PROBABILITY THEORY

Introduction to Probability, Statistics and Random Processes

Hossein Pishro-Nik

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1 Chapter 1

1.1 Set operations

A union of 2 sets is given by the combination of their elements:

$$A \cup B = \{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$$

The intersection of 2 sets is instead given by their shared elements:

$$A \cap B = \{1, 2\} \cap \{2, 3\} = \{2\}$$

Theorem 1 (De Morgan's law)**.** *For any sets $A_1, A_2, \ldots, A_n$ we have*

$$\overline{A_1 \cup A_2 \cup \ldots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}$$
$$\overline{A_1 \cap A_2 \cap \ldots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_n}$$

Theorem 2 (Distributive law)**.** *For any sets A, B and C we have*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

The complement of a set is given by all elements that are in the universal set, but not the set itself:

$$S = \{1, 2, 3, 4, 5\} \qquad \overline{A} = S \setminus A = \{1, 2, 3, 4, 5\} \setminus \{1, 2\} = \{3, 4, 5\}$$

The difference between two sets is given by elements in the first but not the second:

$$A \setminus B = \{1, 2\} - \{2, 3\} = \{1\} \qquad A \setminus B = A \cap \overline{B}$$

Two sets are disjoint if their intersection is an empty set

$$A \cap B = \emptyset$$

Sets can be partitioned into smaller parts. The sets $A_1, A_2, \ldots, A_n$ are a partition of S if they're disjoint and:

$$\bigcup_{i=1}^{n} A_i = S$$

The cartesian product of two sets are given by the ordered pairs of both sets:

$$A \times B = \{1, 2\} \times \{2, 3\} = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$$

Which can be expressed more generally as:

$$A \times B = \{(x, y) | x \in A \text{ and } y \in B\}$$

The number of elements contained in a (finite) sets is given by its cardinality:

$$|A| = |\{1, 2\}| = 2$$

For determining the cardinality of (finite) sets, the inclusion-exclusion principle is often used:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

This can once again be expanded to more sets:

$$|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$$

1.1.1 Cardinality and countable sets

Finite sets are obviously countable, however when we move onto infinite sets they are divided into countable **and** uncountable sets. A countable set is characterised by the ability to write it in one-to-one correspondance with the natural numbers, e.g.:

$$A = \{a_1, a_2, \ldots, a_n\}$$

Meaning you can list the elements, this is true for sets like the natural numbers, $\mathbb{N}$, and the integers, $\mathbb{Z}$, but also the rational numbers, $\mathbb{Q}$. Uncountable sets (such as the real- and complex numbers) on the other hand cannot be written as lists, but instead have to be denoted as intervals.

Definition 1 (Countability of a set)**.** A set, A, is called countable if one of the following is true:

- It is a finite set, $|A| < \infty$.
- The set can be written as a list with one-to-one correspondance with the natural numbers.

This means that any subset of $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are countable, whilst any set containing an interval on the real line is uncountable.

Theorem 3 (Countability of sub- and supersets)**.** *Any subset of a countable set is countable and any superset of an uncountable set is uncountable.*

Proof. Let A be a countable set and $B \subset A$. If A is finite, then it follows that $|B| \leq |A| < \infty$, thus B must be countable as its cardinality cannot exceed that of A, which must be smaller than ∞.

If A is instead countably infinite, then it follows that as B is a subset of A it must be possible to construct it by removing $\overline{B}$ from A, whereby it must also be countable, as it can be constructed as a list.

The opposite can be argued by assuming B is **not** countable, whereby a contradiction would occur in both proofs. □

Theorem 4 (Countability of union). *If $A_1, A_2, \ldots, A_n$ are countable sets, then the union of those must also be countable.*

Proof. As the sets are countable it must be possible to write them in the form

$$A_1 = \{a_{11}, a_{12}, \ldots, a_{1n}\}$$
$$A_2 = \{a_{21}, a_{22}, \ldots, a_{2n}\}$$
$$A_3 = \{a_{31}, a_{32}, \ldots, a_{3n}\}$$

As such the union of those sets must also be possible to construct as a list

$$\bigcup_{i=1}^{m} A_i = \{a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32}, \ldots, a_{mn}\}$$

And as a result must be countable. □

Theorem 5 (Countability of carthesian product). *If A and B are countable, then $A \times B$ is also countable.*

Proof. As A and B are countable it must be possible to write them in the form

$$A = \{a_1, a_2, \ldots, a_n\}$$
$$B = \{b_1, b_2, \ldots, b_n\}$$

In accordance with the definition of the carthesian product, the two sets can be constructed as a list with the form

$$A \times B = \{(a_i, b_j) | i, j \in \mathbb{N}\}$$

Whereby it must be countable as it can be constructed as a list. □

As a result of this proof it also becomes clear that any set that can be written in the form

$$C = \bigcup_i \bigcup_j \{a_{ij}\} \text{ where } i, j \text{ belong to a countable set}$$

Must also be countable, the set of rational numbers is an example of this as it can be written as

$$\mathbb{Q} = \bigcup_{i \in \mathbb{Z}} \bigcup_{j \in \mathbb{N}} \left\{ \frac{i}{j} \right\}$$

1.1.2 Functions

Functions take an input from its domain, apply a rule to said input, whereby an output from the co-domain is produced.

$$f : A \to B \qquad f(x \in A) \in B$$

Definition 2. A function maps elements from the domain set to elements in the co-domain with the property that each input is mapped to exactly one output.

In the same context the range operand is defined, as it is not necessary for a function to be able to output all elements of the codomain:

$$f : \mathbb{R} \xrightarrow{x^2} \mathbb{R}$$

Here both the domain- and co-domain are the real numbers, however it is clear that no value $x \in \mathbb{R}$ would ever produce a negative number, therefore:

$$\text{Range}(f) = \mathbb{R}^+$$

1.1.3 Problems

Problem 3 a) Let $S = \{1, 2, 3\}$. Write all possible partitions of S.

As a partition is any collection of disjoint sets whos union makes up S we have that

1. {1},{2},{3}
2. {1,2},{3}
3. {1},{2,3}
4. {1,3},{2}
5. {1,2,3}

Problem 4 a) Determine whether each of the following sets are countable or countable:

- $A = \{x \in \mathbb{Q}| - 100 \leq x \leq 100\}$
- $B = \{(x, y)|x \in \mathbb{N}, y \in \mathbb{Z}\}$
- $C =]0, 0.1]$
- $D = \left\{\frac{1}{n}|n \in \mathbb{N}\right\}$

As $A \subset \mathbb{Q}$ it is clear that it must be countable.

As B is the carthesian product of 2 countable sets it must be countable.

As C is a range it must be uncountable.

As D can be written in one-to-one correspondance with the naturals it must be countable.

Problem 5 a) Find the range of the function $f : \mathbb{R} \xrightarrow{\sin(x)} \mathbb{R}$.

As $\sin(x)$ has its extrema at $\sin\left(\frac{\pi}{2}\right) = 1$ and $\sin\left(\frac{3\pi}{2} = -1\right)$, it is clear that

$$\text{Range}(f) = [-1, 1]$$

1.2 Random experiments

A random experiment will always have an **outcome** corresponding to an element from the **sample space**, S.

Definition 3. A random experiment is a process by which we observe something uncertain.

When a random experiment is repeated, each repetition is called a **trial**. The goal of analyzing a random experiment is to assign probabilities to **events**, which correspond to subsets of the sample space.

1.2.1 Probability

A probability is assigned to an event, $P(A) \in [0, 1]$. The derivation of probability theorem is based on 3 axioms:

- Axiom 1: For any event A, $1 \geq P(A) \geq 0$.
- Axiom 2: Probability of the sample space, S, is $P(S) = 1$.

- Axiom 3: If $A_1, A_2, \ldots, A_n$ are disjoint events, then $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

Notationally unions and intersections can be read as:

$$P(A \cap B) = P(A \text{ and } B) = P(A, B)$$
$$P(A \cup B) = P(A \text{ or } B)$$

Theorem 6 (Probability of complement)**.** *For any event A, $P(\overline{A}) = 1 - P(A)$.*

Proof. As the complement of a set contains all elements of the sample space that are not in the set

$$\overline{A} = S \setminus A$$

It is clear that their unions must be S and they must be disjoint whereby

$$P(A \cup \overline{A}) = P(S) = 1$$

As they are disjoint we can write the probability of their union as the sum of their probabilities

$$P(A) + P(\overline{A}) = 1 \Leftrightarrow P(A) = 1 - P(\overline{A})$$

□

Theorem 7 (Probability of empty set)**.** *The probability of the empty is zero, $P(\emptyset) = 0$.*

Proof. As the empty set must be the complement of the sample space we have that

$$P(\emptyset) = P(\overline{S}) = 1 - P(S) = 1 - 1 = 0$$

□

Theorem 8 (Probability must be equal to or less than 1)**.** *For any event A, $P(A) \leq 1$.*

Proof. By the first axiom we have that

$$P(\overline{A}) \geq 0$$

It becomes clear that

$$P(A) \leq 1$$

As $P(A) + P(\overline{A}) = 1$. □

Theorem 9 (Probability of a difference)**.** *The probability of a difference is given by* $P(A \setminus B) = P(A) - P(A \cap B)$.

Proof. As $A \cap B$ and $A \setminus B$ must be disjoint, whilst their union must be A

$$(A \cap B) \cup (A \setminus B) = A$$

We have by the third axiom that

$$P(A) = P((A \cap B) \cup (A \setminus B)) = P(A \cap B) + P(A \setminus B)$$

By rearranging it becomes clear that

$$P(A \setminus B) = P(A) - P(A \cap B)$$

□

Theorem 10 (Probability of a union)**.** *The probability of a union is given by* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof. As A and $B \setminus A$ must be disjoint sets whilst their union must be $A \cup B$, it is clear that

$$P(A \cup B) = P(A \cup (B \setminus A))$$

As we know these are disjoint we write

$$P(A \cup B) = P(A) + P(B \setminus A)$$

Rewriting using the previous theorem we then have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

□

Theorem 11 (Probability of a subset must be less than or equal to its superset)**.** *If* $A \subset B$ *then* $P(A) \leq P(B)$.

Proof. As $A \subset B$ it is clear that their union must be B

$$P(B) = P(A \cap B) + P(B \setminus A)$$

As their intersection is A we have that

$$P(B) = P(A) + P(B \setminus A)$$

As

$$P(B \setminus A) \geq 0$$

By the first axiom, we have that

$$P(B) \geq P(A)$$

$\square$

1.2.2 Problems

Problem 2 Write the sample space, S, for the following random experiments:

a) We toss a coin until we see two consecutive tails. We record the number of coin tosses.

$$S = \{n \in \mathbb{N} | n \geq 2\}$$

b) A bag contains 4 balls: one is red, one is blue, one is white and one is green. We choose two distinct balls and record their color in order.

$$S = \{(R, B), (R, W), (R, G), (B, R), (B, W)(B, G), (W, R), \\ (W, B), (W, G), (G, R), (G, B), (G, W)\}$$

c) A customer arrives at a bank and waits in the line. We observe T, which is the total time (in hours) that the customer waits in the line. The bank has a strict policy that no customer waits more than 20 minutes under any circumstances.

$$S = \left[0; \frac{1}{3}\right]$$

Problem 3 Let A, B and C be three events in the sample space S. Suppose we know:

- $A \cup B \cup C = S$
- $P(A) = \frac{1}{2}$
- $P(B) = \frac{2}{3}$
- $P(A \cup B) = \frac{5}{6}$

Answer the following questions:

a) Find $P(A \cap B)$.

We have that

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

Inserting known information we get that

$$P(A \cap B) = \frac{1}{2} + \frac{2}{3} - \frac{5}{6} = \frac{1}{3}$$

b) Do A, B and C form a partition of S?

No, as they aren't disjoint.

c) Find $P(C - (A \cup B))$.

We rewrite the event as

$$C \setminus (A \cup B) = (C \cup (A \cup B)) - (A \cup B)$$

As their union makes up the sample space

$$C \setminus (A \cup B) = S \setminus (A \cup B) = \overline{A \cup B}$$

Which we know is equal to

$$P(C \setminus (A \cup B)) = 1 - \frac{5}{6} = \frac{1}{6}$$

d) If $P(C \cap (A \cup B)) = \frac{5}{12}$, find $P(C)$.

As C must be possible to construct from the intersection of C with A and B and the difference between them we get that

$$P(C) = P(C \cap (A \cup B)) + P(C \setminus (A \cup B))$$

By inserting we get

$$P(C) = \frac{5}{12} + \frac{1}{6} = \frac{7}{12}$$

Problem 4 I roll a fair die twice and obtain two numbers, X_1 = result of the first roll, and X_2 = result of the second roll. Find the probability of the following events.

a) A defined as $X_1 < X_2$.

We write the sample space

$$\begin{aligned} S = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),\\ &(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),\\ &(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),(5,4),(5,5),\\ &(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\} \end{aligned}$$

Is is clear that $|S| = 36$ and $|A| = 15$, as such

$$P(A) = \frac{15}{36} = \frac{5}{12}$$

b) B defined $|\{6\}| \geq 1$.

From the sample space we get that $|B| = 11$, as such

$$P(B) = \frac{11}{36}$$

Problem 5 You purchase a certain product. The manual states that the lifetime, T, of the product defined as the amount of time (in years) the product works properly until it breaks down satisfies:

$$P(T \geq t) = e^{-\frac{t}{5}} \text{ for all } t \geq 0$$

For examples, the probability that the product lasts more than (or equal to) 2 years is

$$P(T \geq 2) = e^{-\frac{2}{5}} = 0.6703$$

a) This is an example of a continuous probability model. Write down the sample space, S.

As $t \geq 0$ we get that

$$S = [0; \infty[$$

b) Check that the statement in the manual makes sense by finding $P(T \geq 0)$ and $\lim_{t\to\infty} P(T \geq t)$

We insert that $t = 0$ and get

$$P(T \geq 0) = e^{-\frac{0}{5}} = e^0 = 1$$

As such it makes sense, we take the limit as $t \to \infty$

$$\lim_{t\to\infty} e^{-\frac{t}{5}} = 0$$

Which also makes sense.

c) Also check that if $t_1 < t_2$ then $P(T \geq t_1) \geq P(T \geq t_2)$. Why does this need to be true?

We assume $t_1 = 1, t_2 = 2$ and get

$$P(T \geq 1) = 0.8187 \qquad P(T \geq 2) = 0.6703$$

Whereby the statement is true. This has to be true as it is an exponential model, resulting in the monotony constantly being negative, as such the value must fall every time t is increased.

d) Find the probability that the product breaks down within three years of the purchase time.

We have that

$$P(T \geq 3) = e^{-\frac{3}{5}} = 0.5488$$

As this is the probability that it last longer than or equal to 3 years, the complement must be

$$P(T < 3) = 1 - P(T \geq 3) = 1 - 0.5488 = 0.4512$$

e) Find the probability that the product breaks down in the second year, i.e., find $P(1 \leq T < 2)$.

We determine the probability of it breaking down within 2 years and then subtract the probability of it breaking down within 1. Using results from c) we have

$$P(T < 1) = 0.1813 \qquad P(T < 2) = 0.3297$$

Subtracting the two we get that

$$P(1 \geq T < 2) = 0.3297 - 0.1813 = 0.1484$$

Problem 6 You get a stick and break it randomly into three pieces. What is the probability that you can make a triangle using the three pieces? You can assume the break points are chosen completely at random, i.e. if the length of the original stick is 1 unit, and x, y, z are the lengths of the three pieces, then (x, y, z) are uniformly chosen from the set

$$\{(x, y, z) \in \mathbb{R}^3 | x + y + z = 1, x, y, z \geq 0\}$$

By the triangle inequality we have that

$$\begin{aligned} x &< y + z \\ y &< x + z \\ z &< x + y \end{aligned}$$

Due to the sum requirement, the only time this is satisfied is when

$$x < \frac{1}{2}$$
$$y < \frac{1}{2}$$
$$z < \frac{1}{2}$$

Furthermore we have that the entire sample space must stretch a 3 dimensional plane from $P_1(1,0,0), P_2(0,1,0), P_3(0,0,1)$, while the possible values for forming a triangle must stretch one from $P_1(0.5,0,0), P_2(0,0.5,0), P_3(0,0,0.5)$. The area of this plane is $\frac{1}{4}$ of the sample spaces plane, and therefore

$$P(A) = \frac{1}{4}$$

1.3 Conditional probability

Conditional probabilities are written as

$$P(A|B)$$

And are read as "*the probability of A, given that B has occurred*". The conditional probability will therefore be given by

$$\begin{aligned} P(A|B) &= \frac{|A \cap B|}{|B|} \\ &= \frac{\frac{|A \cap B|}{|S|}}{\frac{|B|}{|S|}} \text{ dividing by } |S| \\ &= \frac{P(A \cap B)}{P(B)} \end{aligned}$$

This is as when we know B has occured, the sample space of A is shrunk to B, whereby the cardinality of their intersection must be equal to the amount of favourable outcomes.

The earlier established probability axioms can also be formulated for conditional probabilities

- Axiom 1: For any event A, $P(A|B) \geq 0$.
- Axiom 2: Conditional probability of B given B is 1, i.e., $P(B|B) = 1$.

- Axiom 3: If $A_1, A_2, \ldots, A_n$ are disjoint events, then $P(\bigcup_{i=1}^{n} A_i|B) = \sum_{i=1}^{n} P(A_i|B)$.

And the same applies to the established fomulas

Theorem 12. *For any conditional event, $A|C$, $P(\overline{A}|C) = 1 - P(A|C)$.*

Proof. We know that

$$P(A|B) = \frac{P(A|B)}{P(B)}$$

Assuming the theorem is correct we then have that

$$1 - P(\overline{A}|C) = 1 - \frac{P(\overline{A}|C)}{P(B)}$$

We wish to show that

$$\frac{P(A|B)}{P(B)} = 1 - \frac{P(\overline{A}|C)}{P(B)}$$

Multiplying by $P(B)$ gets us

$$P(A|B) = P(B) - P(\overline{A}|C)$$

Adding $P(\overline{A}|C)$ on the LHS we get

$$P(A|B) + P(\overline{A}|C) = P(B)$$

As the two are mutually exclusive by the definition of the complement

$$P((A|B) \cup (\overline{A}|C)) = P(B)$$

Which is true. □

Theorem 13. *The probability of the empty set is zero, $P(\emptyset|C) = 0$.*

Proof. As the empty is the complement of the sample set we have from the previous theorem that

$$P(S|C) = 1$$

By applying previous equation

$$P(\overline{S}|C) = 1 - 1 = 0$$

□

Theorem 14. *The probability of a conditional probability occuring must always be less than or equal to 1,* $P(A|C) \leq 1$.

Proof. From the definition of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

It is clear that the denominator and numerator can never be more than even as $A \cap B = B$ even if AB, resulting in the maximum value being 1.

$$P(A|B) \leq 1$$

□

Theorem 15. *The probability of a difference is given by* $P(A \setminus B|C) = P(A|C) - P(A \cap B|C)$.

Proof. As $A \cap B|C$ and $A \setminus B|C$ must be disjoint whilst their union must be $A|C$, we have that

$$(A \cap B|C)(A \setminus B|C) = A|C$$

By the third axiom we have that

$$P(A|C) = P(A \cap B|C) + P(A \setminus B|C)$$

Rearranging the terms we get

$$P(A \setminus B|C) = P(A|C) - P(A \cap B|C)$$

□

Theorem 16. *The probability of a union is given by* $P(A \cup B|C) = P(A|C) + P(B|C) - P(A \cap B|C)$.

Proof. As $A|C$ and $B \setminus A|C$ must be disjoint and their union must be equal to $A \cup B|C$ we have that

$$P(A \cup B) = P(A|C(B \setminus A|C))$$

As these sets are disjoint we rewrite using the third axiom

$$P(A \cup B) = P(A|C) + P(B \setminus A|C)$$

By the previous theorem the difference is rewritten as

$$P(A \cup B) = P(A|C) + P(B|C) - P(A \cap B|C)$$

□

Theorem 17. *If $A \subset B$ then $P(A|C) \leq P(B|C)$.*

Proof. As $A \subset B$ it is clear that their union must be B

$$P(B|C) = P(A \cap B|C) + P(B \setminus A|C)$$

Since their intersection is A due to it being the subset

$$P(B|C) = P(A|C) + P(B \setminus A|C)$$

By the first axiom

$$P(B \setminus A|C) \geq 0$$

As such

$$P(B|C) \geq P(A|C)$$

□

This introduces some special cases

$$P(A|B) = \frac{P(\emptyset)}{P(B)} = 0, \text{ for } A \cap B = \emptyset$$
$$P(A|B) = \frac{P(B)}{P(B)} = 1, \text{ for } B \subset A$$
$$P(A|B) = \frac{P(A)}{P(B)}, \text{ for } A \subset B$$

Furthermore we also write the chain rule for conditional probability using the definition as a starting point

Theorem 18. *The extended chain rule is given by $P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_2, A_1) \cdots P(A_n|A_{n-1}, A_{n-2}, \ldots, A_1)$*

Proof. From the definition of conditional probability we have that

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

By isolation for $P(A \cap B)$ we get

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(B)$$

Extending to 3 or more events we get that

$$P(A \cap B \cap C) = P(A \cap (B \cap C)) = P(A)P(B \cap C|A)$$

Applying the first equation

$$P(B \cap C) = P(B)P(C|B)$$

By conditioning both sides on A we get

$$P(B \cap C|A) = P(B|A)P(C|A, B)$$

Inserting in the original equation we then get

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A, B)$$

Which can be generalised to

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_2, A_1) \cdots \\ P(A_n|A_{n-1}, A_{n-2}, \ldots, A_1)$$

$\square$

1.3.1 Independence

Conditional probabilities are only relevant if two events are not independent.

Definition 4. Two events A, B are independent if $P(A \cap B) = P(A)P(B)$.

Independence for two or more events requires that all the individual events are independents, as well as all of them together, this means that for 3 events, all of the following must hold

$$\begin{aligned} P(A \cap B) &= P(A)P(B) \\ P(A \cap C) &= P(A)P(C) \\ P(B \cap C) &= P(B)P(C) \\ P(A \cap B \cap C) &= P(A)P(B)P(C) \end{aligned}$$

Theorem 19. *If A and B are independent, then A and $\overline{B}$, $\overline{A}$ and B, $\overline{A}$ and $\overline{B}$ are also independent.*

Proof. The first statement is proven as the others can be concluded from it. As the statement is equivalent to

$$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

As we know A and B are independent we have

$$P(A \cap \overline{B}) = P(A) - P(A)P(B)$$

We factor out $P(A)$ and get that

$$P(A \cap \overline{B}) = P(A)(1 - P(B))$$

And as

$$1 - P(B) = P(\overline{B})$$

It becomes clear that

$$P(A \cap \overline{B}) = P(A)P(\overline{B})$$

□

To determine the probability of several unions of independent events, we make use of De Morgans law

$$P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - P\left(\bigcap_{i=1}^{n} \overline{A_i}\right)$$

Which is equivalent to

$$P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \prod_{i=1}^{n}(1 - P(A_i))$$

1.3.2 Law of Total Probability

The law of total probability states that the probability of an event, A must be the sum of the probability of it occuring in every partition, $B_1, B_2, \ldots, B_n$ of the sample space

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

Proof. As $B_1, B_2, \ldots, B_n$ are partitions of the sample space we write

$$S = \bigcup_{i=1}^{n} B_i$$

Now the event A occuring must be given by its intersection of the sample space

$$A = A \cap \left(\bigcup_{i=1}^{n} B_i \right)$$

By the distributive property it becomes clear that

$$A = \bigcup_{i=1}^{n} (A \cap B_i)$$

Now as the partitions by definition are disjoint we can determine the probability as the sum of probabilities

$$P(A) = P\left(\bigcup_{i=1}^{n} (A \cap B_i) \right) = \sum_{i=1}^{n} P(A \cap B_i)$$

Rewriting using the definition of conditional probability (as $A \in B_i$ can only occur if B_i has occured) we get

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

□

Theorem 20. *Bayes rule states that:* $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

Proof. From the definition of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Multiplying by $P(B)$ on both sides

$$P(A|B)P(B) = P(A \cap B)$$

Dividing by $P(A)$ on both sides

$$\frac{P(A|B)P(B)}{P(A)} = \frac{P(A \cap B)}{P(A)} = P(B|A)$$

□

1.3.3 Conditional Independence

Definition 5. Two events A and B are conditionally independent given an event C if $P(A \cap B|C) = P(A|C)P(B|C)$.

Proof. From the definition of conditional probability we have

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Conditioning both sides on C it becomes apparent that

$$P(A|B, C) = \frac{P(A \cap B|C)}{P(B|C)}$$

Assuming that A and B are conditionally independent we have

$$P(A|B, C) = \frac{P(A|C)P(B|C)}{P(B|C)} = P(A|C)$$

□

1.3.4 Problems

Problem 1 You purchase a certain product. The manual states that the lifetime, T, of the product defined as the amount of time (in years) the product works properly untilk it breaks down satisfies:

$$P(T \geq t) = e^{-\frac{t}{5}} \text{ for all } t \geq 0$$

For example, the probability that the product lasts more than (or equal to) 2 years is

$$P(T \geq 2) = e^{-\frac{2}{5}} = 0.6703$$

a) I purchase the product and use it for two years without any problems. What is the probability that it breaks down in the 3rd year?

We define A as the probability of the product not breaking down in the first two years

$$P(A) = P(T \geq 2) = e^{-\frac{2}{5}} = 0.6703$$

Similarly we define B as the probability of it breaking down in the 3rd year

$$P(B) = P(2 \leq T \leq 3) = P(T \geq 2) - P(T \geq 3) = e^{-\frac{2}{5}} - e^{-\frac{3}{5}} = 0.1215$$

We now wish to determine the conditional probability $P(B|A)$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

As $B \subset A$ it is clear that $A \cap B = B$

$$P(B|A) = \frac{P(B)}{P(A)} = \frac{0.1215}{0.6703} = 0.1812$$

Problem 2 You toss a fair coin three times

a) What is the probability of three heads, HHH?

As the coin tosses are independent we have that

$$P(H \cap H \cap H) = P(H)P(H)P(H) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

b) What is the probability that you observe exactly one heads?

The probability of observing exactly one heads must be the sum of the probabilities of

$$\begin{aligned} A &= \{H, T, T\} \\ B &= \{T, H, T\} \\ C &= \{T, T, H\} \end{aligned}$$

As the likelihood of any sequence is equal we have that

$$P(HHH) = P(A) = P(B) = P(C)$$

As such

$$P(1H) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

c) Given that you have observed at least one heads, what is the probability that you observe at least two heads?

We wish to determine the probability $P(A|B)$ where A is observing ≥ 2 heads and B is knowing there has already been observed at least one. The favourable events for B are

$$P(B) = P(S \setminus \{T, T, T\}) = 1 - \frac{1}{8} = \frac{7}{8}$$

Whilst the favourable events for A are

$$P(A) = P(\{(H,H,T),(H,T,H),(T,H,H),(H,H,H)\}) = \frac{4}{8} = \frac{1}{2}$$

By the definition of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{2}}{\frac{7}{8}} = \frac{4}{7}$$

Problem 3 For three events, A, B and C, we know that

- A and C are independent.
- B and C are independent.
- A and B are disjoint.
- $P(A \cup C) = \frac{2}{3}$,$P(B \cup C) = \frac{3}{4}$ and $P(A \cup B \cup C) = \frac{11}{12}$.

a) Find $P(A), P(B)$ and $P(C)$.

As these are all probabilities of unions we have that

$$\begin{aligned} P(A \cup C) &= P(A) + P(C) - P(A \cap C) = \frac{2}{3} \\ P(B \cup C) &= P(B) + P(C) - P(B \cap C) = \frac{3}{4} \\ P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap C) - P(B \cap C) = \frac{11}{12} \end{aligned}$$

We add the first 2 equations together and get

$$P(A \cup C) + P(B \cup C) = P(A) + P(B) + 2P(C) - P(A \cap C) - P(B \cap C)$$

Subtracting the third equation from these we then get

$$\begin{aligned} &[P(A) + P(B) + 2P(C) - P(A \cap C) - P(B \cap C)] - [P(A) + P(B) + \\ &P(C) - P(A \cap C) - P(B \cap C)] = P(C) = \frac{2}{3} + \frac{3}{4} - \frac{11}{12} = \frac{6}{12} = \frac{1}{2} \end{aligned}$$

Which we then substitute into the other equations

$$\begin{aligned} P(A) + \frac{1}{2} - P(A) \cdot \frac{1}{2} = \frac{2}{3} \Leftrightarrow P(A) = \frac{1}{3} \\ P(B) + \frac{1}{2} - P(B) \cdot \frac{1}{2} = \frac{3}{4} \Leftrightarrow P(B) = \frac{1}{2} \end{aligned}$$

As such $P(A) = \frac{1}{3}, P(B) = \frac{1}{2}$ and $P(C) = \frac{1}{2}$.

Problem 4 Let $C_1, C_2, \ldots, C_M$ be a partition of the sample space, S, and A, B be two events. Suppose we know that

- A and B are conditionally independent given C_i for all $i \in \{1, 2, \ldots, M\}$.
- B is independent of all C_i's.

a) Prove that A and B are independent.

We have that

$$\begin{aligned} P(A \cap B|C) &= P(A|C)P(B|C) \\ P(B \cap C) &= P(B)P(C) \end{aligned}$$

By the law of total probability

$$\begin{aligned} P(A \cap B) &= \sum_{i=1}^{M} P(A \cap B|C_i)P(C_i) \\ &= \sum_{i=1}^{M} P(A|C_i)P(B|C_i)P(C_i) \text{ } A, B \text{ are conditionally independent} \\ &= \sum_{i=1}^{M} P(A|C_i)P(B)P(C_i) \text{ } B \text{ is independet of } C \\ &= P(B)\sum_{i=1}^{M} P(A|C_i)P(C_i) \\ &= P(B)P(A) \end{aligned}$$

Problem 5 In my town, it's rainy one third of the days. Given that it is rainy, there will be heavy traffic with probability $\frac{1}{2}$, and given that it is not rainy, there will be heavy traffic with probability $\frac{1}{4}$. If it's rainy and there is heavy traffic, I arrive late for work with probability $\frac{1}{2}$. On the other hand, the probability of being late is reduced to $\frac{1}{8}$ if it is not rainy and there is no heavy traffic. In other situations (rainy and no traffic, not rainy and traffic) the probability of being late is $\frac{1}{4}$. You pick a random day:

We define the events, R, T, L corresponding to it being rainy, traffic being heavy and the person being late, based on this a probability tree is made.

a) What is the probability that it's not raining and there is heavy traffic and I am not late?

We read from the probability tree that

$$P(\overline{L}|\overline{R}, T) = \frac{1}{8}$$

b) What is the probability that I am late?

We sum the probabilities that result in being late

$$P(L) = \frac{1}{12} + \frac{1}{24} + \frac{1}{24} + \frac{1}{16} = \frac{11}{48}$$

c) Given that I arrived late at work, what is the probability that it rained that day?

$$P(R|L) = \frac{P(R \cap L)}{P(L)} = \frac{\frac{1}{12} + \frac{1}{24}}{\frac{11}{48}} = \frac{6}{11}$$

Problem 6 A box contains three coins: two regular coins and one fake two-headed coin.

a) You pick a coin at random and toss it. What is the probability that it lands heads up?

The probability of getting heads must be the union of the probabilities of getting heads with the individual coins.

$$\begin{aligned} P(H) &= P(H|C_1)P(C_1) + P(H|C_2)P(C_2) + P(H|C_3)P(C_3) \\ &= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

b) You pick a coin at random and toss it, and get heads, what is the probability that it is the two headed coin?

We apply Bayes rule and have that

$$\begin{aligned} P(C_3|H) &= \frac{P(H|C_3)P(C_3)}{P(H)} \\ &= \frac{1 \cdot \frac{1}{3}}{\frac{2}{3}} \\ &= \frac{1}{2} \end{aligned}$$

Problem 7 A family has two children. We ask the father, "Do you have at least one daughter named Lilia?". He replies, "yes", what is the probability that both children are girls? In other words, we want to find the probability that both children are girls, given that the family has at least one daughter named Lilia. Here you can assume that if a child is a girl, her name will be Lilia with a probability $\alpha \ll 1$ independently from other children's names. If the child is a boy, his name will not be Lilia

The sample space is

$$P(S) = \{(B, B), (B, G), (G, G), (G, B)$$

Given that we know they have at least one daughter named Lilia we have

$$\begin{aligned}
&P(L|BB) = 0 \\
&P(L|GB) = P(L|BG) = \alpha \\
&P(L|GG) = \alpha(1-\alpha) + (1-\alpha)\alpha + \alpha^2 = 2\alpha - 2\alpha^2
\end{aligned}$$

Applying Bayes rule we get that

$$\begin{aligned}
P(GG|L) &= \frac{P(L|GG)P(GG)}{P(L)} \\
&= \frac{(2\alpha - \alpha^2)\frac{1}{4}}{(2\alpha - \alpha^2)\frac{1}{4} + \alpha\frac{1}{4} + \alpha\frac{1}{4} + 0\frac{1}{4}} \\
&= \frac{0.5a - 0.25a^2}{(2\alpha - \alpha^2)\frac{1}{4} + \alpha\frac{1}{2}} \\
&= \frac{\frac{1}{4}\alpha(-\alpha + 2)}{\frac{1}{4}\alpha(-\alpha + 4)} \\
&= \frac{2-\alpha}{4-\alpha} \\
&\approx \frac{1}{2} \text{ as } \alpha \ll 1
\end{aligned}$$

Problem 8 A family has two children. We choose one at random and find out that she is a girl. What is the probability that both children are girls?

$$P(GG) = P(GB) = P(BG) = P(BB)$$

$$P(G_r|GG) = 1$$

$$P(G_r|BG) = P(G_r|GB) = \frac{1}{2}P(G_r|BB) = 0$$

Using Bayes rule we determine $P(GG|G_r)$

$$\begin{aligned} P(GG|G_r) &= \frac{P(G_r|GG)P(GG)}{P(G_r)} \\ &= \frac{1 \cdot \frac{1}{4}}{\frac{1}{4}(1+\frac{1}{2}+\frac{1}{2})} \\ &= \frac{1}{2} \end{aligned}$$

Problem 9 (NOT SOLVED AS I DONT GET THE EXPLANATION)
I toss an unfair coin where $P(H) = p$ repeatedly. The game ends the first time two consecutive heads or tails is observed. I win if two heads are observed and lose if two tails are observed.

The probability of tails must be the complement of heads

$$P(T) = 1 - p$$

The probability of winning is given by

$$\begin{aligned} P(W) = P(\{(HH), (HTHH), (HTHTHH), \ldots\}) + \\ P(\{(THH), (THTHH), (THTHTHH, \ldots)\}) \end{aligned}$$

In regards to the above statement we then have that

$$\begin{aligned} P(W) = P(H)^2 + P(H)^3P(T) + P(H)^4P(T)^2 + \ldots + \\ P(T)^1P(H)^2 + P(T)^2P(H)^3 + P(T)^3 + P(H)^4 \end{aligned}$$

1.4 Chapter resume (TO WRITE)

2 Chapter 2

2.1 Counting

For a finite sample space with equal probabilities we recall that

$$P(A) = \frac{|A|}{|S|}$$

As such determining the probability is a counting problem, determining the cardinality of A and S.

Definition 6 (Multiplication principle)**.** Suppose that we perform r experiments such that the k'th experiment has n_k possible outcomes for $k = \{1, 2, \ldots, r\}$. Then there are a total of $n_1 \times n_2 \times \ldots \times n_r$ possible outcomes for the sequence of r experiments.

For counting problems, some general terminology is relevant:

- Sampling: Choosing a random element from a set.
- With replacement, the sampled element is returned to the set and can therefore be drawn multiple times with repeated sampling.
- Without replacement, the sampled element is not returned to the set and can therefore not be drawn multiple times.
- Ordered means that the order at which elements are written matters, $\{a_1, a_2, a_3\} \neq \{a_3, a_1, a_2\}$.
- Unordered means that the order at which elements are written does not matter, $\{a_1, a_2, a_3\} = \{a_3, a_1, a_2\}$.

2.1.1 Ordered sampling with replacement

Suppose we have a set consisting of n elements and we wish to draw k samples from the set, for example, say $A = \{1, 2, 3\}$ where we wish to sample $k = 2$, we then get 9 different possibilities

$$(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)$$

From this its clear that we create a list consisting of k-valued elements where each position has n options for values

$$\begin{array}{cccc} a_1 & a_2 & \ldots & a_k \\ \uparrow & \uparrow & & \uparrow \\ n & n & & n \end{array}$$

Meaning that we can determine the total amount of possibilities as

$$n \times n \times \ldots \times n = n^k$$

2.1.2 Ordered sampling without replacement (Permutations)

As opposed to the previous circumstance, a element is now removed every time we draw, resulting in there being one less options every time we move to the next position, as such, using the same example with A and $k = 2$ we get

$$(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)$$

Here we create a list of k-valued elements where each position has one less option than the previous

$$\begin{array}{cccc} (a_1 & a_2 & \ldots & a_k) \\ \uparrow & \uparrow & & \uparrow \\ n & n-1 & & n-k+1 \end{array}$$

Which is called a k-permutation of the elements in the set, the following notation is used to show the number of k-permutations of an n-element set

$$P_k^n = n \times (n-1) \times \ldots \times (n-k+1)$$

A special case can also occur here when $n < k$, as there then wont be enough options for every position and there therefore are no possible lists. Another special case is an n-permutation where $k = n$ which results in the sequence

$$\begin{aligned} P_n^n &= n \times (n-1) \times (n-2) \times \ldots \times (n-n+1) \\ &= n \times (n-1) \times (n-2) \times \ldots \times 1 \\ &= n! \end{aligned}$$

As such the factorial operator simply denotes the total number of permutations of an n element set, aka the total number of ways you can order n different objects. By definition $0! = 1$, using this we rewrite the formula for P_k^n

Theorem 21. *The amount of k-permutations of an n-element set is given by $P_k^n = \frac{n!}{(n-k)!}$.*

Proof. From the original expression for P_k^n we have that

$$P_k^n = n \times (n-1) \times (n-2) \times \ldots \times (n-k+1)$$

By multiplying by $\frac{(n-k)!}{(n-k)!}$ we get

$$\begin{aligned} &= n \times (n-1) \times (n-2) \times \ldots \times (n-k+1) \cdot \frac{(n-k)!}{(n-k)!} \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

As multiplying by $(n-k)!$ in the numerator results in the sequence “finishing” and being equal to $n!$. □

2.1.3 Unordered sampling without replacement (Combinations)

We now wish to determine the amount of possible lists when we sample k elements from an n-element set. This means that we want to determine the amount of possible k-element subsets of the n-element set. Using the same example as before with A and $k = 2$ we get 3 different combinations

$$(1, 2), (1, 3), (2, 3)$$

We show the number of k-element subsets of A as

$$\binom{n}{k}$$

Which is read as "n choose k", to determine the value of this we compare with P_k^n as the only difference between the two is ordering. This is because for any k-element subset of an n-element set, we can order the elements in $k!$ different ways, as such

$$P_k^n = \binom{n}{k} \times k!$$

Rewriting using our previously established formula for P_k^n and dividing by $k!$ we get

$$\binom{n}{k} = \frac{n!}{k!(n-k)}, \text{ for } 0 \le k \le n$$

This term is used extensively in the binomial theorem, which states that

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Theorem 22. *For any non-negative integers n, k it follows that $\binom{n}{k} = \binom{n}{n-k}$.*

Proof. Assume we wish to determine the amount of possible sequences consisting of k A's and j B's, as such we have $n = j + k$ positions to fill with either A or B, from these positions we need to choose j for A's and whatever is left is filled with B's, as such the amount of ways is

$$\binom{n}{j}$$

If we instead observe this from the point of B's, it is clear that the amount of ways would then be given by

$$\binom{n}{k}$$

As these must be equivalent we have that

$$\binom{n}{j} = \binom{n}{k}$$

From the initial determination of n we get that

$$n = j + k \implies j = n - k \vee k = n - j$$

As such

$$\binom{n}{j} = \binom{n}{n-k} = \binom{n}{k} = \binom{n}{n-j}$$

□

Theorem 23. *For any non-negative integers k, n it follows that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.*

Proof. From the Binomial theorem we know that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

We let $a = b = 1$ and as such get

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

□

Theorem 24. *For non-negative integers $0 \leq k \leq n$ it follows that $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$.*

Proof. We define an arbitrary set, A with $n+1$ elements

$$A = \{a_1, a_2, \ldots, a_n, a_{n+1}\}$$

From this set we wish to choose a $k+1$ element subset, call it B, by combinations we know this is equal to

$$\binom{n+1}{k+1}$$

B can also be constructed as the union of two subsets of B that are defined by either containing- or not containing a_{n+1}

$$B = B_1 \cup B_2, \text{ where } a_{n+1} \notin B_1, a_{n+1} \in B_2, B_1 \cap B_2 = \emptyset$$

To define B_1 we need to choose $k+1$ elements from the set $A \setminus a_{n+1}$ which is equal to

$$\binom{n}{k+1}$$

To complete the set we then need to choose k elements from A which can be done in

$$\binom{n+1}{k}$$

ways. As such the sum of the two must be equal to the initial expression resulting in

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k+1}$$

$\square$

Theorem 25. *Vandermonde's identity states that* $\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$

Proof. We construct a set A with $m+n$ elements, as such

$$A = \{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n\}$$

Determining the number of k-element subsets of A is equal to

$$\binom{m+n}{k}$$

This can also be done by choosing i elements from $\{a_1, a_2, \ldots, a_n$ first, and then $k-i$ elements from $\{b_1, b_2, \ldots, b_n\}$, which then can be done in

$$\binom{m}{i}\binom{n}{k-i}$$

ways. But as i can be any number from $0 \to k$ it is necessary to sum all the possible options whereby we write

$$\binom{m+n}{k} = \sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}$$

□

An important class of random experiments are Bernoulli trials, a random experiemnt where there are two possible outcomes, succes and failure (which can be extended to any experiment with 2 outcomes as we can arbitrarily define success and failure).

In Bernoulli trials the probability of success if usually denoted by p and the probability of failure as its complement $q = 1 - p$. If we perform n independent Bernoulli trials and count the number of successes, it is called a binomial experiment, for example a coin toss where we define success as heads and count the number of heads.

Theorem 26. *The binomial formula is given by* $P(k) = \binom{n}{k} p^k (1-p)^{n-k}$

Proof. Imagine we toss a coin with $P(H) = p$ and $P(T) = 1 - p$ n times, we define C as the event of observing k heads (and $n-k$ tails), the probability of observing k heads will be given by

$$P(k) = |C| p^k (1-p)^{n-k}$$

To determine $|C|$, we realise that we can see the event as ordered sampling without replacement, as such we have that

$$|C| = \binom{n}{k}$$

Which we insert into the previous expression and get

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

□

2.1.4 Unordered sampling without replacement

As opposed to the previous section we're now working with replacement, as such we again use the example of A with $k = 2$ and get 6 possibilities given by

$$(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)$$

One way to represent this sample is to list them as n-element vectors where each position corresponds to a number, eg.

$$\begin{aligned}
(a,b) &\to (x_1,x_2,x_3) = (n_1,n_2,n_3)\\
(1,1) &\to (x_1,x_2,x_3) = (2,0,0)\\
(1,2) &\to (x_1,x_2,x_3) = (1,1,0)\\
(1,3) &\to (x_1,x_2,x_3) = (1,0,1)
\end{aligned}$$

Constructing these vectors a pattern emerges, that is $\sum_{i=1}^{3} x_i = 2$, as such we can determine the amount of possibilities as the amount of integer solutions to $x_1 + x_2 + x_3 = 2$.

Theorem 27. *The number of distinct solutions to the equation* $x_1+x_2+\ldots+x_n = k$ *where* $x_n \in \{0,1,2,3,\ldots\}$ *is equal to* $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$.

Proof. We first define a mapping where an integer $x_n \geq 0$ is replaced with x_n vertical lines. Suppose we now have a solution to the equation where $k = 6$, for example

$$3+0+2+1 \Leftrightarrow |||++||+|$$

Here we realise that the equation can be represented by k vertical lines and $n-1$ plus signs, as such we can use combinations to determine the amount of distinct sequences we can create using k vertical lines and $n-1$ plus signs, as such

$$n_{\text{solutions}} = \binom{k+n-1}{k} = \binom{k+n-1}{n-1}$$

□

2.1.5 Problems

Problem 1 Let A, B be two finite sets with $|A| = m, |B| = n$, how many distinct functions (mappings) can you define from set A to set B, $A \xrightarrow{f} B$?

We let

$$A = \{a_1, a_2, \dots, a_m\}$$
$$B = \{b_1, b_2, \dots, b_n\}$$

For each a_m we have n options as

$$f(a_m) \in B$$

As such we by the multiplication principle have that there exists

$$n^m$$

distinct mappings.

Problem 2 A function is said to be one-to-one if for all $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$. Equivalently we can say a function is one-to-one whenever $f(x_1) = f(x_2) \implies x_1 = x_2$. Let A, B be two finite sets with $|A| = m, |B| = n$, how many distinct one-to-one functions (mappings) can you define from set A to set B, $A \xrightarrow{f} B$.

Again we define

$$A = \{a_1, a_2, \dots, a_m\}$$
$$B = \{b_1, b_2, \dots, b_n\}$$

As opposed to the previous problem we here lose a possible value every time we progress once, this means that a_1 has n options, a_2 has $n-1$ options, etc. as such we're making a permutation resulting in the amount of options that exists being given by

$$P_m^n = \frac{n!}{(n-m)!}$$

Problem 3 An urn contains 30 red balls and 70 green balls. What is the probability of getting exactly k red balls in a sample of size 20 if the sampling is done with replacement? Assume $0 \leq k \leq 20$.

The probability of picking a red ball is

$$P(R) = \frac{30}{100} = 0.3$$

And as such the chance of not drawing one is

$$P(G) = 1 - 0.3 = 0.7$$

As we're working with replacement, this stays constant throughout all samples, as such we can determine the probability of drawing k balls using the binomial formula

$$P(k) = \binom{20}{k} 0.3^k (0.7)^{20-k}$$

Problem 4 An urn contains 30 red balls and 70 green balls. What is the probability of getting exactly k red balls in a sample of size 20 if the sampling is done without replacement?

We let A be the event of getting k red balls, to determine $P(A)$ we need to find

$$P(A) = \frac{|A|}{|S|}$$

Determining $|S|$ is trivial

$$|S| = \binom{100}{20}$$

We now define m as $|R|$ as n as $|G|$, whilst $k = |R|$. As such we make use of

$$|A| = \binom{m}{k}\binom{n}{20-k}$$

Inserting our known information we then get that

$$|A| = \binom{30}{k}\binom{70}{20-k}$$

By using the initial expression we then get

$$P(A) = \frac{\binom{30}{k}\binom{70}{20-k}}{\binom{100}{20}}$$

Problem 5 Assume there are k people in a room and we know that

- $P(k = 5) = \frac{1}{4}$
- $P(k = 10) = \frac{1}{4}$

- $P(k = 15) = \frac{1}{2}$

a) What is the probability that at least two of them have been born in the same month? Assume that all months are equally likely.

We let A be the event that two people are born in the same month, as we the phrase at least suggests determining the complement might be easier we wish to find

$$P(A) = 1 - \frac{|\overline{A}|}{|S|}$$

Determining $|S|$ is trivial by the multiplication principle

$$|S| = 12^k$$

To determine the amount of possible we can use the same principle, except repetition isnt allowed, as such

$$|\overline{A}| = P_k^{12}$$

Whereby it becomes clear that

$$P(A_k) = 1 - \frac{P_k^{12}}{12^k}$$

By the law of total probability we then have that

$$\begin{aligned} P(A) &= P(A_5) \cdot \frac{1}{4} + P(A_{10}) \cdot \frac{1}{4} + P(A_{15})\frac{1}{2} \\ &= \frac{1}{4}\left(1 - \frac{P_5^{12}}{12^5}\right) + \frac{1}{4}\left(1 - \frac{P_{10}^{12}}{12^{10}}\right) + \frac{1}{2} \end{aligned}$$

As $P(A_{15})$ will always be $= 1$ since $k > 12$.

b) Given that we already know there are at least two people that celebrate their birthday in the same month, what is the probability that $k = 10$?

We are asked to determine the conditional probability

$$P(k = 10|A)$$

By Bayes law we have that

$$\begin{aligned}
P(k=10|A) &= \frac{P(A|k=10)P(k=10)}{P(A)} \\
&= \frac{\left(1-\frac{P_{10}^{12}}{10^{12}}\right)\cdot\frac{1}{4}}{\frac{1}{4}\left(1-\frac{P_{5}^{12}}{12^{5}}\right)+\frac{1}{4}\left(1-\frac{P_{10}^{12}}{12^{10}}\right)+\frac{1}{2}} \\
&= \frac{1-\frac{P_{10}^{12}}{10^{12}}}{\left(1-\frac{P_{5}^{12}}{12^{5}}\right)+\left(1-\frac{P_{10}^{12}}{12^{10}}\right)+2}
\end{aligned}$$

Problem 6 How many distinct solutions does the following equation have?

$$x_1 + x_2 +_3 +x_4 = 100, \text{ such that } x_1 \in \{1,2,3,\ldots\},$$
$$x_2 \in \{2,3,4,\ldots\}, x_3, x_4 \in \{0,1,2,3,\ldots\}$$

We know that the number of solutions without the given restrictions is given by

$$\binom{k+n-1}{k} = \binom{k+n-1}{n-1}$$

We wish to rewrite the given equation such that these restrictions are no longer present, as such we rewrite

$$\begin{aligned}
y_1 &= x_1 - 1 \\
y_2 &= x_2 - 2
\end{aligned}$$

As this gives them the same domain, the equation is now

$$y_1 + 1 + y_2 + 2 + x_3 + x_4 = 100, \text{ where } y_1, y_2, x_3, x_4 \in \{0,1,2,3,\ldots\}$$

Rewriting again this gives

$$y_1 + y_2 + x_3 + x_4 = 97$$

Which has the solution

$$\binom{97+4-1}{97} = \binom{100}{3}$$

Problem 7 N guests arrive at a party. Each person is wearing a hat. We collect all hats and then randomly redistribute the hats, giving each person one of the N hats randomly.What is the probability that at least one person receives his/her own hat?

2.2 Chapter resume

Ordered sampling with replacement

$$n^k$$

Ordered sampling without replacement

$$P_k^n = \frac{n!}{(n-k)!}$$

Unordered sampling without replacement

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Unorderered sampling with replacement

$$\binom{k+n-1}{k} = \binom{k+n-1}{n-1}$$

3 Chapter problems

3.1 Chapter 1

Problem 1 Suppose the universal set is defined as $S = \{x \in \mathbb{N} | 1 \geq x \geq 10\}$, $A = \{1, 2, 3\}$, $B = \{x \in S | 2 \geq x \geq 7\}$ and $C = \{7, 8, 9, 10\}$.

a) Find $A \cup B$

$$A \cup B = \{1, 2, 3\} \cup \{2, 3, 4, 5, 6, 7\} = \{1, 2, 3, 4, 5, 6, 7\}$$

b) Find $(A \cup C) \setminus B$

$$(A \cup C) \setminus B = \{1, 2, 3, 7, 8, 9, 10\} \setminus \{2, 3, 4, 5, 6, 7\} = \{1, 8, 9, 10\}$$

c) Find $\overline{A} \cup (B \setminus C)$

$$\overline{A} \cup (B \setminus C) = \{4, 5, 6, 7, 8, 9, 10\} \cup \{2, 3, 4, 5, 6\} = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

d) Do A, B and C form a partition of S?

No as $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$.

Problem 2 When working with real numbers, our universal set is $\mathbb{R}$. Find each of the following sets a) $[6, 8] \cup [2, 7[$

$$[2, 8]$$

b) $[6, 8] \cap [2, 7[$

$$[6, 7[$$

c) $\overline{[0, 1]}$

$$]-\infty, 0[\cup]1, \infty[$$

d) $[6, 8] \setminus]2, 7[$

$$[7, 8]$$

Problem 3 For each of the followpwing Venn diagrams, write the sets denoted by the shaded area

a)

$$A \cup B \setminus A \cap B$$

b)

$$B \setminus C$$

c)

$$A \cap B + A \cap C$$

d)

$$C \cup (A \cap B) \setminus ((C \cap A) \cup (C \cap B))$$

Problem 4 A coin is tossed twice. Let $S = \{H, T\} \times \{H, T\}$. Write the following sets by listing their elements:

a) First toss is heads

$$A = \{(H, T), (H, H)\}$$

b) At least one tails

$$B = \{(T, T), (T, H), (H, T)\}$$

c) Two tosses are different

$$C = \{(H, T), (T, H)\}$$

Problem 5 Let $A = \{1, 2, \ldots, 100\}$. For any $i \in \mathbb{N}$, define A_i as the set of numbers in A that are divisible by i.

a)

$$\begin{aligned}
|A_1| &= \left\lfloor \frac{|S|}{1} \right\rfloor = \left\lfloor \frac{100}{1} \right\rfloor = 100 \\
|A_2| &= \left\lfloor \frac{|S|}{2} \right\rfloor = \left\lfloor \frac{100}{2} \right\rfloor = 50 \\
|A_3| &= \left\lfloor \frac{|S|}{3} \right\rfloor = \left\lfloor \frac{100}{3} \right\rfloor = 33 \\
|A_4| &= \left\lfloor \frac{|S|}{4} \right\rfloor = \left\lfloor \frac{100}{4} \right\rfloor = 25 \\
|A_5| &= \left\lfloor \frac{|S|}{5} \right\rfloor = \left\lfloor \frac{100}{5} \right\rfloor = 20
\end{aligned}$$

b)

By the distributive property we have that

$$A_2 \cap A_3 \cap A_5 = (A_2 \cap A_3) \cap A_5$$

As the intersection of A_2 and A_3 must be the even factors of 3 we get

$$A_2 \cap A_3 = \{x \in A | x = 6n, n \in \mathbb{N}\}$$

The cardinality of this must be half of the original as every other value is valid

$$|A_2 \cap A_3| = \left\lfloor \frac{33}{2} \right\rfloor = 16$$

As only every fifth factor of 6 is a factor of 5 we get that

$$A_2 \cap A_3 \cap A_5 = \{x \in A | x = 30n, n \in \mathbb{N}\}$$

As this is every fifth value of the previous intersection, the cardinality must be one fifth of the previous

$$|A_2 \cap A_3 \cap A_5| = \left\lfloor \frac{16}{5} \right\rfloor = 3$$

Problem 6 As A_1, A_2, A_3 form a partition of the universal set, the cardinality of B must be equal to the sum of the cardinalities in the individual partitions.

$$|B| = \sum_{i=1}^{3} |B \cap A_i| = 10 + 20 + 15 = 45$$

Problem 7 a) As the numbers can be listed in one-to-one correspondance with the natural numbers the set is countable.

b) As the set is made up of the union of 2 countable sets we have that

$$B = \bigcup_{i=\mathbb{Q}} \bigcup_{j=\mathbb{Q}} \{a_i + b_j\sqrt{2}\}$$

As such it must be countable as its constituents are.

c) As the set is a subset of an uncountable set (real numbers), it is not countable.

Problem 8 We take the limit of the upper bound of the interval as $n \to \infty$

$$\lim_{n \to \infty} \frac{n-1}{n} = 1$$

As this limit is the upper bound of the interval, corresponding to the maximal interval included in the union of sets we get that

$$A = \bigcup_{n=1}^{\infty} A_n = [0; 1[$$

Problem 9 Opposed to the previous problem the smallest set will here define the set as the intersection is limited to the smallest component. We take the limit as $n \to \infty$ as the value is inverse proportional to n

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

As such

$$A = \bigcap_{n=1}^{\infty} A_n = \{0\}$$

Problem 10 a)

b)

Problem 11 As the set is given as an interval it is clear that

$$[0,1[\subset \mathbb{R}$$

As subsets of uncountable sets are uncountable, it becomes clear that the range is uncountable.

Problem 12 a)

Reading the function it becomes clear that the domain of the function is given by

$$\{H,T\}^3$$

While the co-domain is given by

$$\mathbb{N} \cup \{0\}$$

b)

As the function is limited by the amount of heads that can appear in the sequence it is clear that

$$\mathrm{Range}(f) = \{0,1,2,3\}$$

c)

Knowing that $x = 2$ tells us that 2 heads are present in the sequence, and as such 1 tails must also be present, as such the possible events are

$$\{(H,H,T),(H,T,H),(T,H,H)\}$$

Problem 13 a)

We logically assume that the events are disjoint as two teams cant win, as such it is clear that

$$0.5 + P(b) + 0.25 = 1 \Leftrightarrow P(b) = 1 - 0.5 - 0.25 = 0.25$$

b) As the events are disjoint we determine the probability as

$$P(b \cup d) = 0.25 + 0.25 = 0.5$$

Problem 14 a)

By inclusion-exclusion principle we have that

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.4 + 0.7 - 0.9 = 0.2$$

b)

We have that

$$P(\overline{A} \cap B) = P(B \setminus A)$$

Expanding this expression we get that

$$P(\overline{A} \cap B) = P(B) - P(A \cap B) = 0.7 - 0.2 = 0.5$$

c)

Expanding the expression we again get

$$P(A \setminus B) = P(A) - P(A \cap B) = 0.4 - 0.2 = 0.2$$

d)

Expanding the expression we get

$$P(\overline{A} \setminus B) = P(\overline{A}) - P(\overline{A} \cap B) = (1 - 0.4) - 0.5 = 0.1$$

e)

By the inclusion-exclusion principle we have

$$P(\overline{A} \cup B) = P(\overline{A}) + P(B) - P(\overline{A} \cap B) = (1 - 0.4) + 0.7 - 0.5 = 0.8$$

f) By the distributive law

$$A \cap (B \cup \overline{A}) = (A \cap B) \cup (A \cap \overline{A})$$

As

$$A \cap \overline{A} = \emptyset$$

by definition. We get that

$$P(A \cap (B \cup \overline{A})) = P((A \cap B) \cup \emptyset) = 0.2$$

Problem 15 a)

We assume the rolls are independent, as such

$$P(X_2 = 4) = \frac{1}{6}$$

b)

The sample space is given by

$$S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),\\ (2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),\\ (4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),(5,4),(5,5),\\ (5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

The outcomes that satisfy the event are

$$\{X_1 + X_2 = 7\} = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

As such

$$P(X_1 + X_2 = 7) = \frac{6}{36} = \frac{1}{6}$$

c)

As they are independent we determine the probability as

$$P(X_1 \neq 2 \cap X_2 \geq 4) = P(X_1 \neq 2)P(X_2 \geq 4) = \frac{5}{6} \cdot \frac{3}{6} = \frac{15}{36} = \frac{5}{12}$$

Problem 16 a)

As the sum of the individual probabilities must equate to 1 we wish to determine a c that satisfies the equation

$$\sum_{k=1}^{\infty} P(k) = \sum_{k=1}^{\infty} \frac{c}{3^k} = 1$$

We apply the formula for a geometric series

$$\sum_{k=0}^{\infty} cr^k = \frac{a}{1-r} \text{ for } |r| < 1$$

As this starts from 0 we subtract c as $P(\{0\}]) = \frac{c}{k^0} = c$

$$\begin{aligned} 1 &= -c + \sum_{k=0}^{\infty} c \left(\frac{1}{3}\right)^k \\ &= -c + \frac{c}{1 - \frac{1}{3}} \end{aligned}$$

By isolation for c we then get that

$$\frac{2}{3} = -\frac{2}{3}c + c \Leftrightarrow \frac{2}{3} = \frac{1}{3}c \Leftrightarrow 2 = c$$

b)

The set is given by the union of the 3, as such

$$\begin{aligned} P(\{2,4,6\}) &= P(\{2\} \cup \{4\} \cup \{6\}) \\ &= P(\{2\}) + P(\{4\}) + P(\{6\}) \\ &= \frac{2}{9} + \frac{2}{81} + \frac{2}{729} \\ &= \frac{182}{729} \end{aligned}$$

c) This must equate to the complement of $P(\{1,2\})$, as such

$$\begin{aligned} P(\{3,4,5,\ldots\}) &= 1 - P(\{1,2\}) \\ &= 1 - \left(\frac{2}{3} + \frac{2}{9}\right) \\ &= 1 - \frac{8}{9} \\ &= \frac{1}{9} \end{aligned}$$

Problem 17 We have that

$$\begin{aligned} P(A) &= P(B) \\ P(C) &= 2P(D) \\ P(A \cup C) &= P(A) + P(C) = 0.6 \end{aligned}$$

We rewrite all terms as functions of $P(A)$, these must equate to 1 as a team has to win, making it the sample space

$$\begin{aligned} 1 &= P(A) + P(A) + (0.6 - P(A)) + \frac{0.6 - P(A)}{2} \\ &= 0.5P(A) + 0.9 \end{aligned}$$

Isolating for $P(A)$ we get

$$P(A) = \frac{1-0.9}{0.5} = 0.2$$

From this we can determine the rest of the probabilities using the requirements set earlier

$$\begin{aligned} P(A) &= P(B) = 0.2 \\ P(C) &= 0.6 - P(A) = 0.4 \\ P(D) &= 0.5P(C) = 0.2 \\ \sum P &= 0.2 + 0.2 + 0.4 + 0.2 = 1 \end{aligned}$$

Problem 18 a)

We insert $t = 1$

$$P(T \leq 1) = \frac{1}{16}1^2 = \frac{1}{16}$$

b)

This must equate to the complement of it taking 2 hours, as such

$$P(2 > t) = 1 - \left(\frac{1}{16}2^2\right) = \frac{3}{4}$$

c)

This equates to the difference between the probability of the job taking more than 1 hour and more than 3 hours

$$P(1 \leq T \leq 3) = P(T \leq 1) - P(T \leq 3) = \frac{1}{16}3^2 - \frac{1}{16}1^2 = \frac{1}{2}$$

Problem 19 The problem has the form of the quadratic equation

$$ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For this to have real solutions it is necessary that

$$b^2 - 4ac \geq 0 \vee a \neq 0$$

Translating this to the given equation we wish to determine values $A, B \in [0, 1]$ that satisfy

$$1^2 - 4AB \geq 0 \vee A \neq 0$$

Seeing this as a function based on the given figure, we write that

$$1^2 - 4xy \geq 0$$

Isolating for y it becomes apparent that

$$\begin{aligned} 1^2 - 4xy &\geq 0 \\ 1^2 &\geq 4xy \\ \frac{1}{4x} &\geq y \end{aligned}$$

As the maximal y-value is 1 we determine the intersection

$$1 = \frac{1}{4x} \Leftrightarrow x = \frac{1}{4}$$

Now we can determine the probability as the proportion of the area that fits under this curve in the interval for which it is inside the unit square

$$\begin{aligned} P(1 - 4AB \geq 0) &= \frac{1}{4} + \frac{1}{4}\int_{\frac{1}{4}}^{1} \frac{1}{4x} \\ &= \frac{1}{4} + \frac{1}{4}\left[\ln x\right]_{\frac{1}{4}}^{1} \\ &= \frac{1}{4} + \frac{1}{4}\left(\ln 1 - \ln\frac{1}{4}\right) \\ &\approx \frac{3}{5} \end{aligned}$$

Problem 20 a)

As every set is a subset of the next, it is clear that A_n will be the largest or equal to the largest set in the sequence, as such

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P(A_n)$$

As all the sets are subsets of A_n and therefore their union will create A_n.

b)

Here it is the opposite where A_n will be the smallest or equal to the smallest set in the sequence as such

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P(A_n)$$

As the intersection will always be limited by the smallest set in the sequence.

Problem 21 DO LATER

Problem 22 Let A be the event that the customer purchased a cup of coffee and B the event they purchased a piece of cake, then

$$\begin{aligned} P(A) &= 0.7 \\ P(B) &= 0.4 \\ P(A \cap B) &= 0.2 \end{aligned}$$

By conditional probability we have that

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.2}{0.4} \\ &= 0.5 \end{aligned}$$

Problem 23 a)

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.2}{0.35} \\ &\approx 0.57 \end{aligned}$$

b)

$$\begin{aligned} P(C|B) &= \frac{P(B \cap C)}{P(B)} \\ &= \frac{0.15}{0.35} \\ &\approx 0.43 \end{aligned}$$

c)

$$\begin{aligned} P(B|A \cup C) &= \frac{P(B \cap (A \cup C))}{P(A \cup C)} \\ &= \frac{0.25}{0.7} \\ &\approx 0.36 \end{aligned}$$

d)

$$\begin{aligned}P(B|A\cap C) &= \frac{P(B\cap(A\cap C))}{P(A\cap C)}\\ &= \frac{0.1}{0.2}\\ &= 0.5\end{aligned}$$

Problem 24 As this is an interval with infinite points, we determine the proportion of the interval that is spanned by the interval were observing, as such

$$P(A \le X \le B) = \frac{B-A}{10}$$

a)

$$P(2 \le X \le 5) = \frac{5-2}{10} = 0.3$$

b)

$$P(X \le 2|X \le 5) = \frac{2}{5} = 0.4$$

c)

$$\begin{aligned}P(3 \le X \le 8|X \ge 4) &= \frac{P(3 \le X \le 8 \cap X \ge 4)}{X \ge 4}\\ &= \frac{P(4 \le X \le 8)}{0.6}\\ &= \frac{\frac{8-4}{10}}{0.6}\\ &= \frac{2}{3}\end{aligned}$$

Problem 25 We define the event A as getting an A in the course and B as living on campus

$$\begin{aligned}P(A) &= \frac{120}{600} = 0.2\\ P(B) &= \frac{200}{600} = \frac{1}{3}\\ P(A|\overline{B}) &= \frac{80}{400} = 0.2P(A|B) & &= \frac{40}{200} = 0.2\end{aligned}$$

For A and B to be independent we would expect

$$P(A|B) = P(A)$$

Which is true, as such, the 2 events are independent.

Problem 26 Define N_1, N_6 as the number of times out of n that a 1 or 6 is rolled, and let X_i be the i'th roll, then

$$P(N_1 \geq 1 \cap N_6 \geq 1) = 1 - P(\overline{N_1 \geq 1 \cap N_6 \geq 1})$$

By De Morgans Law we get

$$\begin{aligned}
&= 1 - P(N_1 = 0 \cap N_6 = 0) \\
&= 1 - (P(X_1 \neq 1, X_2 \neq 1, \ldots, X_n \neq 1) + P(X_1 \neq 6, X_2 \neq 6, \ldots, X_n \neq 6) \\
&- P((X_1 \neq 1, X_2 \neq 1, \ldots, X_n \neq 1) \cap (X_1 \neq 1, X_2 \neq 6, \ldots, X_n \neq 6)))
\end{aligned}$$

As the final term is equivalent to

$$\bigcap_{i=1}^{n} P(X_i \neq 1 \cap X_i \neq 6)$$

We get that

$$\begin{aligned}
&= 1 - \left(2\left(\frac{5}{6}\right)^n - \left(\frac{4}{6}\right)^n\right) \\
&= 1 - \frac{2 \cdot 5^n - 4^n}{6^n}
\end{aligned}$$

Problem 27 a)

Tree is finished on paper.

b)

We sum the probabilities that result in an error

$$\begin{aligned}
P(E) &= P(E|G) + P(E\overline{G}) \\
&= 0.08 + 0.06 \\
&= 0.14
\end{aligned}$$

c)

From the definition of conditional probability we have

$$\begin{aligned}
P(G|\overline{E}) &= \frac{P(G \cap \overline{E})}{P(\overline{E})} \\
&= \frac{0.72}{1 - 0.14} \\
&= \frac{0.72}{0.86} \\
&\approx 0.84
\end{aligned}$$

Problem 28 We define D as the unit being defective, the probability of picking a defective unit on the first draw is

$$P(D_1) = \frac{5}{100}$$

Assuming that one wasnt defective the next draw has

$$P(D_2|\overline{D_1}) = \frac{5}{99}$$

And the 3rd

$$P(D_3|\overline{D_2}, \overline{D_1}) = \frac{5}{98}$$

We wish to determine the probaility of the event

$$P(D=1) = P(\{(D, \overline{D}, \overline{D}), (\overline{D}, D, \overline{D}), (\overline{D}, \overline{D}, D)\})$$

As these are disjoint we determine them as the sum of their individual probailities

$$\begin{aligned}
P(D=1) &= P(\{(D, \overline{D}, \overline{D})\}) + P(\{\overline{D}, D, \overline{D}\}) + P(\{\overline{D}, \overline{D}, D)\}) \\
&= \frac{5}{100} \cdot \frac{95}{99} \cdot \frac{94}{98} + \frac{95}{100} \cdot \frac{5}{99} \cdot \frac{94}{98} + \frac{95}{100} \cdot \frac{94}{99} \cdot \frac{5}{98} \\
&\approx 0.14
\end{aligned}$$

Problem 29 a)

As the components are connected in series, the probability of the system being functional is equal to the complement of one or more of the components not working. Let F be the event that the system is functional and $P(C_i)$ be the event that the i'th component is functional.

$$P(F) = P(C_1 \cap C_2 \cap C_3) = P(C_1)P(C_2)P(C_3)$$

b)

As they're connected in parallel they must all fail for the circuit to be non-functional, as such the probability must be given be the union of them

$$P(F) = P(C_1 \cup C_2 \cup C_3)$$

By the inclusion-exclusion principle

$$\begin{aligned}
P(F) = P(C_1) + P(C_2) + P(C_3) - P(C_1)P(C_2) - P(C_1)P(C_3) - \\
P(C_2)P(C_3) + P(C_1)P(C_2)P(C_3)
\end{aligned}$$

c)

The probability that this circuit works must be given by the probability that either C_1 and C_3 or C_2 and C_3 are functional.

$$\begin{aligned} P(F) &= P((C_1 \cap C_3) \cup (C_2 \cap C_3)) \\ &= P(C_1 \cap C_3) + P(C_2 \cap C_3) - P((C_1 \cap C_3) \cap (C_2 \cap C_3)) \\ &= P(C_1)P(C_3) + P(C_2)P(C_3) - P(C_1 \cap C_2 \cap C_3) \end{aligned}$$

d)

The probability that this circuit works is given by the probability that either C_3 or both C_1 and C_2 work

$$\begin{aligned} P(F) &= P(C_3 \cup (C_1 \cap C_2)) \\ &= P(C_3) + P(C_1 \cap C_2) \\ &= P(C_3) + P(C_1)P(C_2) - P(C_1)P(C_2)P(C_3) \end{aligned}$$

e)

The probability that this circuit works is determined by whether C_1, C_2 and C_5 or C_3, C_4 and C_5 work, therefore

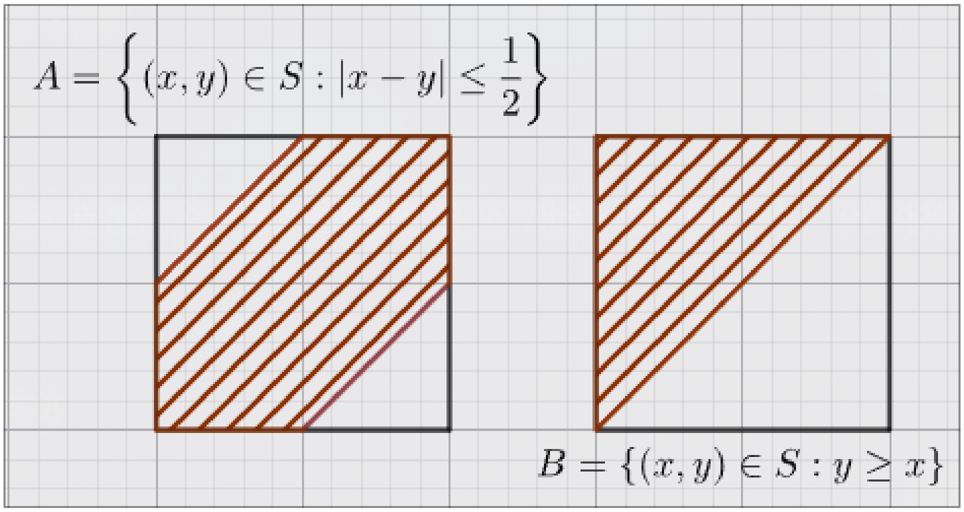
$$\begin{aligned} P(F) &= P(C_1 \cap C_2 \cap C_5) + P(C_3 \cap C_4 \cap C_5) - P(C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5) \\ &= P(C_1)P(C_2)P(C_5) + P(C_3)P(C_4)P(C_5) - \prod_{i=1}^{5} P(C_i) \end{aligned}$$

Problem 30 a)

Rewriting the set as a piecewise function we get

$$A = \begin{cases} x - y \leq \frac{1}{2} & x \geq y \\ y - x \leq \frac{1}{2} & y \geq x \end{cases} \implies \begin{cases} y \leq x - \frac{1}{2} & x \geq y \\ y \leq \frac{1}{2} + x & y \geq x \end{cases}$$

At the same time B is equal to the area above the 45° line in the unit square. Drawing this in Geogebra gives us



b)

Using geometry to calculate the areas we get that

$$P(A) = 1 - 2 \cdot 0.5 \cdot 0.5 \cdot 0.5 = \frac{3}{4}$$
$$P(B) = 1 - 0.5 \cdot 1 \cdot 1 = \frac{1}{2}$$

c)

We determine the probability of the intersection using geometry

$$P(A \cap B) = 1 - 0.5 - 0.5 \cdot 0.5 \cdot 0.5 = \frac{3}{8}$$

For them to be independent

$$P(A \cap B) = P(A)P(B) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$$

As such they are independent.

Problem 31 Let A be the event that an email is spam, B the event that it contains the word refinance, we then have that

$$P(A) = 0.5$$
$$P(B|A) = 0.01$$
$$P(B|\overline{A}) = 0.00001$$

We wish to determine the probability

$$P(A|B)$$

By Bayes rule we have that

$$\begin{aligned} P(A|B) &= \frac{P(B|A)P(A)}{P(B)} \\ &= \frac{0.01 \cdot 0.5}{0.01 \cdot 0.5 + 0.00001 \cdot 0.5} \\ &\approx 0.99 \end{aligned}$$

Problem 32 a)

For the path to be open it is clear that B_1, B_4 or B_2, B_5 or B_1, B_3, B_5 or B_2, B_3, B_4 must be open

$P(A) = P(\{B_1 \cap B_3 \cap B_4\} \cup \{B_1 \cap B_3 \cap B_5\} \cup \{B_2 \cap B_3 \cap B_5\} \cup \{B_2 \cap B_3 \cap B_4\})$

We rename each path to P_i where i is the arbitrary number assigned to the path. By the inclusion exclusion principle we have that

$$\begin{aligned}
P(A) &= P(P_1) + P(P_2)P(P_3) + P(P_4) - P(P_1)P(P_2) - P(P_1)P(P_3) \\
&- P(P_1)P(P_4) - P(P_2)P(P_3) - P(P_2)P(P_4) - P(P_3)P(P_4) \\
&+ P(P_1)P(P_2)P(P_3) + P(P_1)P(P_2)P(P_4) + P(P_1)P(P_3)P(P_4) \\
&+ P(P_2)P(P_3)P(P_4) - P(P_1)P(P_2)P(P_3)P(P_4) \\
&= P(B_1 \cap B_4) + P(B_2 \cap B_5) + P(B_1 \cap B_3 \cap B_5) + P(B_2 \cap B_3 \cap B_4) \\
&- P((B_1 \cap B_4) \cap (B_2 \cap B_5)) - P((B_1 \cap B_4) \cap (B_1 \cap B_3 \cap B_5)) \\
&- P((B_1 \cap B_4) \cap (B_2 \cap B_3 \cap B_4)) - P((B_2 \cap B_5) \cap (B_1 \cap B_3 \cap B_5)) \\
&- P((B_2 \cap B_5) \cap (B_2 \cap B_3 \cap B_4)) - P((B_1 \cap B_3 \cap B_5) \cap (B_2 \cap B_3 \cap B_4)) \\
&+ P((B_1 \cap B_4) \cap (B_2 \cap B_5) \cap (B_2 \cap B_3 \cap B_5)) \\
&+ P((B_1 \cap B_4) \cap (B_2 \cap B_5) \cap (B_2 \cap B_3 \cap B_4)) \\
&+ P((B_1 \cap B_4) \cap (B_1 \cap B_3 \cap B_5) \cap (B_2 \cap B_3 \cap B_4)) \\
&+ P((B_2 \cap B_5) \cap (B_2 \cap B_3 \cap B_4) \cap (B_2 \cap B_3 \cap B_4)) \\
&- P((B_1 \cap B_4) \cap (B_2 \cap B_5) \cap (B_1 \cap B_3 \cap B_5) \cap (B_2 \cap B_3 \cap B_4)) \\
&= P(B_1 \cap B_4) + P(B_2 \cap B_5) + P(B_1 \cap B_3 \cap B_5) + P(B_2 \cap B_3 \cap B_4) \\
&- P(B_1 \cap B_2 \cap B_4 \cap B_5) - P(B_1 \cap B_3 \cap B_4 \cap B_5) \\
&- P(B_1 \cap B_2 \cap B_3 \cap B_4) - P(B_1 \cap B_2 \cap B_3 \cap B_5) \\
&- P(B_2 \cap B_3 \cap B_4 \cap B_5) - P(B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5) \\
&+ P(B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5) + P(B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5) \\
&+ P(B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5) + P(B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5) \\
&- P(B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5) \\
&= P_1P_4 + P_2P_5 + P_1P_3P_5 + P_2P_3P_4 - P_1P_2P_4P_5 - P_1P_3P_4P_5 - P_1P_2P_3P_4 \\
&- P_1P_2P_3P_5 - P_2P_3P_4P_5 - P_1P_2P_3P_4P_5 + 4P_1P_2P_3P_4P_5 - P_1P_2P_3P_4P_5 \\
&= P_1P_4 + P_2P_5 + P_1P_3P_5 + P_2P_3P_4 - P_1P_2P_4P_5 - P_1P_3P_4P_5 - P_1P_2P_3P_4 \\
&- P_1P_2P_3P_5 - P_2P_3P_4P_5 + 2P_1P_2P_3P_4P_5 \\
&= P_1P_4(1 - P_2P_5 - P_3P_5 - P_2P_3 + 2P_2P_3P_5) + P_2P_5 \\
&+ P_1P_3P_5 + P_2P_3(P_4 - P_1P_5 - P_4P_5)
\end{aligned}$$

b) We use Bayes rule to determine the probability, conditioning A on B_3 we effectively cancel out the P_3 terms as $P(P_3|B_3) = 1$

$$\begin{aligned}
P(A|B_3) &= P_1P_4(1 - P_2P_5 - P_5 - P_2 + 2P_2P_5) + P_2P_5 + P_1P_5 \\
&+ P_2(P_4 - P_1P_5 - P_4P_5)
\end{aligned}$$

By Bayes rule we have that

$$P(B_3|A) = \frac{P(A|B_3)P_3}{P(A)}$$

Inserting known information

$$P(B_3|A) = \frac{P_1P_3P_4(1 - P_2P_5 - P_5 - P_2 + 2P_2P_5) + P_2P_3P_5 + P_1P_3P_5 + P_2P_3(P_4 - P_1P_5 - P_4P_5)}{P_1P_4(1 - P_2P_5 - P_3P_5 - P_2P_3 + 2P_2P_3P_5) + P_2P_5 + P_1P_3P_5 + P_2P_3(P_4 - P_1P_5 - P_4P_5)}$$

Problem 33 Before any extra information is given we have that

$$P(H_1) = P(H_2) = P(H_3) = \frac{1}{3}$$

We let C_i be the event that the host opens the i'th door and H_i be the event that the car is behind the i'th door, assuming we opened the 1st door we get the following probabilities

$$\begin{aligned}
P(C_1|H_1) &= 0 \\
P(C_2|H_1) &= \frac{1}{2} \\
P(C_3|H_1) &= \frac{1}{2} \\
P(C_1|H_2) &= 0 \\
P(C_2|H_2) &= 0 \\
P(C_3|H_2) &= 1 \\
P(C_1|H_3) &= 0 \\
P(C_2|H_3) &= 1 \\
P(C_3|H_3) &= 0
\end{aligned}$$

If the host choses to open the 3rd door, we wish to determine whether

$$P(H_2|C_3) > P(H_1)$$

As that would put us in an advantageous situation. We apply Bayes rule and get

$$\begin{aligned}
P(H_2|C_3) &= \frac{P(C_3|H_2)P(H_2)}{P(C_3|H_1)P(H_1) + P(C_3|H_2)P(H_2) + P(C_3|H_3)P(H_3)} \\
&= \frac{1 \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}} \\
&= \frac{2}{3}
\end{aligned}$$

As

$$P(H_2|C_3) > P(H_1)$$

It would be advantageous to switch our guess.

Problem 34 a)

We have that

$$P(A) = \frac{1}{6}$$
$$P(B) = \frac{1}{6}$$
$$P(C) = \frac{1}{6}$$

For A and B to be independent it must be true that

$$P(A \cap B) = P(A)P(B)$$

The intersection of the 2 is only satisfied by the pair $(2, 5)$, as such

$$P(A \cap B) = \frac{1}{36} \stackrel{?}{=} \frac{1}{6} \cdot \frac{1}{6}$$

As the sides are equivalent the two are independent.

b)

Again we determine $P(A \cap C)$, this is again a single pair, $(2, 3)$ as such the rest of the problem is equivalent to the previous. The two are independent.

c)

Again $P(B \cap C)$ is a single pair $(4, 3)$, as such they are once again independent.

d)

As we already know we satisfy 3 of the requirements we must at last determine whether

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

This however is not true as the intersection is empty due to 2 and 3 never adding up to 7, as such

$$P(A \cap B \cap C) = P(\emptyset) = 0$$

Which is not equal to the product of the individual probabilities as

$$0 \neq \left(\frac{1}{6}\right)^3$$

Problem 35

Problem 36 We define $H \dots H$ as the event that n coin tosses resulted in heads, and H_{n+1} as the event that the next coin toss results in heads and F be that we picked the fair coin. We wish to determine the probability

$$P(H_{n+1}|H \dots H)$$

From the law of total probability it becomes clear that

$$P(H_{n+1}) = P(H_{n+1}|F)P(F) + P(n_{+1}|\overline{F})P(\overline{F})$$

Conditioning both sides on $H \dots H$ we then get

$$P(H_{n+1}|H \dots H) = P(H_{n+1}|F, H \dots H)P(F|H \dots H) + P(n_{+1}|\overline{F}, H \dots H)P(\overline{F}|H \dots H)$$

The terms with two conditions are conditionally independent on $H \dots H$ as only the selected coin matters, as such

$$P(H_{n+1}|F, H \dots H) = \frac{1}{2}$$
$$P(H_{n+1}|\overline{F}, H \dots H) = 1$$

By using Bayes rule we can determine $P(F|H \dots H|)$ as

$$\begin{aligned}
P(F|H \dots H) &= \frac{P(H \dots H|F)P(F)}{P(H \dots H|F)P(F) + P(H \dots H|\overline{F})P(\overline{F})} \\
&= \frac{\frac{1}{2}^n \cdot \frac{1}{2}}{\frac{1}{2}^n \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}} \\
&= \frac{\frac{1}{2}^n}{\frac{1}{2}^n + 1} \\
&= \frac{1}{1 + \frac{1}{\frac{1}{2}^n}} \\
&= \frac{1}{2^n + 1}
\end{aligned}$$

And $P(\overline{F}|H \dots H|)$ as its complement

$$P(\overline{F}|H \dots H) = 1 - \frac{1}{2^n + 1}$$

The probability of the n'th toss being must therefore be the sum of these probabitilies

$$
\begin{aligned}
P(H_{n+1}|H\dots H) &= P(F|H\dots H)P(F) + P(\overline{F}|H\dots H) \\
&= \frac{1}{2}\cdot\frac{1}{2^n+1} + \left(1 - \frac{1}{2^n+1}\right) \\
&= \frac{1}{2(2^n+1)} + \left(1 - \frac{1}{2^n+1}\right) \\
&= 1 + \frac{-1}{2(2^n+1)} \\
&= 1 - \frac{1}{2(2^n+1)}
\end{aligned}
$$

Problem 37

Problem 38

Problem 39

3.2 Chapter 2

Problem 1

Problem 2 We make use of combinations with $n = 12, k = 8$

$$\binom{12}{8} = \frac{12!}{8!(12-8)!} = 495$$

Problem 3 a)

The probability of picking exactly 4 black phones will be equal to

$$P(B=4) = \frac{|B=4|}{|S|}$$

As they are not put back we determine $|S|$ as the amount of 10-element subsets of a 50 element set using combinations

$$|S| = \binom{50}{10} = \frac{50!}{10!(50-10)!} = 10272278170$$

Simultaneously we can determine the amount of sets containing 4 black phones and 6 white phones as

$$|B = 4| = \binom{20}{4}\binom{30}{10-4} = \frac{20!}{4!(20-4)!} \cdot \frac{30!}{6!(30-6)!} = 2876839875$$

As such

$$P(B = 4) = \frac{2876839875}{10272278170} \approx 0.28$$

b) We wish to determine $P(B \in \{0,1,2\})$, we use the same method as the previous section and get that

$$\begin{aligned}
|B \in \{0,1,2\}| &= \sum_{i=0}^{2} \binom{20}{i}\binom{30}{10-i} \\
&= \binom{20}{0}\binom{30}{10} + \binom{20}{1}\binom{30}{9} + \binom{20}{2}\binom{30}{8} \\
&= 30045015 + 286143000 + 1112055750 \\
&= 1428243765
\end{aligned}$$

And again we determine the probability as

$$P(B \in \{0,1,2\}) = \frac{1428243765}{10272278170} \approx 0.14$$

Problem 4 a)

We once again make use of combinations, the sample space is given by

$$|S| = \binom{52}{5} = \frac{52!}{5!(52-5)!} = 2598960$$

The amount of 5 element sets consisting of 1 ace must be given by

$$|A = 1| = \binom{4}{1}\binom{48}{4} = \frac{4!}{1!(4-1)!} \cdot \frac{48!}{4!(48-4)!} = 778320$$

Determining the probability is then as simple as

$$P(A = 1) = \frac{778320}{2598960} \approx 0.30$$

b) The phrasing of the question suggests that its easier to determine the complement, as such we find the probability of getting no aces using the same method as in the previous section

$$|A = 0| = \binom{4}{0}\binom{48}{5} = \frac{4!}{0!(4)!} \cdot \frac{48!}{5!(48-5)!} = 1712304$$

The probability is then

$$P(A=0) = \frac{1712304}{2598960} \approx 0.66$$

And we then determine the complement as this is equal to the probability of getting 1 or more aces

$$P(A \in \{1,2,3,4\}) = P(\overline{A=0}) = 1 - P(A=0) \approx 0.34$$

Problem 5 We wish to determine the probability

$$P(A=2|A \geq 1)$$

As the reverse conditional probability is independent of the condition we know that

$$P(A \geq 1|A=2) = 1$$

Using Bayes law we then find that

$$P(A=2|A \geq 1) = \frac{P(A \geq 1|A=2)P(A=2)}{P(A \geq 1)}$$

We determine $|A=2|$ using the same method as in previous section

$$|A=2| = \binom{4}{2}\binom{48}{3} = \frac{4!}{2!(4-2)!} \cdot \frac{48!}{3!(48-3)!} = 103776$$

And then make use of this to determine the probability of getting 2 aces

$$P(A=2) = \frac{103776}{2598960} \approx 0.04$$

As such we have that

$$P(A=2|A \geq 1) = \frac{1 \cdot 0.04}{0.34} \approx 0.12$$

Problem 6 Assuming the cards are dealt in bulk we can determine that

$$n_{\text{cards}} = 52 - 26 = 26$$
$$n_{\text{spades}} = 13 - 7 = 6$$

As such 26 cards containing 6 spades will be dealt to player C, giving us that the sample space is

$$|S| = \binom{26}{13} = \frac{26!}{13!(26-13)!}$$

Whilst the amount of decks containing exactly 4 spades is

$$|C = 4| = \binom{6}{4}\binom{20}{13-4} = \frac{6!}{4!(6-4)!} \cdot \frac{20!}{9!(20-9)!}$$

As such the probability is

$$P(C = 4) = \frac{\binom{6}{4}\binom{20}{9}}{\binom{26}{13}} \approx 0.24$$

Problem 7 The sample space is given by

$$|S| = \binom{50}{15} = \frac{50!}{15!(50-15)!}$$

By inclusion exclusion we have that

$$P(Y \cup J) = P(Y) + P(J) - P(Y \cap J)$$

The amount of sets containing either you or Joe is

$$|Y| = |J| = \binom{1}{1}\binom{49}{14} = \frac{1!}{1!(1-1)!} \cdot \frac{49!}{14!(49-14)!}$$

The amount of sets containing both you and your friend Joe is

$$|Y, J| = \binom{2}{2}\binom{48}{13} = \frac{2!}{2!(2-2)!} \cdot \frac{48!}{13!(48-13)!}$$

As such the probability of both you and Joe being in group is

$$P(Y \cup J) = \frac{\binom{2}{2}\binom{48}{13}}{\binom{50}{15}} \approx 0.09$$

Inserting in the inclusion exclusion principle we then get

$$P(Y \cup J) = 2\frac{\binom{1}{1}\binom{49}{14}}{\binom{50}{15}} - \frac{\binom{2}{2}\binom{48}{13}}{\binom{50}{15}} \approx 0.51$$

Problem 8 For a set with n elements of which r are unique we have that there are

$$N = \frac{n!}{n_1!n_2!\dots n_r!}$$

Where n_i is the amount of repetitions of the given element, in Massachusetts we observe that

$$\begin{aligned} n_m &= 1 \\ n_a &= 2 \\ n_s &= 4 \\ n_c &= 1 \\ n_h &= 1 \\ n_u &= 1 \\ n_e &= 1 \\ n_t &= 2 \end{aligned}$$

As such

$$N = \frac{13!}{2!4!2!} = 64864800$$

Problem 9 a)

We use the binomial theorem and as such have that

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

The probability of it containing 8 heads and 12 tails must simply be given by inserting $k = 8$ and determining the amount of possible sets using combinations

$$P(8) = \binom{20}{8} p^8 (1-p)^{20-8}$$

b)

Observing more than 8 heads and more than 8 tails is only possible when observing either 9, 10 or 11 heads, as such the probability must be given by

$$P(H \geq 8, T \geq 8) = \sum_{k=9}^{11} \binom{20}{k} p^k (1-p)^{20-k}$$

Problem 1

Problem 1

Problem 1

Problem 1

Problem 1

Problem 1

Problem 1

Problem 1

Problem 1

Problem 1

Problem 1

Problem 1