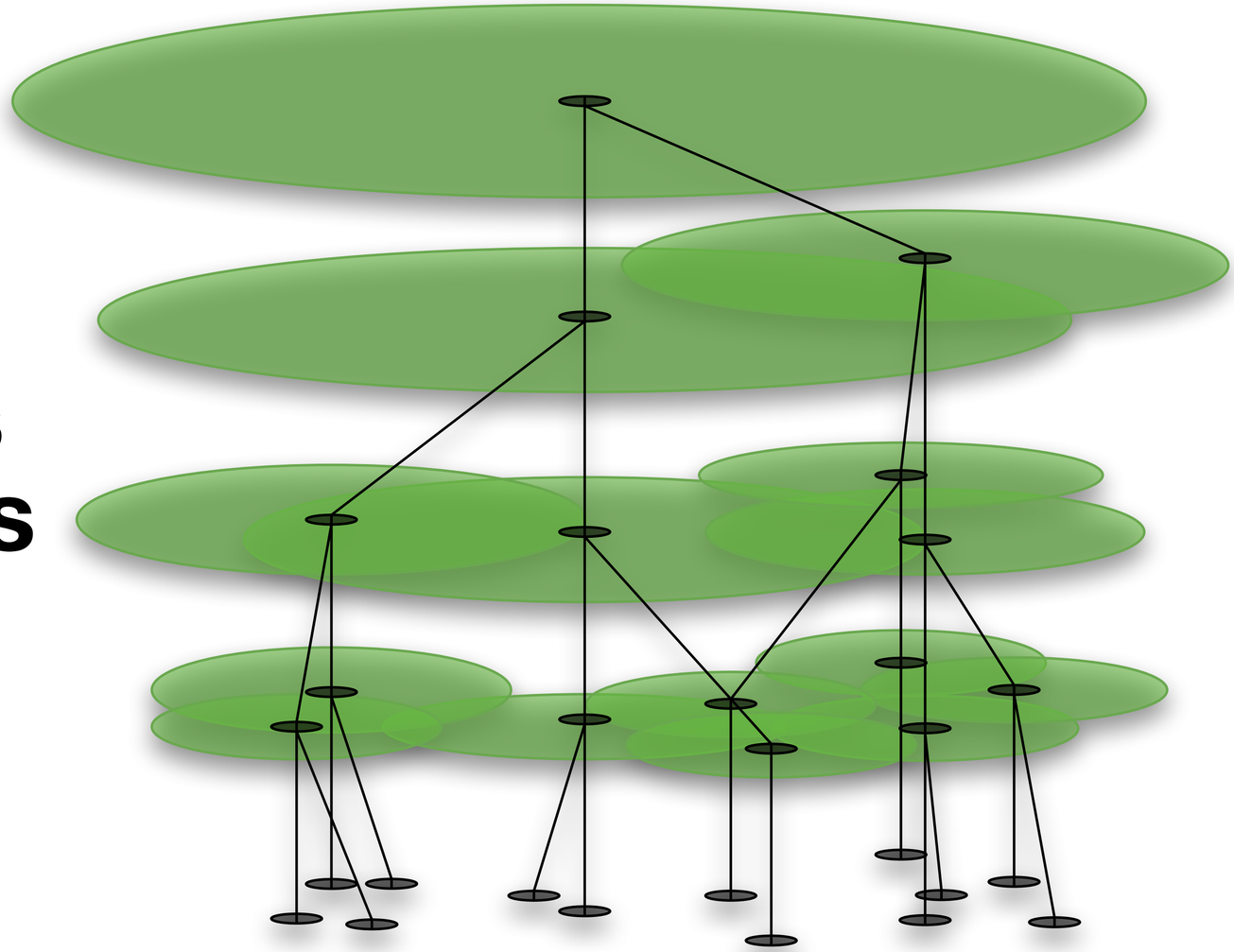
Computing Greedy Trees and Greedy Permutations



Oliver Chubet, FWCG'24

Greedy Permutations

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$$d(x_i, X_i) \geq \max_{j \geq i} d(x_j, X_i) \text{ for all } i > 0.$$

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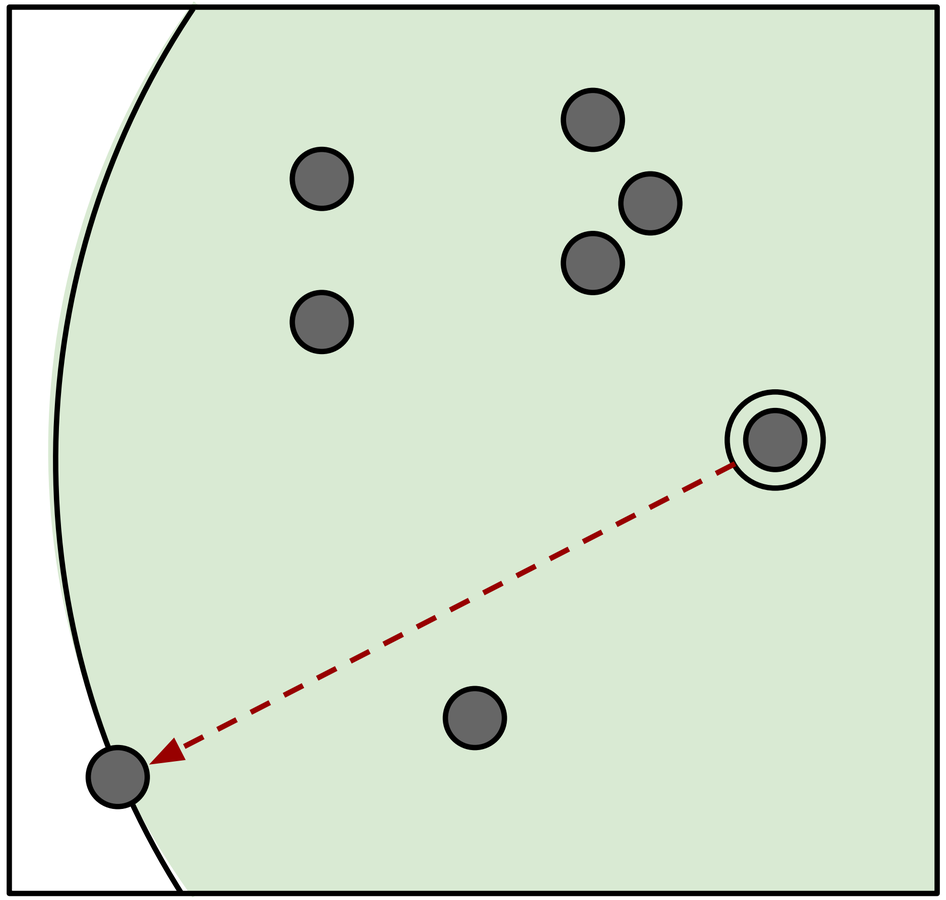
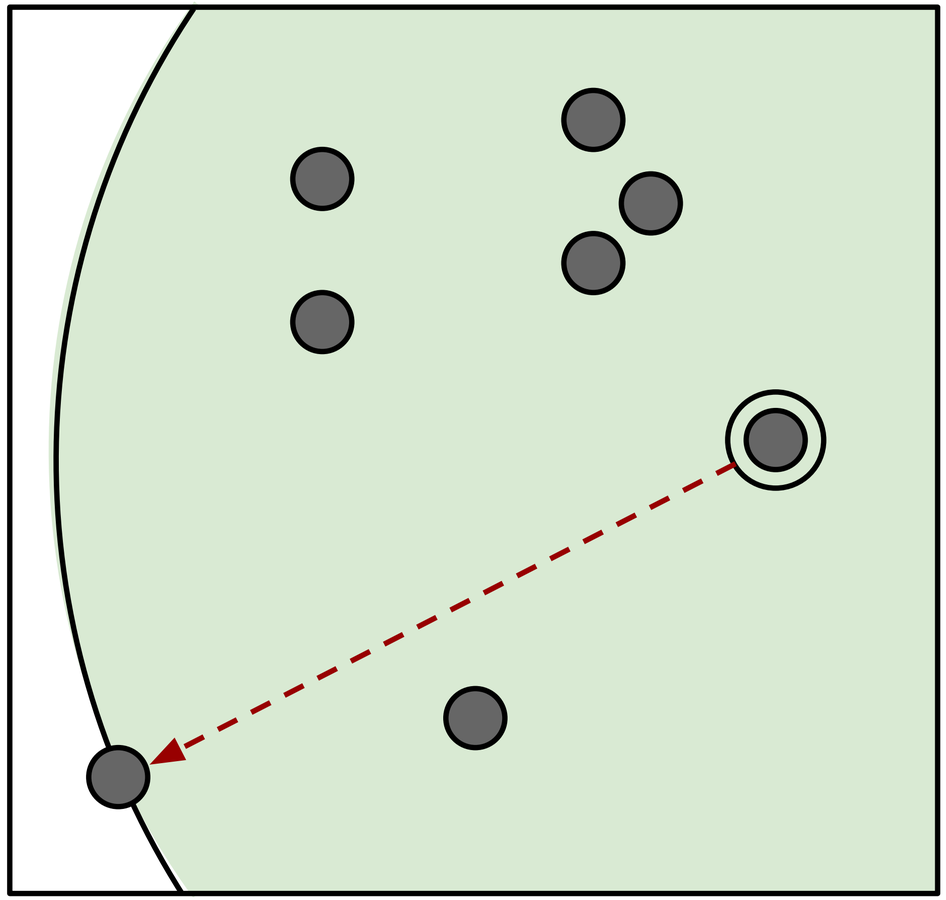
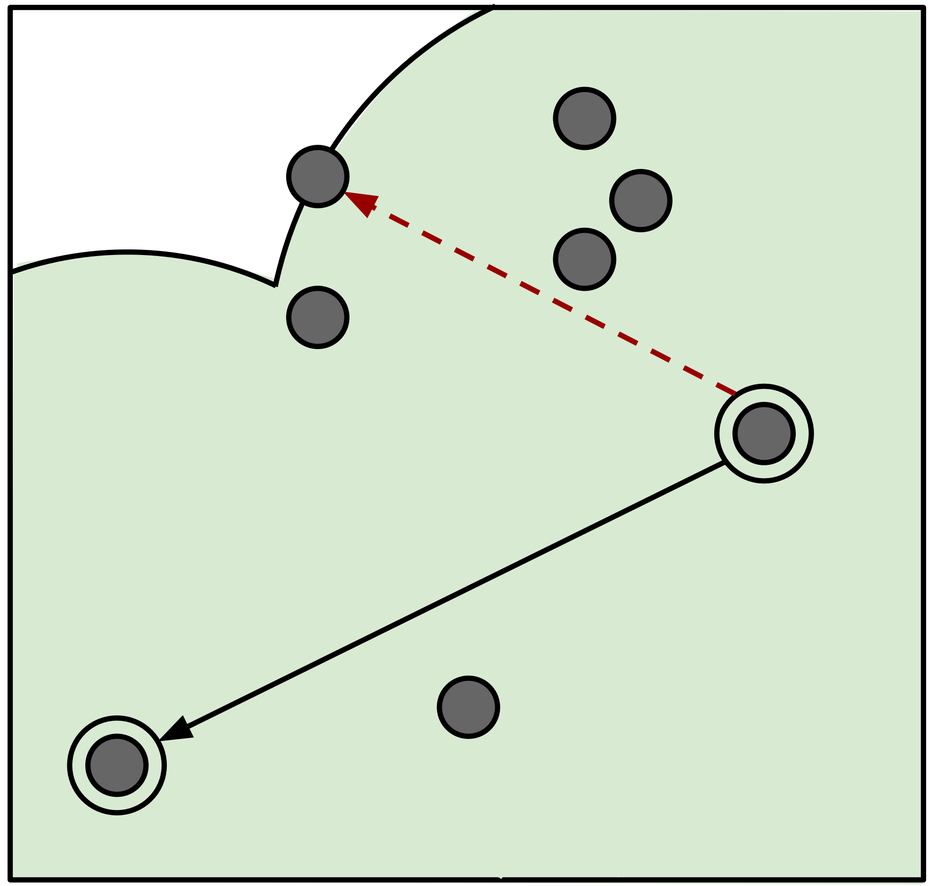


The first point in the permutation can be any point.

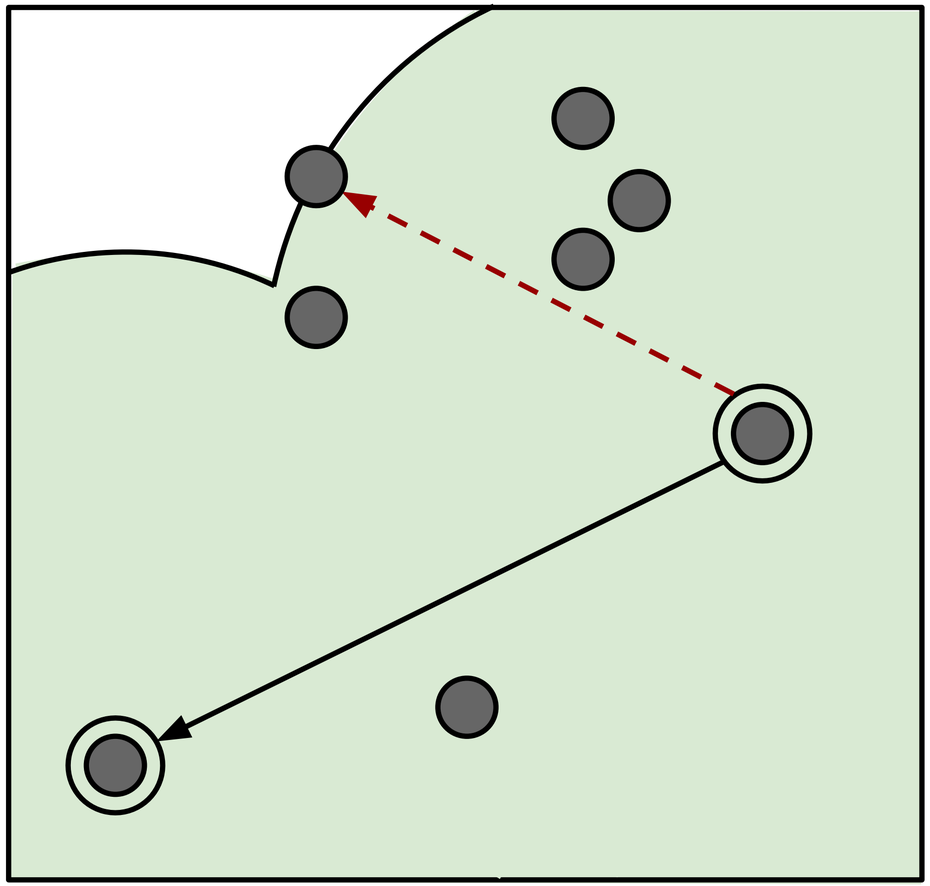
Greedy Permutations

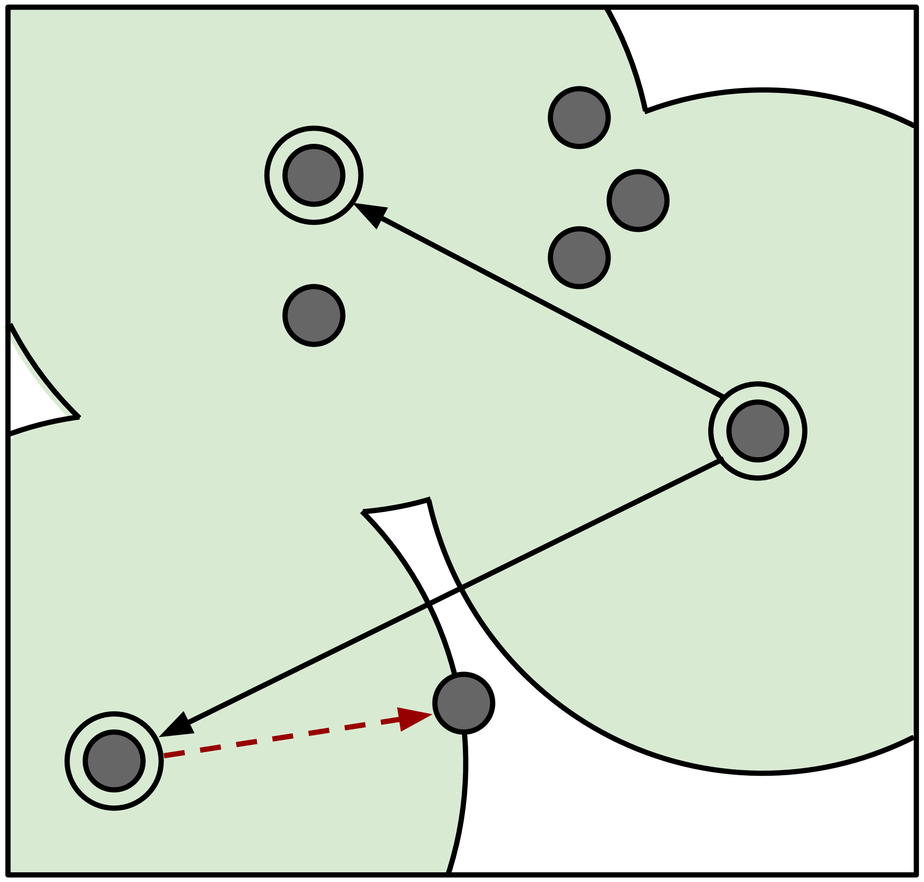




We insert the farthest point from the prefix.

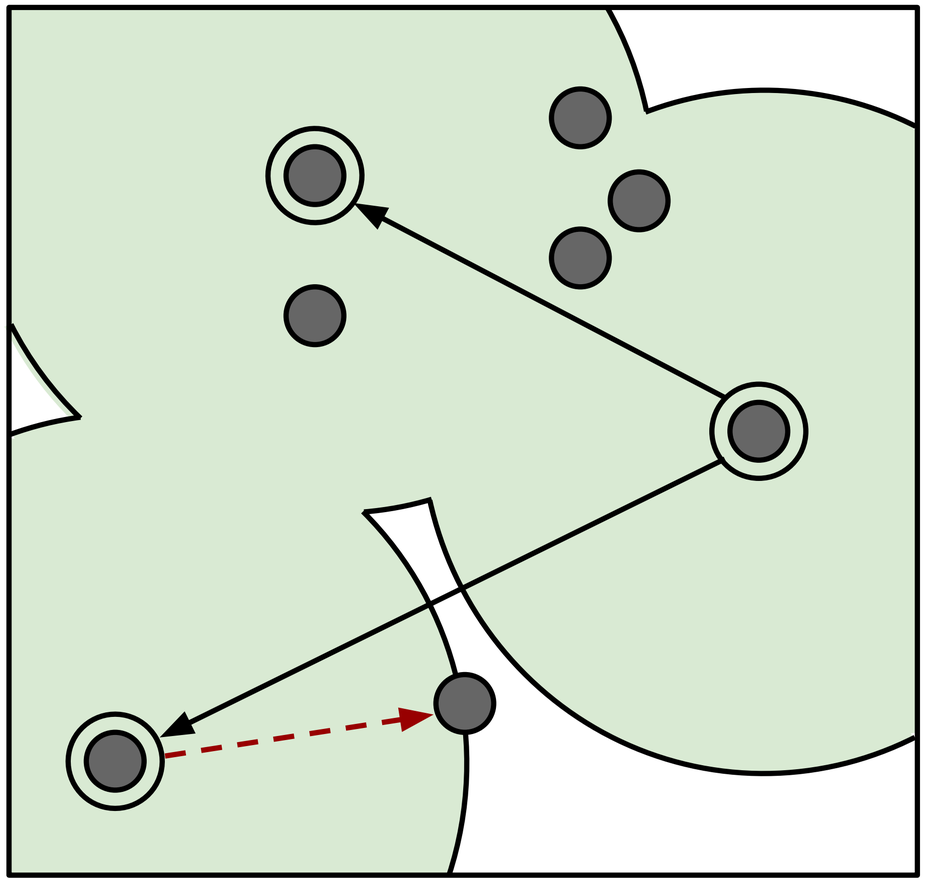
Greedy Permutations

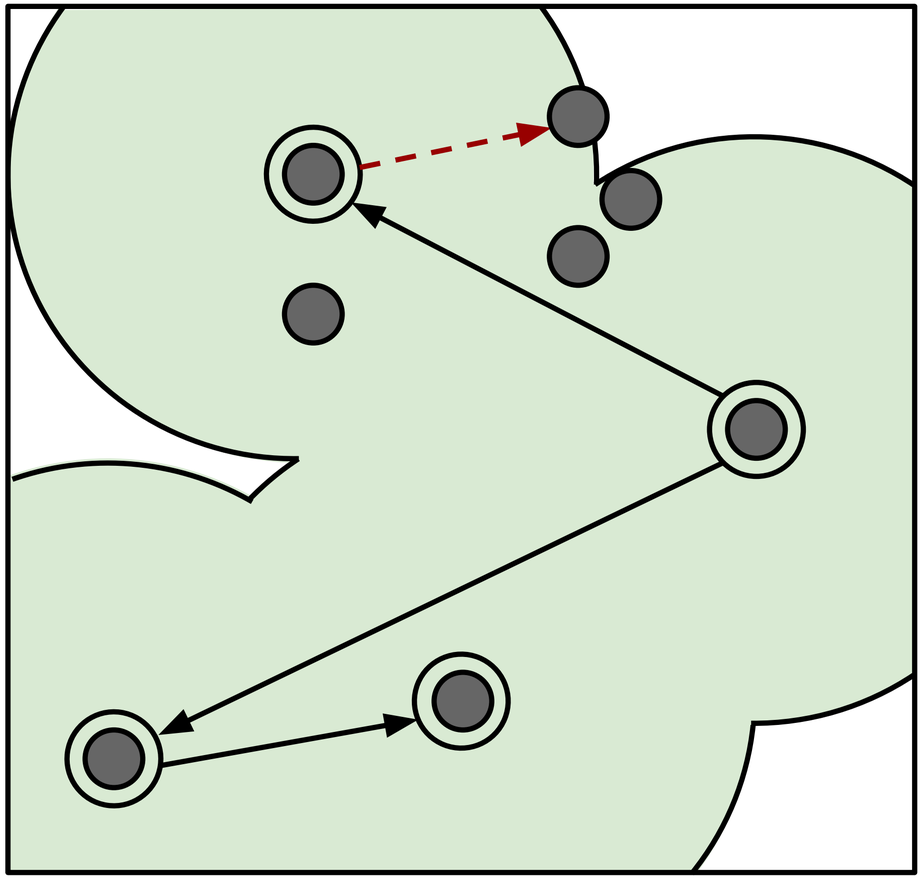




The distance to the next point is also the radius to cover the set.

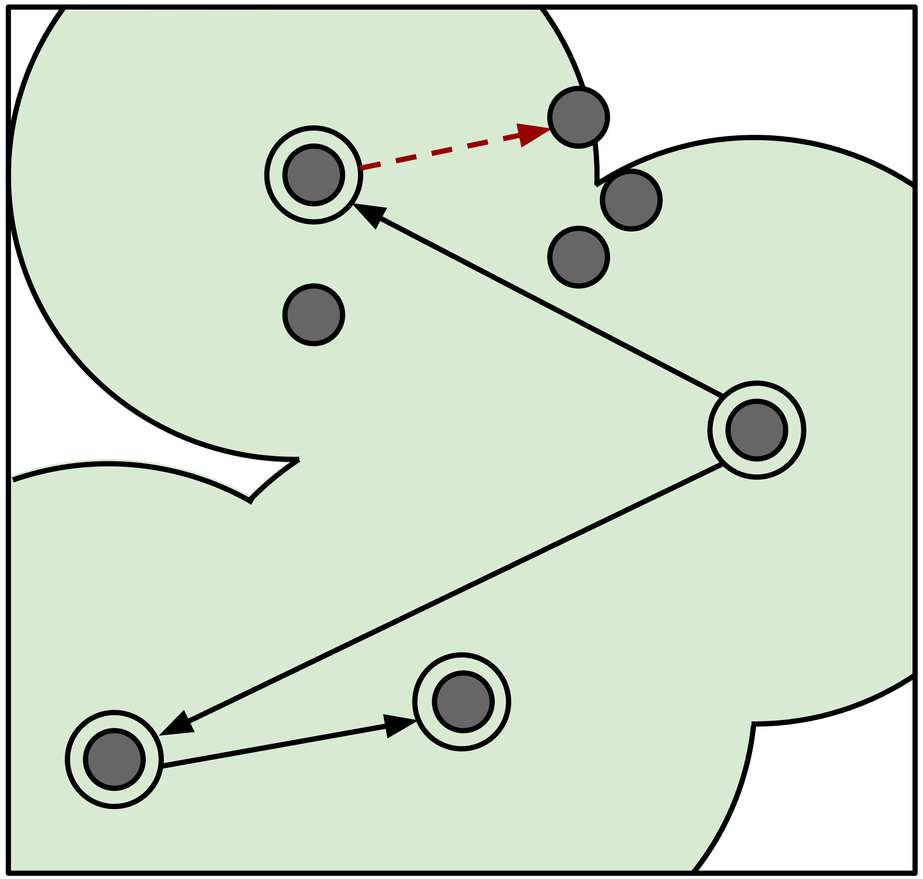
Greedy Permutations

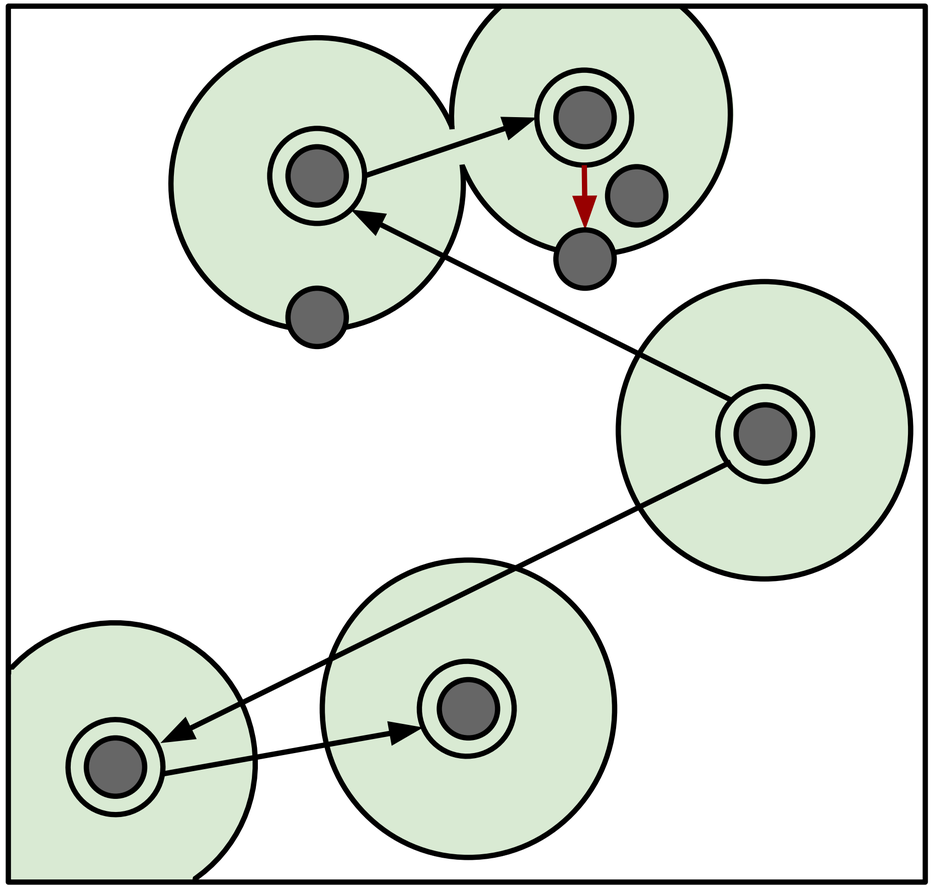




The *insertion distance* of a point is the distance to the prefix.

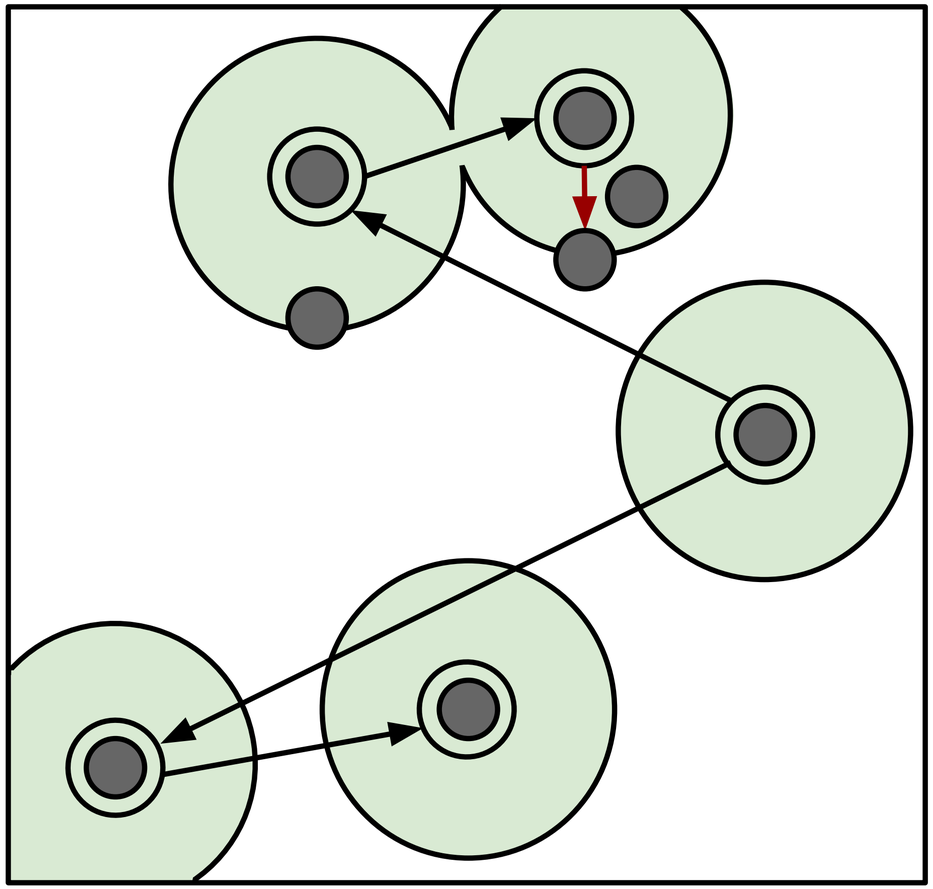
Greedy Permutations

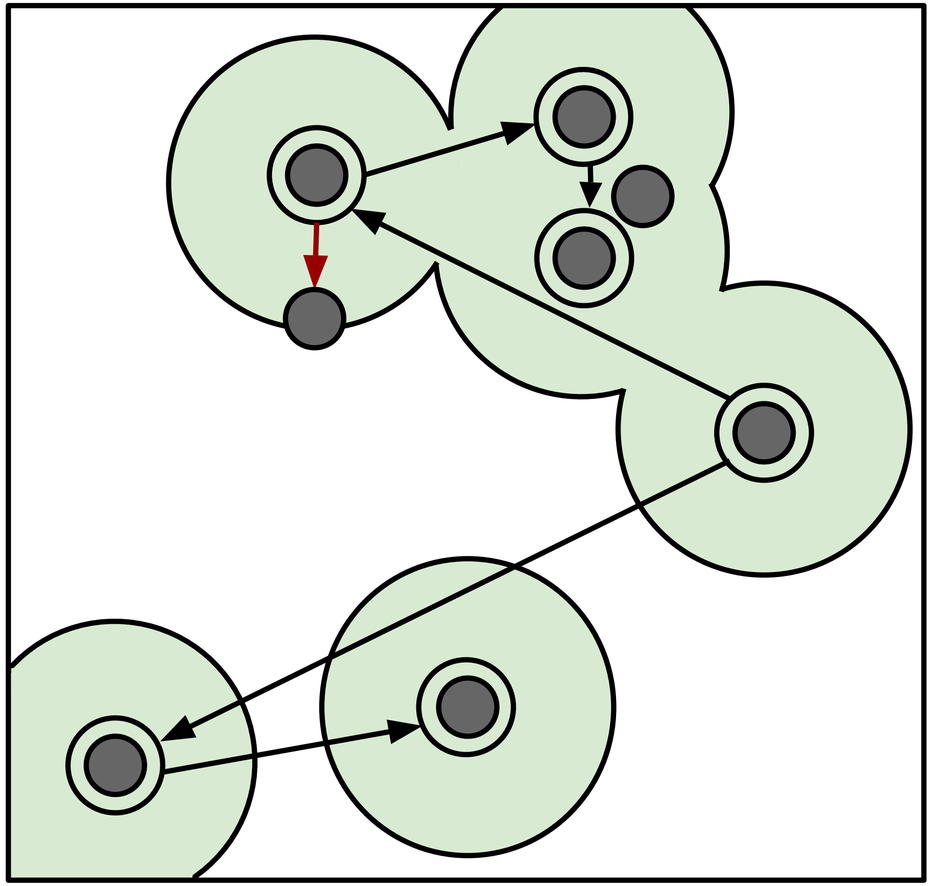




The predecessor of a point is the nearest neighbor in the prefix.

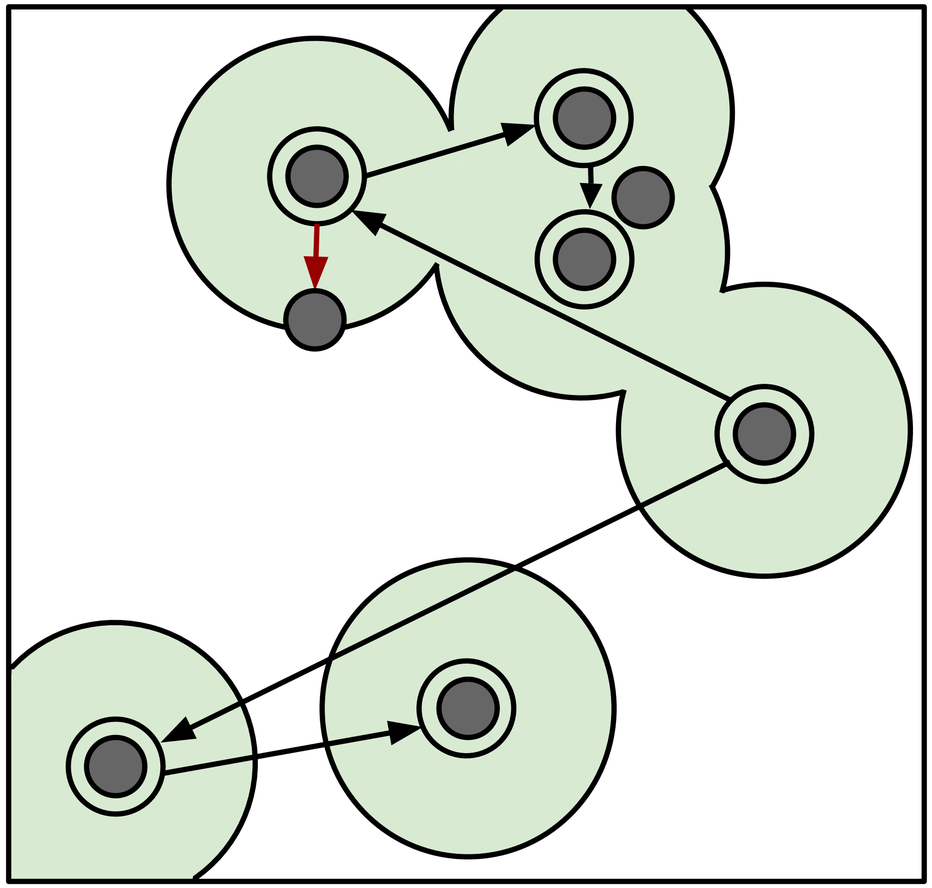
Greedy Permutations

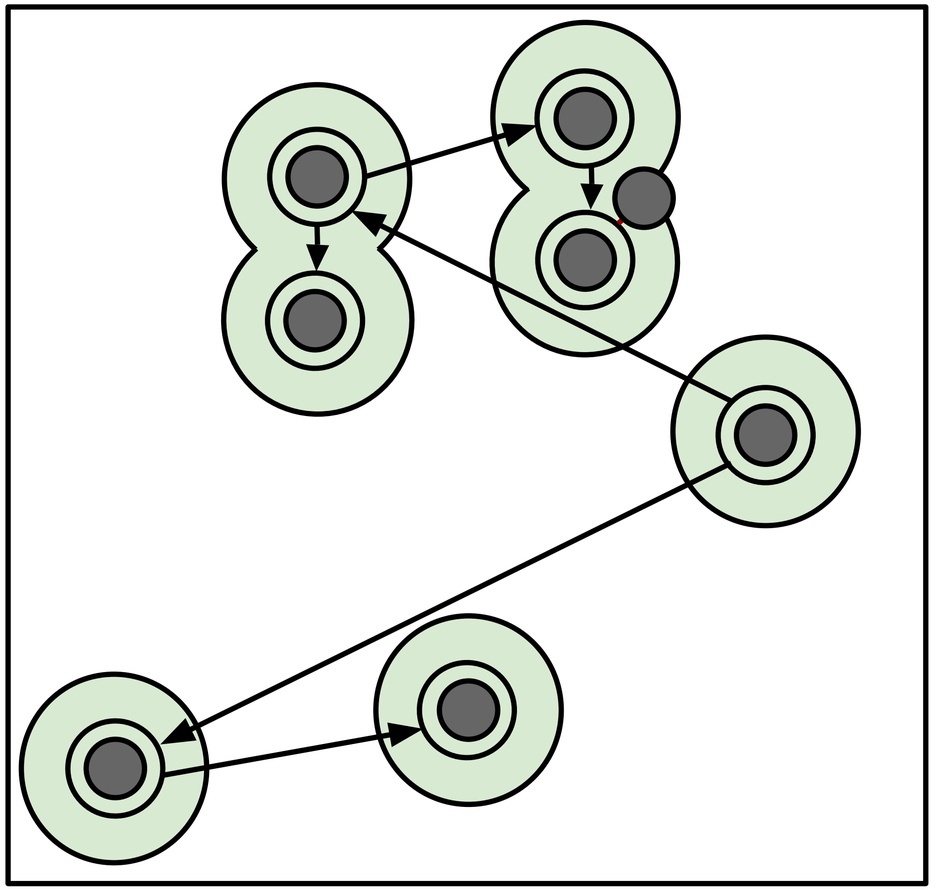




The prefix is packed by the last insertion distance.

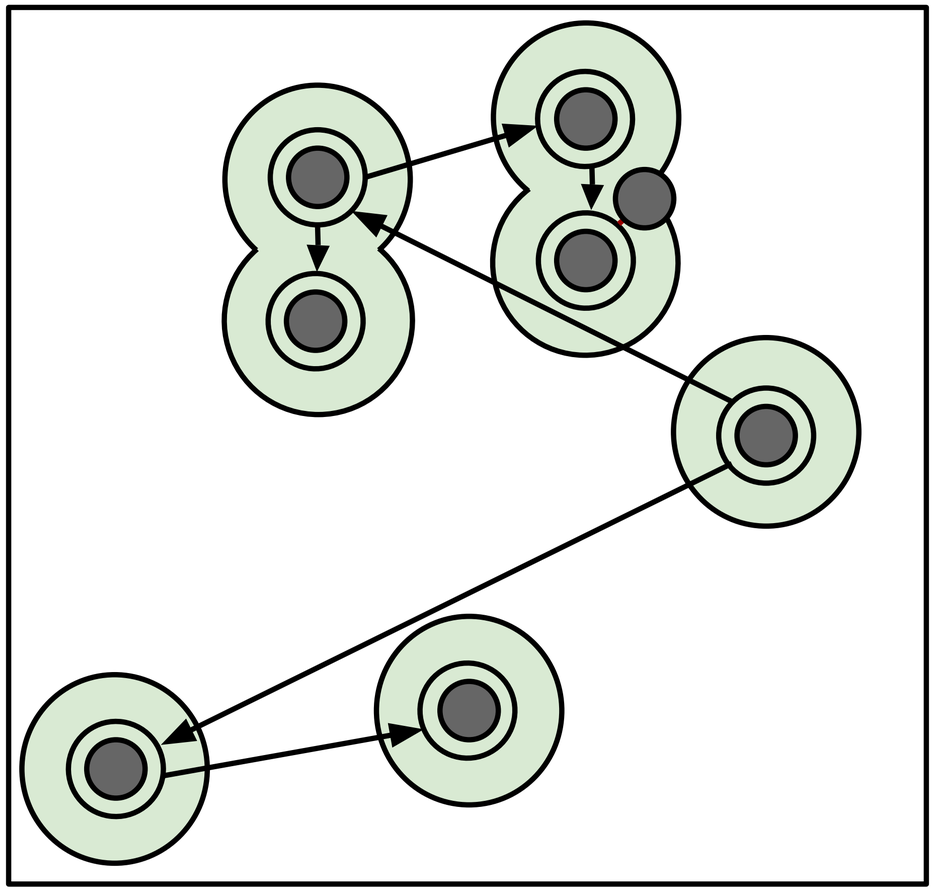
Greedy Permutations

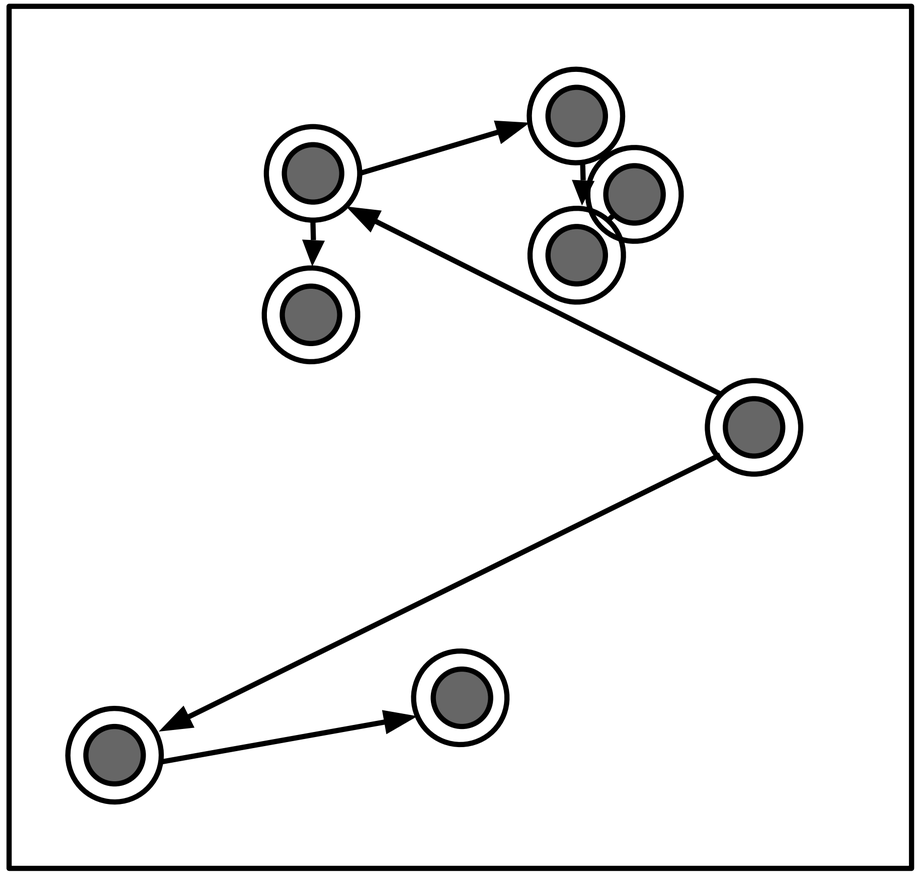




The assignment of points to their predecessor is the predecessor mapping.

Greedy Permutations





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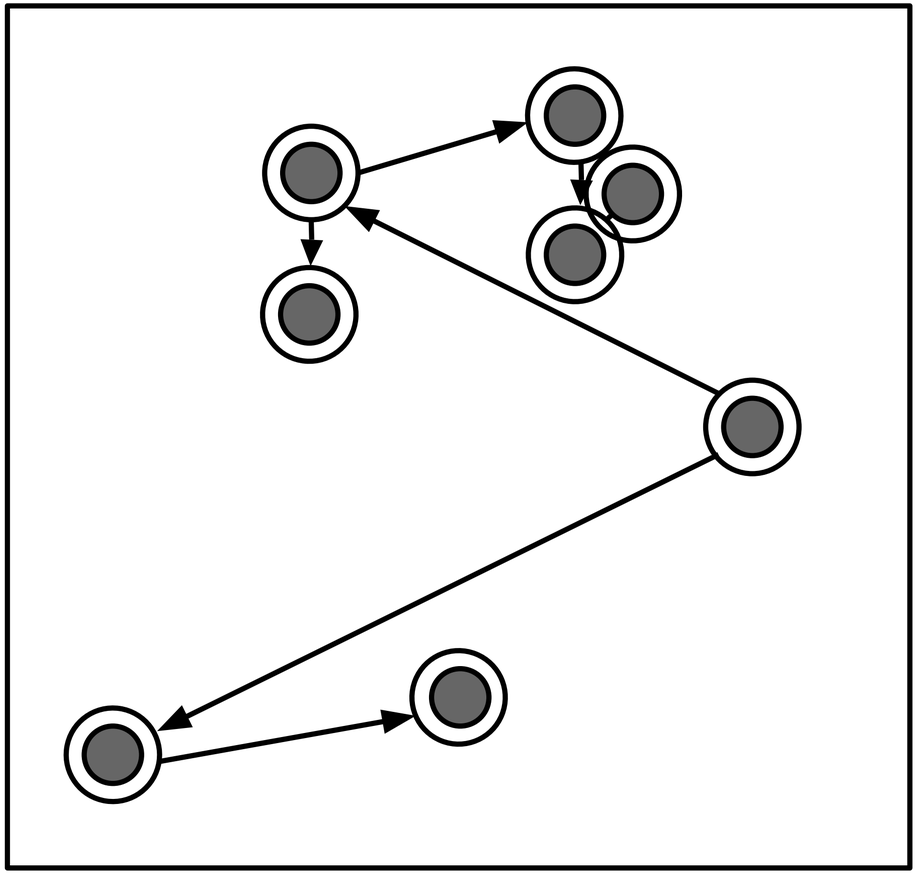
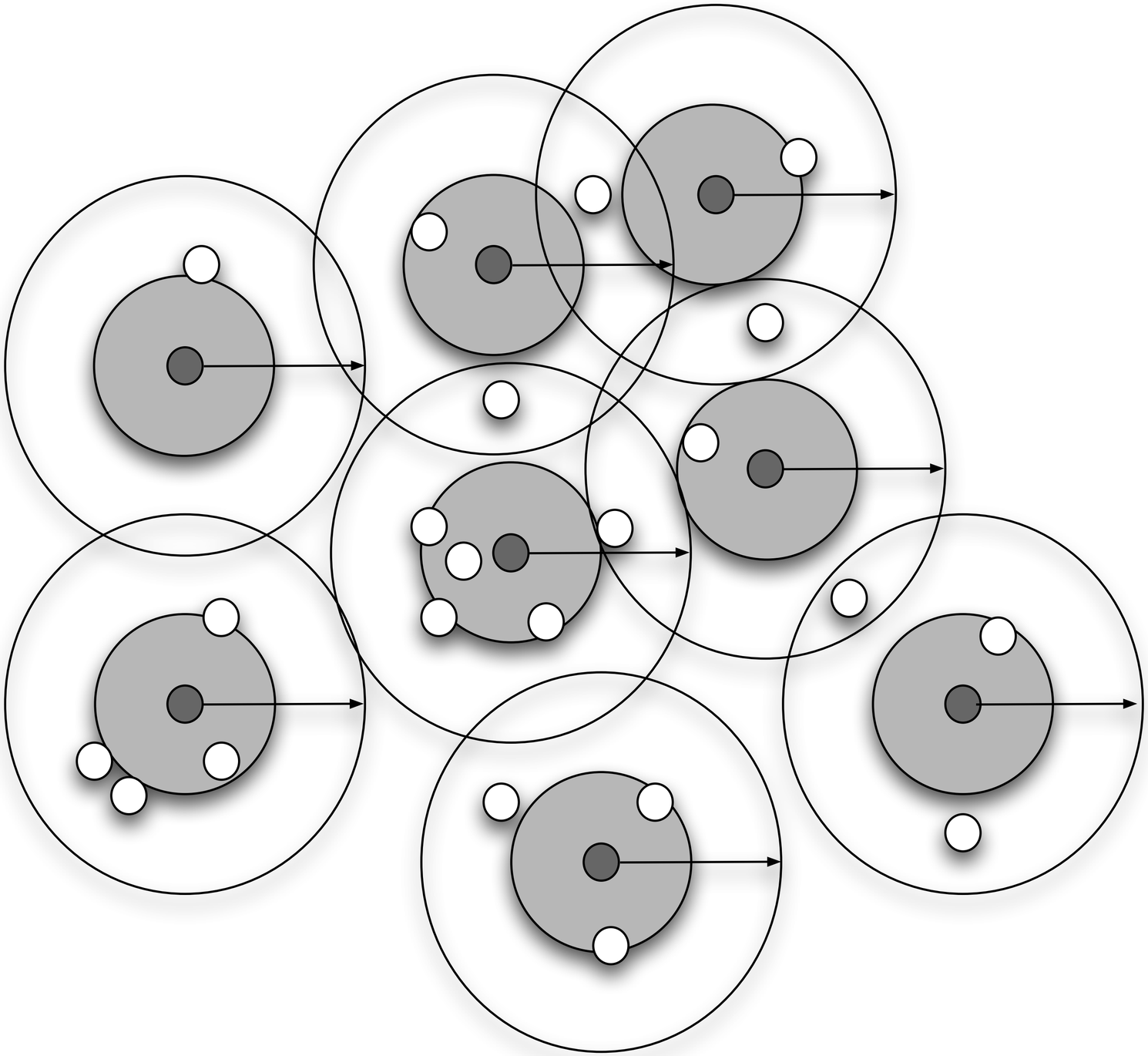


Greedy Permutations

Nets

A Packing and Covering



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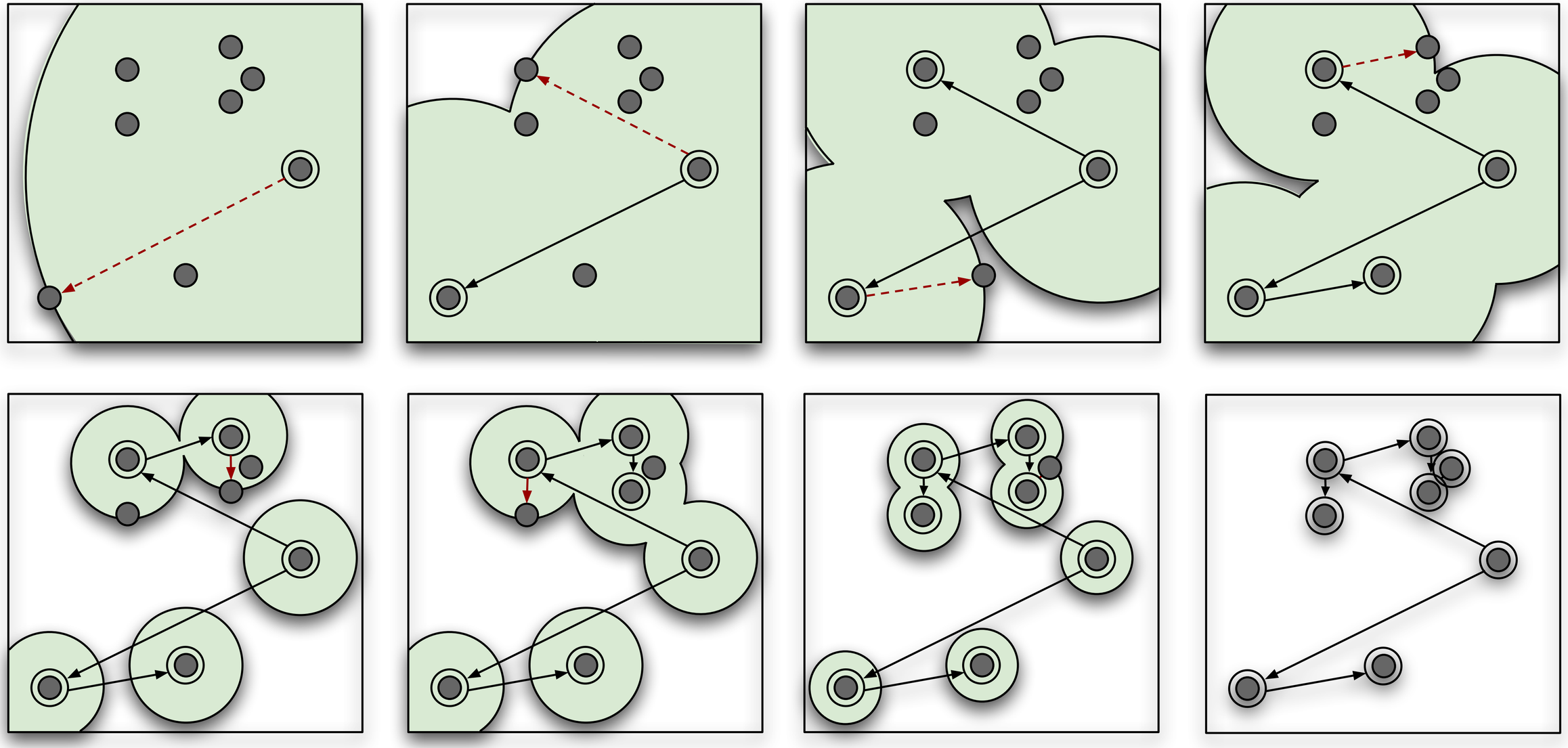

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(Later we can reframe this in terms of Voronoi diagrams…)

Net-trees

"The net-tree can be thought of as a representation of nets from all scales"

- $2^{O(d)} n \log n$ **expected randomized pre-processing**
- $O(n \log n)$ **space data structure**
- $2^{O(d)} \log n + (1/\varepsilon)^{O(d)}$ **query time**
- **Built using a farthest-point traversal of the data set**
  (a.k.a. a greedy permutation or Gonzalez ordering)

[Har-Peled and Mendel 2005]

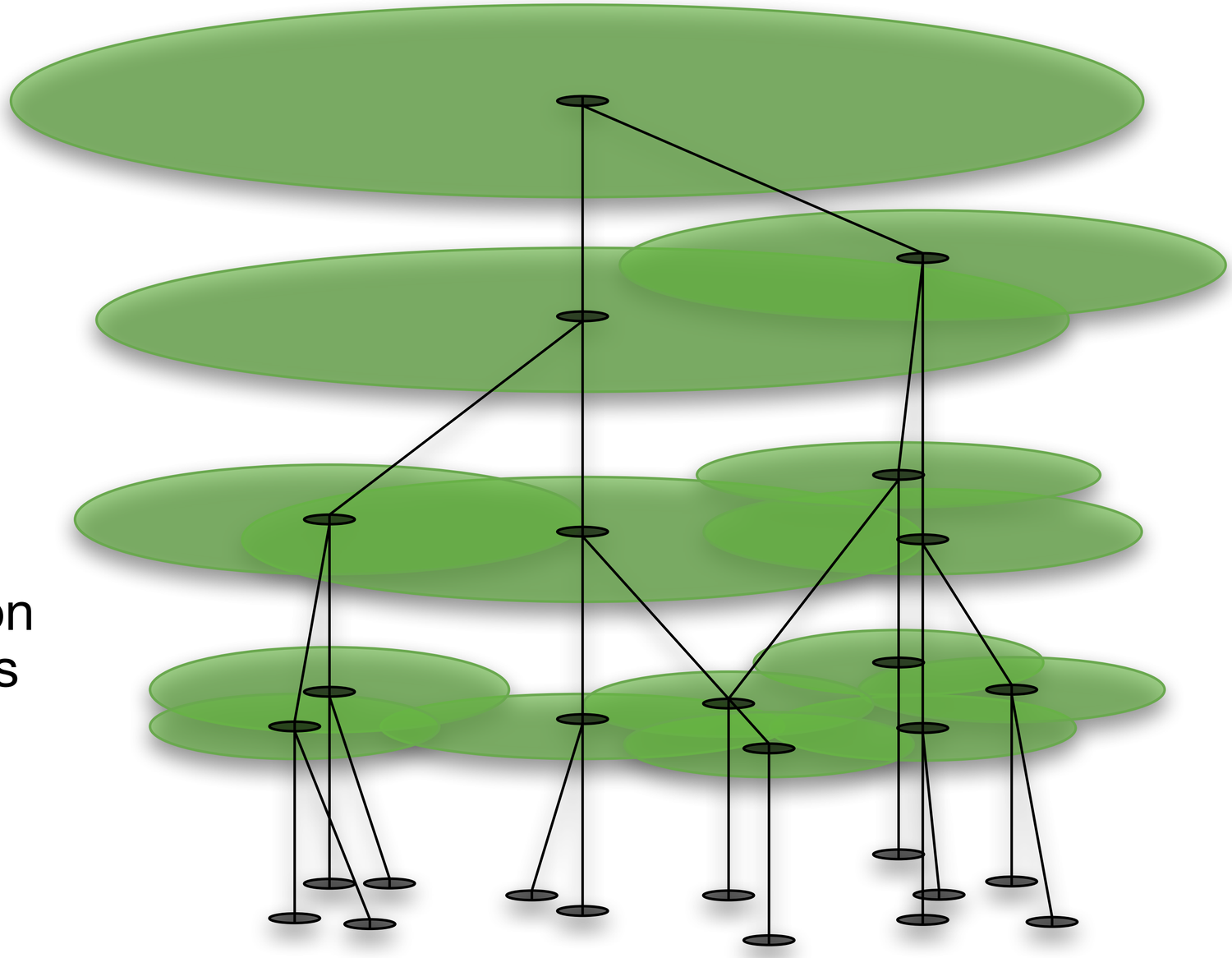
Summary of Results

	Net-trees	Greedy Trees
Construction Time	$2^{O(d)} n \log n$ **expected, randomized**	$2^{O(d)} n \log n$ **deterministic**
Space	$O(n \log n)$	$O(n)$

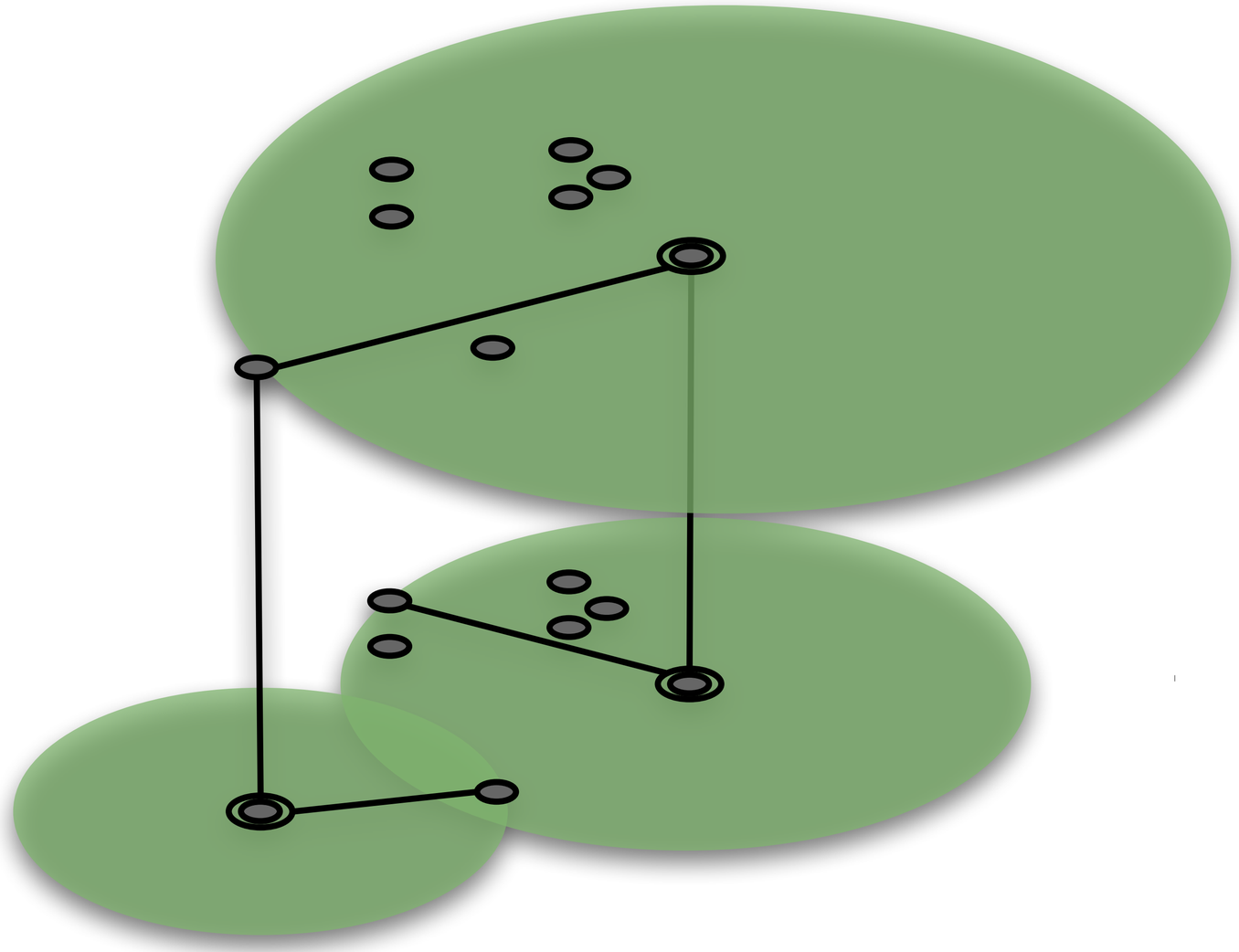
Goal: match or improve on prior work using a simpler algorithm.

Ball Trees

- A binary tree
- Constructed by recursive partitioning of the data set
- Each node covers its partition (the points at the leaves in its subtree) with a ball

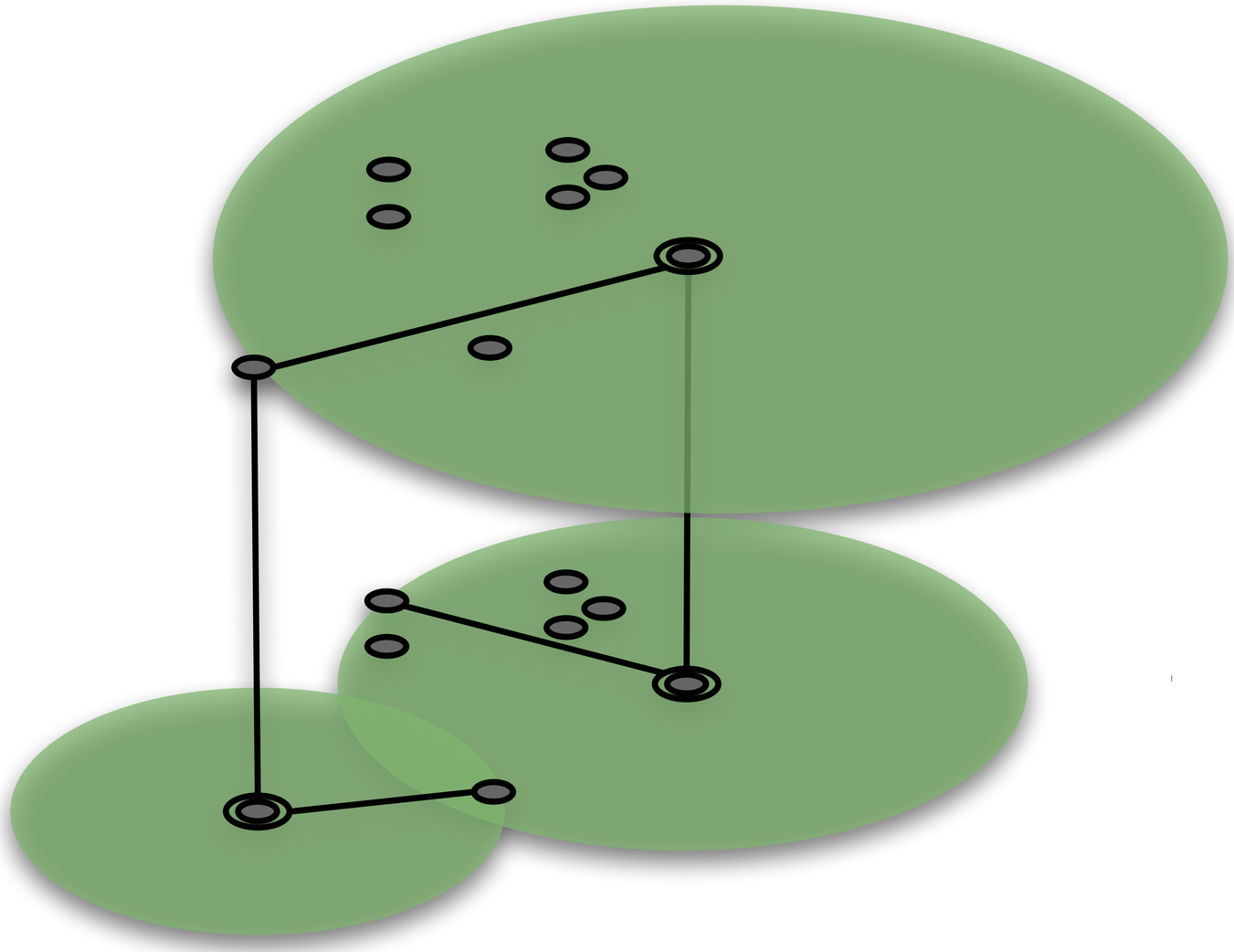


Split a node



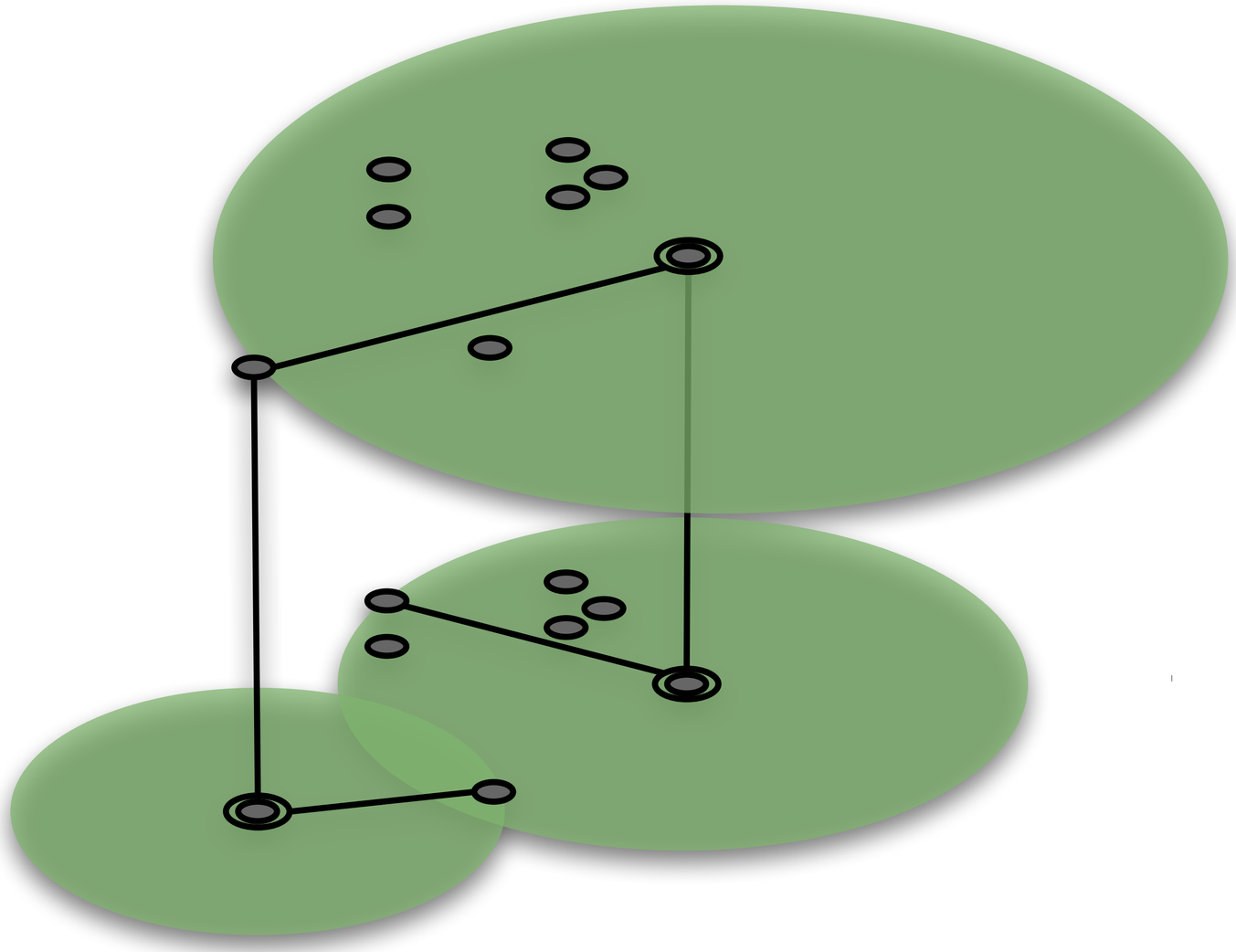
Split a node

- **A node covers all the points in the leaves of its subtree.**



Split a node

- A node covers all the points in the leaves of its subtree.
- The root covers everything.



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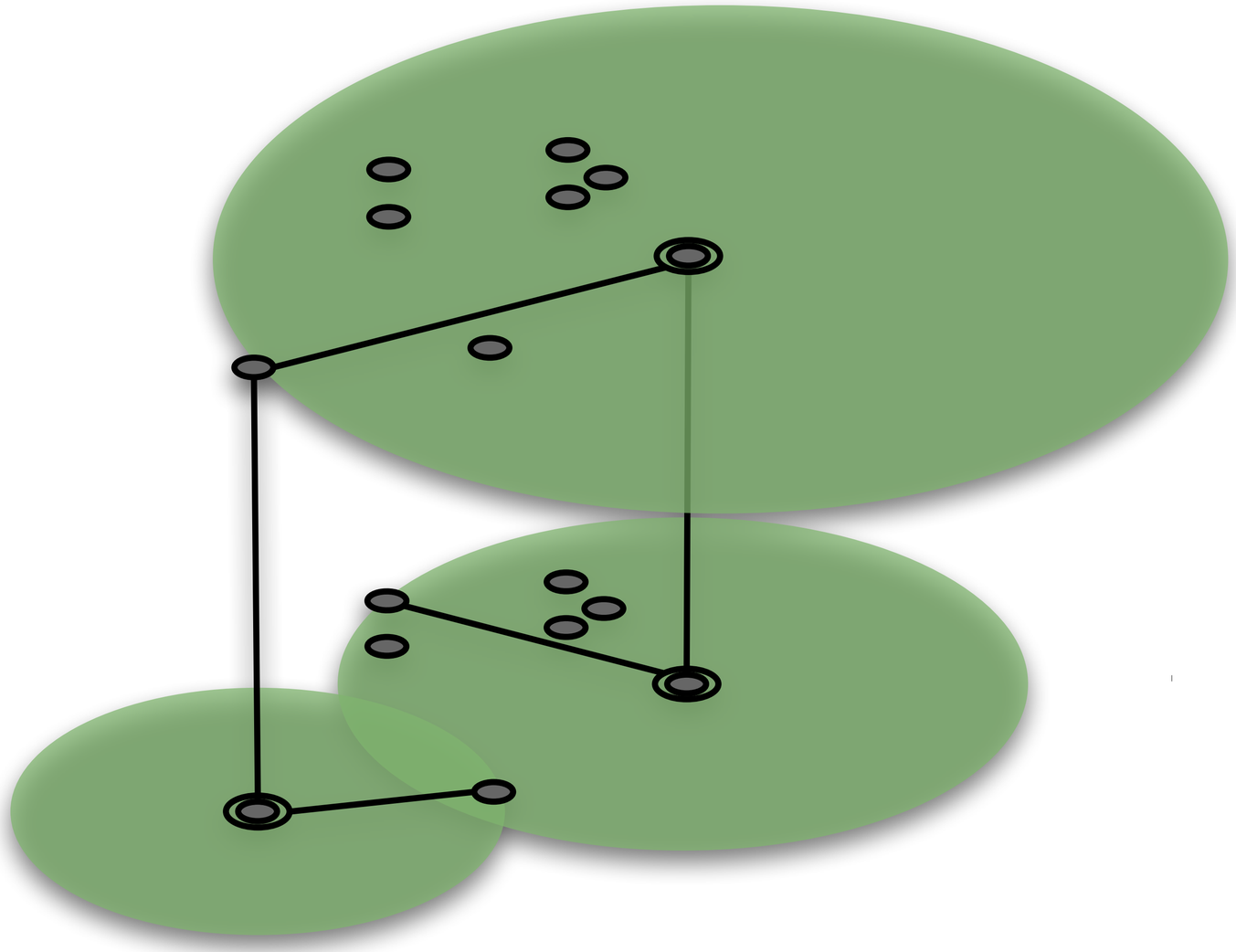

Split a node

- A node covers all the points in the leaves of its subtree.
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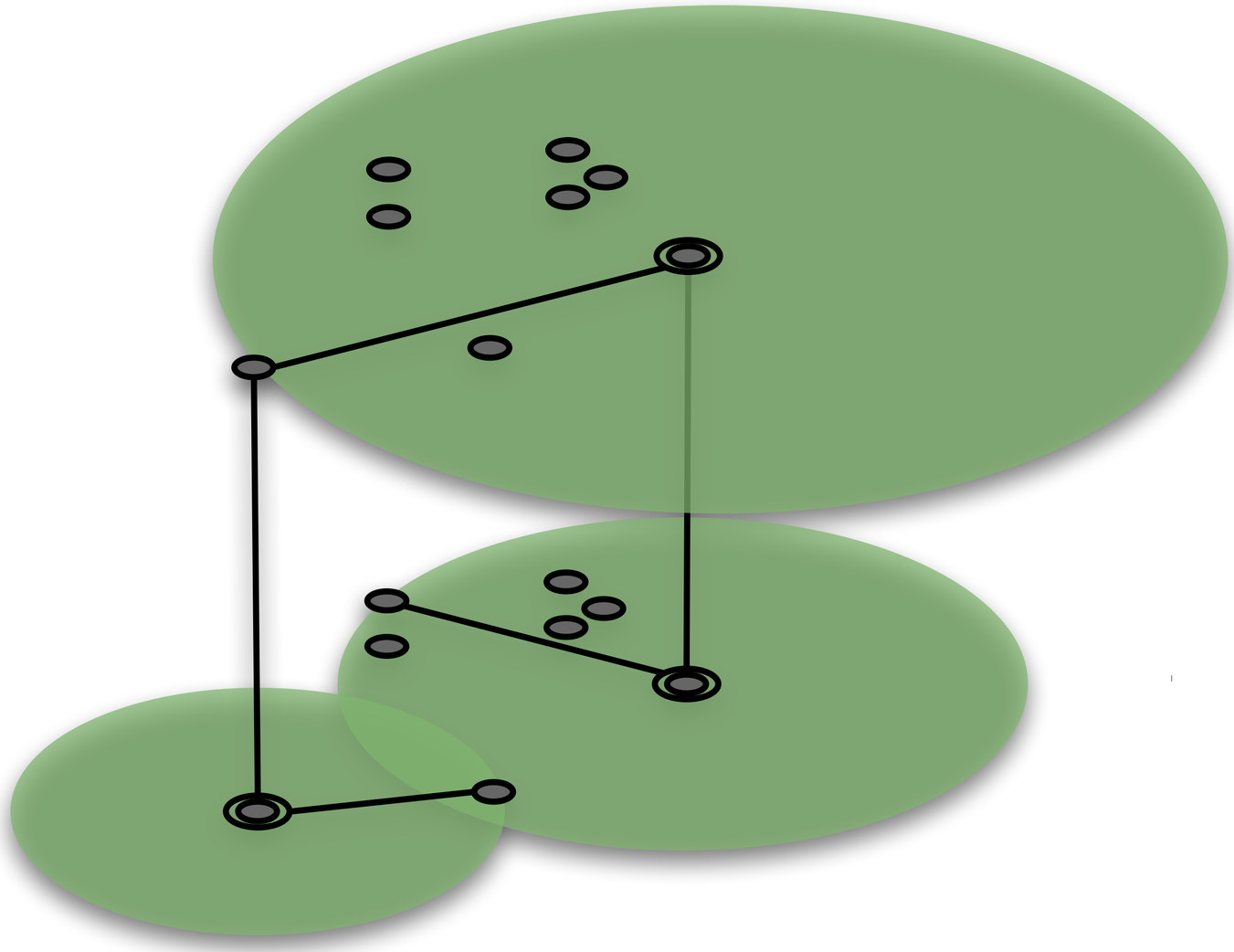
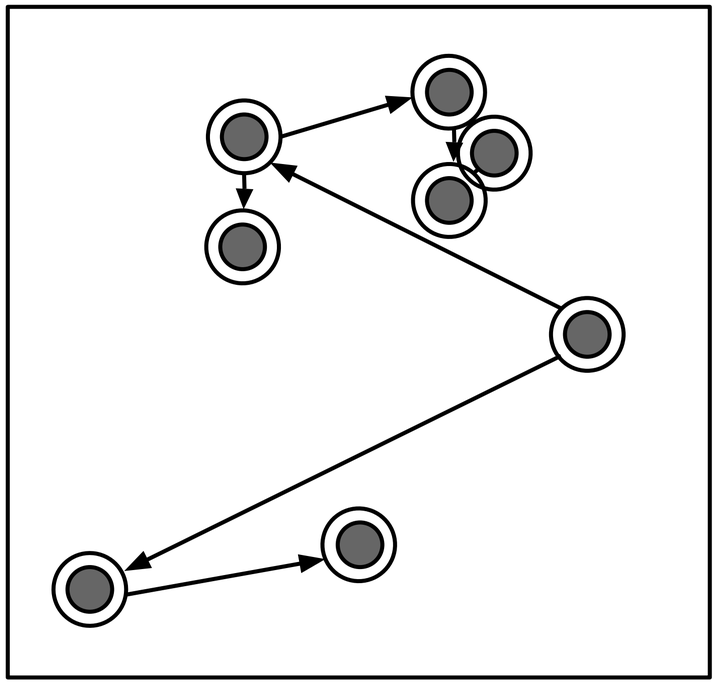
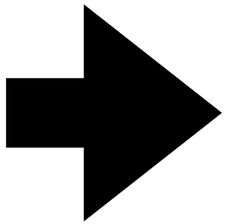
- The children partition the points of their parent.
- Children have smaller radii than their parent.

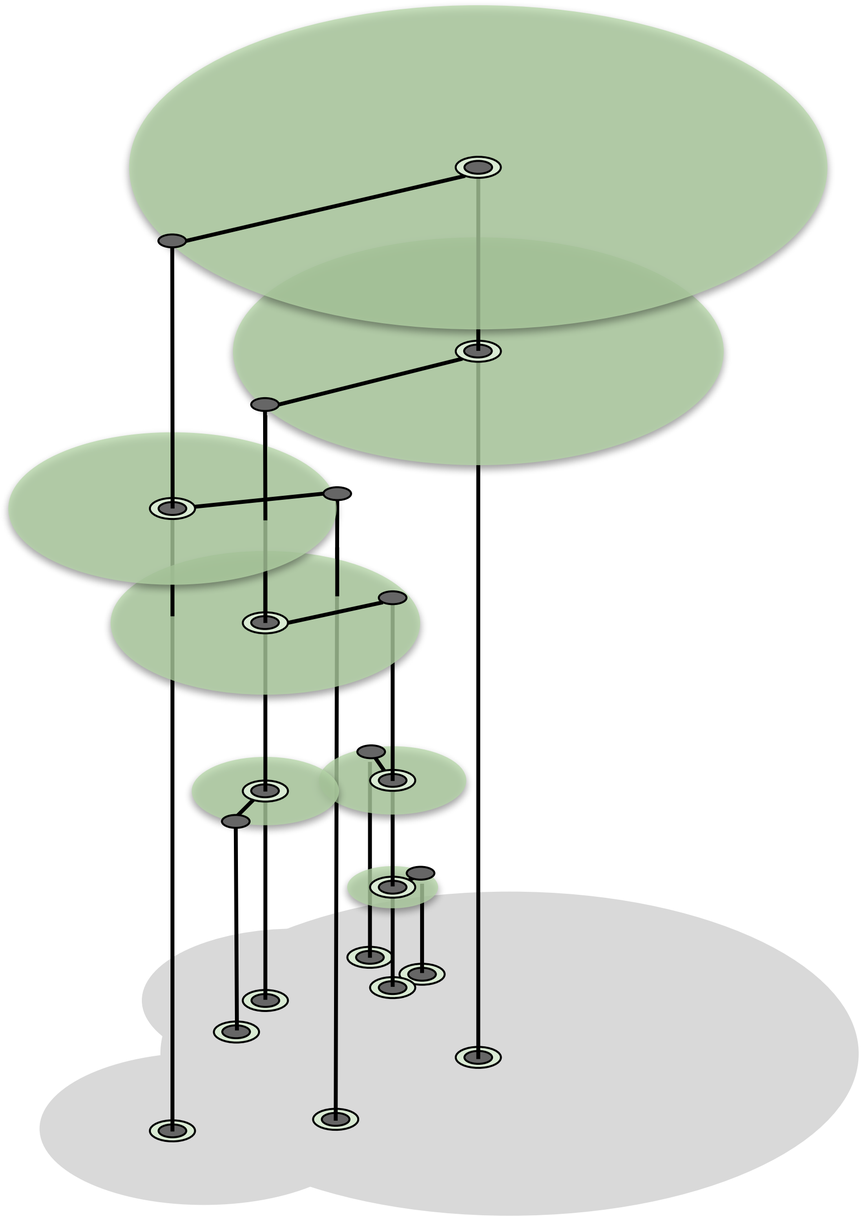


Greedy Trees

A ball tree constructed using a greedy permutation and predecessor mapping







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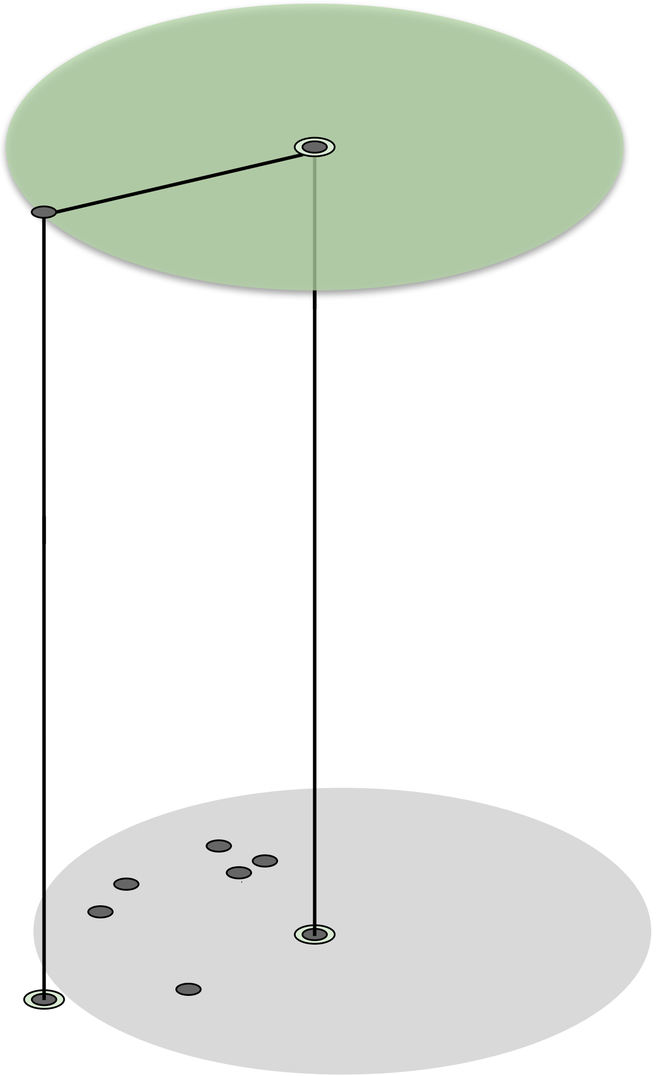

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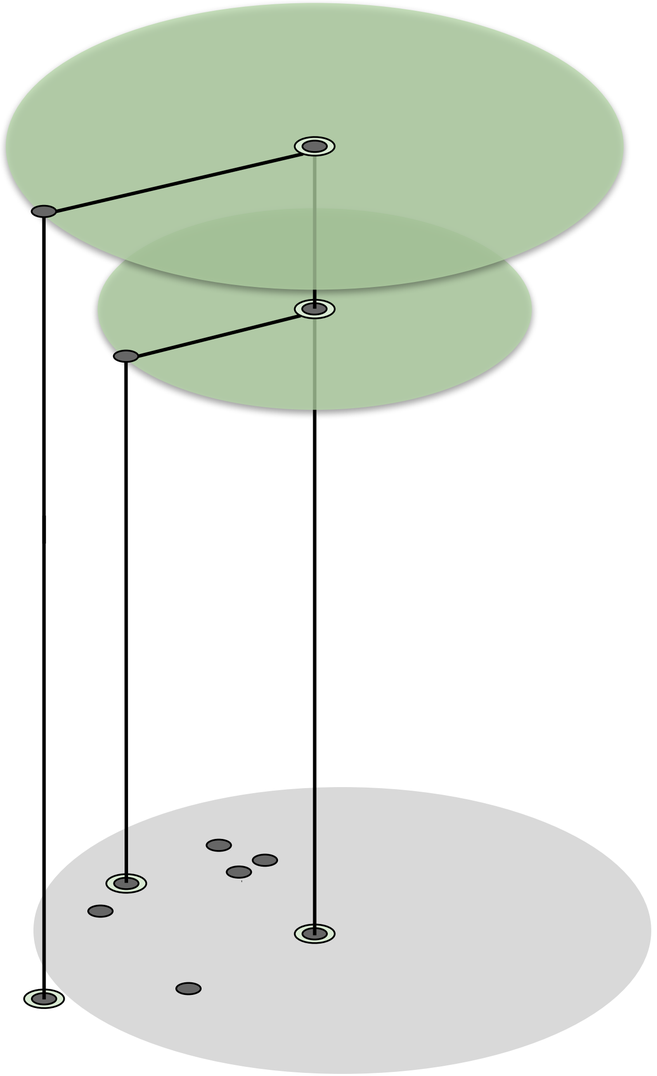

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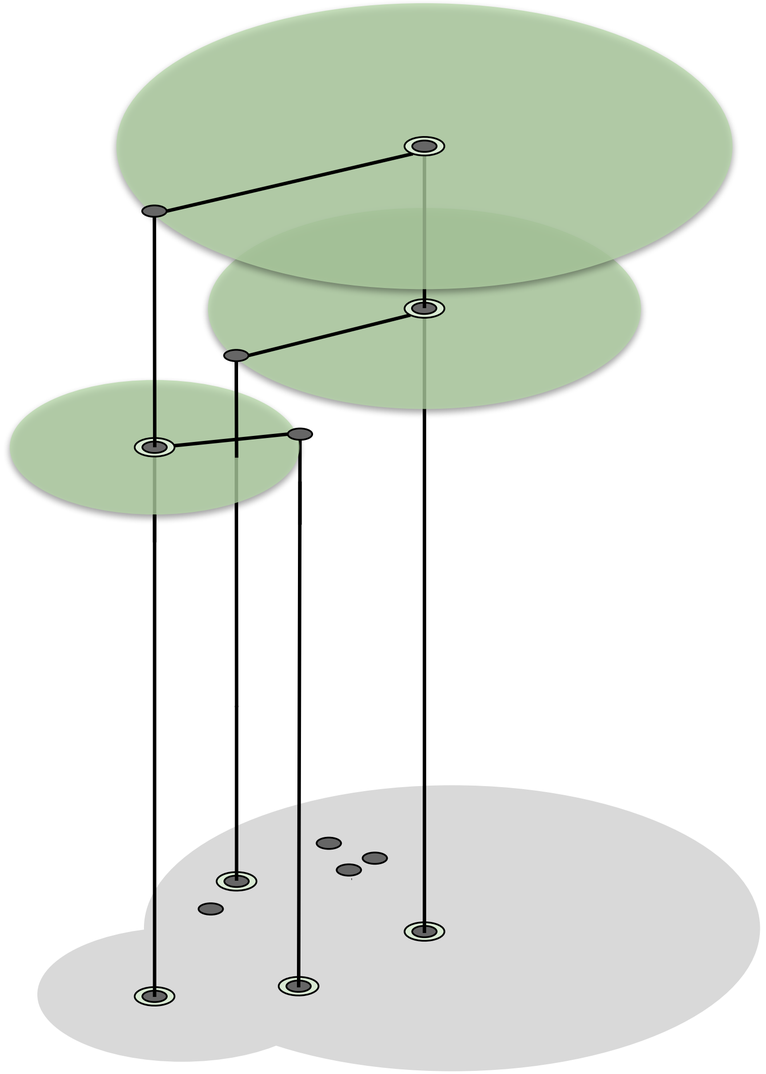

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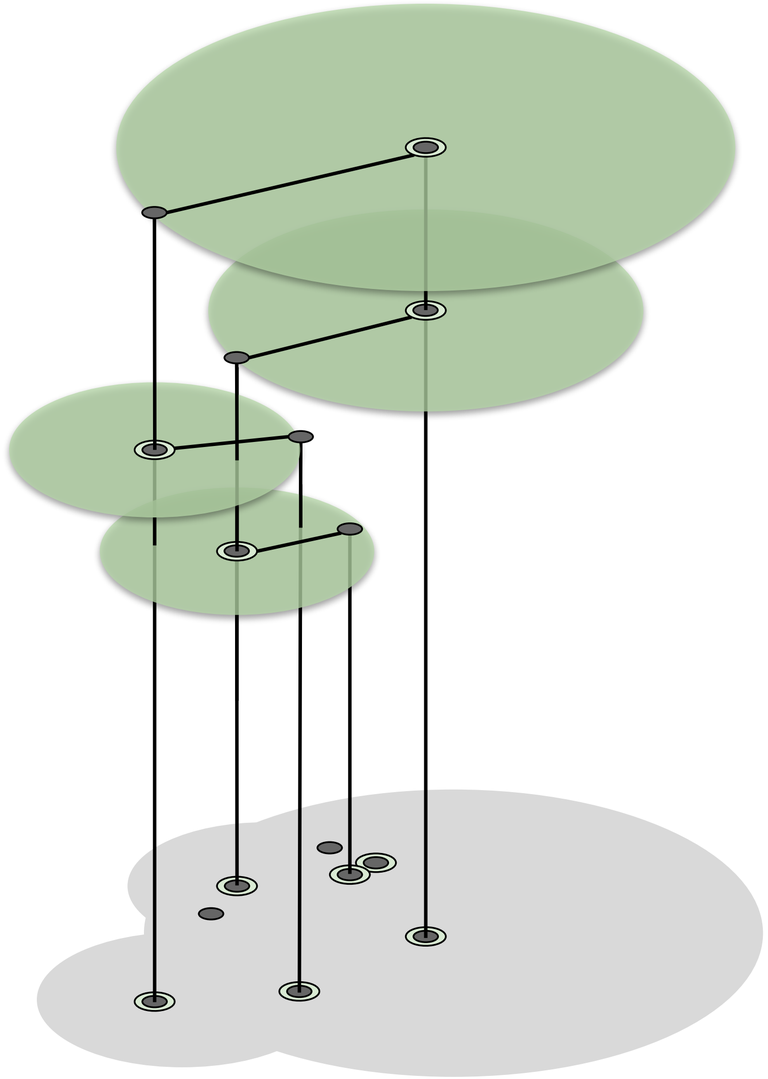

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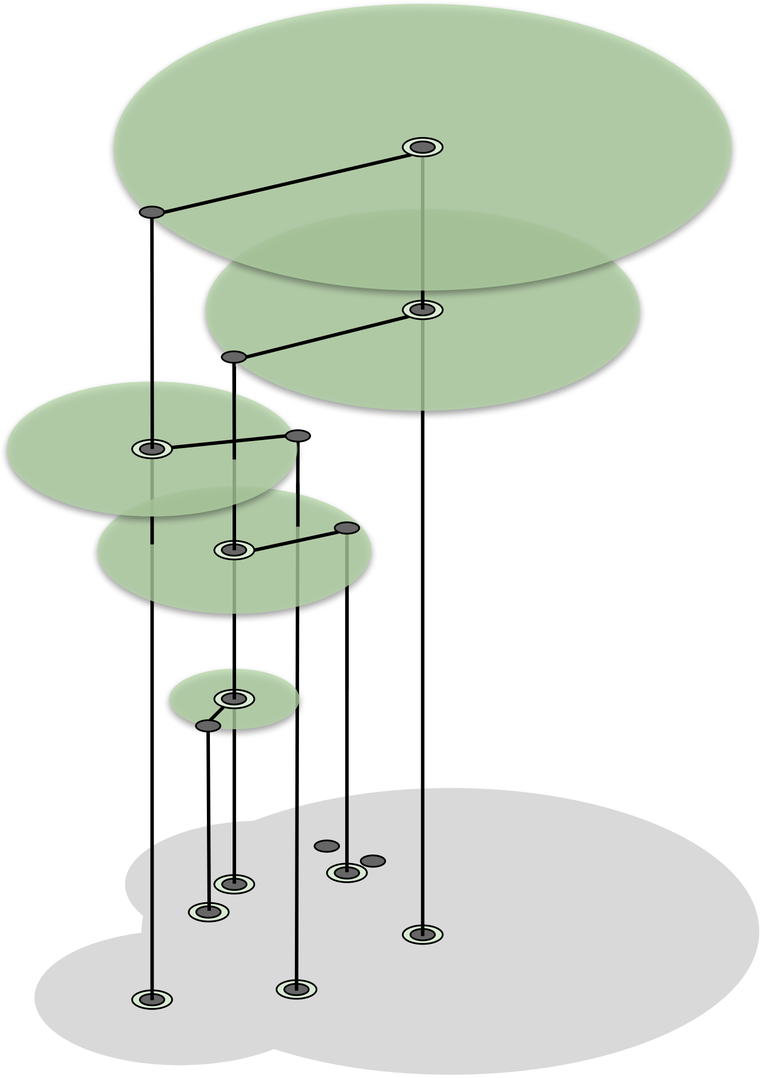

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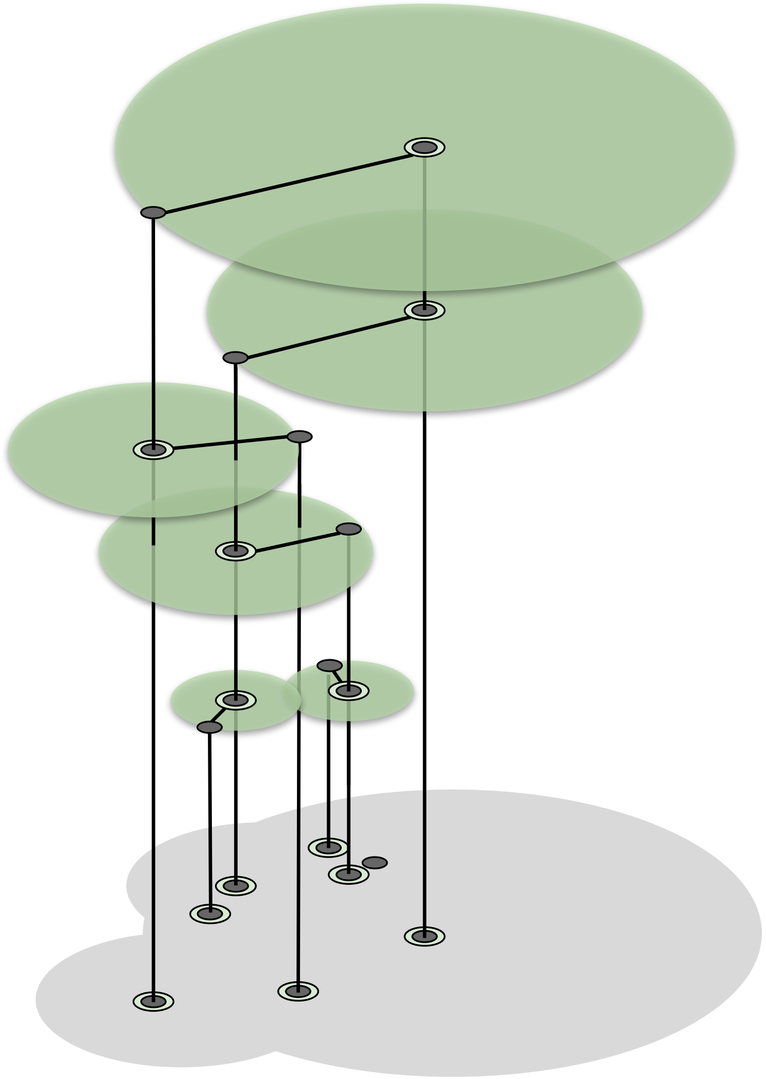

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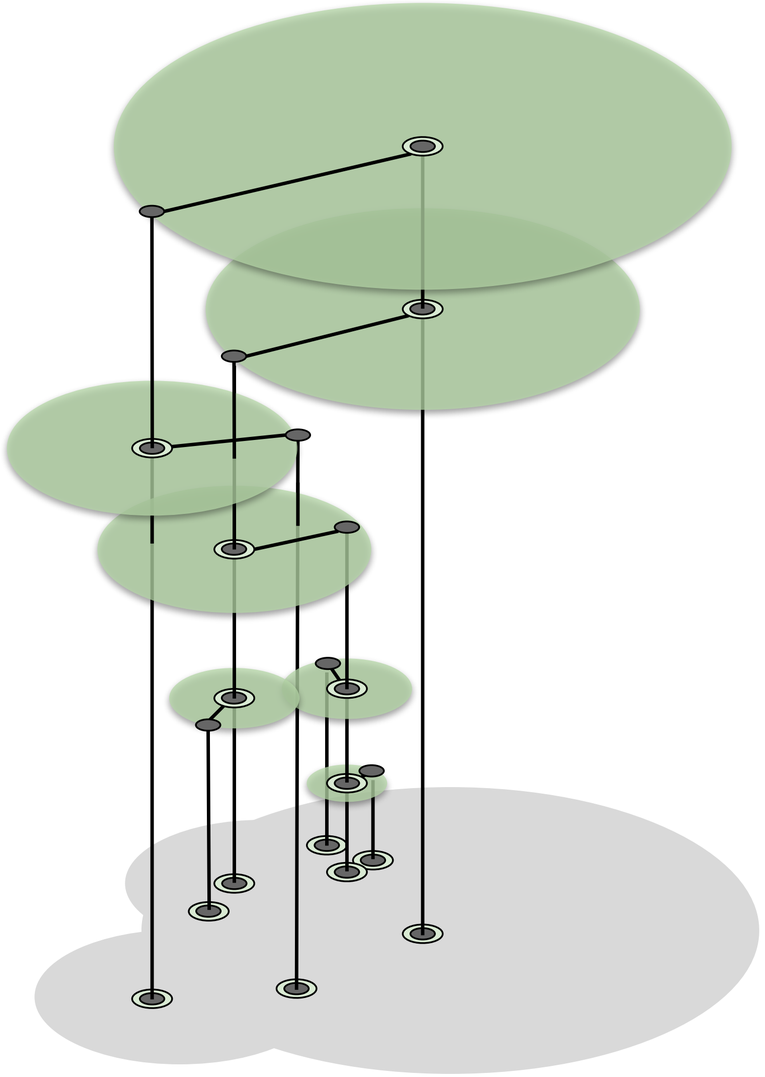
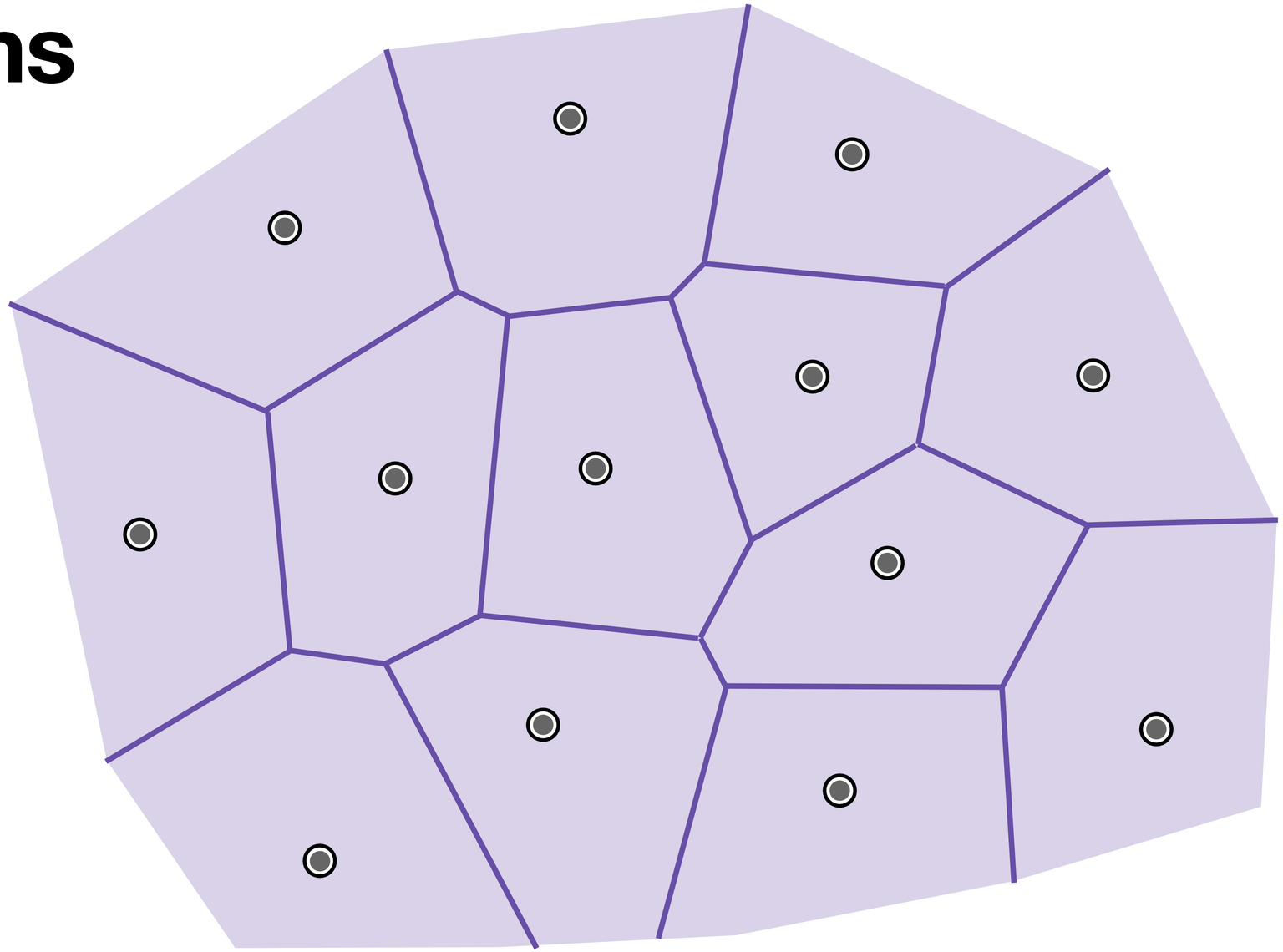


Voronoi Diagrams

There is a set of sites.

Each site is the center of a cell.

Points belong in the cell of the closest site.



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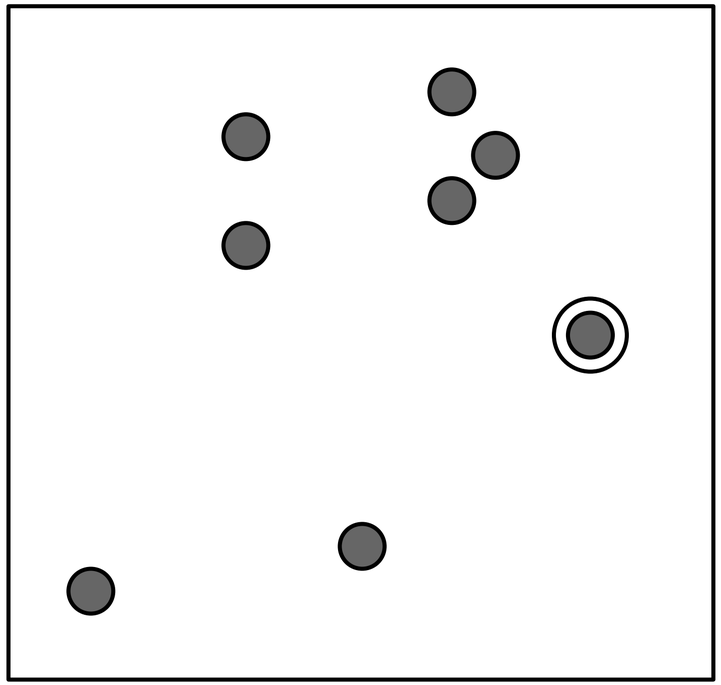
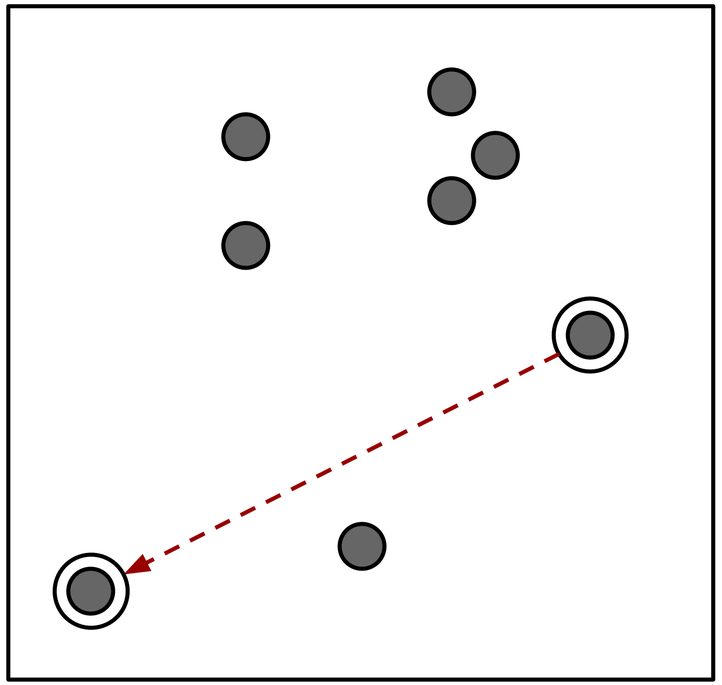
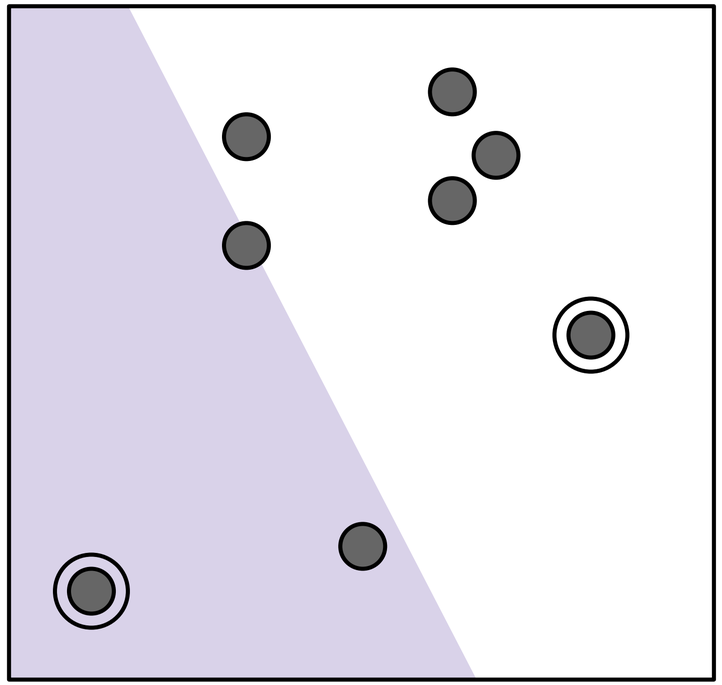


All points start
in the same cell



Insert a point



Point location
(Decide who moves)

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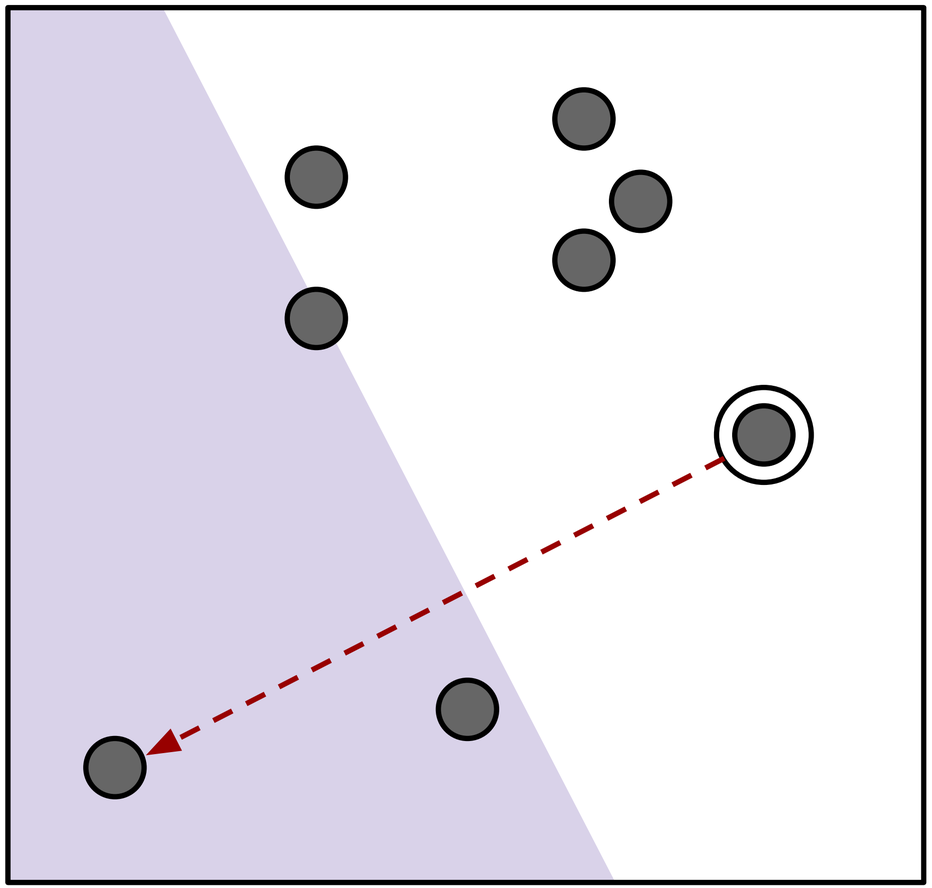

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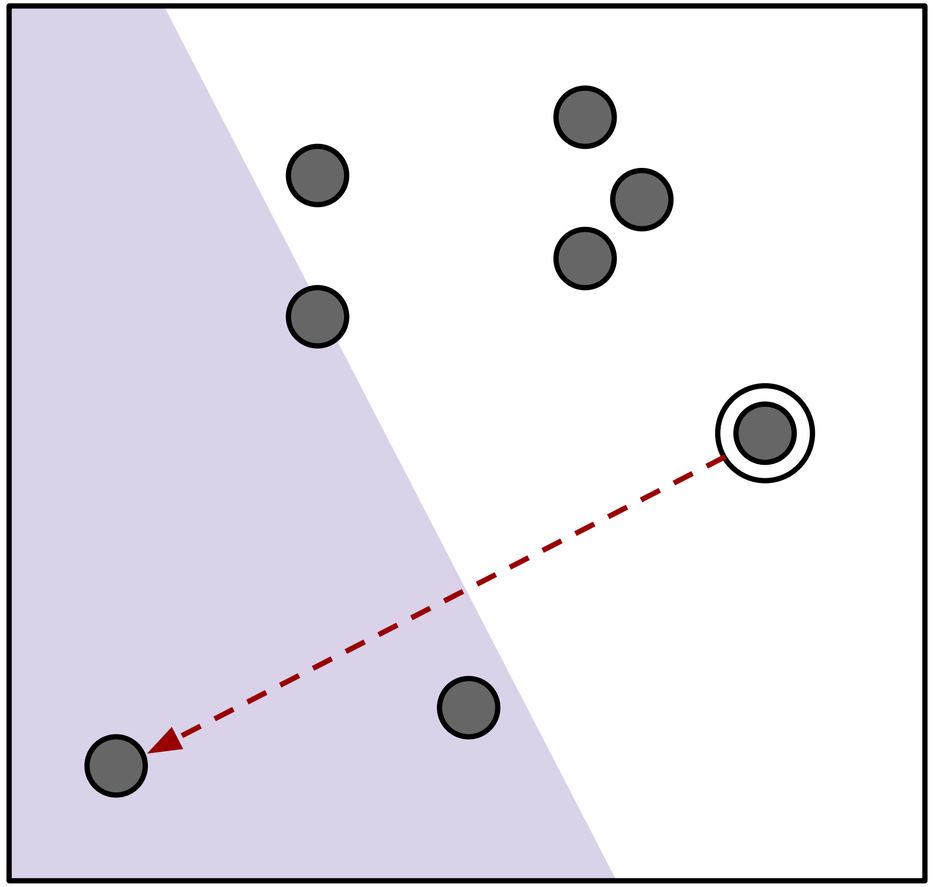

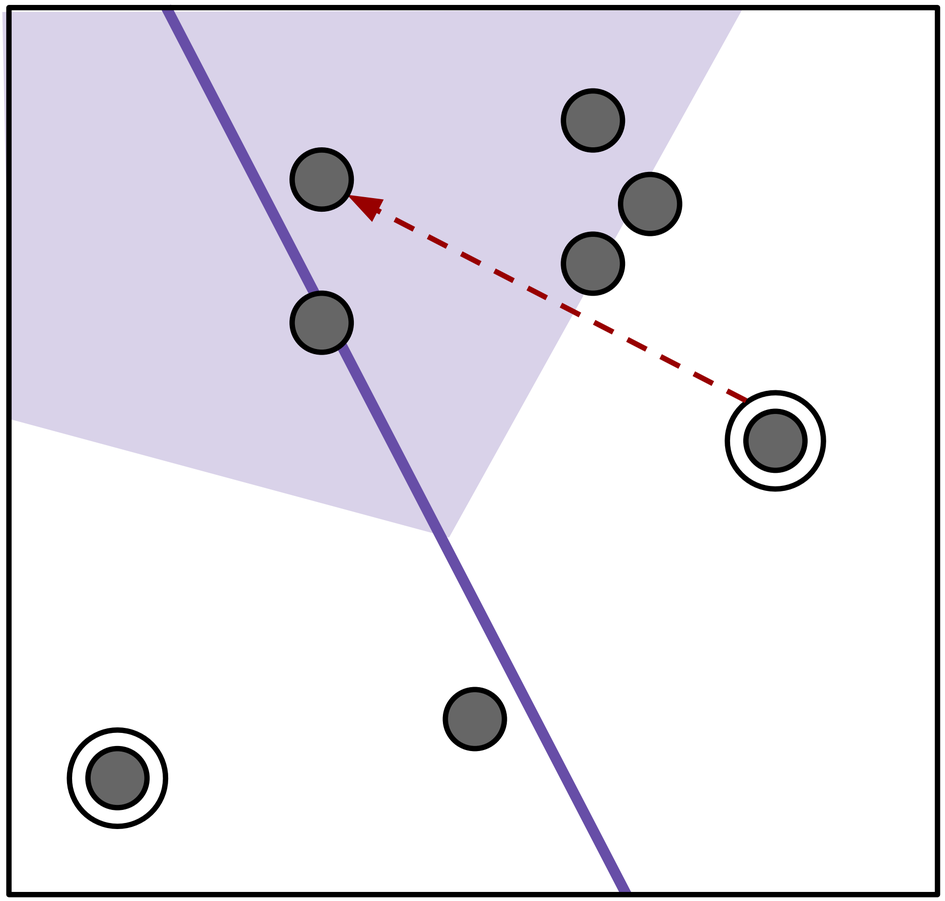

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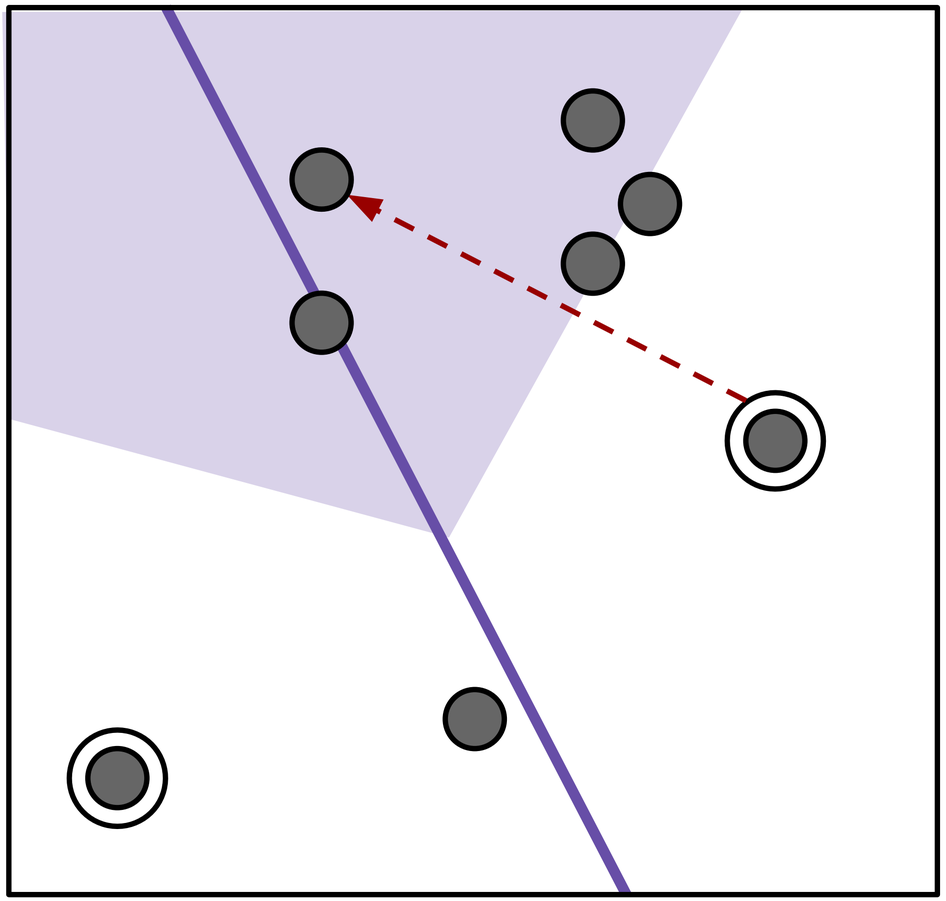

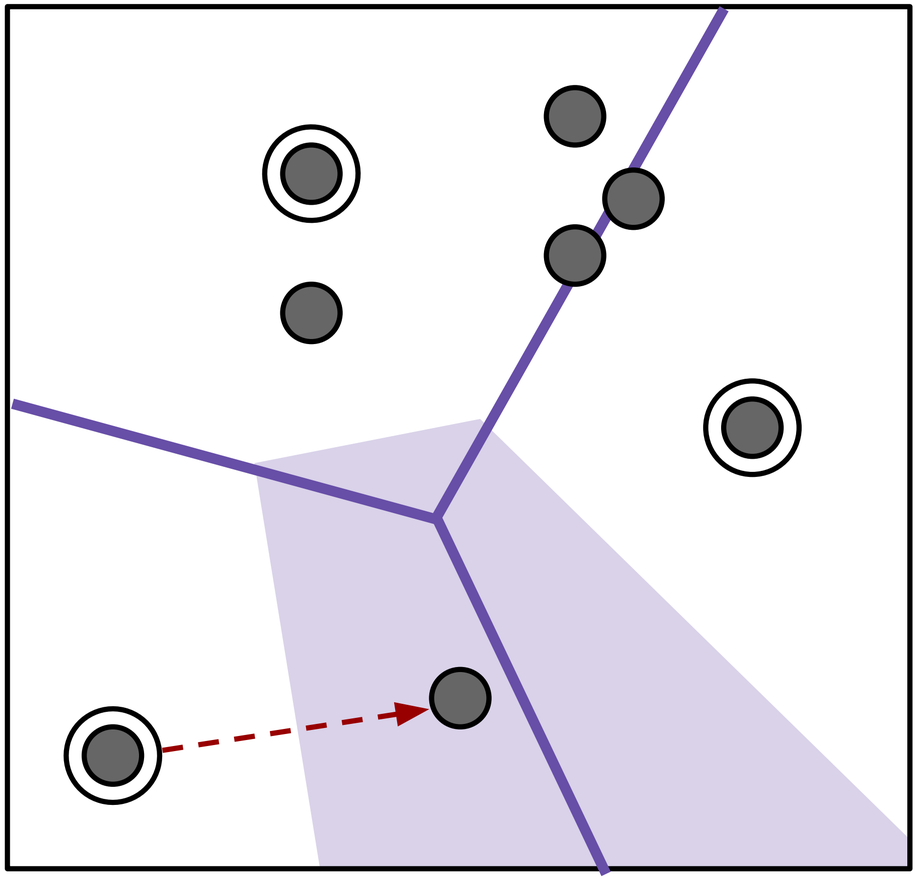

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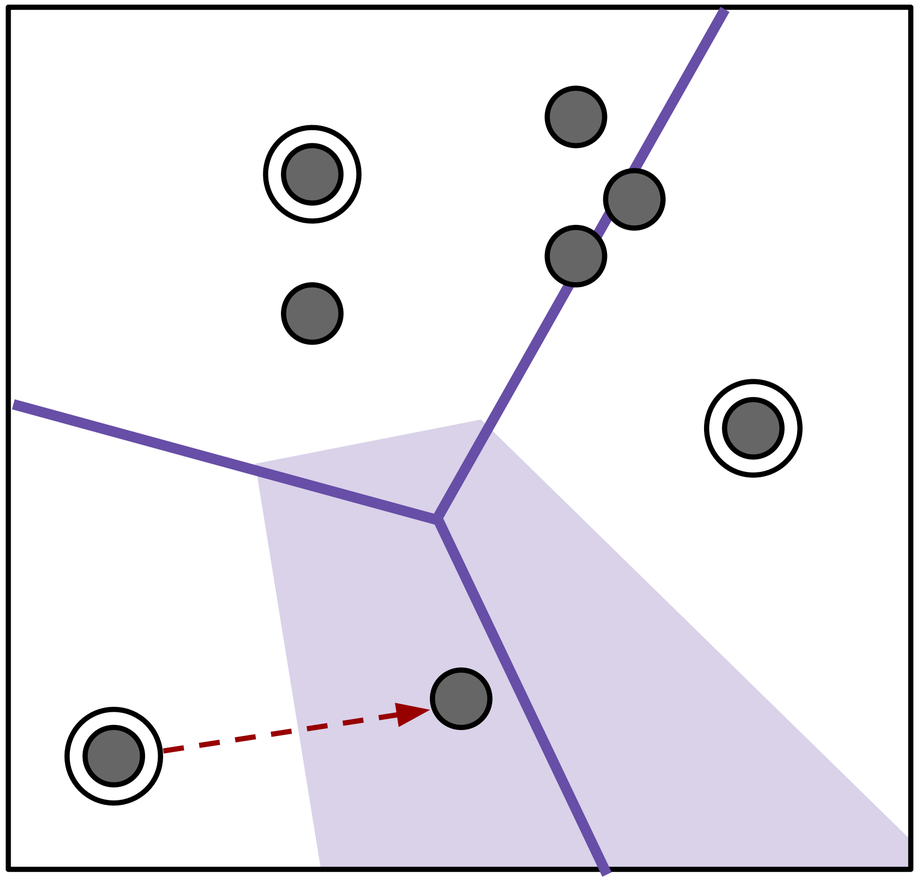

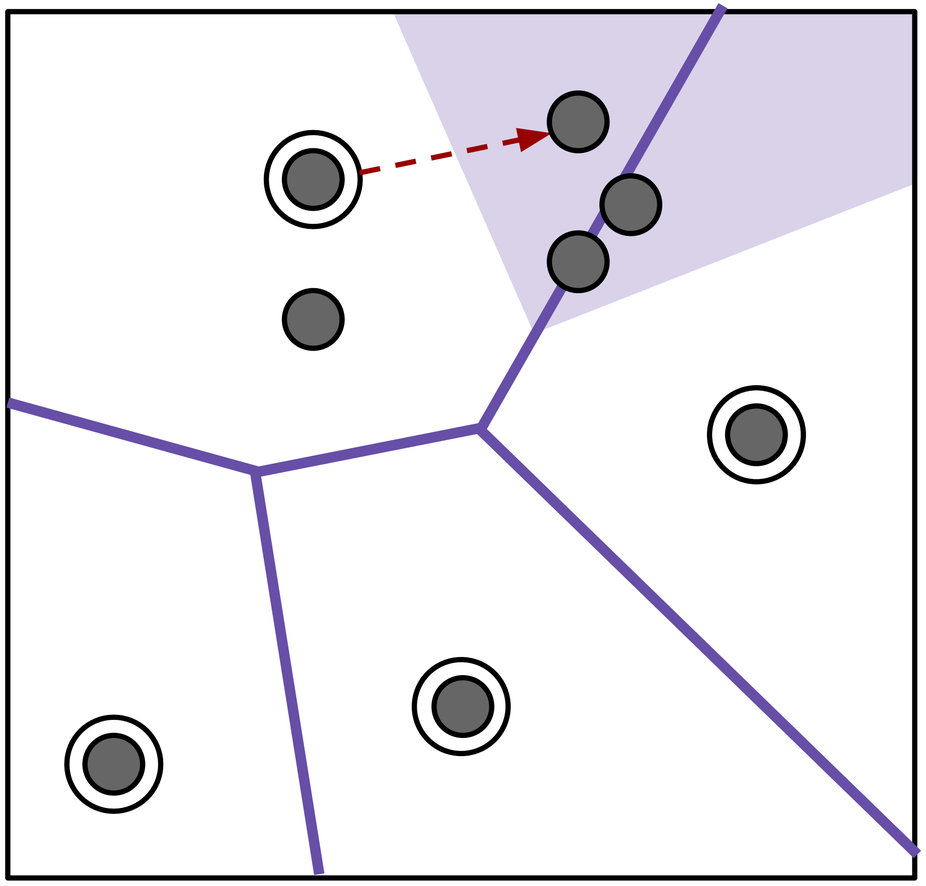

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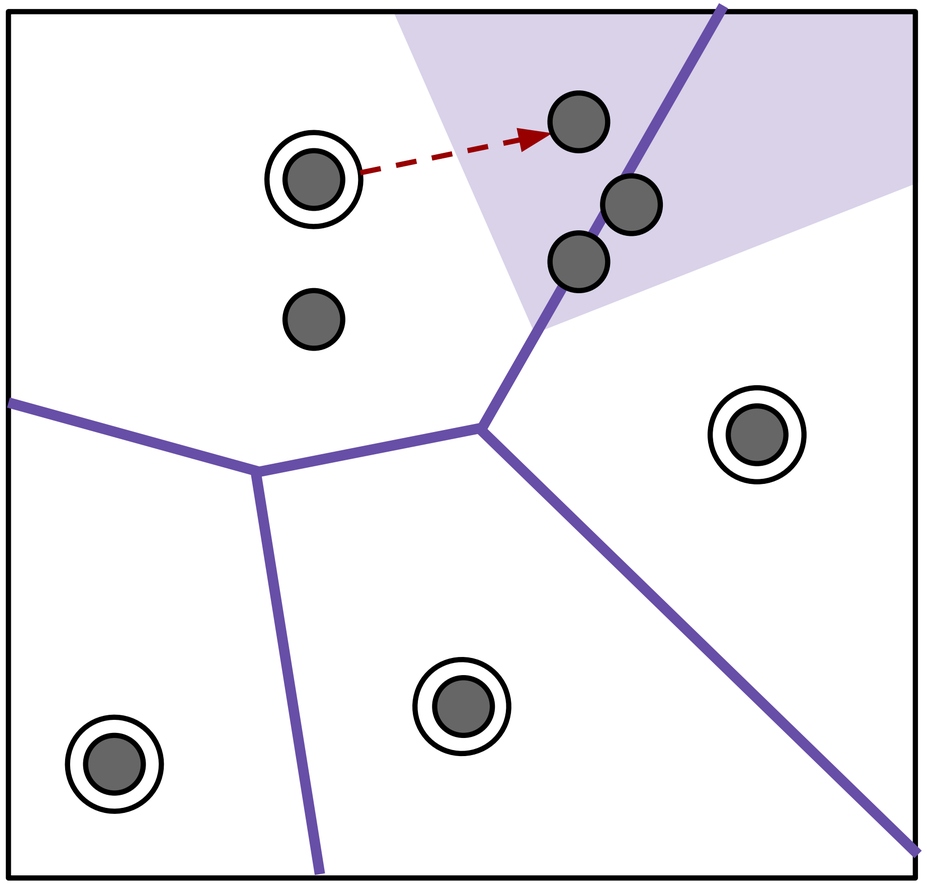

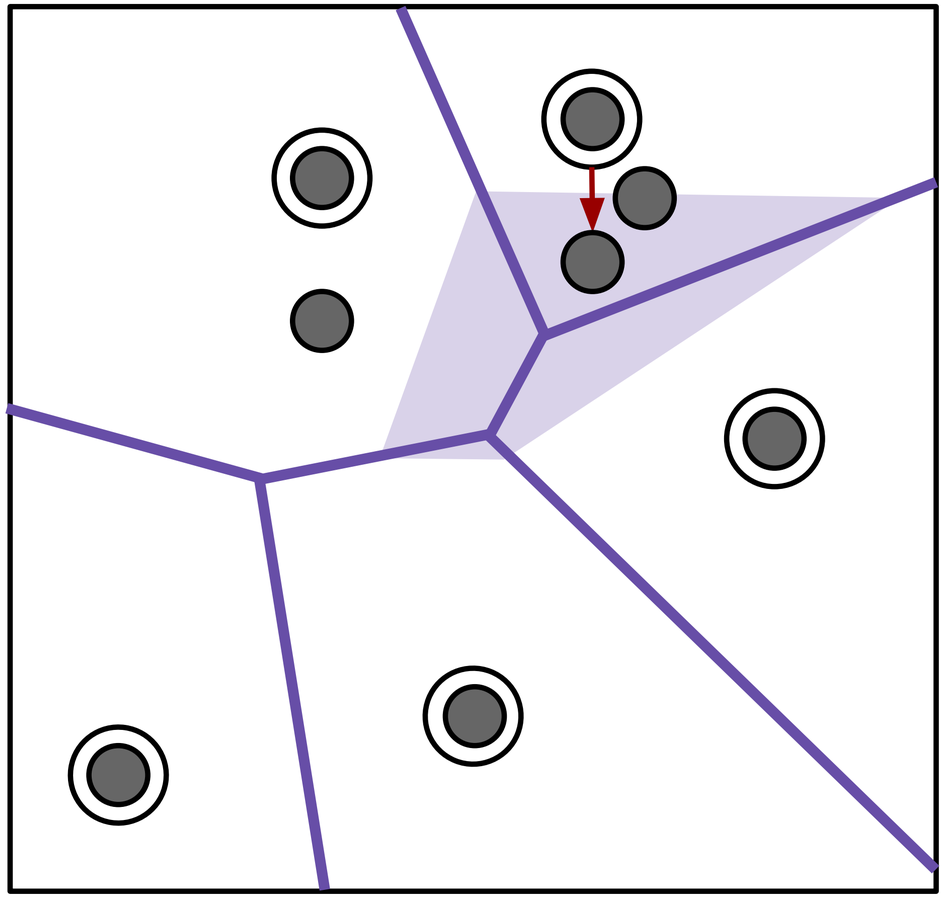

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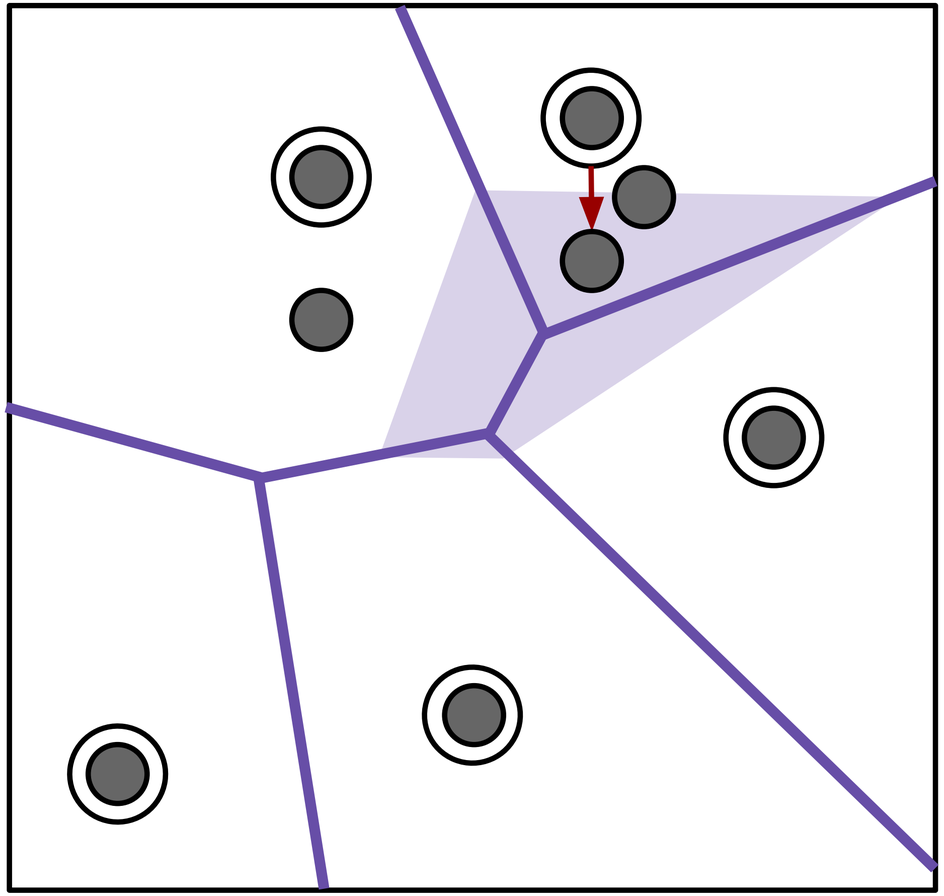

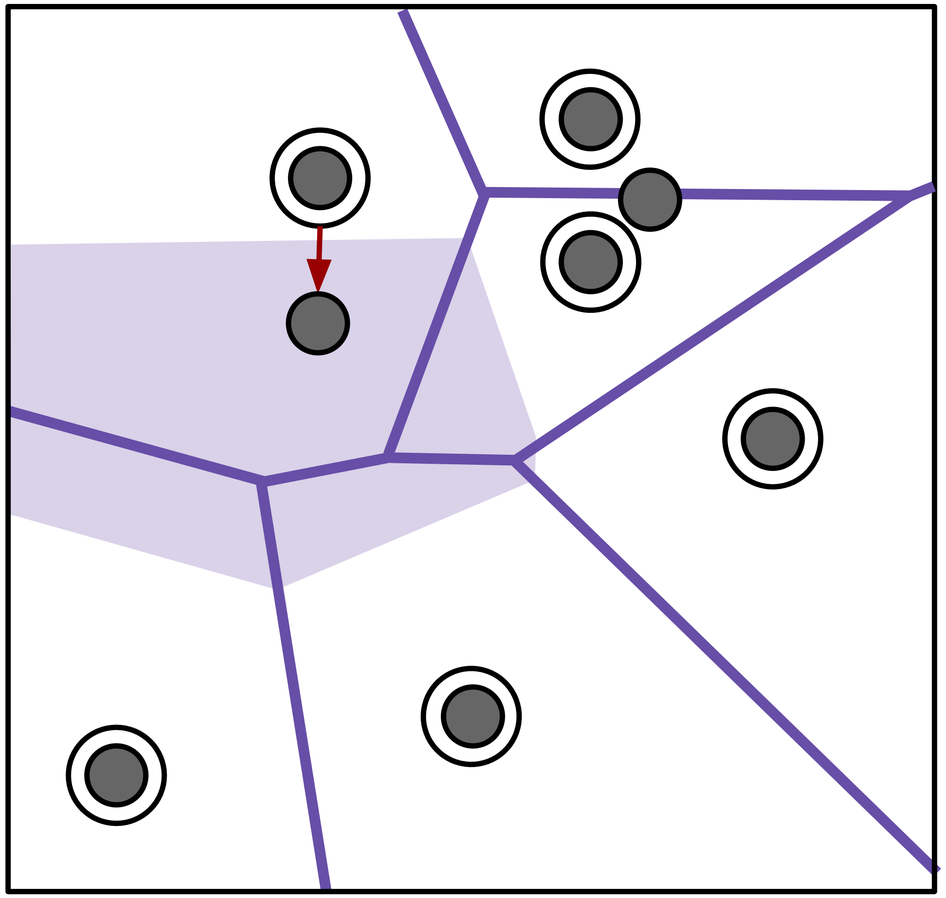

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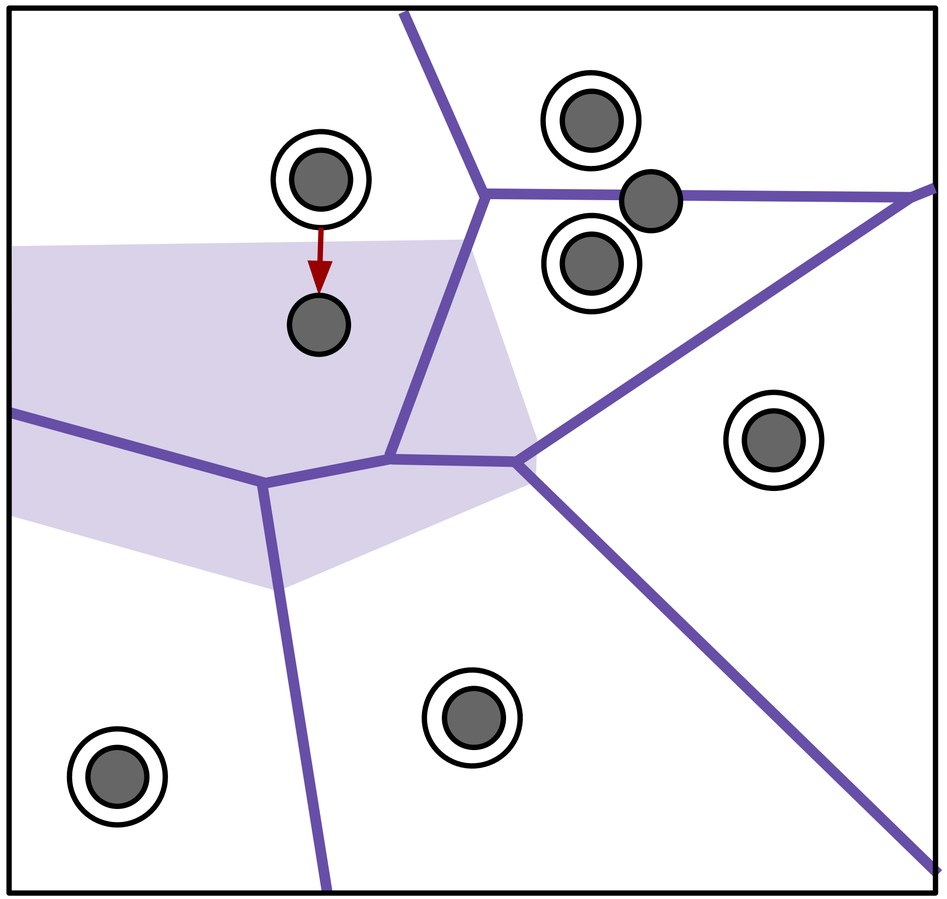

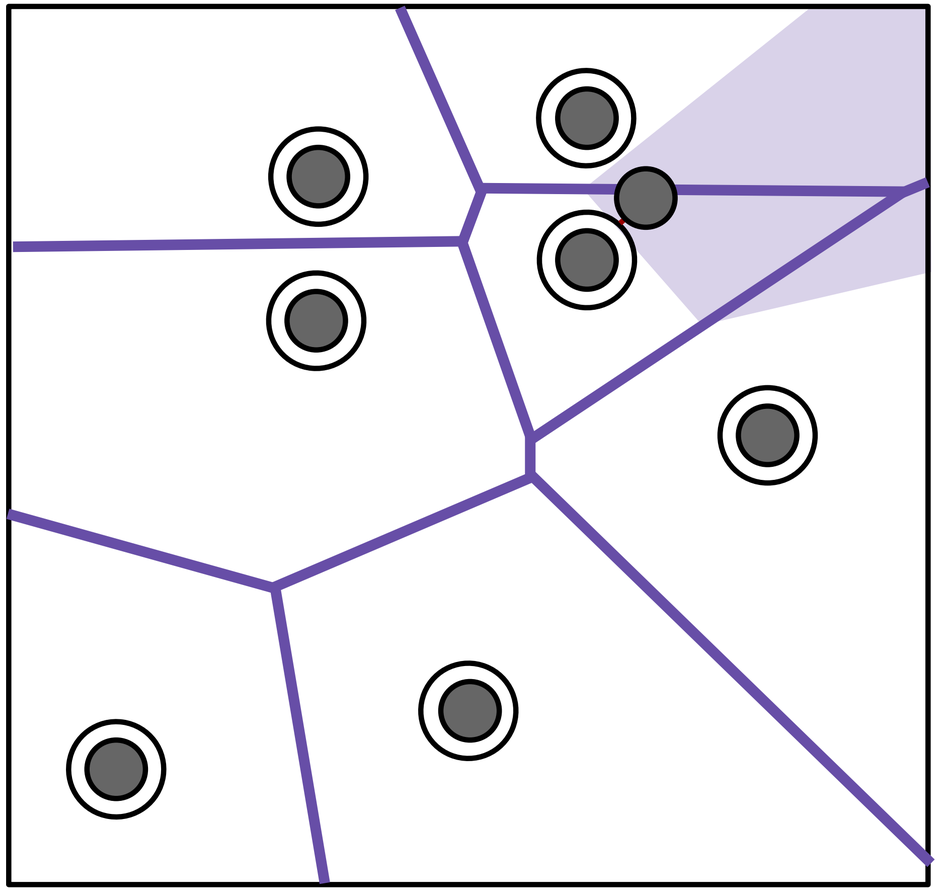

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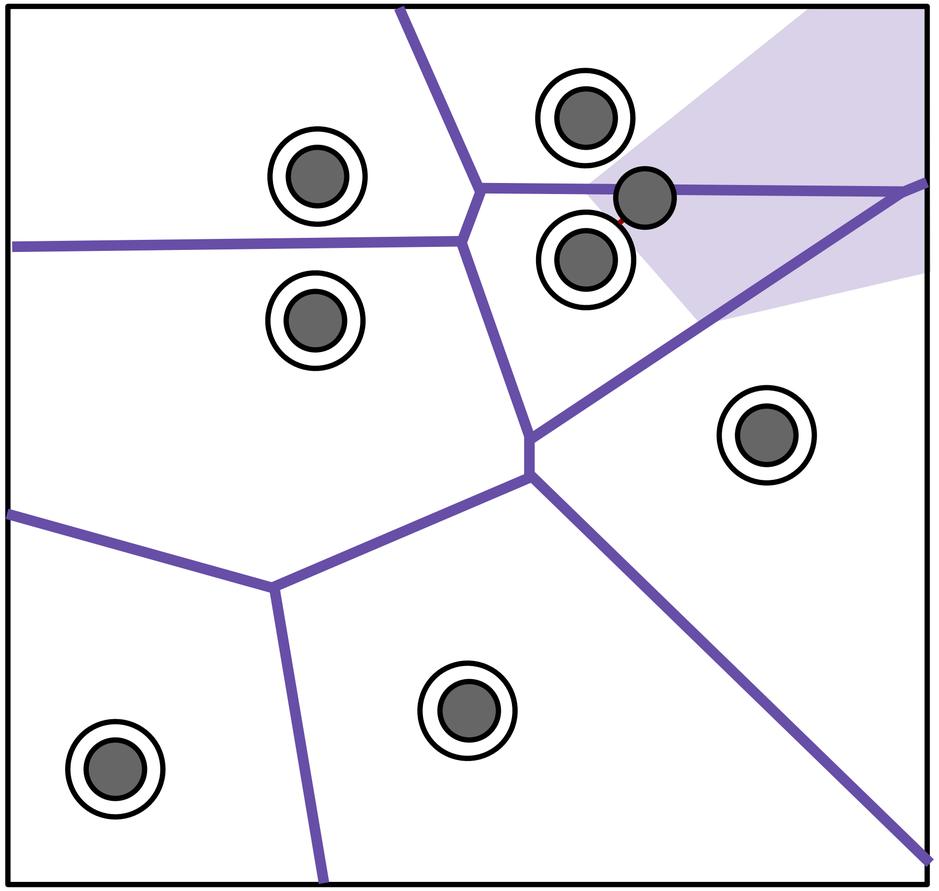

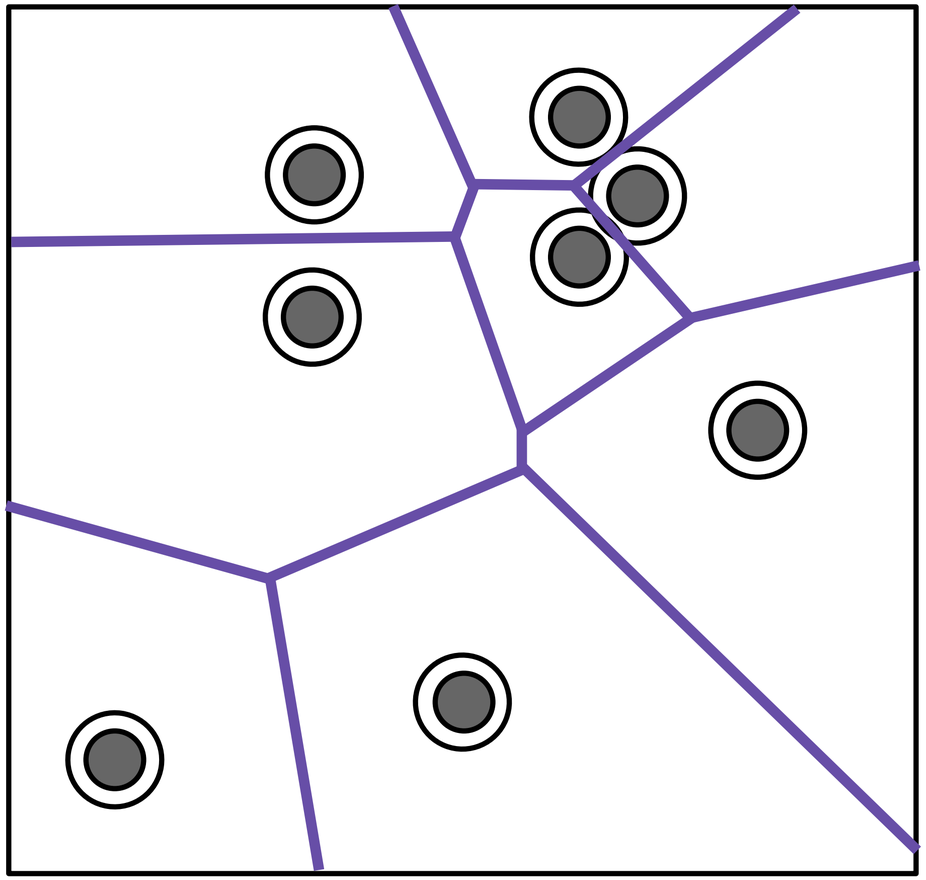

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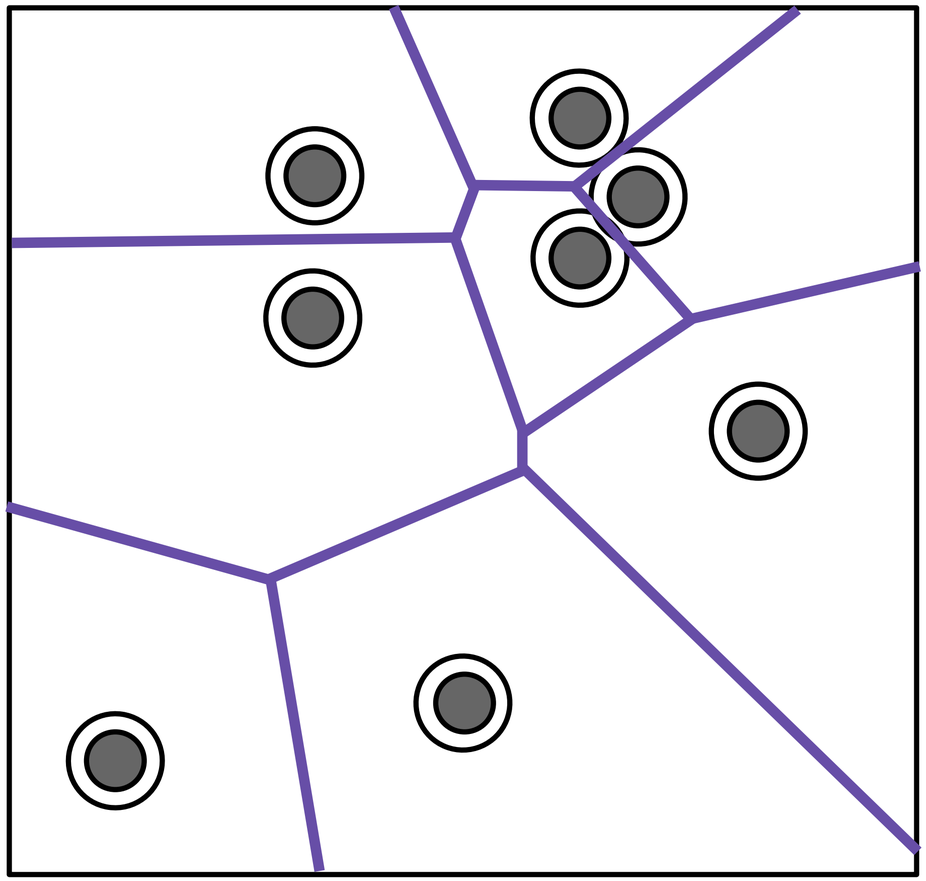

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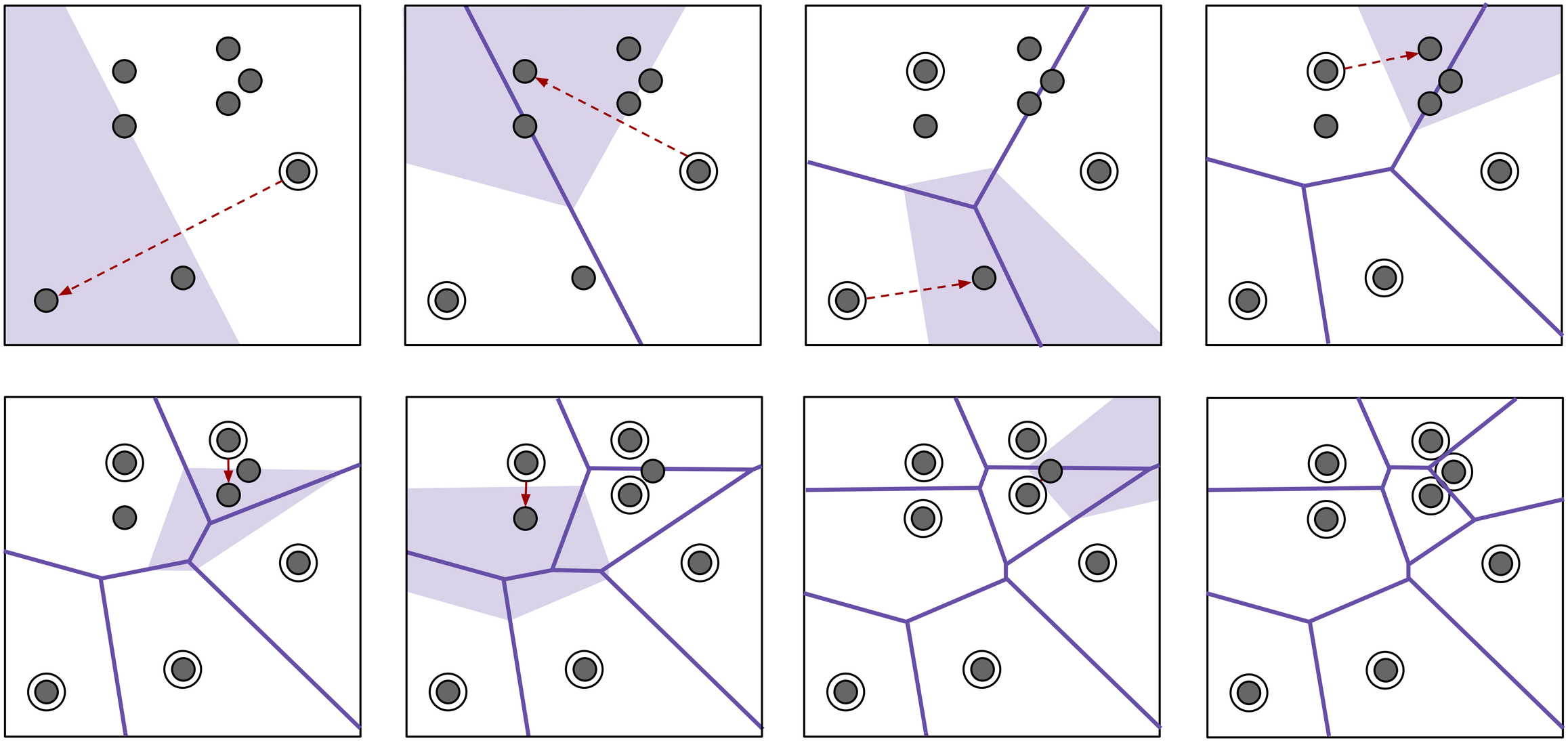

Improving The Naive Algorithm

Improving The Naive Algorithm

1. **Only check points in cells “close enough” to the new site.**

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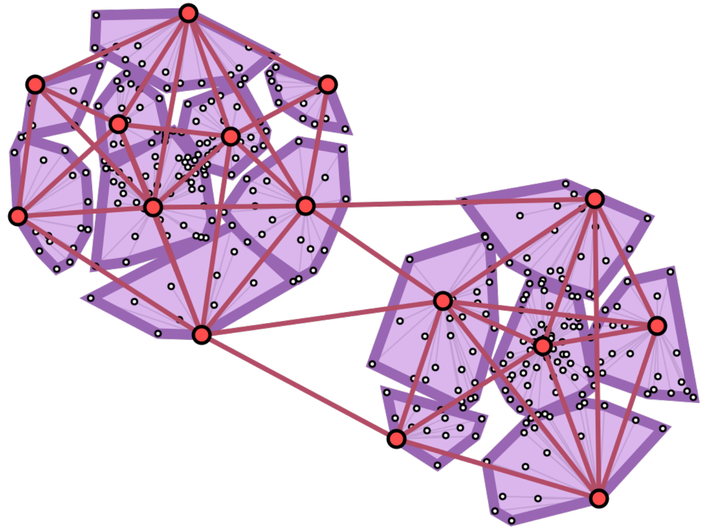
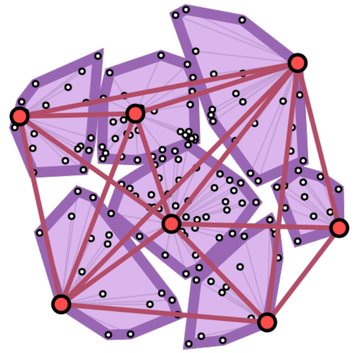
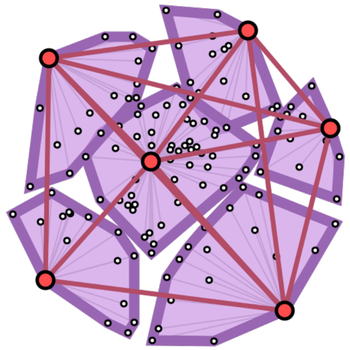
Neighbor Graph

- **In Clarkson's sb data structure, "close enough" information is stored as edges in a graph.**
- **Har-Peled and Mendel show that Clarkson's algorithm runs in $O(n \log \Delta)$.**
    - Δ *is the ratio of the diameter to distance of the closest pair*

Neighbor Graph







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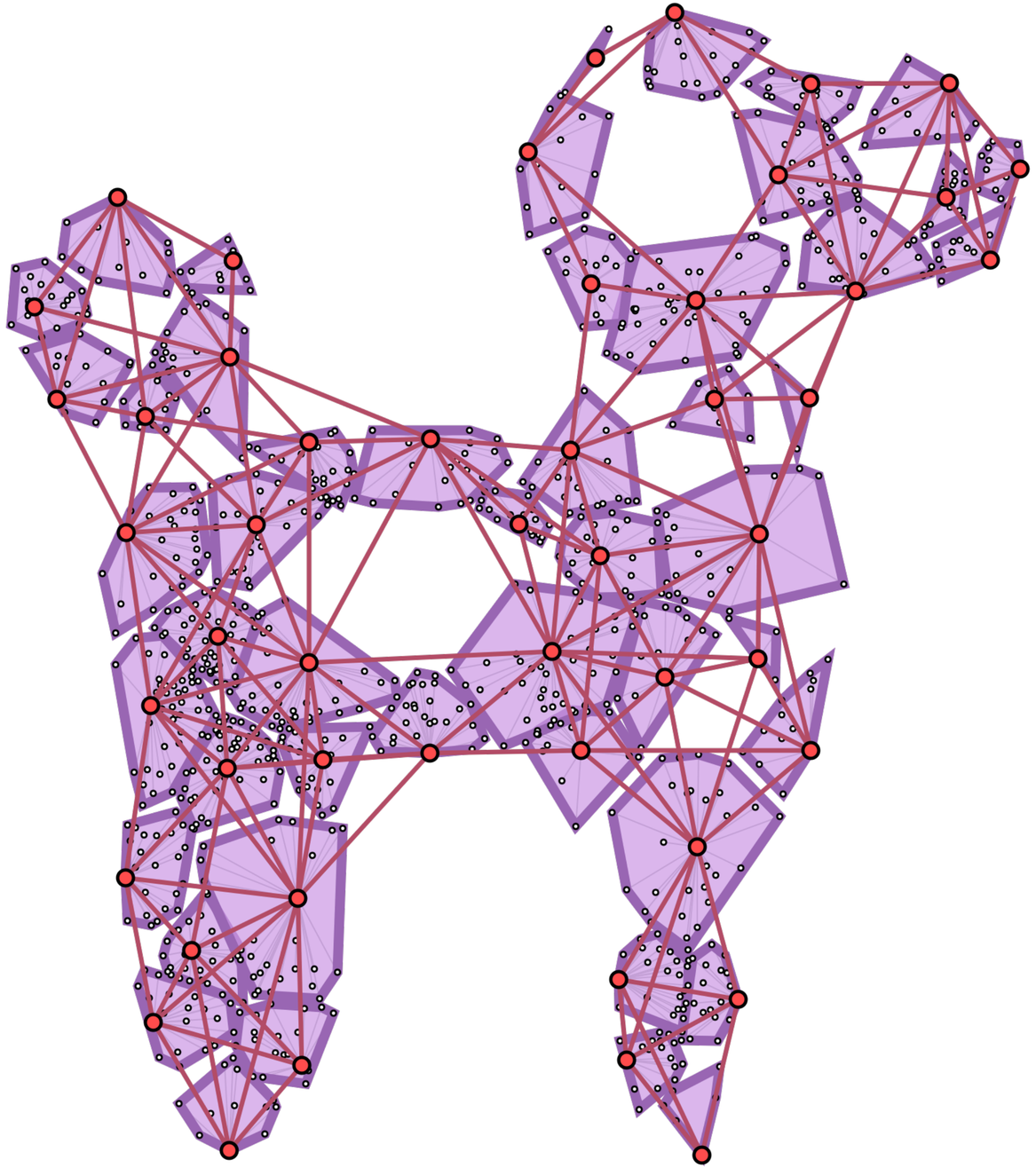
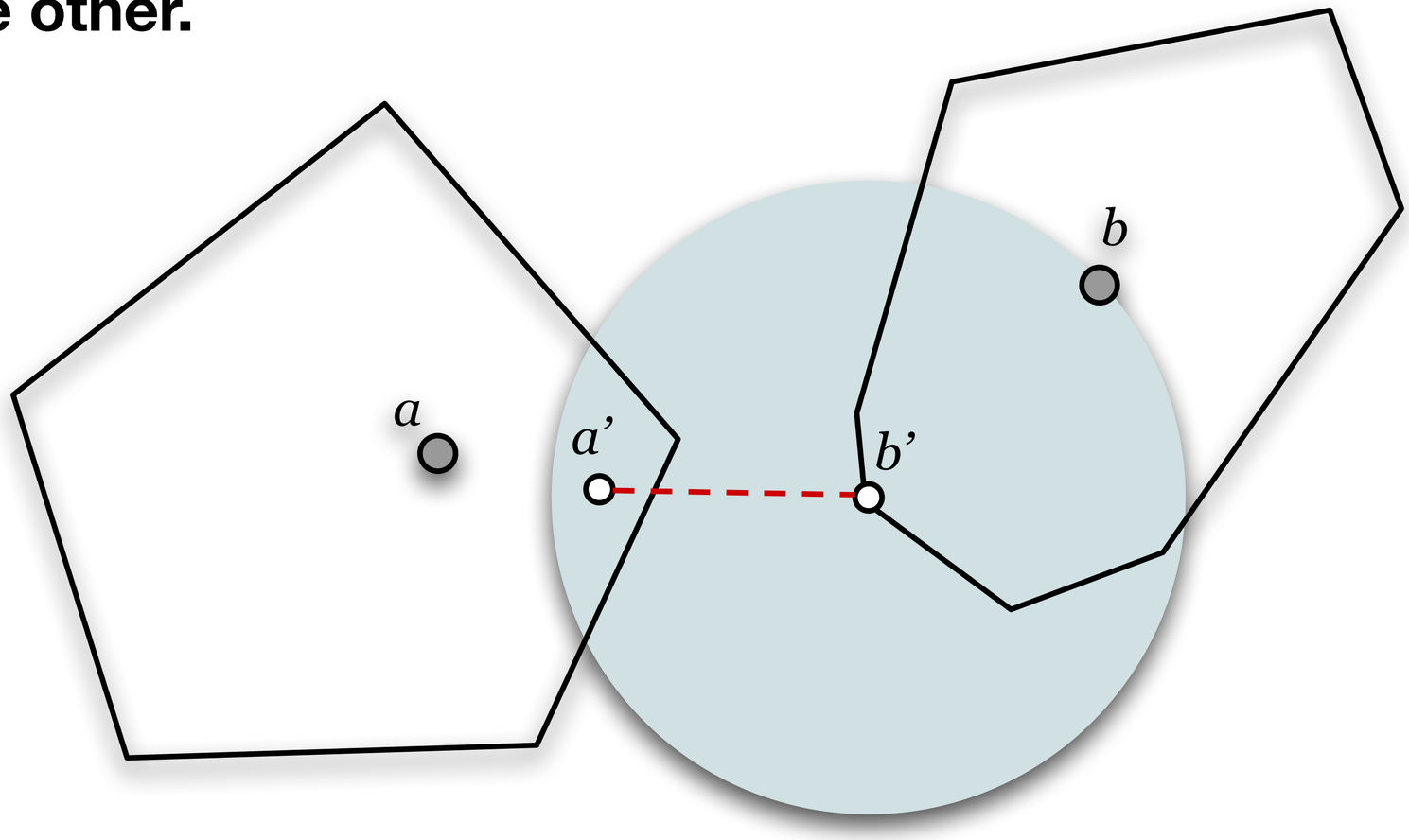


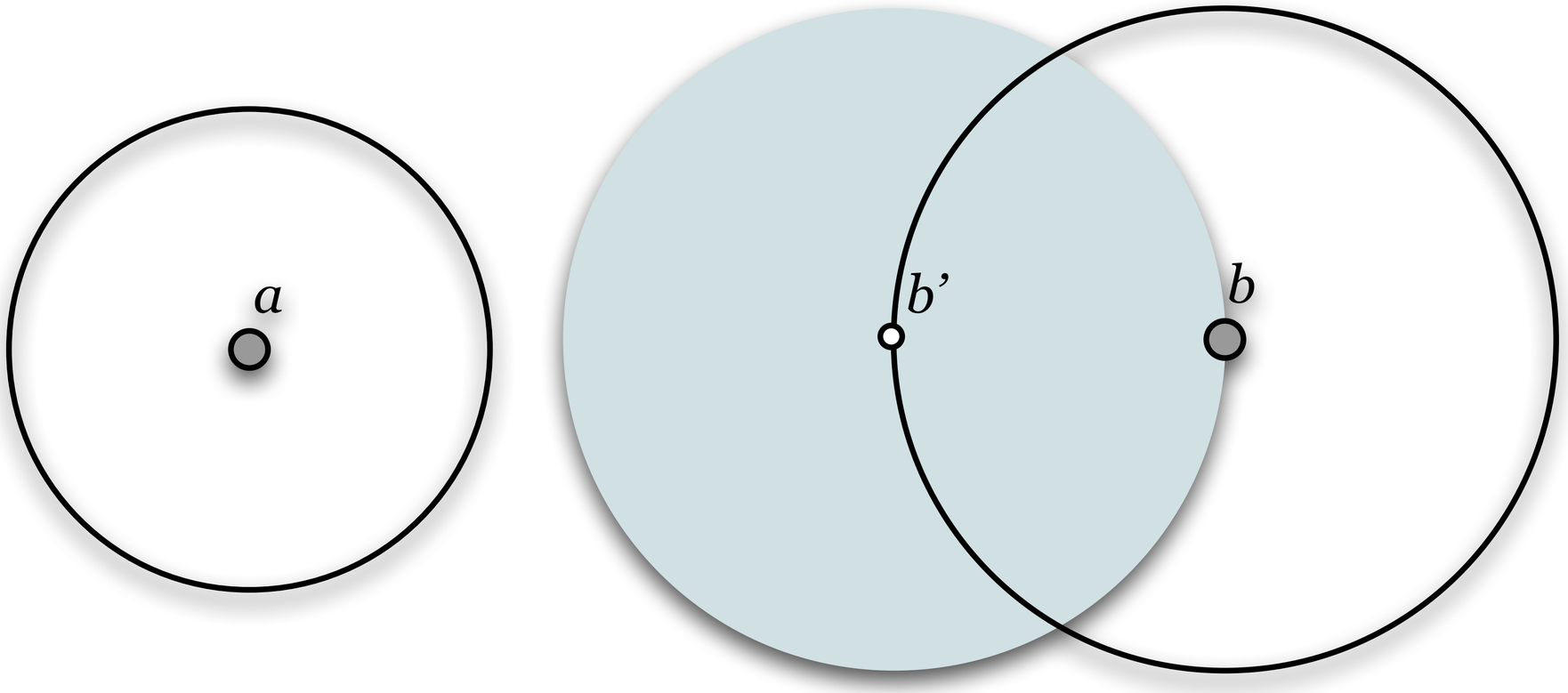
Neighbor Graph

- **Keep an edge if inserting a point from one cell would cause a point to move in the other.**



Pruning Condition

- **Approximate cells by their out-radii to prune unnecessary edges**



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2. We use a bucket queue to achieve constant-time heap operations.
3. Storing clusters in cells introduces some sources of approximation during construction.

Storing Clusters

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- Use a greedy as the input

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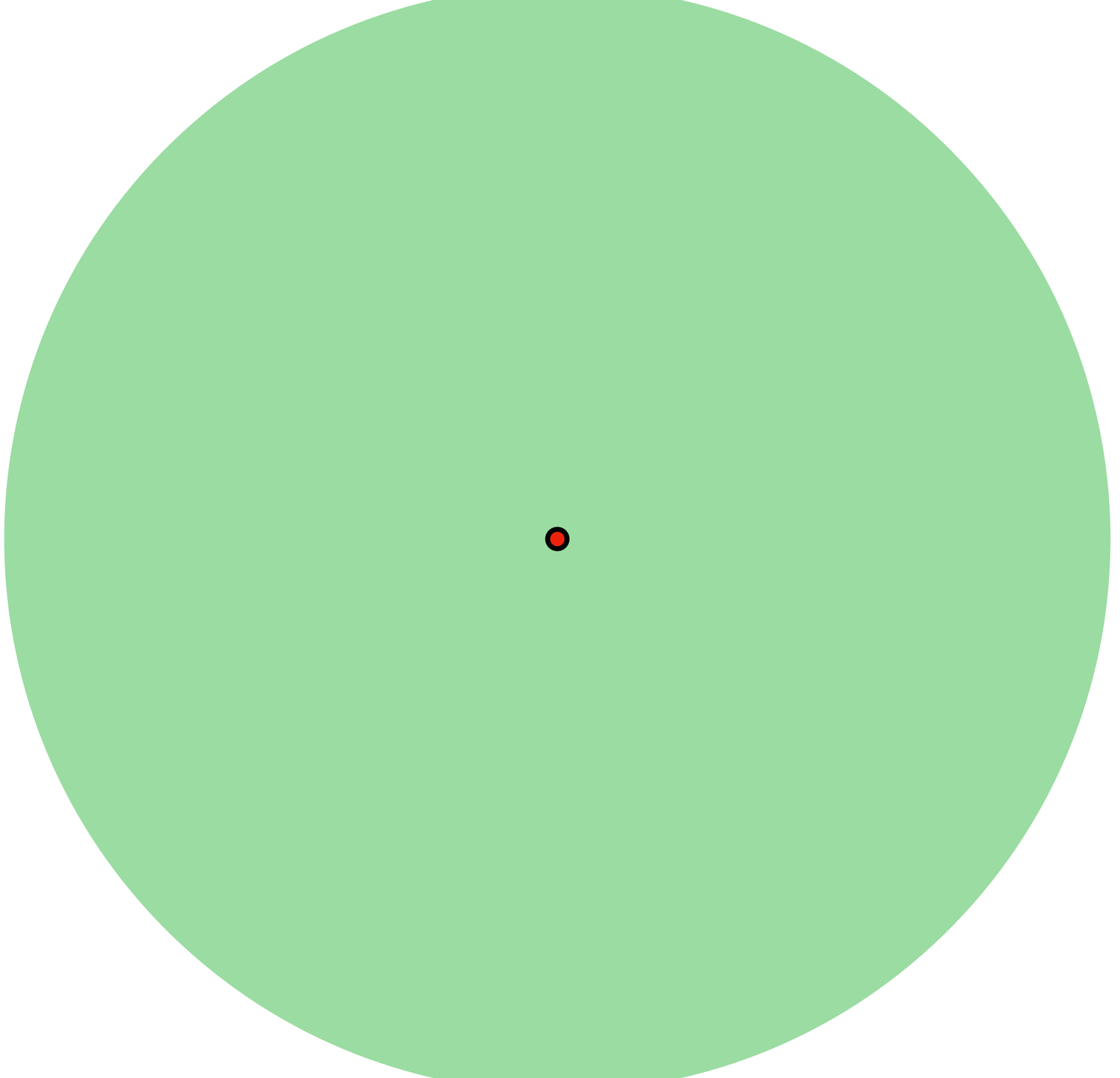

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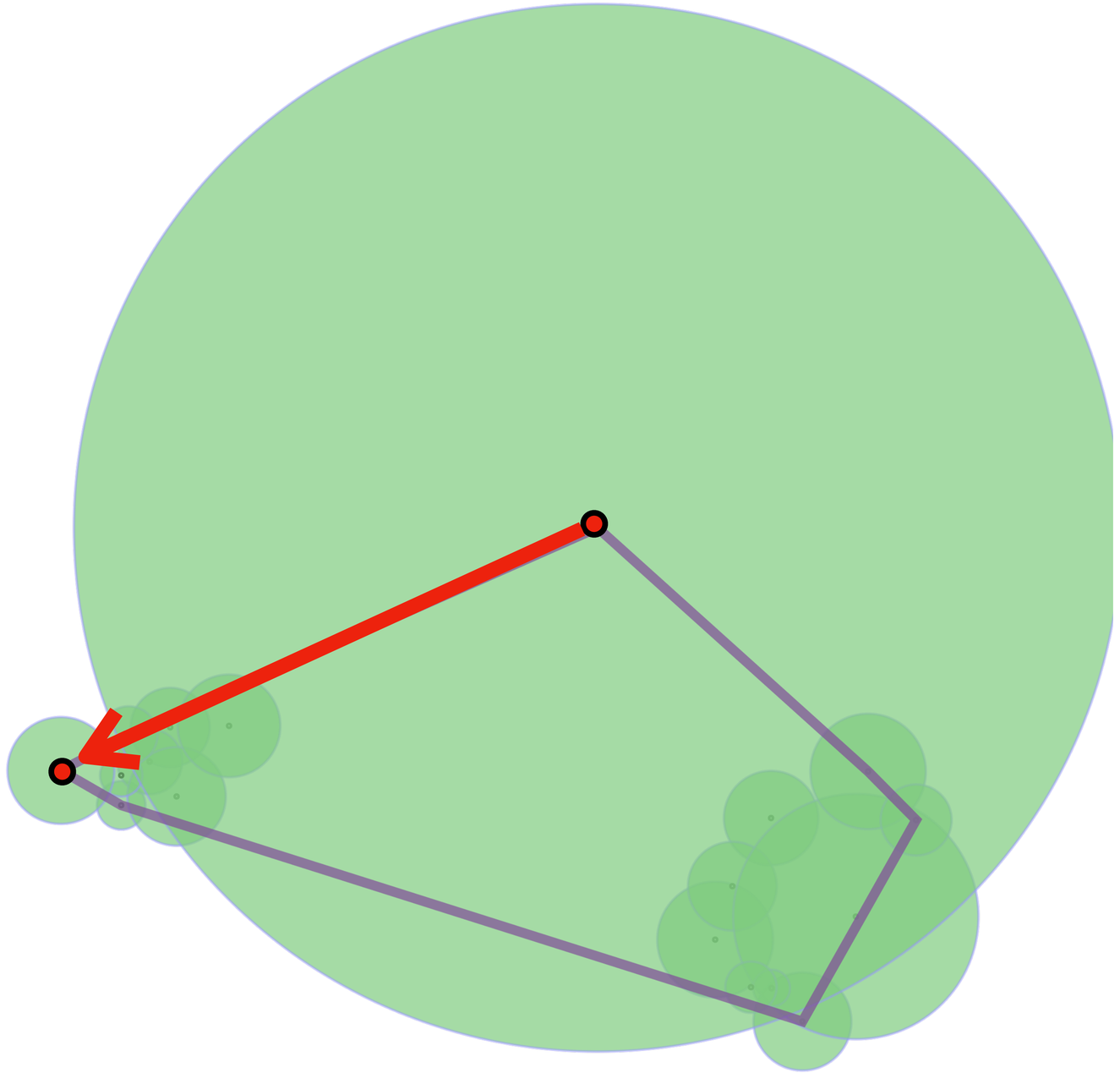

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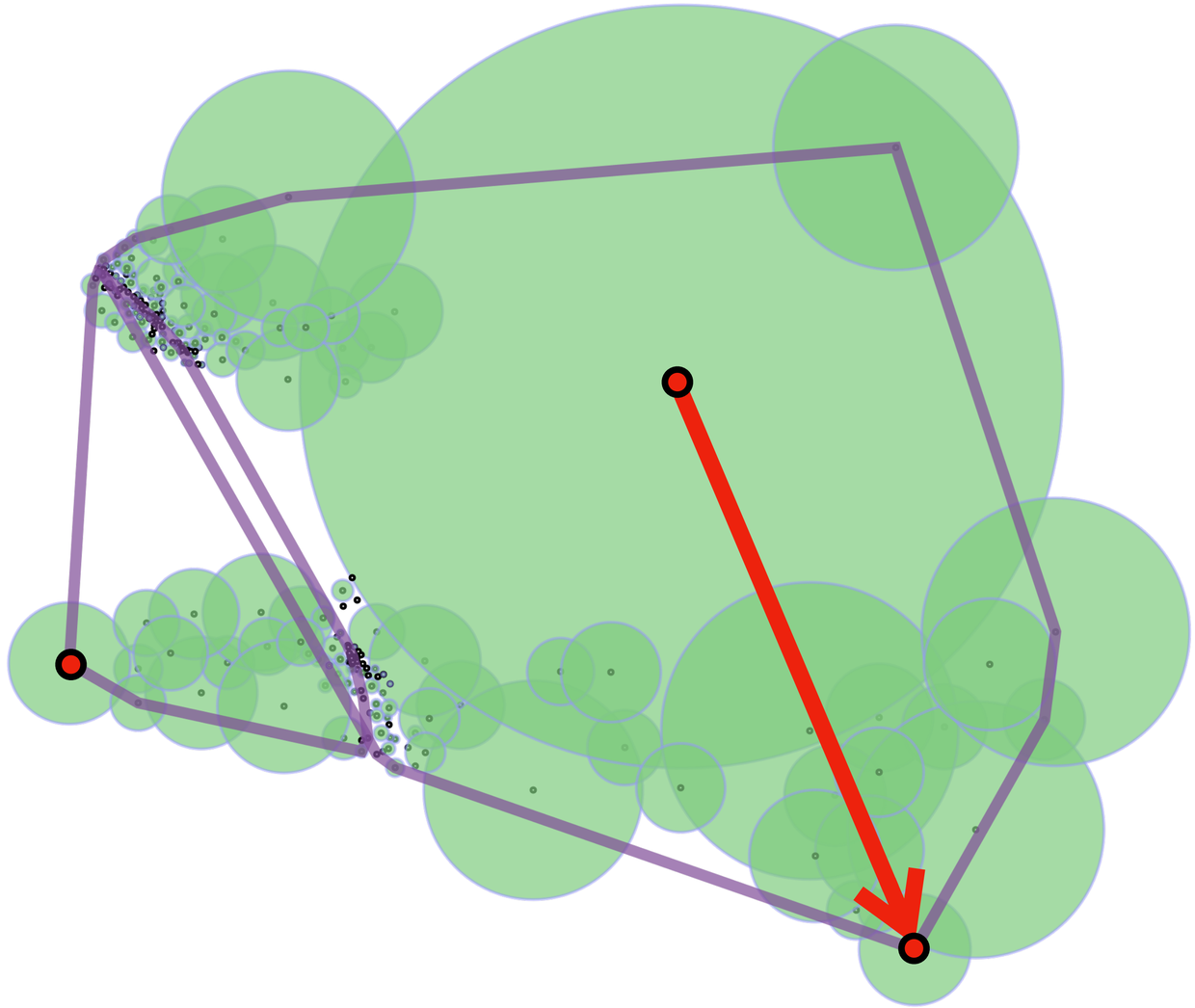

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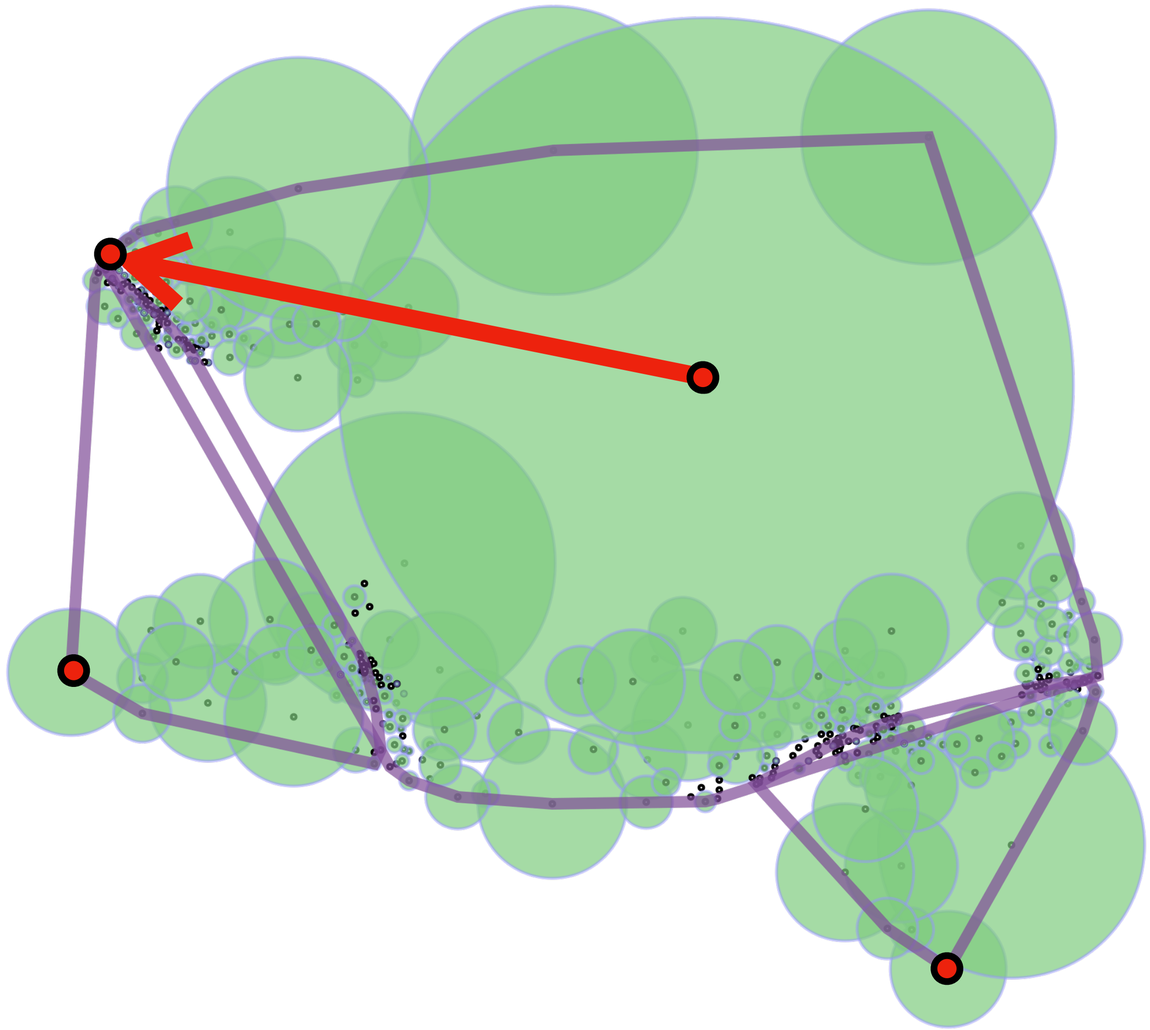

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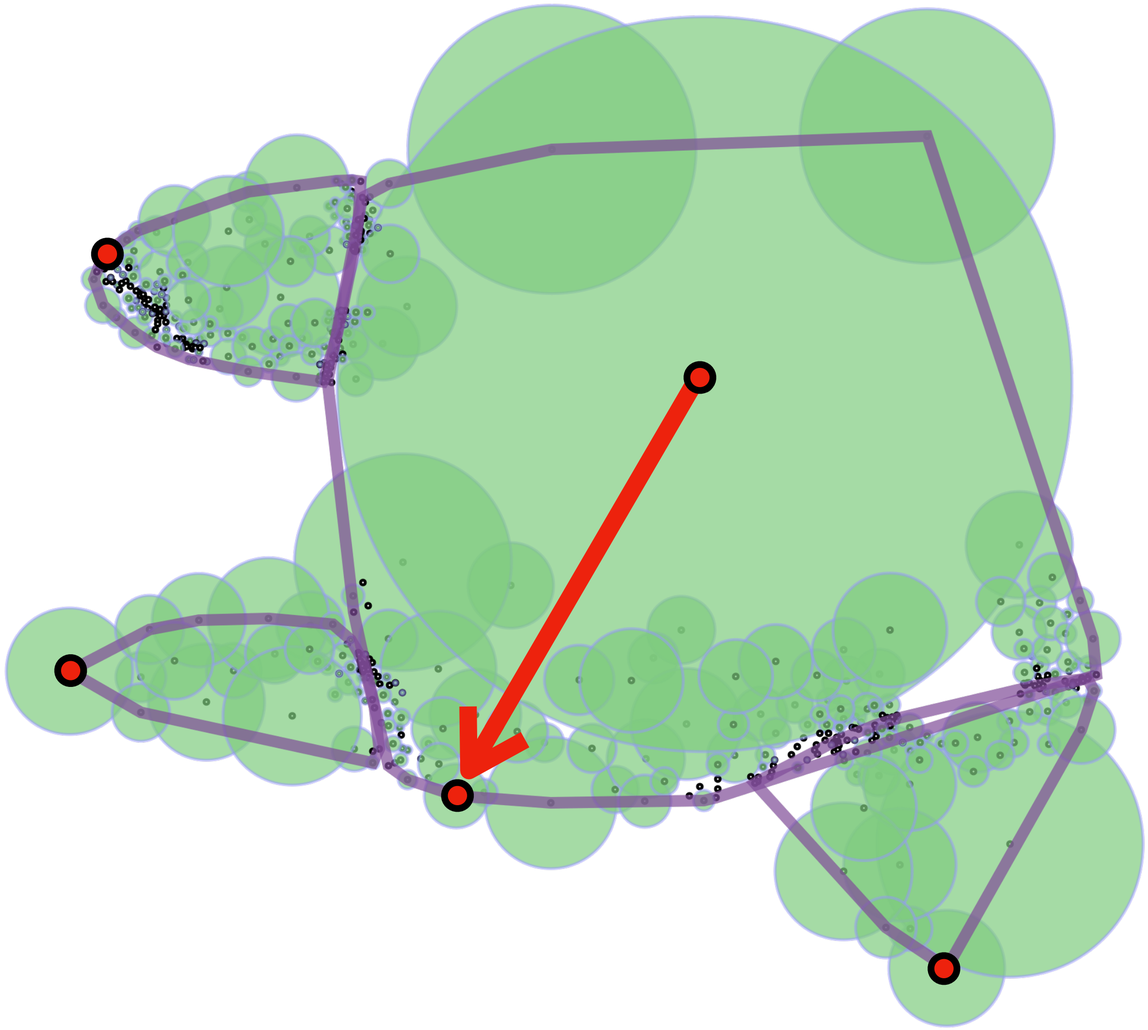

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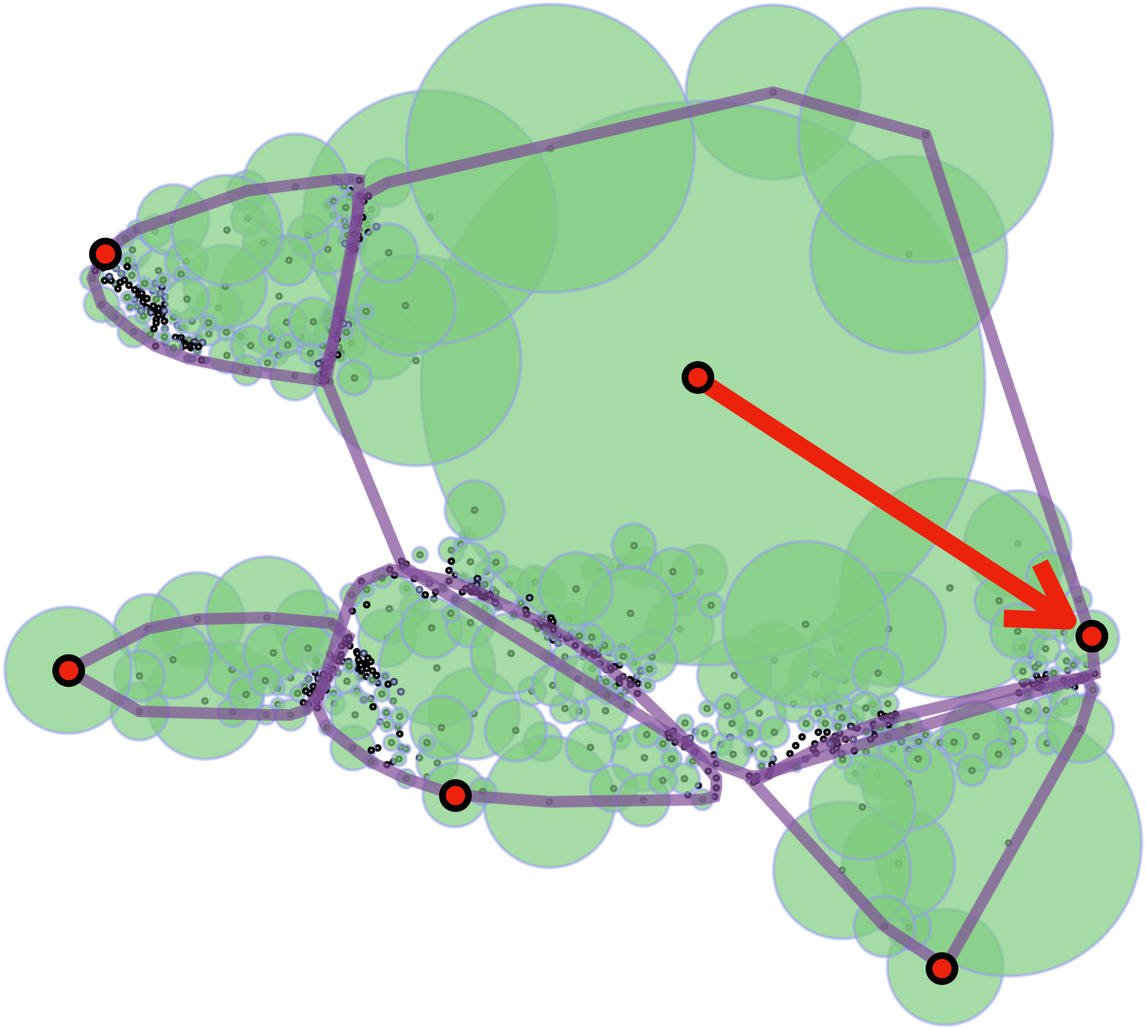

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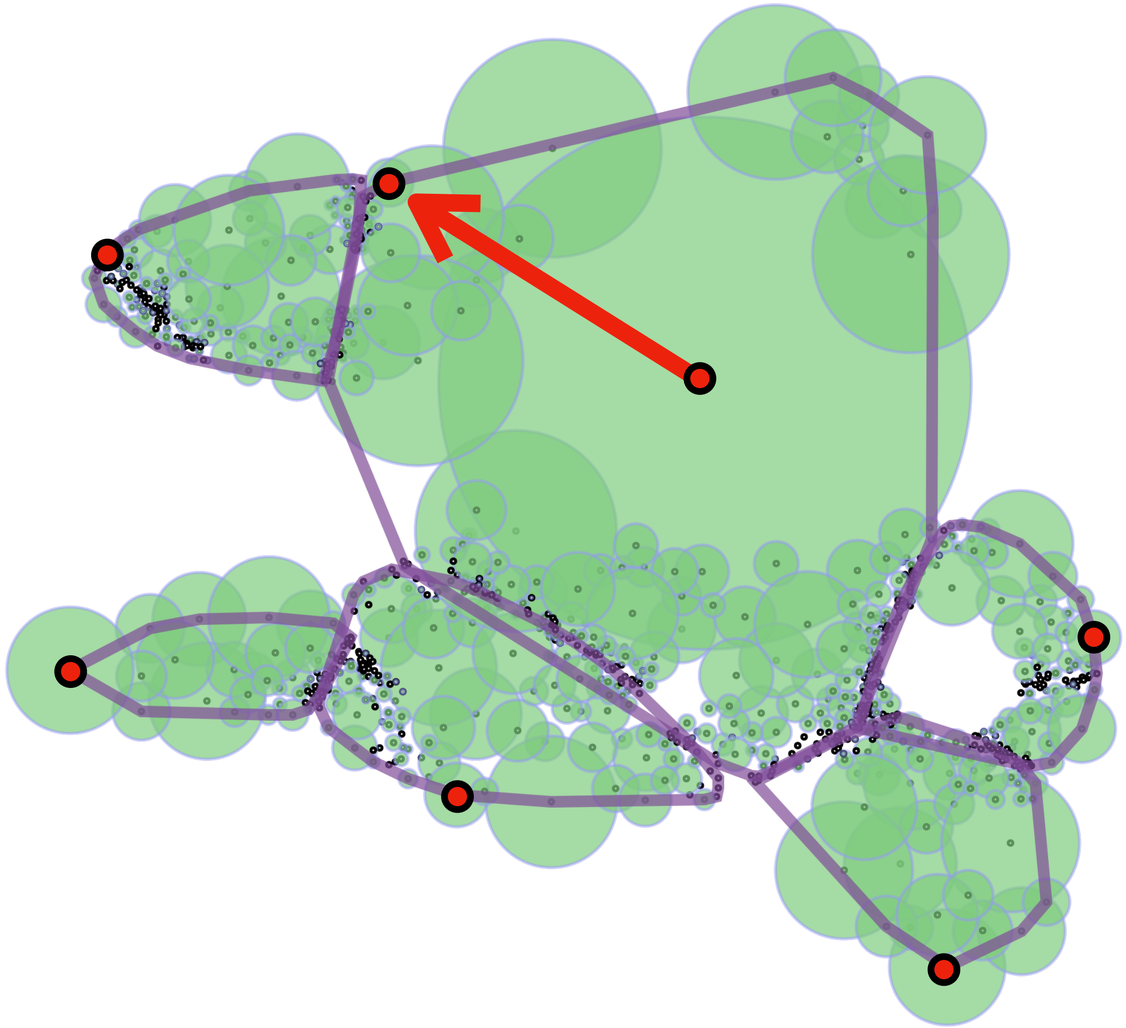

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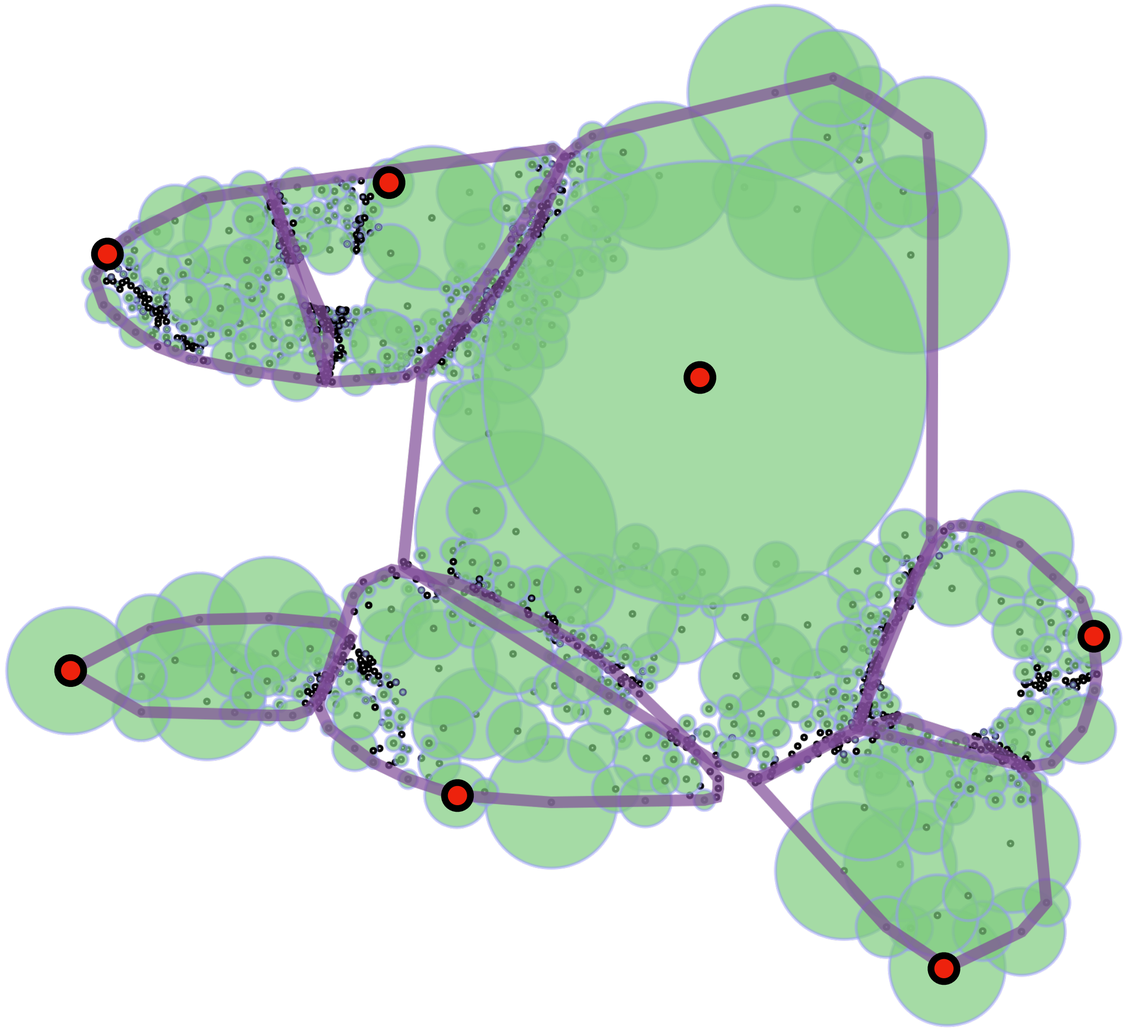

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Recursive Merging:

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- **The input is two greedy trees.**

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This gives us a divide and conquer algorithm.

Conclusion

Theorem:

Using a recursive algorithm, a *greedy permutation* can be constructed deterministically in $O(n \log n)$ time, and in $O(n)$ space.

References

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