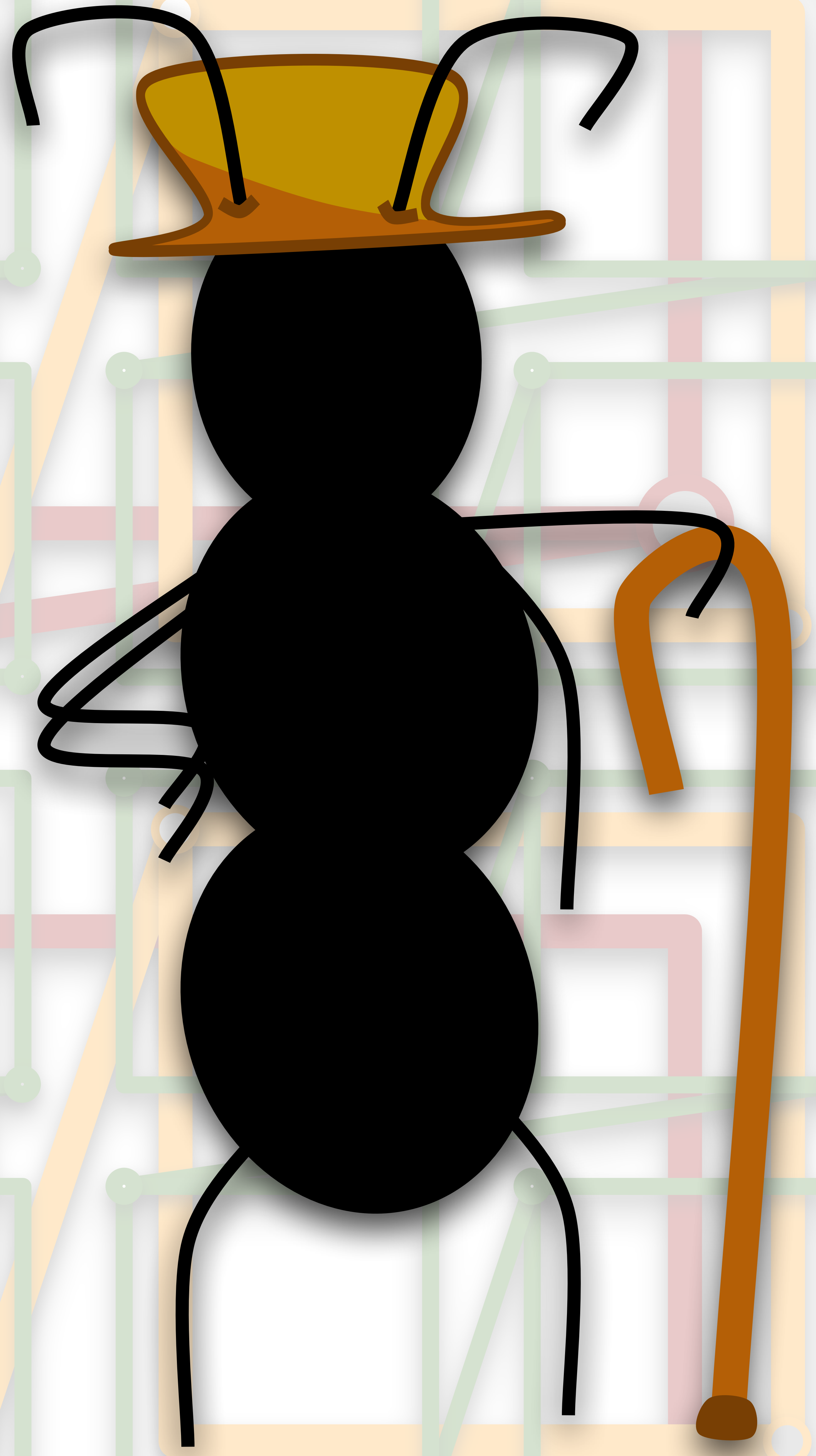
Ants on Manifolds

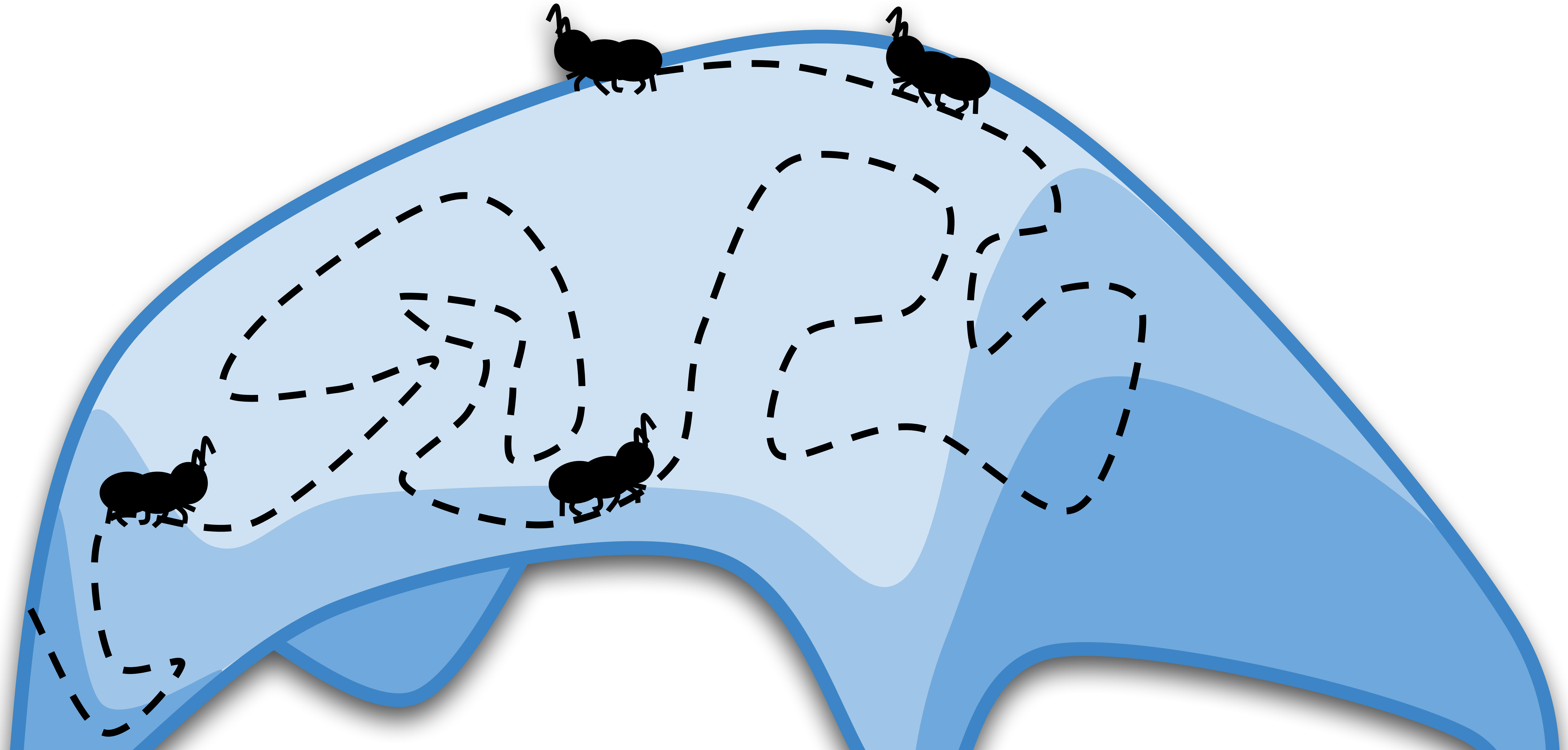
A CS Theory Seminar

Oliver Chubet, 2/21/25



Do space filling curves <u>really</u> exist?

What do we mean by *curve*?



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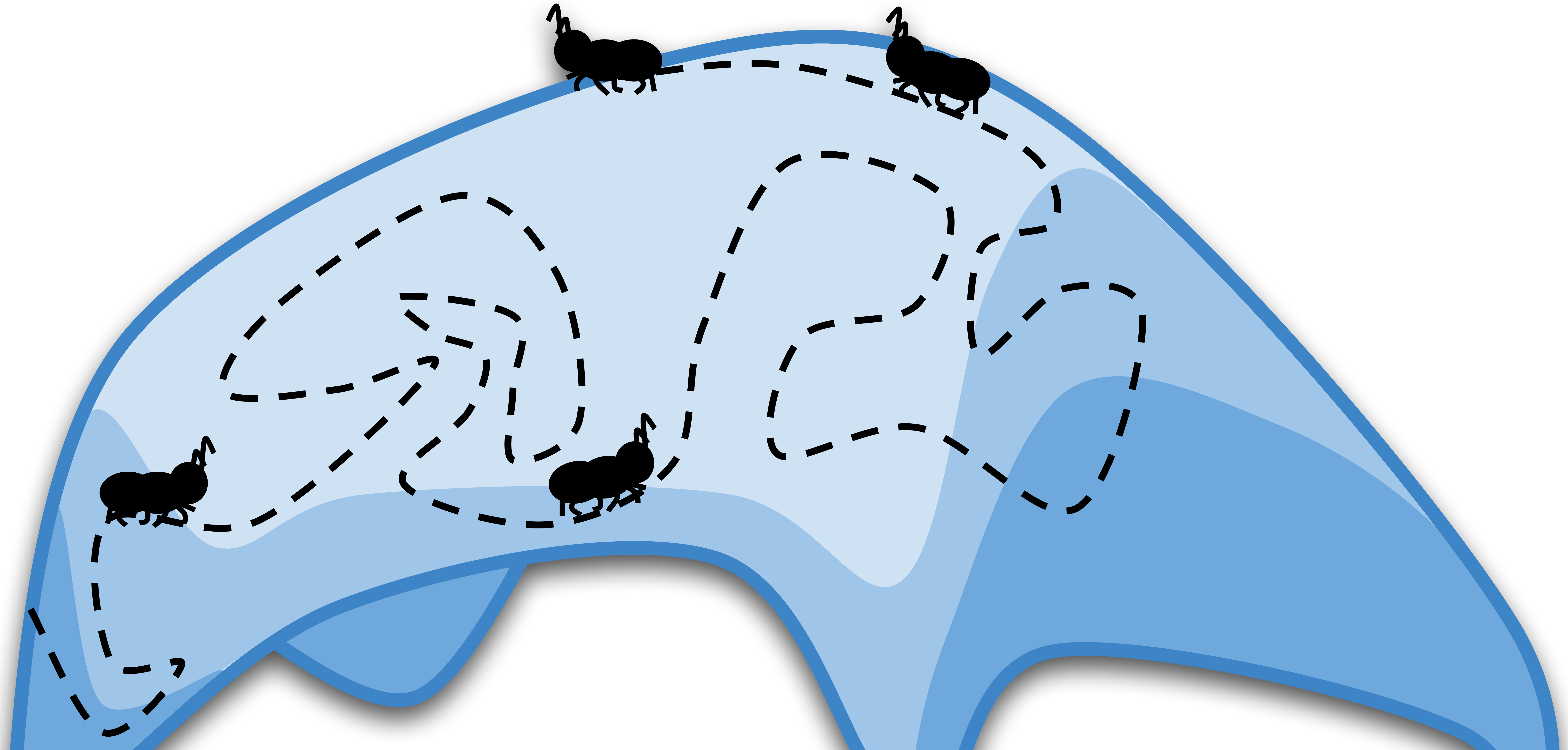

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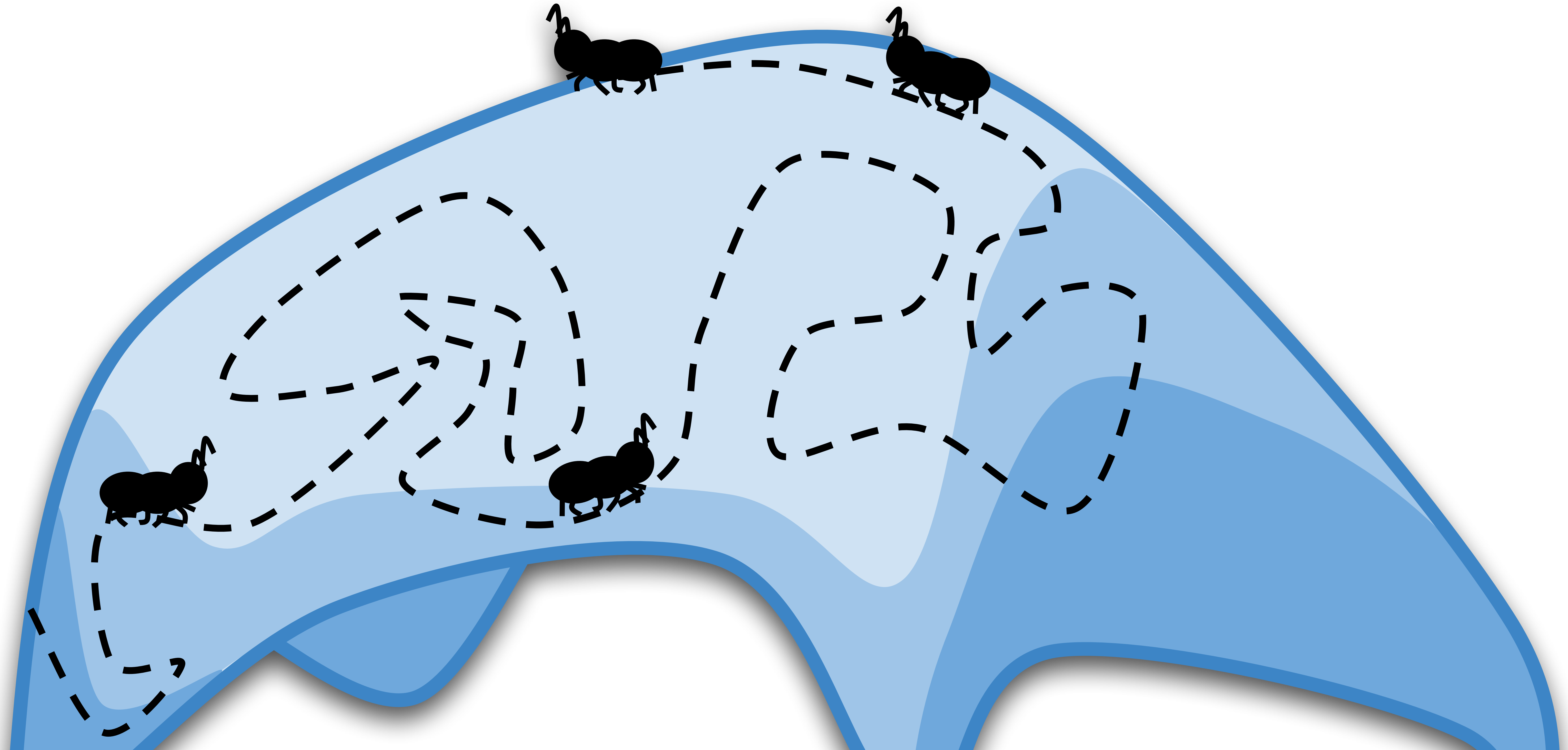

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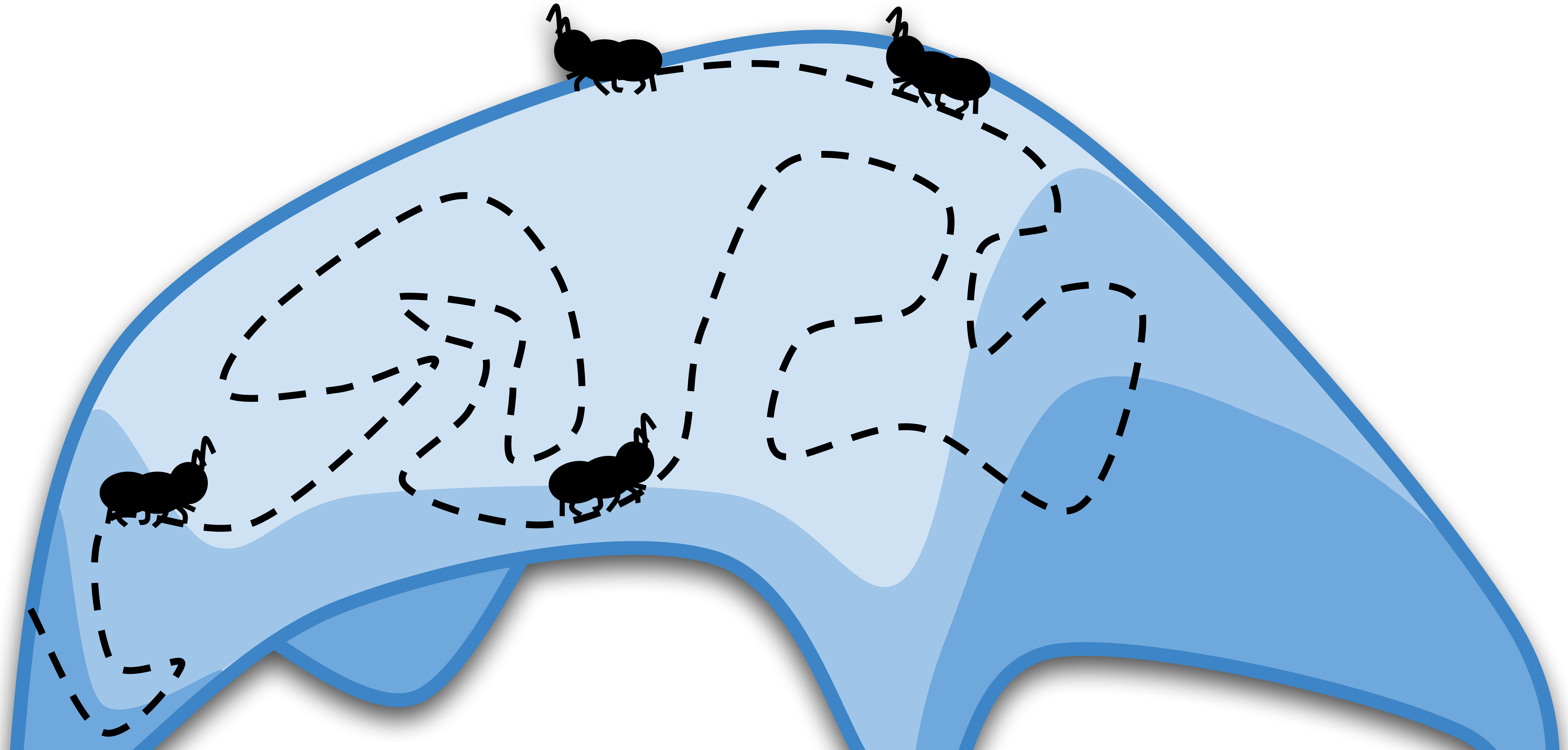
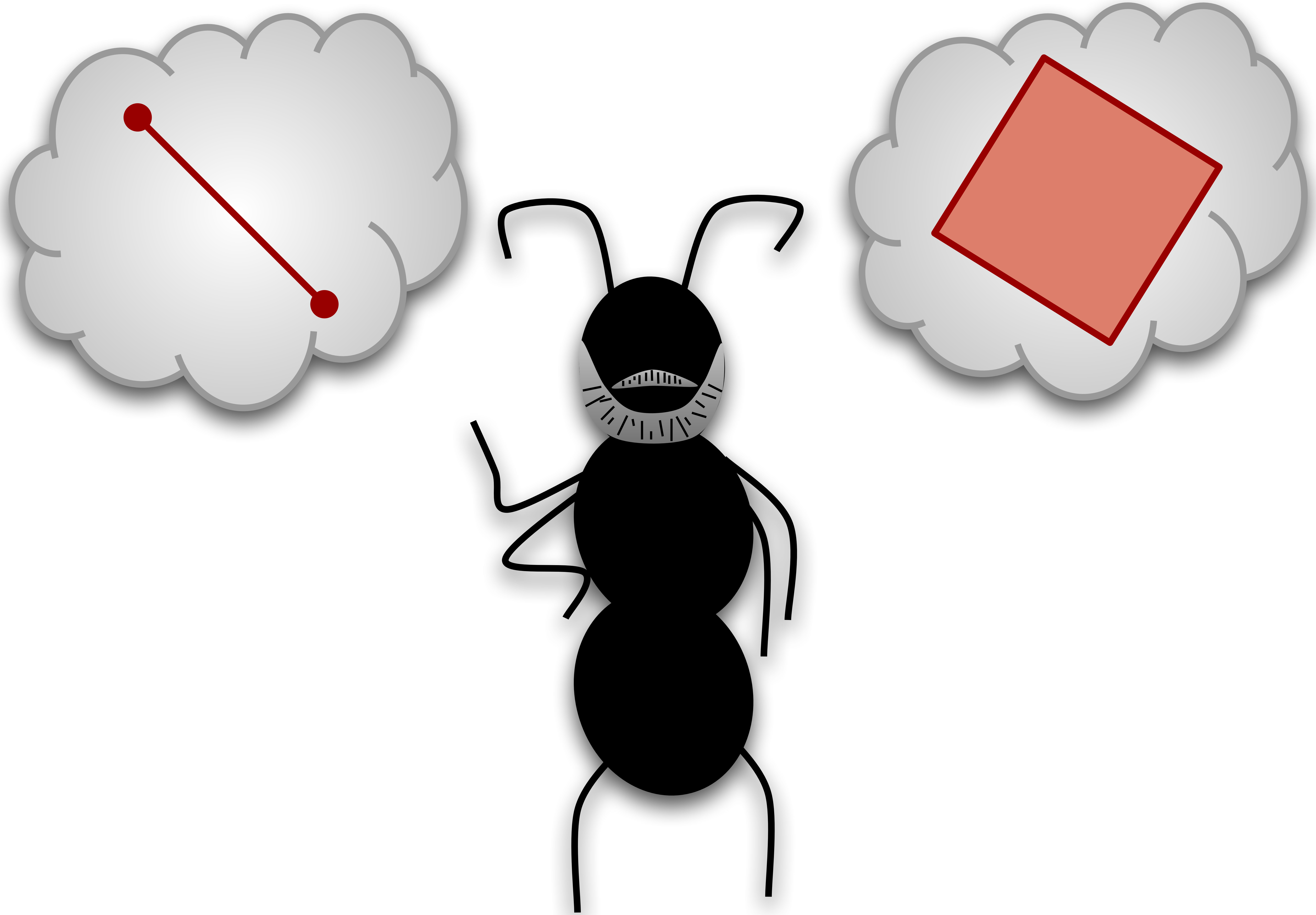
(No crossings, continuous)



Cantor and Cardinality



“In 1878, Georg Cantor demonstrated that any two finite-dimensional smooth manifolds, no matter what their dimensions, have the same cardinality…

… and Mathematics has never been the same since.”

- Hans Sagan

Netto’s Theorem

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(More generally, manifolds of different dimensions cannot be homeomorphic.)

Corollary:

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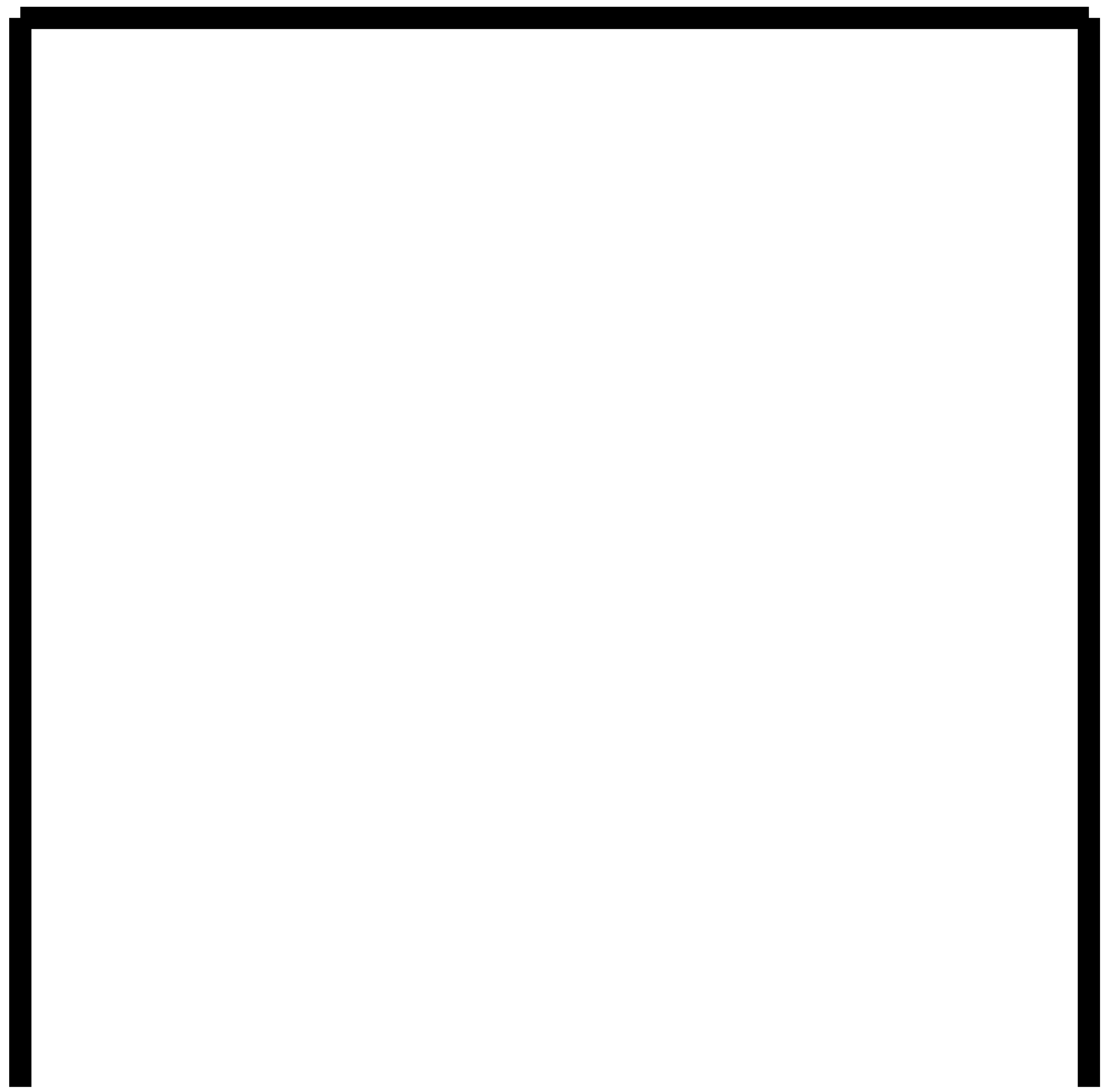
Space filling curves don’t exist.

The Hilbert Curve

(i*th-order*) Hilbert Curves



(ith-order) Hilbert Curves

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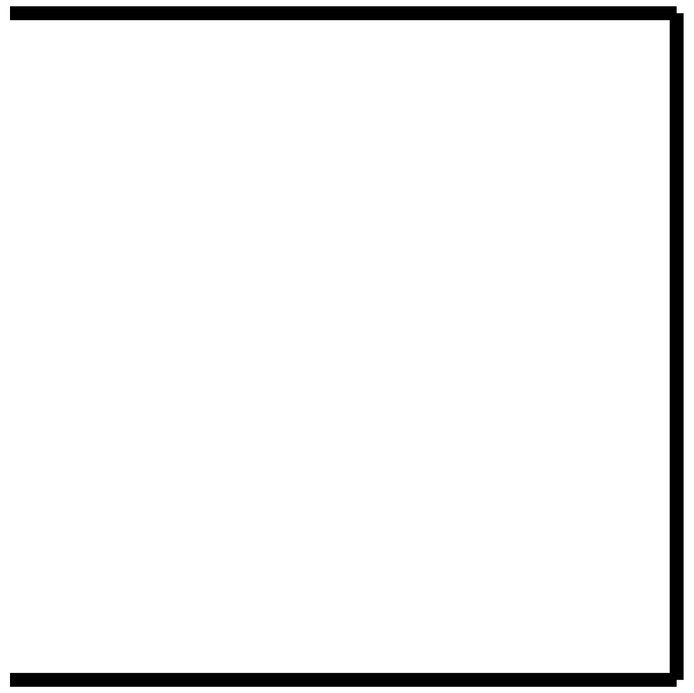

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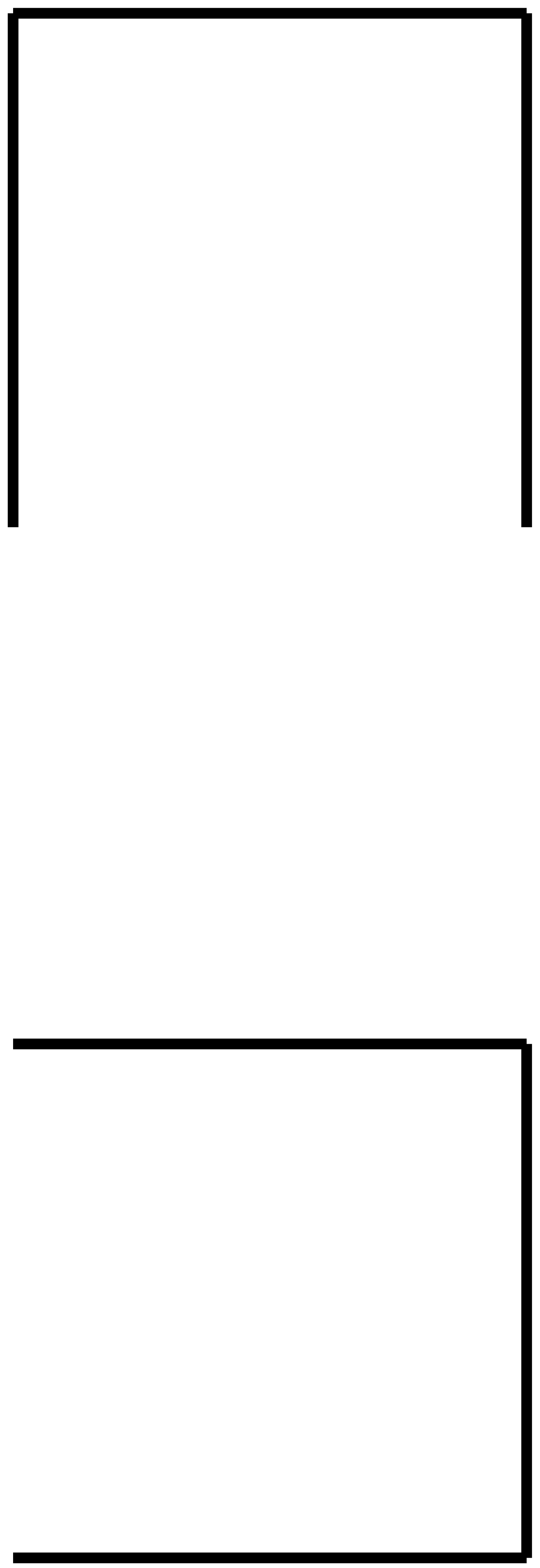

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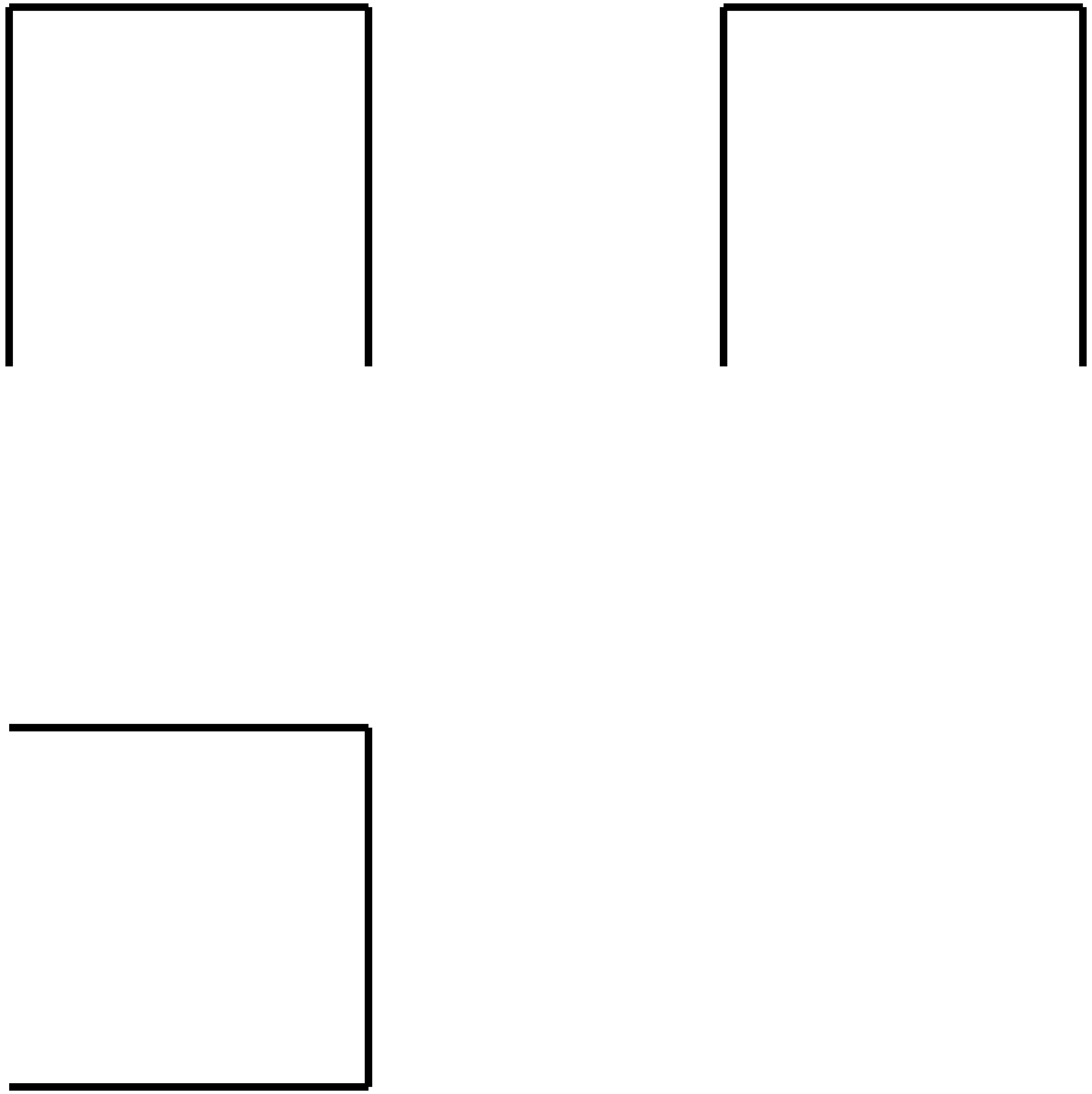

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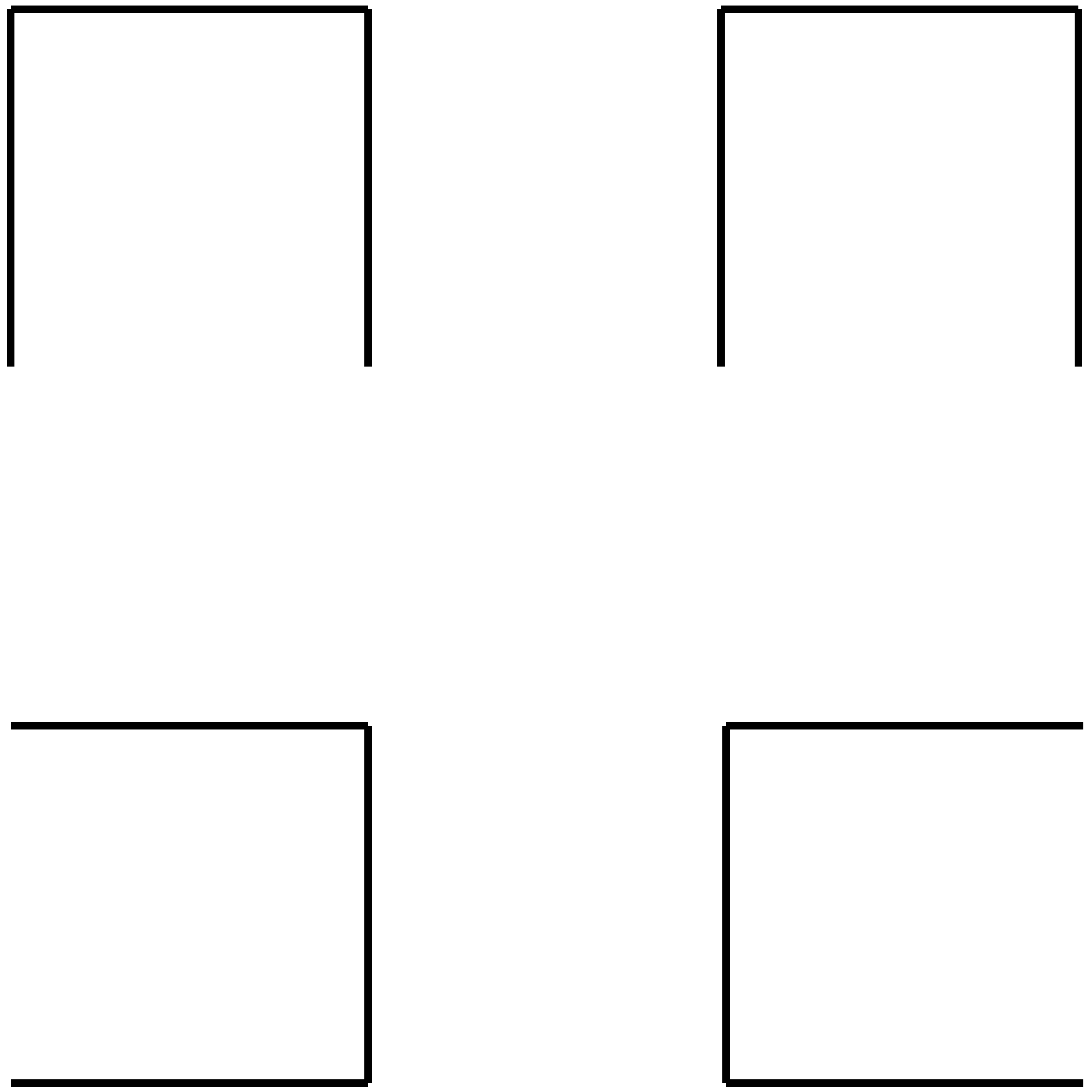

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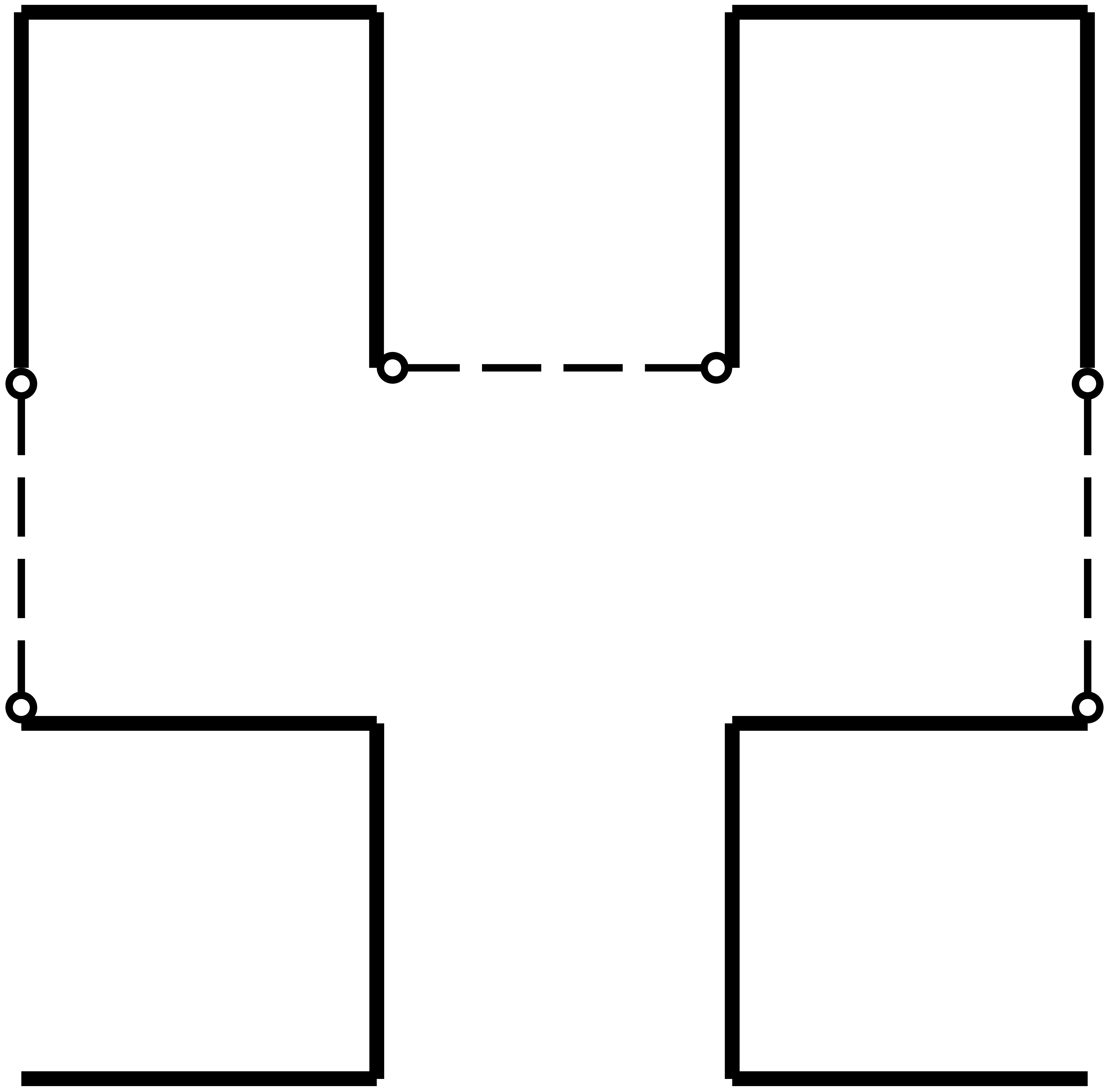
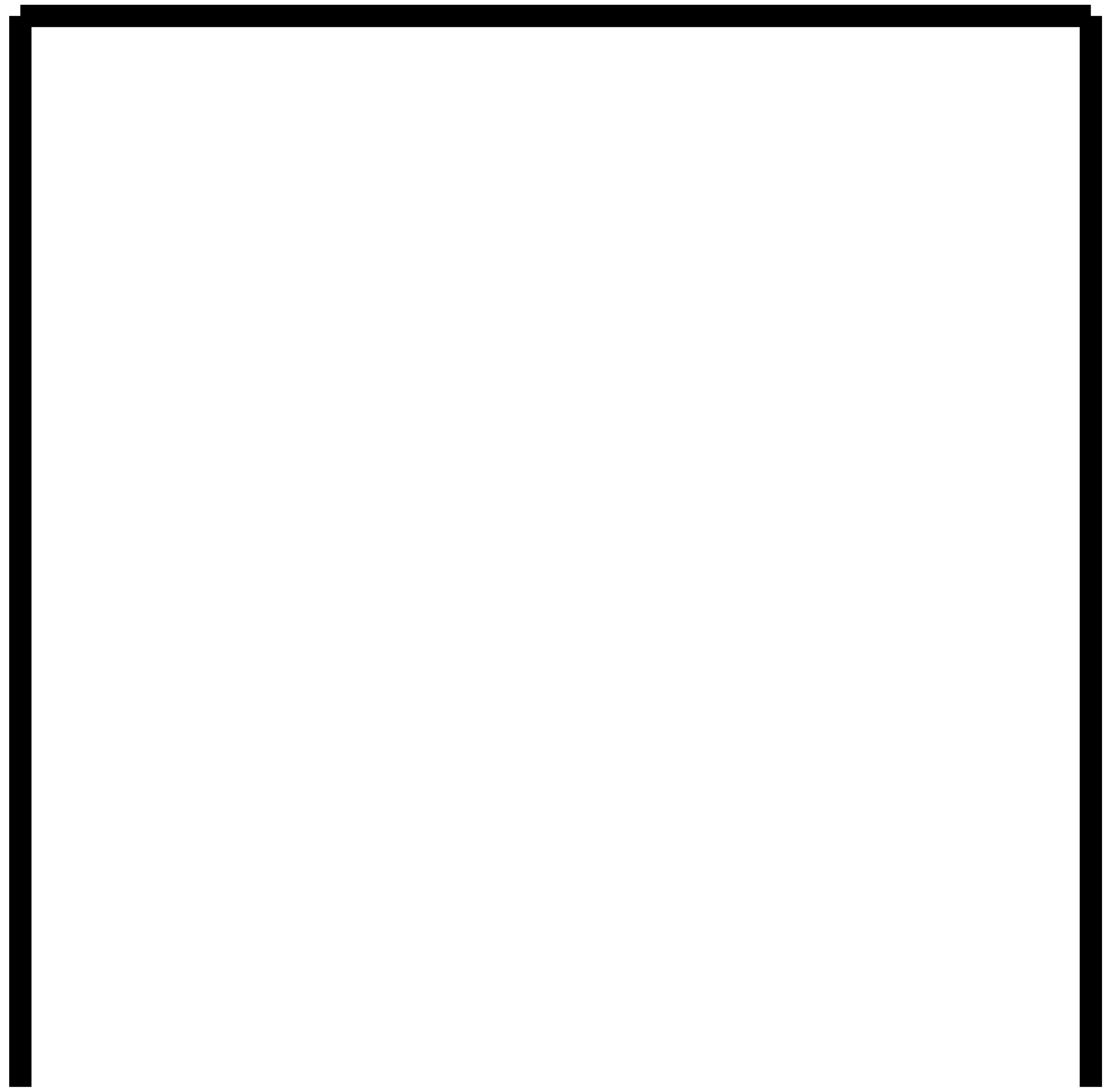


$i = 0$



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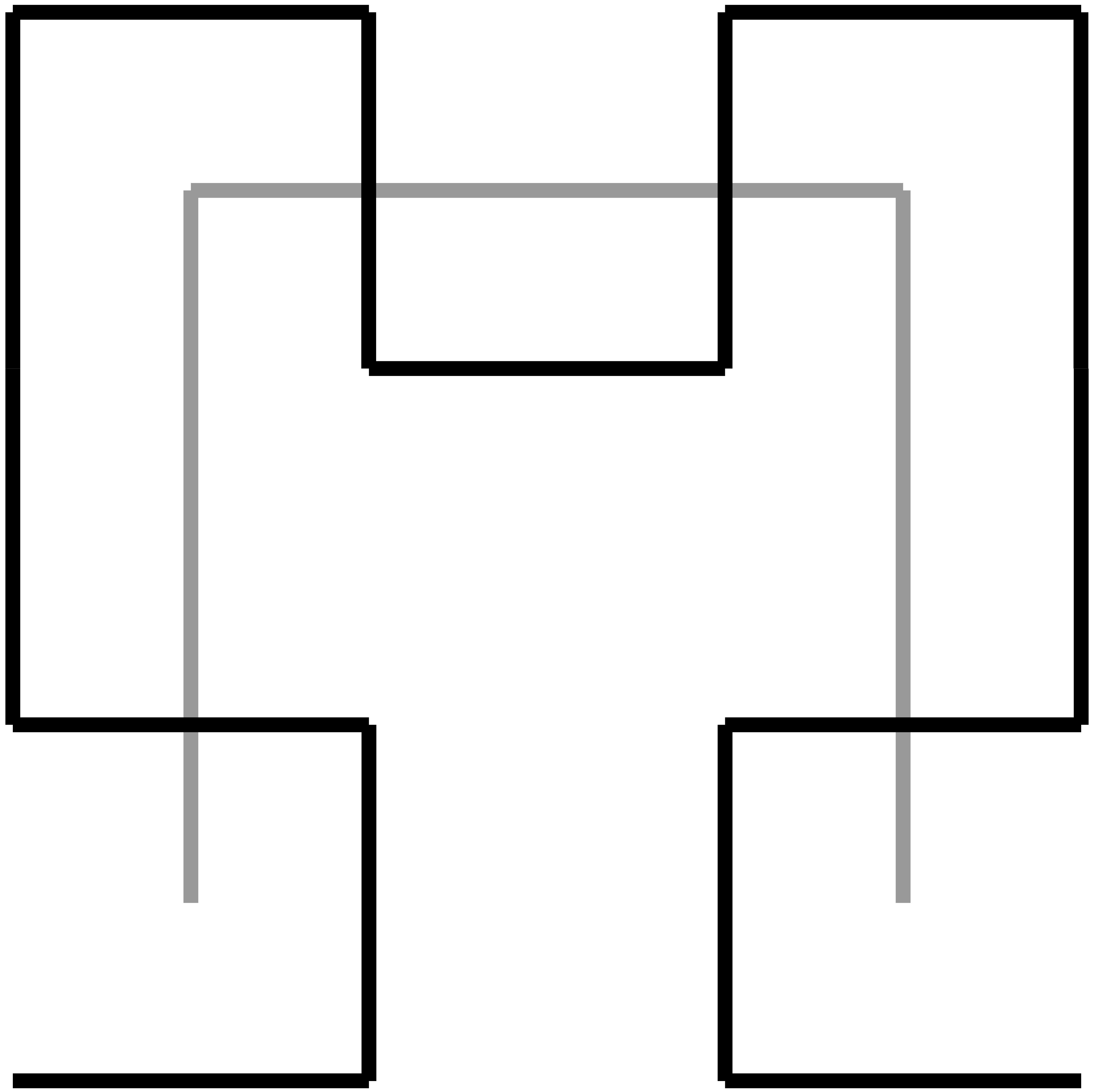

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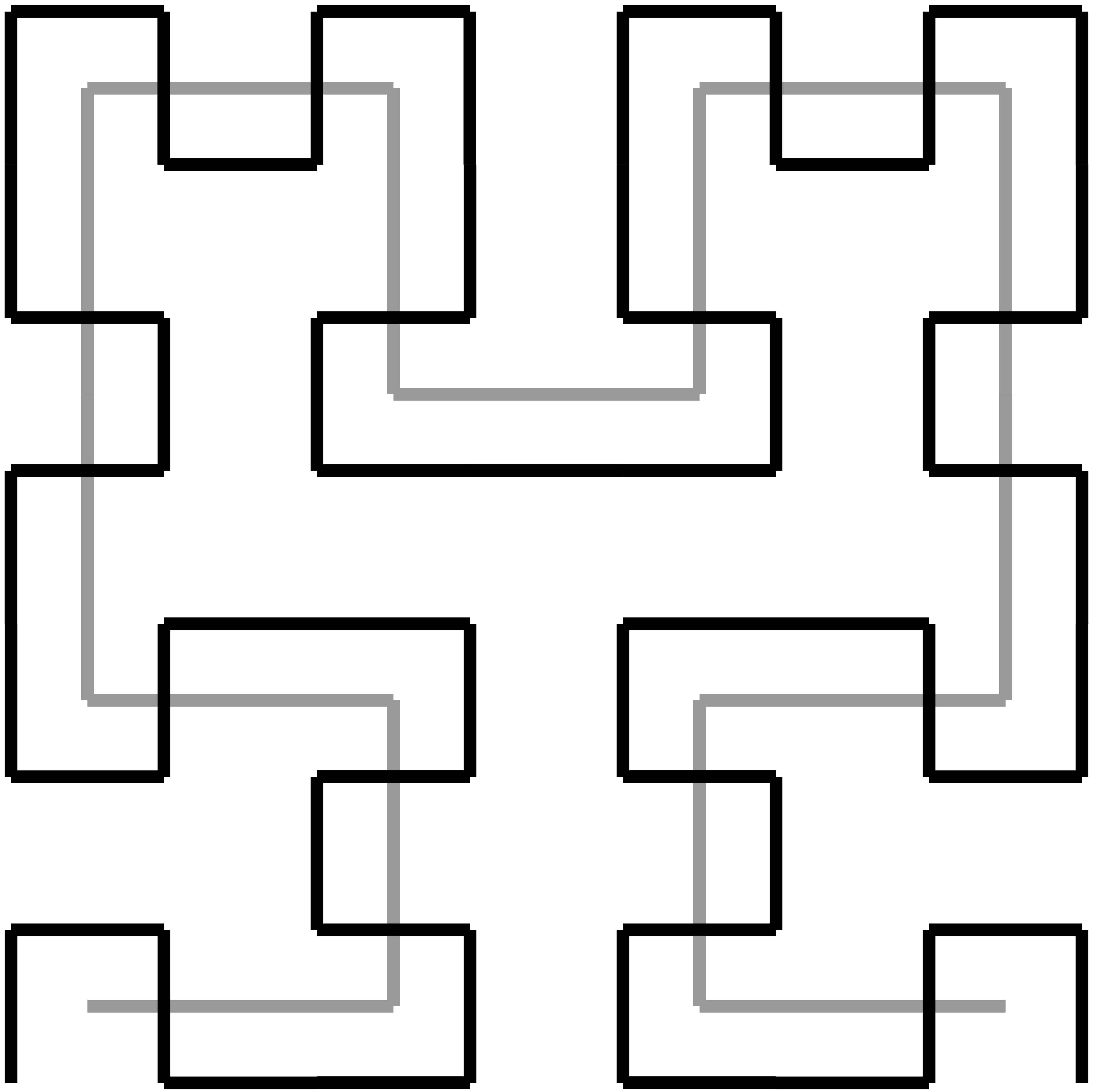

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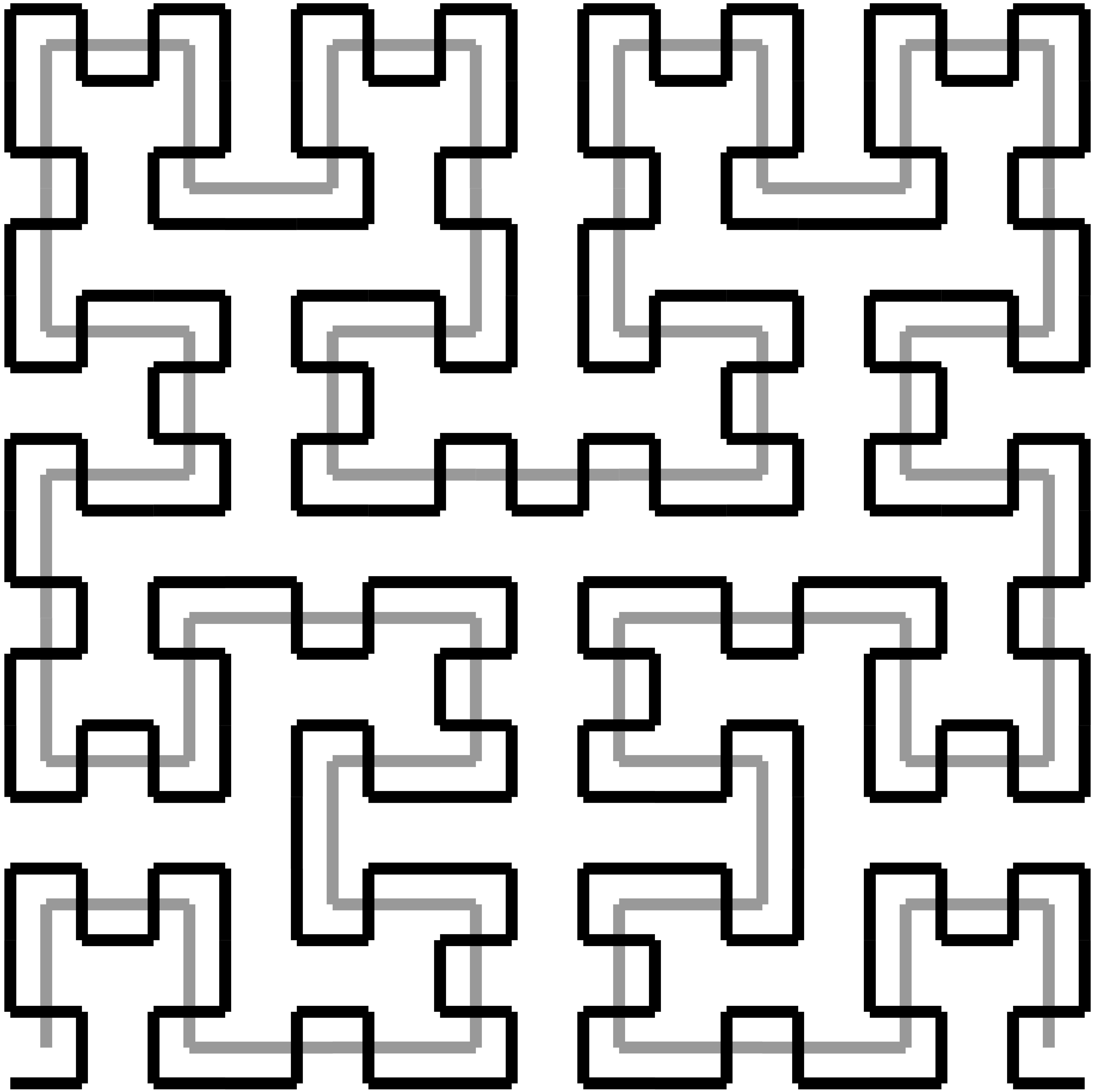

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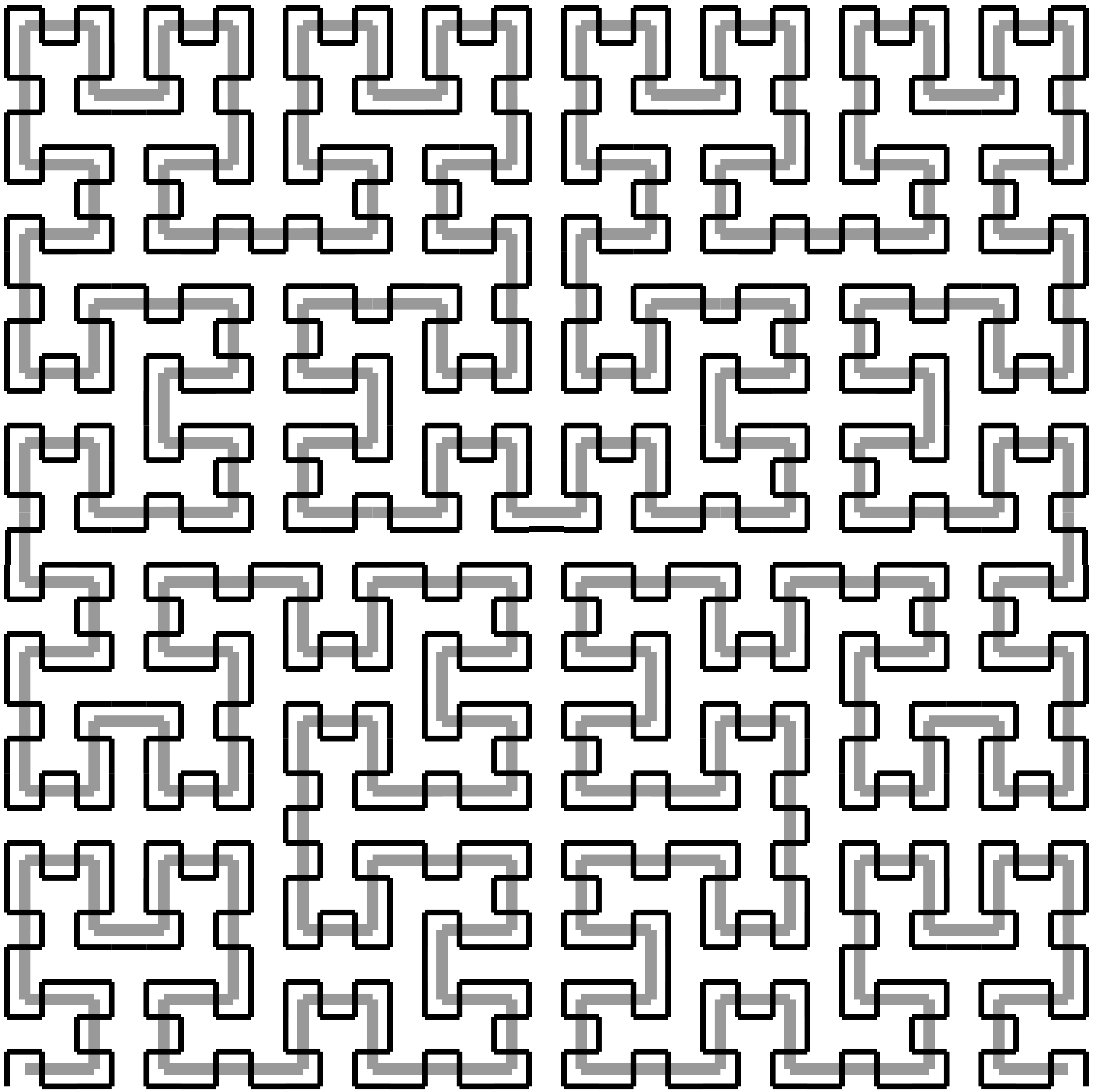

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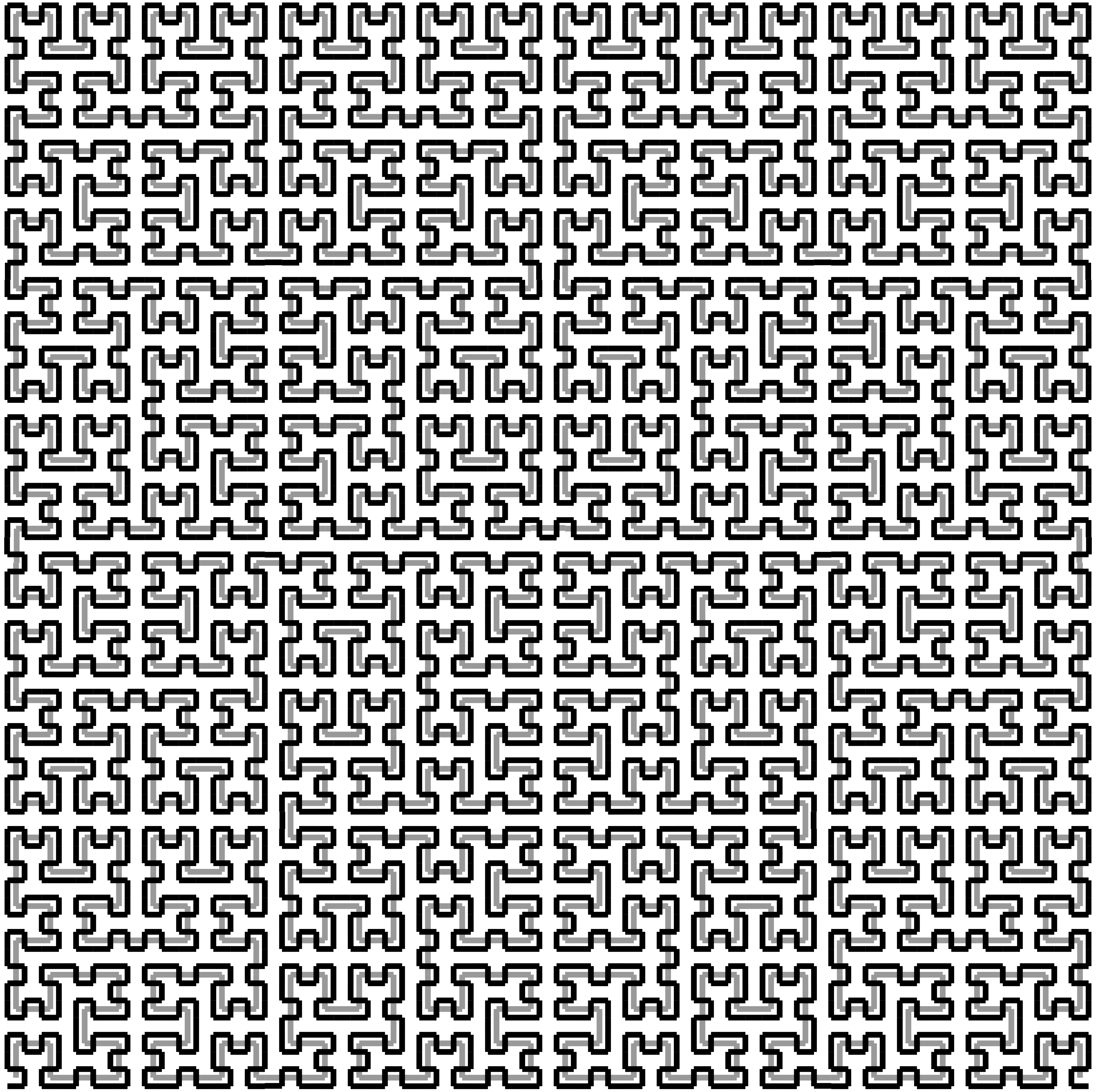

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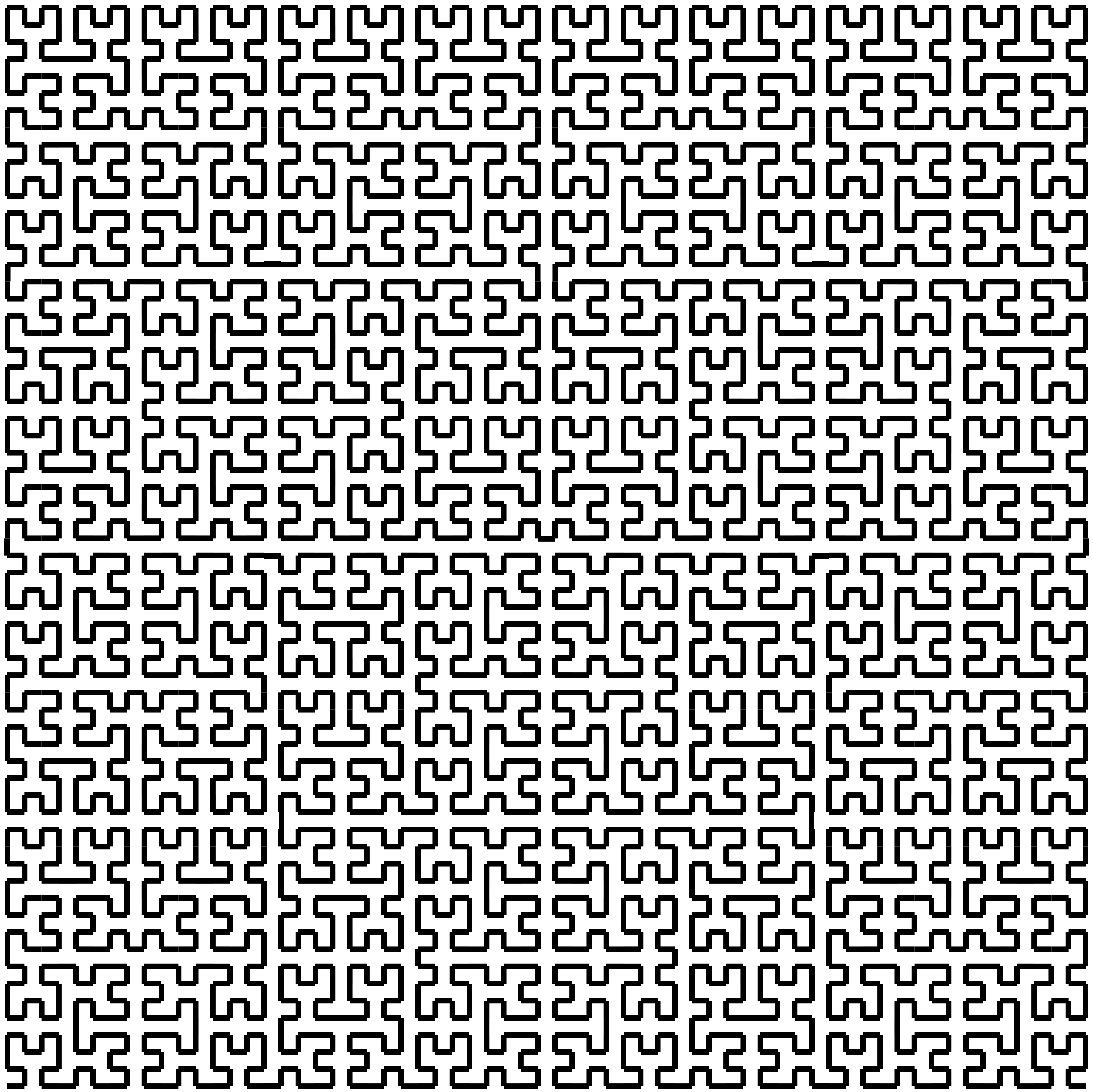

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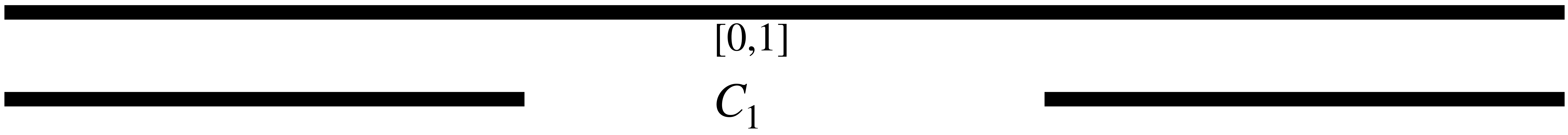

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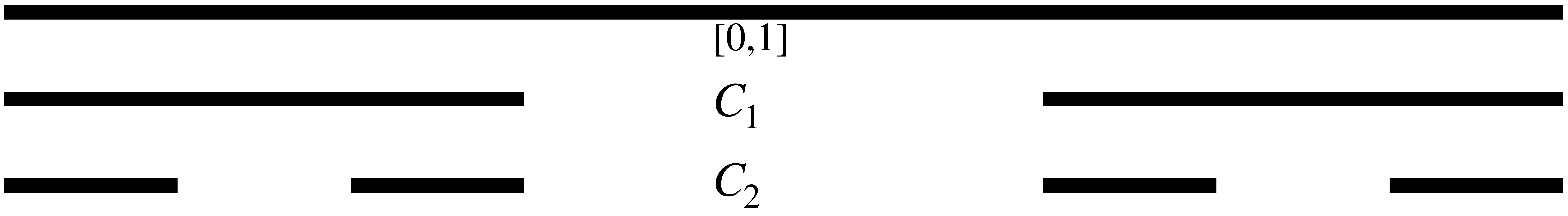

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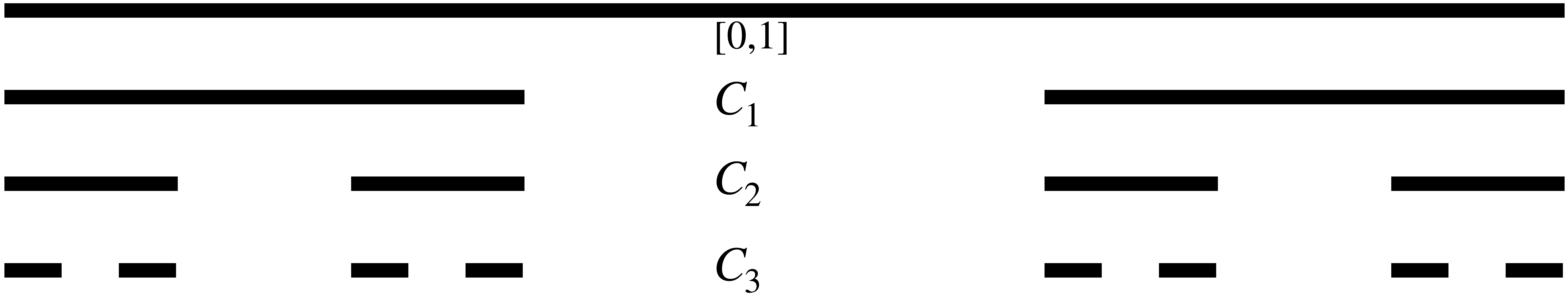

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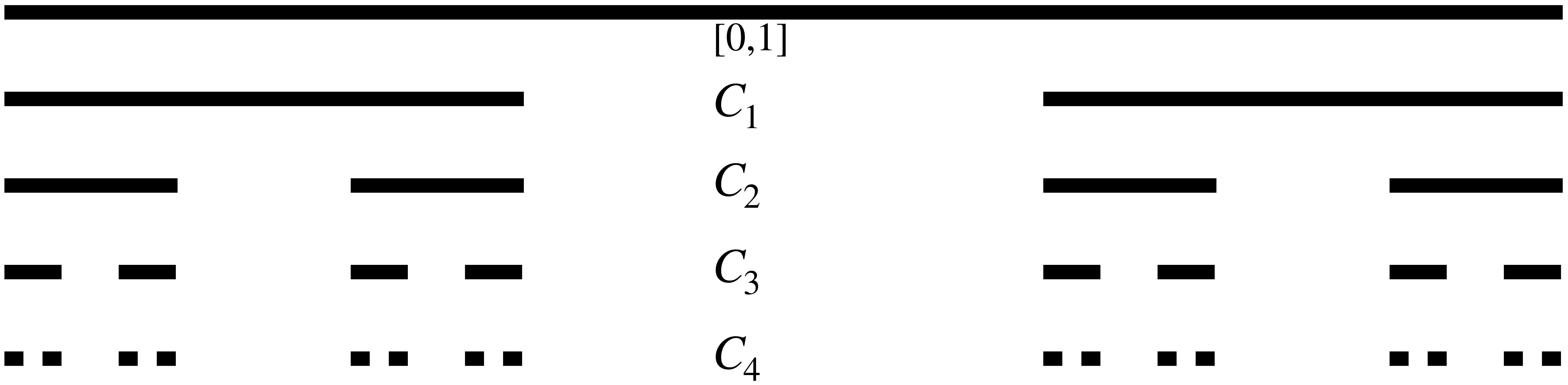

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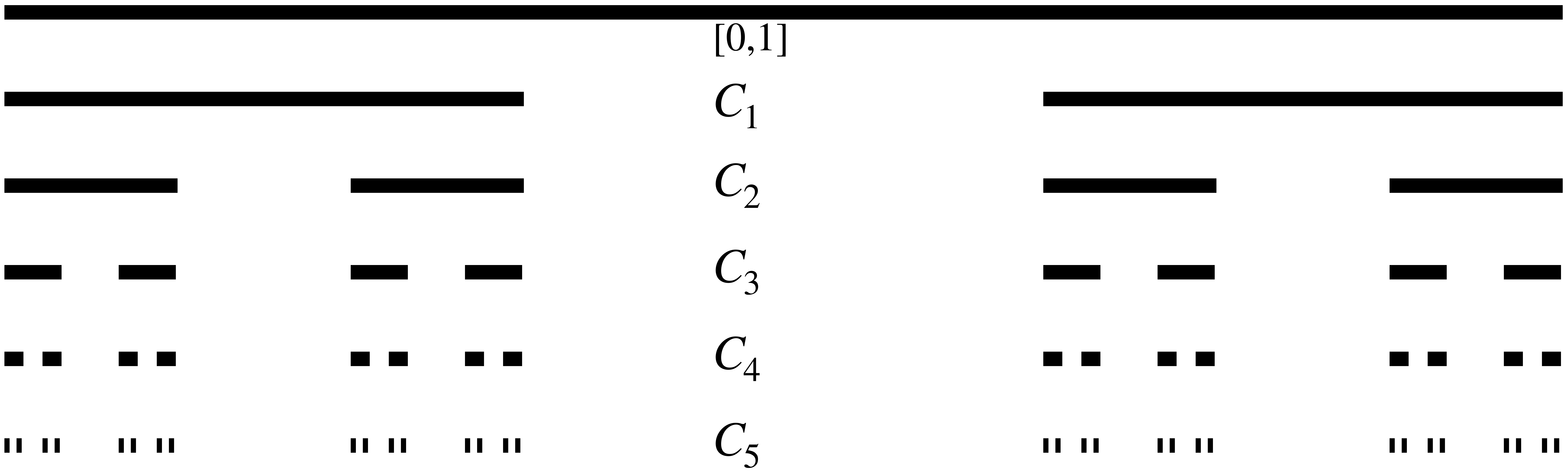

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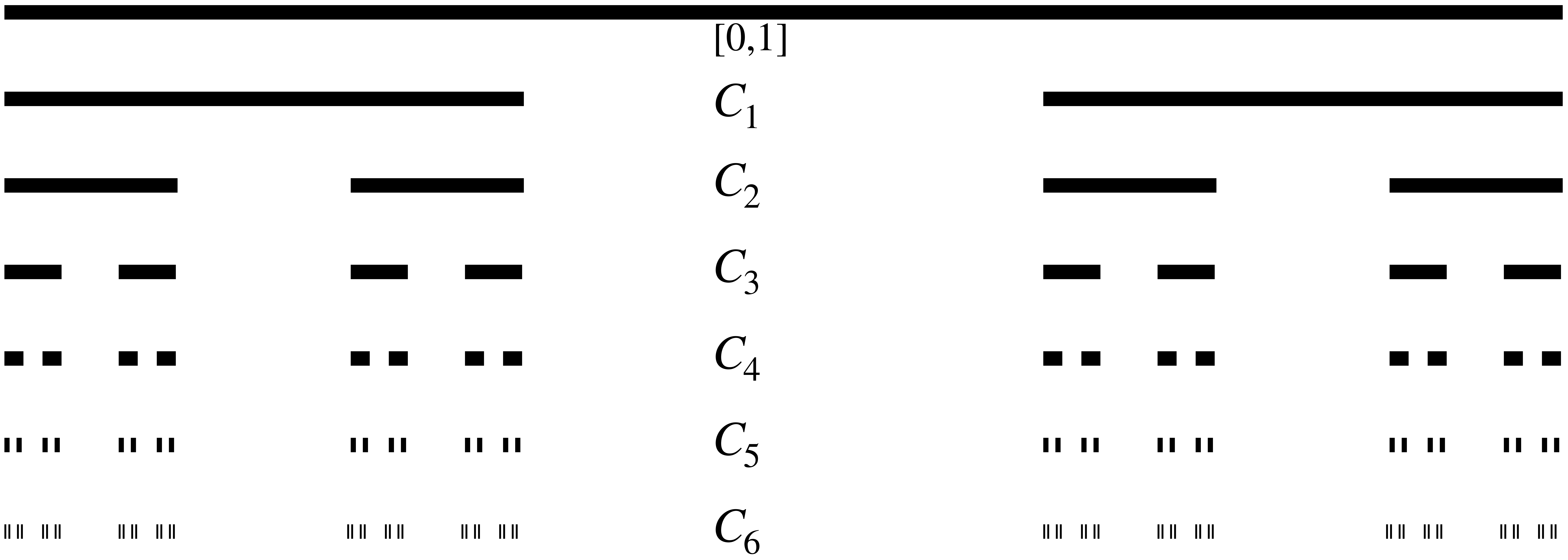

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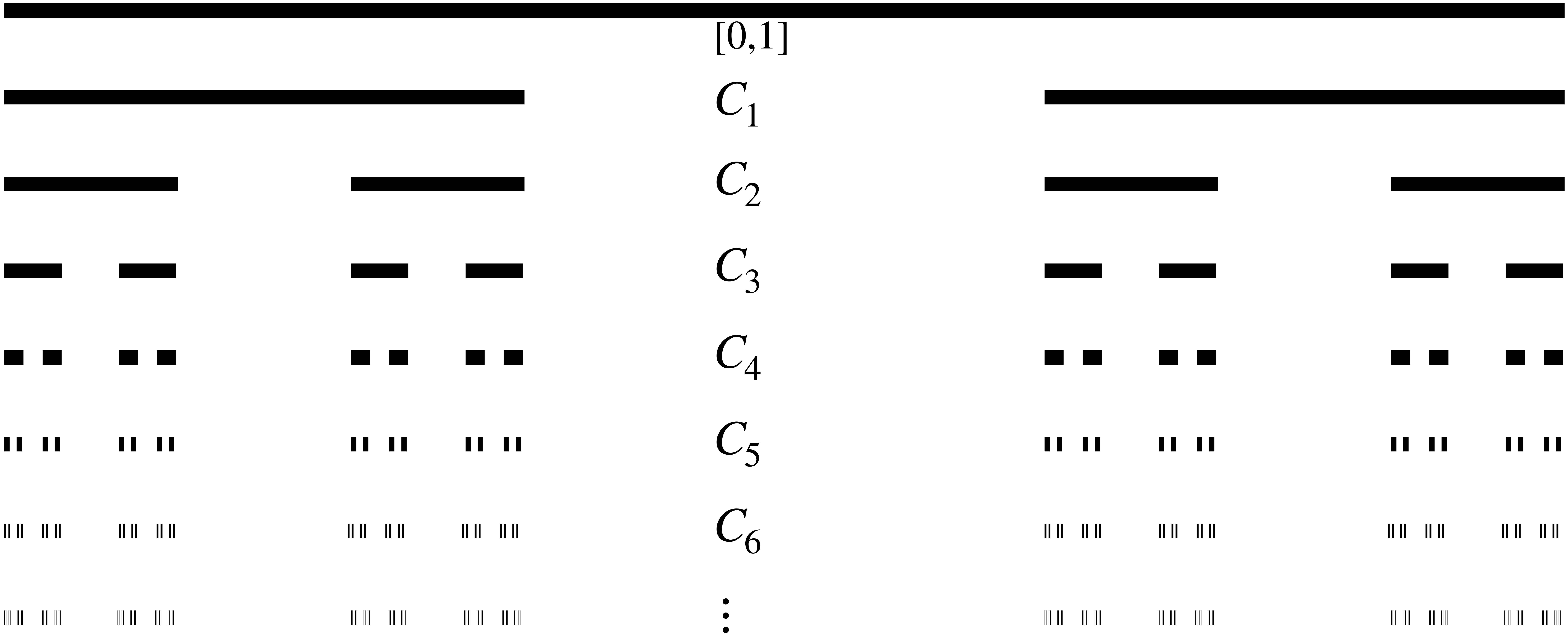

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- $\mathscr{C} \subset [0,1]$ so the cardinality of $[0,1]$ is at least that of $\mathscr{C}$

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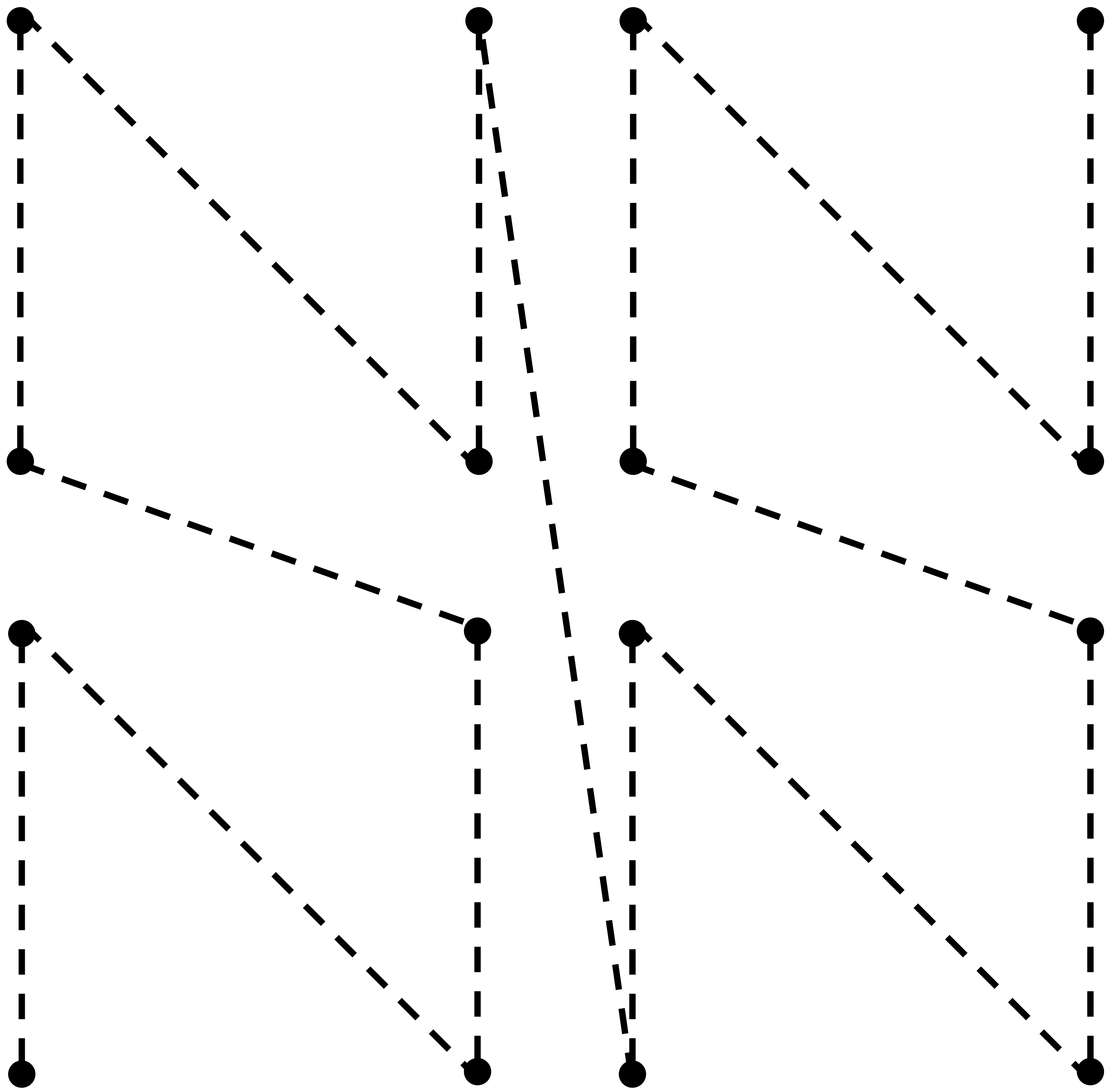
Q: is φ continuous?

The Lebesgue Curve

(ith-order) Lebesgue Curve

Lebegue extends φ continuously to $[0,1]$ by linear interpolation (for each ith-order approximation)

Eg. $\mathscr{L}_2$



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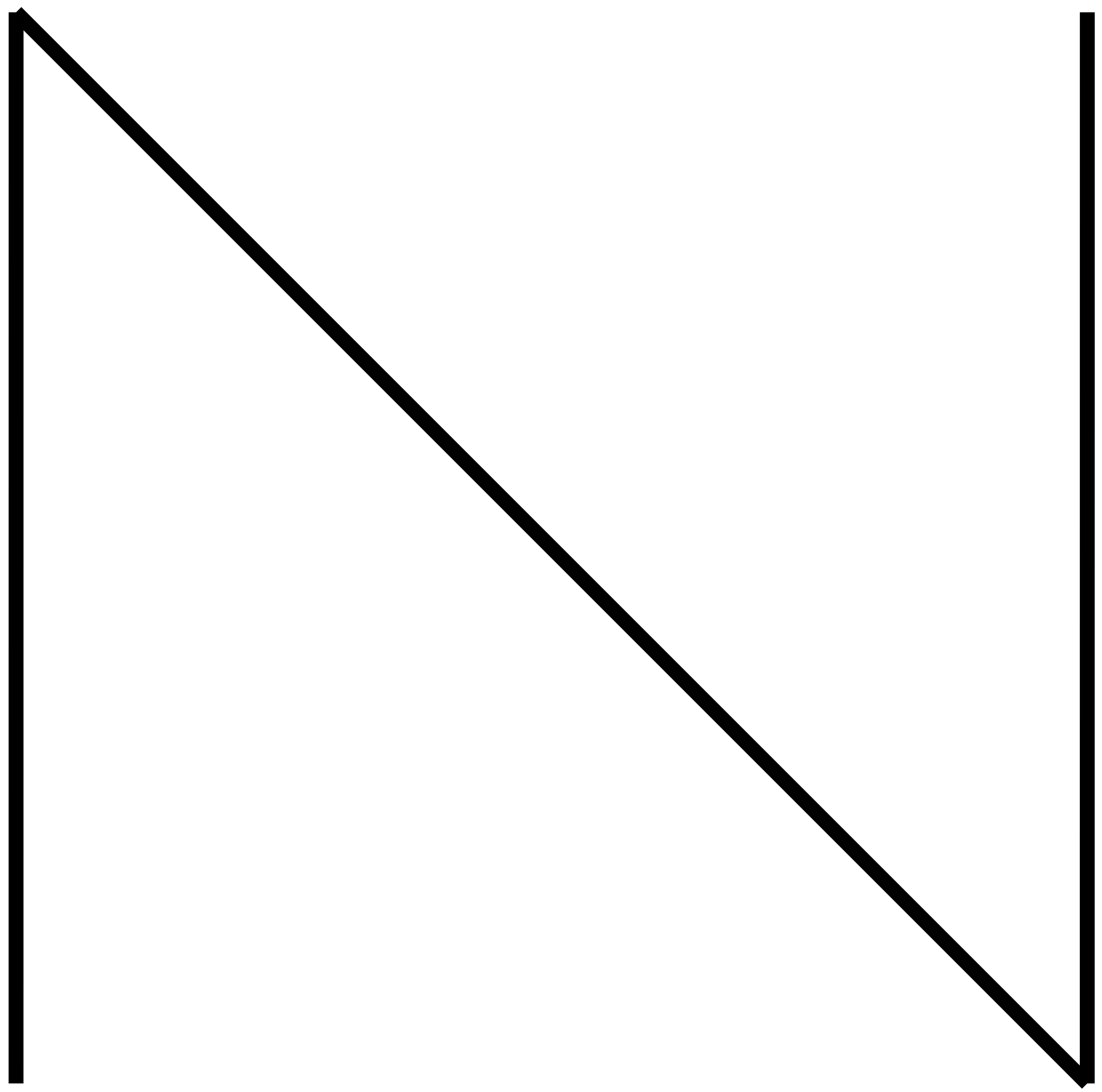

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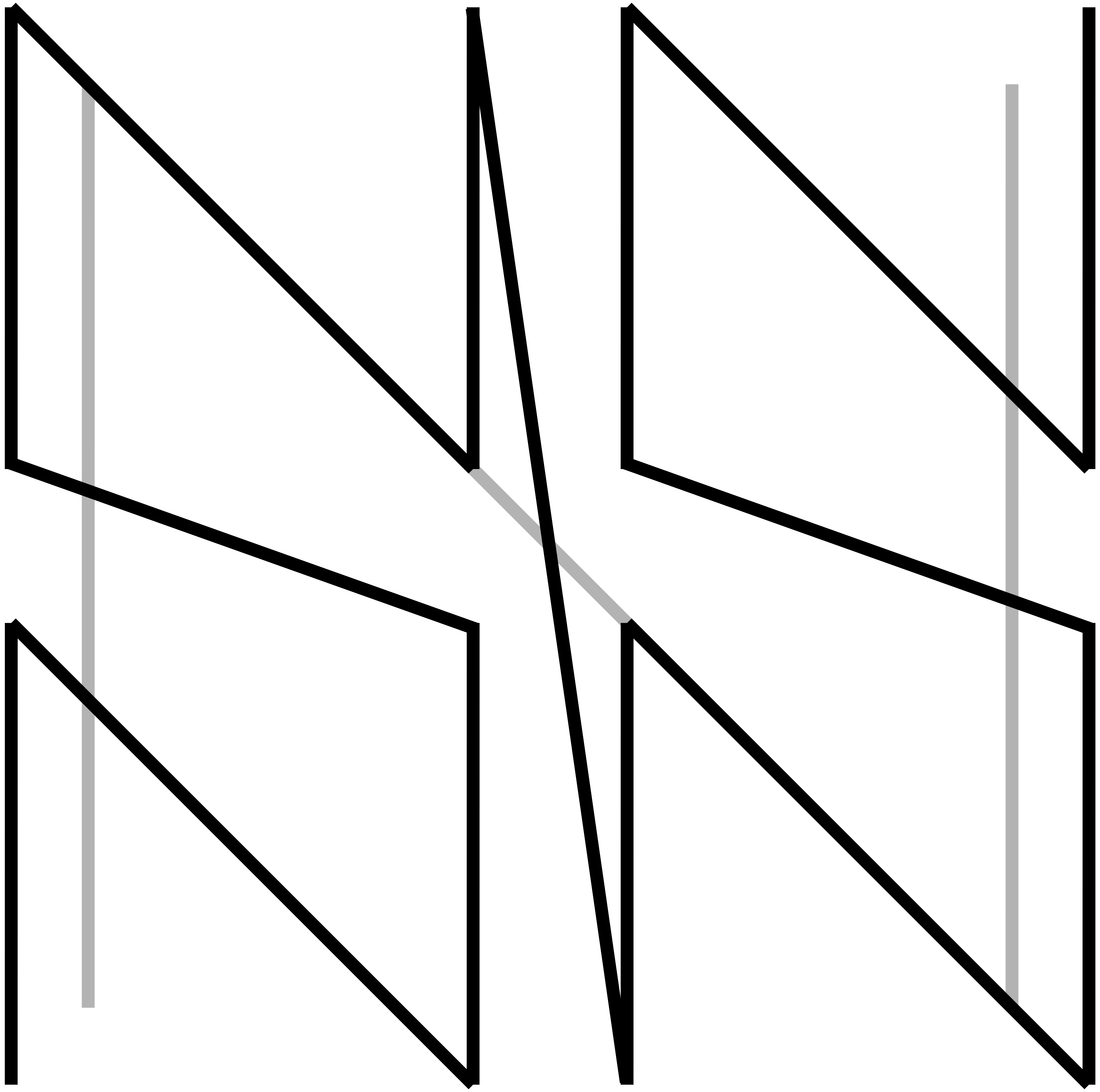

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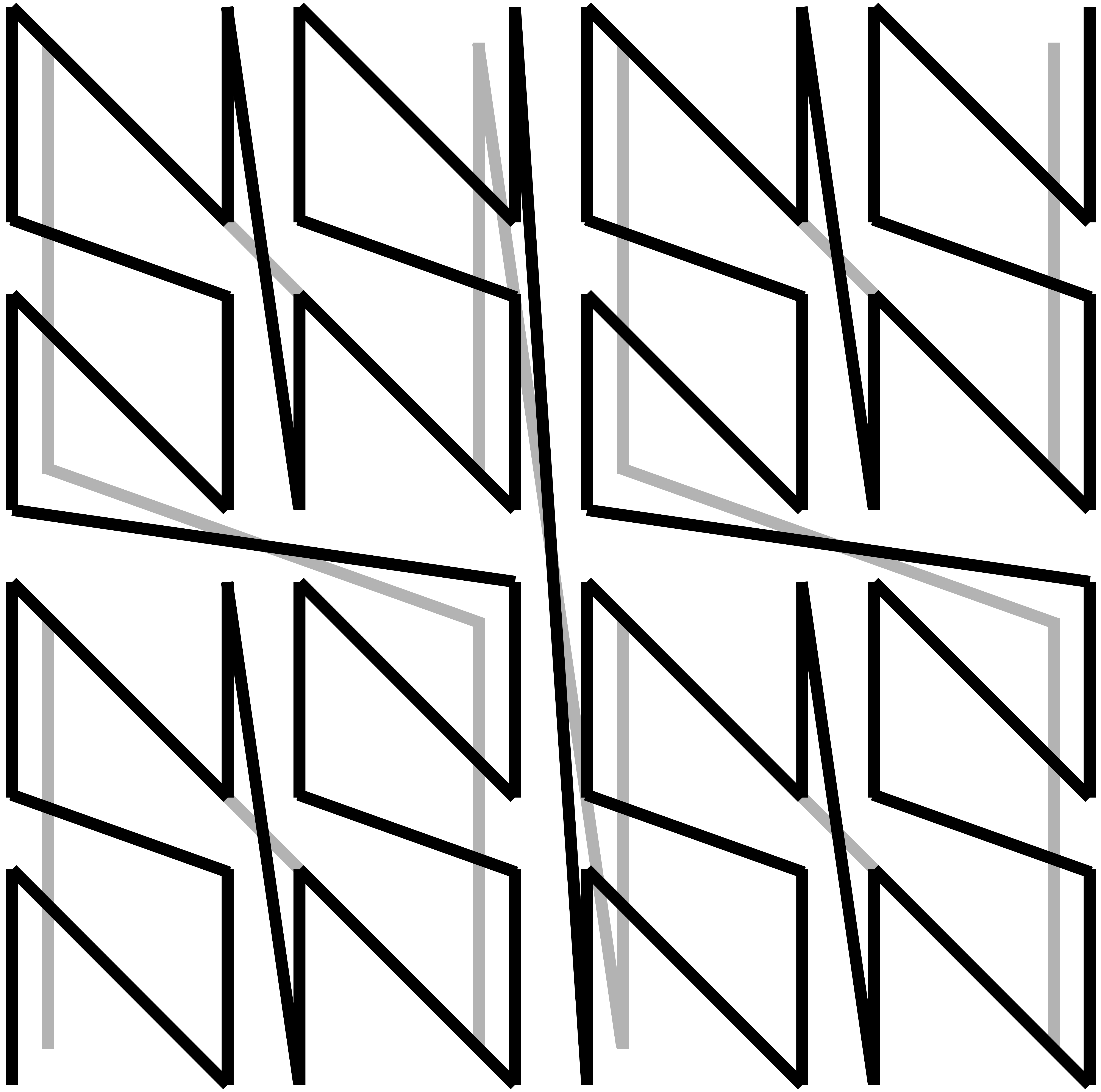

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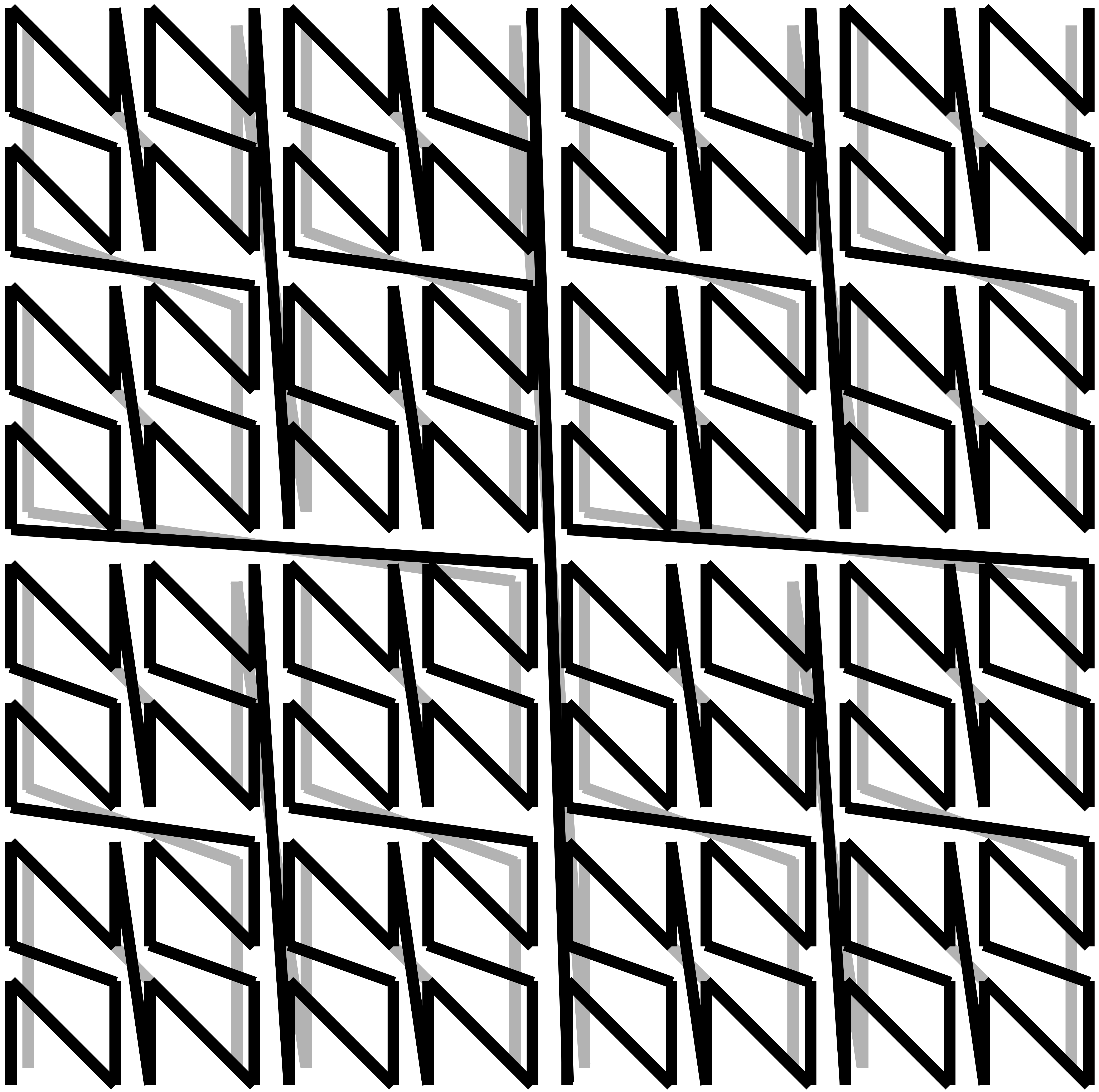

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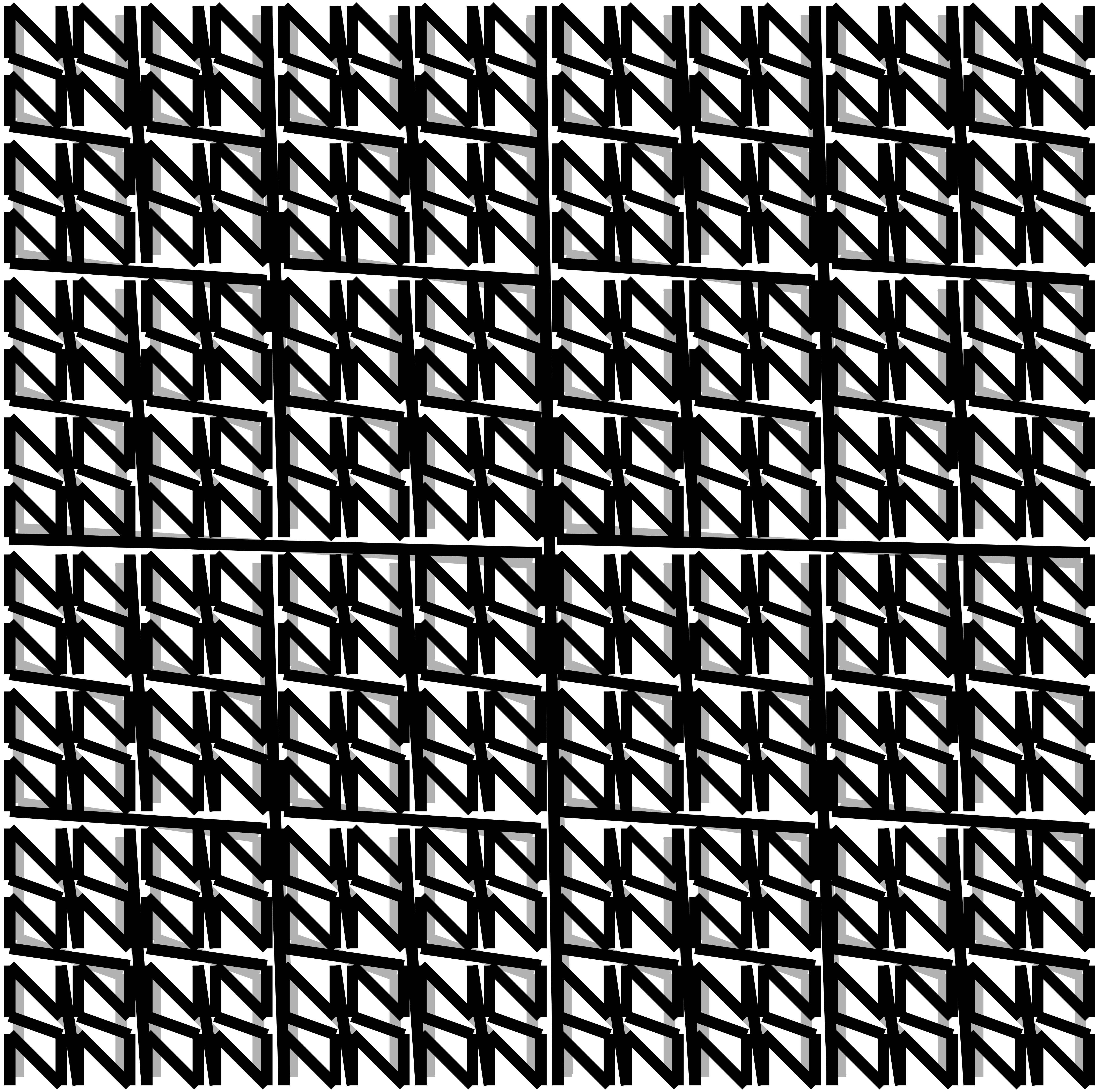

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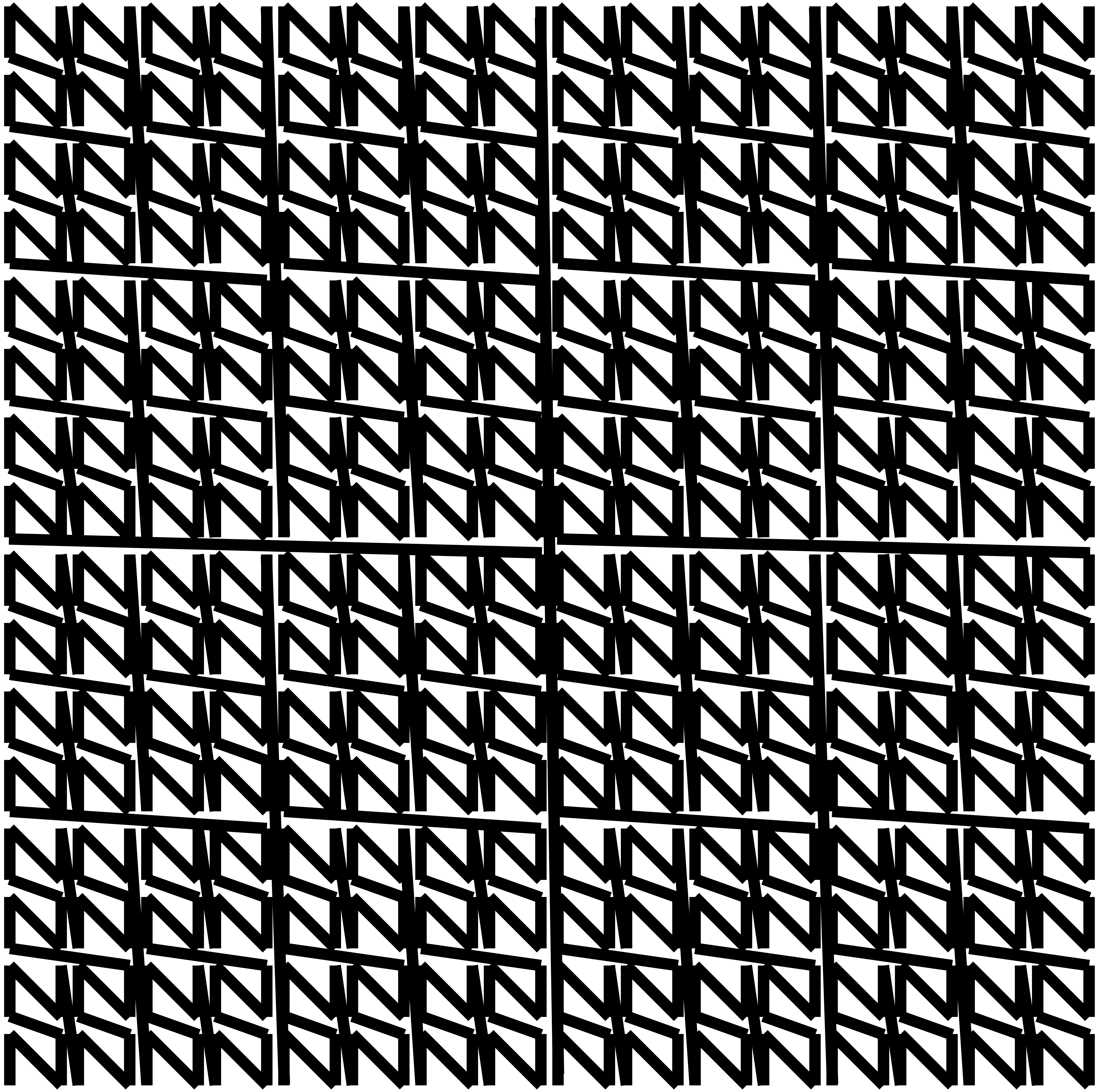

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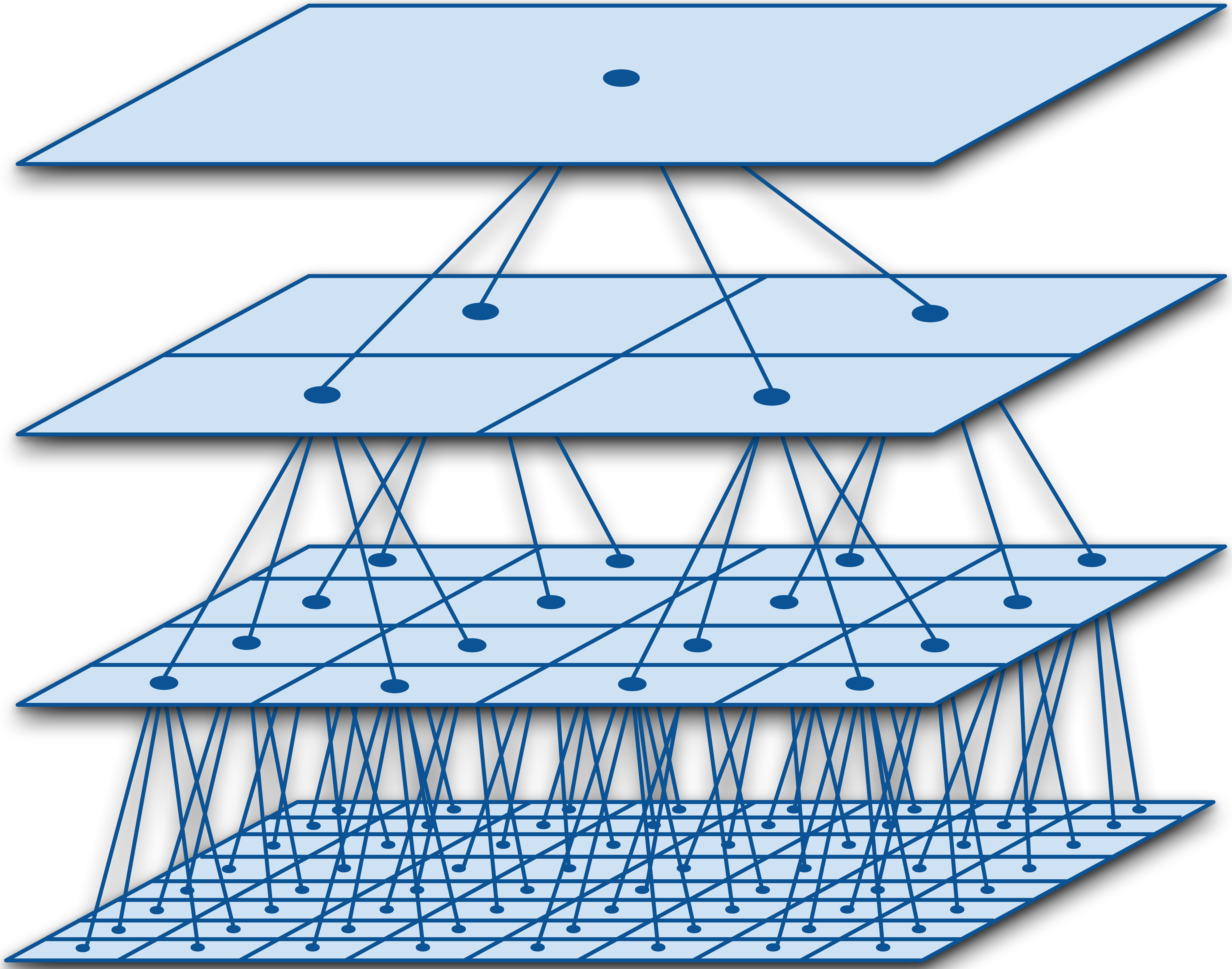
- z is ***<u>not</u> injective***
- z is a ***continuous*** map
- z is ***<u>almost everywhere</u> differentiable***

How to Fill Space

(A paradigm)

1. **Net**
2. **Order**
3. **Recurse**

Example:



Example: Quadtrees