2 \, dS$ because $\mathbf{n}_1 = -\mathbf{n}_2$. This means that chosing one normal or the other simply corresponds to a minus sign in the evaluated integral. This is the notion that there is a choice of orientation inherent with a surface. As a tangible example imagine that the surface has a flow passing it and this flow is determined by a vector field. Then the surface integral would represent the total flow passing the given surface in a given direction.

Curl and divergence

Suppose that $\mathbf{f} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$ is a differentiable vector field.

Definition (curl)

The *curl* of $\mathbf{f}$ is defined as

$$\nabla \times \mathbf{f} = \begin{pmatrix} \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \\ \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \\ \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \end{pmatrix}.$$

Definition (divergence)

The *divergence* of $\mathbf{f}$ is defined as

$$\nabla \cdot \mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}.$$

Often the notation $\operatorname{\mathbf{curl}} \mathbf{f} = \nabla \times \mathbf{f}$ and $\operatorname{\mathbf{div}} \mathbf{f} = \nabla \cdot \mathbf{f}$ is used instead. Note that the symbols"$\times$" and "$\cdot$" used in the notation for curl and divergence are not truly representing the vector and scalar product but are more a convenient way to remember the definitions. These quantities satisfy the following basic properties which can all be proved by basic calculations.

- If $\mathbf{f} = \nabla\varphi$ then $\nabla \times \mathbf{f} = \mathbf{0}$,
- $\nabla \cdot (\nabla \times \mathbf{f}) = 0$,
- $\nabla \times (\nabla \times \mathbf{f}) = \nabla(\nabla \cdot \mathbf{f}) - \nabla^2\mathbf{f}$.

The quantity defined as $\nabla^2\varphi = \nabla \cdot (\nabla\varphi) = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$ is called the Laplacian and occurs in many applications of physics and mathematics.

Some examples:

If $\mathbf{f}(x, y, z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $\nabla \times \mathbf{f} = \mathbf{0}$, $\nabla \cdot \mathbf{f} = 3$.

If $\mathbf{f}(x, y, z) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$ then $\nabla \times \mathbf{f} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$, $\nabla \cdot \mathbf{f} = 0$.

Let $S \subset \mathbb{R}^3$ be convex. Then $\nabla \times \mathbf{f} \equiv \mathbf{0}$ on S if and only if $\mathbf{f}$ is conservative on S. This implies this theorem from Part 5 about conservative fields (the 2D vector fields can be written as 3D vector fields with a zero component).

Stokes' Theorem and Gauss's Theorem

Theorem (Stokes)

Let $S = \mathbf{r}(T)$ be a parametric surface. Suppose that T is simply connected and that the boundary of T is mapped to C, the boundary of S. Let $\boldsymbol{\beta}$ be a counter clockwise parametrization of the boundary of T and let $\boldsymbol{\alpha}(t) = \mathbf{r}(\boldsymbol{\beta}(t))$. Then

$$\iint_S (\nabla \times \mathbf{f}) \cdot \mathbf{n}\, dS = \int_C \mathbf{f} \cdot d\boldsymbol{\alpha}.$$

"Proof"

It's possible to prove this by a similar method to what we used for Green's theorem, but the details are quite complicated and not very interesting.

Just as Green's Theorem holds for regions which can contain holes, as long as they are correctly accounted for, we can extend Stokes' theorem to more general surfaces with the idea of "cutting and gluing" the surface. In particular this allows the extension to surfaces with holes, cylinders, spheres, etc. On the other hand the theorem can't be extended to the Möbius band because the topology of this surface prevents a similar process being completed.

Theorem (Gauss)

Let $V \subset \mathbb{R}^3$ be a solid with boundary the parametric surface S and let $\mathbf{n}$ be the outward normal unit vector. If $\mathbf{f}$ is a vector field then

$$\iiint_V \nabla \cdot \mathbf{f} \, dxdydz = \iint_S \mathbf{f} \cdot \mathbf{n} \, dS.$$

Proof

We start by writing

$$\iiint_V \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \right) dxdydz = \iint_S (f_x n_x + f_y n_y + f_z n_z) \, dS.$$

As such, it suffices to show that $\iiint_V \left(\frac{\partial f_x}{\partial x} \right) dxdydz = \iint_S (f_x n_x) \, dS$. If we suppose the solid V is xy-projectable then we can explicitly write the integral (later to be extended to general solids). We then use the fundamental theorem of calculus to express f_x as the integral of the derivative.

Stokes' Theorem allows us to connect surface integrals (2D) to line integrals (1D). On the other hand Gauss' Theorem allows us to connect volume integrals (3D) to surface integrals (2D). In this way they are similar to each other; in both of them we change one side to the other by lowering the dimension of the integral and removing a derivative. Indeed the fundamental theorem of calculus for line integral also fits into this same pattern. The branch of mathematics called "differential geometry" provides a framework in which all these results can be described in a unified way by the statement

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega.$$

This result is called the "generalized Stokes theorem".

Note that Gauss' Theorem is often called the "divergence theorem". We can use this theorem for the following interpretation of divergence as a limit, similar to the way other versions of derivatives are defined.

Let V_t be the ball of radius $t > 0$ centred at $\mathbf{a} \in \mathbb{R}^3$ and let S_t be its boundary with outgoing unit normal vector $\mathbf{n}$. Then

$$\nabla \cdot \mathbf{f} = \lim_{t \to 0} \frac{1}{\mathrm{Vol}(V_t)} \iint_{S_t} \mathbf{f} \cdot \mathbf{n} \, dS.$$

Using Gauss' theorem.

Curl can also be written as a similar limit. Given the similarity of all the terms, it is not unexpected that there is a relation between curl and divergence with the Jacobian matrix. Recall that

$$\mathbf{Jac}(\mathbf{f}) = \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} & \frac{\partial f_x}{\partial z} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} & \frac{\partial f_y}{\partial z} \\ \frac{\partial f_z}{\partial x} & \frac{\partial f_z}{\partial y} & \frac{\partial f_z}{\partial z} \end{pmatrix}$$

We can immediately see that divergence is the trace of the Jacobian matrix. In order to see the connection with curl, recall that every real matrix A can be written as the sum of a symmetric matrix $\frac{1}{2}(A + A^T)$ and a skew-symmetric matrix $\frac{1}{2}(A - A^T)$. In this case we have that

$$\frac{1}{2}(\mathbf{Jac}(\mathbf{f}) - \mathbf{Jac}(\mathbf{f})^T) = \begin{pmatrix} 0 & \frac{\partial f_x}{\partial y} - \frac{\partial f_y}{\partial x} & \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \\ \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} & 0 & \frac{\partial f_y}{\partial z} - \frac{\partial f_z}{\partial y} \\ \frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z} & \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} & 0 \end{pmatrix}$$

and can see that the terms of the skew-symmetric part of the matrix are exactly the terms of curl.

Additional exercises 1

Exercise 1.1

Let $f(x) = (x, x^2, x^3)$ and $g(x, y, z) = x + y - z$. Identify the domain and codomain of f, g. Determine if $f \circ g$ and $g \circ f$ are well defined. If it is well defined then write the function explicitly and determine the domain and codomain.

Exercise 1.2

Let $f(x, y) = x^2 + 5xy + \ln y$. Compute the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

Exercise 1.3

Let $f(x, y) = x^3 + 5 + e^{xy}$. Calculate ∇f, the gradient of f.

Exercise 1.4

Determine the directional derivative, $D_v f(-1, 4, 6)$ for $f(x, y, z) = e^{xy^2} + 4zy^3$ in the direction of $v = (2, -3, 6)$.

Exercise 1.5

Compute the Jacobian matrix of the transformation

$$(u, v) \mapsto (e^u \cos v, e^u \sin v).$$

Exercise 1.6

Consider the functions,

$$\boldsymbol{\alpha} : [0, \infty) \to \mathbb{R}^2; \quad t \mapsto (t \cos t, t \sin t),$$

$$f : \mathbb{R}^2 \to \mathbb{R}; \quad (x, y) \mapsto 3x^2 + y^2.$$

Let $g = f \circ \boldsymbol{\alpha}$. Calculate g' both by using the chain rule and by first calculating g and then differentiating and confirm that the answer is the same using either method. *Out of curiosity, try to understand what curve $\alpha(t)$ traces out as t varies.*

Exercise 1.7

Let f be defined, for $(x, y) \neq (0, 0)$, as

$$f(x, y) = \frac{xy}{x^2 + y^2},$$

and $f(x, y) = 0$. Calculate the 4 missing values in the following table, assuming that $t \neq 0$.

line	value
$\{x = 0\}$	$f(0, t) = ?$
$\{y = 0\}$	$f(t, 0) = ?$
$\{x = y\}$	$f(t, t) = ?$
$\{x = -y\}$	$f(t, -t) = ?$

What does this say about f when approaching $(0, 0)$ along these different lines?

Exercise 1.8

Consider the surface $x^2 + y^2 - z^2 = 1$. Verify that the point $(1, 1, 1)$ is contained in the surface. Find the tangent plane to this surface at this point.

Hint: write this surface as a level set $\{(x, y, z) : f(x, y, z) = c\}$*, calculate* ∇f *at the specified point and use the known connection between gradient and tangent plane.*

Edit this page on GitHub

Updated at: 13/10/2025, 11:58

Previous page
5. Surface integrals

Next page
2. Extrema

Additional exercises 2

Exercise 2.1

Consider the scalar field

$$f(x, y) = y^4 + 3x^2 - 4xy - 5y + 81.$$

Calculate the Hessian matrix at the point $(2, 1)$.

Exercise 2.2

Consider the scalar field $f(x, y) = 8 \sin x \sin y$. Calculate the second order Taylor approximation of f at the point $(\frac{\pi}{2}, \frac{\pi}{2})$. *Use that:*

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2}(\mathbf{x} - \mathbf{a})\, \mathbf{H} f(\mathbf{a})\, (\mathbf{x} - \mathbf{a})^{\mathrm{T}}.$$

Exercise 2.3

Locate and classify the extrema points of the scalar field

$$f(x, y) = x^4 + 3xy + 2y^2 + 1.$$

Exercise 2.4

Locate and classify the stationary points of the scalar field

$$f(x, y) = 2x^2 - xy - 3y^2 - 3x + 7y + 11.$$

Exercise 2.5

Locate and classify the stationary points of the scalar field

$$f(x, y, z) = 2e^{(x-1)^2}(y^4 - 4yz + 2z^2).$$

Exercise 2.6

Find the absolute minimum and absolute maximum of $f(x, y) = (9x^2 - 1)(1 + 4y)$ on the rectangle given by $-2 \leq x \leq 3$, $-1 \leq y \leq 4$.

Exercise 2.7

Apply the Lagrange multiplier method to find the points on the curve $x^2 + xy + y^2 = 21$ which are closest / furthest to the origin. *Out of curiosity, sketch the curve and identify the extrema points found.*

Exercise 2.8

Apply the Lagrange multiplier method to find the points on the curve of intersection of the two surfaces

$$x^2 - xy + y^2 - z^2 = 4, \qquad x^2 + y^2 = 4$$

which are nearest the origin.

Exercise 2.9

Apply the Lagrange multiplier method to find the extrema of $f(x, y, z) = yz + xy$ subject to the constraints $xy = 1$ and $y^2 + z^2 = 1$.

Edit this page on GitHub

Updated at: 13/10/2025, 11:58

Previous page
1. Higher dimension

Next page
3. Line integrals

Additional exercises 3

Exercise 3.1

Sketch the vector field $F(x,y) = (-y,x)$.

Exercise 3.2

Let C denote the curve $x^2 + y^2 = 4$. Find a clockwise parameterization of the curve C.

Exercise 3.3

Let C denote the line segment from the point $\mathbf{a} = (1,2,3)$ to the point $\mathbf{b} = (5,4,3)$. Find a parameterization of C, starting from $\mathbf{a}$ and finishing at $\mathbf{b}$.

Exercise 3.4

Let C denote the curve $x^2 + y^2 = 4$. Find an anticlockwise parameterization of the curve C.

Exercise 3.5

Let C denote the portion of $y = x^2 + 2$ from $(-1, 3)$ to $(2, 6)$. Find a parameterization of the curve C.

Exercise 3.6

Consider the vector field $F(x, y) = (\sqrt{y}, x^3 + y)$ and the path $\boldsymbol{\alpha}(t) = (t^2, t^3)$ for $t \in (0, 1)$. Evaluate $\int F \cdot d\boldsymbol{\alpha}$.

Exercise 3.7

Consider the vector field $F(x, y) = (2 - y, x)$ and path $\boldsymbol{\alpha}(t) = (t - \sin t, 1 - \cos t)$, $t \in [0, 2\pi]$. Compute the line integral $\int F \cdot d\boldsymbol{\alpha}$.

Exercise 3.8

The vector field $\mathbf{F}(x, y) = \left(3x^2y, x^3\right)$ is conservative on $\mathbf{R}^2$. Find φ such that $\mathbf{F} = \nabla\varphi$.

Exercise 3.9

Consider the vector field

$$\mathbf{f}(x, y) = \begin{pmatrix} -y(x^2+y^2)^{-1} \\ x(x^2+y^2)^{-1} \end{pmatrix}$$

defined on $S = \mathbb{R}^2 \setminus (0, 0)$. Let $\boldsymbol{\alpha}(t)$ denote the path which traverses clockwise the circle of radius $r > 0$ centred at the origin. Evaluate the line integral $\int \mathbf{f} \cdot d\boldsymbol{\alpha}$.

Exercise 3.10

Evaluate $\int (x^2 - 2y)\, d\boldsymbol{\alpha}$ where $d\boldsymbol{\alpha}$ is the path defined as $\boldsymbol{\alpha}(t) = (4t^4, t^4)$ for $t \in [-1, 0]$.

Exercise 3.11

Determine if the vector-field $\mathbf{G}(x, y) = (2y^2, x + 2)$ is conservative on $\mathbb{R}^2$.

Exercise 3.12

Evaluate $\int \nabla f \cdot d\boldsymbol{\alpha}$ where $f(x, y) = ye^{x^2-1} + 4xy$ and the path is $\boldsymbol{\alpha}(t) = (1 - t, 2t^2 - 2t)$ for $0 \leq t \leq 2$.

Additional exercises 4

Exercise 4.1

Evaluate the multiple integral $I = \iint_R f(x,y)\, dxdy$ by iterated integration, where:

1. $R = [0,1] \times [0,1]$, $f(x,y) = 27xy(x+y)$
2. $R = [0,1] \times [0,1]$, $f(x,y) = 4x^3$
3. $R = [0,\pi/2] \times [0,\pi/2]$, $f(x,y) = 7\sin(x+y)$.
4. $R = [0,1] \times [1,2]$, $f(x,y) = xe^{xy}$
5. $R = [0,1] \times [1,2]$, $f(x,y) = xe^{x-y}$

Exercise 4.2

Let $R = [0,\pi] \times [0,\pi]$.

Evaluate the following integral:

$$\iint_R (\sin x \sin y)^2 \, dxdy$$

Hint: integrals of the form $\iint_R f(x)g(y)\, dxdy$ *are equal to the product of integrals.*

Exercise 4.3

Let $S \subset \mathbb{R}^2$ be the triangular region with vertices $(0,0)$, $(\pi, 0)$, (π, π). Evaluate integral

$$\iint_S x \cos(x+y)\, dx dy.$$

Exercise 4.4

Let $S \subset \mathbb{R}^2$ be the region bounded by the curves $y = \sin x$ and the line segment $\{(x, 0) : x \in [0, \pi]\}$.

Sketch S and evaluate the integral

$$\iint_S (x^2 - y^2)\, dx dy.$$

Exercise 4.5

Let $S \subset \mathbb{R}^2$ be the region bounded by the four curves $x - 2y + 8 = 0$, $x + 3y + 5 = 0$, $x = -2$ and $x = 4$. Sketch S and find its centroid.

Exercise 4.6

Find the center of mass of trapezoidal object with corners $(x, y) = (2, 0)$, $(2, 4)$, $(-2, -2)$ and $(-2, 0)$ and density $\rho(x, y) = x^2 + y + 2$.

Exercise 4.7

Use polar coordinates to evaluate

$$\int_0^1 \left[\int_0^x \sqrt{x^2 + y^2}\, dy\right] dx.$$

Hint: $\int \sec^3 x \, dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$.

Exercise 4.8

In this question we will evaluate the integral

$$I = \iiint_V 10x + 8y + 6z \, dxdydz$$

where the integral is over the half ellipsoid

$$V = \left\{(x, y, z) : \frac{x^2}{4} + y^2 + z^2 \leq 1, x \geq 0\right\} \subset \mathbb{R}^3.$$

We choose a change of coordinates $x = r\cos\theta$, $y = \frac{1}{2}r\sin\theta$, $z = z$.

1. Find the Jacobian of this transformation.
2. Find the set of (r, θ, z) which corresponds to V.
3. Using these, evaluate the integral I.

Exercise 4.9

The set $V = \{(x, y, z) : x^2 + y^2 \leq 4^2, 0 \leq z \leq 4 - \sqrt{x^2 + y^2}\}$ is a cone of height 4 with base in the xy-plane.

The set $W = \{(x, y, z) : (x - 2)^2 + y^2 \leq 4\}$ is a cylinder.

Find the volume of the set $D \subset \mathbb{R}^3$ which is the subset of the cone V which is contained within the cylinder W.

Hint: Use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ to write the volume of D as

$$\iint_T r(a - r) \, drd\theta.$$

for some $a \in \mathbb{R}$, $T \subset \mathbb{R}^2$.

Exercise 4.10

Let V be the solid bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the cone $z = \sqrt{x^2 + y^2}$.

Find the volume of V.

Edit this page on GitHub

Updated at: 13/10/2025, 11:58

Previous page
3. Line integrals

Next page
5. Surface integrals

Additional exercises 5

Exercise 5.1

Consider the parametric surface

$$\mathbf{r}(u, v) = (2u\cos v, 3u\sin v, u^2).$$

1. Write the equation for x, y, z which describes this surface.
2. Find the fundamental vector product of $\mathbf{r}$.

Exercise 5.2

Consider the parametric surface

$$\mathbf{r}(u, v) = ((2+\cos u)\sin v, (2+\cos u)\cos v, \sin u).$$

1. Write the equation for x, y, z which describes this surface.
2. Find the fundamental vector product of $\mathbf{r}$.

Exercise 5.3

Let S be the bounded portion of the paraboloid $x^2 + y^2 = 8z$ which is cut off by the plane $z = 4$. Sketch S and choose a parametric representation of this surface. *Hint*:

a possible parametric representation is $\mathbf{r}(u,v) = (u\cos v, u\sin v, u^2/2)$, $(u,v) \in T$ for a suitable choice of $T \subset \mathbb{R}^2$.

Use this to compute the area of S.

Exercise 5.4

Let S denote the plane surface whose boundary is the triangle with vertices at $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Consider the vector field $\mathbf{f}(x,y,z) = 14\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Compute the surface integral $\iint_S \mathbf{f} \cdot \mathbf{n}\, dS$, where $\mathbf{n}$ denotes the unit normal to S which has positive z-component.

Exercise 5.5

Consider the vector field

$$\mathbf{f}(x,y,z) = \begin{pmatrix} x^2 + yz \\ y^2 + xz \\ z^2 + xy \end{pmatrix}.$$

Calculate the curl and divergence of $\mathbf{f}$ and find

$$(\nabla \times \mathbf{f})(1,2,3) \quad \textbf{and} \quad (\nabla \cdot \mathbf{f})(1,2,3).$$

Exercise 5.6

Consider the vector field

$$\mathbf{f}(x,y,z) = \begin{pmatrix} z + \sin y \\ -z + x\cos y \\ 0 \end{pmatrix}.$$

Calculate $(\nabla \times \mathbf{f})(7, \frac{\pi}{2}, 3)$ and $(\nabla \cdot \mathbf{f})(7, \frac{\pi}{2}, 3)$.

Exercise 5.7

Let

$$\mathbf{f}(x, y, z) := \begin{pmatrix} y \\ z \\ x \end{pmatrix}$$

be a vector field. Let S be the portion of the paraboloid $z = 1 - x^2 - y^2$ with $z \geq 0$ and let $\mathbf{n}$ be the unit normal to S with non-negative z-component. Using Stokes' Theorem (to transform the surface integral to a line integral), evaluate $\iint_S (\nabla \times \mathbf{f}) \cdot \mathbf{n}\, dS$.

Exercise 5.8

Let $V \subset \mathbb{R}^3$ be the set of (x, y, z) such that $x^2 + y^2 + z^2 \leq 25$ and $z \geq 3$. This solid is bounded by a closed surface S which is composed of two parts: Let S_1 denote the curved top part and let S_2 denote the planar part.

Consider the vector field

$$\mathbf{f}(x, y, z) := \begin{pmatrix} xz \\ yz \\ 1 \end{pmatrix}.$$

1. Evaluate the surface integral $\iint_{S_2} \mathbf{f} \cdot \mathbf{n}\, dS$.
2. Evaluate the multiple integral $\iiint_V \nabla \cdot \mathbf{f}\, dxdydz$.
3. Evaluate the surface integral $\iint_{S_1} \mathbf{f} \cdot \mathbf{n}\, dS$.

Exercise 5.9

Consider the surface $S = \left\{(x, y, z) : x^2 + y^2 = z, z \leq 9\right\}$. Find a parametric form for S based on polar coordinates $x = r\cos\theta,\ y = r\sin\theta$, and find the associated fundamental vector product.

Use this to find the surface integral $\iint_S \mathbf{f} \cdot \mathbf{n}\, dS$, where $\mathbf{f}$ is the vector field

$$\mathbf{f}(x, y, z) = \begin{pmatrix} y^2 \\ 0 \\ z \end{pmatrix}$$

and $\mathbf{n}$ is the unit normal to S which has *positive* z-component.

Exercise 5.10

Consider the the sphere $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1/4\}$ and the vector-field

$$\mathbf{f}(x, y, z) = \begin{pmatrix} 2x^3 \\ 2y^3 \\ 2z^3 \end{pmatrix}.$$

Calculate $\iint_S \mathbf{f} \cdot \mathbf{n} \, dS$, where $\mathbf{n}$ is the outgoing unit normal on S.

Hint: Use Gauss's theorem to rewrite this as a volume integral.

Exercise 5.11

Let S be the part of the surface $z = 1 - x^2 - 2y^2$ with $z \geq 0$, oriented in the positive z direction. Find

$$I = \iint_S \nabla \times \mathbf{f} \cdot d\mathbf{S},$$

with $\mathbf{f}(x, y, z) = (x, y^2, ze^{xy})$.

Updated at: 13/10/2025, 11:58