Systems of Differential Equations

Considering a system of differential equations of the form:

$$\dot{x} = \frac{1}{2}\left(x - x^3\right) - y - \frac{1}{2} x\, y^2$$
$$\dot{y} = x - \frac{1}{2} x^2\, y + \frac{1}{2} y - \frac{1}{2} y^3$$

Define the functions

In[191]:=
$$f = \frac{1}{2} x - y - \frac{1}{2}\left(x^3 + x\, y^2\right);$$
$$g = x + \frac{1}{2} y - \frac{1}{2}\left(y^3 + x^2\, y\right);$$

Step 1: Find the “Easy” Solutions (Equilibrium Points)

In[159]:=
```
eqPts = Solve[{f == 0, g == 0}, {x, y}]
```

Out[159]= {{x → 0, y → 0}}

Step 2: Linearize about the Equilibrium Points

Let’s first determine the Jacobian.

In[51]:=
```
Df = {{D[f, x], D[f, y]}, {D[g, x], D[g, y]}};
Df // MatrixForm
```

Out[52]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} + \frac{1}{2}\left(-3\, x^2 - y^2\right) & -1 - x\, y \\ 1 - x\, y & \frac{1}{2} + \frac{1}{2}\left(-x^2 - 3\, y^2\right) \end{pmatrix}$$

The Linearization at $(x^*, y^*) \to (0, 0)$

In[160]:=
```
J = Df /. eqPts[[1]];
J // MatrixForm
```

Out[161]//MatrixForm=

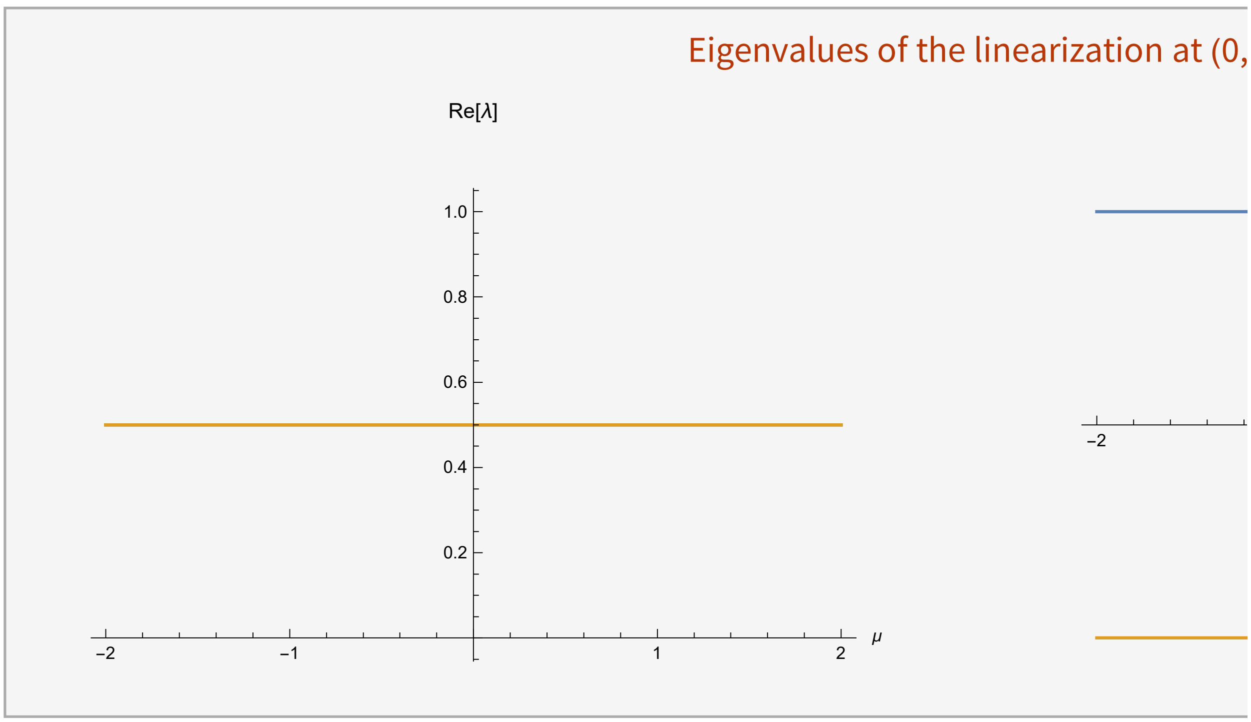
$$\begin{pmatrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{pmatrix}$$

In[162]:=
```
esys1 = Eigensystem[J]
eVal = esys1[[1]]
p1 = Plot[{Re[eVal[[1]]], Re[eVal[[2]]]},
   {μ, -2, 2}, AxesLabel → {μ, ""}, PlotLabel → "Re[λ]"];
p2 = Plot[{Im[eVal[[1]]], Im[eVal[[2]]]}, {μ, -2, 2},
   AxesLabel → {μ, ""}, PlotLabel → "Im[λ]"];
Show[GraphicsRow[{p1, p2}], PlotLabel → Style["Eigenvalues of the linearization at (0,0)",
    "Subsubsection"], ImageSize → 1000] // Panel
```

Out[162]= $\left\{\left\{\frac{1}{2}+i, \frac{1}{2}-i\right\}, \{\{i, 1\}, \{-i, 1\}\}\right\}$

Out[163]= $\left\{\frac{1}{2}+i, \frac{1}{2}-i\right\}$

Out[166]=



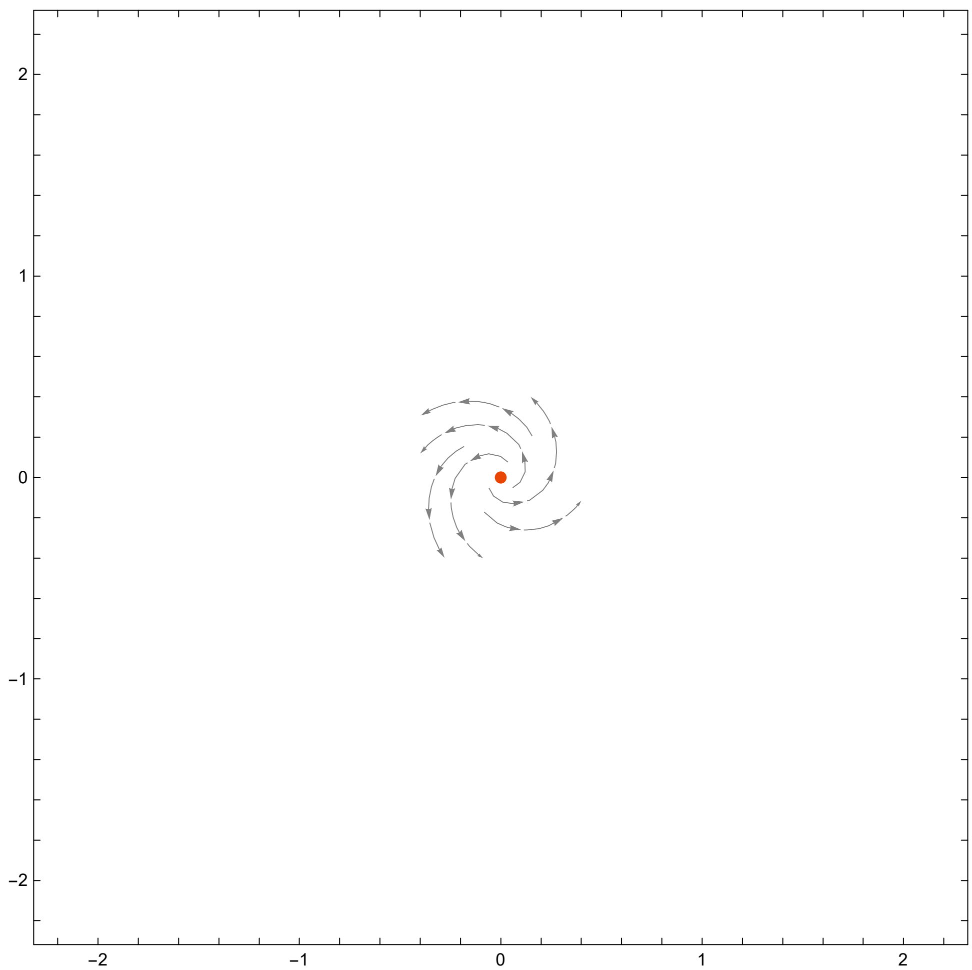
The Phase Plane

Near the equilibrium point (0, 0)

In[154]:=

```
p1 := StreamPlot[{f, g}, {x, -2, 2}, {y, -2, 2},
   ImageSize → 500, StreamStyle → Gray, StreamPoints → 70, StreamScale → 0.05,
   RegionFunction → Function[{x, y, vx, vy, n}, x^10 + y^10 < .4^10]];
p3 := ListPlot[{{0, 0}}, PlotStyle → ColorData["SolarColors"][.4]];
Show[p1, p3]
```

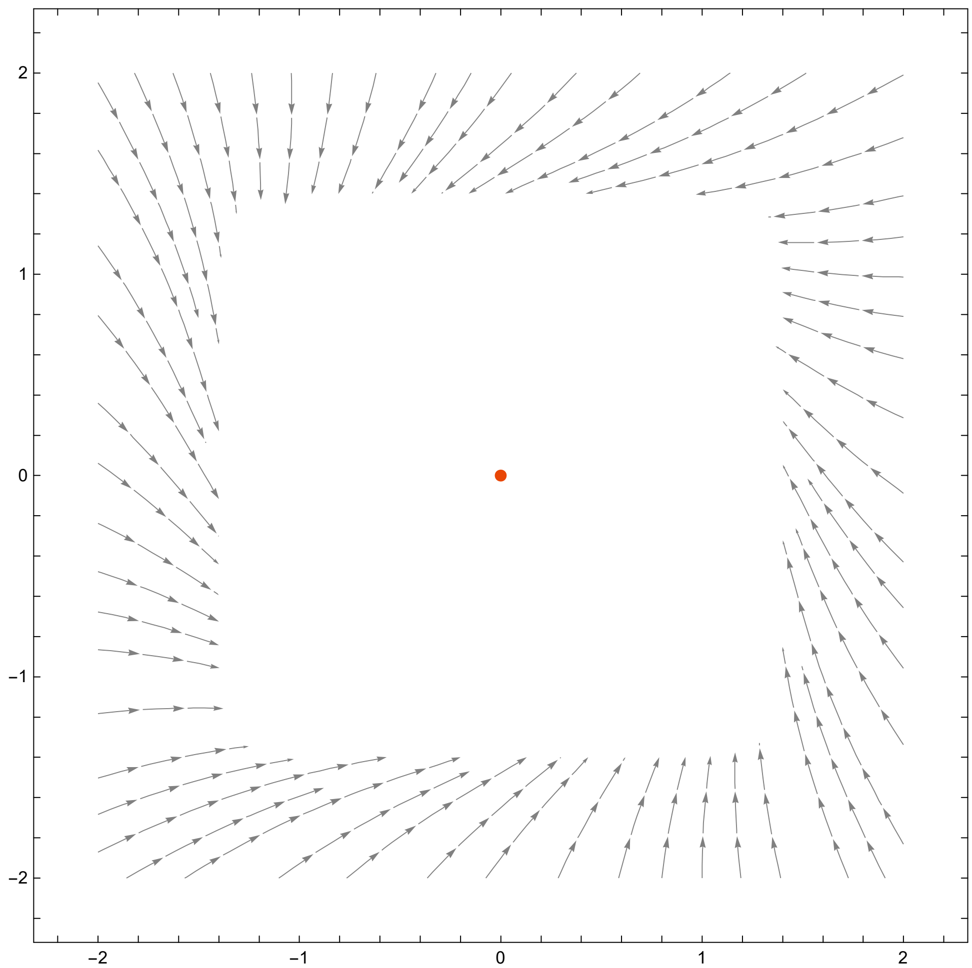
Out[156]=



Away from the equilibrium point

In[138]:=
```
p1 := StreamPlot[{f, g}, {x, -2, 2}, {y, -2, 2},
   ImageSize → 500, StreamStyle → Gray, StreamPoints → 60, StreamScale → 0.05,
   RegionFunction → Function[{x, y, vx, vy, n}, x^10 + y^10 > 1.4^10]];
p2 := ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2 Pi},
   PlotStyle → ColorData["SolarColors"][.8]];
p3 := ListPlot[{{0, 0}}, PlotStyle → ColorData["SolarColors"][.4]];
Show[p1, p3]
```

Out[141]=



Viewing Everything

In[193]:=
```
p1 := StreamPlot[{f, g}, {x, -2, 2}, {y, -2, 2},
   ImageSize → 500, StreamStyle → Gray, StreamPoints → 60, StreamScale → 0.05];
p2 := ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2 Pi},
   PlotStyle → ColorData["SolarColors"][.8]];
p3 := ListPlot[{{0, 0}}, PlotStyle → ColorData["SolarColors"][.4]];
Show[p1, p2, p3]
```

Out[196]=