Controlling the false positive rate of Granger causality tests in fMRI data

Oliver (and Leo, Ben, Mac, Joe)

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Increased false positive rates

Ridiculous question: Does one patient’s fMRI data influence another’s?

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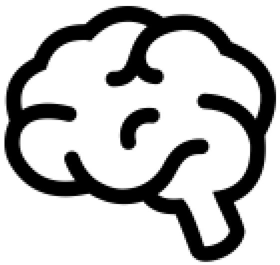

Increased false positive rates

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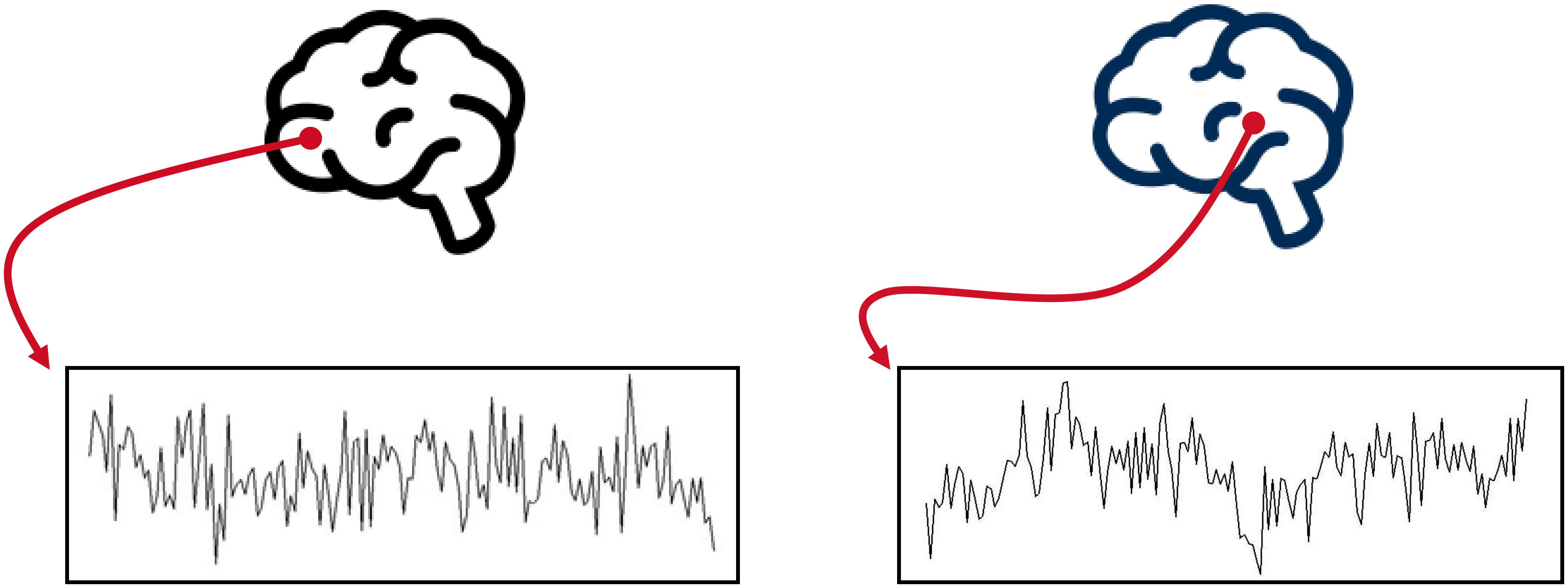

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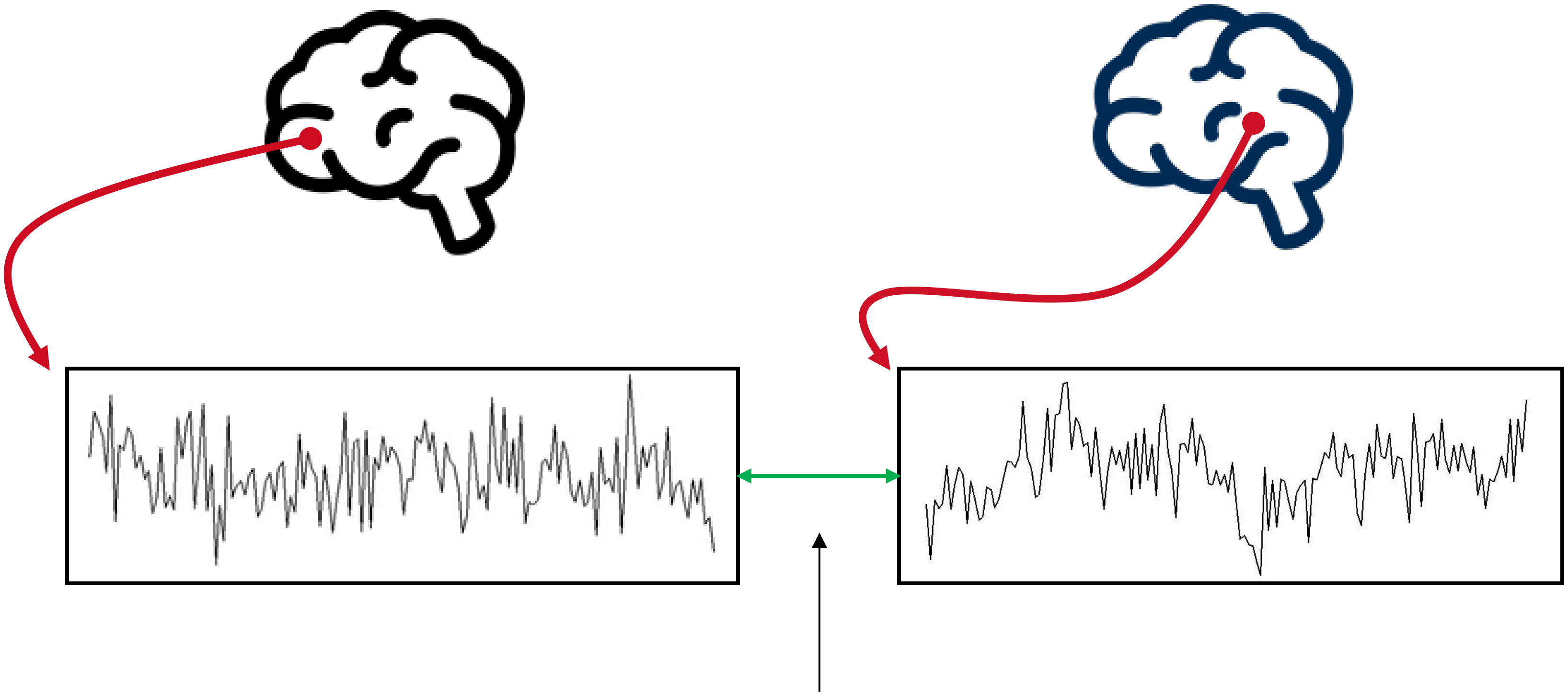
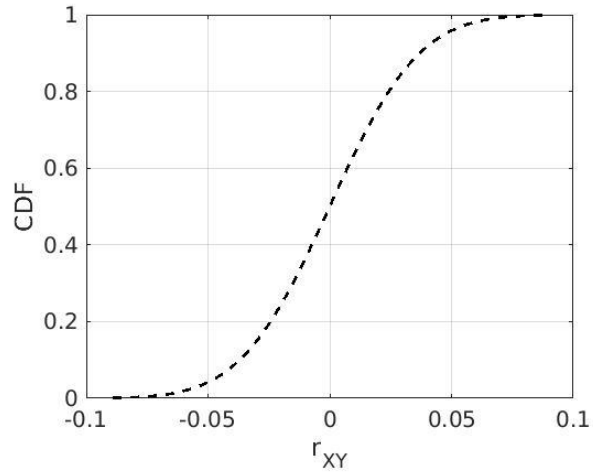


correlation, mutual information, Granger causality, etc.

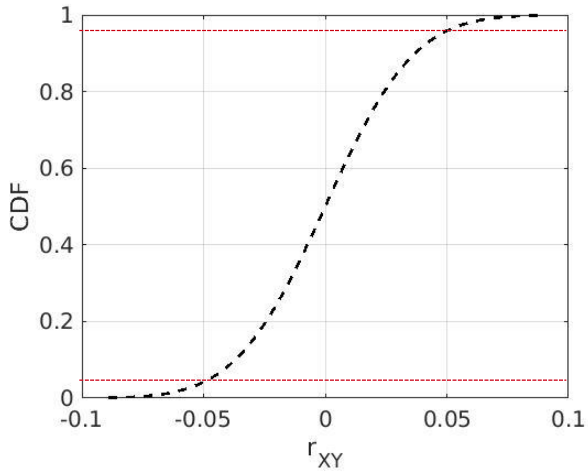
Increased false positive rates

Test correlation against a t distribution…



Increased false positive rates

…at a nominal p-value of 5%



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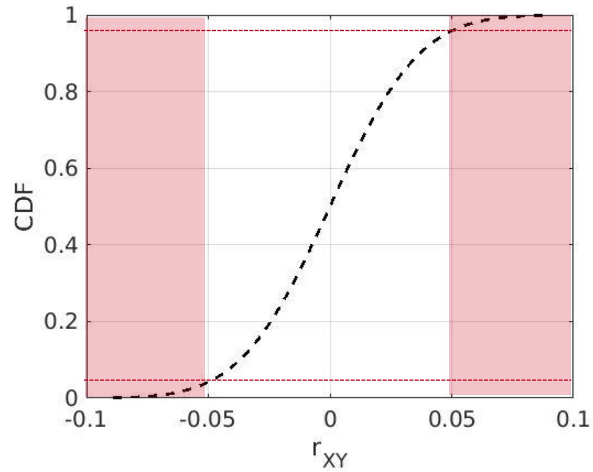
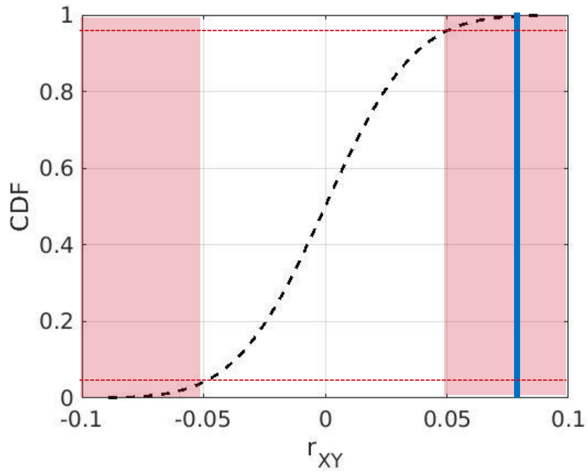
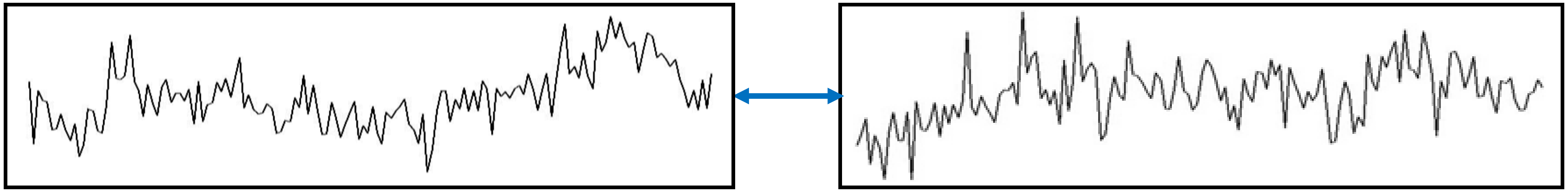


Increased false positive rates

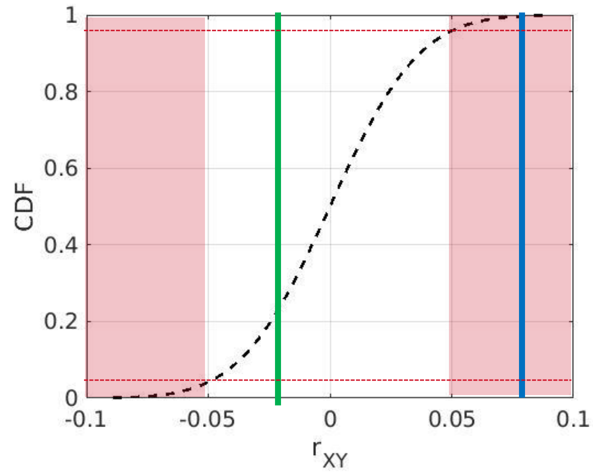
Answer: 1/1 tests indicate true (100%)…

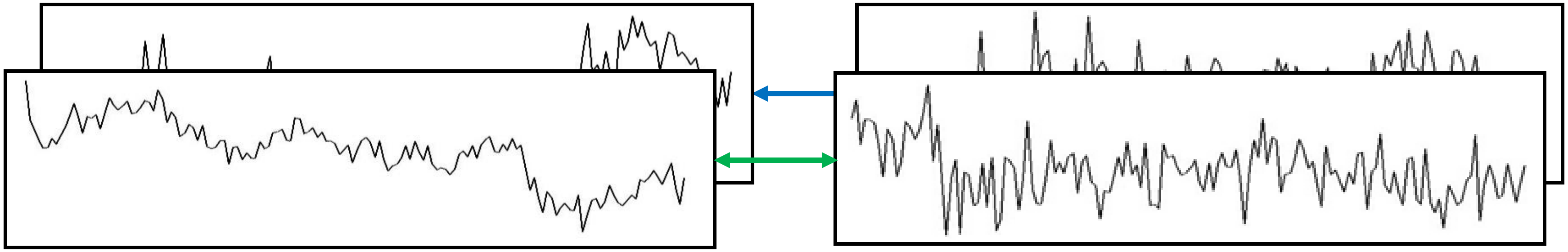




Increased false positive rates

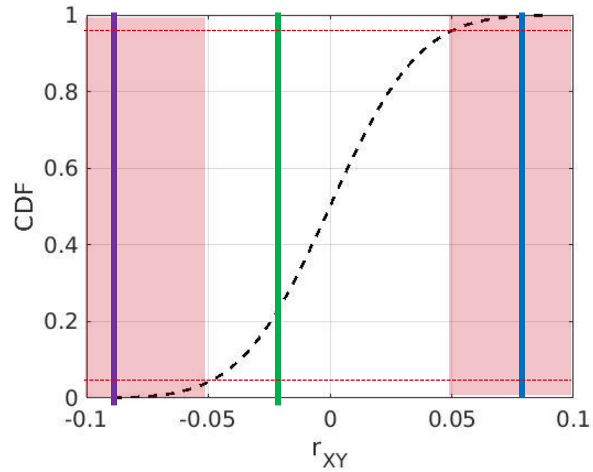
Answer: 1/2 tests indicate true (50%)…

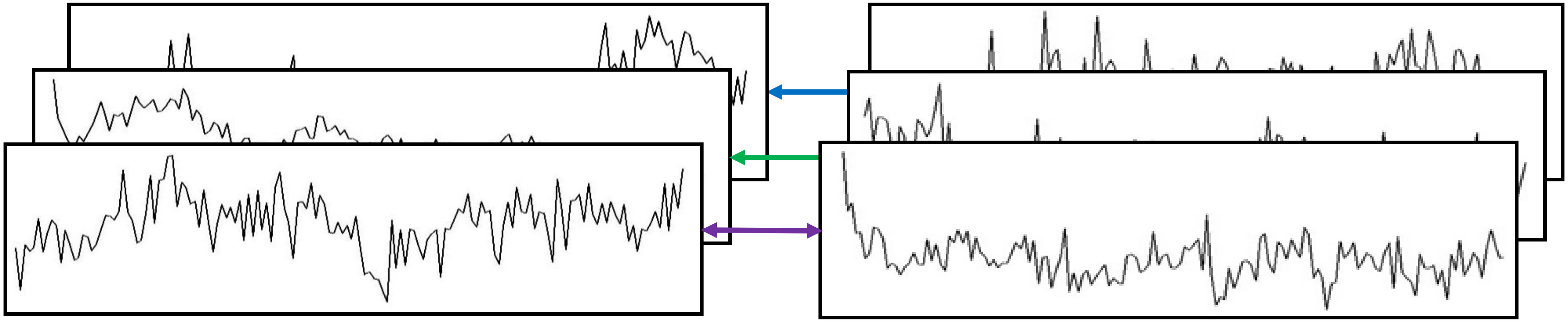




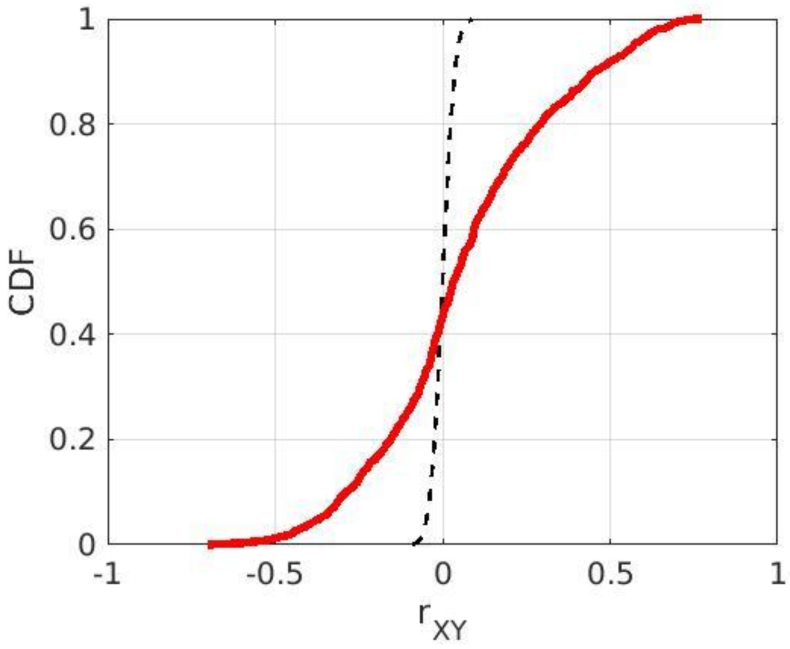
Increased false positive rates

Answer: 2/3 tests indicate true (67%)…





Answer: 781/1000 tests indicate true (78%)



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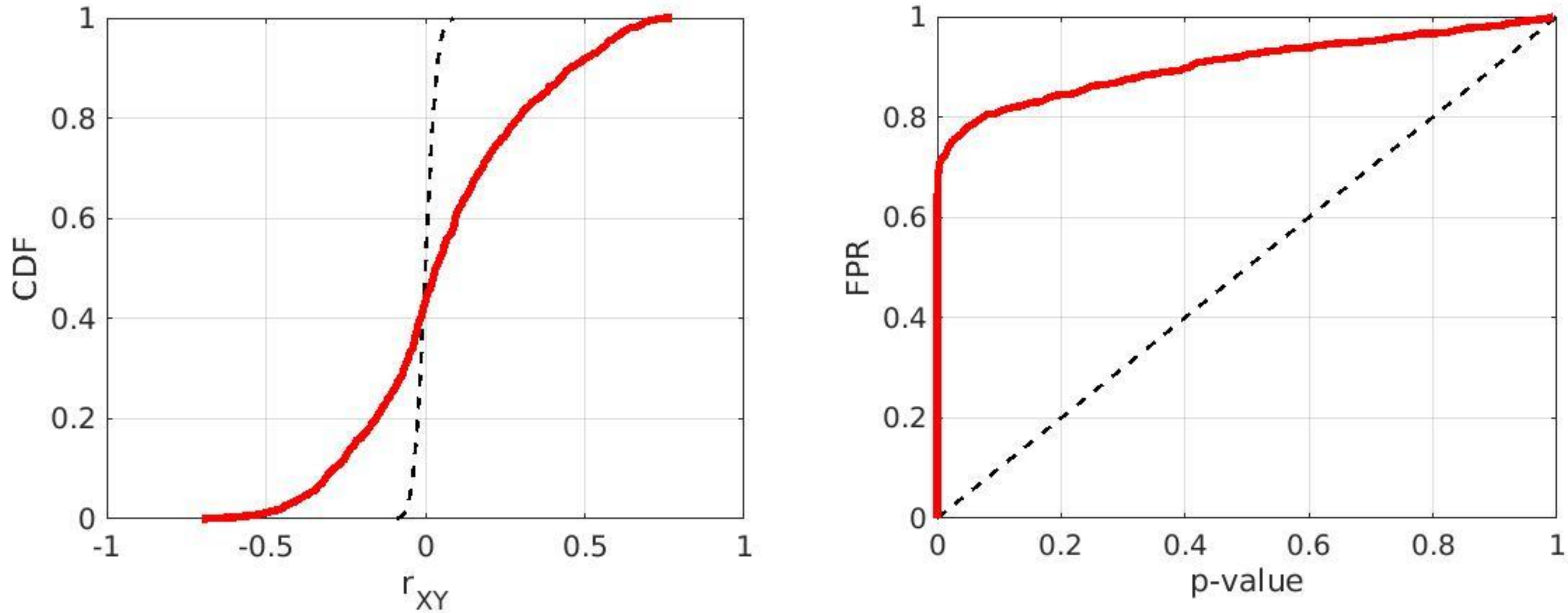

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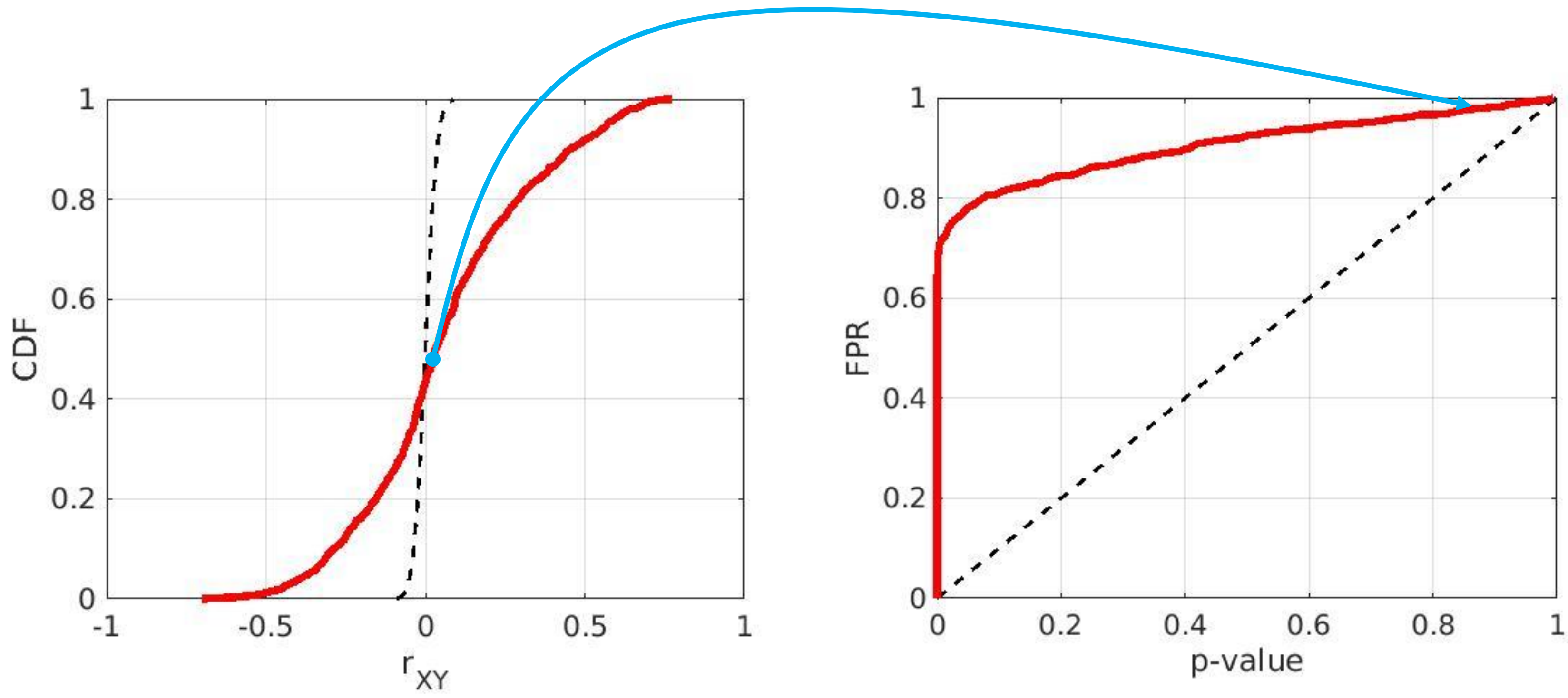

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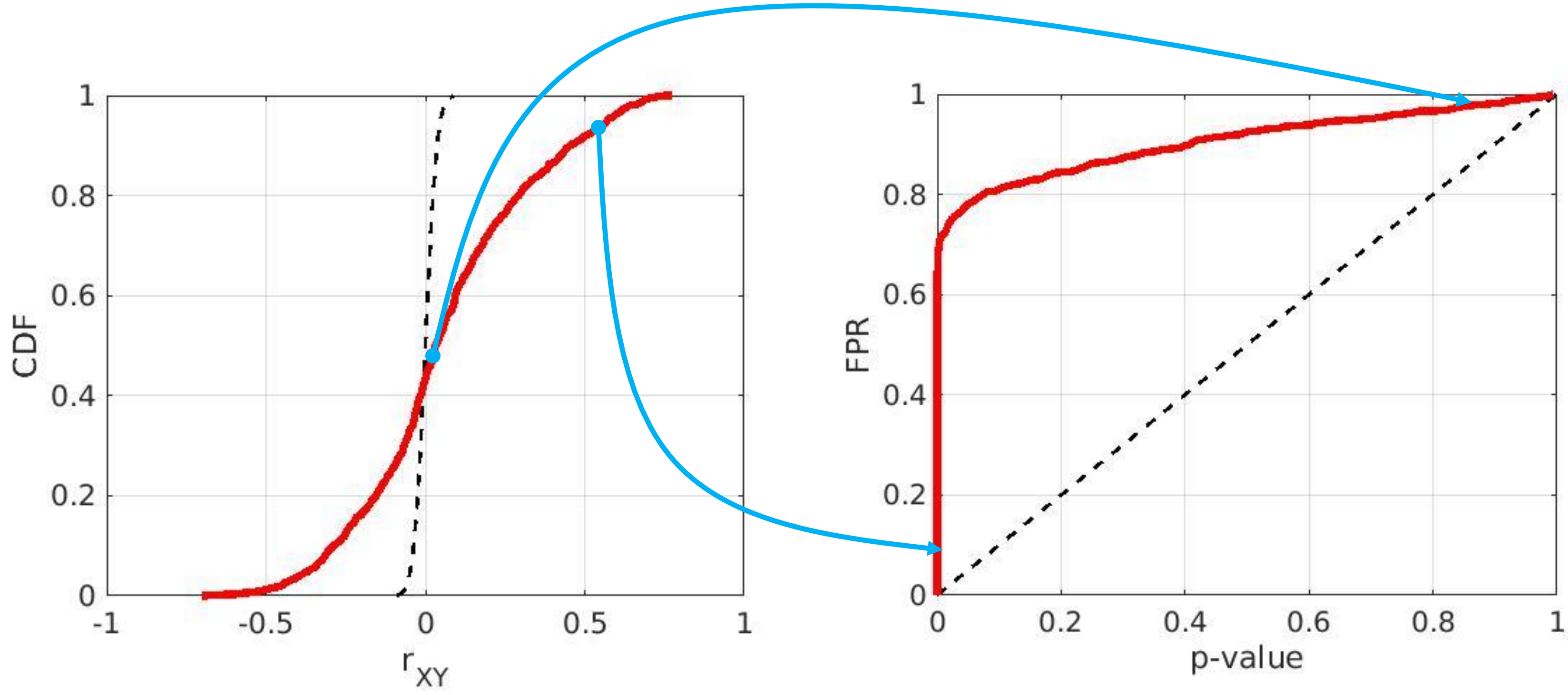
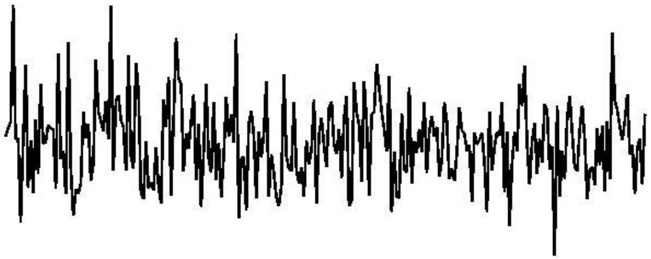


Increased false positive rates

- Pearson correlation assumptions:
  - **Gaussian** (innovations are drawn from bivariate normal dist.)
  - **IID** (no autocorrelation)
  - **Linear**

Increased false positive rates

Test: Gaussian variables using synthetic data (N=256)



Increased false positive rates

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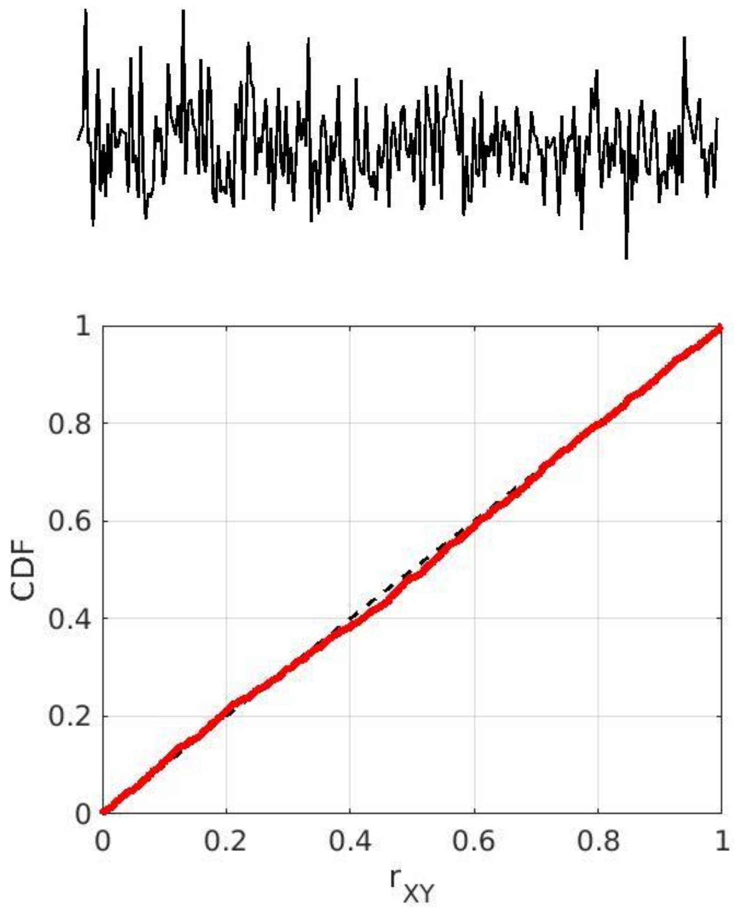
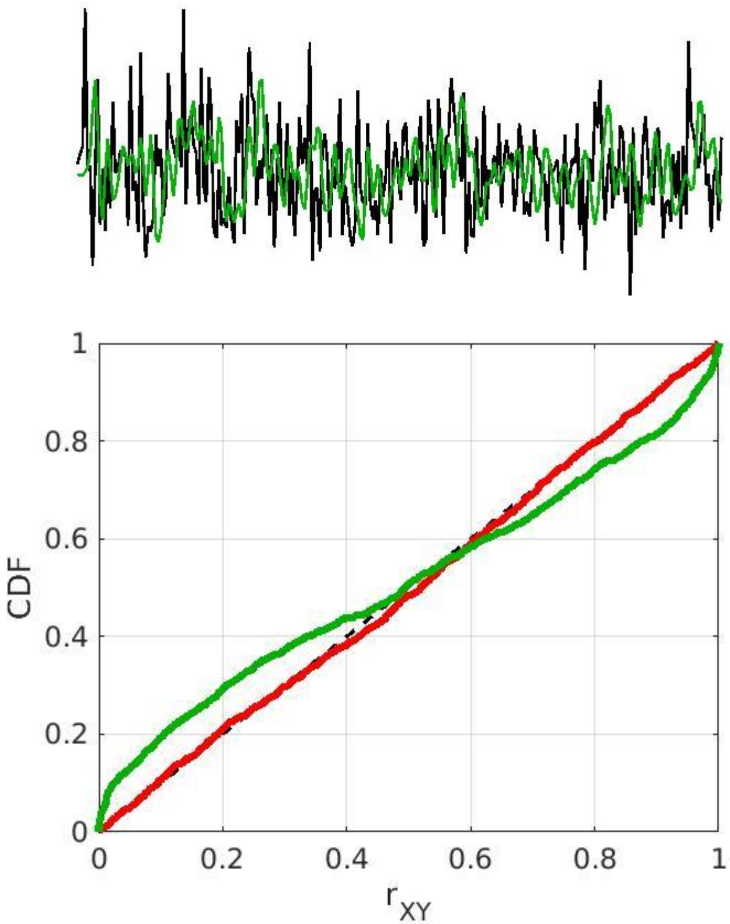


Increased false positive rates

Test: correlated Gaussian variables using synthetic data (N=256)



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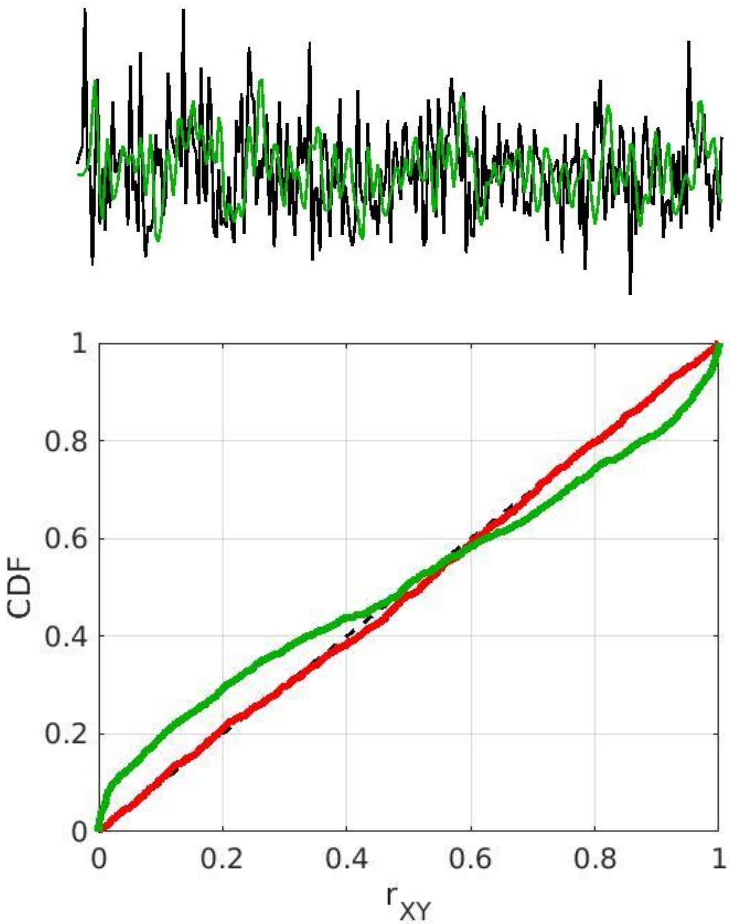

- Weakly stationary processes are fully defined by a constant:
  - Mean
  - Autocorrelation

Increased false positive rates

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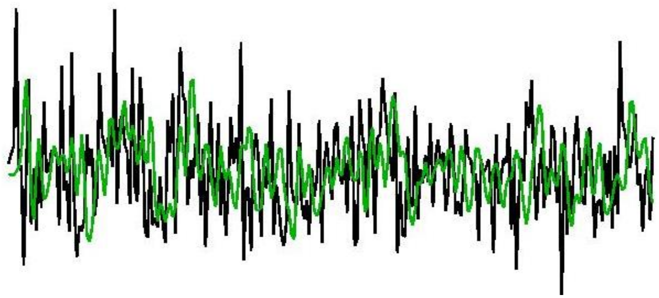

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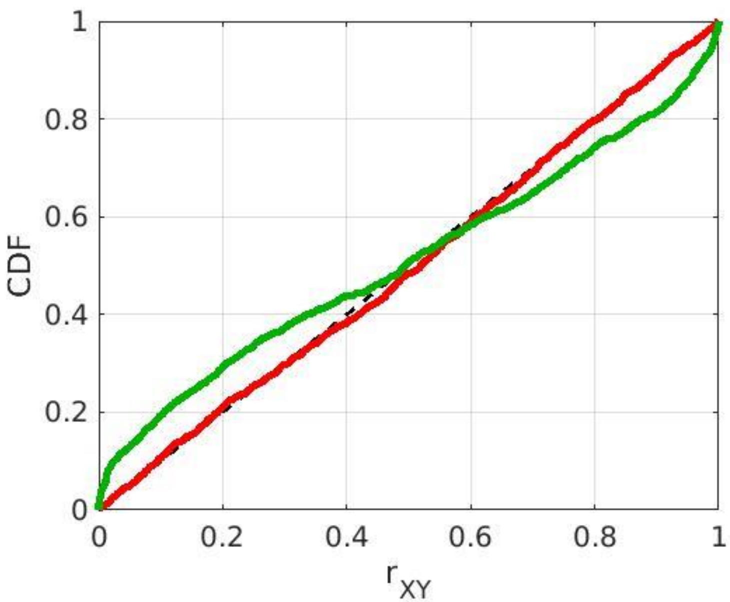

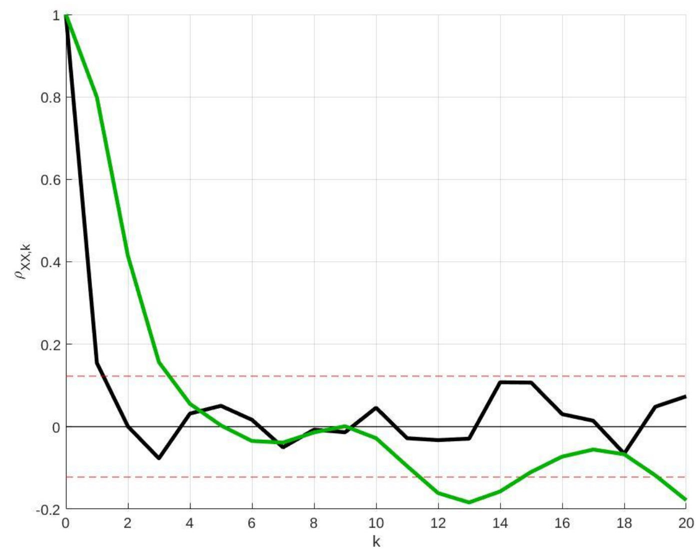

Increased false positive rates

- Higher autocorrelation increases the FPR of sample correlation
  - This has been known since Yule/Pearson's work in the early 20th century

- How do we fix it?
  - **Bartlett's formula**
    - Correct the sample size based on autocorrelation of univariate signals
    - Does the process pass the t-test with effective sample size?
  - **Granger causality**
    - Define process through autocovariance
    - Does adding in another process improve predictability?

Bartlett's formula

- Testing Pearson correlation:
  - On WB

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- Under autocorrelation:
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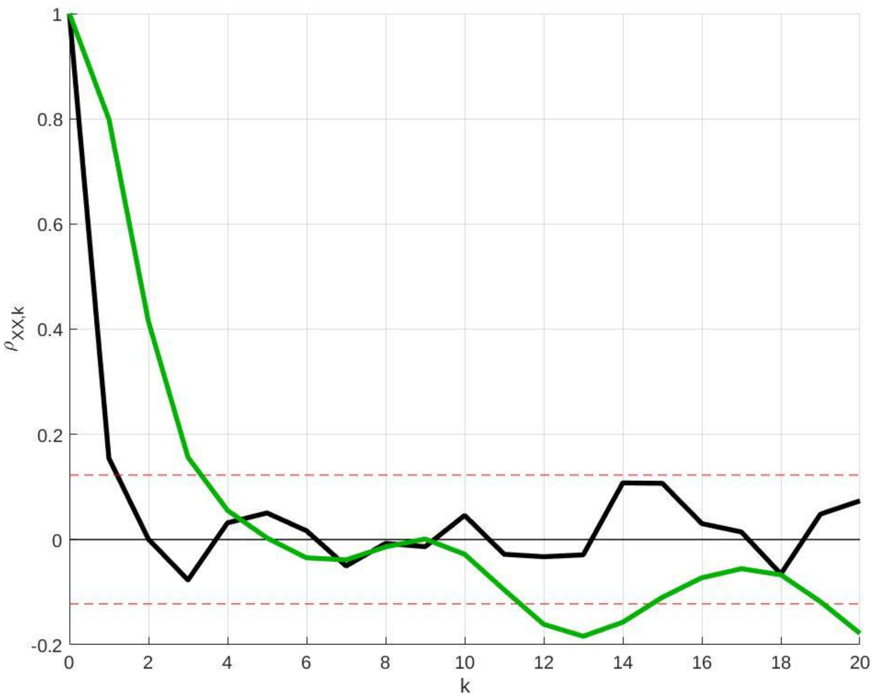

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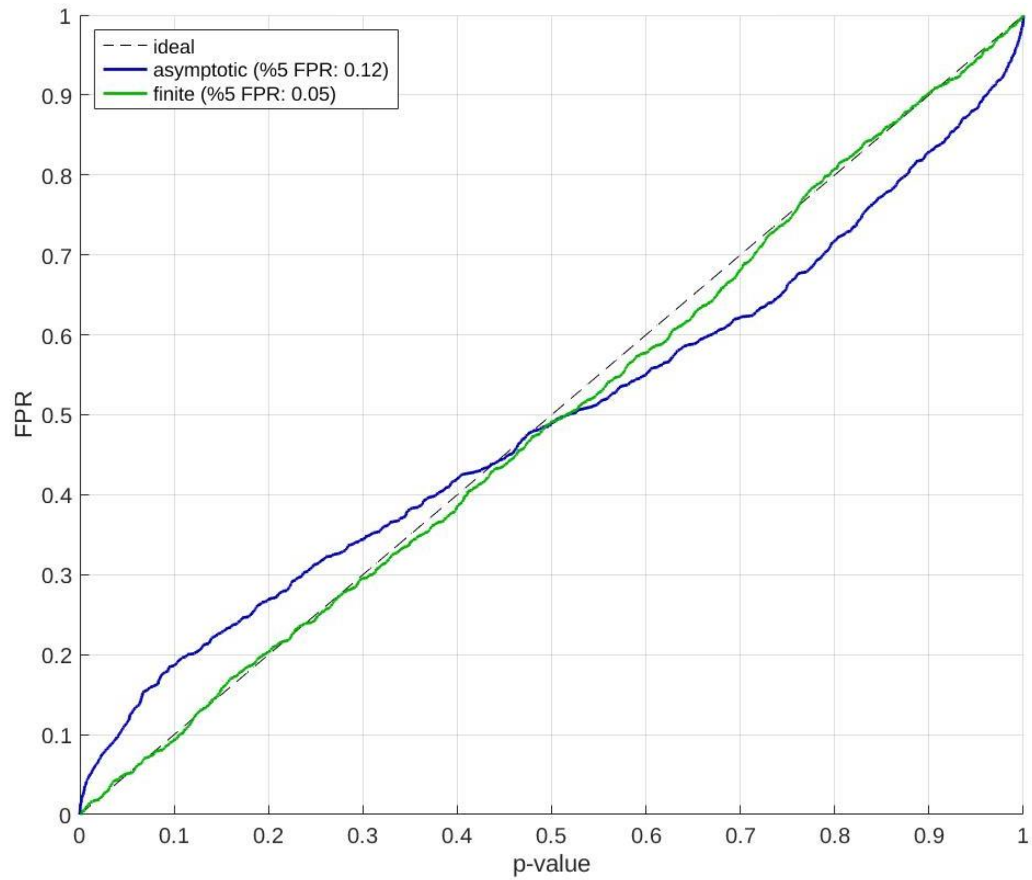

- Asymptotic distribution is **inefficient**
- Bartlett corrected values correct to nominal FPR

Granger causality

- Build a target AR process
  - [On WB]

- (Steal joe's figure for TE)

Granger causality

- Build a target AR process
  - [On WB]
- Include another (source) process
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Granger causality

- Build a target AR process
  - [On WB]
- Include another (source) process
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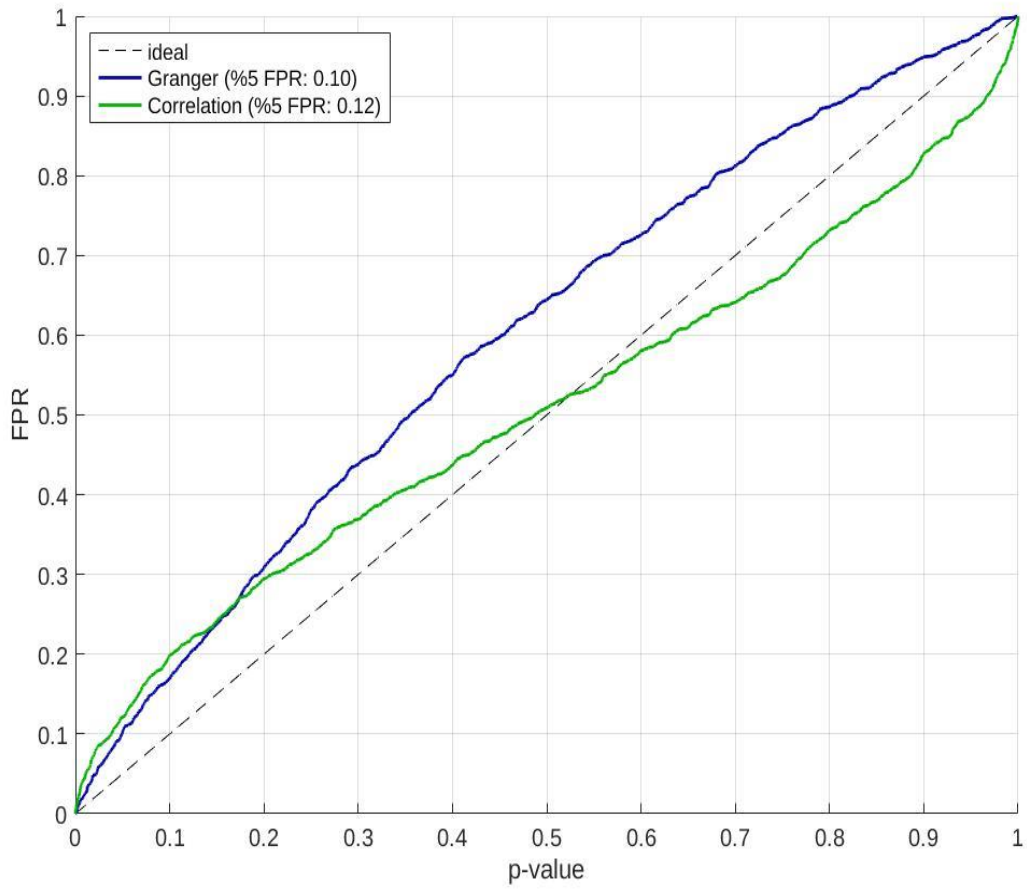
- Test using Wilk’s theorem:
  - [On WB]
- (Steal joe’s figure for TE)

Granger causality



- Asymptotic distribution is **biased**
- Slightly lower FPR than correlation

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Bartlett's formula and Granger causality

- Bartlett's formula controls FPR for univariate processes
  - Multivariate generalization is known as canonical correlations
    - These are also inefficient under serial correlation

- Granger causality reduces but does not control FPR
  - Naturally extends to multiple time series
    - FPR increases for:
      - more dimensions, or
      - higher order AR/filtering

Sampling distribution of Granger causality

- How do we control for the FPR of Granger?

- Key insight: GC can be expressed in terms of correlation
  - Granger causality is equivalent to **conditional mutual information** (Barnett et al., 2009)
  - We show CMI can be expressed as a squared **partial correlation**
  - Through chain rule, multivariate measures are sums of squared PC
  - Bartlett’s formula can be used for the sampling distributions

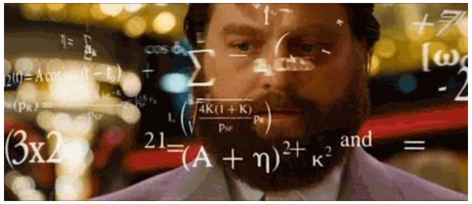
Sampling distribution of log-likelihoods

- Many dependence measures/tests have their sampling distributions derived from Wilk's theorem
  - Theorem assumes IID variables (invalid for stochastic processes)

- No known finite-sample distribution for stochastic processes for:
  - Mutual information
  - Conditional mutual information
  - Transfer entropy/Granger causality

- We provide

Mutual information as squared correlation



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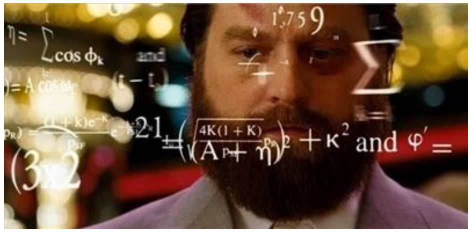

- MI for univariate Gaussian processes:

$$I_{X;Y} = -\frac{1}{2}\log\left(\frac{|\boldsymbol{K}_{XY}|}{|\boldsymbol{K}_X||\boldsymbol{K}_Y|}\right)$$

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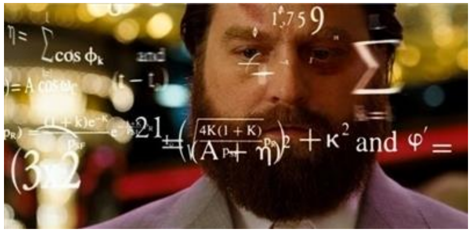

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- Covariance matrices

$$K_X = \mathrm{Var}(X)$$

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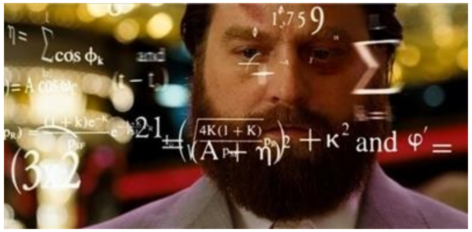

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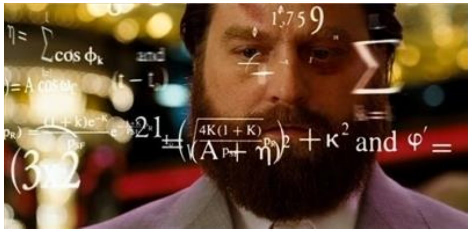

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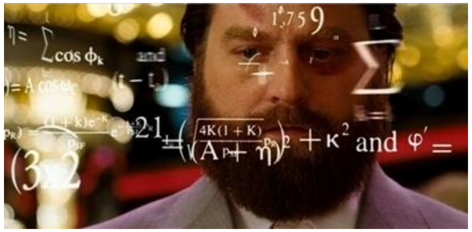
Mutual information as squared correlation



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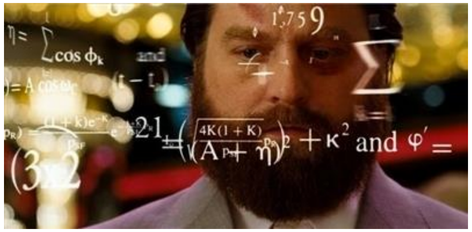
Approximate sampling distribution

Mutual information as squared correlation



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Exact sampling distribution

- First-order Taylor expansion:

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Approximate sampling distribution

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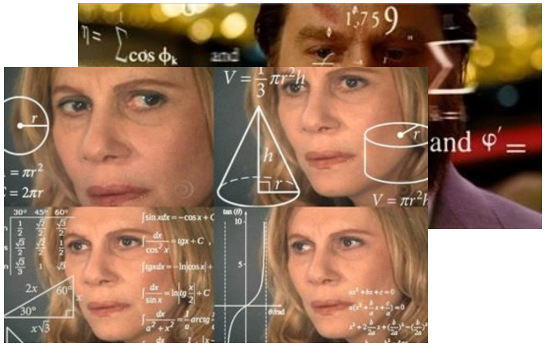
Conditional mutual information as squared partial correlation



- CMI for univariate Gaussian processes:

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$$K_{Y|\boldsymbol{W}} = \mathrm{Var}(Y|\boldsymbol{W})$$

- Taking the determinants gives us:

$$I_{X;Y|\boldsymbol{W}} = -1/2\,\log(1-\rho^2_{XY\cdot\boldsymbol{W}})$$

- First-order Taylor expansion:

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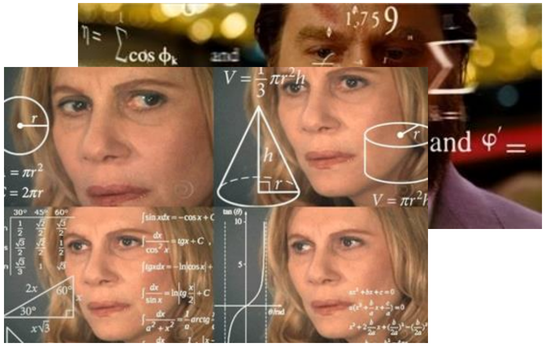
Conditional mutual information as squared partial correlation



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Exact sampling distribution

- First-order Taylor expansion:

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Approximate sampling distribution

What does this mean for GC (and others)?

- Partial correlation is the correlation between two residuals:

$$\rho_{XY\cdot\boldsymbol{W}} = \rho_{e_X e_Y}$$

- Computed w.r.t the conditional:

$$e_X = X - \boldsymbol{W}\beta^*_X$$
$$e_Y = Y - \boldsymbol{W}\beta^*_Y$$

- Approximate sampling distribution:

$$I_{X;Y|\boldsymbol{W}} \approx 1/2\,\rho^2_{XY\cdot\boldsymbol{W}}$$

- Exact sampling distribution:

Change these to this

$$I_{X;Y|\boldsymbol{W}} = -1/2\,\log(1-\rho^2_{XY\cdot\boldsymbol{W}})$$

Sampling distribution of partial correlation and CMI

- Typically, sampling distribution of MI/CMI/Granger assumed chi-squared distribution
  - From Wilk's theorem.

- Partial correlation follows same distribution as correlation with less *effective samples* based on the number of conditionals
  - And CMI is just partial correlation squared

- Squaring t-distribution gives an F-distribution
  - Now we know the exact distribution of univariate CMI
  - What about multivariate measures?

Decomposing the multivariate measures

› Use the chain rule to decompose MV MI

$$
\begin{aligned}
I_{\boldsymbol{X};\boldsymbol{Y}} &= \sum_{i,j} I_{X_i;Y_j|\{\boldsymbol{X}_{1:i-1},\boldsymbol{Y}_{1:j-1}\}} \\
&= -1/2 \sum_{i,j} \log\left(1-\rho^2_{X_iY_j\cdot\{\boldsymbol{X}_{1:i-1},\boldsymbol{Y}_{1:j-1}\}}\right) \\
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F distribution

Decomposing the multivariate measures

- …and MV CMI

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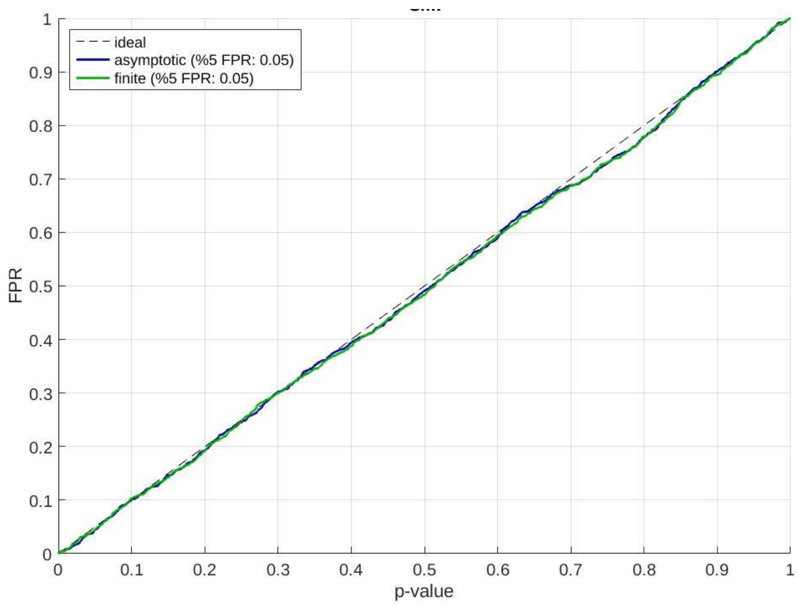
F distribution

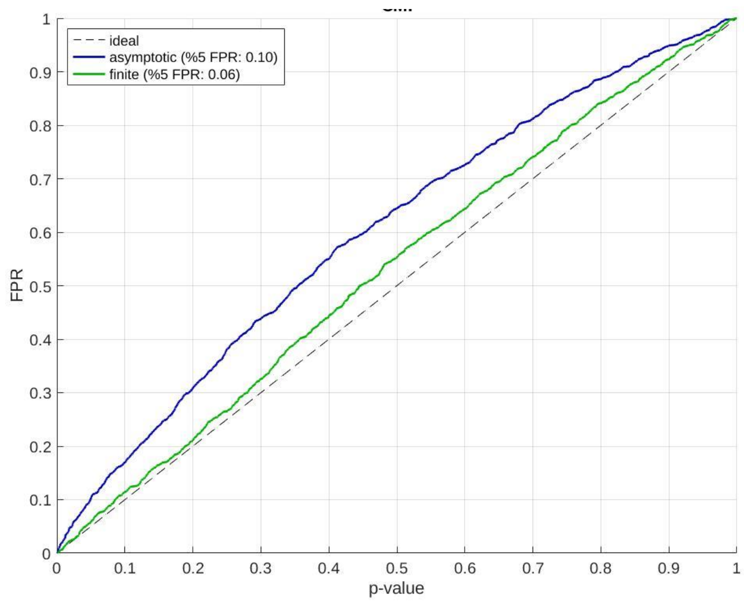
What does it all mean?

Controlling the FPR of GC

IID



~8th Order AR
(FIR filter)



Controlling the FPR of GC

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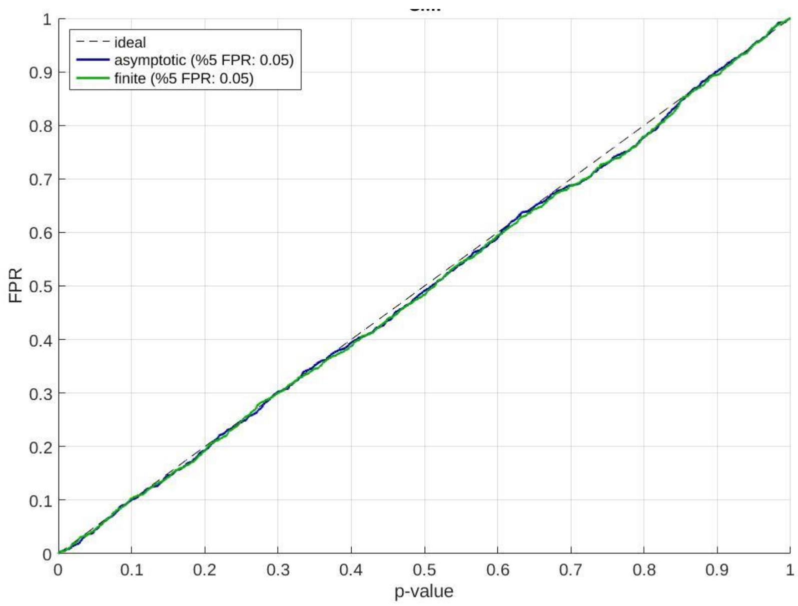
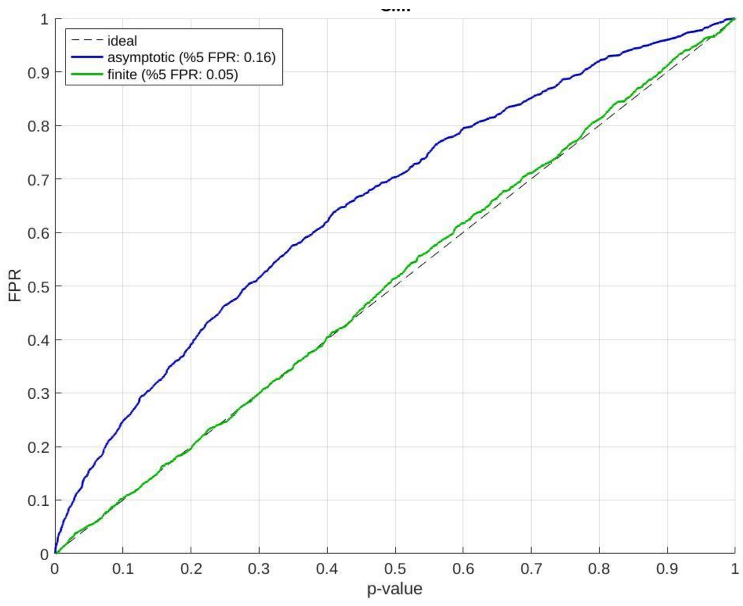


~16th Order AR
(IIR filter)



Take home messages

- Important dependence measures exhibit bias for autocorrelated Gaussian processes
- These measures can be represented as sums of squared partial correlations
- This representation allows us to derive the sampling distribution
- Before our work, these distributions were only valid asymptotically

Take home messages

- Our empirical results confirm the asymptotic sample distribution yields higher FPs for:
  - Higher order AR/filters
  - Higher dimensional processes

- Our sampling distribution controls the FPR of (univariate and multivariate) measures of dependence:
  - Mutual information
  - Conditional mutual information
  - Granger causality

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Thanks!