A Kinematic Notation for Lower-Pair Mechanisms Based on Matrices[1]

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A symbolic notation devised by Reuleaux to describe mechanisms did not recognize the necessary number of variables needed for complete description. A reconsideration of the problem leads to a symbolic notation which permits the complete description of the kinematic properties of all lower-pair mechanisms by means of equations. The symbolic notation also yields a method for studying lower-pair mechanisms by means of matrix algebra; two examples of application to space mechanisms are given.

Introduction

An approach to the problem of rationalizing kinematics into a science by means of a symbolic language was proposed by Reuleaux (2, 3)[4] in 1875. It was his hope that a symbolic language would permit a complete description of the kinematic properties of a mechanism, and that this would be additionally useful not only for the analysis of existing mechanisms but also for the synthesis of new mechanisms. Unfortunately, the symbolism that he devised did not include all variables needed for a complete description of a mechanism, and this limited its usefulness. However, Reuleaux did give several concepts that appear to be fundamental. When exploited, these concepts lead to a manipulative symbolic notation. This notation gives the data necessary for deriving the relative displacements in any lower-pair mechanism. This description is in the form of a symbolic equation, the terms of which are shown to be equivalent to matrices. It is from these matrices that the manipulative properties derive.

The mechanisms to be discussed here are formed from a succession of rigid parts coupled end to end to form a single closed chain. When the motion of each part is constrained or guided in unique fashion the purpose of the mechanism is to transform one motion into another, as the transformation of the linear motion of a piston engine into a continuous circular motion. The connections permitting relative motion are made by means of contact between adjacent boundary surfaces of the parts. The designing of mechanisms consists of finding the proper combination of parts so proportioned that the given input motion yields a desired output motion.

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[4] Numbers in parentheses refer to the Bibliography at the end of the paper.

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Since the connections are made by the contact of adjacent boundary surfaces, two surfaces will be involved in any connection. One surface lies on one part, and one surface lies on the adjacent part. Taken together, these two surfaces are called a pair, i.e., pairs of contact surfaces, each surface being called one element of the pair. If the mutual contact is made by two mating surfaces, sliding with respect to each other, they then constitute a lower pair. If the mutual contact is confined to a line (as in rollers, cams, or gear teeth) or to a point (as in ball bearings), this then constitutes a higher pair.

Because of their apparent relative simplicity, lower pairs have been studied first. The only possible lower pairs are the following:

1 The spheric pair, as a ball-and-socket joint. It allows a rotation about each of three rectangular axes.

2 The plane pair. It allows two translations along the axes defining the plane and a rotation about an axis perpendicular to the plane.

3 The right circular cylinder pair, or cylindric pair. It allows a rotation about the cylinder axis and a translation along the axis.

4 The screw pair. It gives rise to a helical movement in terms of the fixed axial advance per revolution, or lead.

5 The revolute pair. Both contact surfaces are defined as surfaces of revolution, generated by any profile. It allows a rotation about its axis.

6 The prismatic pair or prism pair. Both contact surfaces are cylinders other than the right circular. It allows a translation parallel to a generatrix.

Reuleaux noted that the revolute pair and the prismatic pair represent limiting cases of the screw pair: When the lead of the screw pair becomes zero, a revolute pair results; when the lead becomes infinite, a prismatic pair results. It also may be remarked that each of the three pairs is completely defined with respect to a system of co-ordinates by its axis; in the case of a prismatic pair the axis is an arbitrary line parallel to a generatrix. It will be convenient to use and exploit the symbol S_L to represent a screw pair of lead L. In addition, it will be convenient to indicate the variable u needed to define the unique possible motion, thus $S_L(u)$ where u is either the angle of rotation or the equivalent advance. The symbol for a revolute (pair) may be written $S_0(u)$, but it will simplify the notation to designate the revolute by $R(u)$. Similarly, the symbol for a prism (prismatic pair) may be written $S_\infty(u)$, but to write $P(u)$ will be simpler. No symbols are needed for the spheric pair, the right circular cylinder pair, or the plane pair, for each of these is formed from and described by the appropriate revolute and/or prism pairs needed to give the motion.

Unfortunately, there is no accepted standard nomenclature for kinematics. The names may seem stilted but there is an association between the name and the symbol of the pair by considering the nature of the contact. This is more or less true for English, French, and German, as Grodzinski (4) points out.

The two elements of a pair are distinguished by two additional symbols: S_L^+ means the element given by the surface of the screw, or the full body; and S_L^- means the element that is the nut, or hollow body. The positive and negative signs are used for the element designations of other pairs as well.

Development of the Symbolic Notation

The position of a screw-pair element in a machine part is determined by the position of its axis. The position of the axis may be described in a number of ways, and the apparent number of parameters needed depends on the manner of the description. Thus the line L of Fig. 1 passing through the point $P(x, y, z)$ needs the angles β and γ to complete the description, and it would seem that five parameters constitute the necessary and sufficient number. However, the line also could be defined by the equations $x = Az + B$ and $y = Cz + D$, and here four coefficients suffice.

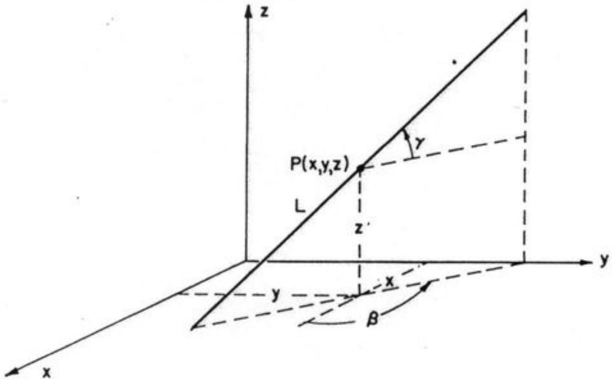

Fig. 1 Parameters x, y, z, β, and γ Defining Line L

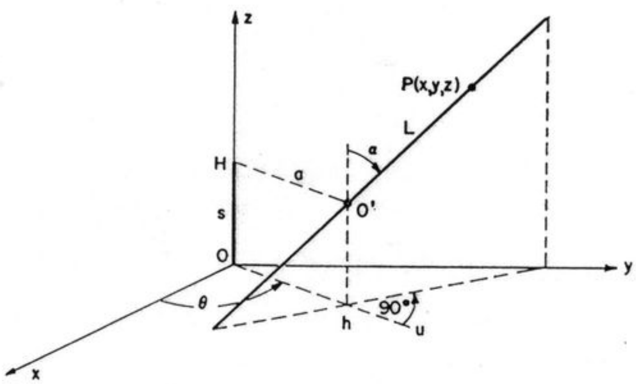

Fig. 2 Parameters a, α, θ, and s Defining Line L, or Axis of a Screw or Revolute Pair

Turning to Fig. 2, the description is based on the use of the unique common perpendicular HO' of length a between Oz and the line L. The angle θ lies between Ox and the common perpendicular, s is the distance from O to H, and α the angle between Oz and L.

Again there are four parameters, and that they are sufficient to define a unique line in the x, y, z-co-ordinate system may be demonstrated by supposing the angle θ given, Fig. 2. The line Ou is then drawn in the xy-plane, and the distance a laid off from O defines the point h. A perpendicular at h of length s defines O'. The plane perpendicular to Oh and containing O' is unique, and in this plane only one straight line L will have an angle α with hO'. The line L or the axis of a screw pair is therefore completely determined by a, α, θ, and s. The axis of a revolute pair also is determined completely by the same four parameters.

Since the axis of a prism pair is an arbitrary line parallel to a generatrix, it may be chosen as going through the origin ($a = 0$, $s = 0$) as shown in Fig. 3. The direction of the axis is then determined completely by θ and α.

A description of a mechanism will involve a description of the relative positions of the successive-pair axes, and this may be done by use of the unique common perpendicular between successive pair axes. The manner of describing the relative positions of successive-pair axes is given by Fig. 4 which depicts two machine parts from a space mechanism. The two parts are connected by the complete revolute pair R_1 with the revolute elements $R_0{}^+$ and $R_2{}^-$ showing where connections are made to the rest of the mechanism. The axes z_0, z_1, and z_2 define the space positions of the revolutes chosen for the illustration but they equally well would define the locations of screw or prism pairs. The axis x_1 is chosen as an extension of the common perpendicular of length a_0 between z_0 and z_1. The axis y_1 is chosen to give a right-handed rectangular system. Similarly, x_2 is the extension of the common perpendicular of length a_1 between z_1 and z_2 with y_2 completing a right-handed system. The position of the pair element $R_2{}^-$ is thus defined completely with respect to the pair R_1 by the four parameters a_1, α_1, θ_1, and s_1.

A compact way of writing the defining relations between the pair elements of a machine part is to assemble all the data in a block, as shown in Fig. 5. This represents part 2 of Fig. 4. Part 1

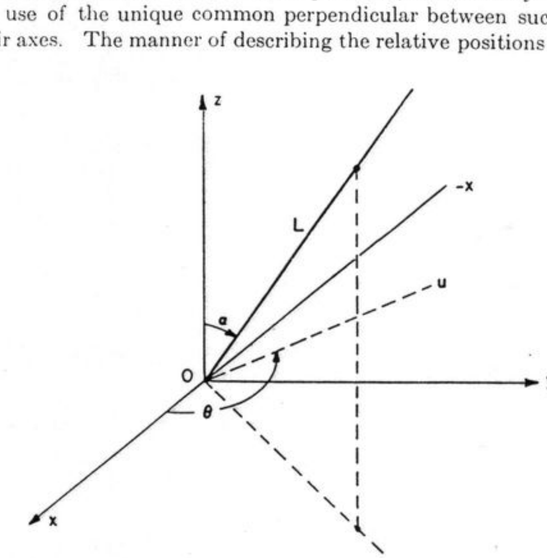

Fig. 3 Parameters α and θ Define Line L or Axis of a Prism Pair ($a = 0$, $s = 0$)

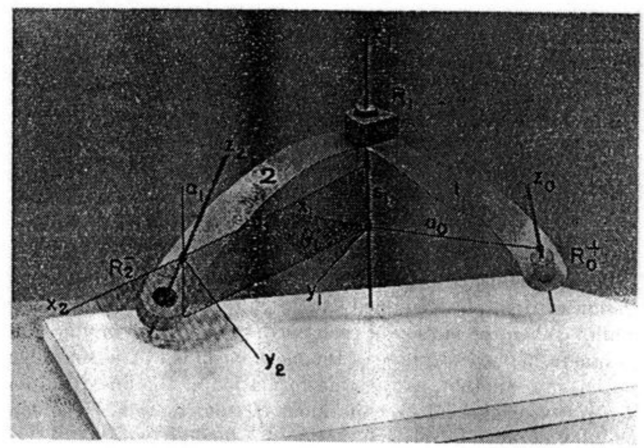

Fig. 4 Parts of a Space Mechanism, Showing Parameters Relating Axes of Three Pairs

cannot be described since the parameters θ_0 and s_0 are dependent on part 0 which precedes part 1.

As an example of how a complete plane mechanism composed of only revolute and prism pairs may be described, consider the slider crank of Fig. 6. The four machine parts are identified for convenience as 1, 2, 3, and 4; they are connected as shown by the pairs P_1, R_2, R_3, and R_4. Each of the parts will be completely described if the geometry of its ends, i.e., its pair elements, is given. The description of the parts leads to a description of the complete mechanism.

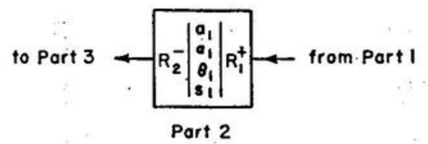

FIG. 5 BLOCK DIAGRAM FOR PART 2 OF FIG. 4

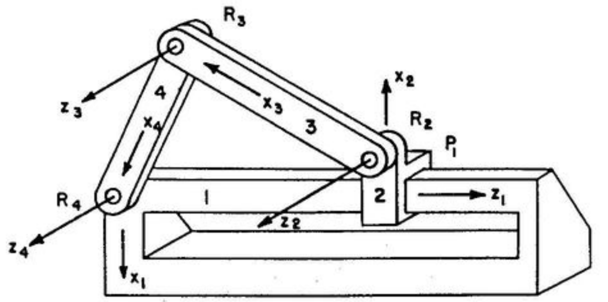

FIG. 6 SLIDER-CRANK MECHANISM, SHOWING PAIRS AND AXES

The location of the prism P_1 is described by the direction of its axis z_1; the revolutes R_2, R_3, and R_4 are described by the location of their axes z_2, z_3, and z_4. In the symbolic description of the mechanism, the angles between successive z-axes is given by α. For the present mechanism the angle between z_1 and z_2 is 90 deg or $\alpha_1 = 90$ deg; the angle between z_2 and z_3 is zero, or $\alpha_2 = 0$; the angle between z_3 and z_4 is also zero, or $\alpha_3 = 0$; and the angle between z_4 and z_1 is 90 deg or $\alpha_4 = 90$ deg.

These angles α are not sufficient for a complete description; obviously certain lengths and other angles are necessary. To find the missing parameters it will be necessary to show the x-axes. The axis x_1 is defined in the machine part 1 along the common perpendicular between z_4 and z_1; x_2 is defined in the part 2 along the common perpendicular between z_1 and z_2; x_3 in part 3 is along the common perpendicular between z_2 and z_3; and x_4 in part 4 is along the common perpendicular between z_3 and z_4. The y-axes are not shown, since no angles are measured from them, but they are considered present to form right-handed systems.

In a symbolic description of the mechanism the perpendicular distances between z-axes will be designated by the latter a. For the present mechanism, the distance between z_4 and z_1 is zero, or $a_4 \doteq 0$; between z_1 and z_2 the distance is a_1; between z_2 and z_3 the distance is a_2; and between z_3 and z_4 the distance is a_3. These distances are illustrated in Fig. 7.

The angles between x-axes are designated by θ. For the present mechanism the angle between x_1 and x_2 is $\theta_1 = 180$ deg, the angles between x_2 and x_3, x_3 and x_4, and x_4 and x_1 are θ_2, θ_3 and θ_4, respectively. These last three angles are the pair variables of the revolutes; their values change with the motion of the mechanism.

The perpendicular distances between x-axes are designated by s, Fig. 7. The perpendicular distance between x_1 and x_2 is s_1, which is also the pair variable of the prism; the distances between x_2 and x_3, x_3 and x_4, x_4 and x_1 are all zero, i.e., $s_2 = 0$, $s_3 = 0$, $s_4 = 0$.

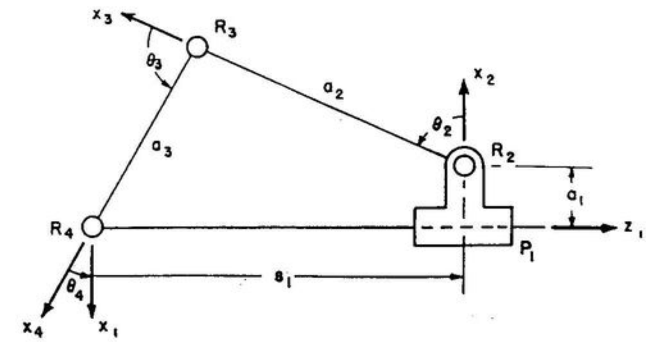

FIG. 7 SCHEMATIC OF SLIDER-CRANK MECHANISM SHOWING PAIR VARIABLES θ AND s

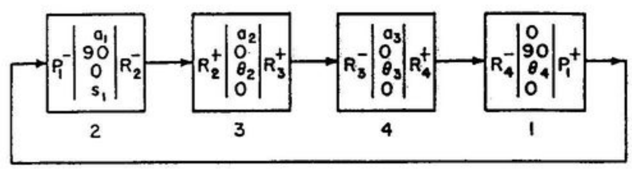

FIG. 8 BLOCK DIAGRAM REPRESENTING SLIDER-CRANK MECHANISMS OF FIGS. 6 AND 7

A representation of this mechanism is given by assembling all these data in blocks, one block to each part, as shown in Fig. 8. Each box represents a machine part; in each block are indicated its pair elements, the parameters defining the machine part itself, and the relations of the part to the preceding machine part. Since the chain of blocks so obtained forms a closed loop, it is seen that the description of the chain may be started at any pair element and may proceed in either direction.

This way of representing a mechanism is cumbersome. A more concise but equally definitive description in the form of an equation would have a manipulative property, particularly if a mathematical interpretation could be shown. Such an equation can be developed from the considerations outlined later. A rigorous justification for the arguments is given under Matrix Interpretation of the Symbolic Equation.

With the aid of Fig. 4 the relations between successive co-ordinate systems were seen to be dependent on the values of the intervening a, α, θ, and s. Such a relation is a linear transformation between the two systems involved, and each linear transformation may be represented by a matrix M; the elements of the matrix are functions of a, α, θ, and s, and each matrix represents the machine part between co-ordinate systems.

On the mechanism of Fig. 6 there are four co-ordinate systems, and hence four linear transformations (taken in succession) would give a return to the original co-ordinate system. The successive linear transformations from one co-ordinate system to another may be effected by multiplying the corresponding matrices. The product of all matrices is the unit matrix when there is a return to the original system. This is the case for a closed chain or mechanism, and for the example of the slider crank the matrix equation would be

$$M_2 \times M_3 \times M_4 \times M_1 = 1 \qquad [1]^5$$

This equation contains the solution to the displacement relations of the mechanism.

The analogy between this equation and Fig. 8 lies in the fact that the matrices of Equation [1] depend on the parameters a, α, θ, and s, appearing between "verticals" in Fig. 8. This, together

[5] In this equation, and all matrix equations to follow, the symbol 1 stands for the unit matrix.

with the fact that a positive and negative-pair element constitute a pair, and that it would be informative to indicate the pair variables, suggests writing the symbolic representation of the closed mechanism as a symbolic equation

$$P_1(s_1)\begin{vmatrix}a_1\\90\\0\\s_1\end{vmatrix}\ R_2(\theta_2)\begin{vmatrix}a_2\\0\\\theta_2\\0\end{vmatrix}\ R_3(\theta_3)\begin{vmatrix}a_3\\0\\\theta_3\\0\end{vmatrix}\ R_4(\theta_4)\begin{vmatrix}0\\90\\\theta_4\\0\end{vmatrix} = 1$$

It will be recognized that this system of symbols can describe any mechanism comprised of lower pairs. The general form of a symbolic equation describing a closed mechanism may be written as

$$S_{L1}(u_1)\begin{vmatrix}a_1\\\alpha_1\\\theta_1\\s_1\end{vmatrix}\ S_{L2}(u_2)\begin{vmatrix}a_2\\\alpha_2\\\theta_2\\s_2\end{vmatrix}\dots S_{Li}(u_i)\begin{vmatrix}a_i\\\alpha_i\\\theta_i\\s_i\end{vmatrix}\dots S_{Ln}(u_n)\begin{vmatrix}a_n\\\alpha_n\\\theta_n\\s_n\end{vmatrix} = 1\quad[2]$$

For a screw $S_{Li}(L_i \neq 0 \text{ or } \infty)$ both parameters θ_i and s_i vary, being related by the lead as

$$\frac{\Delta\theta_i}{2\pi} = \frac{\Delta s_i}{L_i}$$

Here $\Delta\theta_i$ and ΔS_i are the variations, and consequently either θ_i or s_i may be taken as the pair variable u_i.

For a revolute ($L_i = 0$) the parameter θ_i is the pair variable u_i; the other parameters, a_i, α_i, and s_i do not change with movement.

For a prism ($L_i = \infty$) the parameter s_i is the pair variable u_i; the other parameters a_i, α_i, and θ_i do not change with movement.

The right circular cylinder pair is equivalent to a coaxial revolute and prism and is written

$$R_1(\theta_1)\begin{vmatrix}0\\0\\\theta_1\\0\end{vmatrix}\ P_2(s_2)\begin{vmatrix}a_2\\\alpha_2\\0\\s_2\end{vmatrix}$$

The plane pair is equivalent to a combination of a revolute and two prisms at right angles to each other and the revolute axis and is written

$$R_1(\theta_1)\begin{vmatrix}0\\90\\\theta_1\\0\end{vmatrix}\ P_2(s_2)\begin{vmatrix}0\\90\\90\\s_2\end{vmatrix}\ P_3(s_3)\begin{vmatrix}a_3\\\alpha_3\\\theta_3\\s_3\end{vmatrix}$$

The spheric pair is equivalent to a combination of three revolutes whose axes are mutually perpendicular at a common point of intersection and is written

$$R_1(\theta_1)\begin{vmatrix}0\\90\\\theta_1\\0\end{vmatrix}\ R_2(\theta_2)\begin{vmatrix}0\\90\\\theta_2\\0\end{vmatrix}\ R_3(\theta_3)\begin{vmatrix}a_3\\\alpha_3\\\theta_3\\s_3\end{vmatrix}$$

The symbolic notation presented in the foregoing has been developed from the kinematic chain-and-pair-element concepts of Reuleaux by establishing the complete and sufficient mathematical description of the relations between pairs. The matrix interpretation of the data between "verticals" endows the new symbolism with an analytical power that only a mathematical procedure possesses.

The literature of kinematics seems to show no similar approach. Dimentberg (6, 7), who has worked in the same general area of lower-pair space mechanisms, has developed a method for studying displacements based on Clifford's dual numbers (taken from the realm of pure algebra). The abstracts do not mention the development of a systematic, manipulative notation generalized to all lower-pair mechanisms.

Comparison of the New Symbolism With That of Reuleaux

It has been remarked that the symbolism of Reuleaux was incomplete by reason of not including all needed parameters and hence had limited usefulness. By way of comparing Reuleaux's symbolism with that presented here, the universal joint will be taken as an example, since Reuleaux (Kennedy)[6] discusses it at some length as follows:

"We may take for a second illustration a universal joint, or Hooke's joint, of which Fig. 181 (Fig. 9 of text) gives the schematic

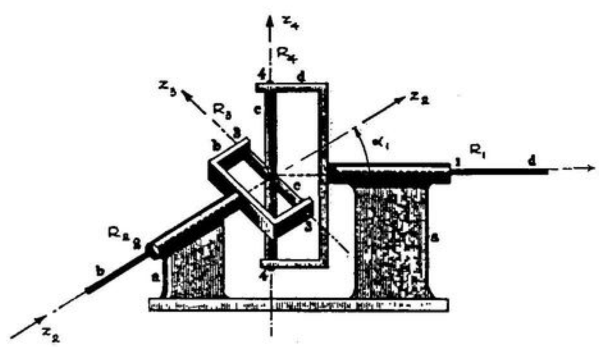

Fig. 9 Reuleaux's Fig. 181, the Universal Joint
(Pairs and axes have been identified in terms of the symbolic notation.)

representation. The chain which constitutes this joint has four links, which are marked on the figure with the letters a, b, c, and d. The link a is paired with b by the turning-pair 2. Normal to this turning-pair is another, 3, which has its open cylinder in the fork of d and its full cylinder in the sloping arm of the piece c; the link b must therefore be written $C^+ \dots \perp \dots C^-$. It must be noted that the lower and upper arms of the fork form together one piece only and must be reckoned as such; the same is true of the two ends of the arm c, which kinematically forms a single element only. The piece c consists of two solid cylinders 3 and 4, having their axes crossing at right angles, and it must therefore be written $C^+ \dots \perp \dots C^+$. The third link, the fork and the spindle d, is similar to b and will be written in the same way. The fourth link a, lastly, consists of two open cylinders, 1 and 2, oblique to each other, and so must be written $C^- \dots \angle \dots C^-$; it is a fixed link, as its form in the figure shows. The complete formula, therefore (to which we have added the letters and numbers that were used in the foregoing to distinguish the links and pairs), runs thus

$$\underbrace{\underset{2}{C^+}\dots\perp\dots C^-}_{b} = \underbrace{\underset{3}{C^+}\dots\perp\dots C^+}_{c} = \underbrace{\underset{4}{C^-}\dots\perp\dots C^+}_{d} = \underbrace{\underset{1}{C^-}\dots\angle\dots \underset{2}{C^-}}_{a} =$$

"There is one geometrical property of the chain which is not shown by our formula, namely, that the axes of the pairs 1, 2, 3, and 4 have a common point of intersection. But unless the chain possessed this property it would not be possible, on our supposi-

[6] Reference (2), pp. 260-261.

tion that all its pairs are closed.[7] No special indication of this property is therefore commonly necessary. Our formula shows, however, that the three links b, c, and d are again identical. This circumstance is very notable, and we shall later on have to deal with it in another form; the common construction of the joint so entirely conceals it as to make it almost unrecognizable."

Reuleaux (Kennedy)[8] remarks that "the formula or symbolic description may in many cases be greatly shortened." The shortened form he calls the contracted formula[9] says:

"The chain forming the universal joint, Fig. 181, —to take another example—allows itself to be written $(C_3^{\perp}\, C^{\angle})$ in words, C normal three C oblique"; . . . . (End of comment on universal joint.)

To describe the universal joint in terms of the new symbolic notation it will be necessary to establish the identities of the four pairs (all revolutes) in terms of the new symbols, as shown in Fig. 9. The z-axes are the axes of the revolutes, and it will be convenient to start numbering at one end of the frame and then proceed to the other end of the frame. Since all axes intersect, the distances a are all zero. The angle of the shafts is given by the angle α_1 which by definition is the angle between the z_1 and z_2-axes. The other angles α are all 90 deg. Each shaft (or revolute) rotation is specified by a value of θ, which is also the pair variable. The four values of s become zero because of mutual axis intersection. Assembling these parameters in the manner of Equation [2] gives the symbolic equation of the universal joint as

$$R_1(\theta_1)\begin{vmatrix}0\\ \alpha_1\\ \theta_1\\ 0\end{vmatrix} R_2(\theta_2)\begin{vmatrix}0\\ 90\\ \theta_2\\ 0\end{vmatrix} R_3(\theta_3)\begin{vmatrix}0\\ 90\\ \theta_3\\ 0\end{vmatrix} R_4(\theta_4)\begin{vmatrix}0\\ 90\\ \theta_4\\ 0\end{vmatrix} = 1 \ldots\ldots [3]$$

This symbolic equation shows all parameters needed to describe completely the universal joint in a geometric manner.

It is to be noted, Reuleaux commented, that the geometrical property of mutual axis intersection was not given in his formula, and that he could write no formula for a mechanism unless he knew beforehand that it moved. Consequently, Reuleaux's symbolism cannot be used to establish the relative displacements of the parts of a mechanism. The new symbolism obviates these shortcomings by indicating all parameters from whose relations the possibility of motion may be deduced.

Matrix Interpretation of the Symbolic Equation

The application of matrix algebra to the description of a mechanism has been outlined in the example of the slider crank. To justify that use of matrices, and to analyze the displacements occurring in a closed mechanism, each matrix must be related to its values of a, α, θ, and s. In the following, the elements of the matrix corresponding to each machine part will be established.

Consider two co-ordinate systems $x_1y_1z_1$ and $x_2y_2z_2$, Fig. 10. A point M is given by its co-ordinates x_2, y_2, z_2 with respect to $x_2y_2z_2$; the problem is to determine the co-ordinates x_1, y_1, z_1 of M with respect to $x_1y_1z_1$. The vector relation

$$\overrightarrow{O_1M} = \overrightarrow{O_1O_2} + \overrightarrow{O_2M}$$

gives by projection on the axes the following system of equations

$$x_1 = (x_0)_1 + x_2 \cos(x_2, x_1) + y_2 \cos(y_2, x_1) + z_2 \cos(z_2, x_1)$$
$$y_1 = (y_0)_1 + x_2 \cos(x_2, y_1) + y_2 \cos(y_2, y_1) + z_2 \cos(z_2, y_1)$$
$$z_1 = (z_0)_1 + x_2 \cos(x_2, z_1) + y_2 \cos(y_2, z_1) + z_2 \cos(z_2, z_1)$$

These equations are nonhomogeneous with respect to the variables x_2, y_2, z_2.

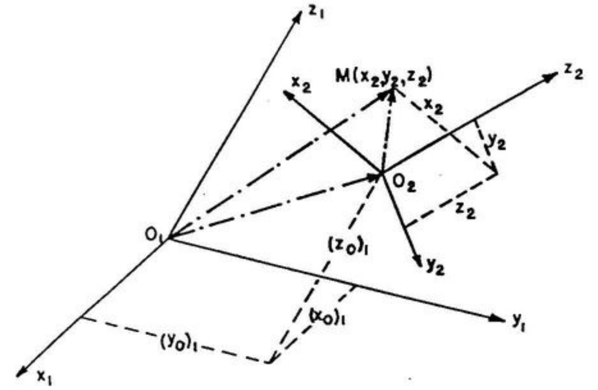

Fig. 10 Transformation of Rectangular Cartesian Co-ordinates

If, however, the equations are put into homogeneous form, they then may be represented by a square matrix. The introduction of a new set of variables defined by

$$X_1 = x_1H_1,\ Y_1 = y_1H_1,\ Z_1 = z_1H_1$$

and $$X_2 = x_2H_2,\ Y_2 = y_2H_2,\ Z_2 = z_2H_2$$

with $$H_1 = H_2$$

will produce the homogeneous system

$$\left.\begin{aligned}
H_1 &= H_2\\
X_1 &= (x_0)_1H_2 + \cos(x_2, x_1)X_2 + \cos(y_2, x_1)Y_2\\
&\qquad + \cos(z_2, x_1)Z_2\\
Y_1 &= (y_0)_1H_2 + \cos(x_2, y_1)X_2 + \cos(y_2, y_1)Y_2\\
&\qquad + \cos(z_2, y_1)Z_2\\
Z_1 &= (z_0)_1H_2 + \cos(x_2, z_1)X_2 + \cos(y_2, z_1)Y_2\\
&\qquad + \cos(z_2, z_1)Z_2
\end{aligned}\right\} \ldots [4]$$

which can be represented by the square matrix of its coefficients

$$M_2 = \begin{vmatrix} 1 & 0 & 0 & 0\\ (x_0)_1 & \cos(x_2, x_1) & \cos(y_2, x_1) & \cos(z_2, x_1)\\ (y_0)_1 & \cos(x_2, y_1) & \cos(y_2, y_1) & \cos(z_2, y_1)\\ (z_0)_1 & \cos(x_2, z_1) & \cos(y_2, z_1) & \cos(z_2, z_1)\end{vmatrix} \ldots [5]$$

Equation [5] was developed for the general case of rectangular Cartesian co-ordinates shown in Fig. 10. When applied to the rectangular systems related by the parameters a, α, θ, and s shown in Fig. 4, the matrix representing part 2 takes the following form

$$M_2 = \begin{vmatrix} 1 & 0 & 0 & 0\\ a_1\cos\theta_1 & \cos\theta_1 & -\sin\theta_1\cos\alpha_1 & -\sin\theta_1\sin\alpha_1\\ a_1\sin\theta_1 & \sin\theta_1 & \cos\theta_1\cos\alpha_1 & \cos\theta_1\sin\alpha_1\\ s_1 & 0 & -\sin\alpha_1 & \cos\alpha_1\end{vmatrix} \ldots [6]$$

The inverse of this matrix, also useful for computation, is

$$M_2^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0\\ -a_1 & \cos\theta_1 & \sin\theta_1 & 0\\ s_1\sin\alpha_1 & -\sin\theta_1\cos\alpha_1 & \cos\theta_1\cos\alpha_1 & -\sin\alpha_1\\ -s_1\cos\alpha_1 & -\sin\theta_1\sin\alpha_1 & \cos\theta_1\sin\alpha_1 & \cos\alpha_1\end{vmatrix} . [7]$$

[7] To avoid confusion, it must be noted that the word "closed" meant two things to Reuleaux, (a) that the last machine part is in contact with the first, i.e., the chain of machine parts constitutes a completed loop, and (b) that the mechanism actually possesses movability. See Reuleaux (Kennedy) (2) pp. 46–49. The authors use closed in only the first sense.

[8] Reference (2), p. 263

[9] Ibid., p. 264.

Examples

To demonstrate the nature of the matrix manipulations involved in establishing the functional relations of the displacement of different parts, two examples of space mechanisms will be presented. The first example, that of the screw chain, is included for demonstration of a simple example. The second example, that of the universal joint, is more complex; the geometric relations of the parts are difficult to draw (as is usual in the case of space mechanisms) because of the three dimensions that are involved. The reasoning of the matrix method needs the support of only a sketch to define the directions of the axes of the revolutes, after which the solution proceeds in formal mathematical manner without the need for scale drawings.

(*a*) *The Screw Chain.* A screw chain is composed of three screws so connected that they have a common axis, as shown in Fig. 11. The three leads are different and the hand of the threads is arbitrary. With a common axis all three z-axes are collinear,

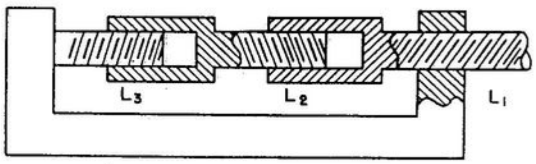

Fig. 11 Screw Chain, General Case

which means that $a_1 = 0$, $a_2 = 0$, and $a_3 = 0$ as well as $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 0$. Each screw has its own θ and s. Assembling these data in the fashion of Equation [2], the symbolic equation describing the screw chain[10] becomes

$$S_{L_1}(u_1)\begin{vmatrix}0\\0\\\theta_1\\s_1\end{vmatrix} S_{L_2}(u_2)\begin{vmatrix}0\\0\\\theta_2\\s_2\end{vmatrix} S_{L_3}(u_3)\begin{vmatrix}0\\0\\\theta_3\\s_3\end{vmatrix} = 1$$

There are three parts, each of which may be represented by a matrix. The matrix equation of the screw chain is then (cf. Equation [1])

$$M_2 \times M_3 \times M_1 = 1$$

Since each matrix is of the form of Equation [6], the expanded matrix equation of the screw chain is given by

$$\begin{vmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta_2 & -\sin\theta_2 & 0\\ 0 & \sin\theta_2 & \cos\theta_2 & 0\\ s_2 & 0 & 0 & 1\end{vmatrix} \times \begin{vmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta_1 & -\sin\theta_1 & 0\\ 0 & \sin\theta_1 & \cos\theta_1 & 0\\ s_1 & 0 & 0 & 1\end{vmatrix}$$
$$\times \begin{vmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta_3 & -\sin\theta_3 & 0\\ 0 & \sin\theta_3 & \cos\theta_3 & 0\\ s_3 & 0 & 0 & 1\end{vmatrix} = 1$$

The result of this multiplication is given in the following and the unit matrix is expressed in terms of its elements

$$\begin{vmatrix}1 & 0 & 0 & 0\\ 0 & \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0\\ 0 & \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0\\ s_1+s_2+s_3 & 0 & 0 & 1\end{vmatrix}$$
$$= \begin{vmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{vmatrix}$$

[10] In Reuleaux's notation this screw chain is given as S'_3 (reference 2, p. 546). The prime sign denotes a common axis.

Identifying the elements of the two matrices leads to the displacement relations

$$\theta_1 + \theta_2 + \theta_3 = 0 \text{ and } s_1 + s_2 + s_3 = 0$$

When L_1, L_2, and L_3 are different from zero and infinity, then either θ or s could be the pair variable u. Choosing θ to represent the pair variable

$$s_1 = L_1\frac{\theta_1}{2\pi},\ s_2 = L_2\frac{\theta_2}{2\pi}, \text{ and } s_3 = L_3\frac{\theta_3}{2\pi}$$

whence

$$L_1\theta_1 + L_2\theta_2 + L_3\theta_3 = 0$$

Taking θ_1 as the independent variable (input), the functional relations become

$$\theta_2 = -\frac{L_1 - L_3}{L_2 - L_3}\theta_1 \text{ and } \theta_3 = -\frac{L_1 - L_2}{L_3 - L_2}\theta_1$$

If s_1 were taken as the independent variable, then the functional relations are given by

$$s_2 = -\frac{\dfrac{1}{L_1} - \dfrac{1}{L_3}}{\dfrac{1}{L_2} - \dfrac{1}{L_3}}s_1 \text{ and } s_3 = -\frac{\dfrac{1}{L_2} - \dfrac{1}{L_1}}{\dfrac{1}{L_2} - \dfrac{1}{L_3}}s_1$$

Similar relations are shown by Beyer.[11]

The foregoing relations apply when the three leads are different from zero and infinity; either θ or s may then be taken as the pair variable. However, if the lead of one screw becomes zero, giving a revolute, then θ is the pair variable; and if the lead of one screw becomes infinitely great to form a prism, then s is the pair variable.

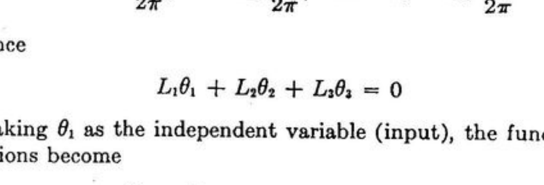

Fig. 12 Screw Chain Transforming Continuous Rotation Into Continuous Translation—or Vice Versa

A special case of practical interest is shown in Fig. 12: it represents the essence of the cross-feed of a lathe, or the feeds of a milling machine table, and so on. Here $L_1 = 0$, $L_2 \neq 0$ or ∞, and $L_3 = \infty$. The functional relations are then

$$\theta_2 = -\frac{\dfrac{L_1}{L_3} - 1}{\dfrac{L_2}{L_3} - 1}\theta_1 = -\theta_1$$

and

$$s_3 = -\frac{\dfrac{1}{L_2} - \dfrac{1}{L_1}}{\dfrac{1}{L_2} - \dfrac{1}{L_3}}\frac{L_1}{2\pi}\theta_1 = -\frac{L_2}{2\pi}\theta_1$$

(*b*) *The Universal Joint.* The symbolic equation has been given already as

[11] Reference (5), p. 46.

$$R_1(\theta_1)\begin{vmatrix}0\\ \alpha_1\\ \theta_1\\ 0\end{vmatrix} R_2(\theta_2)\begin{vmatrix}0\\ 90\\ \theta_2\\ 0\end{vmatrix} R_3(\theta_3)\begin{vmatrix}0\\ 90\\ \theta_3\\ 0\end{vmatrix} R_4(\theta_4)\begin{vmatrix}0\\ 90\\ \theta_4\\ 0\end{vmatrix} = 1 \ldots\ldots [3]$$

In matrix form it is

$$M_2 \times M_3 \times M_4 \times M_1 = 1$$

or with the elements formed from the parameters

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1\cos\alpha & -\sin\theta_1\sin\alpha_1 \\ 0 & \sin\theta_1 & \cos\theta_1\sin\alpha_1 & \cos\theta_1\sin\alpha_1 \\ 0 & 0 & -\sin_1\cdot\alpha_1 & \cos\alpha_1 \end{vmatrix}$$

$$\times \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & \sin\theta_2 & 0 & \cos\theta_2 \\ 0 & 0 & -1 & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_3 & 0 & -\sin\theta_3 \\ 0 & \sin\theta_3 & 0 & \cos\theta_3 \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

$$\times \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_4 & 0 & -\sin\theta_4 \\ 0 & \sin\theta_4 & 0 & \cos\theta_4 \\ 0 & 0 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

The appearance of this last equation, in particular M_2, suggests that the conventional matrix multiplication would be quite tedious, and that this labor would be reduced by the use of the inverse matrix, viz.

$$M_2^{-1} \times M_2 \times M_3 \times M_4 \times M_1 = M_2^{-1} \times 1$$

Since $M_2^{-1} \times M_2 = 1$ the equation becomes

$$M_3 \times M_4 \times M_1 = M_2^{-1}$$

The general form of the inverse matrix M_2^{-1} expressed in terms of a_1, α_1, θ_1, and s_1 has been given as Equation [7]. The matrix multiplication, now reduced to three operations, i.e., $M_3 \times M_4 \times M_1$, is set equal to the inverse matrix M_2^{-1}. This is shown as follows

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2\cos\theta_3\cos\theta_4 + \sin\theta_2\sin\theta_4 & \cos\theta_2\sin\theta_3 & -\cos\theta_2\cos\theta_3\sin\theta_4 + \sin\theta_2\cos\theta_4 \\ 0 & \sin\theta_2\cos\theta_3\cos\theta_4 - \cos\theta_2\sin\theta_4 & \sin\theta_2\sin\theta_3 & -\sin\theta_2\cos\theta_3\sin\theta_4 - \cos\theta_2\cos\theta_4 \\ 0 & -\sin\theta_3\cos\theta_4 & \cos\theta_3 & \sin\theta_3\sin\theta_4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & -\sin\theta_1\cos\alpha_1 & \cos\theta_1\cos\alpha_1 & -\sin\alpha_1 \\ 0 & -\sin\theta_1\sin\alpha_1 & \cos\theta_1\sin\alpha_1 & \cos\alpha_1 \end{vmatrix}$$

Identification of the elements of the two matrices leads to the set of nine equations

$$\begin{aligned}
\sin\theta_3\sin\theta_4 &= \cos\alpha_1 \\
\cos\theta_3 &= \cos\theta_1\sin\alpha_1 \\
\sin\theta_3\cos\theta_4 &= \sin\theta_1\sin\alpha_1 \\
\sin\theta_2\sin\theta_3 &= \cos\theta_1\cos\alpha_1 \\
\cos\theta_2\sin\theta_3 &= \sin\theta_1 \\
\cos\theta_2\cos\theta_3\cos\theta_4 + \sin\theta_2\sin\theta_4 &= \cos\theta_1 \\
-\cos\theta_2\cos\theta_3\sin\theta_4 + \sin\theta_2\cos\theta_4 &= 0 \\
\sin\theta_2\cos\theta_3\cos\theta_4 - \cos\theta_2\sin\theta_4 &= -\sin\theta_1\cos\alpha_1 \\
-\sin\theta_2\cos\theta_3\sin\theta_4 - \cos\theta_2\cos\theta_4 &= -\sin\alpha_1
\end{aligned}$$

Taking θ_1 as the independent variable of the input shaft, the relations for the other three pair variables are found to be

$$\tan\theta_2 = \frac{\cos\alpha_1}{\tan\theta_1},\ \cos\theta_3 = \sin\alpha_1\cos\theta_1,\ \tan\theta_4 = \frac{1}{\tan\alpha_1\sin\theta_1}$$

These relations satisfy all nine equations.

The relation for $\tan\theta_2$ is identical with that given by Beyer.[12]

Summary

This paper presents the development of a symbolic notation that is able to describe completely lower-pair mechanisms. This symbolic notation furnishes a powerful and reliable analytical procedure since the operations are based on matrix algebra. The use of the notation also puts lower-pair problems of kinematics into a form ready for electrical computation methods, as has been demonstrated (8). The matrix manipulation involved in establishing the functional relations of the displacements of different parts of space mechanisms is shown by examples.

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[12] Reference (5), p. 90.