$\boxed{\text{Q1}}$

$$\nabla f(x,y) = \exp\left(\frac{x^2}{3} - 2xy + 2y^3\right)\begin{pmatrix} \frac{2}{3}x - 2y \\ -2x + 6y^2 \end{pmatrix}$$

$$\nabla f(x,y) = 0$$

$$\iff \begin{cases} \frac{2}{3}x - 2y = 0 \\ -2x + 6y^2 = 0 \end{cases} \iff \begin{cases} 2x - 6y = 0 \\ 2x - 6y^2 = 0 \end{cases}$$

$\Rightarrow y^2 - y = 0 \iff y(y-1) = 0 \iff y \in \{0,1\}$

we also know that $x = 3y$.

$\Rightarrow$ 2 solutions: $(0,0), (3,1)$ (the stationary pts)

Hessian:

$$(Hf)_{1,1}(x,y) = \left(\frac{2}{3} + \left(\frac{2x}{3} - 2y\right)^2\right)\exp\left(\frac{x^2}{3} - 2xy + 2y^3\right)$$

$$(Hf)_{1,2}(x,y) = \left(-2 + \left(\frac{2x}{3} - 2y\right)(-2x + 6y^2)\right)\exp\left(\frac{x^2}{3} - 2xy + 2y^3\right)$$

$$(Hf)_{2,2}(x,y) = \left(12y + (-2x + 6y^2)^2\right)\exp\left(\frac{x^2}{3} - 2xy + 2y^3\right)$$

$$(Hf)_{2,1} = (Hf)_{1,2}.$$

$$(Hf)(0,0) = \begin{pmatrix} 2/3 & -2 \\ -2 & 0 \end{pmatrix} \quad \cancel{\exp} \quad (Hf)(3,1) = \begin{pmatrix} 2/3 & -2 \\ -2 & 12 \end{pmatrix}\exp(2-6+2)$$

$$\det\begin{pmatrix} 2/3 & -2 \\ -2 & 0 \end{pmatrix} = -4 < 0 \qquad \det\begin{pmatrix} 2/3 & -2 \\ -2 & 12 \end{pmatrix} = 4 > 0$$

so $(0,0)$ is a saddle pt.

and

$$\operatorname{tr}\begin{pmatrix} 2/3 & -2 \\ -2 & 12 \end{pmatrix} = \frac{2}{3} + 12 > 0$$

So $(3,1)$ is a local minimum.

$\boxed{Q2}$ Change variables $\begin{cases} u = xy \\ v = y/x \end{cases}$

$y/x = 2 \Rightarrow v = 2$ $\quad$ $xy = 2 \Rightarrow u = 2$

$xy = 3 \Rightarrow u = 3$

$y/x = 1 \Rightarrow v = 1$

one-to-one

let $T = \{(u,v) : 2 \le u \le 3,\ 1 \le v \le 2\}$.

let $A : (x,y) \mapsto (xy, y/x)$, $B = A^{-1} : T \to S$

$$JA(x,y) = \begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix}, \quad \det JA(x,y) = \frac{y}{x} + \frac{y}{x} = 2\frac{y}{x}$$

$(= 2v)$

$$JB(u,v) = \frac{1}{2v}$$

$$\iint_S \frac{(xy)^5}{(xy)^3+1}\,dx\,dy = \iint_T \frac{u^5}{u^3+1}\frac{1}{2v}\,du\,dv$$

$$= \int_1^2\int_2^3 \frac{u^5}{u^3+1}\cdot\frac{1}{2v}\,du\,dv = \frac{1}{2}\int_1^2 \frac{1}{v}\,dv \int_2^3 \frac{u^5}{u^3+1}\,du$$

Note that $\displaystyle \frac{u^5}{u^3+1} = \frac{u^2(u^3+1) - u^2}{u^3+1} = u^2 - \frac{u^2}{u^3+1}$

and so $\displaystyle \int_2^3 \frac{u^5}{u^3+1}\,du = \left[\frac{u^3}{3} - \ln(u^3+1)\right]_2^3 = \frac{27-8}{3} + \ln\left(\frac{9}{28}\right)$

$$\int_1^2 \frac{1}{v}\,dv = [\ln v]_1^2 = \ln 2.$$

Consequently:

$$\iint_S \frac{(xy)^5}{(xy)^3+1}\,dx\,dy = \frac{1}{2}\ln 2\left(\frac{19}{3} + \ln\left(\frac{9}{28}\right)\right)$$

$\boxed{\text{Q3}}$

$f_1 \leftrightarrow D$ (periodic in x and y)

$f_2 \leftrightarrow F$ (curves of form $y = c/x$)

$f_3 \leftrightarrow B$ (curves of form $\sqrt{x^2+y^2} = c$)

$f_4 \leftrightarrow E$ (curves of form $y = cx$)

$f_5 \leftrightarrow C$ (curves of form $y = c + x$)

$f_6 \leftrightarrow A$ (curves of form $y^2 = c + x^2$ and when x, y are large this is approximately $y = \pm x$.)

$$\nabla f_1(x,y) = \frac{1}{2}\begin{pmatrix} \cos(x/2)\sin(y/2) \\ \sin(x/2)\cos(y/2) \end{pmatrix}$$

$$\nabla f_2(x,y) = \begin{pmatrix} y \\ x \end{pmatrix}$$

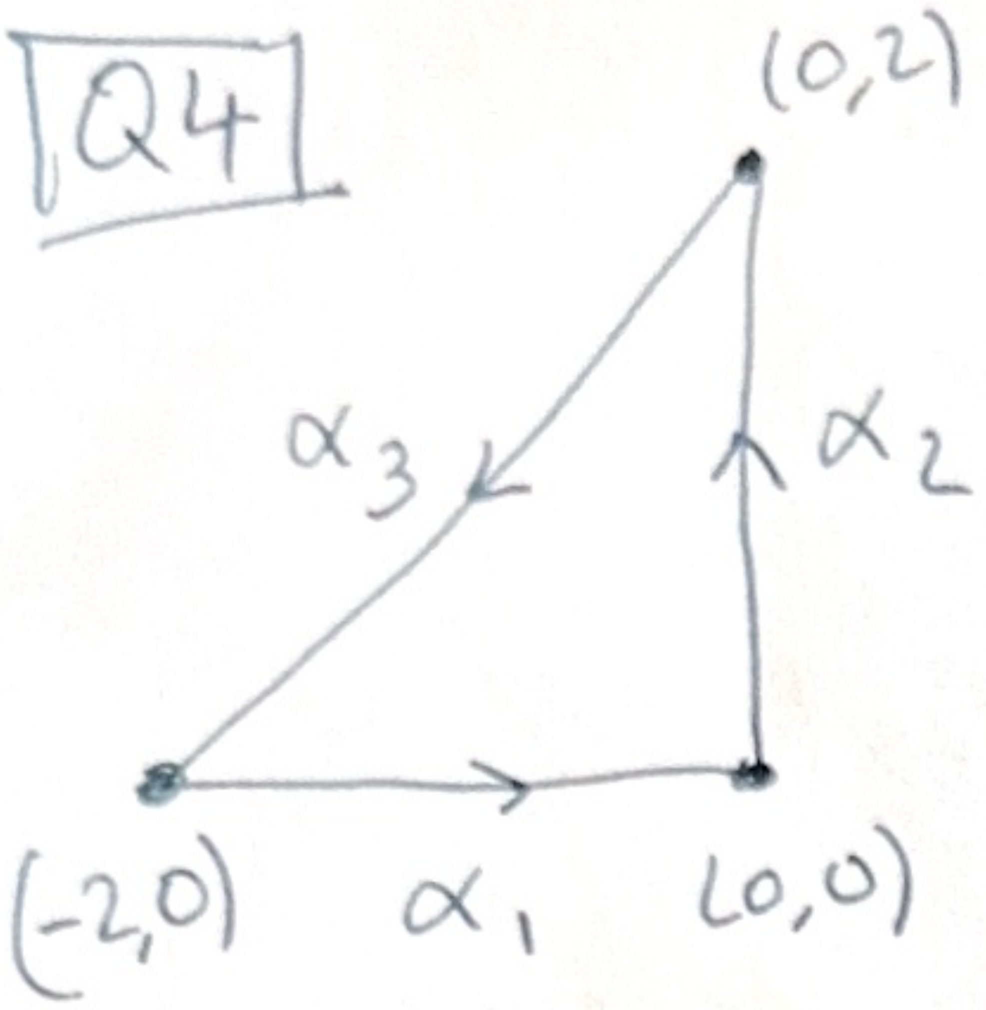
$$\nabla f_3(x,y) = -\frac{\sin\left((x^2+y^2)^{1/2}\right)}{(x^2+y^2)^{1/2}}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\nabla f_4(x,y) = \frac{1}{2}\left(\frac{y}{x}\right)^{-1/2}\begin{pmatrix} -y/x^2 \\ 1/x \end{pmatrix} = \frac{y^{-1/2}x^{-3/2}}{2}\begin{pmatrix} -y \\ x \end{pmatrix}$$

$$\nabla f_5(x,y) = \frac{1}{1+(x-y)^2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\nabla f_6(x,y) = 2\begin{pmatrix} x \\ -y \end{pmatrix}$$

$\boxed{Q4}$



$\alpha_1(t) = (2t-2, 0), \quad t \in [0,1]$

$\alpha_2(t) = (0, 2t), \quad t \in [0,1]$

$\alpha_3(t) = (-2t, 2-2t), \quad t \in [0,1]$

$\underline{\alpha_1}$ $F(\alpha_1(t)) = \begin{pmatrix} 0 \\ 2t-1 \end{pmatrix}, \quad \alpha_1'(t) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad F(\alpha_1(t)) \cdot \alpha_1'(t) = 0$

$$\int F \cdot d\alpha_1 = 0$$

$\underline{\alpha_2}$ $F(\alpha_2(t)) = \begin{pmatrix} 2t \\ -1 \end{pmatrix}, \quad \alpha_2'(t) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad F(\alpha_2(t)) \cdot \alpha_2'(t) = -2$

$$\int F \cdot d\alpha_2 = \int_0^1 (-2)\, dt = -2$$

$\underline{\alpha_3}$ $F(\alpha_3(t)) = \begin{pmatrix} (-2t)^2(2-2t) + 2-2t \\ -2t - 1 \end{pmatrix}, \quad \alpha_3'(t) = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

$$F(\alpha_3(t)) \cdot \alpha_3'(t) = 16t^2(t-1) + 4(t-1) + 4t + 2$$
$$= 16t^3 - 16t^2 + 8t - 2$$

$$\int F \cdot d\alpha_3 = \left[\frac{16}{4}t^4 - \frac{16}{3}t^3 + \frac{8}{2}t^2 - 2t\right]_0^1 = 4 - \frac{16}{3} + 4 - 2 = 6 - \frac{16}{3}$$

$$\int F \cdot d\alpha = 0 - 2 + \left(6 - \frac{16}{3}\right) = 4 - \frac{16}{3} = -\frac{4}{3}$$

$\underline{\text{Green's}}$ Let $T = \{(x,y) : -2 \le x \le 0,\ 0 \le y \le 2+x\}$.

$$\int F \cdot d\alpha = \iint_T (1) - (x^2+1)\, dx\, dy = -\iint_T x^2\, dx\, dy$$

$$= -\int_{-2}^{0} x^2 \left[\int_0^{2+x} (1)\, dy\right] dx = -\int_{-2}^{0} x^2(2+x)\, dx$$

$$= -\left[\frac{2x^3}{3} + \frac{x^4}{4}\right]_{-2}^{0} = -\frac{16}{3} + 4 = -\frac{4}{3}$$

$\boxed{Q5}$ 1. f_n is odd since $f(-x) = -f(x)$.

$$f_n(-1) = f_n(0) = f_n(1) = 0.$$

2. $$f_n'(x) = n\left[(1-x^2)^{2n} + x(2n)(1-x^2)^{2n-1}(-2x)\right]$$

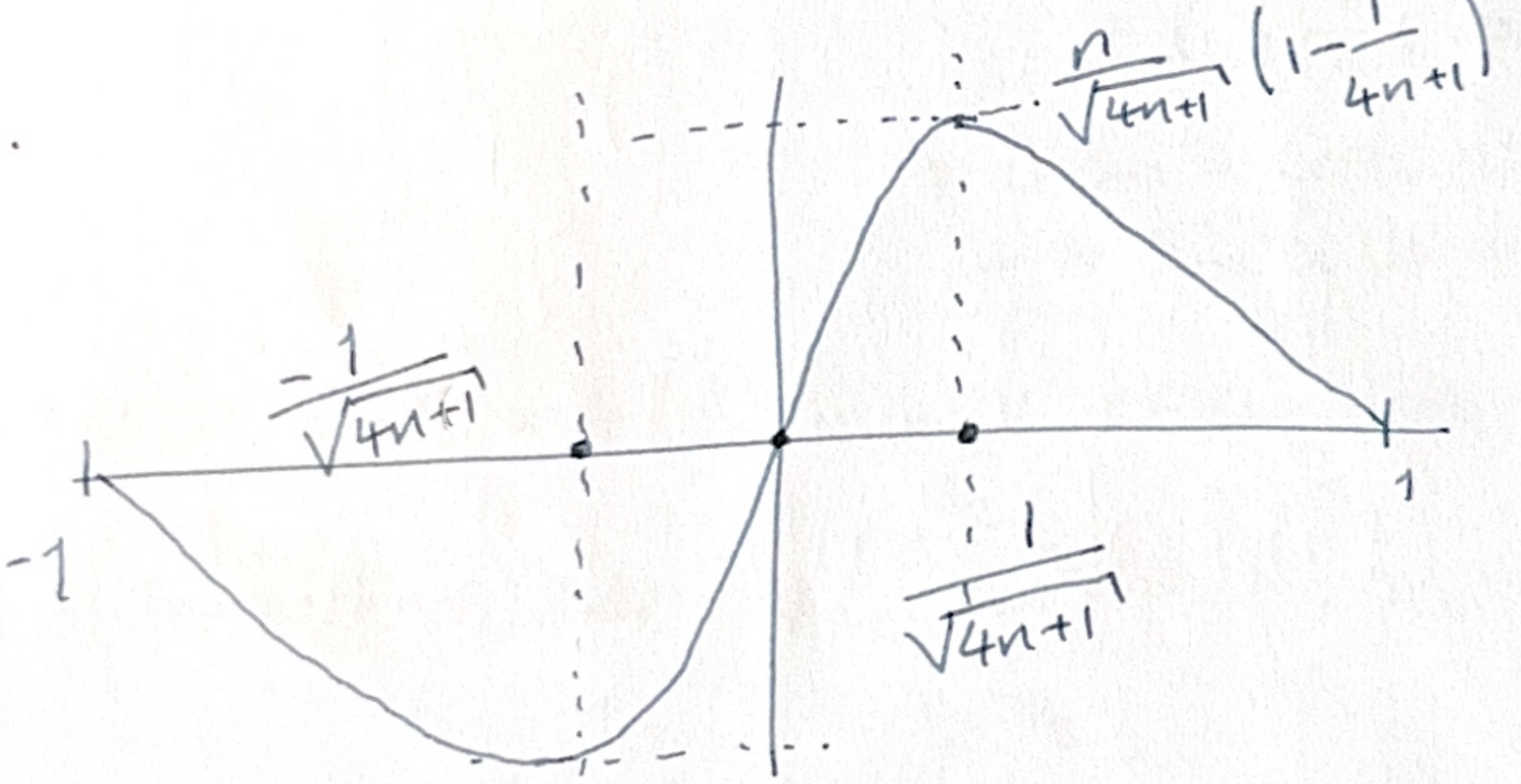
$$= n(1-x^2)^{2n-1}\left[(1-x^2) - 4nx^2\right]$$

$$= n(1-x^2)^{2n-1}\left[1 - (4n+1)x^2\right].$$

$$f_n'(x) = 0 \iff x = \pm 1 \text{ or } x = \pm\frac{1}{\sqrt{4n+1}} \in (-1,1)$$

$$f_n\left(\frac{\pm 1}{\sqrt{4n+1}}\right) = \frac{\pm n}{\sqrt{4n+1}}\left(1 - \frac{1}{4n+1}\right)^{2n} \to \pm\infty \text{ as } n \to \infty.$$

3.



4. $f_n(x)$ converges pointwise to 0. ($\forall x \in (-1,1)$, $(1-x^2) \in (0,1)$)

Cannot converge uniformly since maximum $\to \infty$.

5. $$\int_0^1 f_n(x)\,dx = n\left[\frac{-(1-x^2)^{2n+1}}{2(2n+1)}\right]_0^1 = \frac{n}{2(2n+1)}$$

and so $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \frac{1}{4}$. However $\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 (0)\,dx = 0$

$\boxed{Q6}$ Parametric surface:

$$\sigma(u,v) = (2\sin u \cos v,\ 2\sin u \sin v,\ 2\cos u),$$

$$(u,v) \in T = \{(u,v) : 0 \le u \le \pi/2,\ 0 \le v \le \pi/2\}.$$

$$N(u,v) = \left(\frac{\partial\sigma}{\partial u} \times \frac{\partial\sigma}{\partial v}\right)(u,v) = \begin{pmatrix} 2\cos u \cos v \\ 2\cos u \sin v \\ -2\sin u \end{pmatrix} \times \begin{pmatrix} -2\sin u \sin v \\ 2\sin u \cos v \\ 0 \end{pmatrix}$$

$$= 4\begin{pmatrix} \sin^2 u \cos v \\ \sin^2 u \sin v \\ \sin u \cos u \cos^2 v + \cos u \sin u \sin^2 v \end{pmatrix}$$

$$F(\sigma(u,v)) = \begin{pmatrix} 1 \\ 6\cos u \\ 4\sin u \cos v \end{pmatrix} \qquad F(\sigma(u,v)) \cdot N(u,v) = 4\big(\sin^2 u \cos v$$

$$+ 6\sin^2 u \cos u \sin v \ \cancel{+ 4\sin^2 u \cos u}$$

$$+ 4\sin^2 u \cos u \cos^3 v$$

$$+ 4\sin^2 u \cos u \sin^2 v \cos v\big)$$

$$F(\sigma(u,v)) \cdot N(u,v) = 4\left(\sin^2 u \cos v + 6\sin^2 u \cos u \sin v + 4\sin^2 u \cos u \cos v\right)$$

$$\iint_T F(\sigma(u,v)) \cdot N(u,v))\, du\, dv = 4\left(\int_0^{\pi/2} \sin^2 u\, du\right)\left(\int_0^{\pi/2} \cos v\, dv\right)$$

$$+ 24\left(\int_0^{\pi/2} \sin v\, dv\right)\left(\int_0^{\pi/2} \sin^2 u \cos u\, du\right)$$

$$+ 16\left(\int_0^{\pi/2} \sin^2 u \cos u\, du\right)\left(\int_0^{\pi/2} \cos v\, dv\right)$$

$$\int_0^{\pi/2} \cos v\, dv = \int_0^{\pi/2} \sin v\, dv = 1, \quad \int_0^{\pi/2} \sin^2 u\, du = \pi/4,$$

$$\int_0^{\pi/2} \sin^2 u \cos u\, du = \left[\tfrac{1}{3}\sin^3 u\right]_0^{\pi/2} = 1/3.$$

$$\iint_S F \cdot \hat{n}\, dS = 4\left(\frac{\pi}{4}\right) + 24\left(\tfrac{1}{3}\right) + 16\left(\tfrac{1}{3}\right) = \pi + \frac{40}{3}$$

$\boxed{Q6}$ (alternative solution using divergence theorem)

let $S_x = \{(x,y,z) : x=0,\ y^2+z^2 \le 2^2,\ y,z \geq 0\}$,

$S_y = \{(x,y,z) : y=0,\ x^2+z^2 \le 2^2,\ x,z \geq 0\}$,

$S_z = \{(x,y,z) : z=0,\ x^2+y^2 \le 2^2,\ x,y \geq 0\}$.

Surface formed from S, S_x, S_y, S_z is a closed surface and $\operatorname{div} F = 0$ and so:

$$0 = \iint_S F\cdot\hat{n}\,dS + \int_{S_x} F\cdot\hat{n}\,dS + \iint_{S_y} F\cdot\hat{n}\,dS + \iint_{S_z} F\cdot\hat{n}\,dS$$

$$\iint_{S_x} F\cdot\hat{n}\,dS = \iint_{S_x}(-1)\,dS = -\text{Area}(S_x) = -\frac{\pi\, 2^2}{4} = -\pi.$$

$$\iint_{S_y} F\cdot\hat{n}\,dS = -3\iint_{S_y} z\,dx\,dz,$$

equivalent integrals

$$\iint_{S_z} F\cdot\hat{n}\,dS = -2\iint_{S_z} x\,dx\,dy.$$

$$\iint_{S_z} x\,dx\,dy = \int_0^2\int_0^{\sqrt{4-x^2}} x\,dx\,dy = \int_0^2 x(4-x^2)^{1/2}\,dx$$

$$= \left[-\tfrac{1}{3}(4-x^2)^{3/2}\right]_0^2 = \frac{4^{3/2}}{3} = \frac{8}{3}$$

$$\iint_S F\cdot\hat{n}\,dS = 0 - \left(-\pi - 3\left(\tfrac{8}{3}\right) - 2\left(\tfrac{8}{3}\right)\right) = \pi + \frac{40}{3}$$