

Reconstructing the Free-Surface from Pressure

Time-Dependent Relationships and Experimental Comparisons

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Formulations & Selected Prior Work

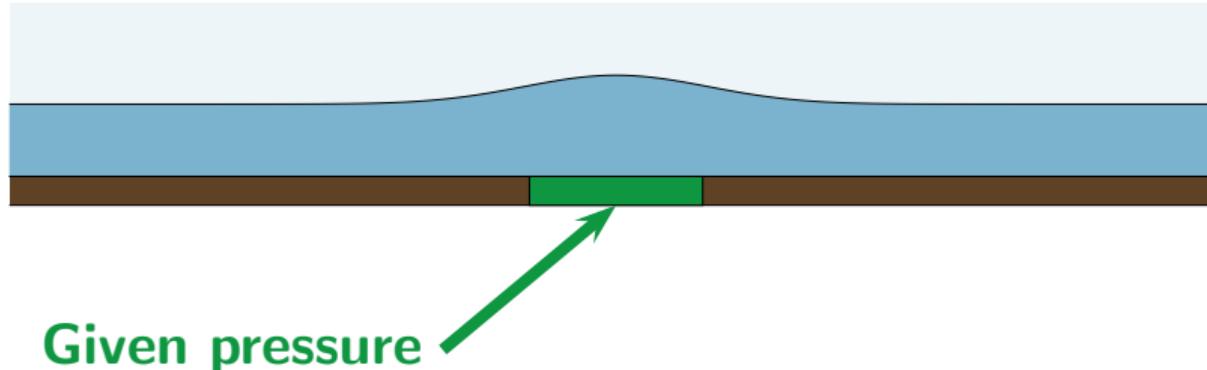
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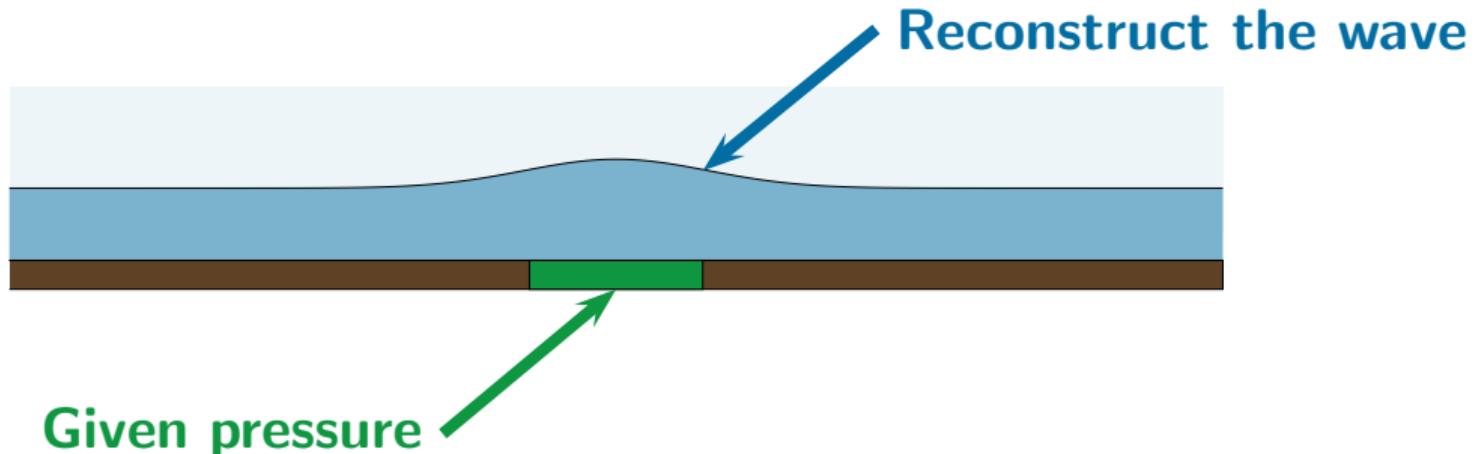
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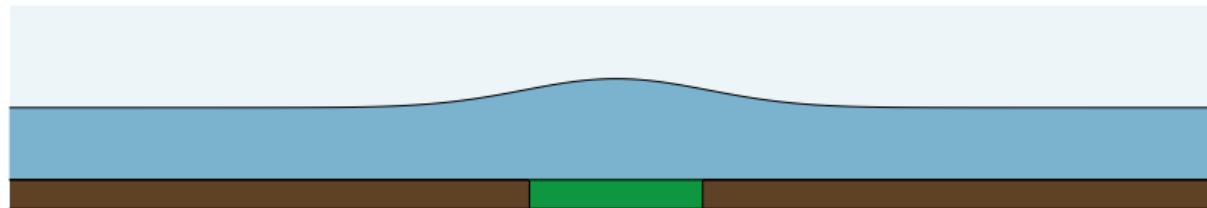
Motivation / Problem Description



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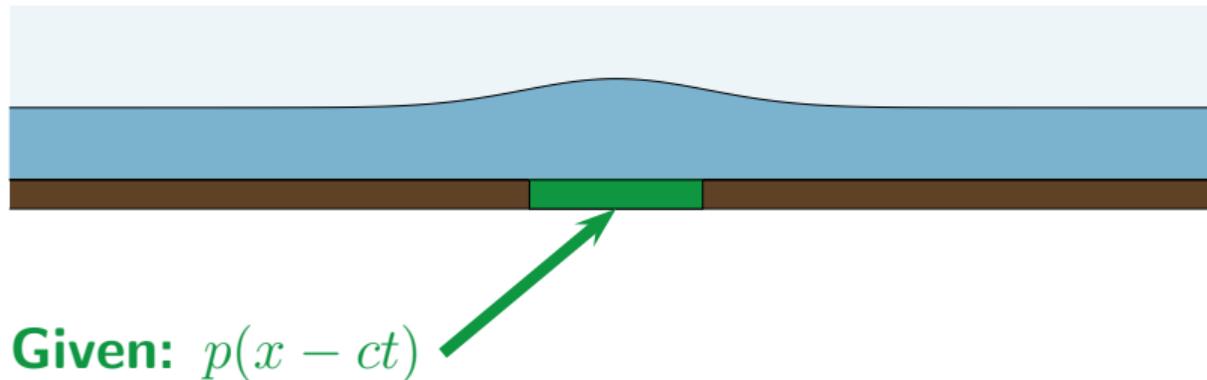


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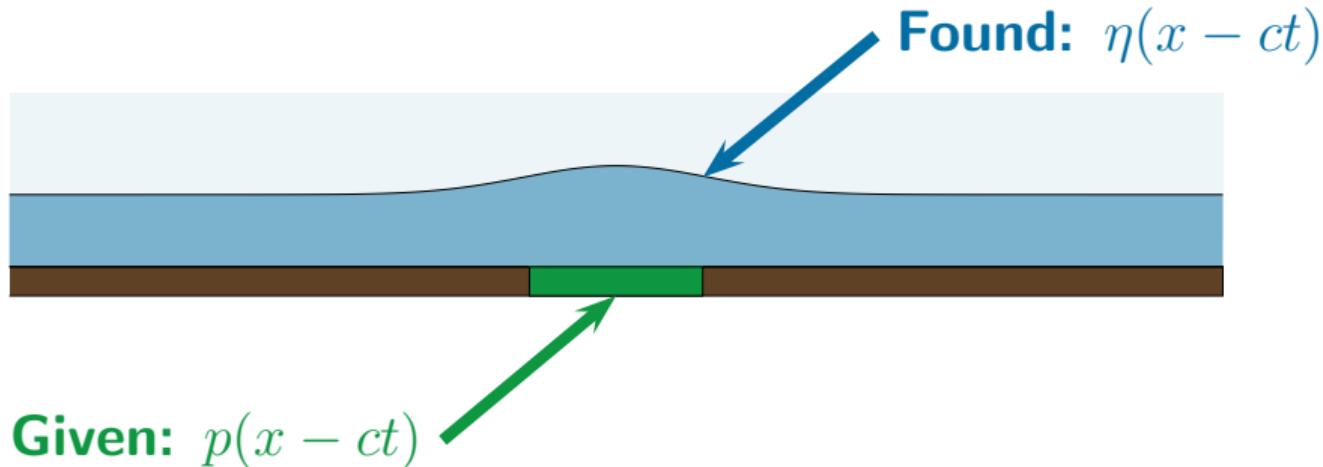
Joint work: Deconinck, Henderson, Oliveras, Vasan

Motivation / Problem Description



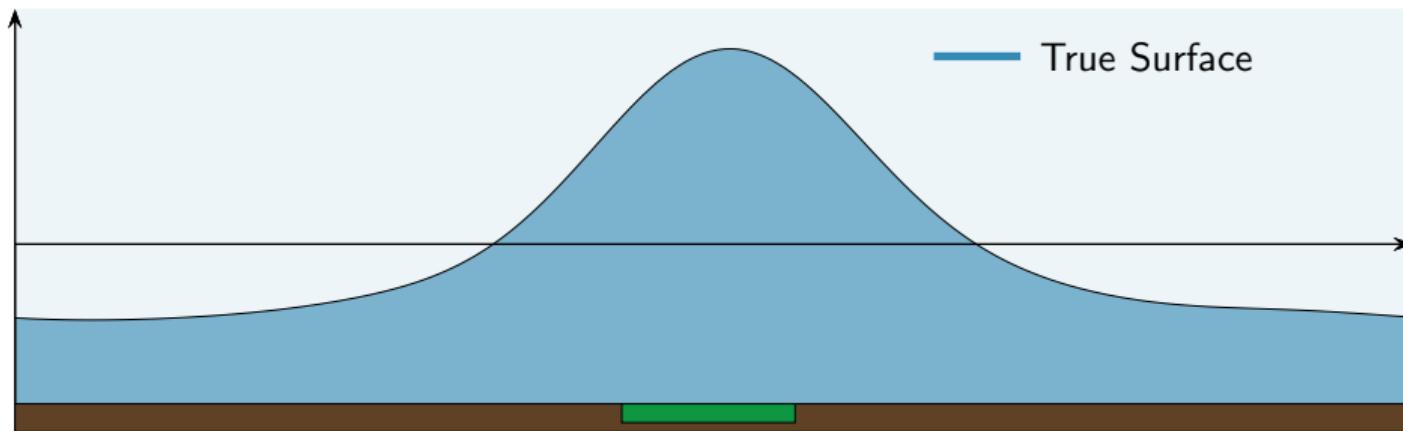
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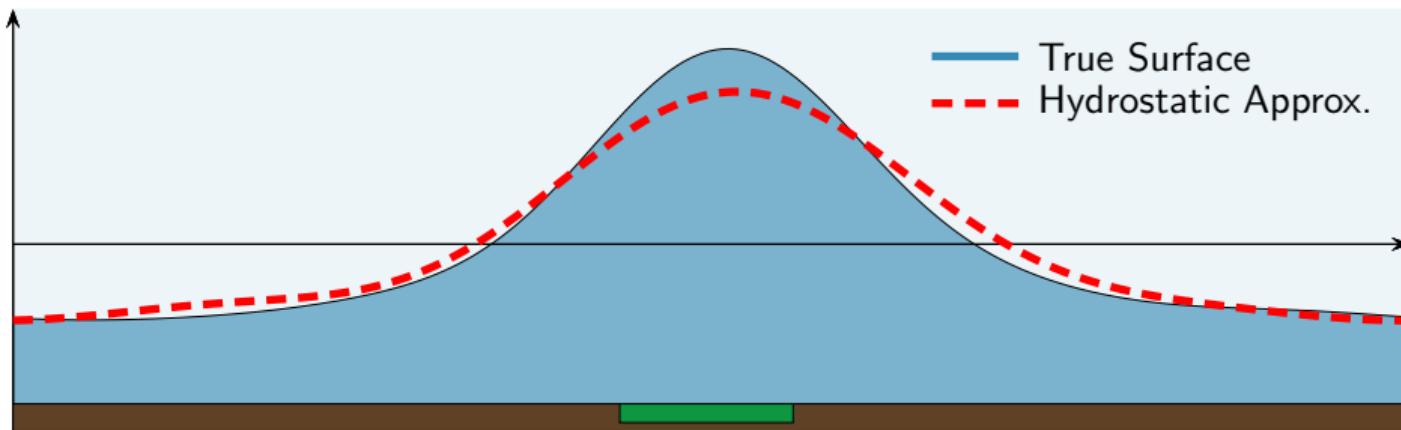


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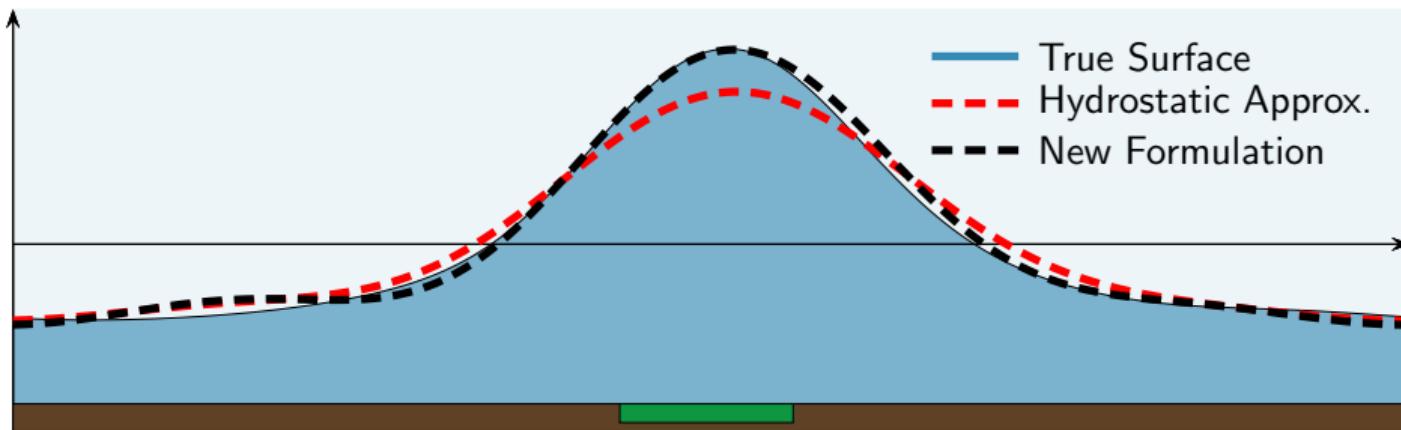
Observations in Experiments



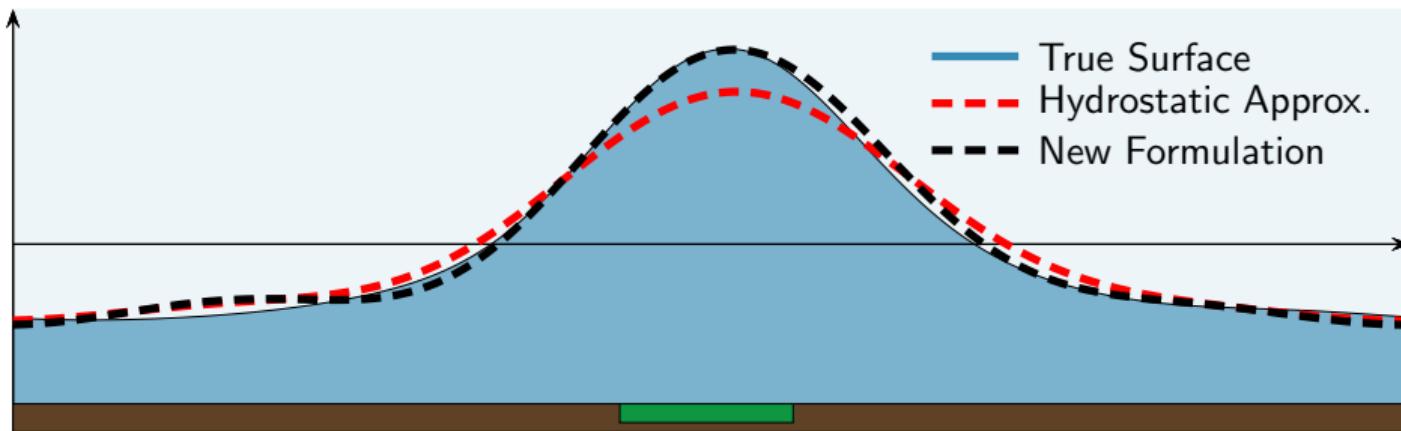
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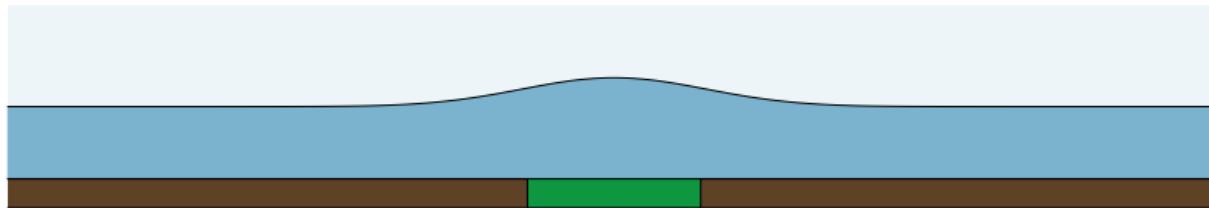


Observations in Experiments



Can we address these challenges and develop a time-series method?

Motivation / Problem Description



Given $p(x_i, t)$

Motivation / Problem Description

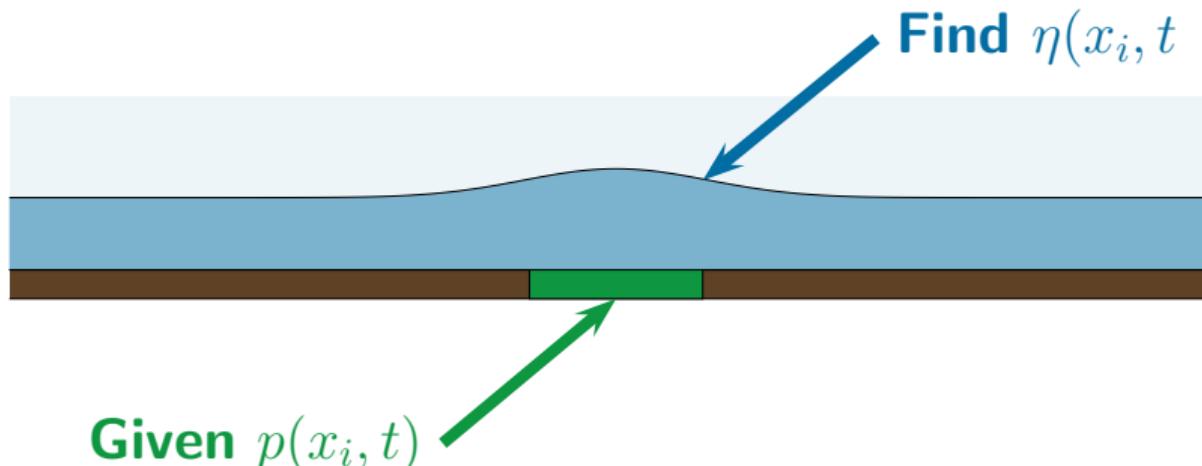


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Equations of Motion - Velocity Potential Formulation

We begin by considering an inviscid, irrotational, fluid with a one-dimensional free-surface on the whole-line:

$$\phi_{xx} + \phi_{zz} = 0, \quad (x, z) \in \mathcal{D},$$

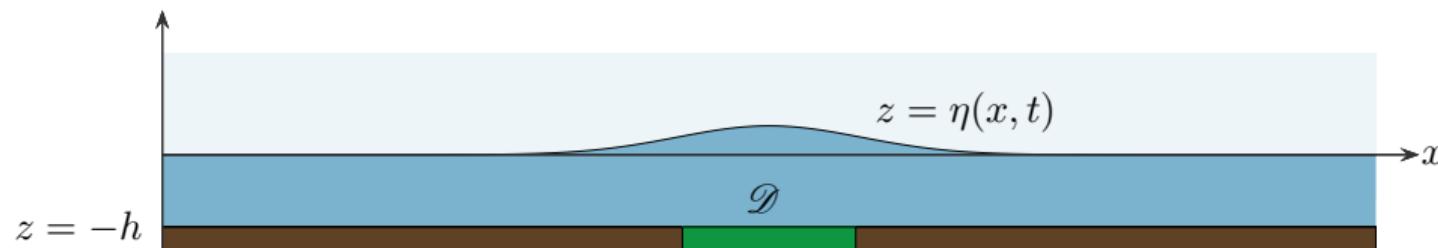
$$\phi_t + \frac{1}{2} |\nabla \phi|^2 + \frac{\rho g z - p(x, z, t)}{\rho} = 0, \quad (x, z) \in \mathcal{D},$$

$$\phi_z = 0, \quad z = -h,$$

$$\eta_t = \phi_z - \eta_x \phi_x, \quad z = \eta(x, t),$$

$$p = 0, \quad z = \eta(x, t),$$

where $\phi(x, z, t)$ represents the velocity potential of the fluid, $\eta(x, t)$ represents the surface elevation.



Equations of Motion - Velocity Potential Formulation

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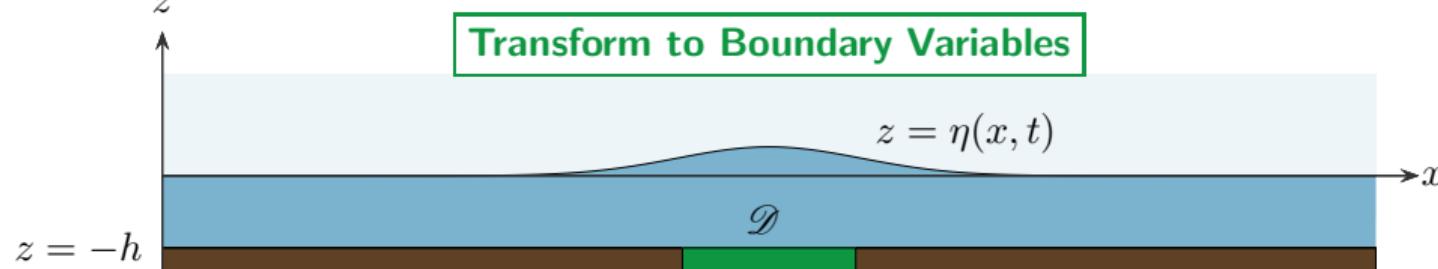
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Equations of Motion - ZCS Formulation

[Zakharov, and Craig & Sulem (ZCS)]:

$$q(x, t) = \phi(x, \eta(x, t), t)$$

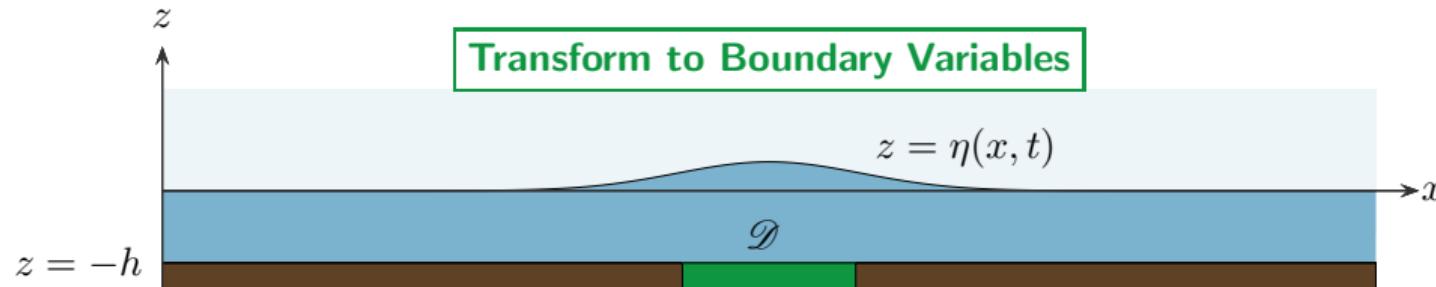
Dynamic Boundary Condition

$$q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2} \frac{(\eta_t + \eta_x q_x)^2}{1 + \eta_x^2} = 0,$$

Kinematic Boundary Condition

$$\eta_t = \mathcal{G}(\eta)q,$$

$$\mathcal{G}(\eta)q = \frac{\partial \phi}{\partial n}, \quad z = \eta$$



Equations of Motion - AFM Formulation

[Ablowitz, Fokas, & Musslimani (AFM)]:

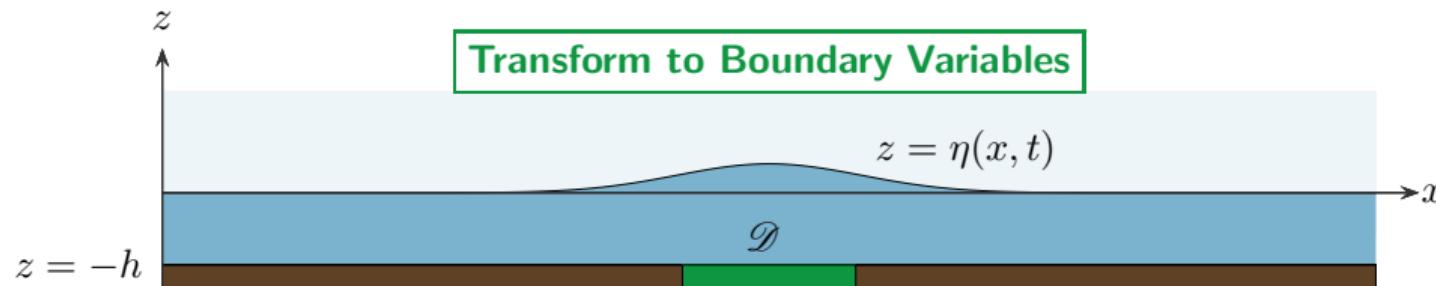
$$q(x, t) = \phi(x, \eta(x, t), t)$$

Dynamic Boundary Condition/Local Equation

$$q_t + \frac{1}{2}q_x^2 + g\eta - \frac{1}{2} \frac{(\eta_t + \eta_x q_x)^2}{1 + \eta_x^2} = 0,$$

Kinematic Boundary Condition/Nonlocal Equation

$$\int_{-\infty}^{\infty} (e^{-ikx} (\eta_t \cosh(k(\eta + h)) + i q_x \sinh(k(\eta + h)))) \, dx = 0$$



Mapping Pressure via AFM

From the previous slide:

A Third Equation

$$Q(x, t) = \phi(x, -h, t)$$

$$\int_{-\infty}^{\infty} \left(e^{-ikx} (\eta_t \sinh(k(\eta + h)) - iq_x \cosh(k(\eta + h))) \right) dx = k\hat{Q}(k, t)$$

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$$\hat{Q}(k, t) \rightarrow Q(x, t) \quad \Rightarrow \quad Q_t + \frac{1}{2}Q_x^2 + \frac{p(x, -h, t) - \rho gh}{\rho} = 0$$

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For traveline waves, you can find the following relationships:

$$p(\xi, -h) - \rho gh = \rho \iint_{\mathbb{R} \times \mathbb{R}} e^{ik(\xi - \xi')} \sqrt{(c^2 - 2g\eta)(1 + \eta_x^2)} \cosh(k(\eta + h)) d\xi' dk$$

or via a separate derivation

$$\int_{-\infty}^{\infty} e^{-ik\xi} \sqrt{c^2 - 2(p - \rho gh)} \cosh(k(\eta + h)) dk = \frac{\sqrt{c^2 - 2g\eta}}{1 + \eta_\xi^2}$$

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Nonlocal/Nonlocal Formulation: The First Integral Relation

For any harmonic function $\varphi(x, z)$, we have

$$\oint_{\partial \mathcal{D}} ((\varphi_z \nabla \phi - \phi \nabla \varphi_z) \cdot \mathbf{n}) \, ds = 0,$$

where we assume that ϕ has sufficient decay properties as $|x| \rightarrow \infty$.

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where we assume that ϕ has sufficient decay properties as $|x| \rightarrow \infty$. Using the **kinematic boundary condition**,

$$\nabla \phi \cdot \mathbf{n} \Big|_{\mathcal{S}} = \eta_t \quad \rightarrow \quad \int_{\mathcal{S}} \varphi_z \eta_t \, dx = \oint_{\partial\mathcal{D}} (\phi \nabla \varphi_z \cdot \mathbf{n}) \, ds$$

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Similarly for ϕ_t :

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Using the **dynamic boundary condition**, along with $\nabla \phi_t \cdot \mathbf{n} \Big|_{\mathcal{S}} = \eta_{tt} + \frac{d}{dx} [\eta_t \phi_x(x, \eta, t)]$

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where we assume that ϕ has sufficient decay properties as $|x| \rightarrow \infty$. Using the **kinematic boundary condition**,

$$\nabla \phi \cdot \mathbf{n} \Big|_{\mathcal{S}} = \eta_t \quad \rightarrow \quad \int_{\mathcal{S}} \varphi_z \eta_t \, dx = \oint_{\partial\mathcal{D}} (\phi \nabla \varphi_z \cdot \mathbf{n}) \, ds$$

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$$\int_{\mathcal{S}} \frac{d}{dt} q \left(\frac{\partial \varphi_z}{\partial \mathbf{n}} - 2\varphi_{zz} \right) \, dx = \int_{\mathcal{S}} g\eta(\varphi_{zz} + \eta_x \varphi_{xz}) - \underbrace{2\eta_t q \varphi_{zzz}}_{(*)} \, dx + \int_{\mathcal{B}} \frac{1}{2} Q_x^2 \varphi_{zz} \, dx$$

Nonlocal/Nonlocal Formulation - Summary / Comments

Integral Relation A

$$\int_{\mathcal{S}} \frac{d}{dt} \varphi \, dx = \int_{\mathcal{S}} \phi (\nabla \varphi_z \cdot \mathbf{n}) \, dx - \int_{\mathcal{B}} Q \varphi_{zz} \, dx$$

$$q = \phi(x, \eta, t), \quad Q = \phi(x, -h, t)$$

Integral Relation B

$$\int_{\mathcal{S}} \frac{d}{dt} q \left(\frac{\partial \varphi_z}{\partial \mathbf{n}} - 2\varphi_{zz} \right) \, dx = \int_{\mathcal{S}} g\eta(\varphi_{zz} + \eta_x \varphi_{xz}) - \underbrace{2\eta_t q \varphi_{zzz}}_{(*)} \, dx + \int_{\mathcal{B}} \frac{1}{2} Q_x^2 \varphi_{zz} \, dx$$

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Remark #1

The **dynamic boundary condition** was only prescribed at the last stage of deriving (B).

Nonlocal/Nonlocal Formulation - Summary / Comments

Integral Relation A

$$\int_{\mathcal{S}} \frac{d}{dt} \varphi \, dx = \int_{\mathcal{S}} \phi (\nabla \varphi_z \cdot \mathbf{n}) \, dx - \int_{\mathcal{B}} Q \varphi_{zz} \, dx$$

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Integral Relation B

$$\int_{\mathcal{S}} \frac{d}{dt} q \left(\frac{\partial \varphi_z}{\partial \mathbf{n}} - 2\varphi_{zz} \right) \, dx = \int_{\mathcal{S}} g\eta(\varphi_{zz} + \eta_x \varphi_{xz}) - \underbrace{2\eta_t q \varphi_{zzz}}_{(*)} \, dx + \int_{\mathcal{B}} \frac{1}{2} Q_x^2 \varphi_{zz} \, dx$$

Remark #2

$\varphi = e^{-ikx} \sinh(k(z+h))$, implies both [ZCS] and [AFM].

If we take $k \rightarrow 0$ in both (A) and (B), we recover (T1) and (T3) immediately.

Nonlocal/Nonlocal Formulation - Summary / Comments

Integral Relation A

$$\int_{\mathcal{S}} \frac{d}{dt} \varphi \, dx = \int_{\mathcal{S}} \phi (\nabla \varphi_z \cdot \mathbf{n}) \, dx - \int_{\mathcal{B}} Q \varphi_{zz} \, dx$$

$$q = \phi(x, \eta, t), \quad Q = \phi(x, -h, t)$$

Integral Relation B

$$\int_{\mathcal{S}} \frac{d}{dt} q \left(\frac{\partial \varphi_z}{\partial \mathbf{n}} - 2\varphi_{zz} \right) \, dx = \int_{\mathcal{S}} g\eta(\varphi_{zz} + \eta_x \varphi_{xz}) - \underbrace{2\eta_t q \varphi_{zzz}}_{(*)} \, dx + \int_{\mathcal{B}} \frac{1}{2} Q_x^2 \varphi_{zz} \, dx$$

Remark #3

We can generate a direct map from the **pressure at the bottom** to the free-surface variables. This can be found by taking the combination

$$\frac{d}{dt}(A) - (B)$$

Pressure Relationship

Integral Relation A

$$\int_{\mathcal{S}} \frac{d}{dt} \varphi \, dx = \int_{\mathcal{S}} \phi (\nabla \varphi_z \cdot \mathbf{n}) \, dx - \int_{\mathcal{B}} Q \varphi_{zz} \, dx$$

$$q = \phi(x, \eta, t), \quad Q = \phi(x, -h, t)$$

Integral Relation B

$$\int_{\mathcal{S}} \frac{d}{dt} q \left(\frac{\partial \varphi_z}{\partial \mathbf{n}} - 2\varphi_{zz} \right) \, dx = \int_{\mathcal{S}} g\eta(\varphi_{zz} + \eta_x \varphi_{xz}) - \underbrace{2\eta_t q \varphi_{zzz}}_{(*)} \, dx + \int_{\mathcal{B}} \frac{1}{2} Q_x^2 \varphi_{zz} \, dx$$

Integral Relation C

$$\int_{\mathcal{B}} P_d \varphi_{zz} = \rho \int_{\mathcal{S}} \eta_{tt} \varphi_z + (\eta_t^2 + g\eta) \varphi_{zz} + (g\eta\eta_x - 2(q_x\eta_t - q_t\eta_x)) \varphi_{xz}$$

$$P_d(x, t) = \frac{p(x, -h, t) - \rho gh}{\rho}$$

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Nondimensionalization

We introduce the nondimensional quantities as follows:

$$\begin{aligned}x &= x^*L, \quad z = z^*h, \quad t = \frac{L}{c_0}t^*, \quad c_0 = \sqrt{gh}, \quad \epsilon = \frac{a}{h}, \quad \mu = \frac{h}{L}, \\ \eta &= a\eta^*, \quad \frac{p(x, -h, t) - \rho gh}{\rho} = \epsilon P_d^*, \quad Q = \frac{Lga}{c_0}Q^*, \quad q = \frac{Lga}{c_0}q^*,\end{aligned}$$

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$$\eta = a\eta^*, \quad \frac{p(x, -h, t) - \rho gh}{\rho} = \epsilon P_d^*, \quad Q = \frac{Lga}{c_0}Q^*, \quad q = \frac{Lga}{c_0}q^*,$$

This yields the **nondimensional relationship**

$$\int_{\mathcal{B}} P_d \varphi_{zz} = \int_{\mathcal{S}} \eta \varphi_{zz} + \mu^2 \eta_{tt} \varphi_z + \epsilon \mu^2 \left(\eta_t^2 \varphi_{zz} + (\eta \eta_x + 2(q_t \eta_x - q_x \eta_t)) \varphi_{xz} \right),$$

where φ solves $\mu^2 \varphi_{xx} + \varphi_{zz} = 0$.

What effect does φ have?

A logical question is **what effect does the choice of φ have?**

$$\int_{\mathcal{B}} P_d \varphi_{zz} = \int_{\mathcal{S}} \eta \varphi_{zz} + \mu^2 \eta_{tt} \varphi_z + \epsilon \mu^2 \left(\eta_t^2 \varphi_{zz} + (\eta \eta_x + 2(q_t \eta_x - q_x \eta_t)) \varphi_{xz} \right),$$

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Remark #1

Choosing $\varphi = e^{-ikx} \sinh(\mu k(z+1))$ clearly eliminates the relationship between the pressure at the bottom. This equation combined with either (A) or (B) *completes* the system in the surface variables η and q .

What effect does φ have?

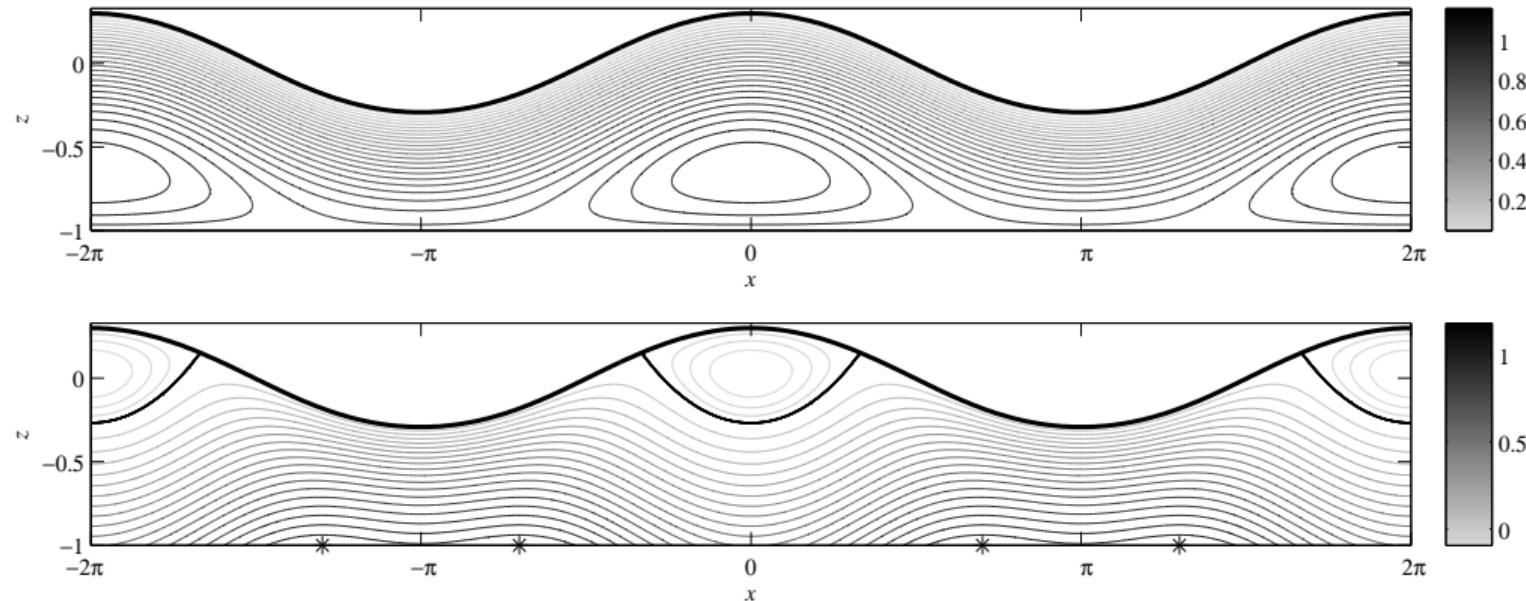
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Remark #2

We have seen the impact of the choice of φ in our previous work - specifically, different choices of φ resulted in more numerically stable algorithms to create pressure contours in the presence of constant vorticity.

What effect does φ have?



What effect does φ have?

A logical question is **what effect does the choice of φ have?**

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Goal

Can we exploit a variety of choices for φ to eliminate the spatial dependence in order to create a direct **time-series** map?

Pressure - choosing $\varphi = e^{-ikx} \cosh(\mu k(z + 1))$

Choosing $\varphi = e^{-ikx} \cosh(\mu k(z + 1))$:

$$\int_{-\infty}^{\infty} e^{-ikx} \left[(\eta + \epsilon\mu^2 \eta_t^2) \mathcal{C} + \left(\frac{\mu}{k} \eta_{tt} - i\epsilon\mu (\eta\eta_x + 2(q_t\eta_x - q_x\eta_t)) \right) \mathcal{S} \right] dx = \int_{-\infty}^{\infty} e^{-ikx} [P_d] dx,$$

where $P_d = (p(x, -h, t) - \rho gh)/\rho$, $\mathcal{S} = \sinh(\mu k(\epsilon\eta + 1))$, and $\mathcal{C} = \cosh(\mu k(\epsilon\eta + 1))$.

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Taking the balance $\epsilon \sim \mu^2$:

$$\hat{P}_d = \hat{\eta} + \epsilon \hat{\eta}_{tt} + \frac{\epsilon k^2}{2} \hat{\eta} + \mathcal{O}(\epsilon^2)$$

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It is worth noting that at leading order, we recover precisely the **hydrostatic approximation**.

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Taking the balance $\epsilon \sim \mu^2$:

$$\hat{P}_d = \hat{\eta} + \epsilon \hat{\eta}_{tt} + \frac{\epsilon k^2}{2} \hat{\eta} + \mathcal{O}(\epsilon^2)$$

It is worth noting that at leading order, we recover precisely the **hydrostatic approximation**. Can we do more?

Comparison of various asymptotic formulae with $\epsilon \sim \mu^2$.

Choice of φ	Resulting Model
$\varphi = e^{-ikx} \cosh(\mu k(z+1))$	$P_d = \eta + \epsilon \left(\eta_{tt} - \frac{1}{2} \eta_{xx} \right) + \mathcal{O}(\epsilon^2)$
$\varphi = e^{-ikx} \cosh(\mu kz)$	$\left(1 - \frac{\epsilon}{2} \partial_x^2 \right) P_d = \eta + \mathcal{O}(\epsilon^2)$
$\varphi = e^{-ikx} \sinh(\mu kz)$	$\left(1 - \frac{\epsilon}{6} \partial_x^2 \right) P_d = \eta - \left(\frac{1}{2} \eta^2 + \epsilon (\partial_x^{-1} \eta_t)^2 \right) + \mathcal{O}(\epsilon^2)$

Three equations for two unknowns?

The three equations are consistent provided:

$$\eta_{tt} - \eta_{xx} = \epsilon \partial_x^2 \left[\frac{1}{3} \eta_{xx} + \frac{1}{2} \eta^2 + (\partial_x^{-1} \eta_t)^2 \right] + \mathcal{O}(\epsilon^2).$$

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So, how do we use this in conjunction with the following?

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Back-substitute the following: $\eta_{xx} = \eta_{tt} + \mathcal{O}(\epsilon)$ and $P_d = \eta + \mathcal{O}(\epsilon)$

Final Asymptotic Relationship

Thus, we end with:

$$\eta = P_d - \frac{\epsilon}{2} P_{d,tt} + \mathcal{O}(\epsilon^2)$$

$$\eta = P_d - \frac{h}{g} P_{d,tt} + \dots$$

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We have a **direct map** from $P_d(x_i, t_j) \rightarrow \eta(x_i, t_j)$ provided that we have a method to approximate $P_{d,tt}(x_i, t_j)$

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Of course, we don't expect that it will perform as well as the fully nonlinear model, but let's see what happens.

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Formulations & Selected Prior Work

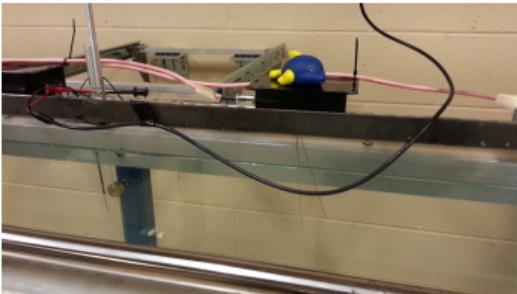
Nonlocal/Nonlocal Formulation

Asymptotic Models

Implementation

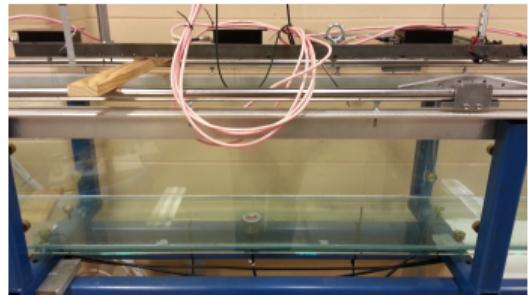
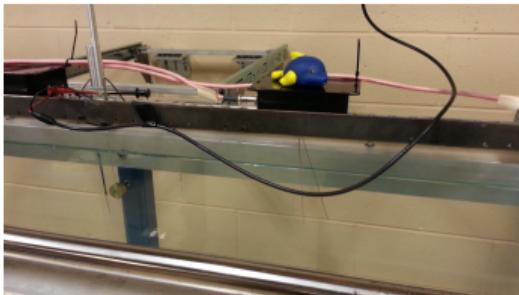
Concluding Remarks

Experimental Set-Up

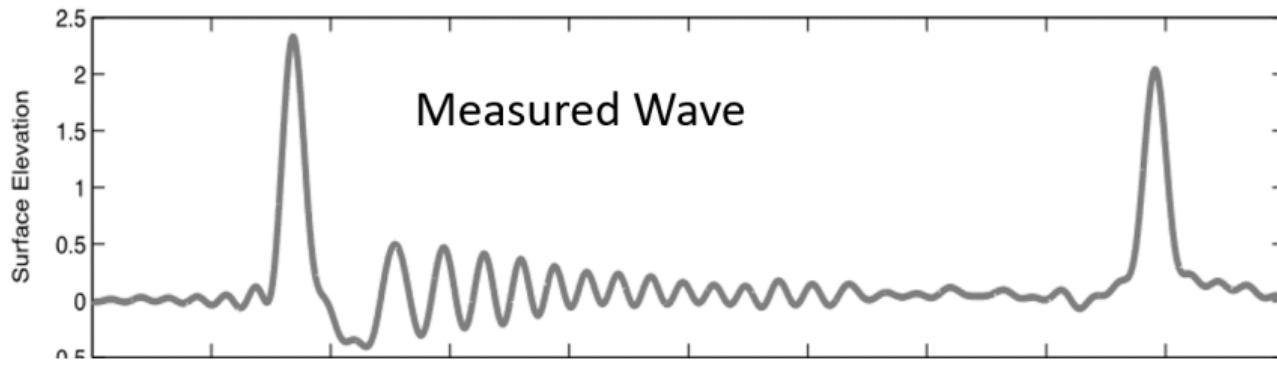
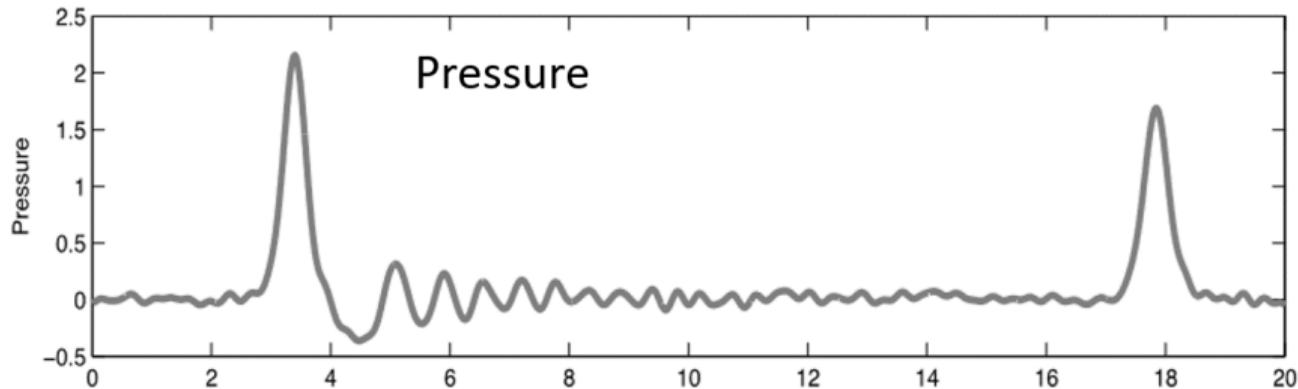


Movie Time!

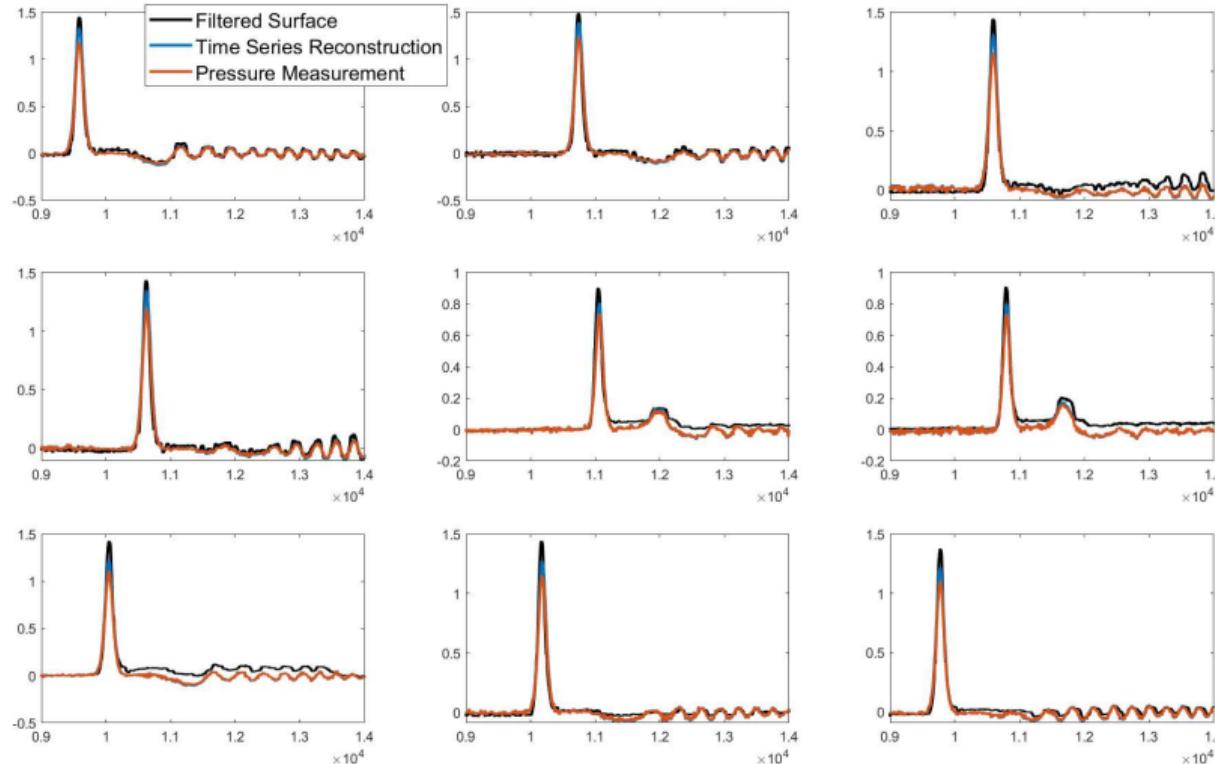
Experimental Set-Up



Experimental Data

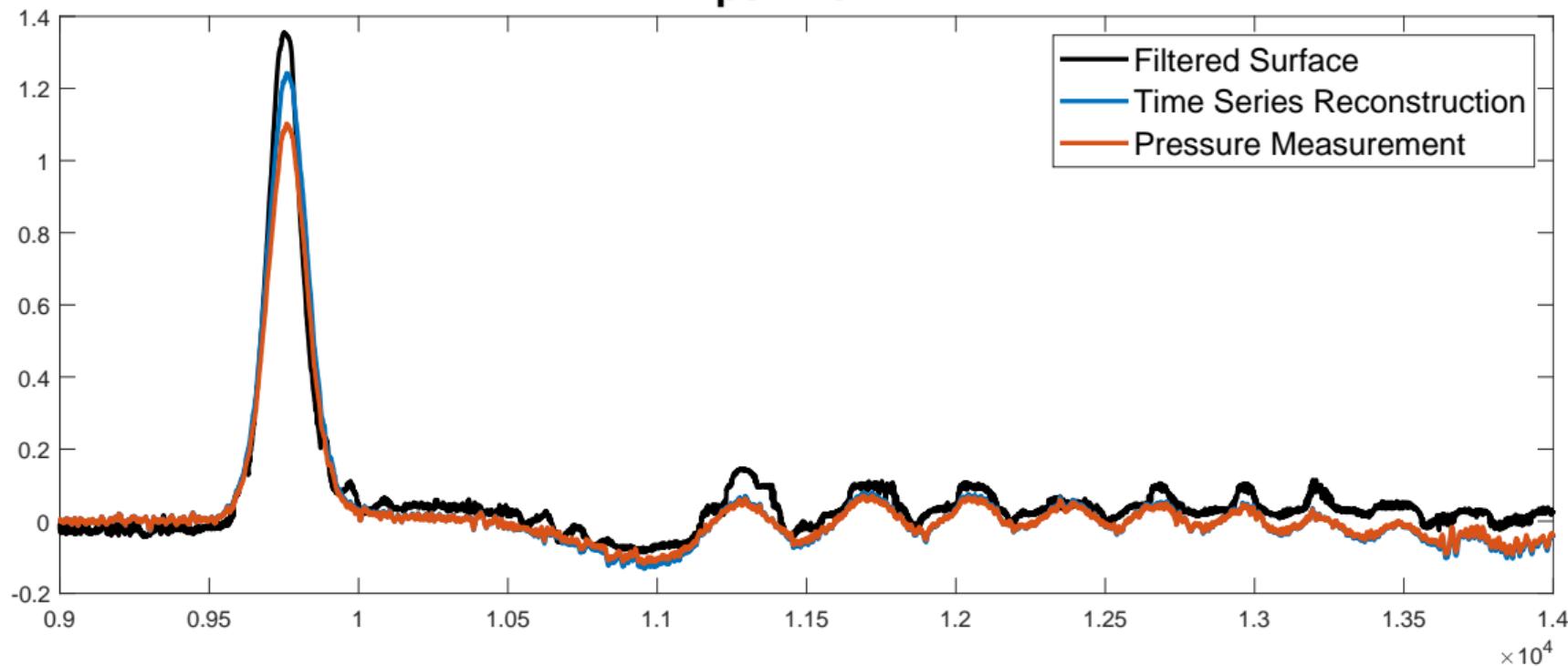


Experimental Results using $\eta = P_d - \frac{h}{g}P_{d,tt}$



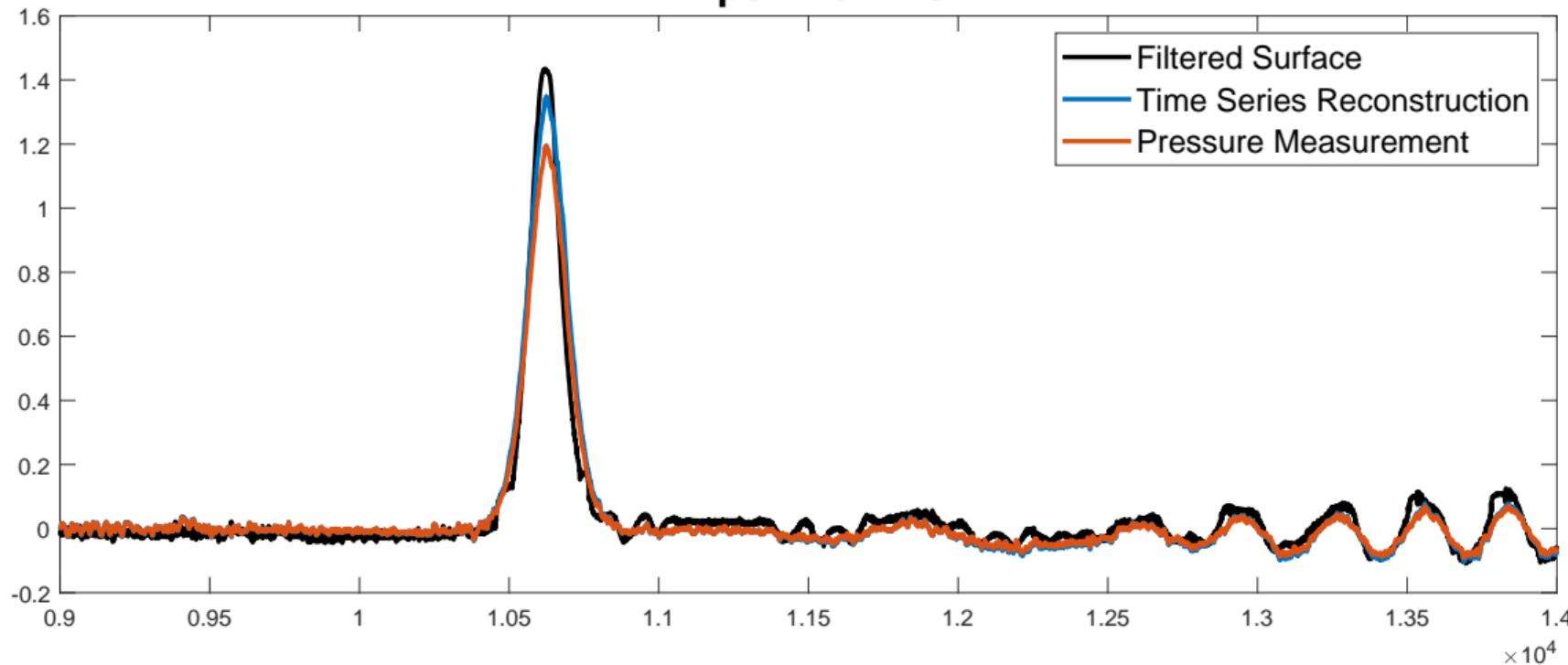
Experimental Results using $\eta = P_d - \frac{h}{g}P_{d,tt}$ with $h = 5.05\text{cm}$

Experiment #1



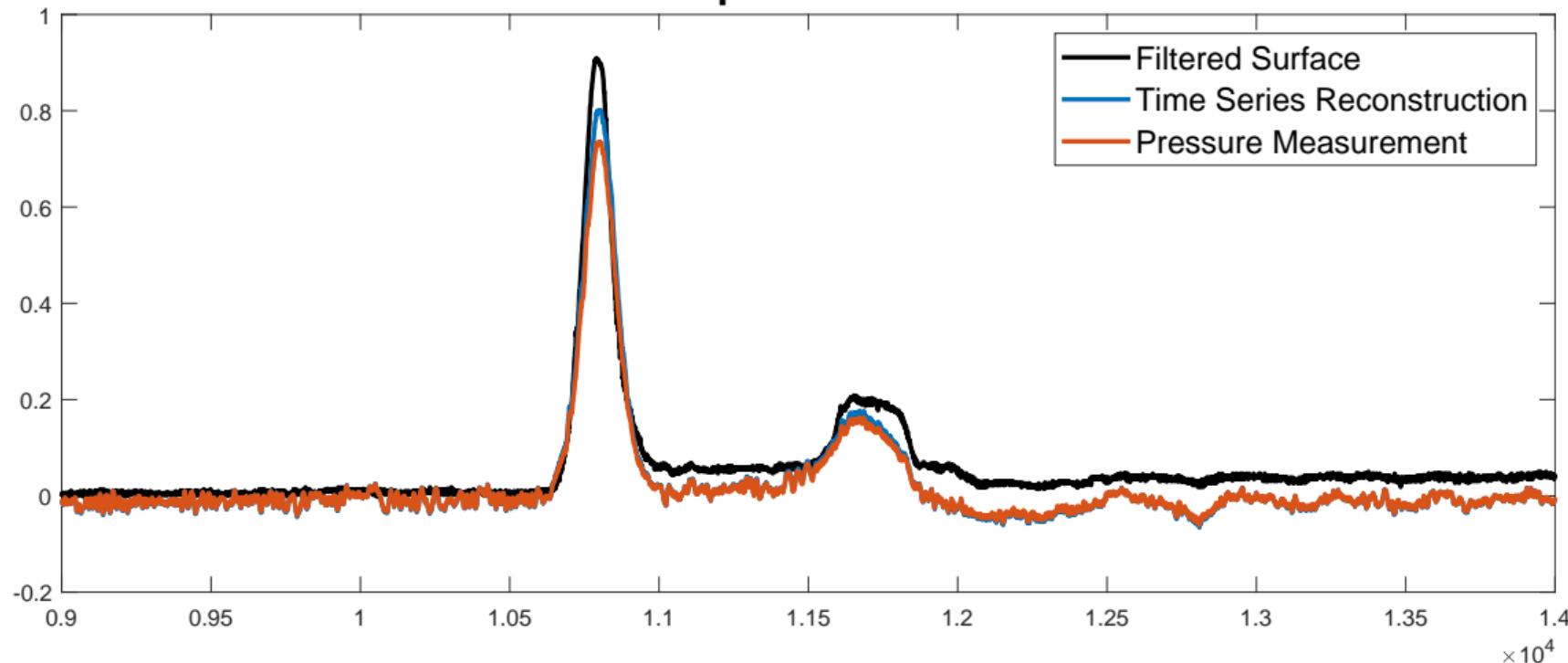
Experimental Results using $\eta = P_d - \frac{h}{g}P_{d,tt}$ with $h = 5.05\text{cm}$

Experiment #5



Experimental Results using $\eta = P_d - \frac{h}{g}P_{d,tt}$ with $h = 3.55\text{cm}$

Experiment #7



Experimental Results using $\eta = P_d - \frac{h}{g}P_{d,tt}$ with $h = 4.10\text{cm}$

Experiment #9

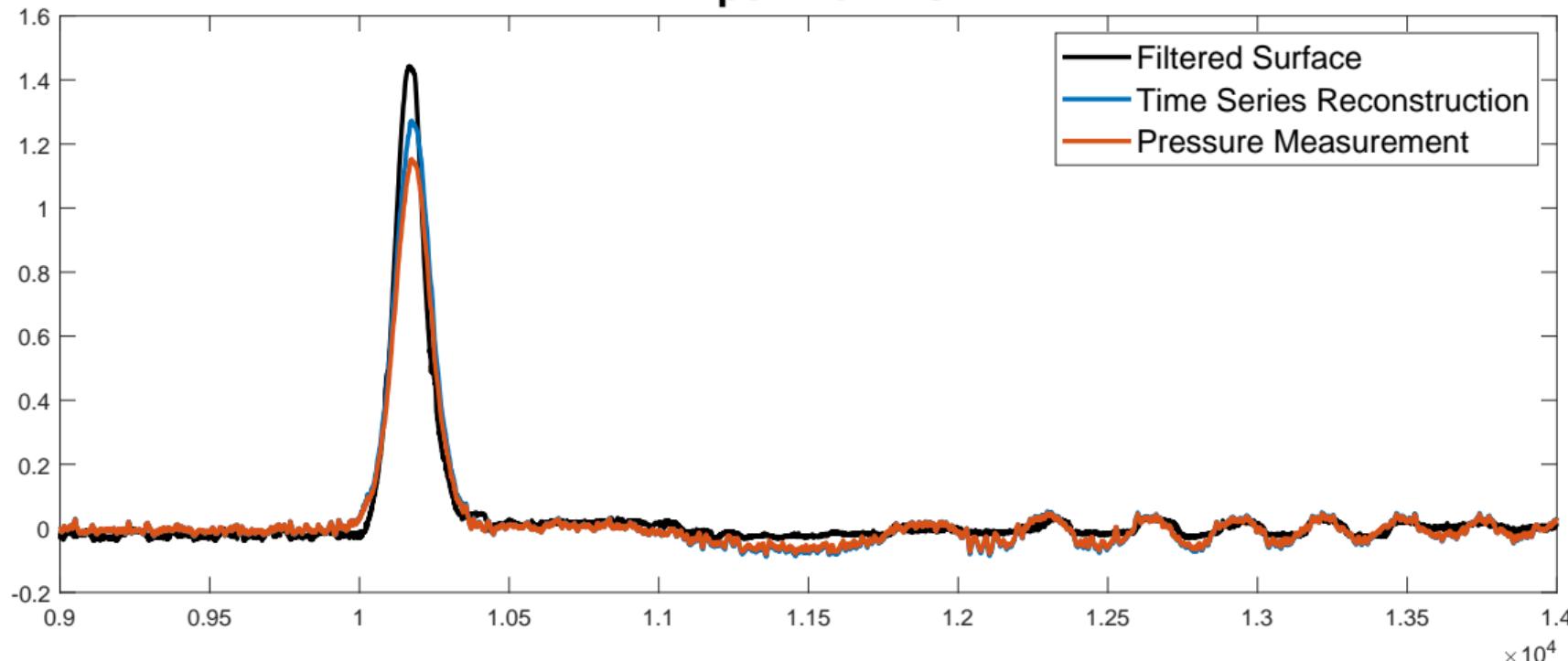


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What's Next?

Comparisons with other known methods.

Can we go to higher order? What is the price that we have to pay?

Can we implement a feedback mechanism to drive the error to zero?

Luenberger Observer / Kalman Filter

Data Assimilation via Titi's work

Can this be paired with the multiple-scales ideas outlined in my previous talk for model-regime prediction?