Reconstructing the Free-Surface from Pressure

Time-Dependent Relationships and Experimental Comparisons

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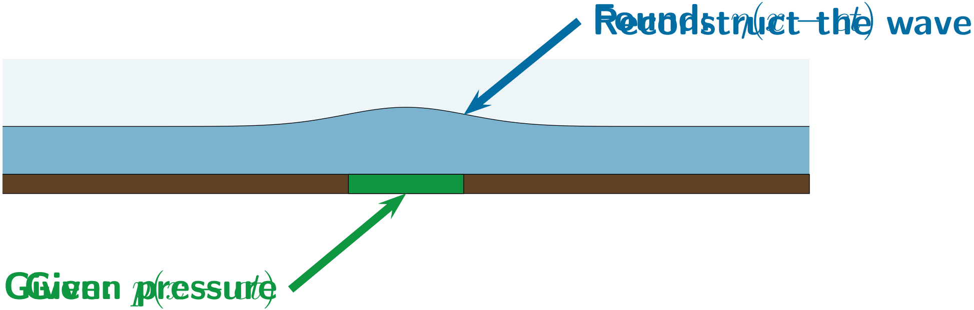
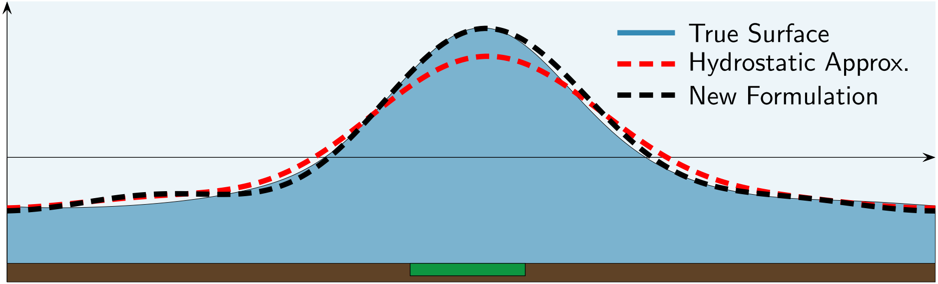


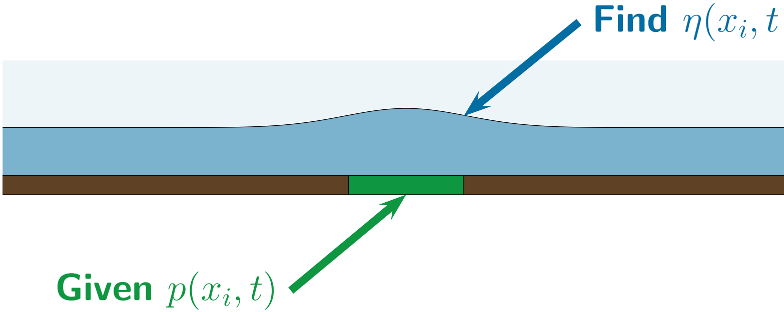
Joint work: Deconinck, Henderson, Oliveras, Vasan

Observations in Experiments



Can we address these challenges and develop a time-series method?

Motivation / Problem Description



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We begin by considering an inviscid, irrotational, fluid with a one-dimensional free-surface on the whole-line:

$$
\begin{aligned}
&\phi_{xx} + \phi_{zz} = 0, && (x,z) \in \mathscr{D},\\
&\phi_t + \frac{1}{2}|\nabla\phi|^2 + \frac{\rho g z - p(x,z,t)}{\rho} = 0, && (x,z) \in \mathscr{D},\\
&\phi_z = 0, && z = -h,\\
&\eta_t = \phi_z - \eta_x\phi_x, && z = \eta(x,t),\\
&p = 0, && z = \eta(x,t),
\end{aligned}
$$

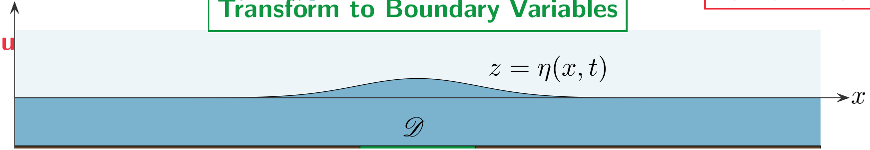
where $\phi(x,z,t)$ represents the velocity potential of the fluid, $\eta(x,t)$ represents the surface elevation.

[Zakharov, and Craig & Sulem (ZCS)]:

$\boxed{q(x,t) = \phi(x,\eta(x,t),t)}$

Transform to Boundary Variables

Dynamic Bou



$z = -h$

Mapping Pressure via AFM

From the previous slide:

A Third Equation

$$\boxed{Q(x,t) = \phi(x,-h,t)}$$

$$\int_{-\infty}^{\infty} \left(e^{-ikx} \left(\eta_t \sinh(k(\eta + h)) - i q_x \cosh(k(\eta + h)) \right) \right) \, dx = k \hat{Q}(k,t)$$

$$\hat{Q}(k,t) \to Q(x,t) \qquad \Rightarrow \qquad Q_t + \frac{1}{2} Q_x^2 + \frac{p(x,-h,t) - \rho g h}{\rho} = 0$$

For traveline waves, you can find the following relationships:

$$p(\xi,-h) - \rho g h = \rho \iint_{\mathbb{R}\times\mathbb{R}} e^{ik(\xi-\xi')} \sqrt{(c^2 - 2g\eta)(1+\eta_x^2)} \cosh(k(\eta+h)) \, d\xi' \, dk$$

or via a separate derivation

$$\int_{-\infty}^{\infty} e^{-ik\xi} \sqrt{c^2 - 2(p - \rho g h)} \cosh(k(\eta + h)) \, dk = \frac{\sqrt{c^2 - 2g\eta}}{1 + \eta_\xi^2}$$

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Nonlocal/Nonlocal Formulation: The First Integral Relation

For **any harmonic function** $\varphi(x,z)$, we have

$$\oint_{\partial\mathscr{D}} \left((\varphi_z \nabla\phi - \phi\nabla\varphi_z) \cdot \mathbf{n} \right) ds = 0,$$

where we assume that ϕ has sufficient decay properties as $|x| \to \infty$. Using the **kinematic boundary condition,**

$$\nabla\phi \cdot \mathbf{n}\big|_{\mathscr{S}} = \eta_t \quad \to \quad \int_{\mathscr{S}} \varphi_z \eta_t \, dx = \oint_{\partial\mathscr{D}} \left(\phi\nabla\varphi_z \cdot \mathbf{n}\right) ds$$

Similarly for ϕ_t:

$$\oint_{\partial\mathscr{D}} \left((\varphi_z \nabla\phi_t - \phi_t\nabla\varphi_z) \cdot \mathbf{n} \right) ds = 0,$$

Using the **dynamic boundary condition,** along with $\nabla\phi_t \cdot \mathbf{n}\big|_{\mathscr{S}} = \eta_{tt} + \dfrac{d}{dx}\left[\eta_t\phi_x(x,\eta,t)\right]$

$$\int_{\mathscr{S}} \frac{d}{dt} q \left(\frac{\partial\varphi_z}{\partial\mathbf{n}} - 2\varphi_{zz} \right) dx = \int_{\mathscr{S}} g\eta(\varphi_{zz} + \eta_x\varphi_{xz}) - \underbrace{2\eta_t\, q\, \varphi_{zzz}}_{(*)} \, dx + \int_{\mathscr{B}} \frac{1}{2} Q_x^2\, \varphi_{zz} \, dx$$

Integral Relation A

$$\int_{\mathscr{S}} \frac{d}{dt}\varphi\, dx = \int_{\mathscr{S}} \phi\left(\nabla\varphi_z \cdot \mathbf{n}\right)\, dx - \int_{\mathscr{B}} Q\varphi_{zz}\, dx$$

$$\boxed{q = \phi(x,\eta,t),\ Q = \phi(x,-h,t)}$$

Integral Relation B

$$\int_{\mathscr{S}} \frac{d}{dt}q\left(\frac{\partial\varphi_z}{\partial\mathbf{n}} - 2\varphi_{zz}\right)\, dx = \int_{\mathscr{S}} g\eta(\varphi_{zz} + \eta_x\varphi_{xz}) - \underbrace{2\eta_t\, q\, \varphi_{zzz}}_{(*)}\, dx + \int_{\mathscr{B}} \frac{1}{2}Q_x^2\, \varphi_{zz}\, dx$$

Remark #1

The **dynamic boundary condition** was only prescribed at the last stage of deriving (B).

Remark #2

$\varphi = e^{-ikx}\sinh(k(z+h))$, implies both **[ZCS]** and **[AFM]**.

Pressure Relationship

Integral Relation A

$$\int_{\mathscr{S}} \frac{d}{dt}\varphi \, dx = \int_{\mathscr{S}} \phi \left(\nabla \varphi_z \cdot \mathbf{n}\right) \, dx - \int_{\mathscr{B}} Q \varphi_{zz} \, dx$$

$$\boxed{q = \phi(x, \eta, t), \;\; Q = \phi(x, -h, t)}$$

Integral Relation B

$$\int_{\mathscr{S}} \frac{d}{dt} q \left(\frac{\partial \varphi_z}{\partial \mathbf{n}} - 2\varphi_{zz} \right) \, dx = \int_{\mathscr{S}} g\eta(\varphi_{zz} + \eta_x \varphi_{xz}) - \underbrace{2\eta_t \, q \, \varphi_{zzz}}_{(*)} \, dx + \int_{\mathscr{B}} \frac{1}{2} Q_x^2 \, \varphi_{zz} \, dx$$

Integral Relation C

$$\boxed{P_d(x,t) = \frac{p(x,-h,t) - \rho g h}{\rho}}$$

$$\int_{\mathscr{B}} P_d \, \varphi_{zz} = \rho \int_{\mathscr{S}} \eta_{tt} \varphi_z + \left(\eta_t^2 + g\eta\right) \varphi_{zz} + \left(g\eta\eta_x - 2\left(q_x \eta_t - q_t \eta_x\right)\right) \varphi_{xz}$$

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Nondimensionalization

We introduce the nondimensional quantities as follows:

$$x = x^* L, \quad z = z^* h, \quad t = \frac{L}{c_0} t^*, \quad c_0 = \sqrt{gh}, \quad \epsilon = \frac{a}{h}, \quad \mu = \frac{h}{L},$$

$$\eta = a\eta^*, \qquad \frac{p(x, -h, t) - \rho g h}{\rho} = \epsilon P_d^*, \qquad Q = \frac{Lga}{c_0} Q^*, \qquad q = \frac{Lga}{c_0} q^*,$$

This yields the **nondimensional relationship**

$$\int_{\mathscr{B}} P_d \, \varphi_{zz} = \int_{\mathscr{S}} \eta\varphi_{zz} + \mu^2 \eta_{tt}\varphi_z + \epsilon\mu^2 \left(\eta_t^2 \varphi_{zz} + \left(\eta\eta_x + 2\left(q_t\eta_x - q_x\eta_t \right) \right) \varphi_{xz} \right),$$

where φ solves $\mu^2 \varphi_{xx} + \varphi_{zz} = 0$.

What effect does φ have?

A logical question is **what effect does the choice of φ have?**

$$\int_{\mathscr{B}} P_d\,\varphi_{zz} = \int_{\mathscr{S}} \eta\varphi_{zz} + \mu^2\eta_{tt}\varphi_z + \epsilon\mu^2\left(\eta_t^2\varphi_{zz} + \left(\eta\eta_x + 2\left(q_t\eta_x - q_x\eta_t\right)\right)\varphi_{xz}\right),$$

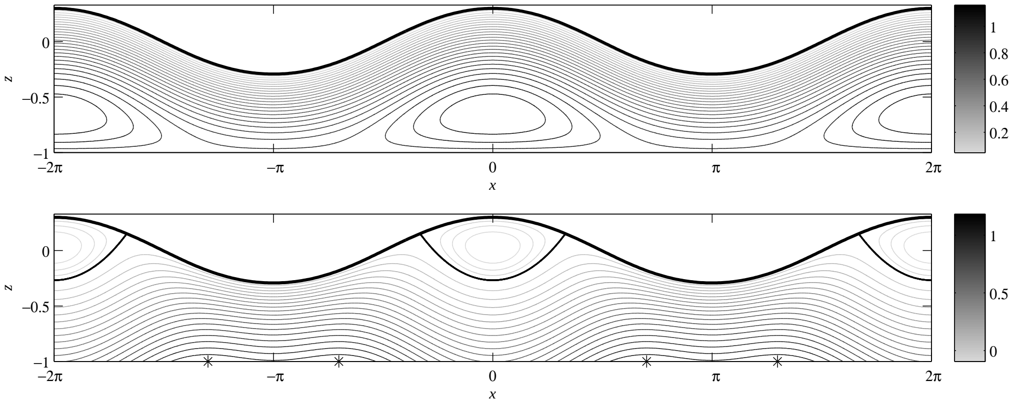
Remark #1

Choosing $\varphi = e^{-ikx}\sinh(\mu k(z+1))$ clearly eliminates the relationship between the pressure at the bottom. This equation combined with either (A) or (B) *completes* the system in the surface variables η and q.

Remark #2

We have seen the impact of the choice of φ in our previous work - specifically, different choices of φ resulted in more numerically stable algorithms to create pressure contours in the presence of constant vorticity.

What effect does φ have?



What effect does φ have?

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$$\int_{\mathscr{B}} P_d \, \varphi_{zz} = \int_{\mathscr{S}} \eta\varphi_{zz} + \mu^2 \eta_{tt}\varphi_z + \epsilon\mu^2 \left(\eta_t^2 \varphi_{zz} + \left(\eta\eta_x + 2 \left(q_t\eta_x - q_x\eta_t \right) \right) \varphi_{xz} \right) ,$$

Goal

Can we exploit a variety of choices for φ to eliminate the spatial dependence in order to create a direct **time-series** map?

Choosing $\varphi = e^{-ikx}\cosh(\mu k(z+1))$:

$$\int_{-\infty}^{\infty} e^{-ikx}\left[\left(\eta + \epsilon\mu^2\eta_t^2\right)\mathcal{C} + \left(\frac{\mu}{k}\eta_{tt} - i\epsilon\mu\left(\eta\eta_x + 2\left(q_t\eta_x - q_x\eta_t\right)\right)\right)\mathcal{S}\right] dx = \int_{-\infty}^{\infty} e^{-ikx}\left[P_d\right] dx,$$

where $P_d = (p(x,-h,t) - \rho g h)/\rho$, $\mathcal{S} = \sinh(\mu k(\epsilon\eta + 1))$, and $\mathcal{C} = \cosh(\mu k(\epsilon\eta + 1))$.

Taking the balance $\epsilon \sim \mu^2$:

$$\hat{P}_d = \hat{\eta} + \epsilon\hat{\eta}_{tt} + \frac{\epsilon k^2}{2}\hat{\eta} + \mathcal{O}(\epsilon^2)$$

It is worth noting that at leading order, we recover precisely the **hydrostatic approximation**. **Can we do more?**

Comparison of various asymptotic formulae with $\epsilon \sim \mu^2$.

Choice of φ	Resulting Model
$\varphi = e^{-ikx}\cosh(\mu k(z+1))$	$P_d = \eta + \epsilon\left(\eta_{tt} - \frac{1}{2}\eta_{xx}\right) + \mathcal{O}(\epsilon^2)$
$\varphi = e^{-ikx}\cosh(\mu k z)$	$\left(1 - \frac{\epsilon}{2}\partial_x^2\right) P_d = \eta + \mathcal{O}(\epsilon^2)$
$\varphi = e^{-ikx}\sinh(\mu k z)$	$\left(1 - \frac{\epsilon}{6}\partial_x^2\right) P_d = \eta - \left(\frac{1}{2}\eta^2 + \epsilon\left(\partial_x^{-1}\eta_t\right)^2\right) + \mathcal{O}(\epsilon^2)$

Three equations for two unknowns?

The three equations are consistent provided:

$$\eta_{tt} - \eta_{xx} = \epsilon \partial_x^2 \left[\frac{1}{3}\eta_{xx} + \frac{1}{2}\eta^2 + \left(\partial_x^{-1}\eta_t\right)^2 \right] + \mathcal{O}(\epsilon^2).$$

So, how do we use this in conjunction with the following?

$$P_d = \eta + \epsilon \left(\eta_{tt} - \frac{1}{2}\eta_{xx} \right) + \mathcal{O}(\epsilon^2)$$

$$\left(1 - \frac{\epsilon}{2}\partial_x^2\right) P_d = \eta + \mathcal{O}(\epsilon^2)$$

$$\left(1 - \frac{\epsilon}{6}\partial_x^2\right) P_d = \eta - \left(\frac{1}{2}\eta^2 + \epsilon \left(\partial_x^{-1}\eta_t\right)^2 \right) + \mathcal{O}(\epsilon^2)$$

Back-substitute the following: $\eta_{xx} = \eta_{tt} + \mathcal{O}(\epsilon)$ and $P_d = \eta + \mathcal{O}(\epsilon)$

Final Asymptotic Relationship

Thus, we end with:

$$\eta = P_d - \frac{\epsilon}{2} P_{d,tt} + \mathcal{O}(\epsilon^2) \qquad\qquad \eta = P_d - \frac{h}{g} P_{d,tt} + \ldots$$

This formulation has **completely eliminated the spatial component**.

We have a **direct map** from $P_d(x_i, t_j) \to \eta(x_i, t_j)$ provided that we have a method to approximate $P_{d,tt}(x_i, t_j)$

Despite being accurate to $\mathcal{O}(\epsilon^2)$, the equation remains **linear**.

Of course, we don't expect that it will perform as well as the fully nonlinear model, but let's see what happens.

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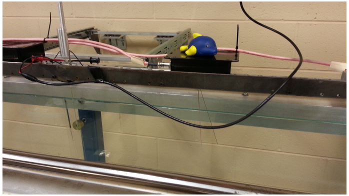
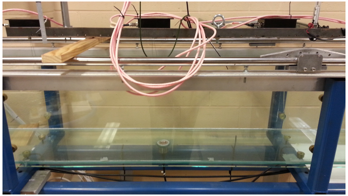
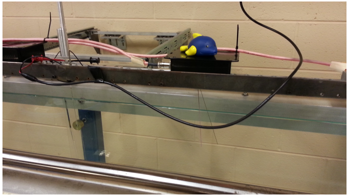
Experimental Set-Up

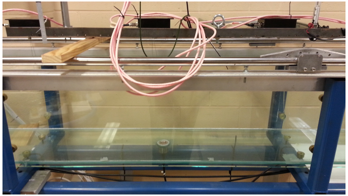




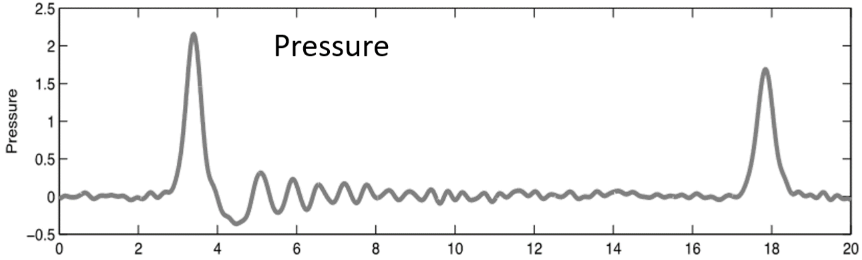
Movie Time!

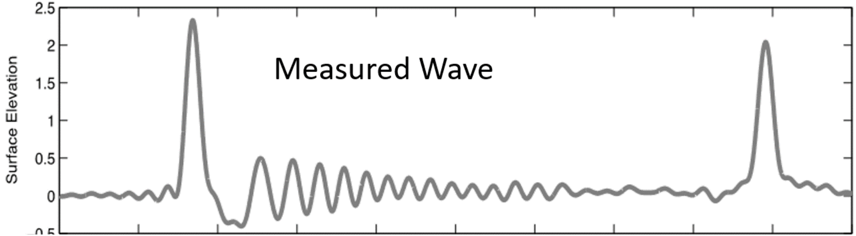
Experimental Set-Up

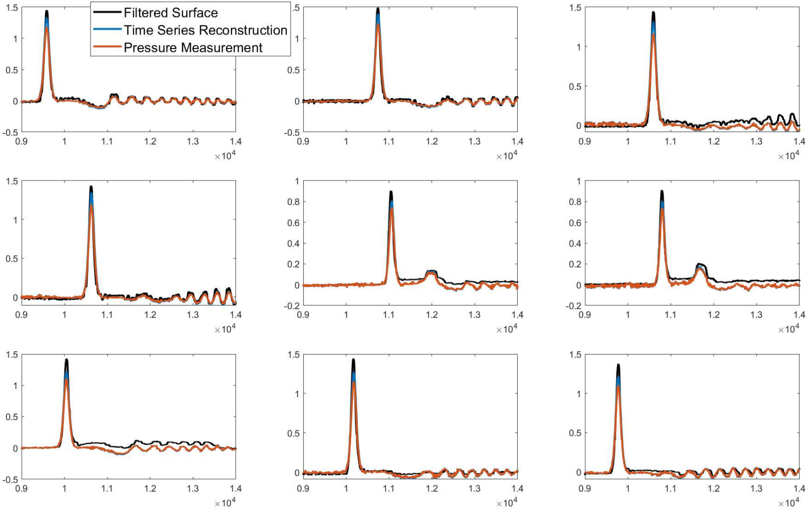




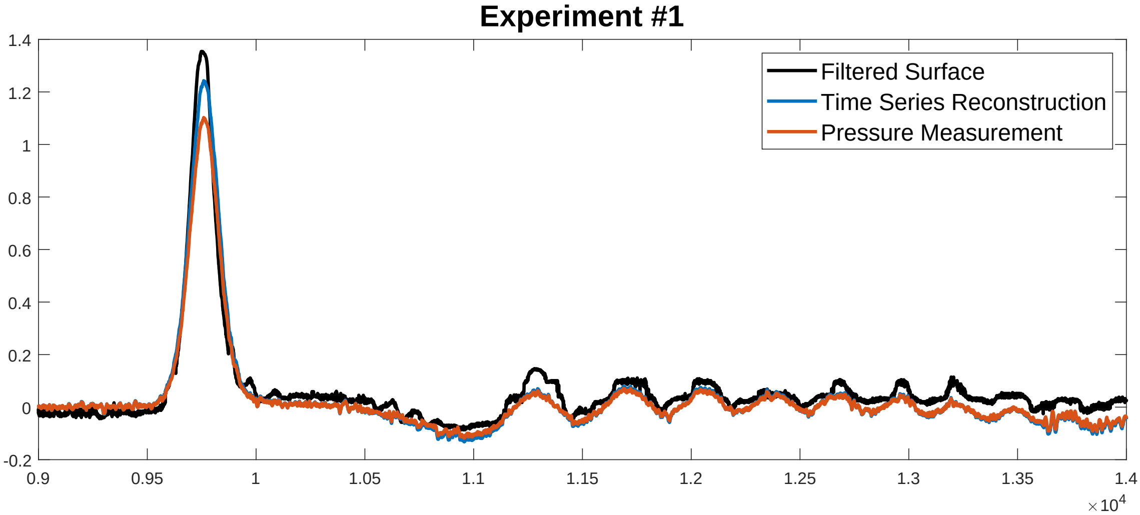
Experimental Data






Filtered Surface
Time Series Reconstruction
Pressure Measurement
$\times 10^4$

Experimental Results using $\eta = P_d - \frac{h}{g} P_{d,tt}$ with $h = 5.05$cm



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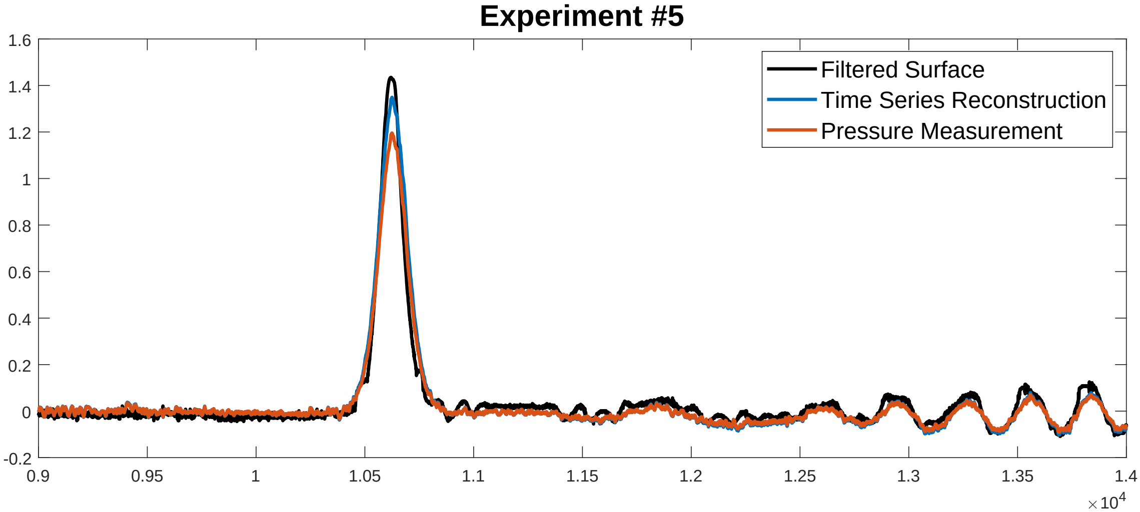
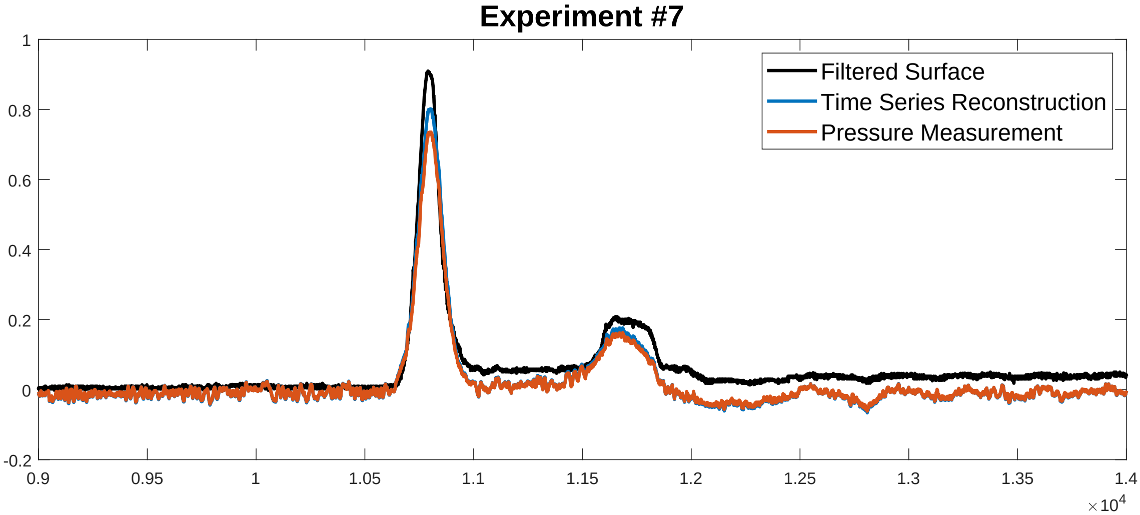
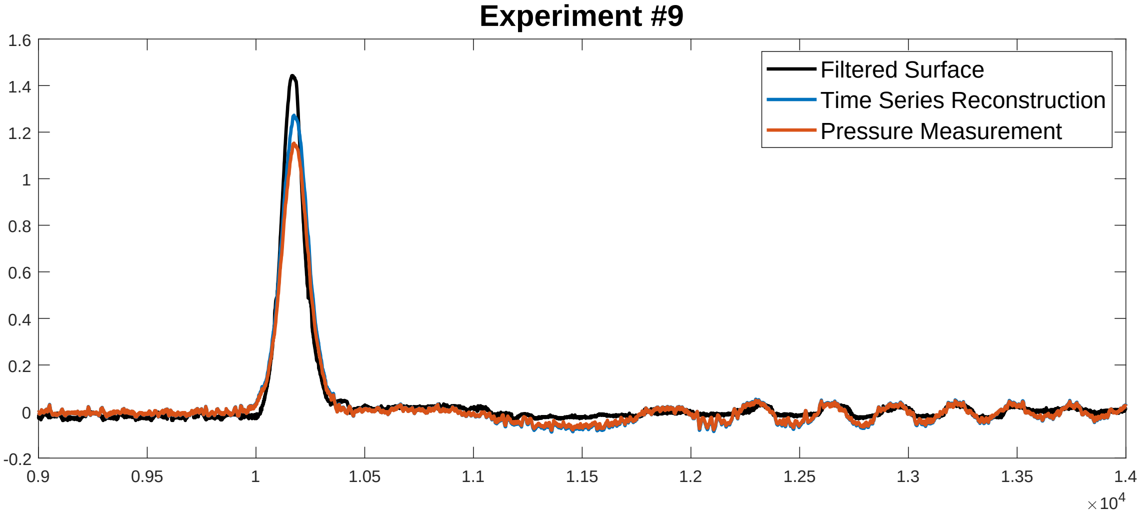


Experimental Results using $\eta = P_d - \frac{h}{g} P_{d,tt}$ with $h = 3.55\text{cm}$



Experimental Results using $\eta = P_d - \frac{h}{g} P_{d,tt}$ with $h = 4.10\text{cm}$



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What's Next?

Comparisons with other known methods.

Can we go to higher order? What is the price that we have to pay?

Can we implement a feedback mechanism to drive the error to zero?

- Luenberger Observer / Kalman Filter
- Data Assimilation via Titi's work

Can this be paired with the multiple-scales ideas outlined in my previous talk for model-regime prediction?