Statistical Properties of Chaotic Flows

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by

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Abstract

Our aim is to further the understanding of statistical properties of chaotic flows. In the theory of dynamical systems the direct functional analytic approach, whereby a suitable functional analytic framework is established so the system may be studied with no need for coding in any way has been shown to be a powerful and far reaching point of view. This approach involves carefully choosing a function space on which the transfer operator associated to the system has good properties. At the moment this approach is mostly limited to uniformly hyperbolic systems. We introduce a class of flows which are a model for a two dimensional centre-unstable bundle where it is not possible to uniformly bound the expanding direction from the flow direction. An anisotropic Banach space is developed on which the transfer operator of such flows acts naturally. A spectral result is given.

All the work is my own unless otherwise acknowledged and referenced.

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Contents

1. Introduction

This work concerns the statistical properties of chaotic flows. The basic setting is a flow $\phi_t : \Omega \to \Omega$ where Ω is a metric space and $\{\phi_t\}$ is a family of maps parametrised by $t \geq 0$ such that $\phi_0 = \mathbf{Id}$ and $\phi_t \circ \phi_s = \phi_{s+t}$ for all $s, t \geq 0$. Note that ϕ_t may or may not be invertible. Subsequently we will consider flows satisfying various degrees of regularity. The parameter t is called the *time* and Ω is called the *state space*. The orbit of a point $x \in \Omega$ is the set $\{\phi_t(x) : t \geq 0\} \subset \Omega$. There is no universal agreement about the meaning of chaotic. For our purposes we consider it to mean that the orbits of the flows are complex and so describing the behaviour of individual orbits is not possible or useful. Rather the more fruitful approach is to study the behaviour of, in some sense, "typical" orbits. Any property concerning the asymptotic behaviour of "typical" orbits is said to be a *statistical property*. Such properties include the existence of a relevant invariant measure, ergodicity, entropy, mixing, rate of decay of correlation, central limit theorem, almost sure invariance principle to name but a few. The prototype of a chaotic flow is an Anosov flow, a class of flows introduced [2] by Anosov in the 1960s. Anosov flows extended the class of geodesic flows in negative curvature, demonstrating that uniform hyperbolicity is a mechanism for producing chaotic behaviour. It has become clear that chaotic behaviour is not limited to these uniformly hyperbolic flows. Despite this progress there are still only limited results for flows outside of this class.

The principal aim of this chapter is threefold. Firstly we introduce several statistical properties of particular interest and give details of the classes of flows for which results are available. Secondly the *direct functional analytic approach* to the study of statistical properties of dynamical systems is introduced. This is a set of methods and ideas which has shown significant development in recent years and has been proven as a powerful and far reaching technique for the study of statistical properties of chaotic dynamical systems. Thirdly and finally we introduce a class of suspension flows which are not uniformly hyperbolic and which we call *unbounded twist flows*. We show that this class represents a significant step towards the understanding of statistical properties for larger classes of flows.

In Chapter 2 we present a new result concerning the unbounded twist flows and which is a major step along the program of the direct functional analytic approach to studying such flows. The unbounded twist flows are two dimensional non-invertible flows which are not uniformly hyperbolic in the sense that it is not possible to separate the tangent space into flow direction and transversal expanding direction where the angle between these directions is bounded from below uniformly for all points in the state space. Later in the chapter, after the statement of the result, further details are given concerning the applicability of such a result and the ways in which it may be extended. Chapter 3 is devoted to the proof of the result. Appendix A contains the article co-authored with Liverani [17] in which smooth Anosov flows are studied with results concerning the way in which the SRB measure varies as the flow is perturbed.

1.1. Statistical Properties

As mentioned above there are various different statistical properties which are important in their own way. In the following we discuss in detail just two such properties as examples. Namely the rate of mixing (decay of correlation) and the differentiability of the invariant measure under perturbations of the flow.

1.1.1. Decay of Correlation

We now assume that Ω is a differentiable manifold. Suppose that μ is a ϕ_t-invariant probability measure on Ω. A flow is said to be *ergodic* if for every measurable $A \subset \Omega$ such that $\phi_t^{-1} A = A$ then $\mu(A) = 0$ or $\mu(A) = 1$. The flow ϕ_t is said to be strong mixing with respect to the measure μ if $\mu(A \cap \phi_t^{-1} B) \to \mu(A)\mu(B)$ as $t \to \infty$ for each measurable $A, B \subset \Omega$. The work of Hopf [41] in 1939 followed in the 1960s by Sinai [80], Anosov and Sinai [3] succeeded in showing that geodesic flows on surfaces of constant negative curvature are ergodic and moreover are strong mixing. In order to quantify the rate of mixing the *correlation* function is introduced as follows. Suppose $h, g \in \mathscr{C}^1(\Omega, \mathbb{R})$. Let

$$\mathrm{cor}_{h,g}(t) := \int_\Omega h \cdot g \circ \phi_t \; d\mu - \int_\Omega h \; d\mu \int_\Omega g \; d\mu \tag{1.1.1}$$

for all $t \geq 0$. Whenever the flow is mixing then $\mathrm{cor}_{h,g}(t) \to 0$ as $t \to \infty$ and so the pertinent question is to determine the rate of this convergence. A flow is said to exhibit exponential decay of correlation with rate $\delta > 0$ if for all $h, g \in \mathscr{C}^1(\Omega, \mathbb{R})$ then there exists $C_{h,g} < \infty$ such that

$$\mathrm{cor}_{h,g}(t) \leq C_{h,g} e^{-\delta t} \quad \text{for all } t \geq 0.$$

Slower rates are also of interest including polynomial and the so called *rapid mixing* which is defined to be faster than any power of t. The first estimates for decay of correlation for flows appeared in the 1980s when Collet, Epstein and Gallavotti [20], Moore [58] and Ratner [69] showed exponential decay of correlation for geodesic flows on two dimensional surfaces of constant negative curvature. The same result was extended by Pollicott [65] to three dimensional surfaces. It had long been known that Anosov flows might not even be mixing. Ruelle [71] showed that mixing Axiom A flows need not mix exponentially fast and Pollicott [64] constructed mixing Axiom A flow examples with arbitrarily slow rates of decay of correlation. In the late 1990s Chernov [18] showed sub-exponential decay of correlation for geodesic flows on surfaces of negative curvature. Using these ideas and the ideas of thermodynamic formalism [64, 82] Dolgopyat [27, 26, 25] showed exponential decay of correlation for mixing Anosov flows with $\mathscr{C}^1$ stable and unstable foliations and other results on the prevalence of rapid mixing amongst certain classes of flows. The new ideas of Dolgopyat were adapted by Liverani [50] to show that all $\mathscr{C}^4$ contact Anosov flows also exhibit exponential decay of correlation. Additionally Pollicott [67] showed exponential decay of correlation for a class of uniformly hyperbolic flows with one dimensional unstable manifolds. Recently Tsujii [85] also studied contact Anosov flows and re-proved exponential decay of correlation with fine results on the rate. In 2007 Field, Melbourne and Török [31] showed that amongst $\mathscr{C}^r$ Axiom A flows, $r \geq 2$ an open and dense set of flows is rapid mixing.

All the above results concern uniformly hyperbolic flows. Outside of this class results on the rate of decay of correlation are extremely limited. In 2005 Luzzatto, Melbourne and Paccaut [54] showed that the Lorenz attractor is mixing but there exist no quantitative estimates on the decay of correlation for these flows. In a different direction Melbourne considered certain classes of flows which are non-uniformly hyperbolic and showed rapid decay of correlation [55] and then polynomial decay of correlation for a larger class of flows [56].

1.1.2. Statistical Properties and Perturbations

Once a particular statistical property has been shown for a given flow the next relevant question is how that particular property behaves when the flow is perturbed. This is an important question for a large range of statistical properties. However in this section we focus on one example, namely the regularity of the SRB measure when the flow is perturbed. At this point we must note that many invariant measures may exist for any given dynamical system. Choosing a relevant one is an important and much studied question. For our present purposes we merely recall that following the work in the 1970s by Sinai, Ruelle and Bowen the so-called SRB measures were introduced and have been defined for a wide class of systems [89].

In 1975 Bowen and Ruelle [16] showed that the Gibbs measure of an Axiom A flow depends continuously on the perturbation. Note that the differentiability of the SRB measure is closely related to the differentiability of pressure and measure theoretic entropy. The connection is given by the thermodynamic formalism. In 1986 de la Llave, Marco and Moriyon [23] considered Anosov diffeomorphisms and flows. They show that there is a $\mathscr{C}^\infty$ family of $\mathscr{C}^\infty$ canonical transformations that reduce the perturbation to the well understood system. Katok, Knieper, Pollicott and Weiss [44, 45] consider the differentiability of both topological and metric entropy for Anosov flows. They show that the topological entropy is $\mathscr{C}^{r-1}$ for a $\mathscr{C}^r$ flow and give a formula for the derivative. Also they show that the metric entropy with respect to some Gibbs state is $\mathscr{C}^{r-2}$. Contreras [21] improved this previous result on flows and showed that the topological entropy is $\mathscr{C}^r$ and gave a result on the differentiability of an equilibrium state. Contreras's result for the differentiability of entropy was extended by Pollicott [66] who gives a formula for both the first and second derivative of topological entropy for Anosov flows. In [75]

Ruelle investigates Axiom A flows, based on his previous work for discrete time Anosov systems [72, 74], and gives a formula for the first derivative of the SRB measure. Dolgopyat [28] shows that any u-Gibbs state for partially hyperbolic systems is differentiable. This is directly applicable to the time-one map of a uniformly hyperbolic flow. Of interest in this work is he doesn't make any assumption on the perturbed system as has been done in all the preceding works. The results are for u-Gibbs states whereas we are interested in SRB measures. The relationship is as follows. If there is a unique u-Gibbs state then it is also the SRB measure. If there are several u-Gibbs states then the SRB measure may not exist and even if it does exist there might be extra u-Gibbs states which are not SRB measures. In joint work with Liverani [17], we give a formula for the derivative of the SRB measure for an Anosov flow to the nth order, reproducing and extending the previous result of Ruelle. We note that as before the results are almost entirely limited to uniformly hyperbolic flows.

1.1.3. Flows Versus Discrete Time Systems

A common approach to studying a flow is to the consider a so called Poincaré section and obtain the return map and return time function. In this way the flow gives rise to a discrete time system. See, for example, the chapter by Chernov of [43] which treats this topic in the Anosov setting. In particular Anosov flows give rise to discrete time Anosov systems in this way. Often this is a fruitful approach and history has shown that the statistical properties of maps are often more easily understood than the statistical properties of flows. The other direction, defining a flow from a discrete time system and return time function, is known as a suspension flow.

Certain statistical properties may be shown as a more or less immediate consequence of the corresponding result for the return map. For example an invariant measure for the return map immediately gives an invariant measure for the flow. Indeed there is a one-to-one correspondence between invariant measures for the flow and invariant measures for the return map to the Poincaré section. Additionally Melbourne and Török [57] give conditions under which the central limit theorem and almost sure invariance principle, for the discrete time system are inherited by the suspension flow.

However many statistical properties, for instance decay of correlation may not be directly deduced by studying the return map. This is clear noting that all topologically transitive Anosov diffeomorphisms mix exponentially whereas topologically transitive Anosov flows may not be mixing at all [2]. Moreover, as demonstrated by Pollicott [64], a flow which admits a Poincaré section with return map exhibiting exponential decay of correlation may, as a flow, have arbitrarily slow decay of correlation. Clearly the question of rate of decay of correlation for flows is significantly different to the question for the corresponding return maps.

1.2. Direct Functional Analytic Study of Dynamical Systems

Historically an important technique for establishing the statistical properties of a dynamical system was to first use Markov partitions and reduce the smooth system to symbolic dynamics. See for example Sinai [81] and Bowen [15]. These constructions have been used widely since the 1960s and this is a technique which has yielded many results. Another important development was that of Young towers introduced in the 1990s by Young [88] which gives another way in which the system is coded as a first step to studying the statistical properties. These Young towers were crucial for the investigation of statistical properties of non-uniformly hyperbolic systems. However here we restrict our discussion to the details of a complementary approach which we call the *direct functional analytic approach.* The spirit of this approach is to allow the study of the system directly without any need for coding.

Associated to the flow $\phi_t : \Omega \to \Omega$ is a family of linear operators $\mathscr{L}_t : \mathbf{L^1}(\Omega) \to \mathbf{L^1}(\Omega)$ for all $t \geq 0$ for which holds

$$\int_\Omega \mathscr{L}_t h \cdot g \; dm = \int_\Omega h \cdot g \circ \phi_t \; dm \quad \text{for all } g \in \mathbf{L}^\infty(\Omega).$$

The $\mathscr{L}_t$ are called *transfer operators* and are bounded linear operators on $\mathbf{L^1}(\Omega)$. Depending on the system being considered there exist various possibilities for defining these operators. In Section 2.1.2 we define them for piecewise smooth flows by means of a simple pointwise formula and show by a calculation that the above duality formula holds. Whereas in Section A.2 we define them on the space of distributions for smooth

Anosov flows and again show that the above duality formula holds but this time for smooth functions in the sense that it holds when considering the injection of the space of functions into the space of distributions.

The point of view is to analyse the spectrum of these operators and then relate the spectrum to the statistical properties. We remark that of course the corresponding operator may also be defined for discrete time systems. Unfortunately the spectrum of $\mathscr{L}_t$ acting on $\mathbf{L^1}(\Omega)$ typically contains no useful information so a major part of this approach is to define a relevant Banach space to act as the domain of the transfer operator. The choice of Banach space is key as the spectrum of the transfer operator will clearly depend on the choice of domain of the operator. Most choices of Banach space are either too small and so perhaps not even invariant under the action of the transfer operator or too large and so the spectrum gives no information. Constructing suitable Banach spaces for different settings is one of the current challenges of the field. The book of Baladi [6] contains an overview of the study of these operators for various dynamical systems.

Early use of these ideas was in the work of Lasota and Yorke [49] which was then extended by Keller in a series of papers with several co-authors in the early 1980s where piecewise expanding interval maps were studied by an investigation of the transfer operator. Further development in the piecewise expanding interval maps case followed. In particular Rychlik [78] allowed countable piecewise smooth parts and merely required the inverse of the derivative to be of bounded variation, then Keller [46, 47] showed that the derivative need only be Hölder continuous and showed how various statistical properties, including zeta functions, may be deduced from the spectral information. Obtaining statistical properties was further developed by Keller and Baladi [10].

The next major step was to tackle the problem of hyperbolicity. To this end various anisotropic norms were constructed in a series of papers clearly demonstrating the success of this methodology in the hyperbolic diffeomorphism setting, in each case constructing an appropriate Banach spaces on which the transfer operator may be studied directly without any form of coding. The first work was by Rugh [76] for analytic hyperbolic maps. In 2002 Blank, Liverani and Keller [14] developed geometrically intuitive Banach space for more general hyperbolic diffeomorphisms. This did indeed allow stronger results due to the smoothness of the systems which would otherwise be lost if the system had been first reduced to a shift map of symbolic dynamics. Gouëzel and Liverani [34] introduced Banach spaces for discrete time Anosov systems to produce results on the decay of correlation for these systems. The use of this family of Banach spaces allows fine results for perturbations of the systems. In particular the Banach spaces are such that perturbations of the system may be studied on the same space. They improve and generalise the set-up in [35] by among other things allowing any Gibbs state. Baladi and Tsujii [11] (extending Baladi [7] which was restricted to the case of $\mathscr{C}^\infty$ stable and unstable foliations) study hyperbolic diffeomorphisms as above and yet again take an approach that does not require coding and does investigate the spectrum of the transfer operator. They however use Banach spaces based on Sobolev spaces. Using the above methods dynamical determinants were investigated for these systems by Liverani [51] and Baladi and Tsujii [12]. A related but different approach was taken by Faure and Roy [30] which only applies to a more limited number of systems.

These ideas for discrete time hyperbolic systems had to be adapted to the setting of flows when Liverani [50] constructed Banach spaces and investigated the spectrum of the generator of the semigroup related to the transfer operator in order to show that all $\mathscr{C}^4$ contact Anosov flows exhibit exponential decay of correlation. The transfer operator for the flow is shown to be a one parameter semigroup and the spectrum of the infinitesimal generator of the flow is studied. Subsequently Tsuji [84], [85] constructed anisotropic Hilbert spaces to study first suspension semiflows of angle multiplying maps and then contact Anosov flows by studying the spectrum of the operator directly without the need to look at the generator. In joint work with Liverani [17] we use a scale of Banach spaces which are a combination of the Banach spaces of Gouëzel and Liverani [34] for discrete time Anosov flows and the way of considering a flow of Liverani [50] for contact Anosov flows. This allows fine results on the behaviour of statistical properties under perturbations of the flow.

As before, outside of the realm of smooth uniformly hyperbolic flows, the development of this technology is more limited. One direction in which the technology could be extended is to deal with discontinuities. Some results are available for certain hyperbolic systems with discontinuities by Demers and Liverani [24] and Baladi and Gouëzel [8, 9].

Certain elements of these ideas may be applied to non-uniformly hyperbolic systems by the operator renewal techniques of Sarig [79] later used by Gouëzel [33].

1.2.1. Detailed Program for the Study of Flows

We now sketch a specific program for investigating the statistical properties of a flow. This is similar to the approach taken by Liverani [50] for contact Anosov flows and joint work with Liverani [17] for more general Anosov flows. We present this overview to show the framework into which the result presented in Chapter 2 fits. Supposing that $\mathbf{B}$ is some complex Banach space let $\mathscr{B}(\mathbf{B},\mathbf{B})$ denote the space of continuous linear operators from $\mathbf{B}$ to $\mathbf{B}$. We recall that a *strongly-continuous one-parameter semigroup* on $\mathbf{B}$ is a family of operators $\mathscr{L}_t \in \mathscr{B}(\mathbf{B},\mathbf{B})$ parametrised by a positive real parameter t such that the following hold. That $\mathscr{L}_0 = \mathbf{Id}$, $\mathscr{L}_{s+t} = \mathscr{L}_s \circ \mathscr{L}_t$ for all $s,t \geq 0$, and $\lim_{t\to 0} \mathscr{L}_t h = h$ for all $h \in \mathbf{B}$. This final condition is called the *strong continuity* of the semigroup. The first and basic step in the proposed program is as follows.

Step 1. *Construct a Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ which contains $\mathscr{C}^1(\Omega,\mathbb{R})$ and on which the family $\{\mathscr{L}_t\}_{t\geq 0}$ forms a strongly-continuous one-parameter semigroup and furthermore $\sup_{t\geq 0} \|\mathscr{L}_t\| < \infty$.*

The above gives very little information concerning the behaviour of the dynamical system being studied. It is however a crucial first step as we may now use the well developed theory of strongly-continuous one-parameter semigroups. The *generator* of a strongly-continuous one-parameter semigroup $\mathscr{L}_t$ is the linear operator defined by

$$Zh := \lim_{t\to 0} \tfrac{1}{t} \left(\mathscr{L}_t h - h\right)$$

the domain of Z being the set of $h \in \mathbf{B}$ for which the limit exists. By standard theory the domain of Z is a dense linear subspace of $\mathbf{B}$, moreover Z is a closed operator. There is no reason to expect Z to be a bounded operator. See [22] for the theory of one parameter semigroups. The estimate $\sup_{t\geq 0} \|\mathscr{L}_t\| < \infty$ of Step 1 immediately implies that the spectrum of Z is contained within the set $\{z \in \mathbb{C} : \Re(z) \leq 0\}$. To proceed we must further investigate the spectrum of Z.

Step 2. *Show there exists $\sigma > 0$ such that the spectrum of Z within the strip $\{z \in \mathbb{C} : -\sigma < \Re(z) \leq 0\}$ consists only of isolated eigenvalues of finite multiplicity.*

For the next step we would like now to consider the behaviour of the flow under perturbation. We suppose that there exists a family of flows $\phi_{\epsilon,t}$ for $\epsilon \in [-a, a]$ where $a > 0$ (and most likely small) and $\phi_t = \phi_{0,t}$. The perturbed flow could be defined as follows. Suppose V_0 denotes the vector field associated to the flow ϕ_t and let V denote some other perturbing vector field. Then we let $V_\epsilon := V_0 + \epsilon V$ and let $\phi_{\epsilon,t}$ denote the flow associated to V_ϵ.

Step 3. *Show that the transfer operators $\mathscr{L}_{\epsilon,t}$ of the perturbed flow, $\epsilon \in [-a, a]$ may be studied on the same Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ as the unperturbed transfer operators and that the conclusions of Step 2 continue to hold.*

In an appropriate setting, i.e. careful control on how $\phi_{\epsilon,t}$ depends on ϵ, the above spectral information allows several important statistical properties to be deduced. This will be discussed below. Even further information concerning the spectrum of Z as follows is useful. This is an estimate in the spirit of Dolgopyat [25]. Note that Step 3 and Step 4 are independent of each other but that both require Step 1 and Step 2.

Step 4. *Show there exists $\delta > 0$, $T < \infty$ such that no part of the spectrum of Z is contained within the set $\{z \in \mathbb{C} : -\delta < \Re(z) \leq 0, |\Im(z)| \geq T\}$.*

This last step is particular to flows and is connected to the delicate mechanism which causes mixing in the flow direction. The statistical properties which may be deduced from each step are discussed in the following.

1.2.2. Statistical Properties from Spectral Results

As hinted at above in the appropriate setting with the appropriate estimates various statistical properties of interest can be read off from the spectrum of the transfer operator with the relevant knowledge. Keller [47, §9] describes a selection of techniques for obtaining various statistical properties from the spectrum of the transfer operator of a discrete time system. If one can study the transfer operator of the time one map of a flow directly as Tsujii does in [84, 85] then these methods apply without modification to the flow case. Otherwise the connection between statistical properties and the spectral picture is more delicate for flows. As described in Liverani [50] one would expect the combination of Step 1, Step 2 and Step 4 to imply exponential decay of correlation for the flow. Alternatively the combination of Step 1, Step 2 and Step 3 should, via the reasoning of [17] based on the perturbation ideas described by Keller and Liverani [48] allow results on the behaviour of a variety of statistical results, including results for the differentiability of the invariant measure under perturbations. Also note Liverani and Tsujii [52] concerning zeta functions and dynamical determinants.

1.3. Beyond Smooth and Uniformly Hyperbolic

In Section 1.1 many known results concerning the statistical properties of flows were recalled. As already noted the majority of these results applied only to uniformly hyperbolic flows. Outside of the realm of uniform hyperbolic very little is known. Here we consider various directions where it could be hoped to extend the results.

1.3.1. Singular Hyperbolic Flows

Introduced in 1963 as a simple model for weather, the Lorenz flow [53] is a smooth three dimensional flow which has long proved elusive to thorough study. It is not uniformly hyperbolic. The class of *singular hyperbolic flows* was introduced and studied in the late 1990s by Morales, Pacifico, Pujals [59, 61, 60]. This class of flows contains the uniform hyperbolic flows and also contains Lorenz flows. Whereas the uniformly hyperbolic flows are identically the flows which are structurally stable as shown by Hayashi [37, 38], the singular hyperbolic flows are identically the flows which are stably transitive. It is known that singular hyperbolic flows are chaotic in that they are expansive and admit an SRB measure [4]. Some further results are known limited to the particular case of the Lorenz attractor. It is know to be mixing [54] and that the central limit theorem and invariance principle hold [40].

Afraĭmovič, Bykov and Silnikov [1] and Guckenheimer and Williams [36, 87] introduced a geometric model of the Lorenz flow and Tucker [86] Showed that the geometric Lorenz flow really was a representative model for the original Lorenz flow. Quotienting along stable manifolds the three dimensional Lorenz flow may be reduced to a suspension over an expanding interval map. Shortly we will return to this topic in further detail.

1.3.2. Suspension Flows

As mentioned earlier a flow gives rise to a return map and return time function by means of a Poincaré section. Here we take the reverse point of view and define the flow that arises from the combination of return map and return time. This will allow us to make some precise observations concerning different classes of two dimensional flows and their properties. We start with the following definition of the basic setting.

Definition 1.3.1 (Piecewise Expanding Suspension Flow)**.** Let $\mathcal{S} \subset [0,1]$ be a finite or countable set of points. Suppose that

$$f \in \mathscr{C}^1\left([0,1] \setminus \mathcal{S}, [0,1]\right),$$

such that $\operatorname{ess\,inf} |f'| > 1$, and that $f([0,1]\setminus\mathcal{S}) = [0,1]$. We call f the *return map*. Further suppose

$$r \in \mathscr{C}^1\left([0,1] \setminus \mathcal{S}, \mathbb{R}_+\right),$$

such that $\inf r > 1$. We call r the *return time function*. Let $\Omega := \{(x,s) : x \in [0,1], 0 \leq s < r(x)\}$ be called the *state space*. For all $t \geq 0$ the flow

$$\phi_t : \Omega \to \Omega$$

is defined by $\phi_t : (x,s) \mapsto (x, s+t)$ subject to the identification $(x, r(x)) \sim (f(x), 0)$. The flow ϕ_t is said to be a *piecewise expanding suspension flow*.

In order to proceed we must add further conditions on f and r. We start with the following definition.

Definition 1.3.2. A piecewise expanding suspension flow $\phi_t : \Omega \to \Omega$ associated to return map f and return time function r is said to have *bounded twist* if

$$\tfrac{r'}{f'} \in \mathbf{L}^\infty([0,1]). \tag{1.3.1}$$

The importance of the above is that if a piecewise suspension flow does have bounded twist than it is uniformly hyperbolic in a sense made precise in Lemma 1.3.3 below. First we discuss the subject of uniform hyperbolicity in the suspension flow setting. For a smooth and invertible flow then uniform hyperbolicity is defined by means of a continuous splitting of the tangent bundle into three invariant subspaces, namely *stable*, *centre* and *unstable* where the centre subspace is one dimensional and coincides with the flow direction and there are uniform estimates for expansion (contraction) in the unstable (stable) subspace. Or equivalently uniform hyperbolicity may be defined in terms of invariant cone-fields, see for example the Hasselblatt chapter of [43]. In the present setting of a piecewise expanding suspension flow there will be no stable direction to consider. Moreover as the suspension flow is not invertible it is not possible to consider an invariant unstable direction however the cone-field characterisation may still be applied.

Lemma 1.3.3. *Suppose that ϕ_t is a piecewise expanding suspension flow with bounded twist* (1.3.1)*. Then there exists $G < \infty$ such that letting*

$$\mathscr{K}_G := \{ \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) : a, b \in \mathbb{R}, |b| \leq G\,|a| \}$$

then for all $x \in [0,1] \setminus \mathcal{S}$ holds

$$D\phi_{r(x)}(x,0)v \in \mathscr{K}_G \quad \text{for all } v \in \mathscr{K}_G \, .$$

Proof of Lemma 1.3.3. As ϕ_t is a piecewise suspension flow then there exists $\lambda < 1$ such that $|f'(x)|^{-1} \leq \lambda$ for all $x \in [0,1] \setminus \mathcal{S}$. Moreover since ϕ_t is supposed to have bounded twist then $B := \| \frac{r'}{f'} \|_{\mathbf{L}^\infty([0,1])} < \infty$. Let $G := B(1-\lambda)^{-1} < \infty$. We now show that the cones $\mathscr{K}_G$ as defined in the statement of the lemma are indeed invariant under the derivative map. Since $\phi_t(x,s) = (f(x), s + t - r(x))$ whenever $(x,s) \in \Omega$ and $r(x) \leq t + s \leq r(x) + r(f(x))$ then differentiating we find that

$$D\phi_{r(x)}(x,0) = \left(\begin{smallmatrix} f'(x) & 0 \\ -r'(x) & 1 \end{smallmatrix}\right).$$

Fixing $v = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathscr{K}$ and $x \in [0,1] \setminus \mathcal{S}$ then

$$D\phi_{r(x)}(x,0)v = \left(\begin{smallmatrix} af'(x) \\ -ar'(x)+b \end{smallmatrix}\right). \tag{1.3.2}$$

Let $\tilde{a} := af'(x)$ and $\tilde{b} := -ar'(x) + b$. Recalling that $\frac{|b|}{|a|} \leq G$ since $v \in \mathscr{K}_G$, holds

$$\frac{|\tilde{b}|}{|\tilde{a}|} \leq \frac{|r'(x)|}{|f'(x)|} + \frac{|b|}{|a|}\frac{1}{|f'(x)|} \leq B + \lambda G.$$

As $B + \lambda G = B + \lambda B(1-\lambda)^{-1} = G$ then $D\phi_{r(x)}(x)v \in \mathscr{K}_G$ as required. □

In the discussion immediately preceding the above lemma it was explained that a invariant cone-field sense of uniform hyperbolicity was most practical in the present context. Actually the invariant cone-field constructed above in Lemma 1.3.3 is only given for the set $\{(x,0) : x \in [0,1]\} \subset \Omega$ but it is easy to see that this cone-field is simply extended for all $(x,s) \in \Omega$ as $D\phi_t(x,s) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ whenever $0 \leq s + t < r(x)$. We remark that in Lemma 1.3.3 the return time was not required to be bounded.

1.3.3. Gauss-Log Suspension

With this example we show that it is possible to have piecewise expanding suspension flows which do have bounded twist but where the return time function is not bounded. Let $\alpha \in (0,1)$.

$$\begin{aligned} f_{\text{gauss}}(x) &:= |x - \tfrac{1}{2}|^{-\alpha} \mod 1, \\ r_{\text{gauss}}(x) &:= -\ln|x - \tfrac{1}{2}|. \end{aligned} \tag{1.3.3}$$

These functions are illustrated in Figure 1.1. Note that for this piecewise expanding suspension flow then the discontinuity set $\mathcal{S}$ is countable but not finite. The return time function is unbounded with $r(x) \to \infty$ as $x \to \frac{1}{2}$. However we may calculate $\frac{r'}{f'}(x) = \frac{1}{\alpha}|x - \frac{1}{2}|^{\alpha}$ and so these suspension flows do indeed have bounded twist. The class studied by Baladi and Vallée [13] are the piecewise expanding suspension flows which may have unbounded return time but bounded twist is required. The above Gauss-Log example (1.3.3) is the prototype of such flows.

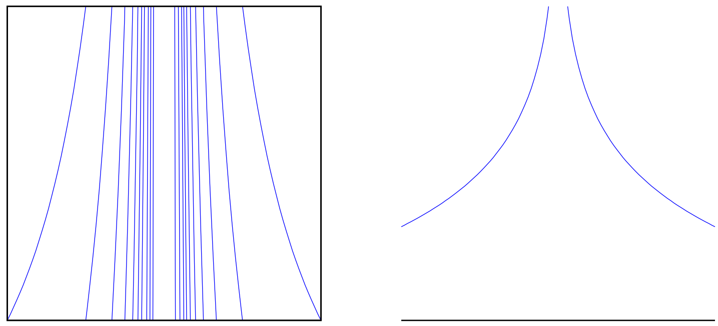

Figure 1.1.: *The Gauss-Log example. On the left the return map* $f_{\text{gauss}} : [0,1] \to [0,1], x \mapsto |x - \frac{1}{2}|^{-\alpha} \mod 1$ *and on the right the return time function* $r_{\text{gauss}} : [0,1] \to (0,\infty), x \mapsto -\ln|x - \frac{1}{2}|$.

1.3.4. Unbounded Twist

The next relevant question is the converse to Lemma 1.3.3, i.e. if a piecewise expanding suspension flow does not have bounded twist does this imply that it is not possible to have an invariant cone-field which is separated from the flow direction. Let f, r be the return map and return time function associated to a piecewise expanding suspension flow. For each $n \in \mathbb{N}$ the nth iterate flow may be defined as follows. Let $r_n := \sum_{i=0}^{n-1} r \circ f^i$. Notice that the pair f^n, r_n satisfy the requirements to be the return map and return time function of a piecewise expanding suspension flow. There is an many-to-1 relationship between the state space of the nth iterate flow and the state space of original flow but the flow remains unchanged. Since

$$\frac{r_n'}{(f^n)'}(x) = \sum_{i=0}^{n-1} \frac{r'}{f'} \circ f^i(x) \frac{1}{(f^{n-i-1})'} \circ f^i(x) \tag{1.3.4}$$

and as f is expanding then it is clear that the property of bounded twist is invariant in the sense that if a piecewise expanding suspension flow has bounded twist then every nth iterate piecewise expanding suspension flow also has bounded twist. The opposite

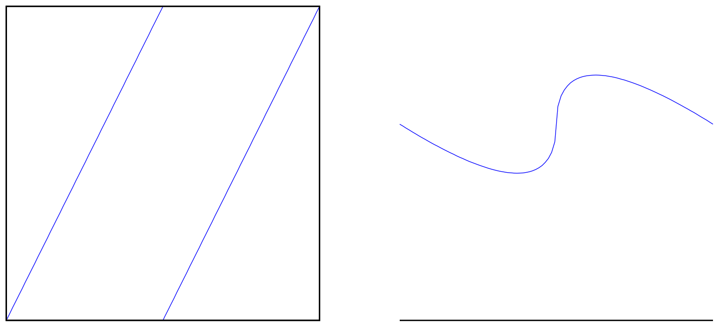

Figure 1.2.: *An example with unbounded twist for every iterate but where the return time is bounded. On the left the return map $f : [0,1] \to [0,1]$ and on the right the return time function $r : [0,1] \to (0,\infty)$ such that $r'(x) \to \infty$ as $x \to \frac{1}{2}$.*

direction is not necessarily true, it is simple to produce examples which do not have bounded twist but some iterate does have. For example construct the flow such that $r'(x) \to \infty$ as $x \to x_0 \in [0,1]$ and $f'(x) \to \infty$ sufficiently fast as $x \to f(x_0)$.

Finding some $n \in \mathbb{N}$ such that the nth iterate suspension does have bounded twist is often not possible. In order to understand this we present the following explicit example. We consider the piecewise expanding suspension flow where the return map and return time function are given by the following respectively.

$$\begin{aligned} f_{\text{twist}}(x) &:= 2x \mod 1, \\ r_{\text{twist}}(x) &:= \begin{cases} 2 - 2x - (1-2x)^{\frac{1}{2}} & \text{if } x < \frac{1}{2} \\ 2 - 2x + (2x-1)^{\frac{1}{2}} & \text{if } x \geq \frac{1}{2} \end{cases} \end{aligned} \tag{1.3.5}$$

These functions are illustrated in Figure 1.2. Note firstly that with this example $\frac{r'}{f'}(x) \to \infty$ as $x \to \frac{1}{2}$. Since $(f^n)'(x) = 2^n$ for all $x \in [0,1]$ it is clear that for any $n \in \mathbb{N}$ then the twist $\frac{r_n'}{(f^n)'}$ will not be bounded. In fact $\frac{r_n'}{(f^n)'} \to \infty$ as $x \to x_0$ for all $x_0 \in [0,1]$ such that $f^n(x_0) = \frac{1}{2}$.

The following lemma presents the converse argument to Lemma 1.3.3

Lemma 1.3.4. *Suppose that $\phi_t : \Omega \to \Omega$ is a piecewise expanding suspension flow of return map f and return time function r and that there is some interval $(x_0, x_1) \subset [0,1] \setminus \mathcal{S}$ and that $\frac{r'}{f'}(x) \to \infty$ as $x \to x_0$ within (x_0, x_1). Further suppose that for each $x \in [0,1]$ there exists some set $\mathscr{K}(x) \subset \{v = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right), a, b \in \mathbb{R}, |\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)| \neq 0\}$ and that these sets are invariant in the following sense. For all $x \in [0,1] \setminus \mathcal{S}$, $v \in \mathscr{K}(x)$ then $D\phi_{r(x)}(x,0)v \in \mathscr{K}(f(x))$. Then for all $L < \infty$ there exists a set $T_L \subset [0,1]$ of non-zero measure such that for each $x \in T_L$ holds $\sup\{\left|\frac{b}{a}\right| : \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathscr{K}(x)\} \geq L$.*

Proof. Let $x \in [0,1]$, $v = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathscr{K}(x)$ and $\tilde{v} = \left(\begin{smallmatrix} \tilde{a} \\ \tilde{b} \end{smallmatrix}\right) := D\phi_{r(x)}(x,0)v$. Recalling (1.3.2) holds

$$\frac{\tilde{b}}{\tilde{a}} = -\frac{r'}{f'}(x) + \frac{b}{a}\frac{1}{f'(x)}. \tag{1.3.6}$$

Let $\tilde{T}_L := \{x \in [0,1] : |\frac{r'}{f'}(x)| \geq 2L\}$. By assumption $\frac{r'}{f'}(x) \to \infty$ as $x \to x_0$ and since $\frac{r'}{f'} \in \mathscr{C}^0([0,1] \setminus \mathcal{S}, \mathbb{R})$ then $\tilde{T}_L$ has non-zero measure. Either there exists a non-zero measure set $T_L \subseteq \tilde{T}_L$ such that $\sup\{\left|\frac{b}{a}\right| : \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathscr{K}(x)\} \geq L$ for all $x \in T_L$ or there exists $\hat{T}_L \subseteq \tilde{T}_L$ such that $\mathrm{m}(\tilde{T}_L) = \mathrm{m}(\hat{T}_L)$ and $\left|\frac{b}{a}\right| \leq L$ for all $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathscr{K}(x)$, $x \in \hat{T}_L$. In the first case this completes the proof of the lemma. In the second case we proceed as follows. The invariance assumption implies that $\left(\begin{smallmatrix} \tilde{a} \\ \tilde{b} \end{smallmatrix}\right) \in \mathscr{K}(f(x))$. However by (1.3.6) noting that $|1/f'(x)| < 1$ then for all $x \in \hat{T}_L$ holds $|\frac{\tilde{b}}{\tilde{a}}| \geq 2L - L \geq L$. We therefore conclude the proof by letting $T_L = f(\hat{T}_L)$. $\square$

In the above we established that the boundedness of the quantity $\frac{r'}{f'}$ (bounded twist) for a piecewise expanding suspension flow is intimately connected with the flow being or not being uniformly hyperbolic. Moreover we established that bounded twist and bounded return time are separate issues: The example given above (1.3.5) had unbounded twist whilst the return time function is bounded, the Gauss-Log example (1.3.3) given previously had unbounded return time and bounded twist.

1.3.5. Lorenz-like suspension flow

We now return to the specifics of Lorenz flows which we introduced above. As already mentioned the Lorenz flow gives rise to a suspension flow [54] which is as follows. The piecewise expanding suspension flow $\phi_t : \Omega \to \Omega$ of return map f and return time r is said to be a *Lorenz-like suspension flow* if the following additional conditions hold. For simplicity we choose $x_s := \frac{1}{2}$ but this particular choice is not necessary.

1. $\mathcal{S} = \{x_s\}$,
2. f' is Hölder continuous on $[0,1] \setminus \mathcal{S}$,
3. There exists $\alpha \in (0,1)$, $0 < C < \infty$ such that for all $x \in [0,1] \setminus \mathcal{S}$ then
$$C^{-1} \left|x - x_s\right|^{\alpha-1} \leq f'(x) \leq C \left|x - x_s\right|^{\alpha-1}.$$
4. $r(x) \to \infty$ as $x \to x_s$,
5. There exists $C < \infty$ such that if both $x, y < x_s$, or if both $x, y > x_s$ then
$$\left|r(x) - r(y)\right| \leq C \left|\ln(x - x_s) - \ln(y - x_s)\right|.$$

An example Lorenz-like suspension flow is shown in Figure 1.3. Note that the return time becomes large as $x \to x_s$ and also $|f'|$ becomes large to partially compensate. Just like the unbounded twist example (1.3.5) the Lorenz-like suspension flows have unbounded twist and will have for any iterate of the suspension. They also have unbounded return time like the Gauss-log example (1.3.3).

In order to deal with such flows which do not have bounded twist we introduce the following, weaker condition on the twist.

Definition 1.3.5 (tempered twist)**.** We say that $\phi_t : \Omega \to \Omega$, a piecewise smooth suspension flow of f and r, has *tempered twist* if

$$\tfrac{r'}{f'} \in \mathbf{L^q}([0,1]) \quad \text{for some } q > 1. \tag{1.3.7}$$

This above property is satisfied by Lorenz-like suspension flows and also the unbounded twist example (1.3.5) given earlier. The lack of bounded twist in a flow suggests difficulties in applying the direct functional analytic approach sketched in Section 1.2 to such systems. There have been no previous works which produce a Banach space for

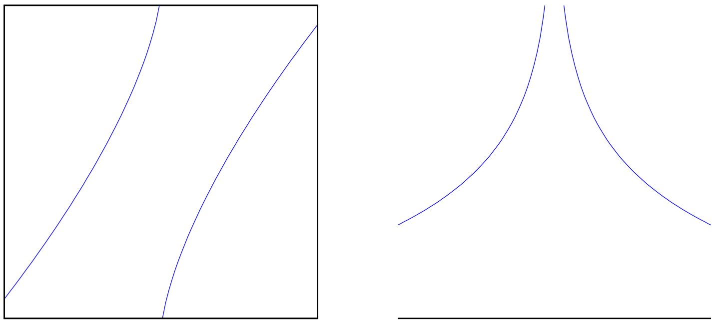

Figure 1.3.: *The Lorenz-like suspension flow with singularity at* $x = \frac{1}{2}$. *On the left the return map* $f : [0,1] \to [0,1]$ *such that* $f(x) \approx Cx^{\alpha} + C'$ *at the singularity and on the right the logarithmic return time function* $r : [0,1] \to (0,\infty)$.

studying dynamical systems where there hasn't been a possibility of uniformly separating unstable and flow directions. The goal and achievement described in Chapter 2 is to successfully develop a Banach space which does allow the investigation of some suspension flows with unbounded twist using only the much weaker requirement of (1.3.7) as introduced above.

1.3.6. Discontinuities

All of the above concerned relaxing the condition of uniform hyperbolicity. A different direction in which to move beyond the class of smooth uniformly hyperbolic flows is to consider flows with discontinuities. The physically relevance of such models is easily argued by considering the situation of objects moving under the action of some external field and sometimes colliding with other objects. See for example the billiards studied by Chernov [19]. At present there exists no previous direct functional analytic study of flows with discontinuities. However there is hope for the ideas to apply as the original setting for the direct functional analytic approach, namely using the space of functions of bounded variation to study piecewise expanding maps, was able to easily deal with the discontinuities of the maps. In higher dimensions, even for expanding maps, the situation is not as simple as demonstrated by the singular example of Tsujii [83]. For piecewise hyperbolic maps the question is even more complicated. Recently there has been good progress in this direction constructing appropriate Banach spaces for the direct function analytic approach by Demers and Liverani [24] in two dimensions and by Baladi and Gouëzel [8, 9] in greater generality.

For flows which are only piecewise smooth the progress in constructing appropriate Banach spaces for direct study of the systems is even slower. In Chapter 2, in addition to allowing unbounded twist as discussed above, we demonstrate that it is possible to apply the direct functional analytic approach to flows which are only piecewise smooth. Note that the piecewise expanding suspension flows introduced above, which help to simplify some issues, are always going to have discontinuities due to allowing the discontinuities in the return map and return time which add more and different complications.

2. Unbounded Twist Flows - Results

In this Chapter we consider in detail the class of suspension flows for which the prototype is the unbounded twist example (1.3.5) of Chapter 1. We allow unbounded twist but require the return time to be bounded. This class of flows is a subclass of the piecewise expanding suspension flows of the previous chapter; however below we define all quantities from scratch in order that this chapter be self contained.

2.1. Statement of Results

Let $\mathcal{S} \subset [0,1]$ be a finite set of points. Suppose that we are given

$$f \in \mathscr{C}^2 \left([0,1] \setminus \mathcal{S}, [0,1]\right),$$

such that $\operatorname{ess\,inf} |f'| > 1$, $\operatorname{ess\,sup} |f'| < \infty$, $\operatorname{ess\,sup} |f''| < \infty$ and $f([0,1] \setminus \mathcal{S}) = [0,1]$. We call f the *return map*. Further suppose that we are given

$$r \in \mathscr{C}^1 \left([0,1] \setminus \mathcal{S}, (0,\infty)\right),$$

such that $\sup r < \infty$ and $\inf r > 1$. We call r the *return time function*. We now introduce a key condition. We say that the return time function r has *tempered twist* if

$$r' \in \mathbf{L}^{\mathbf{q}}([0,1]) \tag{2.1.1}$$

for some $q > 1$. (The measure on $[0,1]$ is always understood to be Lebesgue.) Note that this is equivalent to the previous definition of tempered twist in Chapter 1 as $|f'|$ is here bounded from above and below. We remark that we will never require f to be a Markov map and we will never require $|r'|$ to be bounded.

We fix the phase space $\Omega := [0,1] \times [0,1)$. Note that this is different to Chapter 1. We write points as $x = (x_1, x_2) \in \Omega$. Given a return map f and return time r then in the following we define the flow under time $t \geq 0$

$$\phi_t : \Omega \to \Omega,$$

called a *tempered-twist suspension flow*. This is defined slightly differently to the standard and as used in Chapter 1 where the speed is piecewise constant along flow lines. Precise definition follows. The purpose of this is to make the suspension flow as similar as possible to studying a smooth flow without reducing to a suspension.

2.1.1. Definition of the Flow

First we fix the *speed function* $v : \Omega \to (0, \infty)$ which satisfies the following. There exists $\delta > 0$ and $C < \infty$ such that

1. $v \in \mathscr{C}^1([0,1] \setminus \mathcal{S}) \times [0,1]), (0,\infty))$,
2. $\int_0^1 v(x_1, x_2)^{-1} \, dx_2 = r(x_1)$ for all $x_1 \in [0,1]$,
3. $v(x_1, x_2) = 1$ whenever $x_2 \in [0,\delta] \cup [1-\delta, 1]$,
4. $\sup_{x_2} |\partial_{x_1} v(x_1, x_2)| \leq C \, |r'(x_1)|$ for all $x_1 \in [0,1] \setminus \mathcal{S}$,
5. $\operatorname{ess\,sup} |\partial_{x_2} v| < \infty$.

Remark 2.1.1. Such speed functions can easily be produced given the requirements on r introduced above. Consider the function defined by $v(x_1, x_2) = (1 + 4x_2(r(x_1) - 1)^{-1}$ when $x_2 \in [0, \frac{1}{2}]$ and $v(x_1, x_2) := v(x_1, 1 - x_2)$ when $x_2 \in (\frac{1}{2}, 1]$. The required properties (2), (4) and (5) are immediately satisfied. Moreover $v \in \mathscr{C}^0_f([0,1] \setminus \mathcal{S}) \times [0,1]), (0,\infty))$ and $v(x_1, x_2) = 1$ whenever $x_2 \in \{0, 1\}$. This means that modifying slightly v properties (1) and (3) are also satisfied.

Before giving the precise description let us give an indication of how the flow acts. Each point $(x_1, x_2) \in \Omega \setminus (\mathcal{S} \times [0,1))$ flows in the x_2 coordinate at speed $v(x_1, x_2)$. We call the set $[0,1] \times \{0\} \subset \Omega$ the *base* and we call the set $[0,1] \times \{1\}$ the *roof*. When the point reaches the roof it is mapped back to the base with a shift in the x_1 coordinate according to the return map f. In order to precisely define the flow we introduce some notation. We define the *time function* for $x_1 \in [0,1] \setminus \mathcal{S}$, $x_2, \tilde{x}_2 \in [0,1]$ by

$$\tau(x_1, x_2, \tilde{x}_2) := \int_{x_2}^{\tilde{x}_2} \frac{1}{v(x_1, x_2')} \, dx_2'. \tag{2.1.2}$$

This is the time taken to travel from (x_1, x_2) to $(x_1, \tilde{x}_2)$. We will often consider τ with the second variable zero and so for convenience let $\tau(x_1, x_2) := \tau(x_1, 0, x_2)$. Clearly $\tau(x_1, 1) = r(x_1)$. For each $x_1 \in [0,1] \setminus \mathcal{S}$ we let $x_2 \mapsto \varphi_t(x_1, x_2) \in [0,1]$ denote the flow under time t along the vertical line $\{x_1\} \times [0,1]$ so that φ_t is uniquely determined by the requirements that $\varphi_0(x_1, x_2) = x_2$ and $\frac{d\varphi_t}{dt}(x_1, x_2) = v(x_1, \varphi_t(x_1, x_2))$ for all $x_2 \in [0,1]$. We call φ_t the *local flow* and note that it is well defined for all t such that $-\tau(x_1, 0, x_2) \le t < \tau(x_1, x_2, 1)$. We first define the full flow for limited values of t. For all $(x_1, x_2) \in \Omega$ we define the flow for all $t \in [0, \tau(x_1, x_2, 1)]$ as follows

$$\phi_t(x_1, x_2) := \begin{cases} (x_1, \varphi_t(x_1, x_2)) & \text{if } t < \tau(x_1, x_2, 1) \\ (f(x_1), 0) & \text{if } t = \tau(x_1, x_2, 1). \end{cases} \tag{2.1.3}$$

Notice that $\tau(x_1, x_2, 1) > 0$ for each $(x_1, x_2) \in \Omega$. A consequence of the above definition is that for each $(x_1, x_2) \in \Omega$ then

$$\phi_t \circ \phi_s(x_1, x_2) = \phi_{t+s}(x_1, x_2) \tag{2.1.4}$$

holds for all t, s such that each of the above terms are defined. We define ϕ_t for all $t \ge 0$ by imposing that (2.1.4) continues to hold.

2.1.2. Transfer Operator

We now define the *transfer operator*. For all $t \ge 0$ let $\mathscr{L}_t : \mathbf{L^1}(\Omega) \to \mathbf{L^1}(\Omega)$ be defined by the requirement

$$\int_\Omega \mathscr{L}_t h \cdot \eta \, dx = \int_\Omega h \cdot \eta \circ \phi_t \, dx \qquad \text{for all } \eta \in \mathbf{L}^\infty(\Omega). \tag{2.1.5}$$

In order to understand that the above is well defined we note the following. We could alternatively have defined $\mathscr{L}_t : \mathbf{L}^\infty(\Omega)^* \to \mathbf{L}^\infty(\Omega)^*$ by $\langle \mathscr{L}_t h, \eta \rangle := \langle h, \eta \circ \phi_t \rangle$ for all $\eta \in \mathbf{L}^\infty(\Omega)$. This is clearly well defined. We recall that $\mathbf{L}^\infty(\Omega)^* \supset \mathbf{L}^\mathbf{1}(\Omega)$ and also there is a natural injection $\mathbf{i} : \mathbf{L}^\mathbf{1}(\Omega) \hookrightarrow \mathbf{L}^\infty(\Omega)^*$ defined as $\langle \mathbf{i}h, \eta \rangle := \int_\Omega h \cdot \eta \, dx$. In this way this definition, restricted to $\mathbf{L}^\mathbf{1}(\Omega)$, gives rise to (2.1.5).

$\mathscr{L}_t$ is a bounded linear operator from $\mathbf{L}^\mathbf{1}(\Omega) \to \mathbf{L}^\mathbf{1}(\Omega)$. (See Lemma 2.1.3 below.) However, only by reducing the domain do we obtain good spectral properties. This work concerns the study of $\mathscr{L}_t$ acting on a special Banach space defined below. That $\mathscr{L}_t$ is well defined acting on this space will be part of the statement of the theorem. Notice that

$$\mathscr{L}_{t+s} = \mathscr{L}_t \circ \mathscr{L}_s \quad \text{for all } t, s \geq 0. \tag{2.1.6}$$

This is called the semigroup property of $\mathscr{L}_t$. Some notation: We write dx to signify integration with respect to Lebesgue on Ω. The notation $\mathbf{L}^\mathbf{1}$ when written thus, without mention of measure space, should be understood to mean $\mathbf{L}^\mathbf{1}(\Omega)$. Similarly for $\mathbf{L}^\mathbf{p}$, $\mathbf{L}^\infty$. Later it will become useful to have the following pointwise formula for the transfer operator.

Lemma 2.1.2. *Suppose that $h : \Omega \to \mathbb{R}$, $t \geq 0$ and $y \in \Omega$. Then*

$$\mathscr{L}_t h(y) := \sum_{\phi_t(x) = y} h(x) \cdot \left| \det D\phi_t(x) \right|^{-1}. \tag{2.1.7}$$

Proof. This is a consequence of the definition (2.1.5) of $\mathscr{L}_t$ by a change of variables in the integration. □

Lemma 2.1.3. $\|\mathscr{L}_t h\|_{\mathbf{L}^\mathbf{1}} \leq \|h\|_{\mathbf{L}^\mathbf{1}}$ *for all $t \geq 0$, $h \in \mathbf{L}^\mathbf{1}(\Omega)$.*

Proof. This is a direct consequence of the definition (2.1.5). Note that $\|h\|_{\mathbf{L}^\mathbf{1}} = \sup\{\int_\Omega h \cdot \eta \, dx : \|\eta\|_{\mathbf{L}^\infty} \leq 1\}$. And so $\|\mathscr{L}_t h\|_{\mathbf{L}^\mathbf{1}} = \sup\{\int_\Omega h \cdot \eta \circ \phi_t \, dx : \|\eta\|_{\mathbf{L}^\infty} \leq 1\}$. That $\|\eta\|_{\mathbf{L}^\infty} \leq 1$ implies that $\|\eta \cdot \phi_t\|_{\mathbf{L}^\infty} \leq 1$. □

2.1.3. Banach Space of Densities

In the following we define a Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ which is suitable for studying the statistical properties of this flow. We call the functions in this Banach space *densities*. First it is convenient to define for all $h : [0,1]^2 \to \mathbb{C}$ and $(x_1, x_2) \in \Omega$

$$\mathscr{P}_f h(x_1, x_2) := \sum_{y_1 \in f^{-1}(x_1)} h(y_1, x_2) \cdot \left| f'(y_1) \right|^{-1}. \tag{2.1.8}$$

The symbol $\mathscr{P}_f$ is nothing more than shorthand for the right hand side. It will never be studied as some operator in its own right. Given a smooth suspension flow we make the following definition.

Definition 2.1.4 (Core Densities)**.** Suppose $h \in \mathscr{C}^1(([0,1] \setminus \sigma) \times [0,1], \mathbb{C})$ for some finite set $\sigma \subset [0,1]$ and $\operatorname{ess\,sup} |h| < \infty$. Also, for all $x_1 \in [0,1] \setminus \sigma$

(i) $h(x_1, 0) = \mathscr{P}_f h(x_1, 1)$,

(ii) $(\nabla h)(x_1, 0) = (\nabla \mathscr{P}_f h)(x_1, 1)$.

Also holds

(iii) $\partial_{x_1}(v \cdot h) \in \mathbf{L^1}(\Omega)$,

(iv) The function $(x_1, x_2) \mapsto \partial_{x_2}(v \cdot h)(x_1, x_2)$ admits a $\mathscr{C}^0$ extension to the closure of each connected component of $([0,1] \setminus \sigma) \times [0,1]$.

We define $\mathscr{C}^1_f$, the *core densities*, to be the set of all such h.

Remark 2.1.5. Using the equals sign in item (ii) of the above definition we mean that the two functions agree on Ω in an $\mathbf{L^1}$ sense. We use this convention throughout in all similar circumstances.

Remark 2.1.6. Recall the set $\mathcal{S}$, introduced in the first paragraph of Section 2.1, which denotes the singularities of the flow. A priori there is no connection between $\mathcal{S}$ and the set σ associated to some given $h \in \mathscr{C}^1_f$. In this sense the set $\mathscr{C}^1_f$ may be larger than required in order to be invariant under the action of $\mathscr{L}_t$.

The space $\mathscr{C}_f^1$ is relatively large in the sense that it contains for example all functions $\{h \in \mathscr{C}^1([0,1]^2, \mathbb{C}) : \operatorname{supp}(h) \subset (0,1) \times (0,1)\}$. We now define a norm on the space $\mathscr{C}_f^1$ and then define our Banach space of densities by completion. For any $h \in \mathbf{L}^\infty(\Omega)$ let

$$\|h\|_{\mathbf{S}} := \operatorname*{ess\,sup}_{x_1 \in [0,1]} \int_0^1 |h(x_1, x_2)| \; dx_2.$$

Notice that $\|h\|_{\mathbf{L^1}} \leq \|h\|_{\mathbf{S}}$ for all $h \in \mathscr{C}_f^1$. For any $h \in \mathscr{C}_f^1$ let

$$\|h\|_{\mathbf{F}} := \|\partial_{x_2}(h \cdot v)\|_{\mathbf{S}}. \tag{2.1.9}$$

We call $\|\cdot\|_{\mathbf{F}}$ the *flow direction* component of the norm. For all $x_1 \in [0,1] \setminus \mathcal{S}$ let

$$\lambda(x_1) := \frac{\ln f'(x_1)}{r(x_1)}. \tag{2.1.10}$$

Further more let $\lambda := \operatorname{ess\,inf}_{x_1 \in [0,1]} \lambda(x_1)$. Notice that $\lambda > 0$ since $\operatorname{ess\,inf} |f'| > 1$ and $\sup r < \infty$. For all $x_1 \in [0,1] \setminus \mathcal{S}$, $x_2 \in [0,1]$ let

$$\gamma(x_1, x_2) := \exp[-\lambda(x_1) \cdot \tau(x_1, x_2)]. \tag{2.1.11}$$

Notice that $\gamma(x_1, 0) = 1$ and $\gamma(x_1, 1) = 1/f'(x_1)$ for every $x_1 \in [0,1] \setminus \mathcal{S}$. This is a weighting which exists to distribute the expansion along the orbits to avoid the arguably unrealistic situation where the expansion due to f is only experienced at the moments when the trajectory of the flow crosses the roof. For all $h \in \mathscr{C}_f^1$ we define

$$\|h\|_{\mathbf{T}} := \sup_{\eta \in \mathscr{D}} \left| \int_\Omega h \cdot \partial_{x_1}(\gamma \cdot \eta) \; dx \right| \tag{2.1.12}$$

where $\mathscr{D} := \{\eta \in \mathscr{C}^1([0,1]^2, \mathbb{C}), \|\eta\|_{\mathbf{L}^\infty(\Omega)} \leq 1\}$. We call $\mathscr{D}$ the space of *test functions*. We call $\|\cdot\|_{\mathbf{T}}$ the *transversal* component of the norm. Finally we define the norm which will be the central part of this work.

Definition 2.1.7 (Anisotropic Banach Space)**.** For all $h \in \mathscr{C}_f^1$ let

$$\|h\|_{\mathbf{B}} := \|h\|_{\mathbf{T}} + \|h\|_{\mathbf{F}},$$

and let $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ be the completion of $\mathscr{C}_f^1$ with respect to $\|\cdot\|_{\mathbf{B}}$.

Lemma 2.1.8. *$\exists\, C < \infty$ such that $\|h\|_{\mathbf{S}} \leq C\|h\|_{\mathbf{T}}\ \forall\ h \in \mathbf{B}$.*

Proof in Section 3.1.

Since $\|h\|_{\mathbf{L^1}} \leq \|h\|_{\mathbf{S}}$ the above implies that $\|h\|_{\mathbf{L^1}} \leq C\|h\|_{\mathbf{B}}$ for all $h \in \mathbf{B}$. In particular $\mathbf{B} \subseteq \mathbf{L^1}(\Omega)$. Actually we can improve this embedding result.

Lemma 2.1.9. *$\exists\, C < \infty$ such that $\|h\|_{\mathbf{BV}(\Omega)} \leq C\|h\|_{\mathbf{B}}\ \forall\ h \in \mathbf{B}$.*

Proof in Section 3.1.

Lemma 2.1.10. *$\{h \in \mathbf{B} : \|h\|_{\mathbf{B}} \leq 1\}$ is relatively compact in $\mathbf{L^1}(\Omega)$.*

Proof. Since it is known [29] that $\{h \in \mathbf{B} : \|h\|_{\mathbf{BV}(\Omega)} \leq 1\}$ is relatively compact in $\mathbf{L^1}(\Omega)$ this lemma follows from Lemma 2.1.9. □

Recall the details of strongly-continuous one-parameter semigroups described in Section 1.2.1. In particular the *generator* of a strongly-continuous one-parameter semigroup $\mathscr{L}_t$ is the operator $Zh := \lim_{t\to 0} \frac{1}{t}(\mathscr{L}_t h - h)$ the domain of Z being the set of $h \in \mathbf{B}$ for which the limit exists. The main result is the following theorem. Recall that *tempered-twist suspension flows* were defined at the start of Section 2.1 and that $\lambda > 0$ was defined in (2.1.10).

The Main Theorem. *Let $\phi_t : \Omega \to \Omega$ be a* tempered-twist suspension flow*. Then the family $\{\mathscr{L}_t\}_{t\geq 0}$ restricted to $\mathbf{B}$ forms a strongly-continuous one-parameter semigroup on the Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$. Letting Z be the associated generator, then* $\mathrm{Spec}(Z)$ *is contained within $\{z \in \mathbb{C} : \Re(z) \leq 0\}$. Furthermore, within the strip $\{z \in \mathbb{C} : -\lambda < \Re(z) \leq 0\}$ then* Spec(Z) *consists only of isolated eigenvalues of finite multiplicity.*

Proof in Chapter 3.

We remark that the above theorem corresponds to the combination of Step 1 and Step 2 as described in Section 1.2.1.

2.2. Discussion of Results

We make here some additional remarks to extend what was already discussed in Chapter 1 as relates to the above result. We recall that the return map f was not assumed to be Markov. Additionally the quantity $|r'|$ was allowed to be unbounded. Since $\operatorname{ess\,sup} |f'| < \infty$ this means that these suspension flows have unbounded twist. Indeed the suspension flow is allowed to be one which has unbounded twist for any iterate of the suspension and so is not uniformly hyperbolic as described at length in Section 1.3 of Chapter 1. Related suspensions flows have been studied by Baladi and Vallée [13]. They allow the return time functions to be unbounded but require bounded twist and also require the return map to be Markov. Their results have been applied by Obayashi [63] to certain suspension flows where the return map isn't Markov, but as before the systems are assumed to have bounded twist. Ideally we would eventually like a general theory which encompasses both hyperbolic flows which are associated to non-vanishing vector fields and ones with equilibrium points. There are many problems that must be solved in order to achieve this goal, including understanding how to deal with the unbounded twist. In this work we have successfully undertaken an important first step towards this goal.

One immediate consequence of the above result concerns the Laplace transform of the correlation function. The result on the spectrum of the generator implies that the Laplace transform of the correlation function (1.1.1) is holomorphic on $\{\Re(z) > 0\}$ and meromorphic on $\{\Re(z) > -\lambda\}$. See [17, Remark 2.3] for details. We emphasise that this result concerning the poles of the Laplace transform of the correlation function has already been proven by Baladi [5] and in greater generality. However this work by Baladi does not show the connection between the poles of the Laplace transform of correlation and the eigenvalues of the generator of the flow. The knowledge of the eigenvalues of the generator of the flow is a significantly stronger and more useful result. For example perturbation theory is messy when applied to the suspension flow setting and very clean when applied to a study of generator of the flow acting on a particular Banach space point of view. This is clearly demonstrated by comparing the work of Ruelle [75] with the joint work with Liverani [17], both of which concern the differential of the SRB measure of a perturbed Anosov flow.

In a different direction we make a remark concerning the benefit of studying a system without coding. For flows in higher dimensions there is a problem of a lack of regularity of the stable and unstable foliations. For example restricted to the case of a smooth Anosov flow then in higher dimensions the foliations are not known to differentiable but only Holdër [68]. Coding may be fine for flows in three dimensions but an approach along the lines described in this work is important for obtaining results in greater generality and for obtaining optimal results.

At this stage we emphasise that the key result here is the construction of the Banach space on which such systems may be studied. To allow the study of flows with unbounded twist the anisotropic nature of the space was key, the greater regularity in the flow direction balancing with possibility of the flow to map transversal directions very close to the flow direction. We consider the above setting as a model for the centre-unstable bundle of a singular hyperbolic flow. The hope is that combining with moderately well understood techniques, as discussed previously, for dealing with the stable direction then such flows could be studied directly with no coding. The benefits of such an approach including the ability to study the behaviour of statistical properties under perturbations as demonstrated in joint work with Liverani [17] on Anosov flows and precise description of quantities like the decay of correlation as demonstrated by Tsujii [85], again for contact Anosov flows.

3. Unbounded Twist Flows - Proofs

This chapter contains the proof of The Main Theorem which was sated at the end of Section 2.1 and also the proofs of Lemma 2.1.8 and Lemma 2.1.9. The first section, namely Section 3.1, contains the proofs of Lemma 2.1.8 and Lemma 2.1.9. The second section, namely Section 3.2, contains an overview of the proof of The Main Theorem. The remaining sections contain proofs of results which are required in the overview.

3.1. The Space of Densities

In this section we prove Lemma 2.1.8 and Lemma 2.1.9.

Remark 3.1.1. Throughout we use C to represent some constant dependent only on the system. The actual value of which will vary from place to place. When the value of the constant depends on some parameter we write the parameter as a subscript. For example $C_{a,b}$ for a constant that is dependent on the system and also parameters a, b.

Proof of Lemma 2.1.8. It is sufficient to prove that there exists $C < \infty$ such that $\|h\|_{\mathbf{S}} \leq C\|h\|_{\mathbf{T}}$ for all $h \in \mathscr{C}^1_f$, the result then follows for all $h \in \mathbf{B}$. For any $h \in \mathscr{C}^1_f$ and $x_1 \in [0,1]$ let $\hat{h}(x_1) := \int_0^1 |h|(x_1, x_2)\ dx_2$. Recall that $\{I_i\}_{i=1}^{N_I}$ denotes the connected components of $[0,1] \setminus \mathcal{S}$. This means that $\|h\|_{\mathbf{S}} \leq \sup_i \sup_{x_1 \in I_i} \hat{h}(x_1)$. Fix for the moment some $i \in \{1, \ldots, N_I\}$, write $I_i = (a,b)$ and let $\mathscr{E} := \{\eta \in \mathscr{C}^1(I_i, \mathbb{C}), \eta(a) = 0, \int_a^b |\eta(x_1)|\ dx_1 \leq 1\}$. Notice that for all $h \in \mathscr{C}^1_f$ then

$$\sup_{x_1 \in I_i} \hat{h}(x_1) = \sup_{\eta \in \mathscr{E}} \left| \int_a^b \hat{h}(x_1)\eta(x_1)\ dx_1 \right|. \tag{3.1.1}$$

For each $\eta \in \mathscr{E}$ let $\tilde{\eta}(x_1, x_2) := \int_a^{x_1} \gamma^{-1}(s_1, x_2)\eta(s_1)\ ds_1$ if $x_1 \in (a,b)$ and $\tilde{\eta}(x_1, x_2) = 0$ if $x_1 \in [0,1] \setminus (a,b)$. By definition this function satisfies $(\gamma \cdot \partial_{x_1}\tilde{\eta})(x_1, x_2) = \eta(x_1)$ for every

$(x_1, x_2) \in \Omega$ and that $\tilde{\eta} \in \mathscr{C}^1(\Omega, \mathbb{C})$ and that $\sup|\tilde{\eta}| \leq \|\gamma^{-1}\|_{\mathbf{L}^\infty} \int_a^b |\eta(x_1)| \, dx_1 < \infty$. This means that

$$\sup_{x_1 \in I_i} \hat{h}(x_1) = \sup_{\eta \in \mathscr{E}} \left| \int_a^b \int_0^1 |h|(x_1, x_2)\eta(x_1) \, dx_2 dx_1 \right| \leq C\|h\|_{\mathbf{T}}.$$

This completes the proof of Lemma 2.1.8. □

Before proving Lemma 2.1.9 we first introduce two further lemmas which will be required.

Lemma 3.1.2. *There exists $C < \infty$ such that $\forall \ x_1 \in [0,1] \setminus \mathcal{S}$*

$$\sup_{x_2 \in [0,1]} |\partial_{x_1}\tau(x_1, x_2)| \leq C \left|r'(x_1)\right|.$$

Proof. Since $\tau(x_1, x_2) = \int_0^{x_2} v(x_1, s_2)^{-1} \, ds_2$ then

$$\partial_{x_1}\tau(x_1, x_2) = \int_0^{x_2} (\partial_{x_1}v \cdot v^{-2})(x_1, s_2) \, ds_2.$$

By assumption the speed function satisfies $|\partial_{x_1}v(x_1, s_2)| \leq C|r'(x_1)|$. □

Lemma 3.1.3. *Holds*

$$\int_0^1 \operatorname*{ess\,sup}_{x_2 \in [0,1]} |\partial_{x_1}\gamma(x_1, x_2)| \, dx_1 < \infty$$

Proof. Since by definition $\gamma(x_1, x_2) = \exp[-\ln f'(x_1)\frac{\tau(x_1,x_2)}{r(x_1)}]$ and also $(\ln f'(x_1))' = \frac{f''(x_1)}{f'(x_1)}$ then

$$\partial_{x_1}\gamma(x_1, x_2) = -\gamma(x_1, x_2) \left[\frac{f''(x_1)}{f'(x_1)} \frac{\tau(x_1, x_2)}{r(x_1)} - \ln f'(x_1) \partial_{x_1} \frac{\tau(x_1, x_2)}{r(x_1)} \right].$$

Since $\operatorname{ess\,sup}|f''| < \infty$, $\operatorname{ess\,inf}|f'| > 1$, $\operatorname{ess\,sup}|f'| < \infty$, $\tau(x_1, x_2) \leq r(x_1)$ and also $\operatorname{ess\,sup}|\gamma| < \infty$ then there exists $C < \infty$ such that

$$|\partial_{x_1}\gamma(x_1, x_2)| \leq C + C \left| \partial_{x_1} \frac{\tau(x_1, x_2)}{r(x_1)} \right|$$

for all $x_1 \in [0,1] \setminus \mathcal{S}$, $x_2 \in [0,1]$. As $\partial_{x_1} \frac{\tau(x_1,x_2)}{r(x_1)} = \frac{\partial_{x_1}\tau(x_1,x_2)}{r(x_1)} - r'(x_1)\frac{\tau(x_1,x_2)}{r(x_1)^2}$ and then

$$\operatorname*{ess\,sup}_{x_2\in[0,1]} \left|\partial_{x_1} \frac{\tau(x_1,x_2)}{r(x_1)}\right| \le C \operatorname*{ess\,sup}_{x_2\in[0,1]} |\partial_{x_1}\tau(x_1,x_2)| + C\left|r'(x_1)\right|$$

for all $x_1 \in [0,1] \setminus \mathcal{S}$. Using now the estimate from Lemma 3.1.2 we know that $\operatorname{ess\,sup}_{x_2\in[0,1]} |\partial_{x_1} \frac{\tau(x_1,x_2)}{r(x_1)}| \le C|r'(x_1)|$. We conclude by noticing that since $r' \in \mathbf{L^q}([0,1])$ then $r' \in \mathbf{L^1}([0,1])$. $\square$

Proof of Lemma 2.1.9. Recall that the total variation of $h \in \mathbf{L^1}(\Omega)$ is defined as

$$V(h,\Omega) := \sup\left\{\left|\int_\Omega h\cdot\partial_{x_i}\eta \, dx\right| : \eta \in \mathscr{C}^1_0(\Omega,\mathbb{C}), \|\eta\|_{\mathbf{L^1}} \le 1, i \in \{1,2\}\right\},$$

where $\mathscr{C}^1_0(\Omega,\mathbb{C}^2)$ denotes the $\mathscr{C}^1$ functions of support contained within $(0,1)\times(0,1)$. Fix some $h \in \mathscr{C}^1_f$. Since the support of η is contained within $(0,1)\times(0,1)$ and $x_2 \mapsto h(x_1,x_2)$ is $\mathscr{C}^1$ for each $x_1 \in [0,1]\setminus\mathcal{S}$ then $\int_\Omega h\cdot\partial_{x_2}\eta \, dx = -\int_\Omega \partial_{x_2}h\cdot\eta \, dx$. Additionally $\int_\Omega \partial_{x_2}h\cdot\eta\, dx = \int_\Omega \partial_{x_2}(v\cdot h)\cdot v^{-1}\cdot\eta\, dx - \int_\Omega h\cdot v^{-1}\cdot\partial_{x_2}v\cdot\eta\, dx$. This means that

$$\left|\int_\Omega h\cdot\partial_{x_2}\eta\, dx\right| \le \frac{1}{\inf v}\left(\|h\|_{\mathbf{F}} + \operatorname{ess\,sup}|\partial_{x_2}v|\,\|h\|_{\mathbf{L^1}}\right). \tag{3.1.2}$$

Now we consider the x_1 direction. Notice that

$$\int_\Omega h\cdot\partial_{x_1}\eta\, dx = \int_\Omega h\cdot\gamma^{-1}\cdot\partial_{x_1}(\gamma\cdot\eta)\, dx - \int_\Omega h\cdot\partial_{x_1}\gamma\cdot\gamma^{-1}\cdot\eta\, dx$$

and that

$$\int_\Omega |h\cdot\partial_{x_1}\gamma\cdot\eta|\, dx \le \int_0^1 \operatorname*{ess\,sup}_{x_2\in[0,1]}|\partial_{x_1}\gamma|\, dx_1\|h\|_{\mathbf{S}}.$$

Therefore, recalling Lemma 3.1.3, holds

$$\left|\int_\Omega h\cdot\partial_{x_1}\eta\, dx\right| \le \operatorname{ess\,sup}|\gamma^{-1}|\left(\|h\|_{\mathbf{T}} + C\|h\|_{\mathbf{S}}\right). \tag{3.1.3}$$

Since $x_2 \mapsto \tau(x_1,x_2)$ is monotone then for all $x_1 \in [0,1]\setminus\mathcal{S}$, $x_2 \in [0,1]$ holds $\gamma^{-1}(x_1,x_2) = \exp[\lambda(x_1)\cdot\tau(x_1,x_2)] \le |f'(x_1)|$. And so $\operatorname{ess\,sup}|\gamma^{-1}| < \infty$. Combining the estimates of

(3.1.2) and (3.1.3) we have shown there exists $C < \infty$ such that $\|h\|_{\mathbf{BV}(\Omega)} = V(h, \Omega) + \|h\|_{\mathbf{L}^1} \le C\|h\|_{\mathbf{B}}$ for all $h \in \mathscr{C}^1_f$. By density this estimate extends to all $h \in \mathbf{B}$. □

3.2. Proof of The Main Theorem

In this section we prove The Main Theorem modulo Lemma 3.2.4, Proposition 3.2.7 and Proposition 3.2.9 which are proved in the following sections. We start by introducing further notation to describe the flow. For all $x_1 \in [0, 1]$ and $n \in \mathbb{N}$ let

$$r_n(x_1) := \sum_{i=0}^{n-1} r \circ f^i(x_1).$$

This is the time taken to travel n laps from $(x_1, 0)$ to $(f^n(x_1), 0)$. We define the lap number $n_t(x_1, x_2)$ as the unique $n \in \mathbb{N}$ such that

$$r_n(x_1) \le t + \tau(x_1, x_2) < r_{n+1}(x_1).$$

This is the number of times the point (x_1, x_2) passes the roof before time t. Using this notation and recalling the definition of the flow (2.1.3) we now write an explicit formula for the flow for all time. Suppose $t \ge 0$ and $(x_1, x_2) \in \Omega$. Let $n_t = n_t(x_1, x_2)$ which we note is piecewise constant on Ω. Then

$$\phi_t(x_1, x_2) = \left(x_1^{(t)}(x_1), x_2^{(t)}(x_1, x_2)\right) \tag{3.2.1}$$

where we define

$$\begin{aligned} x_1^{(t)}(x_1) &:= f^{n_t}(x_1), \\ x_2^{(t)}(x_1, x_2) &:= \varphi_{s^{(t)}(x_1,x_2)}(f^{n_t}(x_1), 0), \\ s^{(t)}(x_1, x_2) &:= t - r_{n_t}(x_1) + \tau(x_1, x_2). \end{aligned}$$

Lemma 3.2.1. *Suppose* $t \ge 0$. *Then* $\det D\phi_t = (f^{n_t})' \cdot v \circ \phi_t \cdot v^{-1}$.

Proof of Lemma 3.2.1. Notice that $\det D\phi_t = (D\phi_t)_{11}(D\phi_t)_{22}$ since $(D\phi_t)_{12} = 0$. Differentiating the formula for the flow (3.2.1) we obtain $(D\phi_t)_{11}(x_1,x_2) = (f^{n_t})'(x_1)$ and $(D\phi_t)_{22}(x_1,x_2) = v(x_1^t,x_2^t)\cdot v(x_1,x_2)^{-1}$. In the second case we used that $\frac{d}{dt}\varphi_s(x_1,0) = v(x_1,\varphi_s(x_1,x_2))$ and that $\partial_{x_2}\tau(x_1,x_2) = v(x_1,x_2)^{-1}$. $\square$

We substitute the formula obtained in Lemma 3.2.1 into the definition (2.1.7) of $\mathscr{L}_t$. Suppose $h \in \mathscr{C}^1_f$ and $t \geq 0$. Then

$$\mathscr{L}_t h(y) = v(y)^{-1} \sum_{\phi_t(x)=y} \frac{v\cdot h}{|(f^{n_t})'|}(x). \tag{3.2.2}$$

We introduce some additional notation related to the inverse branches of the return map. Recall that $f \in \mathscr{C}^2\left([0,1]\setminus\mathcal{S},[0,1]\right)$ where $\mathcal{S}$ is a finite set of points. Let $\{I_i\}_{i=1}^{N_I}$ denote the connected components of $[0,1]\setminus\mathcal{S}$. By assumption f is $\mathscr{C}^2$ and $|f'|>1$ and therefore monotone on each I_i. Also $|f'|$ is uniformly bounded from above. This implies that f admits a continuous extension to the closed interval $\bar{I}_i$ with continuous inverse. For each i the map $f: I_i \to [0,1]$ is not necessarily onto. We let

$$\mathcal{S}' := \left\{\overline{f(I_i)}\setminus f(I_i)\right\}_{i=1}^{N_I}. \tag{3.2.3}$$

This is the set of images of endpoints of the intervals I_i under the (continuous extension of) the map f for all $i\in\{1,2,\ldots,N_I\}$. We then let $\{K_k\}_{k=1}^{N_J}$ denote the family of connected components of $[0,1]\setminus\mathcal{S}'$. We define the following index set for the branches

$$\mathscr{I}_k := \{i\in\{1,2,\ldots,N_I\} : f(I_i)\supseteq K_k\}$$

For each $i\in\{1,2,\ldots,N_I\}$ let $F_i : f(I_i)\to[0,1]$ be defined by

$$F_i := \left(f|_{I_i}\right)^{-1}. \tag{3.2.4}$$

It holds that for each k, $i\in\mathscr{I}_k$ then $F_i\in\mathscr{C}^1(K_k,I_i)$ and for each k, $x_1\in K_k$ then $f^{-1}(x_1) = \{F_i(x_1)\}_{i\in\mathscr{I}_k}$.

Recall that we define $\mathscr{P}_f$ above (2.1.8).

Lemma 3.2.2. *Suppose that $t \in (0, \delta)$ and that $h : \Omega \to \mathbb{C}$. Let*

$$\Omega^t_- := [0,1] \times [0,t) \quad \textit{and} \quad \Omega^t_+ := [0,1] \times [t,1].$$

Then for all $(x_1, x_2) \in \Omega$ holds

$$\mathscr{L}_t h(x_1,x_2) = \begin{cases} \mathscr{P}_f h\,(x_1, x_2 - t + 1) & \textit{if } (x_1,x_2) \in \Omega^t_- \\ (v \cdot h)\,(x_1, \varphi_{-t}(x_1,x_2)) \cdot v(x_1,x_2)^{-1} & \textit{if } (x_1,x_2) \in \Omega^t_+. \end{cases}$$

Proof. Fix $t \in (0, \delta)$. Let $\Delta_- := [0,1] \times [1-t, 1)$ and $\Delta_+ := [0,1] \times [0, 1-t)$. We recall the definition of the flow (2.1.3). Take first $(x_1, x_2) \in \Delta_-$. We write $t = t_1 + t_2$ where $t_1 = 1 - x_2$ and $t_2 = t + x_2 - 1$. Recall that the speed of the flow is equal to 1 both δ close to the base and δ close to the roof. This means that $\varphi_{t_1}(x_1, x_2) = t_1 + x_2 = 1$. Also it means that $\varphi_{t_2}(f(x_1), 0) = t_2 = t + x_2 - 1$. For $(x_1, x_2) \in \Delta_+$ we merely write the definition of the flow. We have

$$\phi_t(x_1,x_2) = \begin{cases} (f(x_1), x_2 + t - 1) & \text{if } (x_1,x_2) \in \Delta_- \\ (x_1, \varphi_t(x_1,x_2)) & \text{if } (x_1,x_2) \in \Delta_+ \end{cases} \tag{3.2.5}$$

Both the map $\phi_t : \Delta_+ \to \Omega^t_+$ and the map $\phi_t : \Delta_- \to \Omega^t_-$ are onto. Using the above notation for the inverse branches of the return map we have

$$\phi_t^{-1}(x_1,x_2) = \begin{cases} \{(F_i(x_1), x_2 - t + 1)\}_{i \in \mathscr{I}_k} & \text{if } (x_1,x_2) \in \Omega^t_- \\ \{(x_1, \varphi_{-t}(x_1,x_2))\} & \text{if } (x_1,x_2) \in \Omega^t_+. \end{cases} \tag{3.2.6}$$

Recalling Lemma 3.2.1 it holds

$$\det D\phi_t(x_1,x_2)^{-1} = \begin{cases} 1/f'(x_1) & \text{if } (x_1,x_2) \in \Delta^t_- \\ v(x_1,x_2) \cdot v(x_1, \varphi_t(x_1,x_2))^{-1} & \text{if } (x_1,x_2) \in \Delta^t_+. \end{cases}$$

Substituting the above and (3.2.6) into the definition (2.1.7) of the transfer operator gives the lemma. $\square$

Remark 3.2.3. The return map $f : [0,1] \to [0,1]$ is a piecewise $\mathscr{C}^2$ expanding map with finite branches and $\operatorname{ess\,sup}|f''| < \infty$ and so there exists a unique invariant measure which is absolutely continuous with respect to Lebesgue measure. Let μ_f denote the density of this measure. (This means that $\mu_f(x_1) = \sum_{y_1 \in f^{-1}(x_1)} \mu_f(y_1) \cdot |f'(y_1)|^{-1}$.) For all $(x_1, x_2) \in \Omega$ let

$$\mu(x_1, x_2) := \mu_f(x_1) \cdot v(x_1, x_2)^{-1}.$$

In view of Lemma 3.2.2 it is clear that $\mathscr{L}_t \mu = \mu$ for all $t \geq 0$. In other words μ is the density of an invariant measure for the flow ϕ_t. Also we remark that $\|\mu\|_{\mathbf{F}} = 0$.

It is necessary to check some basic and unsurprising properties of $\mathscr{L}_t$ acting on $\mathscr{C}^1_f$.

Lemma 3.2.4. *For all $t \geq 0$ then $\mathscr{L}_t(\mathscr{C}^1_f) \subset \mathscr{C}^1_f$. Moreover for all $h \in \mathscr{C}^1_f$ then $\|\mathscr{L}_t h - h\|_{\mathbf{B}} \to 0$ as $t \to 0$.*

Proof in Section 3.3.

Lemma 3.2.5. *There exists $C < \infty$ such that*

$$\|\mathscr{L}_t h\|_{\mathbf{S}} \leq C e^{Ct} \|h\|_{\mathbf{S}} \quad \textit{for all } h \in \mathscr{C}^1_f,\ t \geq 0. \tag{3.2.7}$$

Proof. First we estimate $\|\mathscr{L}_t h\|_{\mathbf{S}}$ for $t \in (0, \delta)$. Fix for the moment $h \in \mathscr{C}^1_f$. Recall that associated to h is a finite set $\sigma \subset [0,1]$. Let $x_1 \in [0,1] \setminus \sigma$. Let $s = \varphi_t(x_1, 0)$ and write $\int_0^1 |\mathscr{L}_t h|\, dx_2 = \int_s^1 |\mathscr{L}_t h|\, dx_2 + \int_0^s |\mathscr{L}_t h|\, dx_2$. We use (3.2.2). When $x_2 \in [s, 1]$ then

$$\mathscr{L}_t h(x_1, x_2) = (v \cdot h)(x_1, \varphi_{-t}(x_1, x_2)) \cdot v(x_1, x_2)^{-1}.$$

Using this and performing a change of variables $y_2 = \varphi_{-t}(x_1, x_2)$ and so $\frac{dy_2}{dx_2}(x_1, x_2) = \partial_{x_2} \varphi_{-t}(x_1, x_2) = v(x_1, \varphi_{-t}(x_1, x_2)) \cdot v(x_1, x_2)^{-1}$ holds

$$\int_s^1 |\mathscr{L}_t h|(x_1, x_2)\, dx_2 = \int_0^{\varphi_{-t}(x_1, 1)} |h|(x_1, y_2)\, dy_2 \leq \|h\|_{\mathbf{S}}.$$

Now we consider the other piece $\int_0^s |\mathscr{L}_t h|\, dx_2$. Since $v(x_1, x_2) = 1$ whenever $x_2 \in [0, \delta] \cup$

$[1-\delta, 1]$ then

$$\int_0^s |\mathscr{L}_t h|(x_1, x_2)\ dx_2 = \sum_{y_1 \in f^{-1}(x_1)} |f'(y_1)|^{-1} \int_{1-s}^1 |h|(y_1, x_2 - t + 1)\ dx_2.$$

This is bounded by $\sum_{y_1 \in f^{-1}(x_1)} |f'(y_1)|^{-1} \|h\|_{\mathbf{S}}$. The requirement that $\mathcal{S}$ is a finite set and $|f'| > 1$ implies that $\sum_{y_1 \in f^{-1}(x_1)} < \infty$. We have shown that there exists $C < \infty$ such that $\|\mathscr{L}_t h\|_{\mathbf{S}} \leq C\|h\|_{\mathbf{S}}$ for all $t \in (0, \delta)$, $h \in \mathscr{C}_f^1$. The lemma then follows from the semigroup property of $\mathscr{L}_t$. $\square$

Lemma 3.2.6. *$\exists\, C < \infty$ such that $\|\mathscr{L}_t h\|_{\mathbf{F}} \leq Ce^{Ct}\|h\|_{\mathbf{F}}$ for all $t \geq 0$, $h \in \mathscr{C}_f^1$.*

Proof. Let $h \in \mathscr{C}_f^1$ and $t \geq 0$. First we notice that

$$\partial_{x_2}(v \cdot \mathscr{L}_t h) = \mathscr{L}_t(\partial_{x_2}(v \cdot h)). \tag{3.2.8}$$

This is immediate from (3.2.2) differentiating with respect to x_2 and also noting that $\partial_{x_2} f^{n_t}(x) = 0$. This means that $\|\mathscr{L}_t h\|_{\mathbf{F}} = \|\partial_{x_2}(v \cdot \mathscr{L}_t h)\|_{\mathbf{S}} = \|\mathscr{L}_t(\partial_{x_2}(v \cdot h))\|_{\mathbf{S}}$. We conclude using Lemma 3.2.5. $\square$

Recall the quantity $\lambda > 0$ from (2.1.10). For the transversal component of the norm we have the following result.

Proposition 3.2.7. *For all $\ell \in (0, \lambda)$ there exists $C_\ell < \infty$ such that $\forall\ t \geq 0, h \in \mathscr{C}_f^1$*

$$\|\mathscr{L}_t h\|_{\mathbf{T}} \leq C_\ell e^{-\ell t}\|h\|_{\mathbf{B}} + C_\ell e^{C_\ell t}\left(\|\partial_{x_2}(h \cdot v)\|_{\mathbf{L}^1} + \|h\|_{\mathbf{L}^1}\right).$$

Proof in Section 3.4.

The above estimate is stronger than is needed for our present purposes. We will use the full strength of this estimate only later in the proof of Proposition 3.2.9.

Proposition 3.2.8. *The family of operators $\{\mathscr{L}_t\}_{t\geq 0}$ restricted to $\mathbf{B}$ forms a strongly-continuous one-parameter semigroup. Moreover there exists $C < \infty$, $\alpha < \infty$ such that*

$$\|\mathscr{L}_t\|_{\mathbf{B}} \leq Ce^{\alpha t} \quad \textit{for all } t \geq 0. \tag{3.2.9}$$

Proof. Notice that $\|\partial_{x_2}(h \cdot v)\|_{\mathbf{L^1}} \leq \|\partial_{x_2}(h \cdot v)\|_{\mathbf{S}} = \|h\|_{\mathbf{F}}$. Using this and summing together the estimates from Lemma 3.2.6 and Proposition 3.2.7 also also noting that $\|h\|_{\mathbf{L^1}} \leq C\|h\|_{\mathbf{B}}$ by Lemma 2.1.8, implies the existence of $\alpha < \infty$ such that $\|\mathscr{L}_t h\|_{\mathbf{B}} \leq Ce^{\alpha t}\|h\|_{\mathbf{B}}$ for all $t \geq 0$, $h \in \mathscr{C}^1_f$. Using also Lemma 3.2.4 we have shown that

$$\mathscr{L}_t h \in \mathbf{B} \quad \text{and} \quad \|\mathscr{L}_t h\|_{\mathbf{B}} \leq Ce^{\alpha t}\|h\|_{\mathbf{B}} \quad \forall\, t \geq 0, h \in \mathbf{B}. \tag{3.2.10}$$

It remains to show the strong continuity. This also follows from Lemma 3.2.4. Take a sequence $\{h_n\}_{n=1}^{\infty}$ such that $h_n \in \mathscr{C}^1_f$ for each n and that $h_n \to h$ as $n \to \infty$ in the $\|\cdot\|_{\mathbf{B}}$ norm. Notice that for each n and for each $t \geq 0$

$$\|\mathscr{L}_t h - h\|_{\mathbf{B}} \leq \|\mathscr{L}_t h - \mathscr{L}_t h_n\|_{\mathbf{B}} + \|\mathscr{L}_t h_n - h_n\|_{\mathbf{B}} + \|h_n - h\|_{\mathbf{B}}. \tag{3.2.11}$$

The boundedness (3.2.10) implies that $\|\mathscr{L}_t h_n - \mathscr{L}_t h\|_{\mathbf{B}} \to 0$ as $n \to \infty$. Using the estimate from Lemma 3.2.4 for $\|\mathscr{L}_t h_n - h_n\|_{\mathbf{B}}$ and taking n sufficiently large as t becomes small we have shown that $\|\mathscr{L}_t h - h\|_{\mathbf{B}} \to 0$ as $t \to 0$. $\square$

Now that we have shown $\mathscr{L}_t$ to be a strongly continuous one parameter semigroup we can talk about the generator Z of this semigroup. The above bound immediately implies that $\mathrm{Spec}(Z) \subseteq \{z \in \mathbb{C} : \Re(z) \leq \alpha\}$. Recall that, for every $z \in \mathrm{Res}(Z)$, the *resolvent operator* of Z is defined by $R_z := (z\mathbf{Id} - Z)^{-1}$. Recall $\alpha < \infty$ as given in Proposition 3.2.8. By standard theory of strongly continuous one parameter semigroups [22] Proposition 3.2.8 implies that

$$R_z^n h = \frac{1}{(n-1)!}\int_0^{\infty} t^{n-1}e^{-zt}\mathscr{L}_t h \,\mathrm{dt}, \tag{3.2.12}$$

for all $\Re(z) > \alpha$, $h \in \mathbf{B}$. This integral formula for the resolvent is key for the next stage of the argument.

Proposition 3.2.9. *Suppose $\Re(z) > \alpha$, $\ell \in (0, \lambda)$. Then there exists $C_{z,\ell} < \infty$ such that for all $h \in \mathbf{B}$, $n \in \mathbb{N}$ holds*

$$\|R_z^n h\|_{\mathbf{B}} \leq C_{z,\ell}(\Re(z) + \ell)^{-n}\|h\|_{\mathbf{B}} + C_{z,\ell}\Re(z)^{-n}\|h\|_{\mathbf{L^1}}.$$

Proof in Section 3.5.

By relatively standard arguments, which we include in Section 3.6 for completeness, and which are based on Nussbaum's formula for the essential spectral radius [62], we then show that taken together Proposition 3.2.9 and the compactness property of Lemma 2.1.10 imply the spectral conclusions of The Main Theorem.

3.3. Transfer Operator

In this section we prove Lemma 3.2.4.

Lemma 3.3.1. *Suppose that* $x_1 \in [0,1] \setminus \mathcal{S}$*,* $x_2 \in [0,1]$ *and* $-\tau(x_1,0,x_2) \le t < \tau(x_1,x_2,1)$*. Let* $\tilde{x}_2 := \varphi_t(x_1,x_2)$*. The following equalities hold.*

$$\partial_{x_1}\varphi_t(x_1,x_2) = -v(x_1,\tilde{x}_2)\cdot\partial_{x_1}\tau(x_1,x_2,\tilde{x}_2) \tag{3.3.1}$$

$$\partial_{x_2}\varphi_t(x_1,x_2) = \frac{v(x_1,\tilde{x}_2)}{v(x_1,x_2)}. \tag{3.3.2}$$

Proof. Consider the function $(x_1,x_2,\tilde{x}_2) \mapsto \varphi_{\tau(x_1,x_2,\tilde{x}_2)}(x_1,x_2)$. By definition of the time function and the local flow holds $\varphi_{\tau(x_1,x_2,\tilde{x}_2)}(x_1,x_2) = \tilde{x}_2$ and so differentiating with respect to x_1 the following holds.

$$\partial_t\varphi_{\tau(x_1,x_2,\tilde{x}_2)}(x_1,x_2)\cdot\partial_{x_1}\tau(x_1,x_2,\tilde{x}_2) + \partial_{x_1}\varphi_{\tau(x_1,x_2,\tilde{x}_2)}(x_1,x_2) = 0.$$

Recall that $\partial_t\varphi_t(x_1,\tilde{x}_2) = v(x_1,\varphi_t(x_1,\tilde{x}_2))$ and so $\partial_t\varphi_{\tau(x_1,x_1,\tilde{x}_2)}(x_1,x_2) = v(x_1,\tilde{x}_2)$. Using this in the above equation means that

$$\partial_{x_1}\varphi_{\tau(x_1,x_2,\tilde{x}_2)}(x_1,x_2) = -v(x_1,\tilde{x}_2)\cdot\partial_{x_1}\tau(x_1,x_2,\tilde{x}_2). \tag{3.3.3}$$

This proves (3.3.1). Now we consider (3.3.2). Recall that $\partial_t\varphi_t(x_1,x_2) = v(x_1,\varphi_t(x_1,x_2))$.

This means that

$$\begin{aligned}\frac{v(x_1,\varphi_t(x_1,x_2))}{v(x_1,x_2)} &= \lim_{\epsilon\to 0}\frac{\varphi_{t+\epsilon}(x_1,x_2)-\varphi_t(x_1,x_2)}{\varphi_\epsilon(x_1,x_2)-x_2}\\ &= \lim_{\epsilon\to 0}\frac{\varphi_t(x_1,\varphi_\epsilon(x_1,x_2))-\varphi_t(x_1,x_2)}{\varphi_\epsilon(x_1,x_2)-x_2}\\ &= \lim_{h\to 0}\frac{\varphi_t(x_1,x_2+h)-\varphi_t(x_1,x_2)}{h} = \partial_{x_2}\varphi_t(x_1,x_2).\end{aligned}$$

This completes the proof of (3.3.2). □

3.3.1. Invariance

It suffices to prove the invariance for $t \in (0, \frac{\delta}{2})$. The full result then follows for all $t \geq 0$ by the semigroup property (2.1.4) of $\mathscr{L}_t$. Recall that to each $h \in \mathscr{C}^1_f$ is associated the finite set $\sigma \subset [0,1]$. Recall $\mathcal{S}$, the discontinuity set of the flow, and the image discontinuity set $\mathcal{S}'$ defined earlier (3.2.3). Let

$$\sigma' := \sigma \cup f(\sigma) \cup \mathcal{S} \cup \mathcal{S}'. \tag{3.3.4}$$

This is a finite set and in the following we will show that $\mathscr{L}_t h \in \mathscr{C}^1_f$ with respect to this set. Let I be one of the connected components of $[0,1] \setminus \sigma'$. It holds that $I \subseteq J_k$ for some k and so $F_i \in \mathscr{C}^1(I,[0,1])$ for all $i \in \mathscr{I}_k$.

Lemma 3.3.2. *$\mathscr{L}_t h \in \mathscr{C}^1([0,1]\setminus\sigma',\mathbb{C})$ for all $h \in \mathscr{C}^1_f$, $t \in (0,\frac{\delta}{2})$.*

Proof. We use the explicit formula for $\mathscr{L}_t h$ given in Lemma 3.2.2. As we will repeatedly write these formulae it is convenient to let

$$\begin{aligned} h^-(x_1,x_2) &:= \mathscr{P}_f h\,(x_1, x_2 - t + 1)\\ h^+(x_1,x_2) &:= (v\cdot h)\,(x_1,\varphi_{-t}(x_1,x_2))\cdot v(x_1,x_2)^{-1}.\end{aligned}$$

By inspection of the formulae and the definition of σ' it is clear that $h^+ \in \mathscr{C}^1([0,1]\setminus \sigma' \times [t,1],\mathbb{C})$ and $h^- \in \mathscr{C}^1([0,1]\setminus\sigma'\times[0,t),\mathbb{C})$. It remains to show that the two above formulae, h^- and h^+, are equal and both derivatives are equal on the line $x_2 = t$. At this point we will need to use the branch conditions which h satisfies. First notice that

$$\begin{aligned} h^{-}(x_1,t) &= \mathscr{P}_f h(x_1,1) \\ \nabla h^{-}(x_1,t) &= \nabla \mathscr{P}_f h(x_1,1). \end{aligned} \tag{3.3.5}$$

Now we consider h^{+}. Since v is equal to 1 within a δ sized region of the base and that $t \in (0, \frac{\delta}{2})$ it holds that $v(x_1,0) = v(x_1,t) = 1$ and $\nabla v(x_1,0) = \nabla v(x_1,t) = \binom{0}{0}$. We have

$$\begin{aligned} h^{+}(x_1,t) &= h(x_1,0) \\ \nabla h^{+}(x_1,t) &= \nabla h(x_1,0). \end{aligned} \tag{3.3.6}$$

Combining (3.3.5) and (3.3.6) with the branch conditions which h satisfies completes the proof of the lemma. $\square$

Lemma 3.3.3. *Suppose $h \in \mathscr{C}^1_f$, $t \in (0, \frac{\delta}{2})$. Then for all $x_1 \in [0,1] \setminus \sigma'_h$ holds $\mathscr{L}_t h(x_1,0) = \mathscr{P}_f \mathscr{L}_t h(x_1,1)$ and $\nabla \mathscr{L}_t h(x_1,0) = \nabla \mathscr{P}_f \mathscr{L}_t h(x_1,1)$.*

Proof. When $x_2 \in [0, \frac{\delta}{2})$ then $\mathscr{L}_t h(x_1,x_2) = h^{-}(x_1,x_2)$ and so

$$\mathscr{L}_t h(x_1,x_2) = \mathscr{P}_f h(x_1, x_2 - t + 1) \quad \forall\, x_2 \in [0, \tfrac{\delta}{2}). \tag{3.3.7}$$

When $x_2 \in (1 - \frac{\delta}{2}, 1]$ then $\varphi_{-t}(x_1,x_2) = x_2 - t$ and so $\gamma(x_1, \varphi_{-t}(x_1,x_2))^{-1} = \gamma(x_1,x_2)^{-1}$ and also $\gamma(x_1,x_2)^{-1} = f'(x_1)$. This means that $\mathscr{L}_t h(x_1,x_2) = h^{+}(x_1,x_2) = h(x_1, x_2 - t)$. In turn this means that

$$\mathscr{P}_f \mathscr{L}_t h(x_1,x_2) = \mathscr{P}_f h(x_1, x_2 - t) \quad \forall\, x_2 \in (1 - \tfrac{\delta}{2}, 1]. \tag{3.3.8}$$

Setting $x_2 = 0$ in (3.3.7) and setting $x_2 = 1$ in (3.3.8) gives $\mathscr{L}_t h(x_1,0) = \mathscr{P}_f h(x_1, -t + 1) = \mathscr{P}_f \mathscr{L}_t h(x_1,1)$ as required by the first branch condition. Differentiating and then setting $x_2 = 0$ in (3.3.7) and setting $x_2 = 1$ in (3.3.8) gives $\nabla \mathscr{L}_t h(x_1,0) = \nabla \mathscr{P}_f h(x_1, x_2 - t + 1) = \nabla \mathscr{P}_f \mathscr{L}_t h(x_1,1)$ as required by the second branch condition. $\square$

Lemma 3.3.4. *Suppose $h \in \mathscr{C}^1_f$, $t \in (0, \frac{\delta}{2})$. Then the function $(x_1,x_2) \mapsto \partial_{x_2}(v \cdot \mathscr{L}_t h)(x_1,x_2)$ admits a $\mathscr{C}^0$ extension to the closure of each connected component of* $([0,1]\setminus$

$\sigma') \times [0,1]$.

Proof. Suppose that $h \in \mathscr{C}^1_f$ and $t \in (0, \frac{\delta}{2})$. Using the formula given in Lemma 3.2.2 and differentiating holds

$$\partial_{x_2}(v \cdot \mathscr{L}_t h)(x_1,x_2) = \sum_{i \in \mathscr{I}_k} \partial_{x_2} h\left(F_i(x_1), x_2 - t + 1\right) / f'(F_i(x_1))$$

if $(x_1,x_2) \in \Omega^t_-$, and alternatively if $(x_1,x_2) \in \Omega^t_+$ then

$$\partial_{x_2}(v \cdot \mathscr{L}_t h)(x_1,x_2) = \partial_{x_2}(v \cdot h)\left(x_1, \varphi_{-t}(x_1,x_2)\right) \frac{v(x_1, \varphi_{-t}(x_1,x_2))}{v(x_1,x_2)}.$$

Since $h \in \mathscr{C}^1_f$ then the function $(x_1,x_2) \mapsto \partial_{x_2}(v \cdot h)(x_1,x_2)$ admits a $\mathscr{C}^0$ extension to the closure of each connected component of $([0,1] \setminus \sigma) \times [0,1]$. This, the above two formulae and the piecewise continuity of f', v and F_i concludes the proof of the lemma. $\square$

Before stating and proving the next lemma required for the invariance we isolate a result which will be used again later on.

Lemma 3.3.5. *$\partial_{x_1}\varphi_{-t} \in \mathbf{L^1}(\Omega)$ $\forall\, t \in (0, \frac{\delta}{2})$ and $\|\partial_{x_1}\varphi_{-t}\|_{\mathbf{L^1}} \to 0$ as $t \to 0$.*

Proof. According to Lemma 3.3.1 we have

$$\begin{aligned}\partial_{x_1}\varphi_{-t}(x_1,x_2) &= -v(x_1, \varphi_{-t}(x_1,x_2)) \partial_{x_1}\tau(x_1,x_2,\varphi_{-t}(x_1,x_2)) \\ &= v(x_1, \varphi_{-t}(x_1,x_2)) \int_{\varphi_{-t}(x_1,x_2)}^{x_2} v(x_1,y_2)^{-1}\, dy_2.\end{aligned}$$

Clearly $|\partial_{x_1}\varphi_{-t}| \to 0$ point-wise and recalling Lemma 3.1.2 there exists $C < \infty$ such that $\tau(x_1,x_2,\varphi_{-t}(x_1,x_2)) \leq C\,|r'(x_1)|$ for all $t \in (0, \frac{\delta}{2})$ and $r' \in \mathbf{L^q} \subset \mathbf{L^1}$ by assumption. This implies that $\|\partial_{x_1}\varphi_{-t}\|_{\mathbf{L^1}} \to 0$ as $t \to 0$. $\square$

Lemma 3.3.6. *$\partial_{x_1}(v \cdot \mathscr{L}_t h) \in \mathbf{L^1}(\Omega)$ for all $h \in \mathscr{C}^1_f$, $t \in (0, \frac{\delta}{2})$.*

Proof. Suppose that $h \in \mathscr{C}^1_f$ and $t \in (0, \frac{\delta}{2})$. Using the formula given in Lemma 3.2.2

and differentiating then the following holds. If $(x_1, x_2) \in \Omega_-^t$ then

$$\begin{aligned}\partial_{x_1}(v \cdot \mathscr{L}_t h)(x_1, x_2) = \sum_{i \in \mathscr{I}_k} \Big(&\partial_{x_1} h\left(F_i(x_1), x_2 - t + 1\right) \frac{1}{(f'(F_i(x_1)))^2} \\ &+ h\left(F_i(x_1), x_2 - t + 1\right) \frac{f''(F_i(x_1))}{(f'(F_i(x_1)))^3} \Big).\end{aligned} \tag{3.3.9}$$

Alternatively if $(x_1, x_2) \in \Omega_+^t$ then

$$\begin{aligned}\partial_{x_1}(v \cdot \mathscr{L}_t h)(x_1, x_2) = &\ \partial_{x_1}(v \cdot h)\left(x_1, \varphi_{-t}(x_1, x_2)\right) \\ &+ \partial_{x_2}(v \cdot h)\left(x_1, \varphi_{-t}(x_1, x_2)\right) \partial_{x_1}\varphi_{-t}(x_1, x_2).\end{aligned} \tag{3.3.10}$$

First we show that $\|\partial_{x_1}(v \cdot \mathscr{L}_t h)\|_{\mathbf{L^1}(\Omega_-^t)} < \infty$ using (3.3.9). First note that $\operatorname{ess\,sup}|h| < \infty$ by definition of $\mathscr{C}_f^1$ and that $\operatorname{ess\,sup}|f''| < \infty$ and $\operatorname{ess\,sup}|f'|^{-1} < \infty$ by assumption on the flow. It remains to estimate $\int_0^t \int_0^1 \partial_{x_1} h(F_i(x_1), x_2 - t + 1)\ dx_1 dx_2$ for each i. Changing variables $y_1 = F_i(x_1)$ and $y_2 = x_2 - t + 1$, noting again that $\operatorname{ess\,sup}|f'|^{-1} < \infty$ means that this previous integral is bounded by $C\|\partial_{x_1}(v \cdot h)\|_{\mathbf{L^1}(\Omega)}$ for some $C < \infty$ (we used here that $v(x_1, x_2) = 1$ within a δ sized region of both the roof and the base).

It remains to show that $\|\partial_{x_1}(v \cdot \mathscr{L}_t h)\|_{\mathbf{L^1}(\Omega_+^t)} < \infty$ for which we use (3.3.9). First we consider the first term on the right hand side of the equation. We must calculate the integral $\int_{\Omega_+^t} |\partial_{x_1}(v \cdot h)(x_1, \varphi_{-t}(x_1, x_2))\ dx_1 dx_2$. Changing variables $y_2 = \varphi_{-t}(x_1, x_2)$, noting $dy_2 = dx_2 \cdot v(x_1, \varphi_{-t}(x_1, x_2)) \cdot v(x_1, x_2)^{-1}$ and using that v is bounded both above and below and also that $\partial_{x_1}(v \cdot h) \in \mathbf{L^1}(\Omega)$ means that the first term in (3.3.10) is bounded in $\mathbf{L^1}$ norm. Since $h \in \mathscr{C}_f^1$ then by requirement (iv) of the definition of $\mathscr{C}_f^1$ (Definition 2.1.4) then $(x_1, x_2) \mapsto \partial_{x_2}(v \cdot h)(x_1, x_2)$ admits a continuous extension to the closure of each connected component of $([0,1] \setminus \sigma) \times [0,1]$. This implies that $\operatorname{ess\,sup}_\Omega |\partial_{x_2}(v \cdot h)| < \infty$. Since by Lemma 3.3.5 $\partial_{x_1}\varphi_{-t} \in \mathbf{L^1}(\Omega)$ then the second term in (3.3.10) is bounded in $\mathbf{L^1}$ norm. $\square$

Proof of first part of Lemma 3.2.4. Suppose that $h \in \mathscr{C}_f^1$ and $t \in (0, \frac{\delta}{2})$. Noting the formulae from Lemma 3.2.2 and that $\sup|h| < \infty$, finite number of inverse branches $\{F_i\}_i$, $\operatorname{ess\,sup}|f'|^{-1} < 1$, $\operatorname{ess\,sup}|v| < \infty$ and $\operatorname{ess\,sup}|v|^{-1} < \infty$ implies that $\operatorname{ess\,sup}|\mathscr{L}_t h| < \infty$. In addition to this we recall Lemma 3.3.2, Lemma 3.3.3, Lemma 3.3.4 and Lemma 3.3.6.

We have therefore shown that $\mathscr{L}_t\mathscr{C}_f^1 \subseteq \mathscr{C}_f^1$ for all $t \in (0, \frac{\delta}{2})$. The semigroup property (2.1.6) of $\mathscr{L}_t$ implies that $\mathscr{L}_t\mathscr{C}_f^1 \subseteq \mathscr{C}_f^1$ for all $t \geq 0$. $\square$

3.3.2. Strong Continuity

It remains to show the strong continuity of $\mathscr{L}_t$, i.e. that $\|\mathscr{L}_t h - h\|_{\mathbf{B}} \to 0$ as $t \to 0$. We consider the flow component and the transversal component of the norm separately in the following.

Lemma 3.3.7. *Suppose $h \in \mathscr{C}_f^1$. Then $\|\mathscr{L}_t h - h\|_{\mathbf{F}} \to 0$ as $t \to 0$.*

Proof. Let $h \in \mathscr{C}_f^1$, $t \in (0, \delta)$. Since $h \in \mathscr{C}_f^1$ then by assumption (iv) then $\partial_{x_2}(v \cdot h)$ is uniformly continuous on $([0,1] \setminus \sigma) \times [0,1]$. Substituting $\mathscr{L}_t h - h$ into the definition of $\|\cdot\|_{\mathbf{F}}$ we have

$$\|\mathscr{L}_t h - h\|_{\mathbf{F}} = \operatorname*{ess\,sup}_{x_1 \in [0,1]} \int_0^1 \left|\partial_{x_2}(v \cdot \mathscr{L}_t h) - \partial_{x_2}(v \cdot h)\right| (x_1, x_2)\; dx_2. \tag{3.3.11}$$

Lemma 3.3.4 implies that $\partial_{x_2}(v \cdot \mathscr{L}_t h)$ is also uniformly continuous on $([0,1] \setminus \sigma) \times [0,1]$ for all $t \in (0, \frac{\delta}{2})$. This in particular implies that there exists some $C < \infty$ such that $\operatorname{ess\,sup} |\partial_{x_2}(v \cdot \mathscr{L}_t h)| < C$ for all $t \in (0, \delta/2)$ and so

$$\int_0^t \left|\partial_{x_2}(v \cdot \mathscr{L}_t h) - \partial_{x_2}(v \cdot h)\right|\; dx_2 \leq C \int_0^t\; dx_2 \leq Ct$$

for all $x_1 \in [0,1] \setminus \sigma$. Now we must estimate the other part of the integral in (3.3.11). Given $x_1 \in [0,1] \setminus \sigma$, $x_2 \in (t, 1]$ and differentiating the formula of Lemma 3.2.2 gives

$$\begin{aligned} &v(x_1, x_2) \left|\partial_{x_2}(v \cdot \mathscr{L}_t h) - \partial_{x_2}(v \cdot h)\right| (x_1, x_2) \\ &\qquad\qquad = \left|[v \cdot \partial_{x_2}(v \cdot h)](x_1, \varphi_{-t}(x_1, x_2)) - [v \cdot \partial_{x_2}(v \cdot h)](x_1, x_2)\right|. \end{aligned}$$

We use now the uniform continuity of $\partial_{x_2}(v \cdot h)$, that $\operatorname{ess\,sup} |\partial_{x_2} v| < \infty$ by construction of v and that $\operatorname{ess\,sup} v^{-1} < \infty$ means that as $t \to 0$ then

$$\int_t^1 \left|\partial_{x_2}(v \cdot \mathscr{L}_t h) - \partial_{x_2}(v \cdot h)\right| (x_1, x_2)\; dx_2 \to 0$$

uniformly for all $x_1 \in [0,1] \setminus \sigma$. $\square$

Recall that associated to $h \in \mathscr{C}^1_f$ is a finite set of points $\sigma \subset [0,1]$. We denote by "card" the cardinality of some set.

Lemma 3.3.8. *Suppose $h \in \mathscr{C}^1_f$. Let $N = \mathrm{card}(\sigma) + \mathrm{card}(\mathcal{S})$. Then $\|h\|_{\mathbf{T}} \leq \|\partial_{x_1} h\|_{\mathbf{L}^1} + 2N\|h\|_{\mathbf{L}^\infty}$.*

Proof. Fix $h \in \mathscr{C}^1_f$. We must estimate $\int_\Omega h \cdot \partial_{x_1}(\gamma \cdot \eta)\ dx$ where $\eta \in \mathscr{D}$. Let $\sigma \cup \mathcal{S} = \{a_0, a_1, \ldots, a_n\}$ where $a_0 = 0$, $a_n = 1$ and $a_i < a_j$ whenever $i < j$. The idea is to integrate by parts in x_1 noting that on each interval (a_{i-1}, a_i) and for each $x_2 \in [0,1]$ then $x_1 \mapsto h(x_1, x_2)$ is $\mathscr{C}^1$. We expect h, γ to be discontinuous at $x_1 = a_i$ for each i. For all $\eta \in \mathscr{D}$ and $x_2 \in [0,1]$ we write

$$\int_0^1 (h \cdot \partial_{x_1}(\eta \cdot \gamma))(x_1, x_2)\ dx_1 = \sum_{i=1}^n \int_{a_{i-1}}^{a_i} (h \cdot \partial_{x_1}(\eta \cdot \gamma))(x_1, x_2)\ dx_1.$$

Let $(\eta \cdot h \cdot \gamma)(a_i^+, x_2)$ denote $\lim_{x_1 \uparrow a_i} (\eta \cdot h \cdot \gamma)(x_1, x_2)$ and similarly let $(\eta \cdot h \cdot \gamma)(a_i^-, x_2)$ denote $\lim_{x_1 \downarrow a_i} (\eta \cdot h \cdot \gamma)(x_1, x_2)$. We integrate by parts and so:

$$\begin{aligned}\int_0^1 (h \cdot \partial_{x_1}(\eta \cdot \gamma))(x_1, x_2)\ dx_1 = &-\sum_{i=1}^n \int_{a_{i-1}}^{a_i} (\partial_{x_1} h \cdot \gamma \cdot \eta)(x_1, x_2)\ dx_1 \\ &+ \sum_{i=1}^n \left[(\eta \cdot h \cdot \gamma)(a_i^+, x_2) - (\eta \cdot h \cdot \gamma)(a_{i-1}^-, x_2)\right].\end{aligned}$$

Integrating the above with respect to x_2 gives

$$\begin{aligned}\int_\Omega h \cdot \partial_{x_1}(\eta \cdot \gamma)\ dx = &-\int_\Omega \partial_{x_1} h \cdot \gamma \cdot \eta\ dx \\ &+ \sum_{i=1}^n \int_0^1 (\eta \cdot h \cdot \gamma)(a_i^+, x_2)\ dx_2 - \sum_{i=1}^n \int_0^1 (\eta \cdot h \cdot \gamma)(a_{i-1}^-, x_2)\ dx_2.\end{aligned}$$

Since $\eta \in \mathscr{D}$ then $\|\eta\|_{\mathbf{L}^\infty} \leq 1$ and also $\|\gamma\|_{\mathbf{L}^\infty} \leq 1$ so $|\int_\Omega \partial_{x_1} h \cdot \gamma \cdot \eta\ dx| \leq \|\partial_{x_1} h\|_{\mathbf{L}^1}$. For the other terms we take the simple estimate $|\int_0^1 (\eta \cdot h \cdot \gamma)(a_i^+, x_2)\ dx_2| \leq \|h\|_{\mathbf{L}^\infty}$

and identically when $-$ replaces $+$. There are in total $2(\mathrm{card}(\sigma) + \mathrm{card}(\mathcal{S}))$ of these pieces. $\square$

Lemma 3.3.9. $\|\partial_{x_1}(\mathscr{L}_t h - h)\|_{\mathbf{L^1}} \to 0$ *as* $t \to 0$ *for all* $h \in \mathscr{C}^1_f$

Proof. Recall that $\Omega = \Omega^t_- \cup \Omega^t_+ \mod 0$ where Ω^t_- and Ω^t_+ are as defined in Lemma 3.2.2. First we show show that $\|\partial_{x_1}(\mathscr{L}_t h - h)\|_{\mathbf{L^1}(\Omega^t_-)} \to 0$. Since $\mathrm{m}(\Omega^t_-) \to 0$ as $t \to 0$ and $\partial_{x_1} h$ is a fixed $\mathbf{L^1}$ function then $\|\partial_{x_1} h\|_{\mathbf{L^1}(\Omega^t_-)} \to 0$ as $t \to 0$. According to (3.3.9) then there exists $C < \infty$ such that for all $(x_1, x_2) \in \Omega^t_- = [0,1] \times [0,t)$ and $t \in (0, \frac{\delta}{2})$ holds

$$\begin{aligned}|\partial_{x_1}(v \cdot \mathscr{L}_t h)(x_1, x_2)| \leq C \sum_{i \in \mathscr{I}_k} \Big(&|\partial_{x_1}(F_i(x_1), x_2 - t + 1)| \\ &+ |h(F_i(x_1), x_2 - t + 1)| \Big).\end{aligned}$$

Changing variables $y_2 = x_2 - t + 1$ makes this also a (fixed in t) $\mathbf{L^1}$ function where now the integral is on the set $\widetilde{\Omega}^t_- = [0,1] \times [1-t, 1)$. Since $\mathrm{m}(\widetilde{\Omega}^t_-) \to 0$ as $t \to 0$ we have shown that $\|\partial_{x_1}(\mathscr{L}_t h - h)\|_{\mathbf{L^1}(\Omega^t_-)} \to 0$.

To prove the lemma it remains to show that $\|\partial_{x_1}(\mathscr{L}_t h - h)\|_{\mathbf{L^1}(\Omega^t_+)} \to 0$. According to Lemma 3.2.2 for all $(x_1, x_2) \in \Omega^t_+$ and $t \in (0, \delta)$ holds $(v \cdot \mathscr{L}_t h)(x_1, x_2) = (v \cdot h)(x_1, \varphi_{-t}(x_1, x_2)$. Differentiating with respect to x_1 we have

$$\begin{aligned}\partial_{x_1}(v \cdot \mathscr{L}_t h)(x_1, x_2) &= \partial_{x_1}(v \cdot h)(x_1, \varphi_{-t}(x_1, x_2)) \\ &\quad + \partial_{x_2}(v \cdot h)(x_1, \varphi_{-t}(x_1, x_2)) \cdot \partial_{x_1}\varphi_{-t}(x_1, x_2).\end{aligned}$$

As $\operatorname{ess\,sup} |\partial_{x_2}(v \cdot h)| < \infty$ the above implies that there exists $C < \infty$ such that

$$\begin{aligned}\left|\partial_{x_1}(v \cdot (\mathscr{L}_t h - h)\right|(x_1, x_2) \leq{}& |\partial_{x_1}(v \cdot h)(x_1, \varphi_{-t}(x_1, x_2)) - \partial_{x_1}(v \cdot h)(x_1, x_2)| \\ &+ C\left|\partial_{x_1}\varphi_{-t}\right|(x_1, x_2).\end{aligned} \tag{3.3.12}$$

Lemma 3.3.5 implies that $\|\partial_{x_1}\varphi_{-t}\|_{\mathbf{L^1}} \to 0$ as $t \to 0$. This deals with the second term of (3.3.12). For all $\epsilon > 0$ let $\Delta_\epsilon := \{x_1 \in [0,1] : \mathrm{dist}(x_1, \sigma) \geq \epsilon\}$. Holds that $\partial_{x_1}(v \cdot h)$ is uniformly continuous on $\Delta_\epsilon \times [t, 1)$. This means that, the first term of (3.3.12), $|\partial_{x_1}(v \cdot h)(x_1, \varphi_{-t}(x_1, x_2)) - \partial_{x_1}(v \cdot h)(x_1, x_2)| \to 0$ uniformly as $t \to 0$ on the set $\Delta_\epsilon \times [t, 1)$. Additionally $\mathrm{m}(\Delta_\epsilon) \to 0$ as $\epsilon \to 0$. Taken together, letting $\epsilon \to 0$ as $t \to 0$

this implies that $\|\partial_{x_1}(\mathscr{L}_t h - h)\|_{\mathbf{L^1}(\Omega_+^t)} \to 0$ as required. This completes the proof of the lemma. $\square$

Lemma 3.3.10. *Suppose $h \in \mathscr{C}_f^1$. Then $\|\mathscr{L}_t h - h\|_{\mathbf{L}^\infty} \to 0$ as $t \to 0$.*

Proof. We will show that $\|\mathscr{L}_t h - h\|_{\mathbf{L}^\infty(\Omega_+^t)} \to 0$ and then, completely separately, we will show that $\|\mathscr{L}_t h - h\|_{\mathbf{L}^\infty(\Omega_-^t)} \to 0$ as $t \to 0$. If $(x_1, x_2) \in \Omega_+^t$ then the formula for $\mathscr{L}_t h$ from Lemma 3.2.2 implies that

$$[v \cdot (\mathscr{L}_t h - h)](x_1, x_2) = (v \cdot h)(x_1, \varphi_{-t}(x_1, x_2)) - (v \cdot h)(x_1, x_2).$$

By definition of $\mathscr{C}_f^1$ the quantity $\partial_{x_2}(v \cdot h)$ admits a continuous extension to the closure of each connected component of $([0,1] \setminus \sigma) \times [0,1])$ and so in particular $\operatorname{ess\,sup} |\partial_{x_2}(v \cdot h)| < \infty$. Therefore holds that $\|v \cdot (\mathscr{L}_t h - h)\|_{\mathbf{L}^\infty(\Omega_+^t)} \to 0$ and so $\|\mathscr{L}_t h - h\|_{\mathbf{L}^\infty(\Omega_+^t)} \to 0$. Now, for the other case, let $(x_1, x_2) \in \Omega_-^t$. Lemma 3.2.2 implies that

$$(\mathscr{L}_t h - h)(x_1, x_2) = \mathscr{P}_f h(x_1, x_2) - h(x_1, x_2).$$

Recall condition (i) in the definition of $\mathscr{C}_f^1$ (Definition 2.1.4). Combined with the above this means that

$$\begin{aligned}(\mathscr{L}_t h - h)(x_1, x_2) = {} & \mathscr{P}_f h(x_1, x_2) - \mathscr{P}_f h(x_1, 1) \\ & + h(x_1, 0) - h(x_1, x_2).\end{aligned}$$

These four terms will be estimated pairwise. As noted above $\operatorname{ess\,sup} |\partial_{x_2}(v \cdot h)| < \infty$ and in this case $v = 1$ and so $\|\mathscr{L}_t h - h\|_{\mathbf{L}^\infty(\Omega_-^t)} \to 0$. $\square$

Proof of second part of Lemma 3.2.4. We start by fixing $h \in \mathscr{C}_f^1$. We must estimate $\|\mathscr{L}_t h - h\|_{\mathbf{B}} = \|\mathscr{L}_t h - h\|_{\mathbf{T}} + \|\mathscr{L}_t h - h\|_{\mathbf{F}}$ as $t \to 0$. The estimate from Lemma 3.3.7 that $\|\mathscr{L}_t h - h\|_{\mathbf{F}} \to 0$ means that it remains to consider the transversal component of the norm. Notice that, as proved earlier in this section, for all $t \in (0, \delta)$ then $\mathscr{L}_t h \in \mathscr{C}_f^1$. This means that associated to $\mathscr{L}_t h$ is a set $\sigma' \subset [0,1]$ defined in (3.3.4). Moreover this set is fixed for $t \in (0, \delta)$ and so we let $N := \operatorname{card}(\sigma') + \operatorname{card}(\mathcal{S})$. According to Lemma

3.3.8 holds

$$\|\mathscr{L}_t h - h\|_{\mathbf{T}} \leq \|\partial_{x_1}(\mathscr{L}_t h - h)\|_{\mathbf{L^1}} + 2N\|\mathscr{L}_t h - h\|_{\mathbf{L^\infty}}.$$

Using the estimate from Lemma 3.3.9 and Lemma 3.3.10 in the above equation implies that $\|\mathscr{L}_t h - h\|_{\mathbf{T}} \to 0$ as $t \to 0$ completing the proof of the lemma. □

3.4. Transversal Regularity

In this section we estimate $\|\mathscr{L}_t h\|_{\mathbf{T}}$ and so prove Proposition 3.2.7. By the definition of $\|\cdot\|_{\mathbf{T}}$ and the duality (2.1.5) of the transfer operator we have:

$$\|\mathscr{L}_t h\|_{\mathbf{T}} = \sup_{\eta \in \mathscr{D}} \left| \int_\Omega h \cdot \partial_{x_1}(\gamma \cdot \eta) \circ \phi_t \; dx \right|. \tag{3.4.1}$$

To proceed we must consider the pull back of the derivative. For all $\eta \in \mathscr{D}$, $t \geq 0$ holds

$$\partial_{x_1}(\gamma \cdot \eta) \circ \phi_t = (D\phi_t^{-1})_{11} \cdot \partial_{x_1}((\gamma \cdot \eta) \circ \phi_t) + (D\phi_t^{-1})_{21} \cdot \partial_{x_2}((\gamma \cdot \eta) \circ \phi_t).$$

Substituting into (3.4.1) we have for all $h \in \mathscr{C}_f^1$ and all $\eta \in \mathscr{D}$

$$\|\mathscr{L}_t h\|_{\mathbf{T}} \leq \sup_{\eta \in \mathscr{D}} \; |\Psi_{t,h,\eta}^{(A)}| + |\Psi_{t,h,\eta}^{(B)}| \tag{3.4.2}$$

where we define

$$\begin{aligned} \Psi_{t,h,\eta}^{(A)} &:= \int_{\Omega_t} h \cdot (D\phi_t^{-1})_{11} \cdot \partial_{x_1}((\gamma \cdot \eta) \circ \phi_t) \; dx \\ \Psi_{t,h,\eta}^{(B)} &:= \int_{\Omega_t} h \cdot (D\phi_t^{-1})_{21} \cdot \partial_{x_2}((\gamma \cdot \eta) \circ \phi_t) \; dx. \end{aligned} \tag{3.4.3}$$

In the remainder of this section we estimate these two terms separately to complete the proof of Proposition 3.2.7. First we introduce notation and consider the way the discontinuities for the return map propagate under the action of the flow. For $t \geq 0$ let

$$\mathcal{S}^{(t)} := \{\phi_s^{-1}(x_1, 0) : x_1 \in \mathcal{S}, s \in [0, t]\}$$

the preimage of discontinuities $\mathcal{S}$ and let

$$\Lambda^{(t)} := \{\phi_t^{-1}(x_1, 0), x_1 \in [0,1]\}$$

denote the preimage of the base under the flow. Note that $n_t(x_1, x_2)$ is constant on each connected component of $\Omega \setminus (\mathcal{S}^{(t)} \cup \Lambda^{(t)})$. For all $t \geq 0$ let

$$\sigma^{(t)} := \{x_1 \in [0,1] : (x_1, x_2) \in \mathcal{S}^{(t)} \text{ for some } x_2 \in [0,1]\}$$

We define sets of *vertical strips* at time $t \geq 0$ as follows. Let

$$\mathscr{J}^{(t)} = \{J_i\}_{i=1}^{\theta(t)} \tag{3.4.4}$$

denote the connected components of $((0,1) \setminus \sigma^{(t)}) \times [0,1])$. For each $i \in \{1, 2, \ldots, \theta(t)\}$ we write $J_i = (a_i, b_i) \times [0,1]$ and let $|J_i| := b_i - a_i$.

3.4.1. Expansion

We define the following key quantity. Let

$$A_t := \frac{\gamma \circ \phi_t}{\gamma \cdot (f^{n_t})'} = \gamma^{-1} \cdot \gamma \circ \phi_t \cdot (D\phi_t^{-1})_{11}. \tag{3.4.5}$$

The importance of the quantity A_t is that holds

$$\begin{aligned} \Psi_{t,h,\eta}^{(A)} = &\int_{\Omega_t} h \cdot \partial_{x_1} \left(\gamma \cdot A_t \cdot \eta \circ \phi_t\right) \, dx \\ &- \int_{\Omega_t} h \cdot \partial_{x_1} (D\phi_t^{-1})_{11} \cdot (\gamma \cdot \eta) \circ \phi_t \, dx. \end{aligned}$$

Before estimating $\Psi_{t,h,\eta}^{(A)}$. it is convenient to define a piecewise-linear approximation to $A_t \cdot \eta \circ \phi_t$ as follows. Recall the vertical strips defined in (3.4.4). We expect both A_t and $\eta \circ \phi_t$ to be discontinuous on the vertical lines $\sigma^{(t)} \times [0,1]$.

Lemma 3.4.1. *Suppose that $t \geq 0$. For all $i \in \{1, \ldots, \theta(t)\}$ then*

$$A_t \in \mathscr{C}^0(J_i^{(t)}, \mathbb{R}) \quad \textit{and} \quad A_t \in \mathscr{C}^1(J_i^{(t)} \setminus \Lambda^{(t)}, \mathbb{R}).$$

Proof. That $A_t \in \mathscr{C}^1(J_i^{(t)} \setminus \Lambda^{(t)}, \mathbb{R})$ is immediate from the definition (3.4.5) of A_t. It remains to show that $A_t \in \mathscr{C}^0(J_i^{(t)}, \mathbb{R})$ by showing that the continuity holds across the curves of $\Lambda^{(t)}$. Fixing $t \geq 0$ and taking some $(x_1, x_2) \in \Lambda^{(t)}$ and letting $n = n_t(x_1, x_2)$ we have

$$\lim_{\epsilon \to 0} \phi_t(x_1, x_2 - \epsilon) = (f^{n-1}(x_1), 1),$$
$$\phi_t(x_1, x_2 - \epsilon) = (f^n(x_1), 0)$$

and $\lim_{\epsilon \to 0} n_t(x_1, x_2 - \epsilon) = n - 1$. We now compare $\lim_{\epsilon \to 0} A_t(x_1, x_2 - \epsilon)$ and $A_t(x_1, x_2)$ using the formula for A_t given in (3.4.5).

$$\lim_{\epsilon \to 0} A_t(x_1, x_2 - \epsilon) = \frac{\gamma(f^{n-1}(x_1), 1)}{\gamma(x_1, x_2)} \frac{1}{(f^{n-1})'(x_1)}.$$

As $\gamma(f^{n-1}(x_1), 1) = 1/f'(f^{n-1}(x_1))$ and so $\gamma(f^{n-1}(x_1), 1)/(f^{n-1})'(x_1) = 1/(f^n)'(x_1)$. $\square$

We now define the piecewise linear approximation. Suppose $\eta \in \mathscr{D}$, $t \geq 0$ and $i \in \{1, 2, \ldots, \theta(t)\}$. Recall that $J_i = (a_i, b_i) \times [0, 1]$ and let $g_{\eta,t}^i : J_i \to \mathbb{C}$ denote the function which satisfies the following

1. $x_1 \mapsto (\gamma \cdot g_{\eta,t}^i)(x_1, x_2)$ is linear for $(x_1, x_2) \in J_i$,
2. $g_{\eta,t}^i$ is equal to the continuous extension of $A_t \cdot \eta \circ \phi_t$ on the set $\{a_i, b_i\} \times [0, 1]$
3. $g_{\eta,t}^i(x_1, x_2) = 0$ for all $(x_1, x_2) \in \Omega \setminus J_i$.

Thereby define $g_{\eta,t} : \Omega \to \mathbb{C}$ as

$$g_{\eta,t} := \sum_{i \in \{1,2,\ldots,\theta(t)\}} g_{\eta,t}^i. \tag{3.4.6}$$

Lemma 3.4.2. *Suppose $t \geq 0$, $\eta \in \mathscr{D}$ and $h \in \mathscr{C}_f^1$. Then*

$$|\Psi_{t,h,\eta}^{(A)}| \leq 2\|A_t\|_{\mathbf{L}^\infty}\|h\|_{\mathbf{T}} + \|h\|_{\mathbf{L}^1}\left(\|\partial_{x_1}(\gamma \cdot g_{\eta,t})\|_{\mathbf{L}^\infty} + \|\partial_{x_1}(D\phi_t^{-1})_{11}\|_{\mathbf{L}^\infty}\right).$$

Proof. Fix $h \in \mathscr{C}_f^1$, $t \geq 0$ and $\eta \in \mathscr{D}$. We must estimate $|\Psi_{t,h,\eta}^{(A)}| = |\int_{\Omega_t} h \cdot (D\phi_t^{-1})_{11} \cdot \partial_{x_1}((\gamma \cdot \eta) \circ \phi_t)\ dx|$. Using the piecewise linear function defined above (3.4.6) and the quantity A_t (3.4.5) holds

$$\Psi_{t,h,\eta}^{(A)} = \int_{\Omega_t} h \cdot \partial_{x_1}\left[\gamma \cdot (A_t \cdot \eta \circ \phi_t - g_{\eta,t})\right]\ dx \\ + \int_{\Omega_t} h \cdot \partial_{x_1}(\gamma \cdot g_{\eta,t})\ dx - \int_{\Omega_t} h \cdot \partial_{x_1}(D\phi_t^{-1})_{11} \cdot (\gamma \cdot \eta) \circ \phi_t\ dx.$$

Notice that by construction of $g_{\eta,t}$ then $\|A_t \cdot \eta \circ \phi_t - g_{\eta,t}\|_{\mathbf{L}^\infty} \leq 2\|A_t\|_{\mathbf{L}^\infty}$. From Lemma 3.4.1 we know that for each $i \in \{1, 2, \ldots, \theta(t)\}$ then $A_t \in \mathscr{C}^0(J_i, \mathbb{C})$ and so $A_t \cdot \eta \circ \phi_t - g_{\eta,t} \in \mathscr{C}^0(\Omega, \mathbb{C})$. Moreover $A_t \cdot \eta \circ \phi_t - g_{\eta,t}$ is $\mathscr{C}^1$ on the set $\Omega \setminus (\Lambda^{(t)} \cup (\sigma^{(t)} \times [0,1]))$. With regard to Lemma 3.4.3 this completes the proof of the lemma. □

Lemma 3.4.3. *Let $\mathscr{G} \subset [0,1]^2$. Let $\Delta_\epsilon := \{(x_1, x_2) \in [0,1]^2 : \mathrm{d}((x_1, x_2), \mathscr{G}) \geq \epsilon\}$ for each $\epsilon > 0$. We suppose that each Δ_ϵ is a compact subset of $\mathbb{R}^2$ and that $\mathrm{m}([0,1]^2 \setminus \Delta_\epsilon) \to 0$ as $\epsilon \to 0$. Let $\zeta \in \mathscr{C}^0([0,1]^2, \mathbb{R})$ and furthermore that $\zeta \in \mathscr{C}^1([0,1]^2 \setminus \mathscr{G}, \mathbb{R})$ and $\sup_{i \in \{1,2\}} \operatorname{ess\,sup}_{[0,1]^2} |\partial_{x_i}\eta| < \infty$. Let $h \in \mathscr{C}_f^1$. Then $\left|\int_\Omega h \cdot \partial_{x_1}(\gamma \cdot \zeta)\ dx\right| \leq \|\zeta\|_{\mathbf{L}^\infty}\|h\|_{\mathbf{T}}$.*

Proof. Let ζ_ϵ denote the restriction of ζ to Δ_ϵ. By the Whitney Extension Theorem there exists $\bar{\zeta}_\epsilon \in \mathscr{C}^1([0,1]^2, \mathbb{R})$ such that $\bar{\zeta}_\epsilon = \zeta_\epsilon$ on Δ_ϵ. Since for all $x \in \mathscr{G}$ and $y \in \Delta_\epsilon$ by Taylor holds $\bar{\zeta}_\epsilon(x) = \zeta(y) + R_0(x, y)$ and the uniform bound on the derivatives of η means that there exists $M < \infty$, independent of choice of $\epsilon > 0$ such that $|R_0(x, y)| \leq M\,|x - y|$ for all $x \in \mathscr{G}$ and $y \in \Delta_\epsilon$. This means that $\|\bar{\zeta}_\epsilon\|_{\mathbf{L}^\infty([0,1]^2)} \to \|\zeta\|_{\mathbf{L}^\infty([0,1]^2)}$ as $\epsilon \to 0$. By construction $\partial_{x_1}\bar{\zeta}_\epsilon \in \mathscr{C}^0([0,1]^2, \mathbb{R})$ and since $\mathrm{m}([0,1]^2 \setminus \Delta_\epsilon) \to 0$ as $\epsilon \to 0$ means that $\|\partial_{x_1}(\bar{\zeta}_\epsilon - \zeta)\|_{\mathbf{L}^1([0,1]^2)} \to 0$ as $\epsilon \to 0$.

Fix $h \in \mathscr{C}^1_f$. For each $\epsilon > 0$ holds

$$\int_\Omega h \cdot \partial_{x_1}(\gamma \cdot \zeta)\ dx = \int_\Omega h \cdot \partial_{x_1}(\gamma \cdot \bar{\zeta}_\epsilon)\ dx \\ + \int_\Omega h \cdot \gamma \cdot \partial_{x_1}(\zeta - \bar{\zeta}_\epsilon)\ dx + \int_\Omega h \cdot \partial_{x_1}\gamma \cdot (\zeta - \bar{\zeta}_\epsilon)\ dx.$$

Since $\bar{\zeta}_\epsilon \in \mathscr{C}^1_f$ and $\|\bar{\zeta}_\epsilon\|_{\mathbf{L}^\infty([0,1]^2)} \to \|\zeta\|_{\mathbf{L}^\infty([0,1]^2)}$ as $\epsilon \to 0$ then

$$\Big| \int_\Omega h \cdot \partial_{x_1}(\gamma \cdot \bar{\zeta}_\epsilon)\ dx \Big| \to \|\zeta\|_{\mathbf{L}^\infty([0,1]^2)} \|h\|_{\mathbf{T}} \quad \text{as } \epsilon \to 0.$$

Since $\|\partial_{x_1}(\bar{\zeta}_\epsilon - \zeta)\|_{\mathbf{L^1}([0,1]^2)} \to 0$ as $\epsilon \to 0$ then $|\int_\Omega h \cdot \gamma \cdot \partial_{x_1}(\zeta - \bar{\zeta}_\epsilon)\ dx| \to 0$. Since $\partial_{x_1}\gamma \in \mathbf{L^1}$ and $\mathrm{m}([0,1]^2 \setminus \Delta_\epsilon) \to 0$ as $\epsilon \to 0$ then $|\int_\Omega h \cdot \partial_{x_1}\gamma \cdot (\zeta - \bar{\zeta}_\epsilon)\ dx| \to 0$. □

Lemma 3.4.4. $\|A_t\|_{\mathbf{L}^\infty} \leq e^{-\lambda t}$ *for all* $t \geq 0$.

Proof. Fix $(x_1, x_2) \in \hat{\Omega}$ and $t \geq 0$. The definition (2.1.10) of $\lambda(x_1)$ implies that $f'(x_1) = \exp[r(x_1) \cdot \lambda(x_1)]$. The definition (2.1.11) of γ implies that $\gamma(x_1, x_2) = \exp[-\tau(x_1, x_2) \cdot \lambda(x_1)]$. Letting $t_1 := t - r_n(x_1) + \tau(x_1, x_2) \geq 0$ then

$$\gamma \circ \phi_t(x_1, x_2) = \exp\left[-t_1 \cdot \lambda(f^{n_t}(x_1))\right]. \tag{3.4.7}$$

Letting $t_2 := r(x_1) - \tau(x_1, x_2) \geq 0$ then

$$\begin{aligned} \gamma(x_1, x_2) \cdot f'(x_1) &= \exp\left[-\tau(x_1, x_2) \cdot \lambda(x_1)\right] \cdot \exp\left[r(x_1) \cdot \lambda(x_1)\right] \\ &= \exp\left[t_2 \cdot \lambda(x_1)\right]. \end{aligned} \tag{3.4.8}$$

Also

$$\left(f^{n_t}\right)'(x_1)/f'(x_1) = \prod_{i=1}^{n_t-1} f'(f^i(x_1)) = \exp\Big[\sum_{i=1}^{n_t-1} (r \cdot \lambda)(f^i(x_1))\Big] \tag{3.4.9}$$

Substituting (3.4.7), (3.4.8), (3.4.9) into the definition (3.4.5) of A_t holds

$$A_t(x_1, x_2) = \exp\Big[- t_1\lambda(f^{n_t}(x_1)) - t_2\lambda(x_1) - \sum_{i=1}^{n_t-1} (r \cdot \lambda)(f^i(x_1)) \Big].$$

Noticing that $\lambda(x_1) \geq \lambda$ for all $x_1 \in [0,1]$ and that

$$t_1 + t_2 + \sum_{i=1}^{n_t-1} r(f^i(x_1))) = t,$$

we have shown that $|A_t|(x_1, x_2) \leq \exp[-t\lambda]$ as required. $\square$

Lemma 3.4.5. *For all $t \geq 0$ there exists $C < \infty$ such that*

$$\sup_{\eta \in \mathscr{D}} \|\partial_{x_1}(\gamma \cdot g_{\eta,t})\|_{\mathbf{L}^\infty} \leq C_t.$$

Proof. Notice that for all $(x_1, x_2) \in J_i$ then $\partial_{x_1}(\gamma \cdot g^i_{\eta,t})(x_1, x_2) = ((\gamma \cdot A_t \cdot \eta \circ \phi_t)(b_1, x_2) - (\gamma \cdot A_t \cdot \eta \circ \phi_t)(a_1, x_2))/(b_i - a_i)$ because $g^i_{\eta,t}$ was defined to be linear in this sense. Since $|\eta| \leq 1$ and $|\gamma| \leq 1$ then holds

$$\sup_{(x_1,x_2)\in J_i} |\partial_{x_1}(\gamma \cdot g^i_{\eta,t})(x_1, x_2)| \leq 2|J_i|^{-1} \|A_t\|_{\mathbf{L}^\infty}. \tag{3.4.10}$$

Let $C_t := 2\sup_{i \in \{1,2,\dots,\theta(t)\}} |J_i|^{-1}$. $\square$

Lemma 3.4.6. *There exists $C < \infty$ such that for all $t \geq 0$*

$$\|\partial_{x_1}(D\phi_t^{-1})_{11}\|_{\mathbf{L}^\infty} \leq C.$$

Proof. Since $(D\phi_t^{-1})_{11}(x_1, x_2) = \frac{1}{(f^{n_t})'}(x_1)$ then $\partial_{x_1}(D\phi_t^{-1})_{11}(x_1, x_2) = \frac{(f^{n_t})''}{[(f^{n_t})']^2}(x_1)$. For each $n \in \mathbb{N}$ holds

$$(f^n)'' = \sum_{i=0}^{n-1} f'' \circ f^i \cdot (f^i)' \cdot \frac{(f^n)'}{f' \circ f^i},$$

and so

$$\frac{(f^n)''}{[(f^n)']^2} = \sum_{i=0}^{n-1} \frac{f''}{(f')^2} \circ f^i \cdot \prod_{j=i+1}^{n-1} \frac{1}{f' \circ f^i}.$$

Since $\operatorname{ess\,sup} |f''| < \infty$ and $\operatorname{ess\,inf} |f'| > 1$ then the above quantity is bounded from above independently of n. $\square$

Combining Lemma 3.4.2, Lemma 3.4.4, Lemma 3.4.5 and Lemma 3.4.6 implies that for all $t \geq 0$ there exists $C_t < \infty$ such that

$$\sup_{\eta \in \mathscr{D}} |\Psi^{(A)}_{t,h,\eta}| \leq 2e^{-\lambda t} \|h\|_{\mathbf{T}} + C_t \|h\|_{\mathbf{L}^1} \quad \text{for all } h \in \mathscr{C}^1_f. \tag{3.4.11}$$

3.4.2. Twisting

Now we estimate $\Psi^{(B)}_{t,h,\eta}$, the other term in (3.4.2). It is convenient at this stage to define and then study the following quantity. For $t \geq 0$ let

$$B_t := -\frac{\gamma \circ \phi_t \cdot (D\phi_t)_{21}}{v \circ \phi_t \cdot (D\phi_t)_{11}}. \tag{3.4.12}$$

The importance of the quantity B_t is that holds

$$\begin{aligned} \Psi^{(B)}_{t,h,\eta} = &\int_\Omega h \cdot v \cdot \partial_{x_2} (B_t \cdot \eta \circ \phi_t) \; dx \\ &- \int_\Omega h \cdot v \cdot \partial_{x_2} \Big(\tfrac{(D\phi_t^{-1})_{21}}{v} \Big) \cdot (\gamma \cdot \eta) \circ \phi_t \; dx. \end{aligned}$$

Furthermore, for $t \geq 0$, $x_1 \in [0,1]$ let

$$b_t(x_1) := \sup_{x_2 \in [0,1]} |B_t(x_1, x_2)| \,. \tag{3.4.13}$$

Lemma 3.4.7. *Suppose $(x_1, x_2) \in \Omega$, $t \geq 0$ such that $t + \tau(x_1, x_2) \neq r_m(x_1)$ for any $m \in \mathbb{N}$. Let $(x_1^{(t)}, x_2^{(t)}) := \phi_t(x_1, x_2)$ and $n = n_t(x_1, x_2)$. Then*

$$\frac{(D\phi_t)_{21}}{v \circ \phi_t}(x_1, x_2) = -r_n'(x_1) + (\partial_{x_1} \tau)(x_1, x_2) - (\partial_{x_1} \tau)(x_1^{(t)}, x_2^{(t)}) \cdot (f^n)'(x_1).$$

Proof. We differentiate the formula for the flow (3.2.1). Holds

$$(D\phi_t)_{21}(x_1,x_2) = \partial_t \varphi_{s^{(t)}(x_1,x_2)}(x_1^{(t)},0)\cdot \partial_{x_1} s^{(t)}(x_1,x_2) + \partial_{x_1}\varphi_{s^{(t)}(x_1,x_2)}(x_1^{(t)},0)\cdot \partial_{x_1} x_1^{(t)}(x_1).$$

By definition of the local flow holds $\partial_t \varphi_{s^{(t)}(x_1,x_2)}(x_1^{(t)},0) = v(x_1^{(t)},x_2^{(t)})$. Differentiating the definition of $s^{(t)}$ holds

$$\partial_{x_1} s^{(t)}(x_1,x_2) = -r_n'(x_1) + \partial_{x_1}\tau(x_1,x_2).$$

Using (3.3.1) of Lemma 3.3.1 holds

$$\partial_{x_1}\varphi_{s^{(t)}(x_1,x_2)}(x_1^{(t)},0) = -v(x_1^{(t)},x_2^{(t)})\cdot \partial_{x_1}\tau(x_1^{(t)},x_2^{(t)}).$$

Lastly $\partial_{x_1} x_1^{(t)}(x_1) = (f^n)'(x_1)$. Combining the above calculations gives the required formula. □

Lemma 3.4.8. *Suppose $t \geq 0$. For Lebesgue almost every $x_1 \in [0,1]$ the function $x_2 \mapsto B_t(x_1,x_2)$ is $\mathscr{C}^0$ and $B_t(x_1,1) = B_t(f(x_1),0)$.*

Proof. First we compare $B_t(x_1,1)$ and $B_t(f(x_1),0)$. Let $n = n_t(x_1,x_2)$ and $m = n_t(f(x_1),0)$ and note that $m+1 = n$. Let $(x_1^{(t)},x_2^{(t)}) = \phi_t(x_1,1) = \phi_t(f(x_1),0)$. By Lemma 3.4.7 holds

$$\begin{aligned} B_t(f(x_1),0) &= \gamma(x_1^{(t)},x_2^{(t)})\cdot\Big(\frac{r_m'\circ f(x_1)}{(f^m)'\circ f(x_1)} + \partial_{x_1}\tau(x_1^{(t)},x_2^{(t)})\Big),\\ B_t(x_1,1) &= \gamma(x_1^{(t)},x_2^{(t)})\cdot\Big(\frac{r_n'(x_1) - \partial_{x_1}\tau(x_1,1)}{(f^n)'(x_1)} + \partial_{x_1}\tau(x_1^{(t)},x_2^{(t)})\Big). \end{aligned} \tag{3.4.14}$$

Since $\tau(x_1,0) = r(x_{1})$ then $\partial_{x_1}\tau(x_1,1) = r'(x_1)$ and since $n = m+1$ then $(f^n)'(x_1) = (f^m)'\circ f(x_1)\cdot f'(x_1)$. Also note that $r_n(x_1) = r(x_1) + r_m\circ f(x_1)$ and so $r_n'(x_1) = r'(x_1) + r_m'\circ f(x_1)\cdot f'(x_1)$. Combining these calculations implies that

$$\frac{r_m'\circ f(x_1)}{(f^m)'\circ f(x_1)} = \frac{r_n'(x_1) - \partial_{x_1}\tau(x_1,1)}{(f^n)'(x_1)}$$

which, in view of (3.4.14), implies that $B_t(x_1, 1) = B_t(f(x_1), 0)$ as required.

We must check the continuity of $x_2 \mapsto B_t(x_1, x_2)$ for $(x_1, x_2) \in \Lambda^{(t)}$. As before let $n = n_t(x_1, x_2)$. Suppose that $(x_1, x_2) \in \Lambda^{(t)}$ and note that this implies $t + \tau(x_1, x_2) - r_n(x_1) = 0$. We must compare $B_t(x_1, x_2)$ with $B_t(x_1, x_2 - \epsilon)$ as $\epsilon \to 0$. Note that $\phi_t(x_1, x_2) = (f^n(x_1), 0)$ whilst $\phi_t(x_1, x_2 - \epsilon) = (f^{n-1}(x_1), 1 - \epsilon)$ for all $\epsilon > 0$. Again by Lemma 3.4.7 holds

$$B_t(x_1, x_2) = \gamma(f^n(x_1), 0) \cdot \Big(\frac{r_n'(x_1) - \partial_{x_1}\tau(x_1, x_2)}{(f^n)'(x_1)} + \partial_{x_1}\tau(f^n(x_1), 0)\Big).$$

Since $\gamma(f^n(x_1), 0) = 1$ and $\partial_{x_1}\tau(f^n(x_1), 0) = 0$ then holds

$$B_t(x_1, x_2) = \frac{r_n'(x_1) - \partial_{x_1}\tau(x_1, x_2)}{(f^n)'(x_1)}. \tag{3.4.15}$$

Yet again by Lemma 3.4.7

$$B_t(x_1, x_2 - \epsilon) = \gamma(f^{n-1}(x_1), 1 - \epsilon) \cdot \Big(\frac{r_{n-1}'(x_1) - \partial_{x_1}\tau(x_1, x_2 - \epsilon)}{(f^{n-1})'(x_1)} + \partial_{x_1}\tau(f^{n-1}(x_1), 1 - \epsilon)\Big).$$

Since in the limit $\epsilon \to 0$ then $\gamma(f^{n-1}(x_1), 1-\epsilon) \to 1/f'(f^{n-1}(x_1))$ and $\partial_{x_1}\tau(x_1, x_2-\epsilon) \to \partial_{x_1}\tau(x_1, x_2)$ and $\partial_{x_1}\tau(f^{n-1}(x_1), 1 - \epsilon) \to r'(f^{n-1}(x_1))$ then

$$B_t(x_1, x_2 - \epsilon) \to \frac{r_{n-1}'(x_1) - \partial_{x_1}\tau(x_1, x_2) + r'(f^{n-1}(x_1)) \cdot (f^{n-1})'(x_1)}{f'(f^{n-1}(x_1)) \cdot (f^{n-1})'(x_1)}$$

as $\epsilon \to 0$. Comparing the above formula to (3.4.15) we have shown that $B_t(x_1, x_2-\epsilon) \to B_t(x_1, x_2)$ as $\epsilon \to 0$ as required. ☐

Lemma 3.4.9. *Suppose $1 < p < \infty$. There exists $C < \infty$ such that for all $n \in \mathbb{N}$, $g \in \mathbf{L^p}([0, 1])$ then $\|g \circ f^n\|_{\mathbf{L^p}([0,1])} \leq C\|g\|_{\mathbf{L^p}([0,1])}$.*

Proof. Let $\mathscr{Q} : \mathbf{L^1}([0, 1]) \to \mathbf{L^1}([0, 1])$ denote the usual transfer operator associated to f. I.e. $\mathscr{Q}g(x) := \sum_{y\in f^{-1}(x)}(g/f')(y)$ for all $g \in \mathbf{L^1}([0, 1])$. For each $n \in \mathbb{N}$ and $g \in \mathbf{L^p}([0, 1])$

it holds

$$\|g \circ f^n\|^p_{\mathbf{L^p}([0,1])} = \int_0^1 |g|^p \circ f^n \, dx = \int_0^1 |g|^p \cdot \mathscr{Q}^n 1 \, dx.$$

This means that $\|g \circ f^n\|^p_{\mathbf{L^p}([0,1])} \leq \|\mathscr{Q}^n 1\|_{\mathbf{L^\infty}([0,1])} \|g\|^p_{\mathbf{L^p}([0,1])}$. It therefore suffices to show the existence of $C < \infty$ such that $\|\mathscr{Q}^n 1\|_{\mathbf{L^\infty}([0,1])} \leq C$ for all $n \in \mathbb{N}$. We conclude by noting that $\|\cdot\|_{\mathbf{L^\infty}([0,1])} \leq \|\cdot\|_{\mathbf{BV}([0,1])}$ (a speciality of $\mathbf{BV}$ in one dimension) and recalling that it is known that the spectral radius of $\mathscr{Q}$ acting on $\mathbf{BV}([0,1])$ is equal to 1 and $\|1\|_{\mathbf{BV}([0,1])} = 1$. $\square$

Recall the quantity $q > 1$ from the tempered twist condition (2.1.1).

Lemma 3.4.10. $\sup_{t \geq 0} \|b_t\|_{\mathbf{L^q}([0,1])} < \infty$.

Proof. Suppose $(x_1, x_2) \in \Omega$. Let $(x_1^{(t)}, x_2^{(t)}) = \phi_t(x_1, x_2)$, and let $n = n_t(x_1, x_2)$. Using the formula from Lemma 3.4.7 holds

$$B_t(x_1, x_2) = \gamma(x_1^{(t)}, x_2^{(t)}) \cdot \Big(\frac{r_n'(x_1)}{(f^n)'(x_1)} - \frac{\partial_{x_1}\tau(x_1, x_2)}{(f^n)'(x_1)} + \partial_{x_1}\tau(x_1^{(t)}, x_2^{(t)}) \Big)$$

Using the estimate from Lemma 3.1.2 and recalling that $\beta := \operatorname{ess\,inf} |f'| > 1$ we have shown that there exists $C < \infty$, independent of x_1 such that

$$b(x_1) = \sup_{x_2 \in [0,1]} |B_t(x_1, x_2)| \leq C \sum_{j=0}^{n} |r'| \circ f^j(x_1) \cdot \beta^{-(n-j)}.$$

Bounding $\||r'| \circ f^n\|_{\mathbf{L^p}([0,1])}$ using Lemma 3.4.9 completes the estimate. $\square$

Lemma 3.4.11. *There exists $C < \infty$ such that*

$$|\Psi^{(B)}_{t,h,\eta}| \leq C\epsilon \|h\|_{\mathbf{F}} + C\epsilon^{-p} \|\partial_{x_2}(v \cdot h)\|_{\mathbf{L^1}} + \|h \cdot \partial_{x_2}(\tfrac{(D\phi_t^{-1})_{21}}{v})\|_{\mathbf{L^1}}$$

for all $t \geq 0$, $\epsilon > 0$, $h \in \mathscr{C}^1_f$, $\eta \in \mathscr{D}$.

Proof. Fixing $t \geq 0$, $h \in \mathscr{C}^1_f$ and $\eta \in \mathscr{D}$ we will estimate $\Psi^{(B)}_{t,h,\eta} = \int_{\Omega_t} h \cdot (D\phi_t^{-1})_{21} \cdot$

$\partial_{x_2}((\gamma \cdot \eta) \circ \phi_t)\ dx$. Recalling B_t defined above (3.4.12) then holds

$$\begin{aligned}\Psi^{(B)}_{t,h,\eta} &= \int_\Omega h \cdot v \cdot \partial_{x_2}(B_t \cdot \eta \circ \phi_t)\ dx \\ &\quad - \int_\Omega h \cdot v \cdot \partial_{x_2}\Big(\tfrac{(D\phi_t^{-1})_{21}}{v}\Big) \cdot (\gamma \cdot \eta) \circ \phi_t\ dx.\end{aligned}$$

It remains to estimate $\left|\int_\Omega h \cdot v \cdot \partial_{x_2}(B_t \cdot \eta \circ \phi_t)\ dx\right|$. Due to Lemma 3.4.8 we know that $B_t \cdot \eta \circ \phi_t$ has the branch property, i.e. $(B_t \cdot \eta \circ \phi_t)(x_1, 1) = (B_t \cdot \eta \circ \phi_t)(f(x_1), 0)$. This means that we may integrate by parts with no boundary term. We have

$$\begin{aligned}\left|\int_\Omega h \cdot v \cdot \partial_{x_2}(B_t \cdot \eta \circ \phi_t)\ dx\right| &= \left|\int_\Omega \partial_{x_2}(h \cdot v) \cdot B_t \cdot \eta \circ \phi_t\ dx\right| \\ &\leq \int_0^1 \int_0^1 |\partial_{x_2}(h \cdot v) \cdot B_t|\ dx_2 dx_1.\end{aligned}$$

Let $\hat{h}(x_1) := \int_0^1 |\partial_{x_2}(h \cdot v)| (x_1, x_2)\ dx_2$. Recall the quantity b_t defined above (3.4.13). Let $b := \sup_{t \geq 0} \|b_t\|_{\mathbf{L^q}([0,1])} < \infty$, the finite bound was shown in Lemma 3.4.10. Holds, for all $\epsilon > 0$ that

$$\begin{aligned}\int_0^1 \int_0^1 |\partial_{x_2}(h \cdot v) \cdot B_t|\ dx_2 dx_1 &\leq \int_0^1 \hat{h}(x_1) \cdot b_t(x_1)\ dx_1 \\ &\leq b\|\hat{h}\|_{\mathbf{L^q}([0,1])} \\ &\leq b\epsilon\|\hat{h}\|_{\mathbf{L^\infty}([0,1])} + b\epsilon^{-q}\|\hat{h}(x_1)\|_{\mathbf{L^1}([0,1])}.\end{aligned}$$

Remark 3.4.12. The final line is simply Chebyshev's inequality as follows. Let $I := [0, 1]$. Let $\mathbb{A}_\epsilon$ denote the set where $|g| \geq C_\epsilon := \epsilon^{-p}\|g\|_{\mathbf{L^1}(I)}$. Let $g_\epsilon := g \cdot \chi_{\mathbb{A}_\epsilon}$. Notice that $C_\epsilon \mathbf{m}(\mathbb{A}_\epsilon) \leq \|g\|_{\mathbf{L^1}(I)}$ and so $\mathbf{m}(\mathbb{A}_\epsilon) \leq \epsilon^p$. This means that $\|g_\epsilon\|_{\mathbf{L^q}(I)} \leq \|g\|_{\mathbf{L^\infty}(I)} \mathbf{m}(\mathbb{A}_\epsilon)^{1/p} \leq \epsilon\|g\|_{\mathbf{L^\infty}(I)}$. Additionally $\|g - g_\epsilon\|_{\mathbf{L^p}(I)} \leq C_\epsilon = \epsilon^{-p}\|g\|_{\mathbf{L^1}(I)}$. This means that $\|g\|_{\mathbf{L^q}(I)} \leq \|g_\epsilon\|_{\mathbf{L^q}(I)} + \|g - g_\epsilon\|_{\mathbf{L^q}(I)} \leq \epsilon\|g\|_{\mathbf{L^\infty}(I)} + \epsilon^{-p}\|g\|_{\mathbf{L^1}(I)}$.

Now note that $\|\hat{h}\|_{\mathbf{L^\infty}([0,1])} = \|h\|_{\mathbf{F}}$ and $\|\hat{h}(x_1)\|_{\mathbf{L^1}([0,1])} = \|\partial_{x_2}(v \cdot h)\|_{\mathbf{L^1}}$ to complete the proof. □

Lemma 3.4.13. $\|h \cdot \partial_{x_2}(\frac{(D\phi_t^{-1})_{21}}{v})\|_{\mathbf{L^1}} < C\epsilon\|h\|_{\mathbf{S}} + \epsilon^{-q}\|h\|_{\mathbf{L^1}}$.

Proof. First note that as $(D\phi_t^{-1})_{21} = -(D\phi_t)_{21}(D\phi_t)_{11}^{-1}(D\phi_t)_{22}^{-1}$ and using the formula of Lemma 3.4.7 holds

$$\tfrac{(D\phi_t^{-1})_{21}}{v}(x_1, x_2) = \frac{r_n'(x_1)}{(f^n)'(x_1)} - \frac{(\partial_{x_1}\tau)(x_1, x_2)}{(f^n)'(x_1)} + (\partial_{x_1}\tau) \circ \phi_t(x_1, x_2).$$

Let $U_t(x_1, x_2) := \partial_{x_2}((D\phi_t^{-1})_{21} \cdot v^{-1})(x_1, x_2)$. Differentiating with respect to x_2 we obtain

$$U_t(x_1, x_2) = -\frac{\partial_{x_2}\partial_{x_1}\tau(x_1, x_2)}{(f^n)'(x_1)} + \partial_{x_2}\partial_{x_1}\tau \circ \phi_t(x_1, x_2) \cdot (D\phi_t)_{22}(x_1, x_2).$$

The crucial observation is that the derivatives may be exchanged and

$$\partial_{x_2}\partial_{x_1}\tau(x_1, x_2) = \partial_{x_1}v(x_1, \varphi_t(x_2)).$$

By assumption on the speed function for each x_1 then $\sup_{x_2} |\partial_{x_1}v(x_1, x_2)| \leq C\,|r'(x_1)|$. Increasing $C < \infty$ as required we have shown that $|U_t(x_1, x_2)| \leq C\,|r'(x_1)|$ for all $t \geq 0$, $x_1 \in [0, 1] \setminus \sigma$, $x_2 \in [0, 1]$. Let

$$u_t(x_1) := \operatorname*{ess\,sup}_{x_2 \in [0,1]} |U_t(x_1, x_2)|\,.$$

Recalling the tempered twist assumption (2.1.1) the above calculation implies that $\sup_{t\geq 0} \|u_t\|_{\mathbf{L^q}([0,1])} < \infty$. We conclude as before and as described in Remark 3.4.12. $\square$

Proof of Proposition 3.2.7. According to (3.4.2) by definition we have the upper bound $\|\mathscr{L}_t h\|_{\mathbf{T}} \leq \sup_{\eta\in\mathscr{D}}\ |\Psi_{t,h,\eta}^{(A)}| + |\Psi_{t,h,\eta}^{(B)}|$. We have an estimate (3.4.11) for $\sup_{\eta\in\mathscr{D}} |\Psi_{t,h,\eta}^{(A)}|$. To estimate $\sup_{\eta\in\mathscr{D}} |\Psi_{t,h,\eta}^{(B)}|$ we combine Lemma 3.4.11 with Lemma 3.4.13 setting $\epsilon = \epsilon(t) = e^{-\lambda t}$ in the second estimate. Summing together the above statements we have the following. There exists $C < \infty$ and for all $t \geq 0$ there exists $C_t < \infty$ such that

$$\|\mathscr{L}_t h\|_{\mathbf{T}} \leq Ce^{-\lambda t}\|h\|_{\mathbf{T}} + C\epsilon\|h\|_{\mathbf{F}} + C\epsilon^{-p}\|\partial_{x_2}(v \cdot h)\|_{\mathbf{L^1}} + C_t\|h\|_{\mathbf{L^1}} \tag{3.4.16}$$

for all $\epsilon > 0$ and $h \in \mathscr{C}_f^1$. The idea is to complete the proof by iterating the above expression using that $\|\mathscr{L}_t h\|_{\mathbf{L^1}} \leq \|h\|_{\mathbf{L^1}}$ and $\|\partial_{x_2}(v \cdot \mathscr{L}_t h)\|_{\mathbf{L^1}} \leq \|\partial_{x_2}(v \cdot h)\|_{\mathbf{L^1}}$ for all

$t \geq 0$ as shown in Lemma 2.1.3 and (3.2.8). Fix $\tilde{\lambda} \in (0, \lambda)$. Choose $T < \infty$ such that $2Ce^{-\lambda T} \leq e^{-\tilde{\lambda} T}$ where the C in question the one which appears in (3.4.16) above. We make the following claim which we will then prove by induction on k. There exists $H, G, g < \infty$ such that for all $k \in \mathbb{N}$ then

$$\|\mathscr{L}_{kT} h\|_{\mathbf{T}} \leq e^{-kT\tilde{\lambda}} \|h\|_{\mathbf{B}} + Ge^{gkT} \|\partial_{x_2}(v \cdot h)\|_{\mathbf{L^1}} + H\|h\|_{\mathbf{L^1}}. \tag{3.4.17}$$

We let $H := C_T(1 - \frac{1}{2}e^{-\tilde{\lambda}T})^{-1}$, $g := (C + \tilde{\lambda})p$ and $G := (2C)^p e^{-CT}(1 - \frac{1}{2}e^{-T(\tilde{\lambda}+g)})^{-1}$. For the case $k = 1$ then that (3.4.17) holds true is immediate from (3.4.16) choosing $\epsilon = e^{-\lambda T}$. We now suppose that (3.4.17) holds true for k and will show this implies the same for $k+1$. Notice that $\|\mathscr{L}_t h\|_{\mathbf{L^1}} \leq \|h\|_{\mathbf{L^1}}$ and $\|\partial_{x_2}(v \cdot \mathscr{L}_t h)\|_{\mathbf{L^1}} \leq \|\partial_{x_2}(v \cdot h)\|_{\mathbf{L^1}}$ for all $t \geq 0$ as shown in Lemma 2.1.3 and (3.2.8). Also recall Lemma 3.2.6 that $\|\mathscr{L}_t h\|_{\mathbf{F}} \leq Ce^{Ct}\|h\|_{\mathbf{F}}$ for all $t \geq 0$. Therefore summing the estimates (3.4.17) and (3.4.16) we have

$$\begin{aligned}\|\mathscr{L}_{(k+1)T} h\|_{\mathbf{T}} \leq \tfrac{1}{2}e^{-\tilde{\lambda}T}\Big[&e^{-kT\tilde{\lambda}}\|h\|_{\mathbf{B}} + Ge^{gkT}\|\partial_{x_2}(v \cdot h)\|_{\mathbf{L^1}} + H\|h\|_{\mathbf{L^1}}\Big] \\ &+ \epsilon Ce^{CkT}\|h\|_{\mathbf{F}} + C\epsilon^{-p}\|\partial_{x_2}(v \cdot h)\|_{\mathbf{L^1}} + C_T\|h\|_{\mathbf{L^1}},\end{aligned}$$

where this estimate holds for any choice of $\epsilon > 0$. We choose $\epsilon = \frac{1}{2}C^{-1}e^{-CkT}e^{-(k+1)\tilde{\lambda}T}$. This means that

$$\|\mathscr{L}_{(k+1)T} h\|_{\mathbf{T}} \leq e^{-\tilde{\lambda}T(k+1)}\|h\|_{\mathbf{B}} + H'\|h\|_{\mathbf{L^1}} + G'\|\partial_{x_2}(v \cdot h)\|_{\mathbf{L^1}},$$

where $H' = \frac{1}{2}e^{-\tilde{\lambda}T}H + C_T$ and $G' = (2Ce^{CkT}e^{(k+1)\tilde{\lambda}T})^p + \frac{1}{2}e^{-\tilde{\lambda}T}Ge^{gkT}$. Because H, G and g were chosen appropriately then $G' \leq Ge^{g(k+1)T}$ and $H' \leq H$. We have thereby shown that (3.4.17) holds for $k+1$ which completes the proof by induction. Increasing the constants as required means that the formula holds for all $t \geq 0$ rather than just $t \in \{kT, k \in \mathbb{N}\}$. □

3.5. Resolvent Operator

In this section we estimate $\|R_z^n h\|_{\mathbf{B}}$ and so prove Proposition 3.2.9. We achieve this by taking advantage of the smoothness of $R_z h$ in the flow direction.

Lemma 3.5.1. *Suppose* $t \geq 0$, $h \in \mathscr{C}_f^1$. *Then* $\partial_{x_2}(v \cdot \mathscr{L}_t h) = -\frac{d}{dt}\mathscr{L}_t h$.

Proof. Let $(x_1, x_2) \in \Omega \setminus ([0,1] \times \{0\})$. According to Lemma 3.2.2 there exists $s > 0$ such that for all $t \in [0, s]$ holds $\mathscr{L}_t h(x_1, x_2) = (v \cdot h)(x_1, \varphi_{-t}(x_1, x_2)) \cdot v(x_1, x_2)^{-1}$. Differentiating this expression with respect to t and noting that $\frac{d}{dt}\varphi_t(x_1, x_2) = v(x_1, \varphi_t(x_1, x_2))$ we have shown that

$$\left.\frac{d}{dt}\mathscr{L}_t h(x_1, x_2)\right|_{t=0} = -\partial_{x_2}(v \cdot h)(x_1, x_2).$$

As $\frac{d}{dt}\mathscr{L}_t h(x_1, x_2)|_{t=s} = \frac{d}{dt}\mathscr{L}_t(\mathscr{L}_s h)(x_1, x_2)|_{t=0} = -\partial_{x_2}(v \cdot \mathscr{L}_s h)(x_1, x_2)$ this implies the claimed result for all $t \geq 0$. □

Lemma 3.5.2. *Suppose* $\Re(z) > \alpha$, $h \in \mathbf{B}$. *Then*

$$\begin{aligned} \|R_z h\|_{\mathbf{F}} &\leq |z| \, \|R_z h\|_{\mathbf{S}} + \|h\|_{\mathbf{S}}, \\ \|\partial_{x_2}(R_z h \cdot v)\|_{\mathbf{L^1}} &\leq |z| \, \|R_z h\|_{\mathbf{L^1}} + \|h\|_{\mathbf{L^1}}. \end{aligned} \tag{3.5.1}$$

Proof. Recall the integral formula (3.2.12) for R_z and the result of Lemma 3.5.1. This implies that for all $\Re(z) > \alpha$ and $h \in \mathscr{C}_f^1$ then

$$\partial_{x_2}(v \cdot R_z) = -\int_0^\infty e^{-zt}\left(\tfrac{d}{dt}\mathscr{L}_t h\right) \, dt.$$

Integrating by parts in the above we have shown that

$$\partial_{x_2}(v \cdot R_z h) = h - zR_z h \tag{3.5.2}$$

for all $h \in \mathscr{C}_f^1$, $\Re(z) > \alpha$. □

Lemma 3.5.3. *Exists* $C < \infty$ *such that for all* $n \in \mathbb{N}$, $h \in \mathbf{B}$

$$\begin{aligned} \|R_z^n h\|_{\mathbf{L^1}} &\leq \Re(z)^{-n}\|h\|_{\mathbf{L^1}}, \\ \|R_z^n h\|_{\mathbf{B}} &\leq C(\Re(z) - \alpha)^{-n}\|h\|_{\mathbf{B}}. \end{aligned}$$

Proof. Recall the estimate $\|\mathscr{L}_t h\|_{\mathbf{L^1}} \leq \|h\|_{\mathbf{L^1}}$ from Lemma 2.1.3 and furthermore the estimate $\|\mathscr{L}_t h\|_{\mathbf{B}} \leq Ce^{\alpha t}\|h\|_{\mathbf{B}}$ from Proposition 3.2.8. Using the integral formula for the resolvent (3.2.12) in each case and integrating gives the required result. □

Lemma 3.5.4. *Suppose $\Re(z) > \alpha$. There exists $C_z < \infty$ such that*

$$\|R_z^n h\|_{\mathbf{T}} \leq C_z(\Re(z) + \lambda)^{-n}\|h\|_{\mathbf{B}} + C_z\|h\|_{\mathbf{L^1}}$$

for all $n \in \mathbb{N}$, $h \in \mathbf{B}$.

Proof. Recall that in Proposition 3.2.7 we have the estimate $\|\mathscr{L}_t h\|_{\mathbf{T}} \leq Ce^{-\lambda t}\|h\|_{\mathbf{B}} + Ce^{\alpha t}\left(\|\partial_{x_2}(h \cdot v)\|_{\mathbf{L^1}} + \|h\|_{\mathbf{L^1}}\right)$. Integrating holds

$$\|R_z^n h\|_{\mathbf{T}} \leq C(\Re(z) + \lambda)^{-n}\|h\|_{\mathbf{B}} + C(\Re(z) - \alpha)^{-n}\left(\|\partial_{x_2}(h \cdot v)\|_{\mathbf{L^1}} + \|h\|_{\mathbf{L^1}}\right).$$

Using now Lemma 3.5.2 to estimate $\|\partial_{x_2}(R_z h \cdot v)\|_{\mathbf{L^1}}$ we have

$$\begin{aligned}\|R_z^{n+1} h\|_{\mathbf{T}} \leq C(\Re(z) + \lambda)^{-n}\|R_z h\|_{\mathbf{B}} \\ + C(\Re(z) - \alpha)^{-n}\left(|z|\,\|R_z h\|_{\mathbf{L^1}} + \|h\|_{\mathbf{L^1}} + \|R_z h\|_{\mathbf{L^1}}\right).\end{aligned}$$

We conclude using Lemma 3.5.3 to estimate $\|R_z h\|_{\mathbf{B}}$ and $\|R_z h\|_{\mathbf{L^1}}$ and then choosing C_z. □

Lemma 3.5.5. *Suppose $\Re(z) > \alpha$. There exists $C_z < \infty$ such that*

$$\|R_z^n h\|_{\mathbf{F}} \leq C_z(\Re(z) + \lambda)^{-n}\|h\|_{\mathbf{B}} + C_z\|h\|_{\mathbf{L^1}}$$

for all $n \in \mathbb{N}$, $h \in \mathbf{B}$.

Proof. Combining the estimates from Lemma 3.5.3, Lemma 2.1.8 and Lemma 3.5.2 means that $\|R_z h\|_{\mathbf{F}} \leq |z|\,\|R_z h\|_{\mathbf{S}} + \|h\|_{\mathbf{S}} \leq C_z\|h\|_{\mathbf{S}} \leq C_z\|h\|_{\mathbf{T}}$. We conclude by using the estimate for $\|h\|_{\mathbf{T}}$ from Lemma 3.5.4. □

Proof of Proposition 3.2.9. Suppose $\Re(z) > \alpha$. Combining the estimates from Lemma 3.5.4 and Lemma 3.5.5 we have shown that there exists $C_z < \infty$ such that

$$\|R_z^n h\|_{\mathbf{B}} \leq C_z(\Re(z) + \lambda)^{-n}\|h\|_{\mathbf{B}} + C_z\|h\|_{\mathbf{L^1}}$$

for all $n \in \mathbb{N}$, $h \in \mathbf{B}$. Suppose that $\ell \in (0, \lambda)$ and choose $m < \infty$ such that $C_z(\Re(z) + \lambda)^{-m} \leq (\Re(z) + \ell)^{-m}$. Holds

$$\|R_z^m h\|_{\mathbf{B}} \leq (\Re(z) + \ell)^{-m} \|h\|_{\mathbf{B}} + C_z \|h\|_{\mathbf{L^1}} \tag{3.5.3}$$

for all $h \in \mathbf{B}$. Let $\tilde{C}_z := C_z[\Re(z)^{-m} - (\Re(z) + \ell)^{-m}]^{-1} < \infty$. We claim that

$$\|R_z^{km} h\|_{\mathbf{B}} \leq (\Re(z) + \ell)^{-km} \|h\|_{\mathbf{B}} + \tilde{C}_z \Re(z)^{-km} \|h\|_{\mathbf{L^1}} \tag{3.5.4}$$

for all $k \in \{1, 2, \ldots\}$, $h \in \mathbf{B}$. We prove this claim by induction on k. The case $k = 1$ is exactly (3.5.3). Applying (3.5.3) to R_z^{km} and then using the inductive hypothesis and recalling that $\|R_z^n h\|_{\mathbf{L^1}} \leq \Re(z)^{-n} \|h\|_{\mathbf{L^1}}$ as estimated in Lemma 3.5.3 we have

$$\begin{aligned}
\|R_z^{(k+1)m} h\|_{\mathbf{B}} &\leq (\Re(z) + \ell)^{-m} \|R_z^{km} h\|_{\mathbf{B}} + C_z \|R_z^{km} h\|_{\mathbf{L^1}} \\
&\leq (\Re(z) + \ell)^{-(k+1)m} \|h\|_{\mathbf{B}} \\
&\quad + [\tilde{C}_z (\Re(z) + \ell)^{-m} + C_z] \Re(z)^{-km} \|h\|_{\mathbf{L^1}}.
\end{aligned}$$

We see that $\tilde{C}_z$ was appropriately chosen as $\tilde{C}_z(\Re(z) + \ell)^{-m} + C_z = \tilde{C}_z \Re(z)^{-m}$ which completes the proof of (3.5.4) and also completes the proof of the proposition. □

3.6. Essential Spectrum

In this section we complete the proof of the main theorem. This is Hennion's argument [39] based on the Nussbaum's formula for the essential spectral radius [62]. We recall that the *spectral radius* of $R_z : \mathbf{B} \to \mathbf{B}$ is defined as $r_{\mathrm{spec}}(R_z) := \lim_{n \to \infty} \|R_z^n\|_{\mathbf{B}}^{1/n}$ and the *essential spectral radius* is defined as

$$r_{\mathrm{ess}}(R_z) := \inf \{ r_{spec}(R_z - K) : K \in \mathscr{B}(\mathbf{B}, \mathbf{B}) \text{ is compact } \}.$$

We say that an operator is *quasi-compact* if the essential spectral radius is strictly smaller than the spectral radius. The following Lemma is a consequence of the Lasota-Yorke type estimate of Proposition 3.2.9 and the compactness property shown in Lemma 2.1.10.

Lemma 3.6.1. *Suppose $\Re(z) > \alpha$. Then for $R_z : \mathbf{B} \to \mathbf{B}$ holds*

$$r_{\text{spec}}(R_z) \leq \Re(z)^{-1} \quad \textit{and} \quad r_{\text{ess}}(R_z) \leq (\Re(z) + \lambda)^{-1}.$$

Proof. The estimate of Proposition 3.2.9 implies that for all $\Re(z) > \alpha$ there exists $C_z < \infty$ such that $\|R_z^n h\|_{\mathbf{B}} \leq C_z \Re(z)^{-n}$ for all $n \in \mathbb{N}$ sufficiently large. This immediately implies that $r_{\text{spec}}(R_z) \leq \Re(z)^{-1}$. To bound the essential spectral radius we define

$$B_n := \{R_z^n h \in \mathbf{B} : \|h\|_{\mathbf{B}} \leq 1\}$$

and let r_n denote the infimum of the r such that the set B_n may be covered by a finite number of balls of radius r. The formula of Nussbaum [62] says that

$$r_{ess}(R_z) = \liminf_{n \to \infty} \sqrt[n]{r_n}. \tag{3.6.1}$$

Notice that if B_n is compact then $r_n = 0$. As we don't expect R_z to be compact then we don't expect B_n to be compact. However, we have by Lemma 2.1.10 that B_0 is relatively compact in the $\|\cdot\|_{\mathbf{L}^1}$ norm and therefore, for each $\epsilon > 0$, there exists a finite set $\{G_i\}_{i=1}^{N_\epsilon}$ of subsets of B_0 whose union covers B_0 and such that

$$\|h - \tilde{h}\|_{\mathbf{L}^1} \leq \epsilon \quad \text{for all } h, \tilde{h} \in G_i. \tag{3.6.2}$$

Notice that r_n can be bounded above by the supremum of the diameters of the elements of any given cover of B_n. Since the union of $\{G_i\}_{i=1}^{N_\epsilon}$ is a cover of B_0, then $\{R_z^n(G_i)\}_{i=1}^{N_\epsilon}$ is a cover of B_n and therefore it is sufficient to obtain an upper bound for the maximum diameter of the $R_z^n(G_i)$. We use the estimate on $\|R_z^n h\|_{\mathbf{B}}$ from Proposition 3.2.9. Holds for all $h, \tilde{h} \in G_i$, $\tilde{\lambda} \in (0, \lambda)$ and $n \in \mathbb{N}$ sufficiently large that

$$\|R_z^n h - R_z^n \tilde{h}\|_{\mathbf{B}} \leq C_z \left[(\Re(z) + \tilde{\lambda})^{-n} \|h - \tilde{h}\|_{\mathbf{B}} + \|h - \tilde{h}\|_{\mathbf{L}^1} \right].$$

Substituting (3.6.2) we have shown that $r_n \leq C_z[(\Re(z) + \tilde{\lambda})^{-n} + \epsilon]$. We choose $\epsilon = \epsilon(n)$ small enough so that $r_n \leq C_z(\Re(z) + \tilde{\lambda})^{-n}$ and so (3.6.1) implies the desired estimate on the essential spectral radius. □

We now consider the spectrum of the generator Z. Lemma 3.6.1 implies that for all $\Re(z) > \alpha$ then $\|R_z h\|_{\mathbf{B}} \leq \Re(z)$. As $\operatorname{dist}(z, \operatorname{Spec}(Z)) \geq \|R_z\|_{\mathbf{B}}^{-1} \geq \Re(z)$ this estimate implies that $\operatorname{Spec}(Z)$ is contained in $\{\Re(z) \leq 0\}$. Fix for the moment $\Re(z) > 0$. Then for all $\eta \in \mathbb{C}$ holds

$$(\eta \mathbf{Id} - R_z)(z\mathbf{Id} - Z) = \eta\left((z + \eta^{-1})\mathbf{Id} - Z\right). \tag{3.6.3}$$

From the essential spectral radius estimate of Lemma 3.6.1 we know that the operator valued function $\eta \mapsto (\eta \mathbf{Id} - R_z)^{-1}$ is meromorphic on the set $\{|\eta| > (\Re(z) + \lambda)^{-1}\}$. Letting $\zeta := z + \eta^{-1}$ the relationship (3.6.3) implies that the operator valued function $\zeta \mapsto (\zeta \mathbf{Id} - Z)^{-1}$ is meromorphic on $\{|\zeta - z| \leq \Re(z) - \lambda\}$. Now letting z vary in the set $\{\Re(z) > 0\}$ we see that $\zeta \mapsto (\zeta \mathbf{Id} - Z)^{-1} = R_\zeta$ is meromorphic on $\{\Re(\zeta) > -\lambda\}$. This implies that the spectrum in this region consists of isolated points which are eigenvalues of finite multiplicity. This completes the proof of The Main Theorem.

A. Smooth Anosov flows: Correlation spectra and stability.

This chapter contains verbatim the published article co-authored with Liverani [17]. By introducing appropriate Banach spaces one can study the spectral properties of the generator of the semigroup defined by an Anosov flow. Consequently, it is possible to easily obtain sharp results on the Ruelle resonances and the differentiability of the SRB measure.

A.1. Introduction

In the last years there has been a growing interest in the dependence of the SRB measures on the parameters of the system. In particular, G.Gallavotti [32] has argued the relevance of such an issue for non-equilibrium statistical mechanics.

On a physical basis (linear response theory) one expects that the average behaviour of an observable changes smoothly with parameters. Yet the related rigorous results are very limited and the existence of very irregular dependence from parameters (think, for example, to the quadratic family) shows that, in general, *smooth dependence* must be properly interpreted to have any chance to hold.

The only cases in which some simple rigorous results are available are smooth uniformly hyperbolic systems and some partially hyperbolic systems. In particular, Ruelle [72] has proved differentiability and has provided an explicit (in principle computable) formula for the derivative in the case of SRB measures for smooth hyperbolic diffeomorphisms. Subsequently, D.Dolgopyat has extended such results to a large class of partially hyperbolic systems [28]. More recently Ruelle has obtained similar results for Anosov flows [75]. Ruelle's proofs of the above results use the classical thermodynamic formalism and precise structural stability results which, although reasonably efficient for diffeomorphisms, produce a quite cumbersome proof in the case of flows. It should also be remarked that much of the results concerning statistical properties of dynamical systems are related to the analytical properties of the Ruelle zeta function [70, 6]. In the context of Anosov flows such properties have been first elucidated by Pollicott in [64].

In more recent years, several authors have attempted to put forward a different approach to the study of hyperbolic dynamical systems based on the direct study of the transfer operator (see [6] for an introduction to the theory of transfer operators in dynamical systems). Starting with [76, 14] it has become clear that it is possible to construct appropriate functional spaces such that the statistical properties of the systems are accurately described by the spectral data of the operator acting on such spaces. The recent papers [50, 48, 34, 7, 11, 51, 30, 24, 52, 12, 35], have shown that such an approach yields a simpler and far reaching alternative to the more traditional point of view based on Markov partitions.

In this paper we present an application of these methods to the above mentioned issue: the differentiability properties of the SRB measure for Anosov flows. Not only the formulae in [72] are easily recovered, but higher differentiability is obtained as well whereby making rigorous some of the results in [73]. In addition, the method employed yields naturally precise information on the structure of the Ruelle resonances extending the results in [64, 77].

Note that the same strategy can be used to prove differentiability (and obtain in principle computable formulae) for many other physically relevant quantities (at least for $\mathscr{C}^\infty$ flows) such as: Ruelle's resonances and eigendistributions, the variance in the central limit theorem (diffusion constant), the rate in the large deviations. Also a small generalization of the present approach, that is considering transfer operators with real potential, would apply to general Gibbs measures. This would allow, for example, to obtain an easy alternative proof of the results in [45].

The key reason for the straightforwardness of the present approach is that, once the proper functional setting is established, the usual formal manipulations to compute the derivative are rigorously justified whereby making the argument totally transparent.

The spaces used here are the ones introduced in [34] although similar results could, most likely, be obtained by using the spaces introduced in [11, 12].

Recently some new results have been obtained on the stability of mixing [31]. It would be interesting to investigate the relationship between such qualitative results and the quantitative theory in this paper.

Finally, it should be remarked that the approach of the present paper is based on the study of the resolvent, rather than the semigroup, in the spirit of [51]. Nevertheless, a recent paper by M.Tsujii [84] has shown that it is possible to introduce Banach spaces allowing the direct study of the semigroup, although limited to the case of suspensions over an expanding endomorphism. Such an approach yields much stronger results. To construct similar spaces for flows and, possibly, other classes of partially hyperbolic systems is one of the current challenges of the field.

The plan of the paper is as follows: Section A.2 details the systems we consider, introduces the norms we use and corresponding Banach spaces and states the results. In section A.3 we precisely define the Banach spaces relevant for our approach and study some of their properties. In section A.4 we look at the the properties of the transfer operator in this setting and discuss the spectral decomposition of its generator. In section A.5 we give results on the behaviour of the part of the spectrum close to the imaginary axis and in A.6 discuss specifically the behaviour of the SRB measure as the dynamical system is perturbed and, in the course of this, the Ruelle formula for the derivative is established. In section A.7 the main dynamical inequalities are proven for the transfer operator while in section A.8 the corresponding inequalities are established for the resolvent of the generator of the flow. The paper also includes an appendix in which some necessary technical (but intuitive) facts are proven.

Remark A.1.1. In the present paper we will use C to designate a generic constant depending only on the Dynamical Systems $(\mathscr{M}, T_t)$, while $C_{a,b,\dots}$ will be used for a generic constant depending also on the parameters $a, b, \dots$. Accordingly, the actual numerical value of C may vary from one occurrence to the next.

A.2. Statements and results

Let us consider the $\mathscr{C}^\infty$ d-dimensional compact Riemannian manifold $\mathscr{M}$ and the Anosov flow $T_t \in \mathrm{Diff}(\mathscr{M}, \mathscr{M})$. In other words the following conditions are satisfied.

Condition 1. *T satisfies the following*

$$
\begin{aligned}
&T_0 = \mathbf{Id}, \\
&T_p \circ T_q = T_{p+q} \quad \text{for each } p, q \in \mathbb{R}.
\end{aligned}
$$

That is T_t is a flow.

Condition 2. *At each point $x \in \mathscr{M}$ there exists a splitting of tangent space $T_x\mathscr{M} = E^s(x) \oplus E^f(x) \oplus E^u(x)$, $x \in \mathscr{M}$. The splitting is continuous and invariant with respect to T_t. E^f is one dimensional and coincides with the flow direction. In addition, for each $\nu \in E^f$, $DT_t\nu = 0 \implies \nu = 0$ and there exist $\lambda > 0$ such that*

$$
\begin{aligned}
\|DT_t\nu\| < e^{-\lambda t}\|\nu\| \quad &\text{for each } \nu \in E^s \text{ and } t \geq 0, \\
\|DT_{-t}\nu\| < e^{-\lambda t}\|\nu\| \quad &\text{for each } \nu \in E^u \text{ and } t \geq 0.
\end{aligned}
$$

That is the flow is Anosov.[1]

A smooth flow naturally defines a related vector field V. Often the vector field is a more fundamental object than the flow, we will thus put our smoothness requirement directly on the vector field.

Condition 3. *We assume $V \in \mathscr{C}^{r+1}$, $r > 1$.[2] This implies $T_t \in \mathscr{C}^{r+1}$.*

To study the statistical properties of such systems it is helpful to study the action of

[1] In general one can have a $Ce^{-\lambda t}$ instead of $e^{-\lambda t}$ in the first two inequalities, yet it is always possible to change the Riemannian structure in order to have $C = 1$ by losing a little bit of hyperbolicity (e.g., define $\langle v, w\rangle_L := \int_{-L}^{L} e^{2\lambda'|s|}\langle DT_t v, DT_t w\rangle ds$ with $\lambda' < \lambda$ and L such that $Ce^{(\lambda'-\lambda)L} < 1$).

[2] The reason for such a condition, instead of the more natural $r > 0$, is purely technical and rests in the limitation $p \in \mathbb{N}$ for the spaces $\mathbf{B}^{p,q}$ used in the following. Most likely it could be removed either using the spaces in [11] or generalizing the present spaces.

the dynamics on distributions. To this end let us define $\mathscr{L}_t : \mathscr{D}'_{r+1} \to \mathscr{D}'_{r+1}$ by[3]

$$\langle \mathscr{L}_t h, \varphi \rangle := \langle h, \varphi \circ T_t \rangle, \qquad \text{for all } \varphi \in \mathscr{C}^{r+1}. \tag{A.2.1}$$

It is easy to see that the $\mathscr{L}_t$ are continuous.

Remark A.2.1. Given the standard continuous embedding[4] $\mathbf{i} : \mathscr{C}^r \hookrightarrow \mathscr{D}'_r$ we can, and we will, view functions as distributions. In particular, if $h \in \mathscr{C}^r$, then it can be viewed as the density of the absolutely continuous measure $\mathbf{i}h$. In such a case a simple computation shows that, setting

$$\widetilde{\mathscr{L}_t} h := [h \det(DT_t)^{-1}] \circ T_t^{-1}, \tag{A.2.2}$$

holds $\mathbf{i}\widetilde{\mathscr{L}_t} = \mathscr{L}_t \mathbf{i}$. Formula (A.2.2) provides a more common expression for the transfer operator.

Unfortunately it turns out that the spectral properties of $\mathscr{L}_t$ on the above spaces bear not clear relation with the statistical properties of the system. To establish such a connection in a fruitful way it is necessary to introduce Banach spaces that embody in their inner geometry the key properties of the system (that is the hyperbolicity).

The first step is to define appropriate norms on $\mathscr{C}^\infty(\mathscr{M}, \mathscr{C})$ and then take the closure in the relative topology. The exact definition of the norms can be found in section A.3, yet let us give here a flavor of the construction.

For each $p \in \mathbb{N}$, $q \in \mathbb{R}_+$, consider a set Σ of manifolds of roughly uniform size and *close* to the strong stable manifolds and let $\mathcal{V}$ be the set of smooth vector fields (see section A.3 for precise definitions). For each $W \in \Sigma$, $v_1, \dots, v_p \in \mathcal{V}$ and $\varphi \in \mathscr{C}_0^{p+q}(W, \mathscr{C})$ we can then define the linear functionals on $\mathscr{C}^\infty(\mathscr{M}, \mathscr{C})$,[5]

$$\ell_{W, v_1, \dots, v_p, \varphi}(h) := \int_W \varphi v_1 \cdots v_p h$$

[3]In the following we will use indifferently $\langle h, \varphi \rangle$ and $h(\varphi)$ to designate the action of the distribution h on the smooth function φ.

[4]If $g, f \in \mathscr{C}^r$, then $\langle \mathbf{i}f, g \rangle := \int_{\mathscr{M}} fg$.

[5]Here, and in the following, the integrals are meant with respect to the induced Riemannian metric. Moreover, given a vector field v and a function h, by vh or $v(h)$ we mean the Lie derivative of h along v.

and the *dual* ball

$$\mathscr{U}_{p,q} := \left\{ \ell_{W,v_1,\dots,v_p,\varphi} \mid W \in \Sigma,\ |\varphi|_{\mathscr{C}_0^{q+p}} \le 1,\ |v_i|_{\mathscr{C}^{q+p}} \le 1 \right\}.$$

We can finally define the norms we are interested in:

$$\begin{aligned} \|h\|^-_{p,q} &:= \sup_{\ell \in \mathscr{U}_{p,q}} \ell(h) \qquad && \forall p \in \mathbb{N},\ q \in \mathbb{R}_+ \\ \|h\|_{p,q} &:= \sup_{n \le p} \|h\|^-_{n,q} \qquad && \forall p \in \mathbb{N},\ q \in \mathbb{R}_+, \end{aligned} \tag{A.2.3}$$

where the parameter $A \in (0,1)$ will be chosen later. We define the spaces $\mathbf{B}^{p,q} := \overline{\mathscr{C}^\infty(\mathscr{M},\mathscr{C})}^{\|\cdot\|_{p,q}}$. Note that such spaces are equivalent to the ones defined in Section 2 of [34], the only difference being in their use: there they depend on the stable cone of an Anosov diffeomorphism, here they depend on the *strong* stable cone of an Anosov flow. Consequently we will often refer to results proved in [34].

A first relevant property of the spaces $\mathbf{B}^{p,q}$ has been proven in [34, Lemma 2.1]:

Lemma A.2.2. *For each $p \in \mathbb{N}_*, q \in \mathbb{R}_+$ holds $\|\cdot\|_{p-1,q+1} \le C_{p,q,A}\|\cdot\|_{p,q}$. In addition, the unit ball of $\mathbf{B}^{p,q}$ is relatively compact in $\mathbf{B}^{p-1,q+1}$.*

It is easy to show that $\mathscr{L}_t : \mathbf{B}^{p,q} \to \mathbf{B}^{p,q}$, with $p+q<r$, is a bounded strongly continuous semigroup (Lemma A.4.2), in addition the semigroup is uniformly bounded in t, Lemma A.4.1. Accordingly, by general theory, the generator X of the semi-group is a closed operator. Clearly, the domain $D(X) \supset \mathscr{C}^{r+1}(\mathscr{M},\mathscr{C})$ and, restricted to $\mathscr{C}^{r+1}(\mathscr{M},\mathscr{C})$, X is nothing else but the action of the adjoint of the vector field defining the flow, that is

$$Xh = -V(h) - h \operatorname{div} V \ \in \mathscr{C}^r. \tag{A.2.4}$$

Obviously, the spectral properties of the generator depend on the resolvent $R(z) = (z\mathbf{Id} - X)^{-1}$. It is well known (e.g. see [22]) that for uniformly bounded semigroup (Lemma A.4.1) the spectrum of X is contained in $\{z \in \mathscr{C} \ :\ \Re(z) \le 0\}$. That is, for all $z \in \mathscr{C}$, $\Re(z) > 0$, the resolvent $R(z)$ is a well defined bounded operator on $\mathbf{B}^{p,q}$ and, moreover, holds true the formula

$$R(z)f = \int_0^\infty e^{-zt}\mathscr{L}_t f\,\mathrm{dt}\,. \tag{A.2.5}$$

The above facts allow us to establish several facts concerning the spectrum of the generator.

Theorem A.2.3. *For each $p \in \mathbb{N}, q \in \mathbb{R}_+$, $p + q \le r$, the spectrum of the generator, acting on $\mathbf{B}^{p,q}$, in the strip $0 \ge \Re(z) > -\min\{p, q\}\lambda$ consists only of isolated eigenvalues of finite multiplicity. Such eigenvalues correspond to the Ruelle resonances (see Remark A.2.4 for more details). In addition, the eigenspace associated to the eigenvalue zero is the span of the SRB measures.[6] The SRB measure is unique iff the eigenvalue is simple and it is mixing iff zero is the only eigenvalue on the imaginary axis.*

The first statement is proven in Lemma A.4.5, the second, and more, in Lemma A.5.1. The above theorem extends the well known results of Pollicott and Rugh [64, 77] to the higher regularity and higher dimensional setting. Indeed we can connect the above results to physically relevant quantities: the *correlations spectrum.*

Let $f, g \in \mathscr{C}^\infty$, then one is interested in $C_{f,g}(t) := \int g \circ T_t f - \int f \int g$ where the integral may be with respect to Lebesgue or to the SRB measure depending on whether one is observing the system in equilibrium or out of equilibrium starting from a state properly prepared.

Remark A.2.4. A typical information that can be obtained on the quantity $C_{f,g}$ is its

[6] Here we adopt the following definition of SRB measure: a measure ν is SRB if there exists a positive Lebesgue measure open set U such that $\forall\, \varphi \in \mathscr{C}^0$ and Lebesgue a.-e. $x \in U$

$$\frac{1}{T}\int_0^T \varphi \circ T_t(x)\,\mathrm{dt} \to \nu(\varphi).$$

The above implies, in the present setting, all the usual properties of SRB measures (e.g. absolute continuity along weak unstable manifold) that we do not detail as they will not be used in the following. We will only use, at the end of the proof of Lemma A.5.1, that the union of the basins of all the SRB measures is of full Lebesgue measure, that is: for each continuous function the forward ergodic average exists Lebesgue-a.s.

Fourier transform

$$\hat{C}_{f,g}(k) := \int_0^\infty e^{-ikt} C_{f,g}(t)\,\mathrm{dt} = \int \left(g - \int g\right) R(ik)f.$$

The above results imply thus that the quantity $\hat{C}_{f,g}$ has a meromorphic extension in the strip $0 \geq \Re(z) > -\min\{p,q\}\lambda$. In addition, in such a region, the poles (the so called *Ruelle resonances*) and their residues describe (and are described by) exactly the spectrum of X. In particular this means that the spectral data of X on the Banach spaces $\mathbf{B}^{p,q}$ are not a mathematics nicety but physically relevant quantities.

Given such a spectral interpretation it is then easy to apply the perturbation theory of [34] and obtain our other main result.

Let us consider a vector field $V_\epsilon := V + \epsilon V_1 \in \mathscr{C}^{r+1}$ and the associated flow $T_{\epsilon,t}$. Suppose, for simplicity, that $T_{0,t}$ has a unique SRB measure. The issue is to show that $T_{\epsilon,t}$ has a unique SRB measure μ_ϵ as well, that such a measure is a smooth function of ϵ and finally to establish a formula for its derivative.

Let us define $\mu_\epsilon^{(n)} := \frac{d^n}{d\epsilon^n}\mu_\epsilon$. In section A.6 we prove the following.

Theorem A.2.5. *There exists $\epsilon_0 > 0$ such that, if the flow $T_{0,t}$ has a unique SRB measure, then the same holds for the flows $T_{\epsilon,t}$ for $|\epsilon| \leq \epsilon_0$. Calling μ_ϵ such an SRB measure the function $\epsilon \mapsto \mu_\epsilon$ belongs to $\mathscr{C}^{r-2}([-\epsilon_0,\epsilon_0],\mathbf{B}^{0,r})$. In addition, for all $\epsilon \in [-\epsilon_0,\epsilon_0]$ and $\varphi \in \mathscr{C}^r$, it holds the formula*

$$\mu_\epsilon^{(n)}(\varphi) = \lim_{a\to 0^+} \int_0^\infty n e^{-at} \mu_\epsilon^{(n-1)}(V_1(\varphi \circ T_{\epsilon,t}))\,\mathrm{dt}\,.$$

Remark A.2.6. The convergence of the integral in the above formula is far from obvious and it is part of the statement of the Theorem. Notice that for $n = 1$ Theorem A.2.5 yields Ruelle's result [75] while, for $n > 0$, it makes rigorous some of the results in [73]. In addition, if operators X_ϵ has a spectral gap (as may happen for geodesic flows in negative curvature [50]), then from the proof of Theorem A.2.5 follows that the above integral is converging also for $a = 0$ and one has the formula

$$\mu_\epsilon^{(n)}(\varphi) = \int_0^\infty n\mu_\epsilon^{(n-1)}(V_1(\varphi \circ T_{\epsilon,t}))\,\mathrm{dt}\,.$$

A.3. The Banach spaces

To define the norms it is convenient to consider a fixed $\mathscr{C}^{r+1}$ atlas $\{U_i, \Psi_i\}_{i=1}^N$ such that $\Psi_i U_i = B(0, 4\delta)$ and $\cup_i \Psi_i^{-1}(B(0,\delta)) = \mathscr{M}$.[7] In addition, we can require $D_0\Psi_i^{-1}\{(0,u,0) : u \in \mathbb{R}^{d_u}\} = E^u(\Psi_i^{-1}(0))$, $D_0\Psi_i^{-1}\{(s,0,0) : s \in \mathbb{R}^{d_s}\} = E^s(\Psi_i^{-1}(0))$, and $\Psi_i^{-1}((s,u,t)) = T_t\Psi_i^{-1}((s,u,0))$.

Next we wish to define a set of *(strong) stable leaves*. For each $\rho > 0$, small enough, $M > 0$ large enough and $\xi \in B(0,\delta)$ let us define

$$\mathcal{F} := \{F : B(0,3\delta) \subset \mathbb{R}^{d_s} \to \mathbb{R}^{d_u+1} \ : \ F(0) = 0\,;\ |F|_{\mathscr{C}^1} \leq \rho\,;\ |F|_{\mathscr{C}^r} \leq M\}.$$

For each $F \in \mathcal{F}$, let us define $G_{x,F}(\xi) := x + (\xi, F(\xi))$. Also let us define $\widetilde{\Sigma} := \{G_{x,F} : x \in B(0,\delta), F \in \mathcal{F}\}$. To each $i \in \{1,\dots,N\}$, $G \in \widetilde{\Sigma}$ we associate the leaf $W_{i,G} = \{\Psi_i^{-1}G(\xi)\}_{\xi\in B(0,2\delta)}$, which form our set of stable leaves Σ, and its reduced and enlarged version $W_{i,G}^{\pm} = \{\Psi_i^{-1}G(\xi)\}_{\xi\in B(0,(2\pm1)\delta)}$.

Integrating on such leaves we can define linear functionals on $\mathscr{C}^r(\mathscr{M},\mathbb{R})$. More precisely, for each $i \in \{1,\dots,N\}$, $s \in \mathbb{N}$, $G \in \widetilde{\Sigma}$, $\varphi \in \mathscr{C}_0^0(\overline{W_{i,G}},\mathscr{C})$ and $\mathscr{C}^s$ vector fields $v_1,\dots,v_s$, defined in a neighbourhood of $W_{i,G}^+$, we define

$$\ell_{i,G,\varphi,v_1,\dots,v_s}(h) := \int_{W_{i,G}} \varphi\, v_1 \cdots v_s h\,; \quad \forall\, h \in \mathscr{C}^r(\mathscr{M},\mathscr{C}).$$

We use the above functionals to define a set that can be intuitively interpreted as the unit ball of the dual of the space we wish to define. For $p \in \mathbb{N}$, $q \in \mathbb{R}_+$, let[8]

$$\mathscr{U}_{p,q} := \left\{\ell_{i,G,\varphi,v_1,\dots,v_p} \mid 1 \leq i \leq N,\, G \in \widetilde{\Sigma},\, |\varphi|_{\mathscr{C}_0^{q+p}} \leq 1,\, |v_j|_{\mathscr{C}^{q+p}} \leq 1, \right\},$$

The norms $\|\cdot\|_{p,q}$ are then defined in A.2.3.

Remark A.3.1. Note that for each $h \in \mathscr{C}^\infty(\mathscr{M},\mathscr{C})$ and $q \in \mathbb{R}_+$, $p \in \mathbb{N}$ holds true

[7] Here, and in the following, by $\mathscr{C}^n$ we mean the Banach space obtained by closing $\mathscr{C}^\infty$ with respect to the norm $|f|_{\mathscr{C}^n} := \sup_{k\leq n} |f^{(k)}|_\infty 2^{n-k}$. Such a norm has the useful property $|fg|_{\mathscr{C}^n} \leq |f|_{\mathscr{C}^n}|g|_{\mathscr{C}^n}$, that is $(\mathscr{C}^n, |\cdot|_{\mathscr{C}^n})$ is a Banach algebra.

[8] By $|v_j|_{\mathscr{C}^{q+p}} \leq 1$ we mean that there exists $U = \overset{\circ}{U} \supset W_{i,G}^+$ such that v_j is defined on U and $|v_j|_{\mathscr{C}^{q+p}(U)} \leq 1$.

$\|h\|_{p,q} \leq |h|_{\mathscr{C}^p}$.

We have the following characterization of $\mathbf{B}^{p,q}$, see [34, Proposition 4.1].

Lemma A.3.2. *The embedding* $\mathbf{i}$ *extends to a continuous injection from* $\mathbf{B}^{p,q}$ *to* $\mathscr{D}'_q \subset \mathscr{D}'$*, the distributions of order* q*.*

Remark A.3.3. In the following we will often identify h and $\mathbf{i}h$ if this causes no confusion.

A.4. The transfer operator

A first property of the transfer operators is detailed by the following lemma whose proof is the content of section A.7.

Lemma A.4.1. *For each* $p \in \mathbb{N}$*,* $q \in \mathbb{R}_+$*,* $p + q \leq r$*,* $t \in \mathbb{R}_+$ *and* $h \in \mathscr{C}^r$ *holds true*

$$\|\mathscr{L}_t h\|_{p,q} \leq C_{p,q}\|h\|_{p,q}. \tag{A.4.1}$$

As an immediate consequence we have the following first result.

Lemma A.4.2. *The operators* $\mathscr{L}_t$*, restricted to* $\mathbf{B}^{p,q}$*, form a bounded strongly continuous semigroup on the Banach space* $(\mathbf{B}^{p,q}, \|\cdot\|_{p,q})$*.*

Proof. For all $h \in \mathbf{B}^{p,q}$ there exists, by definition, a sequence $\{h_n\} \subset \mathscr{C}^r$ converging to h in the $\|\cdot\|_{p,q}$ norm. By Lemma A.3.2 the sequence converges in the spaces of distributions as well and, due to the continuity of $\mathscr{L}_t$, $\{\mathscr{L}_t h_n\}$ converges to $\mathscr{L}_t h$ in $\mathscr{D}'_q$. On the other hand, by Lemma A.4.1, $\{\mathscr{L}_t h_n\}$ is a Cauchy sequence in $\mathbf{B}^{p,q}$, hence it converges and, by Lemma A.3.2 again, it must converge to $\mathscr{L}_t h$. Thus $\mathscr{L}_t h \in \mathbf{B}^{p,q}$ and

$$\|\mathscr{L}_t h\|_{p,q} \leq C_{p,q}\|h\|_{p,q} \qquad \forall\, h \in \mathbf{B}^{p,q}.$$

We have thus a semigroup of bounded operators. The strong continuity follows from the fact that, for all $h \in \mathscr{C}^r$, holds

$$\lim_{t\to 0} |\mathscr{L}_t h - h|_{\mathscr{C}^r} = \lim_{t\to 0} \left|[h \det(DT_t)^{-1}] \circ T_t^{-1} - h\right|_{\mathscr{C}^r} = 0.$$

Next, for $h \in \mathbf{B}^{p,q}$ let $\{h_n\} \subset \mathscr{C}^r$ be converging to h, then, using Remark A.3.1,

$$\|\mathscr{L}_t h - h\|_{p,q} \leq \|\mathscr{L}_t h_n - h_n\|_{p,q} + C_{p,q}\|h - h_n\|_{p,q} \leq C_A|\mathscr{L}_t h_n - h_n|_{\mathscr{C}^r} + C_{p,q}\|h - h_n\|_{p,q},$$

taking first n sufficiently large and then t small, one can make the right hand side arbitrarily small, that is

$$\lim_{t\to 0} \|\mathscr{L}_t h - h\|_{p,q} = 0 \qquad \forall\, h \in \mathbf{B}^{p,q}.$$

□

In addition we have the following result, proved in section A.8.

Lemma A.4.3. *For each $p \in \mathbb{N}$, $q \in \mathbb{R}_+$, $p + q \leq r$, $z \in \mathscr{C}$, $\Re(z) = a > 0$, holds*

$$\|R(z)^n\|_{p,q} \leq C_{p,q} a^{-n}.$$

For each $\lambda' \in (0, \lambda)$, $p, n \in \mathbb{N}$, $q \in \mathbb{R}_+$ and $z \in \mathscr{C}$, $a := \Re(z) \geq a_0 > 0$ it holds true

$$\|R(z)^n h\|_{p,q} \leq C_{p,q,\lambda'}(a + \bar{p}\lambda')^{-n}\|h\|_{p,q} + a^{-n} C_{p,q,\lambda',a_0}|z|\, \|h\|_{p-1,q+1},$$

where $\bar{p} := \min\{p, q\}$.

The above means that the spectral radius of $R(z) \in L(\mathbf{B}^{p,q}, \mathbf{B}^{p,q})$, $\Re(z) = a > 0$, is bounded by a^{-1}, and in fact equals it if $z = a$ since $\int R(z)h = a^{-1}\int h$ implies that a^{-1} is an eigenvalue of the dual. Since Lemma A.4.3 implies that $R(z)$ is a bounded operator from $\mathbf{B}^{p,q}$ to itself and since Lemma A.2.2 implies that a bounded ball in the $\|\cdot\|_{p,q}$ norm is relatively compact in $\mathbf{B}^{p-1,q+1}$, it readily follows:

Lemma A.4.4. *For each $p \in \mathbb{N}$, $q \in \mathbb{R}_+$, $p + q < r$, and $z \in \mathscr{C}$, $\Re(z) > 0$ the operator $R(z) : \mathbf{B}^{p,q} \to \mathbf{B}^{p-1,q+1}$ is compact.*

The above implies, via a standard argument [39], that the essential spectral radius of $R(z)$ is bounded by $(a + \lambda\bar{p})^{-1}$. This readily implies the following (see [50, Section2] if details are needed).

Lemma A.4.5. *The spectrum $\sigma(X)$ of the generator is contained in the left half plane. The set $\sigma(X) \cap U_{\bar{p}\lambda'} := \{z \in \mathscr{C} \mid \Re(z) > -\bar{p}\lambda'\}$ consists of, at most, countably many isolated points of point spectrum with finite multiplicity.*

Thanks to the above result we can connect the spectral properties of the generator to the statistical properties of the flow. First of all, by the spectral decomposition of closed operators on Banach spaces (see [42, sections 3.6.4 and 3.6.7]), if we select N isolated eigenvalues from the spectrum we have that

$$X = X_r + \sum_{j=1}^{N} (\zeta_{k_j} S_{k_j} + N_{k_j})$$

where the operators S_k, N_k, X_r commute, the S_k, N_k are finite rank and $S_k S_j = \delta_{kj} S_k$, $N_k S_j = \delta_{kj} N_k$ and N_k is nilpotent. Finally, if the selected eigenvalues are the ones with imaginary part in the interval $[-L, L]$, for some $L > 0$, then X_r is a closed operator with spectrum contained in the set $\{z \in \mathscr{C} \ : \ \Re(z) \leq -p\bar{\lambda}\} \cup \{z \in \mathscr{C} \ : \ \Re(z) \leq 0 \,; |Im(z)| > L\} \cup \{0\}$ where the eigenspaces corresponding to zero is the union of the ranges of the S_k.

A.5. The peripheral spectrum

Here we analyze the meaning of the spectrum on the imaginary axis.

Lemma A.5.1. *The SRB measures belong to $\mathbf{B}^{p,q}$, $p + q \leq r$; $0 \in \sigma(X)$ and it is simple iff the SRB measure is unique. Moreover, the SRB measure is mixing iff 0 is the only eigenvalue on the imaginary axis. Finally, $\sigma(X) \cap i\mathbb{R}$ is a group and the associated eigenfunctions are all measures absolutely continuous with respect to a convex combination of the SRB measures.*

Proof. If $Xh = ibh$, then $\mathscr{L}_t h = e^{ibt} h$. On the other hand there cannot be Jordan blocks, indeed if $Xf = ibf + h$, then $\frac{d}{dt} e^{-ibt} \mathscr{L}_t f = h$, thus $e^{-ibt} \mathscr{L}_t f = f + th$ which, since $\mathscr{L}_t$ is uniformly bounded (Lemma A.4.1), is a contradiction.

Moreover we have[9]

$$\widetilde{S}_b := \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-ibt}\mathscr{L}_t\,\mathrm{dt} = \begin{cases} 0 & \text{if } ib \text{ is not an eigenvalue}\\ S_k & \text{if } ib = \zeta_k \end{cases} \tag{A.5.1}$$

To prove the above note the following. If ib is not an eigenvalue,

$$\begin{aligned}\int_0^T e^{-ibt}\mathscr{L}_t &= \lim_{a\to 0}\int_0^T e^{-(a+ib)t}\mathscr{L}_t = \lim_{a\to 0}\left[R(a+ib) - \int_T^\infty e^{-(a+ib)t}\mathscr{L}_t\right]\\ &= \lim_{a\to 0}\left[R(a+ib) - e^{-(a+ib)T}\mathscr{L}_T\int_0^\infty e^{-(a+ib)t}\mathscr{L}_t\right]\\ &= \lim_{a\to 0}(\mathbf{Id} - e^{-(a+ib)T}\mathscr{L}_T)R(a+ib)\\ &= (\mathbf{Id} - e^{-ibT}\mathscr{L}_T)R(ib),\end{aligned}$$

which is uniformly bounded in T. On the other hand if $ib = \zeta_k$, then $R(a+ib) = (a+ib-\zeta_k)^{-1}S_k + R_1(a+ib)$, where $R_1(z)$ is an analytic function in a neighbourhood of ib [42, 3.6.5 p. 180]. The result then follows by the same computations as above.[10]

Let ν be an SRB measure and let m be the Riemannian (Lebesgue) measure. By definition (cf. footnote 6) there exists an open set A such that, for each $\varphi \in \mathscr{C}^0$ and Lebesgue a.-e. $x \in A$, $\frac{1}{T}\int_0^T \varphi\circ T_t(x)\,\mathrm{dt} \to \nu(\varphi)$. Thus, given $h \in \mathscr{C}^\infty$, $\operatorname{supp} h \subset A$,

[9] The integral must be interpreted in the strong topology.

[10] For further use note that the convergence in (A.5.1) takes place not only in $\mathbf{B}^{p,q}$, $p > 0$, where we have non trivial spectral informations, but also in $\mathbf{B}^{0,q}$. To see it first notice that Lemma A.4.1 implies that for each $h \in \mathbf{B}^{1,q}$, $\|S_0 h\|_{0,q} \le C_q\|h\|_{0,q}$, hence S_0 has a unique continuous extension to $\mathbf{B}^{0,q}$. Next, consider $h \in \mathbf{B}^{0,q}$. There exists $\{h_n\} \subset \mathbf{B}^{1,q}$ such that $\lim_{n\to\infty}\|h - h_n\|_{0,q} = 0$. Moreover, by Lemma A.4.1, $\|T^{-1}\int_0^T \mathscr{L}_t(h_n - h)\|_{0,q} \le C_q\|h - h_n\|_{0,q}$. Thus

$$\limsup_{T\to\infty}\left\|\frac{1}{T}\int_0^T \mathscr{L}_t h - S_0 h_n\right\|_{0,q} \le C_q\|h - h_n\|_{0,q}.$$

To conclude note that the range of S_0 is finite dimensional, hence there exists a convergent subsequence $S_0 h_{n_j}$, let $\bar{h}$ be the limit, then, taking the limit $j \uparrow \infty$ follows $S_0 h = \bar{h}$ and

$$\lim_{T\to\infty}\left\|\frac{1}{T}\int_0^T \mathscr{L}_t h - S_0 h\right\|_{0,q} = 0.$$

$m(h) = 1$, $\forall\, \varphi \in \mathscr{C}^r$ by the Lebesgue dominated convergence and Fubini Theorems

$$\mu_h(\varphi) := S_0 h(\varphi) = \lim_{T\to\infty} \int_{\mathscr{M}} \frac{1}{T} \int_0^T h(x)\varphi(T_t x)\, \mathrm{dt} = \nu(\varphi).$$

In view of Lemma A.3.2, the above implies that $\mu_h = \nu$, that is $\nu \in \mathbf{B}^{p,q}$. In other words the SRB measures belong to the space and are eigenfunction, corresponding to the eigenvalue zero, of X.

Next, let us define $\mu := S_0 1$. The inequality

$$|\mu(\phi)| \leq \lim_{T\to\infty} \frac{1}{T} \int_0^T m(|\phi| \circ T_t)\, \mathrm{dt} \leq |\phi|_\infty$$

shows that μ is a measure. In addition, if $Xh = ibh$ and S is the corresponding projector, since $\mathscr{C}^r$ is dense in $\mathbf{B}^{p,q}$ and $S\mathscr{C}^r$ is finite dimensional, it follows that $S\,\mathbf{B}^{p,q} = S\,\mathscr{C}^r$. Hence there exists $f \in \mathscr{C}^r$ such that $h = Sf$. Accordingly,

$$|h(\varphi)| = |Sf(\varphi)| \leq \lim_{T\to\infty} \int_{\mathscr{M}} \frac{1}{T} \int_0^T \varphi \mathscr{L}_t |f| \leq |f|_\infty \mu(\varphi). \tag{A.5.2}$$

Therefore all the eigenvalues on the imaginary axis are measures and such measures are absolutely continuous with respect to μ and with bounded density.

Consequently, if $Xh = ibh$, then h is a measure and there exists $f \in L^\infty(\mathscr{M}, \mu)$ such that $dh = f d\mu$. But then

$$f\mu = h = e^{-ibt}\mathscr{L}_t h = e^{-ibt}\mathscr{L}_t f\mu = e^{-ibt} f \circ T_{-t}\mathscr{L}_t\mu = e^{-ibt} f \circ T_{-t}\mu,$$

hence $f \circ T_{-t} = e^{ibt} f$ μ-a.s.. The above argument shows that the peripheral spectrum of $\mathscr{L}_t$ on $\mathbf{B}^{p,q}$ is contained, with multiplicity, in the point spectrum of the Koopman operator $U_t f := f \circ T_{-t}$ acting on $L^2(\mathscr{M}, \mu)$. In fact, the two objects coincide as we are presently going to see.

Let $t \in \mathbb{R}_+$ and $f \in L^2(\mathscr{M}, \mu)$ such that $U_t f = e^{ibt} f$. Note that, since $U_t|f| = |f|$, the sets $\{x \in \mathscr{M} \;:\; |f(x)| \leq L\}$ are μ a.s. invariant. Thus we can consider, without loss of generality, the case $f \in L^\infty(\mathscr{M}, \mu)$. By Lusin theorem and the density of $\mathscr{C}^r$ in $\mathscr{C}^0$, for each $\varepsilon > 0$ there exists $f_\varepsilon \in \mathscr{C}^r$, $|f_\varepsilon|_\infty \leq |f|_\infty$, such that $\mu(|f_\varepsilon - f|) \leq \varepsilon$. Next, let us define, for each $f \in L^2(\mathscr{M}, \mu)$, $R'(z)f := \int_0^\infty e^{-zt} U_t f$. A direct computation shows that $R(z)(f\mu) = (R'(z)f)\mu$, $R'(1+ib)f = f$ and $\|f_\varepsilon \mu\|_{0,q} \leq C|f|_\infty$. Accordingly, Lemma A.4.3 implies

$$\begin{aligned}
&\|R(1+ib)^n(f_\varepsilon\mu)\|_{p,q} \leq C_{p,q,\lambda',\varepsilon}(1+\lambda')^{-n} + C_{p,q,\lambda'}|f|_\infty|1+ib| \\
&\mu(|f - R'(1+ib)^n f_\varepsilon|) \leq \mu(R'(1)^n|f - f_\varepsilon|) = \mu(|f - f_\varepsilon|) \leq \varepsilon.
\end{aligned}$$

For each ε we choose n_ε such that $\|R(1+ib)^{n_\varepsilon}(f_\varepsilon\mu)\|_{p,q} \leq 2C_{p,q,\lambda'}|f|_\infty|1+ib|$, thus Lemma A.2.2 implies that the set $\Xi := \{R(1+ib)^{n_\varepsilon}(f_\varepsilon\mu)\}$ is compact in $\mathbf{B}^{p-1,q+1}$. Let us consider a convergent subsequence ε_j, let $\mu_f \in \mathbf{B}^{p-1,q+1}$ be the limit, then for all $\varphi \in \mathscr{C}^{p+q}$,

$$f\mu(\varphi) = \mu(f\varphi) = \lim_{j\to\infty} \mu(R'(1+ib)^{n_{\varepsilon_j}} f_{\varepsilon_j}\varphi) = \lim_{j\to\infty} [R(1+ib)^{n_{\varepsilon_j}} f_{\varepsilon_j}\mu](\varphi) = \mu_f(\varphi).$$

The fact that the spectrum is an additive subgroup of $i\mathbb{R}$, follows then from well known facts about positive operators [22, section 7.4].

To conclude it suffices to prove that all the eigenfunctions of zero are SRB measure. First of all, since the range of S_0 is finite dimensional, $S_0\mathbf{B}^{0,q+p} = S_0\mathbf{B}^{p,q}$, $\mathscr{C}^0$ is dense in $\mathbf{B}^{0,p+q}$, and remembering footnote 10 we have $S_0\mathscr{C}^0 = S_0\mathbf{B}^{p,q}$. Hence for each $\nu \in \mathbf{B}^{p,q}$ there exists $f \in \mathscr{C}^0$ such that $\nu = S_0 f$. On the other hand, setting $f_\pm := \max\{\pm f, 0\} \in \mathscr{C}^0$, $\nu_\pm := S_0 f_\pm$ are invariant positive measures and $\nu = \nu_+ - \nu_-$, thus the range of S_0 has a base of positive probability measures. Next, we can assume, without loss of generality, that ν is an ergodic probability measure. Then, for each $\phi \in \mathscr{C}^0$, $\phi \geq 0$, such that $\int_{\mathscr{M}} f\phi = 1$, we can define $\nu_\phi := S_0(\phi f)$. By a computation similar to (A.5.2), ν_ϕ is a probability measure absolutely continuous with respect to ν, hence, by ergodicity, $\nu = \nu_\phi$. Then for each $\phi \in \mathscr{C}^0$, $\phi > 0$, and $\varphi \in \mathscr{C}^q$, since Lebesgue a.e. point has forward ergodic average (see footnote 6),

$$\begin{aligned}\int_{\mathscr{M}} f\phi[\varphi_+ - \nu(\varphi)] :=& \int_{\mathscr{M}} f\phi\left[\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi\circ T_t - \nu(\varphi)\right]\\ =& \lim_{T\to\infty}\frac{1}{T}\int_{\mathscr{M}} f\phi\int_0^T[\varphi-\nu(\varphi)]\circ T_t\\ =& S_0(f\phi)(\varphi-\nu(\varphi))\int_{\mathscr{M}} f\phi = (\nu(\varphi-\nu(\varphi))\int_{\mathscr{M}} f\phi = 0.\end{aligned}$$

Taking the sup over ϕ, the above yields $\int_{\mathscr{M}} f|\varphi_+ - \nu(\varphi)| = 0$ Accordingly, for Lebesgue almost every point in the support of f the forward average of φ is $\nu(\varphi)$, that is ν is SRB. □

A.6. Differentiability of the SRB measures

It is possible to state very precise results on the dependence of the eigenfunction on a parameter of the system. To give an idea of the possibilities let us analyze, limited to Anosov flows, a situation discussed by Ruelle in [75].

Calling $\mathscr{L}_{\epsilon,t}$ the transfer operator associated to the flow $T_{\epsilon,t}$, X_ϵ its generator and setting $R_\epsilon := (z\mathbf{Id} - X_\epsilon)^{-1}$, it follows that the SRB measure μ_ϵ is an eigenfunction of $R_\epsilon(a)$ corresponding to the eigenvalue a^{-1}. Taking $X_\epsilon = X + \epsilon X_1$ one can prove, by induction,

$$R_\epsilon(a) = \sum_{k=0}^{n} \epsilon^k [R_0(a)X_1]^k R_0(a) + \epsilon^{n+1}[R_0(a)X_1]^{n+1}R_\epsilon(a). \tag{A.6.1}$$

In addition, we know that a^{-1} is an isolated eigenvalue of $R_\epsilon(a)$. We can thus apply the perturbation theory developed in [34, section 8] to the operator $R_\epsilon(a)$,[11] where we choose $\mathbf{B}^s := \mathbf{B}^{s,q+r-1-s}$ with $q \in (0,1)$ and $s \in \{0,\dots,r-1\}$, it follows that there

[11] Such a theory applies since $R_\epsilon(a)$ satisfies a uniform Lasota-Yorke inequality, (A.6.1) allows to estimate the closeness of $R_0(a)$ and $R_\epsilon(a)$ in the appropriate norms and since the X_ϵ are bounded operators from $\mathbf{B}^{p,q}$ to $\mathbf{B}^{p-1,q+1}$. In particular this means that the domain of X_ϵ, viewed as a closed operator on $\mathbf{B}^{p,q}$, contains $\mathbf{B}^{p+1,q-1}$.

exists $\epsilon_0 > 0$ such that $\mu_\epsilon \in \mathscr{C}^{r-2}((-\epsilon_0, \epsilon_0), \mathbf{B}^0)$. Moreover

$$\frac{d^n}{d\epsilon^n}\mu_\epsilon\bigg|_{\epsilon=0} \in \mathbf{B}^{r-1-n}.$$

We use the natural normalization $\mu_\epsilon(1) = 1$ so that $\mu_\epsilon^{(n)}(1) = 0$. We can thus differentiate the equation $X_\epsilon \mu_\epsilon = 0$, $n \leq r-2$ times with respect to ϵ, obtaining[12]

$$X_\epsilon \mu_\epsilon^{(n)} + nX_1\mu_\epsilon^{(n-1)} = 0. \tag{A.6.2}$$

From [42, 3.6.5 p. 180] and remembering that there are no Jordan blocks we have that $R_\epsilon(z) = z^{-1}S_{0,\epsilon} + Q_\epsilon(z)$ where $Q_\epsilon(z)$ is analytic in a neighbourhood of zero and $S_{0,\epsilon}$ is the spectral projector associated to the eigenvalue zero. In addition,

$$R_\epsilon(z)X_\epsilon = R_\epsilon(z)(X_\epsilon - z) + zR_\epsilon(z) = -\mathbf{Id} + zR_\epsilon(z).$$

Therefore

$$\lim_{z\to 0} R_\epsilon(z)X_\epsilon\mu_\epsilon^{(n)} = -\mu_\epsilon^{(n)} + S_{0,\epsilon}\mu_\epsilon^{(n)} = -\mu_\epsilon^{(n)} \tag{A.6.3}$$

where we have used that $S_{0,\epsilon}\nu(\phi) = \mu_\epsilon(\phi)\cdot\nu(1)$ and so $S_{0,\epsilon}\mu_\epsilon^{(n)} = 0$. Combining equations (A.6.2) and (A.6.3) we may write

$$\mu_\epsilon^{(n)} = \lim_{z\to 0} nR_\epsilon(z)X_1\mu_\epsilon^{(n-1)} = \lim_{a\to 0^+}\int_0^\infty ne^{-at}\mathscr{L}_{\epsilon,t}X_1\mu_\epsilon^{(n-1)}\,\mathrm{dt}\,.$$

This completes the proof of Theorem A.2.5.

Remark A.6.1. Note that the perturbation theory in [48] and [34] allows to investigate, by similar arguments, also the behaviour of the other eigenvalues of X_ϵ, with the related eigenspaces, outside the essential spectrum.

[12] Remembering again that X, X_1 are a bounded operators from $\mathbf{B}^{p,q}$ to $\mathbf{B}^{p-1,q+1}$, we can exchange X_0, X_1 with the derivative with respect to ϵ provided that $n \leq r-2$.

A.7. Lasota-Yorke type inequalities–the transfer operator

Here we prove Lemma A.4.1. But first let us introduce some convenient notation.

Remark A.7.1. We will use the notation $\prod_{i=1}^{n} v_i$ to write the action of many vector fields. That is

$$\prod_{i=1}^{n} v_i h := v_1 \dots v_n h.$$

Note that this suggestive notation does not mean that the vector fields commute.

Let $0 < n \leq p$, $0 \leq l \leq n$, and let $v_1, \dots, v_n$ be $\mathscr{C}^{q+n}$ vector fields defined on a neighbourhood of W with $|v_i|_{\mathscr{C}^{q+n}} \leq 1$, and $\varphi \in \mathscr{C}_0^{n+q}(W)$ with $|\varphi|_{\mathscr{C}^{n+q}(W)} \leq 1$. We need to estimate

$$\int_W v_1 \dots v_n (\mathscr{L}_t h) \cdot \varphi.$$

The basic idea is to decompose each v_i as a sum $v_i = w_i^u + w_i^f + w_i^s$ where w_i^s is tangent to W, w_i^f points in the flow direction and w_i^u is "almost" in the unstable direction cross the flow direction. We will state precisely what we mean by "almost" in lemma A.7.4. The w_i^s may then be dealt with by an integration by parts and then noting that w_i^u, w_i^f are not expanded by DT_{-t} allows us to conclude.

We wish to look at the problem locally and so we use a partition of unity as given in the following lemma ([34, Lemma 3.3]):

Lemma A.7.2. *For any admissible leaf W and $t \in \mathbb{R}^+$, there exist leaves $W_1, \dots, W_\ell$, whose number ℓ is bounded by a constant depending only on t, such that*

1. $T_{-t}(W) \subset \bigcup_{j=1}^{\ell} W_j^-$.
2. $T_{-t}(W^+) \supset \bigcup_{j=1}^{\ell} W_j^+$.
3. *There exists a constant C (independent of W and t) such that a point of $T_{-t}W^+$ is contained in at most C sets W_j.*
4. *There exist functions $\rho_1, \dots, \rho_\ell$ of class $\mathscr{C}^{r+1}$ and compactly supported on W_j^- such that $\sum \rho_j = 1$ on $T_{-t}(W)$, and $|\rho_j|_{\mathscr{C}^{r+1}} \leq C$.*

Remark A.7.3. Note that the construction in Lemma A.7.2 can be easily modified to ensure that there exists $c > 0$ such that for all $t \in \mathbb{R}_+$ and $|s - t| \leq c\delta$, the leaves $T_s W_i$ and the partition $\rho_i \circ T_{-s}$ still satisfy properties (1-4).

Take some index j, we will estimate

$$\left| \int_{T_t(W_j)} v_1 \dots v_n(\mathscr{L}_t h) \cdot \varphi \cdot \rho_j \circ T_{-t} \right|. \tag{A.7.1}$$

The needed decomposition of v_i is given by the following lemma whose proof can be found in appendix A.9:

Lemma A.7.4. *Fix $\lambda' \in (0, \lambda)$. Let v be a vector field on a neighbourhood of W^+ with $|v|_{\mathscr{C}^a} \leq 1$, $a \leq r$ and $t \in \mathbb{R}_+$. Then there exists $c > 0$ such that, for each j, there exists a neighbourhood U_j of $\cup_{s \in [t - c\delta, t + c\delta]} T_t(W_j^+)$ and $\mathscr{C}^a(U_j)$ vector fields w^f, w^u and w^s satisfying, for all $|s - t| \leq c\delta$:*

1. *for all $x \in T_s(W_j)$, holds $v(x) = w^s(x) + w^f(x) + w^u(x)$.*
2. *for all $x \in T_s(W_j)$, $w^s(x)$ is tangent to $T_s(W_j)$.*
3. *for all $x \in T_s(W_j)$, $w^f(x)$ is proportional to the flow direction V.*
4. *$|w^s|_{\mathscr{C}^a(U_j)} \leq C_t$, $|w^u|_{\mathscr{C}^a(U_j)} \leq C_t$ and $|w^f|_{\mathscr{C}^a(U_j)} \leq C_t$.*
5. *$|w^s \circ T_s|_{\mathscr{C}^a(W_j)} \leq C$.*
6. *$|(T_s^* w^u)|_{\mathscr{C}^a(T_{-s} U_j)} \leq C e^{-\lambda' s}$ and $|w^f \circ T_s|_{\mathscr{C}^a(T_{-s} U_j)} \leq C$.*

Where $(T_t^* w^u) = DT_t(x)^{-1} w^u(T_t x)$ is the pull back of w^u by T_t.

The fundamental remark in the following computations is that, since the commutator of two $\mathscr{C}^{n+q}$ vector fields is a $\mathscr{C}^{n+q-1}$ vector field, if we exchange two vector fields, the difference consists of terms with $n-1$ $\mathscr{C}^{n-1+q}$ vector fields, hence it can be bounded by $C_{n,q} \|\mathscr{L}_t h\|^-_{n-1,q}$. For each j in (A.7.1) we can then write $w_1^s + w_1^f + w_1^u$ instead of v_1 since they agree on $T_t W_j$. After that we can commute such vector fields with the vector fields v_j, $j \in \{2, \dots, n\}$, as explained above. At this point we can decompose v_2 and so until (A.7.1) is bounded by

$$\sum_{\sigma \in \{s,f,u\}^n} \left| \int_{T_t(W_j)} w_1^{\sigma_1} \dots w_n^{\sigma_n} (\mathscr{L}_t h) \cdot \varphi \cdot \rho_j \circ T_{-t} \right| + C_{n,q,t} \|\mathscr{L}_t h\|^-_{n-1,q}$$

Take $\sigma \in \{s, f, u\}^n$, and let $k = \#\{i \mid \sigma_i = s\}$ and $l = \#\{i \mid \sigma_i = f\}$. Let π be a permutation of $\{1, \dots, n\}$ such that $\pi\{1, \dots, k\} = \{i \mid \sigma_i = s\}$ and $\pi\{n-l+1, \dots, n\} = \{i \mid \sigma_i = f\}$. Therefore

$$\begin{aligned} &\left| \int_{T_t(W_j)} w_1^{\sigma_1} \dots w_n^{\sigma_n} (\mathscr{L}_t h) \cdot \varphi \cdot \rho_j \circ T_{-t} \right| \leq \\ &\quad \left| \int_{T_t(W_j)} \prod_{i=1}^{k} w^s_{\pi(i)} \prod_{i=k+1}^{n-l} w^u_{\pi(i)} \prod_{i=n-l+1}^{n} w^f_{\pi(i)} (\mathscr{L}_t h) \cdot \varphi \cdot \rho_j \circ T_{-t} \right| + C_{n,q,t} \|\mathscr{L}_t h\|^-_{n-1,q}. \end{aligned}$$

By definition $w_i^f(g) = \alpha_i V(g)$, where $\alpha_i \in \mathscr{C}^{n+q}$. Thus $w_i^f(g) = -\alpha_i X g - \alpha_i g \operatorname{div} V$, where X, for the time being, is defined by (A.2.4). The terms coming from taking derivatives of the α_i or the terms involving the divergence of the vector fields are bounded by the $\| \cdot \|^-_{n-1,q}$. In particular $\|X^l h\|^-_{n-l,q} \leq \|h\|^-_{n,q} + C_{n,q} \|h\|_{n-1,q}$. Hence, setting $\bar{\alpha} := (-1)^l \prod_{i=k+1}^{k+l} \alpha_i$, for $k > 0$ we have

$$\begin{aligned} &\left| \int_{T_t(W_j)} w_1^{\sigma_1} \dots w_n^{\sigma_n} (\mathscr{L}_t h) \cdot \varphi \cdot \rho_j \circ T_{-t} \right| \leq \\ &\left| \int_{T_t(W_j)} \prod_{i=1}^{k} w^s_{\pi(i)} \prod_{i=k+1}^{n-l} w^u_{\pi(i)} X^l (\mathscr{L}_t h) \cdot \varphi \cdot \rho_j \circ T_{-t} \cdot \bar{\alpha} \right| + C_{n,q,t} \|\mathscr{L}_t h\|_{n-1,q}. \end{aligned} \tag{A.7.2}$$

Next, we integrate by parts with respect to the vector fields $w^s_{\pi(i)}$. These vector fields are tangent to the manifold W, hence $\int_W w^s_{\pi(i)} f \cdot g = -\int_W f \cdot w^s_{\pi(i)} g + \int_W f g \cdot \operatorname{div} w^s_{\pi(i)}$. Since $w^s_{\pi(i)}$ is $\mathscr{C}^{q+n}$ and the manifold W is $\mathscr{C}^{r+1}$ with a $\mathscr{C}^{r+1}$ volume form, the divergence terms are bounded by $C_{n,q,t}\|\mathscr{L}_t h\|_{n-1,q}$. This yields

$$\left|\int_{T_t(W_j)} w_1^{\sigma_1}\dots w_n^{\sigma_n}(\mathscr{L}_t h)\cdot\varphi\cdot\rho_j\circ T_{-t}\right| \le$$
$$\left|\int_{T_t(W_j)}\prod_{i=k+1}^{n-l} w^u_{\pi(i)}X^l(\mathscr{L}_t h)\cdot\prod_{i=1}^{k} w^s_{\pi(i)}(\varphi\cdot\rho_j\circ T_{-t}\cdot\bar\alpha)\right| + C_{n,q,t}\|\mathscr{L}_t h\|_{n-1,q}.$$

By Lemma A.7.4 it follows that $\prod_{i=1}^{k} w^s_{\pi(i)}(\varphi\cdot\rho_j\circ T_{-t}\cdot\bar\alpha)$ is a $\mathscr{C}^{q+n-k}$ test function while on $\mathscr{L}_t h$ act only $n-k$ vector fields. Thus the above integral can be bounded by the $\|\cdot\|_{n-1,q}$ norm unless $k=0$.

Next we need to analyze the case $k=0$ in more detail. For each $h\in\mathscr{C}^r$, $X^l\mathscr{L}_t h = \mathscr{L}_t X^l h = (X^l h)\circ T_{-t}\det(DT_t)^{-1}\circ T_{-t}$.[13] If we differentiate $\det(DT_t)^{-1}\circ T_{-t}$ we obtain terms that are bounded by $C_{n,q,t}\|X^l\mathscr{L}_t h\|_{n-l-1,q+1}\le C_{n,q,t}\|\mathscr{L}_t h\|_{n-1,q}$. Hence

$$\left|\int_{T_t(W_j)} w_1^{\sigma_1}\dots w_n^{\sigma_n}(\mathscr{L}_t h)\cdot\varphi\cdot\rho_j\circ T_{-t}\right| \le$$
$$\left|\int_{T_t(W_j)}\prod_{i=1}^{n-l} w^u_{\pi(i)}(X^l h)\circ T_{-t}\cdot\varphi\cdot\left[\rho_j\cdot\det(DT_t)^{-1}\right]\circ T_{-t}\cdot\bar\alpha\right| + C_{n,q,t}\|\mathscr{L}_t h\|_{n-1,q}.$$

Let $\bar w_i^u(x) = DT_t(x)^{-1}w_i^u(T_t x)$. This is a vector field on a neighbourhood of W_j^+. We can then write the above integral as

$$\int_{T_t(W_j)}\left(\prod_{i=1}^{n-l}\bar w^u_{\pi(i)}X^l h\right)\circ T_{-t}\cdot\rho_j\circ T_{-t}\cdot\det(DT_t)^{-1}\circ T_{-t}\cdot\bar\alpha\cdot\varphi$$

and, changing variables, we obtain

$$\int_{W_j}\prod_{i=1}^{n-l}\bar w^u_{\pi(i)}X^l h\cdot(\bar\alpha\varphi)\circ T_t\cdot\rho_j\cdot\det(DT_t)^{-1}\cdot J_W T_t, \tag{A.7.3}$$

[13]Since for smooth ϕ holds $VT_t\phi = T_tV\phi$ and we have used (A.2.2).

where $J_W T_t$ is the Jacobian of $T_t : W_j \to W$. Note that

$$|(\bar{\alpha}\varphi) \circ T_t|_{\mathscr{C}^{q+n}} \leq C_{p,q} \, |\varphi|_{\mathscr{C}^{q+n}} \leq C_{p,q},$$

because of Lemma A.7.4. Moreover, $\left|\bar{w}^u_{\pi(i)}\right|_{\mathscr{C}^{q+n}} \leq C_{p,q} e^{-\lambda' t}$ (see Lemma A.7.4) and so:

$$\prod_{i=k+1}^{n-l} |\bar{w}^u_{\pi(i)}|_{\mathscr{C}^{q+n}} \leq C_{p,q} e^{-\lambda'(n-k-l)t}. \tag{A.7.4}$$

Putting together all the above estimates we finally obtain[14]

$$\begin{aligned} \left|\int_W v_1 \dots v_n (\mathscr{L}_t h) \cdot \varphi\right| \leq \sum_{\substack{0 \leq l \leq n \\ j \leq \ell}} \left|\int_{W_j} V^l \prod_{i=1}^{n-l} \bar{w}^u_{\pi(i)} h \cdot (\bar{\alpha}\varphi) \circ T_t \frac{\rho_j \cdot J_W T_t}{\det(DT_t)}\right| \\ + C_{p,q,t}(\|\mathscr{L}_t h\|_{n-1,q} + \|h\|_{n-1,q}). \end{aligned} \tag{A.7.5}$$

To conclude we need the following distortion lemma:[15]

Lemma A.7.5 ([34] Lemma 6.2)**.** *Given $W \in \Sigma$ and leaves W_j such that $W \subset \bigcup_{j \leq \ell} W_j$ and $W^+ \supset \bigcup_{j \leq \ell} W_j$ we have the following control:*

$$\sum_{j \leq \ell} \left|J_W T_t \cdot \det(DT_t)^{-1}\right|_{\mathscr{C}^r(W_j, \mathbb{R})} \leq C. \tag{A.7.6}$$

Lemma A.7.5 together with (A.7.4) and (A.7.5) implies, for all $0 < n \leq p$,

$$\begin{aligned} \|\mathscr{L}_t h\|^-_{0,q} &\leq C\|h\|^-_{0,q} \\ \|\mathscr{L}_t h\|^-_{n,q} &\leq C e^{-\lambda' t}\|h\|^-_{n,q} + C\|V^n h\|_{0,q+n} + C_{p,q,t}(\|\mathscr{L}_t h\|_{n-1,q} + \|h\|_{n-1,q}). \end{aligned} \tag{A.7.7}$$

The idea is to finish the proof by induction. For $n = 0$ the first inequality of (A.7.7) is the same as $\|\mathscr{L}_t h\|_{0,q} \leq C_{p,q}\|h\|_{0,q}$. On the other hand if $\|\mathscr{L}_t h\|_{m,q} \leq C_{p,q}\|h\|_{m,q}$ for

[14] Where we have used again the possibility to commute the vector fields by paying an error bound in the $\|\cdot\|_{n-1,q}$ norm and we have recalled (A.2.4).

[15] In fact, [34] applies to hyperbolic maps, yet the proof holds also for flows with the only change of thickening $T_t W_j$ by ρ also in the flow direction.

each $m \le n < p$, then the second inequality of (A.7.7) yields

$$
\begin{aligned}
\|\mathscr{L}_t h\|^-_{n+1,q} &\le Ce^{-\lambda' t}\|h\|^-_{n+1,q} + C\|X^{n+1}h\|_{0,q+n+1} + C_{p,q,t}(\|\mathscr{L}_t h\|_{n,q} + \|h\|_{n,q})\\
&\le Ce^{-\lambda' t}\|h\|^-_{n+1,q} + C\|X^{n+1}h\|_{0,q+n+1} + C_{p,q,t}\|h\|_{n,q}.
\end{aligned}
$$

Next, choose t_0 such that $Ce^{-\lambda' t_0} \le \sigma < 1$ Then

$$
\begin{aligned}
\|\mathscr{L}_{t_0+t} h\|^-_{n+1,q} &\le \sigma\|\mathscr{L}_t h\|^-_{n+1,q} + C\|\mathscr{L}_t X^{n+1}h\|_{0,q+n+1} + C_{p,q}\|\mathscr{L}_t h\|_{n,q}\\
&\le \sigma\|\mathscr{L}_t h\|^-_{n+1,q} + C_{p,q}\|X^{n+1}h\|_{0,q+n+1} + C_{p,q}\|h\|_{n,q}.
\end{aligned}
$$

Writing t as $mt_0 + s$, $s \in (0, t_0)$, and iterating the above equation yields

$$
\begin{aligned}
\|\mathscr{L}_t h\|^-_{n+1,q} &\le \sigma^m\|\mathscr{L}_s h\|^-_{n+1,q} + (1-\sigma)^{-1}C_{p,q}\left[\|X^{n+1}h\|_{0,q+n+1} + \|h\|_{n,q}\right]\\
&\le C_{p,q}\|h\|^-_{n+1,q} + C_{p,q}\|h\|_{n,q}.
\end{aligned}
$$

Finally we have

$$
\|\mathscr{L}_t h\|_{n+1,q} \le \|\mathscr{L}_t h\|^-_{n+1,q} + \|\mathscr{L}_t h\|_{n,q} \le C_{p,q}\|h\|_{n+1,q}.
$$

This completes the proof of Lemma A.4.1.

A.8. Lasota-Yorke type estimates–the resolvent

In this section we prove Lemma A.4.3. In order to do this note that the following may be shown by induction from equation (A.2.5):

$$
R(z)^m h = \frac{1}{(m-1)!}\int_0^\infty t^{m-1}e^{-zt}\mathscr{L}_t h \,\mathrm{dt}\,. \tag{A.8.1}
$$

The first inequality of lemma A.4.3 follows directly from equations (A.4.1) and (A.8.1) by integration over t. Analogously we can use (A.4.1) to cut the domain of integration.

Indeed for each $z := a + ib$, $a \ge a_0 > 0$, $\beta \ge 16$ and $L := \frac{m\beta}{a}$, we have[16]

[16]Indeed, setting $I(m) := \int_L^\infty t^m e^{-at}$, integrating by parts yields $I(m) = L^m a^{-1}e^{-aL} + ma^{-1}I(m-1)$.

$$
\begin{aligned}
\left\|\frac{1}{(m-1)!}\int_L^\infty t^{m-1}e^{-zt}\mathscr{L}_t h\,\mathrm{d}t\right\|_{p,q} &\leq \frac{1}{(m-1)!}\int_L^\infty t^{m-1}e^{-at}C_{p,q}\|h\|_{p,q} \\
&\leq C_{p,q}a^{-m}e^{-\frac{m\beta}{2}}\|h\|_{p,q}.
\end{aligned}
\tag{A.8.2}
$$

Accordingly, to prove the second part of lemma A.4.3 it suffices to fix $n \leq p$, $|v_i|_{\mathscr{C}^{q+n}} \leq 1$, $|\varphi|_{\mathscr{C}_0^{n+q}} \leq 1$ and estimate

$$
\frac{1}{(m-1)!}\int_0^L t^{m-1}e^{-zt}\int_W v_1\dots v_n(\mathscr{L}_t h)\cdot\varphi\,\mathrm{d}t\,.
$$

To do so it is convenient to localize in time by introducing a smooth partition of unity $\{\phi_i\}$ of $\mathbb{R}_+$ subordinated to the partition $\{[(s-1/2)t_*,(s+3/2)t_*]\}_{s\in\mathbb{N}}$ where $t_* = c\delta$ and c is specified in Remark A.7.3. In fact, it is possible to have such a partition of the form $\phi_s(t) := \phi(t - st_*)$ for some fixed function ϕ.

We will use the notation of section A.7 and the formula (A.7.5) where the families of submanifolds are chosen for each $t = st_*$, $s \in \mathbb{N}$, according to Lemma A.7.2 and for $t \neq st_*$ the families of submanifolds are constructed as described in Remark A.7.3. We

---

Hence, by induction, we can prove the formula

$$
\frac{1}{(m-1)!}I(m-1) = \sum_{j=0}^{m-1}\frac{L^j}{a^{m-j}j!}e^{-aL} = a^{-m}\sum_{j=0}^{m-1}\frac{m^j\beta^j}{j!}e^{-m\beta} \leq a^{-m}\sum_{j=0}^{m-1}\left(\frac{m}{j}e\right)^j\beta^j e^{-m\beta},
$$

since $j! \geq j^j e^{-j}$. Next, since the maximum of $\left(\frac{m}{j}e\right)^j$ is achieved for $j = m$, hence $\left(\frac{m}{j}e\right)^j \leq e^m$,

$$
\frac{1}{(m-1)!}I(m-1) \leq \frac{a^{-m}\beta^m}{\beta-1}e^{-m(\beta-1)} \leq Ca^{-m}e^{-m\frac{\beta}{2}}.
$$

can then write, for each $s \in \mathbb{N}$ and setting $t_s := st_* - t$,[17]

$$\begin{aligned}
&\left|\int_0^L t^{m-1}e^{-zt}\phi_s(t)\int_W v_1\dots v_n(\mathscr{L}_t h)\cdot\varphi\,\mathrm{dt}\right| \\
&\leq \sum_{\substack{0\leq l\leq n\\ j\leq \ell}}\left|\int_0^L \frac{t^{m-1}\phi_s(t)}{e^{zt}}\int_{T_{t_s}W_j} V^l\prod_{i=1}^{n-l}\bar{w}^u_{\pi(i)}h\cdot\frac{(\bar{\alpha}\varphi)\circ T_t\cdot\rho_j\circ T_{t_s}\cdot J_W T_t}{\det(DT_t)}\right| \\
&\quad + C_{p,q,L}L^m m^{-1}\|h\|_{n-1,q},
\end{aligned} \tag{A.8.3}$$

where we have used equations (A.7.5), (A.7.7). Changing variables and using Fubini's theorem on all the right hand side integrals and setting $t_s^+ := st_* + t$ yields

$$\int_{W_j}\int_{\mathbb{R}}(t_s^+)^{m-1}e^{-zt_s^+}\phi(t)V^l\left(\left(\prod_{i=1}^{n-l}\bar{w}^u_{\pi(i)}h\right)\circ T_t\right)\frac{(\bar{\alpha}\varphi)\circ T_{st_*}\cdot\rho_j\cdot J_W T_{st_*}}{\det(DT_{t_s})^{-1}\circ T_t}.$$

For $l \neq 0$, we can integrate by parts, since $V(\Psi\circ T_t) = \frac{d}{dt}\Psi\circ T_t$, obtaining

$$C\|h\|_{n-1,q}|z|\int_{\mathbb{R}_+} t^{m-1}e^{-at}\phi_s(t) \leq C\|h\|_{n-1,q}|z|a^{-m}. \tag{A.8.4}$$

For $l = 0$ and $n = p$, remembering (A.7.4), we have

$$\begin{aligned}
&\sum_{\substack{s\in\mathbb{N}\\ j\leq\ell}}\frac{1}{(m-1)!}\left|\int_0^L t^{m-1}e^{-zt}\phi_s(t)\int_{W_j}\prod_{i=1}^{p}\bar{w}^u_{\pi(i)}h\cdot(\bar{\alpha}\varphi)\circ T_t\cdot\frac{\rho_j\cdot J_W T_t}{\det(DT_t)}\right| \\
&\quad\leq \frac{C_{p,q}}{(m-1)!}\int_{\mathbb{R}_+} t^{m-1}e^{-(a+\lambda p)t}\|h\|^-_{p,q} \leq C_{p,q}(a+\lambda' p)^{-m}\|h\|^-_{p,q}.
\end{aligned} \tag{A.8.5}$$

In the case $l = 0$, $n < p$ we must use a regularization trick in order to have the wanted decay in the norm. Since the composition with T_t decreases the derivatives one can take advantage of such a fact by smoothening the test function.

For $\varepsilon \leq \delta$ and $\bar{\varphi} \in \mathscr{C}^a_0(W,\mathbb{R})$, let $\mathbf{A}_\varepsilon\bar{\varphi} \in \mathscr{C}^{a+1}_0(W^+,\mathbb{R})$ be obtained by convolving $\bar{\varphi}$ with a $\mathscr{C}^\infty$ mollifier whose support is of size ε. We will use the following, standard,

[17] By construction, the manifolds $\{W_j\}$ in the formula (A.8.3) depend on s but not on t.

result.

Lemma A.8.1. *For each $n \in \mathbb{N}, q \in \mathbb{R}_+$ and $\bar{\varphi} \in \mathscr{C}^{q+n}$,*

$$
\begin{aligned}
&|\mathbf{A}_\varepsilon \bar{\varphi}|_{\mathscr{C}^{q+n}} \leq C|\bar{\varphi}|_{\mathscr{C}^{q+n}}; \qquad |\mathbf{A}_\varepsilon \bar{\varphi}|_{\mathscr{C}^{q+1+n}} \leq C\varepsilon^{-1}|\bar{\varphi}|_{\mathscr{C}^{q+n}};\\
&|\mathbf{A}_\varepsilon \bar{\varphi} - \bar{\varphi}|_{\mathscr{C}^{q+n}} \leq C\varepsilon|\bar{\varphi}|_{\mathscr{C}^{q+n+1}}.
\end{aligned}
$$

Hence, setting $\Delta\varphi = (\varphi - \mathbf{A}_\varepsilon\varphi) \circ T_t$, by the action of T_t on the derivatives follows $|\Delta\varphi|_{\mathscr{C}^{q+n}} \leq Ce^{-\lambda(q+n)t}$, provided one chooses $\varepsilon \leq Ce^{-\lambda(q+n)t}$. Thus, using (A.7.4) as well, we have

$$
\begin{aligned}
&\sum_{\substack{s\in\mathbb{N}\\ j\leq\ell}} \frac{1}{(m-1)!}\left|\int_0^L t^{m-1}e^{-zt}\phi_s(t)\int_{W_j}\prod_{i=1}^n \bar{w}^u_{\pi(i)} h\cdot\varphi\circ T_t\cdot\frac{\rho_j\cdot J_W T_t}{\det(DT_t)}\right|\\
&\leq \sum_{\substack{s\in\mathbb{N}\\ j\leq\ell}}\int_0^L \frac{t^{m-1}e^{-zt}\phi_s(t)}{(m-1)!}\left|\int_{W_j}\prod_{i=1}^n \bar{w}^u_{\pi(i)} h\cdot\frac{\Delta\varphi\cdot\rho_j\cdot J_W T_t}{\det(DT_t)}\right|\\
&\quad + \frac{C_{p,q,a_0,L}L^m\|h\|^-_{n,q+1}}{m!}\\
&\leq C_{p,q}(a+\lambda(q+n))^{-m}\|h\|_{p,q} + C_{p,q,a_0,L}\frac{L^m}{m!}\|h\|^-_{n,q+1}.
\end{aligned}
\tag{A.8.6}
$$

Collecting equations (A.8.2), (A.8.3), (A.8.4), (A.8.5) and (A.8.6) yields, for each $n \leq p$,

$$
\begin{aligned}
\|R(z)^m h\|^-_{n,q} \leq &C_{p,q}\left[a^{-m}e^{-\frac{m\beta}{2}} + (a+\lambda' p)^{-m} + (a+\lambda q)^{-m}\right]\|h\|_{n,q}\\
&+ (a^{-m}|z| + C_{p,q,L}m^{-1})\|h\|_{n-1,q} + C_{p,q,a_0,L}\frac{L^m}{m!}\|h\|_{n-1,q+1}.
\end{aligned}
$$

To conclude it is convenient to introduce, for each $0 < A < 1$, the equivalent weighted norms[18]

$$
\|h\|_{p,q,A} := \sum_{n\leq p} A^n\|h\|^-_{n,q}.
$$

[18]The advantage of using weighted norms has been pointed out to us by Sébastien Gouëzel.

Using such a norm we can write

$$\begin{aligned}\|R(z)^m h\|_{p,q,A} \leq & C_{p,q}\left[a^{-m}e^{-\frac{m\beta}{2}} + (a+\lambda' p)^{-m} + (a+\lambda q)^{-m}\right]\|h\|_{p,q,A}\\ & + A(a^{-m}|z| + C_{p,q,L}m^{-1})\|h\|_{p-1,q,A} + C_{p,q,a_0,L}\frac{L^m}{m!}\|h\|_{n-1,q+1,A}.\end{aligned}$$

For each $\lambda'' < \lambda'$, calling $\bar{p} := \min\{p,q\}$, there exists $m_a \in \mathbb{N}$, e.g. $m_a = C_{\lambda'',p,q}\, a$ will do, such that $C_{p,q}(a+\lambda'\bar{p})^{-m_a} \leq \frac{1}{4}(a+\lambda''\bar{p})^{-m_a}$. Choosing then β, and hence L, large enough[19] and A small enough we have

$$\|R(z)^{m_a}h\|_{p,q,A} \leq (a+\lambda''\bar{p})^{-m_a}\|h\|_{p,q,A} + C_{p,q,a_0}a^{-m_a}|z|\|h\|_{p-1,q+1,A},$$

which can be iterated to yield the wanted estimate (given the equivalence of the norms).

A.9. Appendix

Proof of Lemma A.7.4. Our aim is to write the vector field as $v = w^s + w^u + w^f$. We start by making a $\mathscr{C}^{r+1}$ change of variables in the charts[20] so that W_j^+ and W^+ are subsets of $\mathbb{R}^{d_s} \times \{0\} \times \{0\}$ while $T_t(s,u,\tau) = (s,u,\tau+t)$. In addition, chosen $z \in W_j$, we can assume, without loss of generality, that $E^u(z) = \{(0,0,u) \ : \ u \in \mathbb{R}^{d_u}\}$ and $E^u(T_t z) = \{(0,0,u) \ : \ u \in \mathbb{R}^{d_u}\}$. We can then consider the foliation $E = \{E(s,\tau,u)\}$ of a neighbourhood of W_j^+ made by the leaves $E(s,\tau,u) := \{(s,\tau,u+v) \ : \ v \in \mathbb{R}^{d_u};\ |v| \leq \delta\}$ and define the foliation $F = T_t E$.

The idea is to first define the splitting on $T_{t+s}W_j$ and then extend it to a neighbourhood. We thus define the splitting on $\{(s,\tau,0)\}$ as follows: $\langle w^s, (0,\tau,u)\rangle = 0$, for each $u \in \mathbb{R}^{d_u}$, $\tau \in \mathbb{R}$; w^f is in the flow direction; w^u belongs to the tangent spaces of the leaves of the foliation F.

To verify that the splitting satisfies the wanted properties we need to write the differential of T_t in the chosen coordinates. For each x in a neighbourhood of W_j, by the

[19] For example, $\beta \geq 2\lambda\bar{p}a^{-1}$ will do, notice that this choice implies that L can be chosen uniformly bounded with respect to a.

[20] A point in the charts will be written as $(s,\tau,u) \in \mathbb{R}^d$ with $s \in \mathbb{R}^{d_s}$, $\tau \in \mathbb{R}$ and $u \in \mathbb{R}^{d_u}$.

requirement that the flow direction is mapped into the flow direction it follows

$$DT_t(x) = \begin{pmatrix} A_t(x) & 0 & B_t(x) \\ a_t(x) & 1 & b_t(x) \\ C_t(x) & 0 & D_t(x) \end{pmatrix}.$$

Moreover, if $x \in W_j^+$, then it must be $a_t(x) = 0$; $C_t(x) = 0$ and, finally $b_t(z) = 0$ and $B_t(z) = 0$. In addition, due to the uniform hyperbolicity of the flow, we have that, for each $x \in W_j^+$, $\|A_t(x)\| \leq Ce^{-\lambda t}$, while, for each x in a neighbourhood of W_j, $\|(B_t(x)u, \langle b_t(x), u\rangle, D_t(x)u)\| \geq Ce^{\lambda t}\|u\|$.[21] Notice as well that the size of the neighbourhood we are interested in can be chosen arbitrarily, thus, by continuity, we can assume $\|C_t\|_{\mathscr{C}^r} + \|a_t\|_{\mathscr{C}^r}$ arbitrarily small.[22] Finally, since the foliation F must be close to the unstable direction, it must follow $\|B_t(x)u\| + |\langle b_t(x)u\rangle| \leq \frac{1}{2}\|D_t(x)u\|$, for all $u \in \mathbb{R}^{d_u}$.

By construction the tangent space to the leaves of the foliation F has the form $\{(B_t(x)D_t(x)^{-1}u, \langle b_t(x), D_t(x)^{-1}u\rangle, u) \; : \; u \in \mathbb{R}^{d_u}\}$. Accordingly, setting $v =: (v_s, v_f, v_u)$, we have

$$\begin{aligned} w^s &= (v_s - (B_tD_t^{-1}) \circ T_{-t}v_u, 0, 0) \\ w^f &= (0, v_f - (b_tD_t^{-1}) \circ T_{-t}v_u, 0) \\ w^u &= ((B_tD_t^{-1}) \circ T_{-t}v_u, (b_tD_t^{-1}) \circ T_{-t}v_u, v_u). \end{aligned} \tag{A.9.1}$$

By construction such vector fields satisfy points (a-d) of the Lemma; moreover they belong to $\mathscr{C}^r(T_t(W_j))$. To estimate the $\mathscr{C}^r$ norm we must study the $\mathscr{C}^r$ norm of $U_t(x) := B_t(x)D_t(x)^{-1}$ and $\beta_t(x) := b_t(x)D_t(x)^{-1}$.[23]

To do so it is convenient to break up the trajectory in pieces of finite length t_0 and, at all the points $T_{kt_0}x$, introduce the same type of coordinates already defined. By the

[21] The latter follows from the possibility to choose δ small enough so that all the tangent spaces to the foliations E lay in the unstable cone.

[22] Given a function A with values in the matrices we define $\|A\|_{\mathscr{C}^n} := \sup_k \sum_j |A_{kj}|_{\mathscr{C}^n}$. Such a definition has the useful consequence that if $A = BD$, then $\|A\|_{\mathscr{C}^n} \leq \|B\|_{\mathscr{C}^n}\|D\|_{\mathscr{C}^n}$.

[23] Note that, within a chart, the matrices do not depend on x_f

hyperbolicity assumption, given $\lambda' \in (0,\lambda)$, it is possible to choose $t_0 \le C$ so that $nt_0 = t$ and $\|D_{t_0}(T_{kt_0}x)^{-1}\| \le e^{-\lambda' t_0}$, $\|A_{t_0}(T_{kt_0}x)\| \le e^{-\lambda' t}$, $\|T_{kt_0}x - T_{kt_0}y\| \le e^{-\lambda' t}\|T_{(k-1)t_0}x - T_{(k-1)t_0}y\|$ for each $k \le n$ and $x, y \in W_j$. Accordingly, since $D_{(k+1)t_0}(x) = D_{t_0}(T_{kt_0}x)D_{kt_0}(x)$

$$\|D_t^{-1}\|_{\mathscr{C}^r} = \|D_{nt_0}^{-1}\|_{\mathscr{C}^r} \le (e^{-\lambda' t_0} + Ce^{-\lambda'(n-1)t_0})\|D_{(n-1)t_0}^{-1}\|_{\mathscr{C}^r} \le Ce^{-\lambda' t}. \tag{A.9.2}$$

Next, notice that

$$\begin{aligned}&\begin{pmatrix} A_{(k+1)t_0}(x) & 0 & B_{(k+1)t_0}(x) \\ 0 & 1 & b_{(k+1)t_0}(x) \\ 0 & 0 & D_{(k+1)t_0}(x)\end{pmatrix} \\ &= \begin{pmatrix} A_{t_0}(T_{kt_0}x)A_{kt_0}(x) & 0 & A_{t_0}(T_{kt_0}x)B_{kt_0}(x) + B_{t_0}(T_{kt_0}x)D_{kt_0}(x) \\ 0 & 1 & b_{t_0}(T_{kt_0}x)D_{kt_0}(x) + b_{kt_0}(x) \\ 0 & 0 & D_{t_0}(T_{kt_0}x)D_{kt_0}(x)\end{pmatrix}.\end{aligned}$$

Thus, setting $U_k := B_{kt_0}D_{kt_0}^{-1}$, holds

$$U_{k+1} = A_{t_0}(T_{kt_0}x)U_kD_{t_0}(T_{kt_0}x)^{-1} + B_{t_0}(T_{kt_0}x)D_{t_0}(T_{kt_0}x)^{-1}.$$

Hence,

$$\|U_n\|_{\mathscr{C}^r} \le (e^{-\lambda' t_0} + e^{-\lambda'(n-1)t_0})^2\|U_{n-1}\|_{\mathscr{C}^r} + C.$$

Iterating the above equation yields $\|U_t\|_{\mathscr{C}^r(W_j)} \le C$. By a similar argument it follows $\|\beta_t\|_{\mathscr{C}^r(W_j)} \le C$. Applying the above estimates to (A.9.1) yields $|w^s \circ T_t|_{\mathscr{C}^a(W_j)} \le C$, $|w^u \circ T_t|_{\mathscr{C}^a(W_j)} \le C$ and $|w^f \circ T_t|_{\mathscr{C}^a(W_j)} \le C$, which proves (e).

To tackle (f) we need to extend the vector fields smoothly, this is easily done by taking them constant along the leaves of F. Since on W_j we have $DT_t^{-1}w^u \circ T_t = (0,0,D_t^{-1}v^u \circ T_t)$ and $w^f \circ T_t = (0, v^f \circ T_t - bD_t^{-1}v^u \circ T_t, 0)$, the above estimates imply $|T_t^* w^u|_{\mathscr{C}^a(W_j)} \le Ce^{-\lambda' t}$ and $|w^f \circ T_t|_{\mathscr{C}^a(W_j)} \le C$. Since the vector fields have been extended by keeping them constant on the leaves of F, if follows that their preimages are constant along the leaf of E, that is they do not depend on u. This means that the above bounds on the norms does not increase when they are considered on the neighbourhood $T_{-t}U_j$, hence point (e). $\square$

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