PROBABILITY THEORY

Introduction to Probability, Statistics and Random Processes

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1 Chapter 1

1.1 Set operations

A union of 2 sets is given by the combination of their elements:

$$A \cup B = \{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$$

The intersection of 2 sets is instead given by their shared elements:

$$A \cap B = \{1, 2\} \cap \{2, 3\} = \{2\}$$

Theorem 1 (De Morgan's law)**.** *For any sets $A_1, A_2, \ldots, A_n$ we have*

$$\begin{aligned}\overline{A_1 \cup A_2 \cup \ldots \cup A_n} &= \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n} \\ \overline{A_1 \cap A_2 \cap \ldots \cap A_n} &= \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_n}\end{aligned}$$

Theorem 2 (Distributive law)**.** *For any sets A, B and C we have*

$$\begin{aligned}A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C)\end{aligned}$$

The complement of a set is given by all elements that are in the universal set, but not the set itself:

$$S = \{1, 2, 3, 4, 5\} \qquad \overline{A} = S \setminus A = \{1, 2, 3, 4, 5\} \setminus \{1, 2\} = \{3, 4, 5\}$$

The difference between two sets is given by elements in the first but not the second:

$$A \setminus B = \{1, 2\} - \{2, 3\} = \{1\} \qquad A \setminus B = A \cap \overline{B}$$

Two sets are disjoint if their intersection is an empty set

$$A \cap B = \emptyset$$

Sets can be partitioned into smaller parts. The sets $A_1, A_2, \ldots, A_n$ are a partition of S if they're disjoint and:

$$\bigcup_{i=1}^{n} A_i = S$$

The cartesian product of two sets are given by the ordered pairs of both sets:

$$A \times B = \{1, 2\} \times \{2, 3\} = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$$

Which can be expressed more generally as:

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

The number of elements contained in a (finite) sets is given by its cardinality:

$$|A| = |\{1, 2\}| = 2$$

For determining the cardinality of (finite) sets, the inclusion-exclusion principle is often used:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

This can once again be expanded to more sets:

$$|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j|$$
$$+ \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$$

1.2 Cardinality and countable sets

Finite sets are obviously countable, however when we move onto infinite sets they are divided into countable **and** uncountable sets. A countable set is characterised by the ability to write it in one-to-one correspondance with the natural numbers, e.g.:

$$A = \{a_1, a_2, \ldots, a_n\}$$

Meaning you can list the elements, this is true for sets like the natural numbers, $\mathbb{N}$, and the integers, $\mathbb{Z}$, but also the rational numbers, $\mathbb{Q}$. Uncountable sets (such as the real- and complex numbers) on the other hand cannot be written as lists, but instead have to be denoted as intervals.

Definition 1 (Countability of a set)**.** A set, A, is called countable if one of the following is true:

- It is a finite set, $|A| < \infty$.

- The set can be written as a list with one-to-one correspondance with the natural numbers.

This means that any subset of $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are countable, whilst any set containing an interval on the real line is uncountable.

Theorem 3 (Countability of sub- and supersets)**.** *Any subset of a countable set is countable and any superset of an uncountable set is uncountable.*

Proof. Let A be a countable set and $B \subset A$. If A is finite, then it follows that $|B| \leq |A| < \infty$, thus B must be countable as its cardinality cannot exceed that of A, which must be smaller than ∞.

If A is instead countably infinite, then it follows that as B is a subset of A it must be possible to construct it by removing $\overline{B}$ from A, whereby it must also be countable, as it can be constructed as a list.

The opposite can be argued by assuming B is **not** countable, whereby a contradiction would occur in both proofs. $\square$

Theorem 4 (Countability of union)**.** *If $A_1, A_2, \ldots, A_n$ are countable sets, then the union of those must also be countable.*

Proof. As the sets are countable it must be possible to write them in the form

$$\begin{aligned} A_1 &= \{a_{11}, a_{12}, \ldots, a_{1n}\} \\ A_2 &= \{a_{21}, a_{22}, \ldots, a_{2n}\} \\ A_3 &= \{a_{31}, a_{32}, \ldots, a_{3n}\} \end{aligned}$$

As such the union of those sets must also be possible to construct as a list

$$\bigcup_{i=1}^{m} A_i = \{a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32}, \ldots, a_{mn}\}$$

And as a result must be countable. $\square$

Theorem 5 (Countability of carthesian product)**.** *If A and B are countable, then $A \times B$ is also countable.*

Proof. As A and B are countable it must be possible to write them in the form

$$A = \{a_1, a_2, \ldots, a_n\}$$
$$B = \{b_1, b_2, \ldots, b_n\}$$

In accordance with the definition of the carthesian product, the two sets can be constructed as a list with the form

$$A \times B = \{(a_i, b_j) \mid i, j \in \mathbb{N}\}$$

Whereby it must be countable as it can be constructed as a list. □

As a result of this proof it also becomes clear that any set that can be written in the form

$$C = \bigcup_i \bigcup_j \{a_{ij}\} \text{ where } i, j \text{ belong to a countable set}$$

Must also be countable, the set of rational numbers is an example of this as it can be written as

$$\mathbb{Q} = \bigcup_{i \in \mathbb{Z}} \bigcup_{j \in \mathbb{N}} \left\{ \frac{i}{j} \right\}$$

1.3 Functions

Functions take an input from its domain, apply a rule to said input, whereby an output from the co-domain is produced.

$$f : A \to B \qquad f(x \in A) \in B$$

Definition 2. A function maps elements from the domain set to elements in the co-domain with the property that each input is mapped to exactly one output.

In the same context the range operand is defined, as it is not necessary for a function to be able to output all elements of the codomain:

$$f : \mathbb{R} \xrightarrow{x^2} \mathbb{R}$$

Here both the domain- and co-domain are the real numbers, however it is clear that no value $x \in \mathbb{R}$ would ever produce a negative number, therefore:

$$\text{Range}(f) = \mathbb{R}^+$$

1.4 Problems

1.4.1 Problem 3

a) Let $S = \{1, 2, 3\}$. Write all possible partitions of S.

As a partition is any collection of disjoint sets whos union makes up S we have that

1. {1},{2},{3}
2. {1,2},{3}
3. {1},{2,3}
4. {1,3},{2}
5. {1,2,3}

1.4.2 Problem 4

a) Determine whether each of the following sets are countable or countable:

- $A = \{x \in \mathbb{Q} \mid -100 \leq x \leq 100\}$
- $B = \{(x, y) \mid x \in \mathbb{N}, y \in \mathbb{Z}\}$
- $C =]0, 0.1]$
- $D = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$

As $A \subset \mathbb{Q}$ it is clear that it must be countable.

As B is the carthesian product of 2 countable sets it must be countable.

As C is a range it must be uncountable.

As D can be written in one-to-one correspondance with the naturals it must be countable.

1.4.3 Problem 5

a) Find the range of the function $f : \mathbb{R} \xrightarrow{\sin(x)} \mathbb{R}$.

As $\sin(x)$ has its extrema at $\sin\left(\frac{\pi}{2}\right) = 1$ and $\sin\left(\frac{3\pi}{2} = -1\right)$, it is clear that

$$\text{Range}(f) = [-1, 1]$$

1.5 Random experiments

A random experiment will always have an **outcome** corresponding to an element from the **sample space**, S.

Definition 3. A random experiment is a process by which we observe something uncertain.

When a random experiment is repeated, each repetition is called a **trial**. The goal of analyzing a random experiment is to assign probabilities to **events**, which correspond to subsets of the sample space.

1.6 Probability

A probability is assigned to an event, $P(A) \in [0,1]$. The derivation of probability theorem is based on 3 axioms:

- Axiom 1: For any event A, $1 \geq P(A) \geq 0$.
- Axiom 2: Probability of the sample space, S, is $P(S) = 1$.
- Axiom 3: If $A_1, A_2, \ldots, A_n$ are disjoint events, then $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

Notationally unions and intersections can be read as:

$$P(A \cap B) = P(A \text{ and } B) = P(A, B)$$
$$P(A \cup B) = P(A \text{ or } B)$$

Theorem 6 (Probability of complement)**.** *For any event A, $P(\overline{A}) = 1 - P(A)$.*

Proof. As the complement of a set contains all elements of the sample space that are not in the set

$$\overline{A} = S \setminus A$$

It is clear that their unions must be S and they must be disjoint whereby

$$P(A \cup \overline{A}) = P(S) = 1$$

As they are disjoint we can write the probability of their union as the sum of their probabilities

$$P(A) + P(\overline{A}) = 1 \Leftrightarrow P(A) = 1 - P(\overline{A})$$

□

Theorem 7 (Probability of empty set)**.** *The probability of the empty is zero,* $P(\emptyset) = 0$.

Proof. As the empty set must be the complement of the sample space we have that

$$P(\emptyset) = P(\overline{S}) = 1 - P(S) = 1 - 1 = 0$$

□

Theorem 8 (Probability must be equal to or less than 1)**.** *For any event* A*,* $P(A) \leq 1$.

Proof. By the first axiom we have that

$$P(\overline{A}) \geq 0$$

It becomes clear that

$$P(A) \leq 1$$

As $P(A) + P(\overline{A}) = 1$. □

Theorem 9 (Probability of a difference)**.** *The probability of a difference is given by* $P(A \setminus B) = P(A) - P(A \cap B)$.

Proof. As $A \cap B$ and $A \setminus B$ must be disjoint, whilst their union must be A

$$(A \cap B) \cup (A \setminus B) = A$$

We have by the third axiom that

$$P(A) = P((A \cap B) \cup (A \setminus B)) = P(A \cap B) + P(A \setminus B)$$

By rearranging it becomes clear that

$$P(A \setminus B) = P(A) - P(A \cap B)$$

□

Theorem 10 (Probability of a union)**.** *The probability of a union is given by* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof. As A and $B \setminus A$ must be disjoint sets whilst their union must be $A \cup B$, it is clear that

$$P(A \cup B) = P(A \cup (B \setminus A))$$

As we know these are disjoint we write

$$P(A \cup B) = P(A) + P(B \setminus A)$$

Rewriting using the previous theorem we then have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

□

Theorem 11 (Probability of a subset must be less than or equal to its superset)**.** *If* $A \subset B$ *then* $P(A) \leq P(B)$*.*

Proof. As $A \subset B$ it is clear that their union must be B

$$P(B) = P(A \cap B) + P(B \setminus A)$$

As their intersection is A we have that

$$P(B) = P(A) + P(B \setminus A)$$

As

$$P(B \setminus A) \geq 0$$

By the first axiom, we have that

$$P(B) \geq P(A)$$

□