Math 3910: Optimal Estimation and Control | Example LQR Code

Problem/Assignment Overview

Consider the following optimization problem:

$$\begin{aligned} &\text{Minimize} && J[u,x] = \int_{t_0}^{t_f} \left[\frac{1}{2}\vec{x}^\top Q\,\vec{x} + \frac{1}{2}\vec{u}^\top R\,\vec{u}\right]\,dt + \frac{1}{2}\vec{x}^\top G\,\vec{x}\Big|_{t=t_f} \\ &\text{Subject to} && \dot{\vec{x}} = A\vec{x} + B\vec{u}, \\ & && \vec{x}(t_0) = \vec{x}_0, \\ & && \text{where } t_0 \text{ and } t_f \text{ are known.} \end{aligned}$$

Summary/Outline from Class

Using Pontryagin's Maximum Principle, we can solve the optimal control problem by forming the Hamiltonian H which is given by

$$H = \frac{1}{2}\vec{x}^\top Q\,\vec{x} + \frac{1}{2}\vec{u}^\top R\,\vec{u} + \vec{p}^\top\left(A\vec{x} + B\vec{u}\right)$$

where $\vec{p}$ represents our Lagrange multipliers. Following the **Pontryagin's Principle Summary Handout**, and the work done in class, we have the following steps:

Step 1: We know the optimal control $\vec{u}$ in terms of $\vec{p}$ from solving the equation

$$\delta u: \qquad H_u = 0 \quad \longrightarrow \quad \vec{u} = -R^{-1}B^\top\vec{p}$$

Step 2: We would like to solve for $\vec{p}$ and $\vec{u}$ by setting up the following equations:

$$\begin{aligned} \delta x: &\qquad \dot{\vec{p}} + -Q\vec{x} - A^\top\vec{p}, \\ \delta u: &\qquad \dot{\vec{x}} = A\vec{x} + B\vec{u}, \\ \delta x_f: &\qquad \vec{p}\big|_{t_f} = G\vec{x}\big|_{t_f}, \\ \text{given}: &\qquad \vec{x}\big|_{t_0} = \vec{x}_0. \end{aligned}$$

Since the optimal control is given by $\vec{u} = -R^{-1}B^\top\vec{p}$, we can replace $\vec{u}$ in the above equations to express the above as a system of ODEs for the two unknowns $\vec{x}$ and $\vec{p}$ given by:

$$\begin{bmatrix} \dot{\vec{x}} \\ \dot{\vec{p}} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{bmatrix}\begin{bmatrix} \vec{x} \\ \vec{p} \end{bmatrix}, \qquad \begin{bmatrix} \vec{x}(t_0) \\ \vec{p}(t_0) \end{bmatrix} = \begin{bmatrix} \vec{x}_0 \\ \vec{p}_0 \end{bmatrix}$$

The solution to the above system can be expressed in terms of the state-transition matrix $\Phi(t, t_0)$ as

$$\begin{bmatrix} \vec{x}(t) \\ \vec{p}(t) \end{bmatrix} = \Phi(t, t_0)\begin{bmatrix} \vec{x}_0 \\ \vec{p}_0 \end{bmatrix},$$

where the challenge is that we don't have an initial condition for $\vec{p}$, nor do we know the value of $\vec{x}$ when $t = t_f$
.

Step 3: We can at least determine $\vec{x}_f$ via the $\delta\vec{x}_f$ variation to find that

$$\vec{p} = G\vec{x}_f.$$

Evaluating the solution to the system when $t = t_f$, we find

$$\begin{bmatrix} \vec{x}_f \\ \vec{p}_f \end{bmatrix} = \Phi(t_f, t_0) \begin{bmatrix} \vec{x}_0 \\ \vec{p}_0 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} \vec{x}_f \\ G\vec{x}_f \end{bmatrix} = \Phi(t_f, t_0) \begin{bmatrix} \vec{x}_0 \\ \vec{p}_0 \end{bmatrix}$$

Step 4: Despite only knowing $\vec{x}_0$ and how $\vec{p}_f$ relates to $\vec{x}_f$, we can find the state transition matrix $\Phi(t, t_0)$. The bulk of the document is about finding these matrices and thus the solution.

The Code

Preliminaries and Matrix Definitions

In [80]:
```
from matplotlib import pyplot as plt
import numpy as np
plt.rcParams['text.usetex'] = True
```

Now, let's define the matrices A, B, Q, R, and G.

In [113…
```
A = np.array([[0., 1.],[0., 0.]]);
B = np.array([[0.],[1.]]);

w1 = 1.;
w2 = 1.;

Q = np.array([[w1,0.],[0., w2]]);
R = np.array([[1.]]);

G = Q;
```

Let's also set the initial and final times, t_0 and t_f respectively. Finally, we can input the initial condition x_0.

In [86]:
```
t0 = 0;
tf = 10;
x0 = np.array([[3.],[1.]])
```

At this point, we will now define the number of time-steps (`numSteps`), as well the vector of t values at which we will compute the solution (`tV`), and the width of each time step (`dt`).

In [87]:
```
numSteps = 100;
tV = np.linspace(t0,tf,numSteps);
dt = tV[1]-tV[0];
```

Numerically Solving for the Transition Matrix $\Phi(t, t_0)$

Recall that the state transition matrix $\Phi(t, t_0)$ satisfies the differential equation

$$\frac{d\Phi}{dt} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{bmatrix}}_{\footnotesize K} \Phi, \quad \Phi(t_0, t_0) = I.$$

We can use Euler's method to numerically integrate and time-step the solution for Φ. Recall that Euler's method says that

$$\Phi(t_j + \Delta t, t_0) \approx \Phi(t_n, t_0) + \Delta t \cdot K\Phi(t_n, t_0) \approx (I + \Delta t \cdot K)\Phi(t_j, t_0)$$

In the following block of code, we initializes an array that will store Φ at each time step.

In [ ]:
```
Phi = np.zeros((numSteps,4,4));
Phi[0,:,:] = np.eye(4);
```

Now, we will take the prescribed number of steps (`numSteps`) to approximate the transition matrix Φ from t_0 to t_f.

In [119…
```
K = np.vstack((np.hstack((A, -B@R**(-1)@B.transpose())),np.hstack((-Q, -A.transpose()))))

for j in range(0,numSteps-1):
    t = tV[j];
    Phi[j+1,:,:] = (np.eye(4) + dt*K)@Phi[j,:,:]

# The following simply determines Phi(t_f,t_0) by taking the final computed value of Phi
PhiF = Phi[-1,:,:]
PhiF11 = PhiF[0:2,0:2];
PhiF12 = PhiF[0:2,2:4];
PhiF21 = PhiF[2:4,0:2];
PhiF22 = PhiF[2:4,2:4];
```

Finding $\vec{p}_0$ and $\vec{x}_f$ using $\Phi(t_f, t_0)$.

Recall that in class, we set up the system

$$\begin{bmatrix} \vec{x}_f \\ \vec{p}_f \end{bmatrix} = \begin{bmatrix} \Phi_{1,1}(t_f, t_0) & \Phi_{1,2}(t_f, t_0) \\ \Phi(_{2,1} t_f, t_0) & \Phi 2, 2(t_f, t_0) \end{bmatrix} \begin{bmatrix} \vec{x}_0 \\ \vec{p}_0 \end{bmatrix} \quad \text{where} \quad \vec{p}_f = G\vec{x}_f.$$

Considering the unknowns to be $\vec{x}_f$ and $\vec{p}_0$, we can rearrange the system to find

$$\underbrace{\begin{bmatrix} I & -\Phi_{1,2}(t_f, t_0) \\ G & -\Phi_{2,2} \end{bmatrix}}_{\text{\footnotesize{lhs}}} \begin{bmatrix} \vec{x}_f \\ \vec{p}_0 \end{bmatrix} = \underbrace{\begin{bmatrix} \Phi_{1,1}(t_f, t_0) & 0 \\ \Phi_{1,2}(t_f, t_0) & 0 \end{bmatrix}}_{\text{\footnotesize{rhs}}} \begin{bmatrix} \vec{x}_0 \\ \vec{p}_f \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} \vec{x}_f \\ \vec{p}_0 \end{bmatrix} = \begin{bmatrix} I & -\Phi_{1,2}(t_f, t_0) \\ G & -\Phi_{2,2} \end{bmatrix}^{-1} \begin{bmatrix} \Phi_{1,1}(t_f, t_0 \\ \Phi_{1,2}(t_f, t_0 \end{bmatrix}$$

Because computing inverses of matrices via commands such as `inv` are computational expensive, we will use `np.linalg.solve` to solve the above system using more stable, and computational efficient methods (such as LU decomposition, etc).

In [117…
```
lhs1 = np.hstack((np.eye(2),-PhiF12));
lhs2 = np.hstack((G,-PhiF22));
lhs = np.vstack((lhs1,lhs2));

rhs = np.vstack((PhiF11@x0,PhiF21@x0));
output = np.linalg.solve(lhs,rhs)

# peel off the last two terms as this represents the initial condition for p.
p0 = output[2:4];

# now, create the vector of initial conditions for z = [x  p]^T
z0 = np.vstack((x0,p0));
```

Now that we have the initial conditions $\vec{z}_0 = [\vec{x}_0 \;\; \vec{p}_0]^\top$ and the state-transition matrix $\Phi(t, t_0)$ (computed for each time step), we can find $\vec{z}$ at each time step by noting that the solution to the problem is given by

$$\vec{z}(t) = \Phi(t, t_0)\vec{z}_0 \quad \text{and} \quad u(t) = -R^{-1}B^\top \vec{p}(t)$$

In [118…

```
z = np.zeros((numSteps,4,1))
u = np.zeros((numSteps))

for j in range(0,numSteps-1):
    t = tV[j]
    z[j,:,:] = Phi[j,:,:]@z0;
    u[j] = -np.linalg.solve(R,B.transpose())@z[j,2:4,:];
```

Plotting the Solution/Output

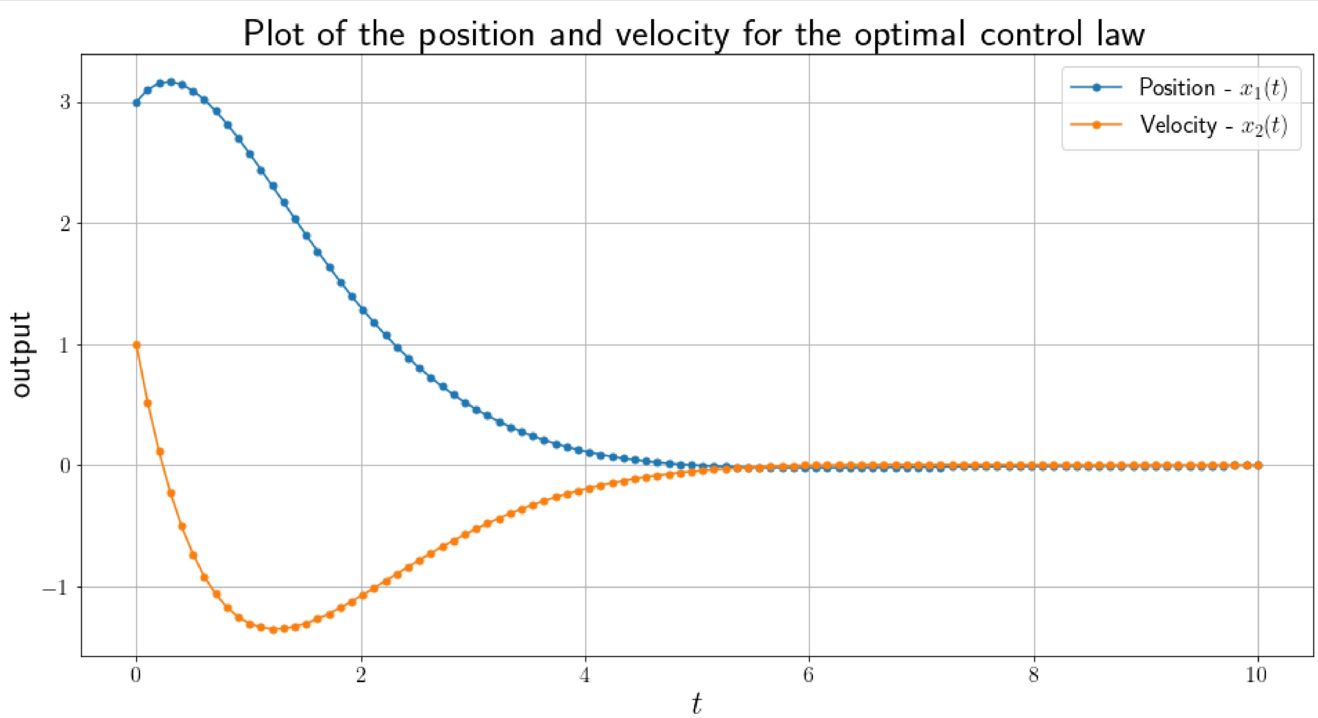
Now that we have computed the control, let's see what happens when we implement this.

In [110…

```
plt.figure(figsize=(16, 8))


plt.rc('xtick', labelsize=16)
plt.rc('ytick', labelsize=16)

plt.plot(tV,z[:,0:2,:].reshape(numSteps,2),'.-',markersize=10)
plt.legend([r'Position - $x_1(t)$','Velocity - $x_2(t)$'])
plt.xlabel(r'$t$', fontsize=24)
plt.ylabel(r'output', fontsize=24)
plt.title(r'Plot of the position and velocity for the optimal control law',fontsize=28)
plt.grid()
plt.show()
```

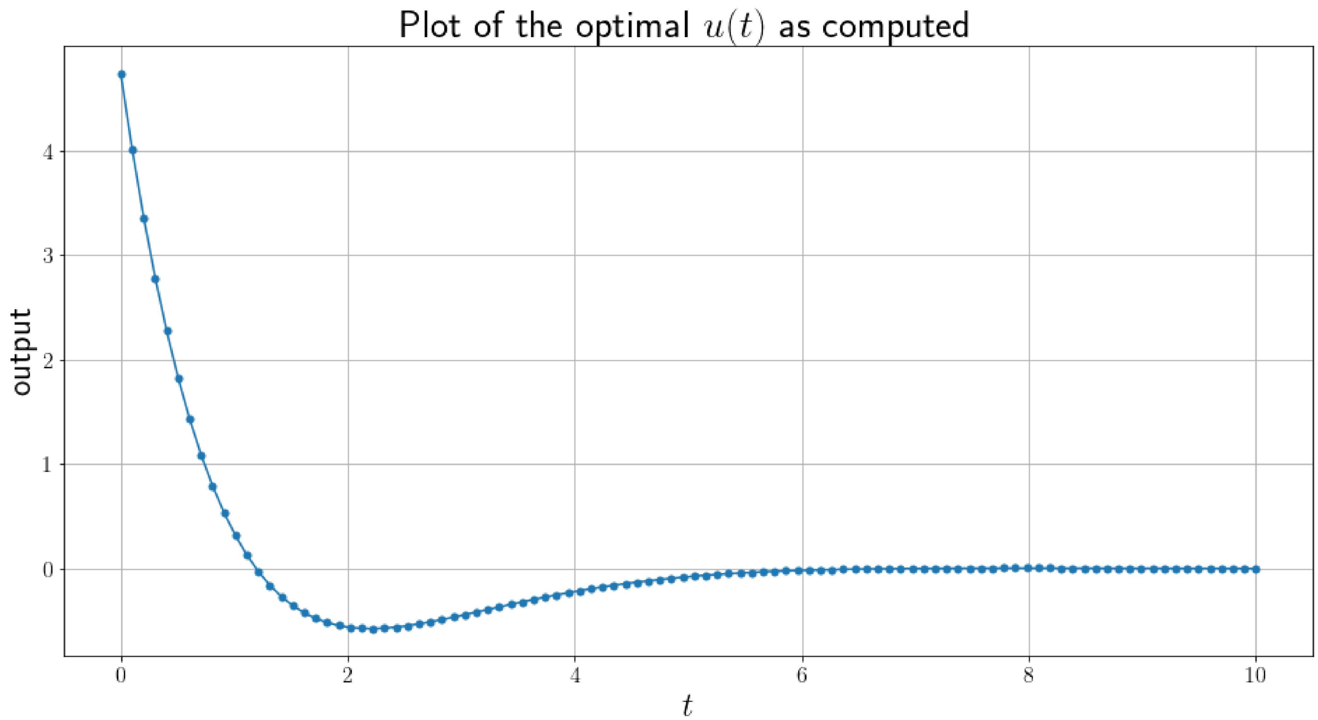


In [112…

```
plt.figure(figsize=(16, 8))
```

```python
plt.rc('xtick', labelsize=16)
plt.rc('ytick', labelsize=16)

plt.plot(tV,u,'.-',markersize=10)
plt.xlabel(r'$t$', fontsize=24)
plt.ylabel(r'output', fontsize=24)
plt.title(r'Plot of the optimal $u(t)$ as computed',fontsize=28)
plt.grid()
plt.show()
```



In [ ]: