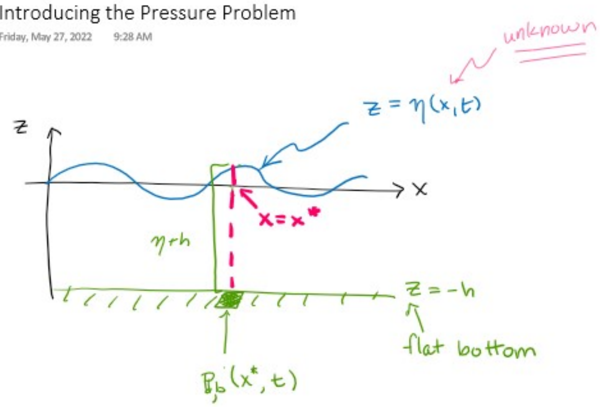
Introducing the Pressure Problem

Friday, May 27, 2022 9:28 AM



$$p_{d,b} = \eta + \epsilon\left(\eta_{tt} - \frac{1}{2}\eta_{xx}\right) + \mathcal{O}(\epsilon^2)$$

$\eta_{xx} = \eta_{tt} + \mathcal{O}(\varepsilon)$

$$\left(1 - \frac{\epsilon}{2}\partial_x^2\right)p_{d,b} = \eta + \mathcal{O}(\epsilon^2)$$

$p_{xx} = p_{tt} + \mathcal{O}(\varepsilon)$

$\partial_x^{-1}\eta_t = \int_{-\infty}^{x} \eta_t(s,t)\,ds$

$$\left(1 - \frac{\epsilon}{6}\partial_x^2\right)p_{d,b} = \eta - \left(\frac{1}{2}\eta^2 + \epsilon\left(\partial_x^{-1}\eta_t\right)^2\right) + \mathcal{O}(\epsilon^2)$$

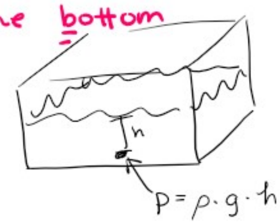
$p_{d,b} - \frac{\varepsilon}{2}\partial_x^2 p_{d,b}$

$p_{d,b} - \frac{\varepsilon}{2}\frac{\partial^2 p_{d,b}}{\partial x^2} = p_{d,b} - \frac{\varepsilon}{2}\frac{\partial^2 p_{d,b}}{\partial t^2}$

Goal: given $p_{d,b}(x^*, t)$ find or estimate $\eta(x^*, t)$

dynamic pressure at the bottom

Physics: $p = \rho g h$



hydrostatic pressure

only true when water is still.

$p = \rho \cdot g \cdot (\eta + h)$ ← approximation

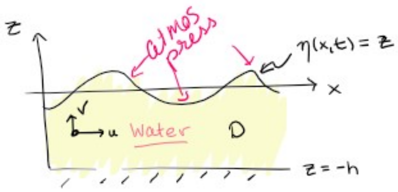
$p = \underbrace{\rho g \eta}_{\text{dynamic pressure}} + \rho g h$

dynamic pressure is the total pressure with the hydrostatic part factored out.

ϕ = velocity potential

$$\nabla\phi = \begin{bmatrix}\phi_x \\ \phi_z\end{bmatrix} = \begin{bmatrix}u(x,z,t) \\ v(x,z,t)\end{bmatrix} \begin{array}{l}\text{horiz velocity} \\ \text{vert velocity}\end{array}$$

$$\nabla = \begin{bmatrix}\partial_x \\ \partial_z\end{bmatrix}$$



PDE: $\phi_{xx} + \phi_{zz} = 0$ $\quad (x,z) \in D$

$\phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_z^2 + gz + p(x,z,t) = 0 \quad (x, z \in D)$

BC: $\phi_z = 0$ at $z = -h$

$\phi_z = \eta_t + \phi_x\eta_x$ or $\phi_z - \eta_x\phi_x = \eta_t$ at $z = \eta(x,t)$

$\phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_z^2 + g\eta = 0$ at $z = \eta(x,t)$

↳ given: $ax^2 + bx + c = 0$

$x^2 - (4 + \varepsilon) = 0$ guess $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$

$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)^2 - 4 - \varepsilon = 0$

$x_0^2 - 4 + 2\varepsilon x_0 x_1 - \varepsilon + \mathcal{O}(\varepsilon^2) = 0$ $\quad \mathcal{O}(\varepsilon^0)$ $x_0^2 - 4 = 0 \rightarrow x_0^2 = 4 \quad x_0 = \pm 2$

$\mathcal{O}(\varepsilon^1)$ $2x_0 x_1 - 1 = 0 \quad x_1 = \frac{1}{2x_0}$

$x_0 = 2 \rightarrow x_1 = \frac{1}{4}$ $\quad x \approx 2 + \frac{1}{4}\varepsilon + \mathcal{O}(\varepsilon^2) = \sqrt{4 + \varepsilon}$

$\eta_{tt} = \eta_{xx} + \mathcal{O}(\varepsilon)$

$$p_{d,b} = \eta + \epsilon\left(\eta_{tt} - \frac{1}{2}\eta_{xx}\right) + \mathcal{O}(\epsilon^2)$$

$p_{db} = \eta + \varepsilon \frac{\eta_{tt}}{2} + \mathcal{O}(\varepsilon^2)$

$$\left(1 - \frac{\epsilon}{2}\partial_x^2\right) p_{d,b} = \eta + \mathcal{O}(\epsilon^2)$$

$(1 - \varepsilon$

$$\cancel{\left(1 - \frac{\epsilon}{6}\partial_x^2\right) p_{d,b} = \eta - \left(\frac{1}{2}\eta^2 + \epsilon\left(\partial_x^{-1}\eta_t\right)^2\right) + \mathcal{O}(\epsilon^2)}$$

$\eta_{tt} - \eta_{xx} = \mathcal{O}(\varepsilon)$

$\eta_{xx} = \eta_{tt} + \mathcal{O}(\varepsilon)$

$$p_{db} = \eta + \varepsilon\left(\eta_{tt} - \tfrac{1}{2}\eta_{xx}\right) + \mathcal{O}(\varepsilon^2)$$

$$= \eta + \varepsilon\left(\eta_{tt} - \tfrac{1}{2}\eta_{tt} + \mathcal{O}(\varepsilon)\right) + \mathcal{O}(\varepsilon^2)$$

$$p_{db} = \eta + \varepsilon\left(\frac{\eta_{tt}}{2}\right) + \underbrace{\mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon^2)}_{\mathcal{O}(\varepsilon^2)}$$

$$\longrightarrow \quad p_{db} - \frac{\varepsilon}{2} p_{db,tt} = \eta$$

$$p = \eta + \frac{\varepsilon}{2}\eta_{tt} \qquad p - \frac{\varepsilon}{2} p_{tt} = \eta$$

$\boxed{\text{Step 1}}$

write as :

$$\dot{x} = Ax + \cancel{Bu}^{\,0}$$

$$y = Cx$$

$$x = \begin{bmatrix} \eta \\ \dot{\eta} \\ p \\ \dot{p} \end{bmatrix}$$ one option

$$x = \begin{bmatrix} \eta \\ \dot{\eta} \\ p \\ \dot{p} \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} \dot{\eta} \\ \ddot{\eta} \\ \dot{p} \\ \ddot{p} \end{bmatrix} \qquad \begin{bmatrix} \dot{\eta} \\ \ddot{\eta} \\ \dot{p} \\ \ddot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} \eta \\ \dot{\eta} \\ p \\ \dot{p} \end{bmatrix}$$

Objective : Find A, and C. Then, determine if $\text{rank}\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = 4$ (Is it observable ?)

let $\varepsilon = 0.1$

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import control as ctrl

O = ctrl.obsv(A,C)
rank = np.linalg.matrix_rank(O)
```

Simulate the system.