Systems of Differential Equations

Using a Change of Variables and Nullclines to analyze systems

Considering a system of differential equations of the form:

$$\dot{x} = \frac{1}{2}\left(x - x^3\right) - y - \frac{1}{2} x y^2$$
$$\dot{y} = x - \frac{1}{2} x^2 y + \frac{1}{2} y - \frac{1}{2} y^3$$

Here, we would like to determine how solutions behave by investigating regions of the phase-plane.

Define the functions

In[1]:=
```
f[x_, y_] = 1/2 x - y - 1/2 (x^3 + x y^2);
g[x_, y_] = x + 1/2 y - 1/2 (y^3 + x^2 y);
```

$$f[x\_, y\_] = \frac{1}{2} x - y - \frac{1}{2}\left(x^3 + x y^2\right);$$
$$g[x\_, y\_] = x + \frac{1}{2} y - \frac{1}{2}\left(y^3 + x^2 y\right);$$

Now, let's define the vec

```
F[x_, y_] = {f[x, y], g[x, y]};
```

Out[10]//MatrixForm=

$$\begin{pmatrix} \frac{x}{2} - y + \frac{1}{2}\left(-x^3 - x y^2\right) \\ x + \frac{y}{2} + \frac{1}{2}\left(-x^2 y - y^3\right) \end{pmatrix}$$

Step 1: Find the “Easy” Solutions (Equilibrium Points)

In[11]:=
```
eqPts = Solve[F[x, y] == {0, 0}, {x, y}]
```

Out[11]=

$\{\{x \to 0, y \to 0\}\}$

Step 2: Linearize about the Equilibrium Points

Let’s first determine the Jacobian.

In[22]:=
```
Df = Grad[F[x, y], {x, y}];
Df // MatrixForm
```

Out[23]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} + \frac{1}{2}\left(-3x^2 - y^2\right) & -1 - x y \\ 1 - x y & \frac{1}{2} + \frac{1}{2}\left(-x^2 - 3y^2\right) \end{pmatrix}$$

The Linearization at $(x^*, y^*) \to (0, 0)$

In[35]:=
```
J = Df /. eqPts[[1]];
J // MatrixForm
esys1 = Eigensystem[J];
% // MatrixForm
eVal = esys1[[1]];
```

Out[36]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{pmatrix}$$

Out[38]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} + i & \frac{1}{2} - i \\ \{i, 1\} & \{-i, 1\} \end{pmatrix}$$

Note: The following part will be useful for Homework #4.

```
p1 =
  Plot[{Re[eVal[[1]]], Re[eVal[[2]]]}, {μ, -2, 2}, AxesLabel → {μ, ""}, PlotLabel → "Re[λ]"];
p2 =
  Plot[{Im[eVal[[1]]], Im[eVal[[2]]]}, {μ, -2, 2}, AxesLabel → {μ, ""}, PlotLabel → "Im[λ]"];
Show[GraphicsRow[{p1, p2}],
  PlotLabel → Style["Eigenvalues of the linearization at (0,0)", "Subsubsection"],
  ImageSize → 1000] // Panel
```

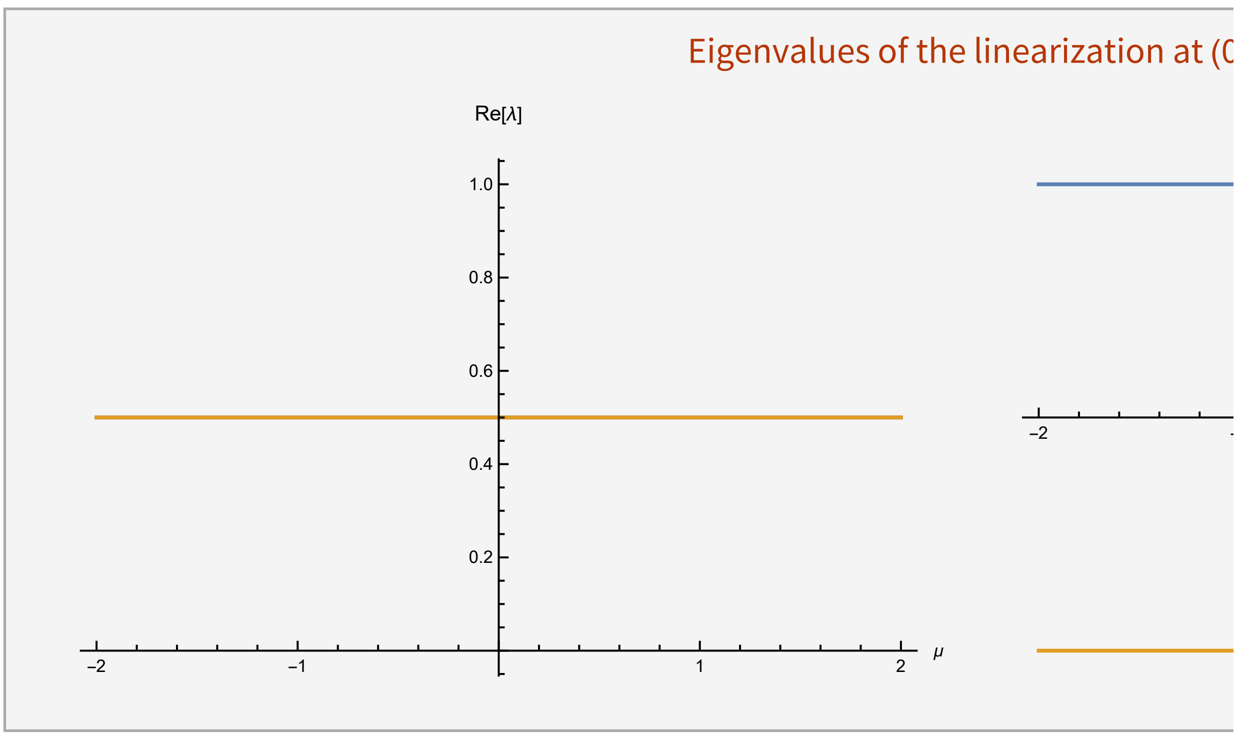
Out[26]=

$$\left\{\left\{\frac{1}{2}+i, \frac{1}{2}-i\right\}, \{\{i, 1\}, \{-i, 1\}\}\right\}$$

Out[27]=

$$\left\{\frac{1}{2}+i, \frac{1}{2}-i\right\}$$

Out[30]=



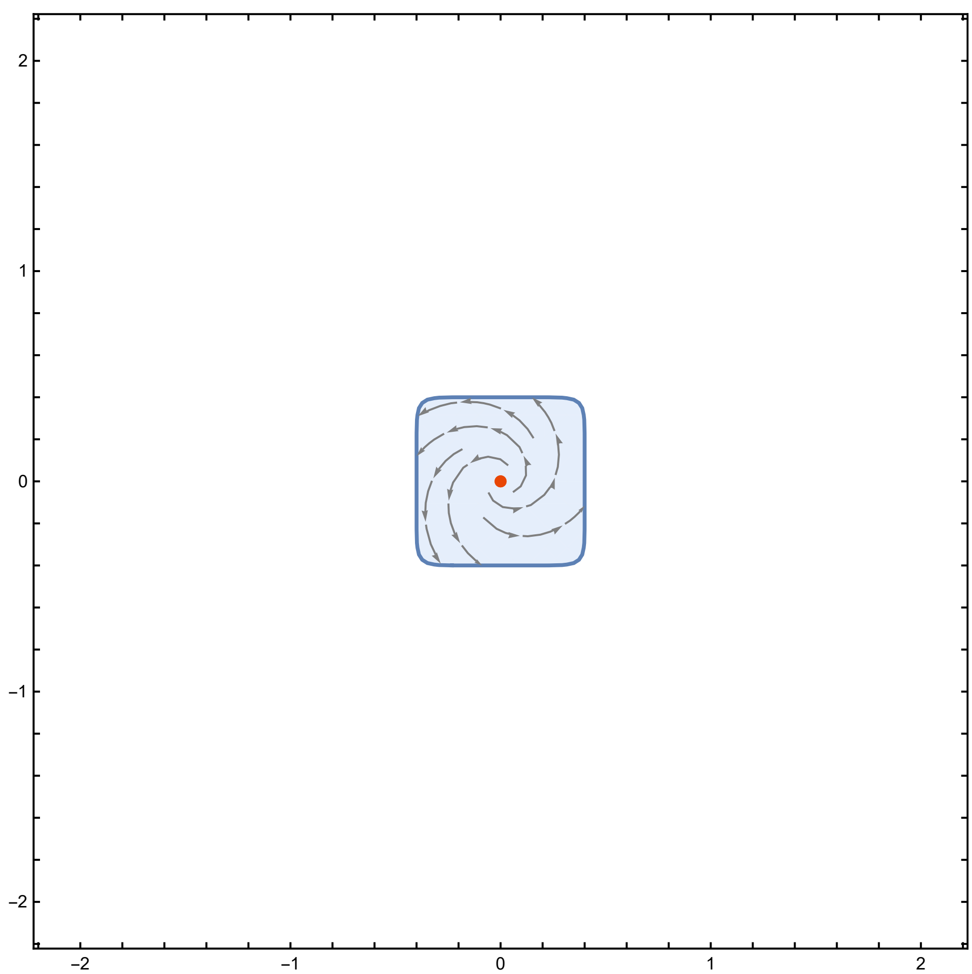
The Phase Plane - Visualizing portions to illustrate a point

Near the equilibrium point (0, 0)

In[127]:=

```
p1 := StreamPlot[{f[x, y], g[x, y]}, {x, -2, 2}, {y, -2, 2},
   ImageSize → 500,
   StreamColorFunction → None,
   StreamStyle → Gray,
   StreamPoints → 70,
   StreamScale → 0.05,
   RegionFunction → Function[{x, y, vx, vy, n}, x^10 + y^10 < .4^10]];
(*This last part restricts the plotting region*)
p3 := ListPlot[{{0, 0}}, PlotStyle → ColorData["SolarColors"][.4]];
Show[p1, p3]
```

Out[129]=

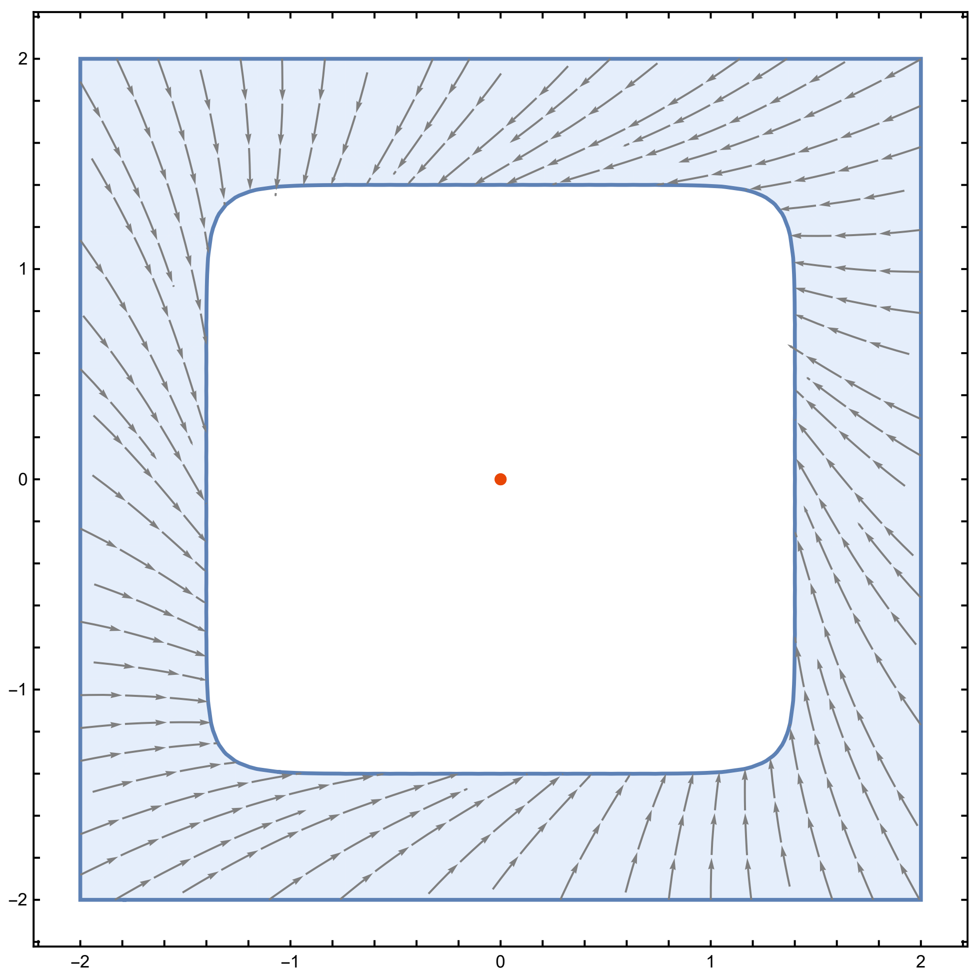


Away from the equilibrium point

In[130]:=

```
p1 := StreamPlot[{f[x, y], g[x, y]}, {x, -2, 2}, {y, -2, 2},
   ImageSize → 500,
   StreamColorFunction → None,
   StreamStyle → Gray,
   StreamPoints → 70,
   StreamScale → 0.05, RegionFunction → Function[{x, y, vx, vy, n}, x^10 + y^10 > 1.4^10]];
p2 := ParametricPlot[{Cos[t], Sin[t]},
   {t, 0, 2 Pi}, PlotStyle → ColorData["SolarColors"][.8]];
p3 := ListPlot[{{0, 0}}, PlotStyle → ColorData["SolarColors"][.4]];
Show[p1, p3]
```

Out[133]=

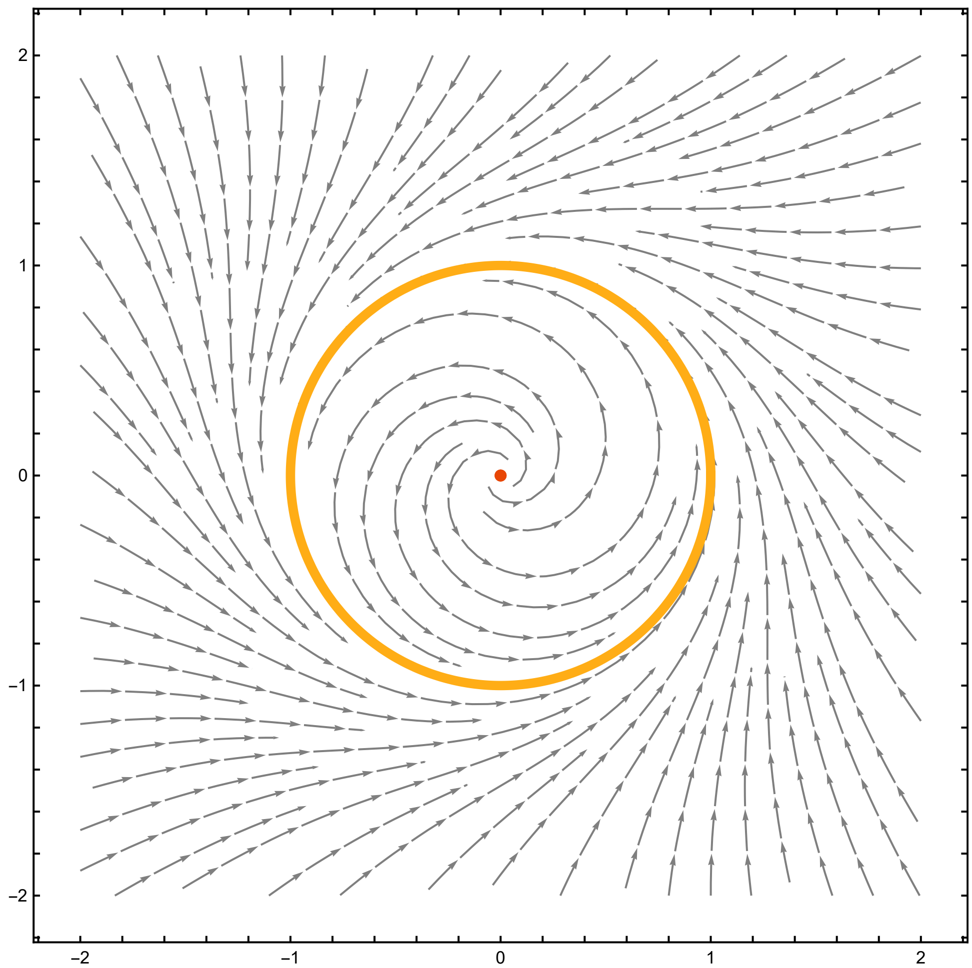


Viewing Everything

In[134]:=

```
p1 := StreamPlot[{f[x, y], g[x, y]}, {x, -2, 2}, {y, -2, 2},
    ImageSize → 500,
    StreamColorFunction → None,
    StreamStyle → Gray,
    StreamPoints → 70,
    StreamScale → 0.05];
p2 := ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2 Pi},
    PlotStyle → {{ColorData["SolarColors"][.8], Thickness[0.01]}}];
p3 := ListPlot[{{0, 0}}, PlotStyle → ColorData["SolarColors"][.4]];
Show[p1, p2, p3]
```

Out[137]=



Analyzing the System in Polar Coordinates

Let's define the change of variables given by $(x, y) \to (r, \theta)$ where $x = r\cos(\theta)$ and $y = r\sin(\theta)$

```
varChange = {x → r Cos[θ], y → r Sin[θ]}
```

Now, we know that $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. This will allow us to determine $\dot{r}$ and $\dot{\theta}$ in a slightly different manner than we did in class. For example, since $r^2 = x^2 + y^2$, we can use implicit differentiation to determine $2\, r\, \dot{r} = 2\, x\, \dot{x} + 2\, y\, \dot{y}$ or in a more compact form, we have that $r\, \dot{r} = x\, f\,(x,\, y) + y\, g\,(x,\, y)$ or $\dot{r} = \frac{x\, f\,(x,\, y) + y\, g\,(x,\, y)}{r}$. In the line below, we will simplify this expression for $\dot{r}$ where we will let $x = r\,(t)\cos\,(\theta(t))$ and $y = r\,(t)\sin\,(\theta(t))$

```
In[71]:= rDot = (x f[x, y] + y g[x, y]) / r;
rDot = rDot /. varChange // FullSimplify
```

Out[72]=

$$\frac{1}{2}\left(r - r^3\right)$$

Likewise, since $\tan\,(\theta) = \frac{y}{x}$, we have $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ so that $\dot{\theta} = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{\dot{y}\, x - \dot{x}\, y}{y^2}$ which yields $\dot{\theta} = \frac{\dot{y}\, x - \dot{x}\, y}{x^2 + y^2} = \frac{g\,(x,\, y)\, x - f\,(x,\, y)\, y\ \text{â-..}}{x^2 + y^2}$

```
thetaDot = (g[x, y] x - f[x, y] y) / (x^2 + y^2);
thetaDot = thetaDot /. varChange // FullSimplify
```

Out[68]=

$$-1$$

Thus, we now know that we have the system $\dot{r} = \frac{1}{2}\left(r - r^3\right)$ and $\dot{\theta} = 1$. Doing the streamplot in the $(r,\, \theta)$ plane, we find the following:

```
p1 = StreamPlot[{rDot, thetaDot}, {r, 0, 3}, {θ, -Pi, Pi},
   ImageSize → 500,
   StreamColorFunction → None,
   StreamStyle → Gray,
   StreamPoints → 70,
   StreamScale → 0.075];
p2 = ParametricPlot[{{0, t}, {1, t}}, {t, -Pi, Pi},
   PlotStyle → {
     {ColorData["SolarColors"][.4], Thickness[0.01]},
     {ColorData["SolarColors"][.8], Thickness[0.01]}
    },
   PlotLegends → {
     "r = 0  (r-nullcline)",
     "r = 1  (r nullcline)"
    }];
Show[p1, p2, FrameLabel → {"r", "θ"}]
```

Out[171]=