Exact Inference of Linear Dependence between Multiple Autocorrelated Time Series

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Centre for Complex Systems and Brain and Mind Centre collaboration
USYD

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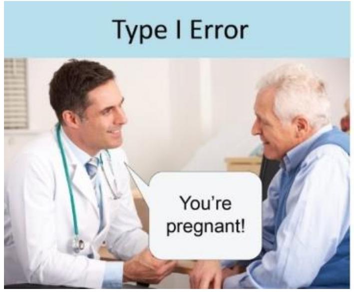
Type I Error
You're
pregnant!
Type II Error
You're not
pregnant!



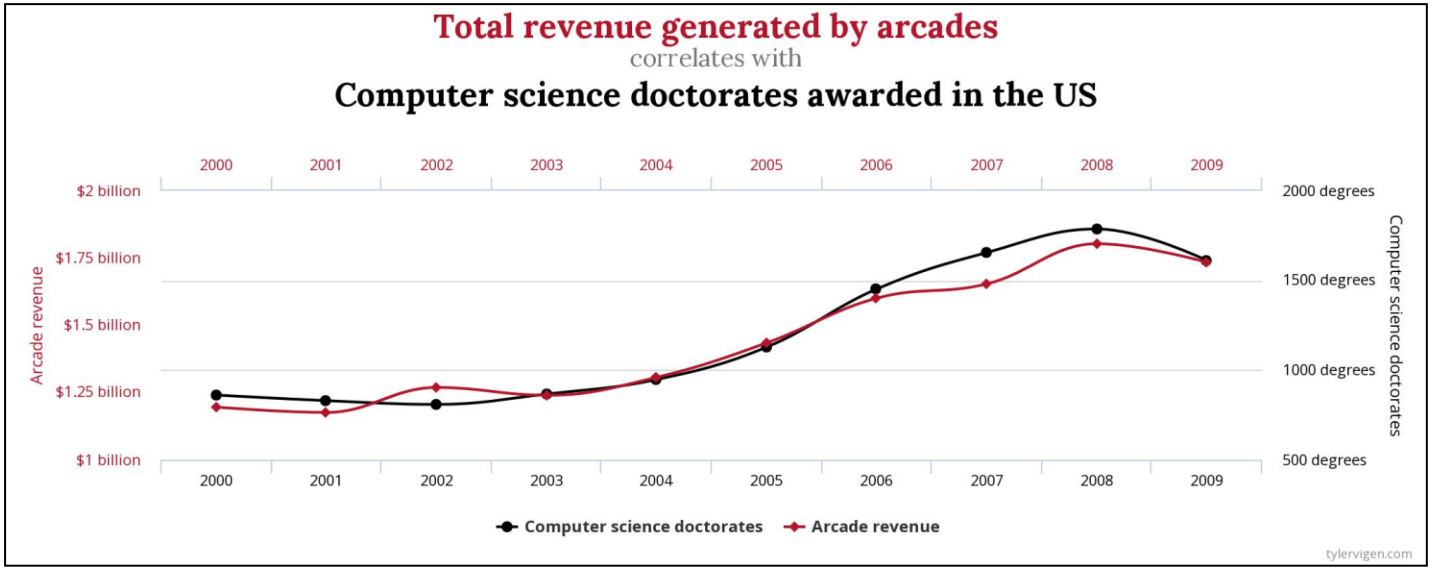
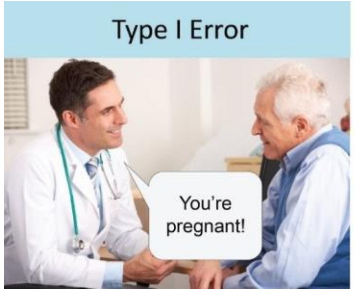
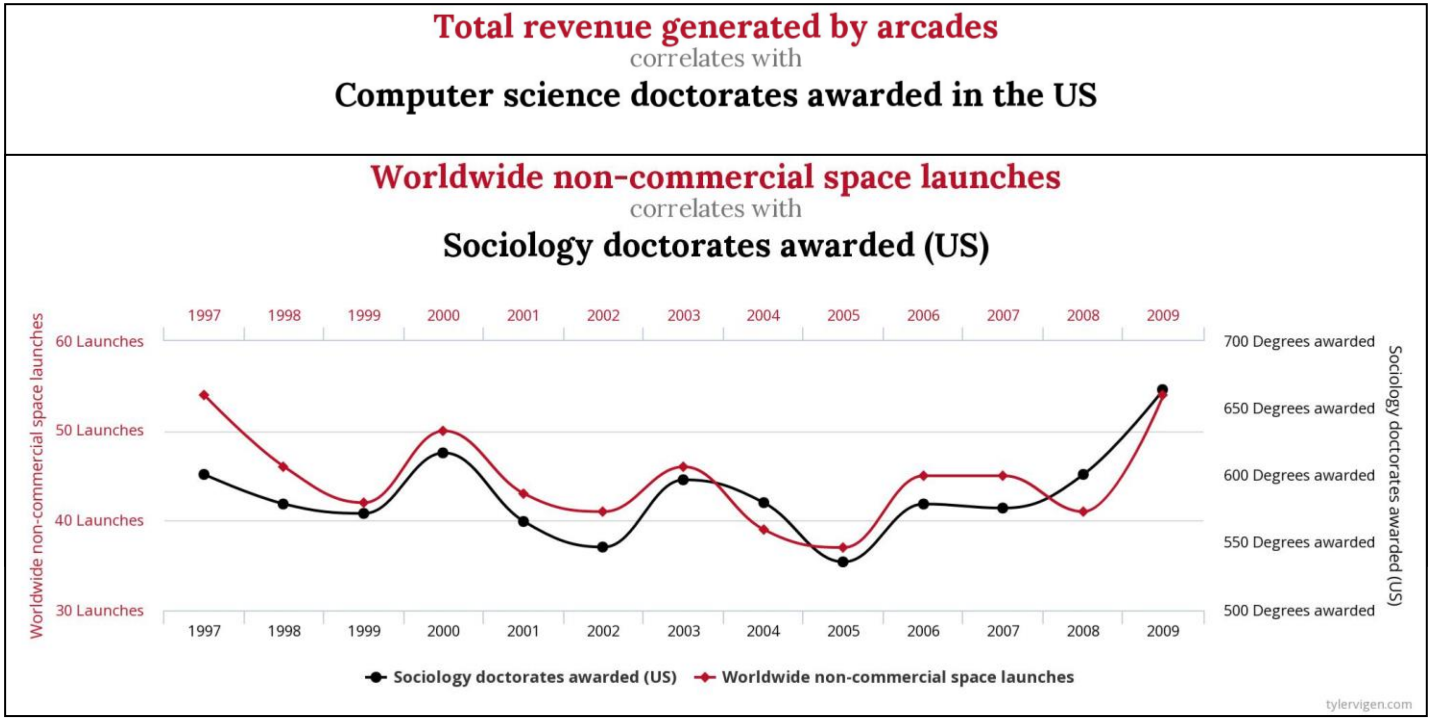


Image source: http://www.tylervigen.com/spurious-correlations





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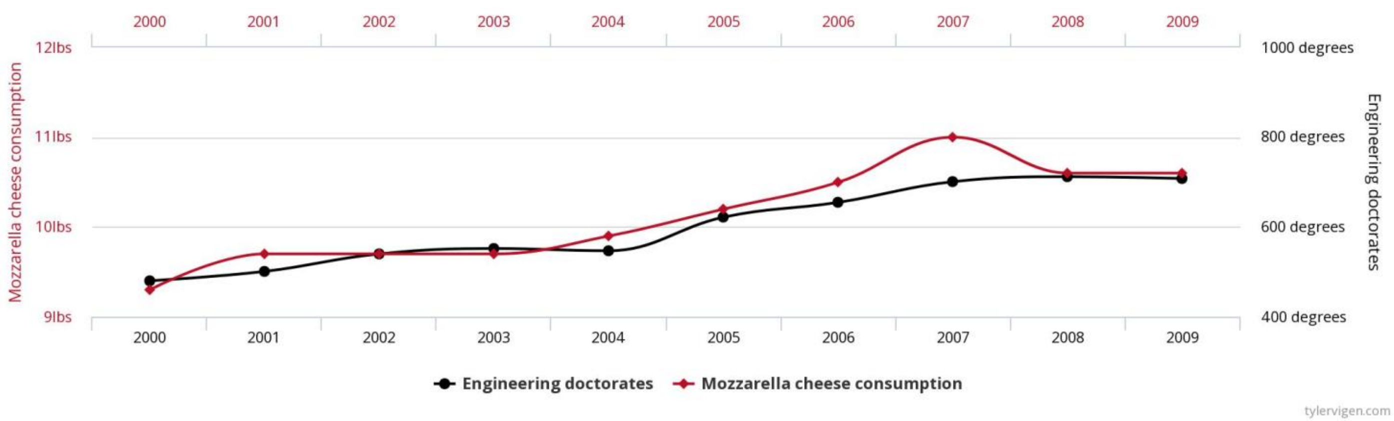
Total revenue generated by arcades
correlates with
Computer science doctorates awarded in the US

Worldwide non-commercial space launches
correlates with
Sociology doctorates awarded (US)

Per capita consumption of mozzarella cheese
correlates with
Civil engineering doctorates awarded



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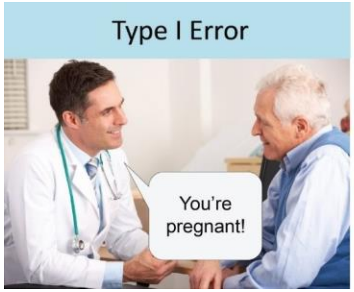

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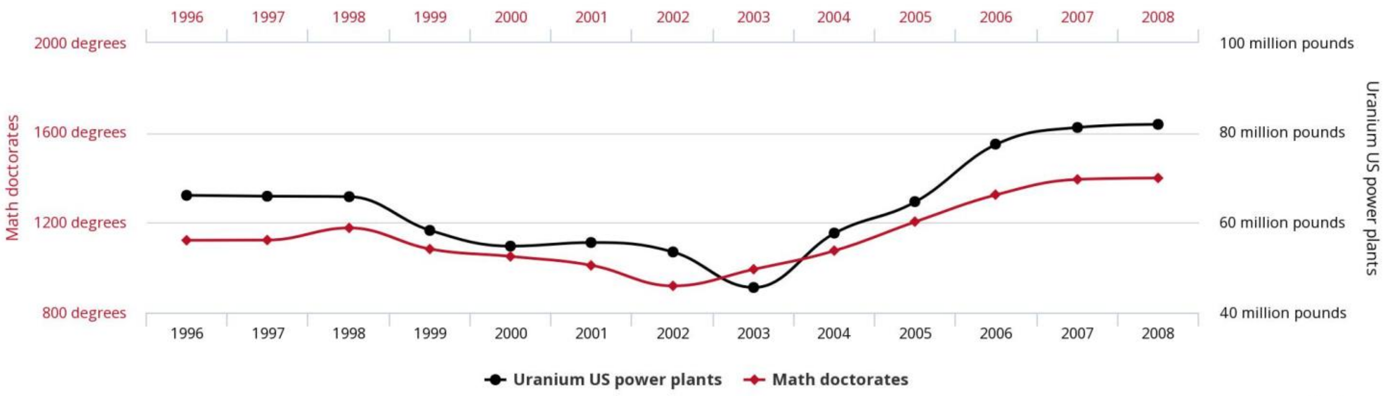
Uranium stored at US nuclear power plants



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False positives in neuroscience

- Many causes for false positives in neuroscience
  - Spatial autocorrelation in the brain [Burt et al., NeuroImage (2020)]
  - Co-expression of genes (for GSEA) [Fulcher et al., bioRxiv (2020)]
  - **Temporal autocorrelation in neuroimaging** (fMRI, M/EEG, etc.) [Afyouni et al., NeuroImage (2019)]

- It is known that Pearson correlation is affected but so are others:
  - Canonical correlation analysis
  - **(Conditional) mutual information** (undirected and multivariate)
  - **Granger causality** (directed and multivariate)

- We provide the exact null distributions for these measures under autocorrelation

Ridiculous question: Does one patient’s fMRI data influence another’s?

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Neuroimaging case study

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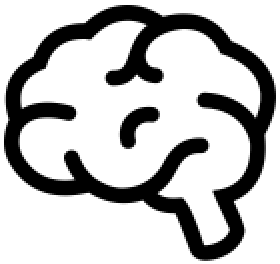

Human connectome project

- Resting-state fMRI data
- Independent subjects

Neuroimaging case study

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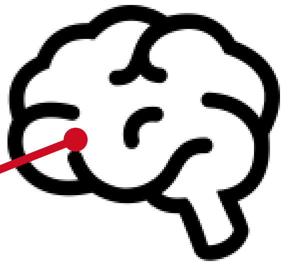

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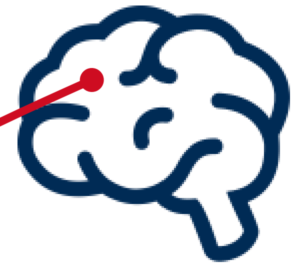
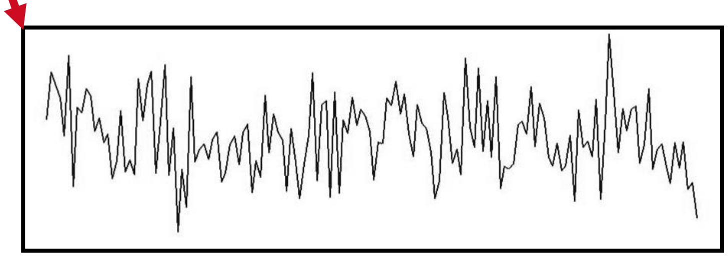
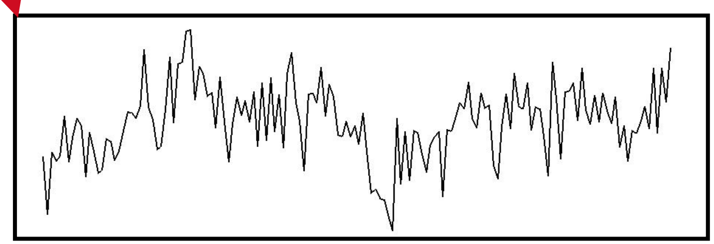


- Independent regions
- Sample size = 800
- Detrended and z-scored.





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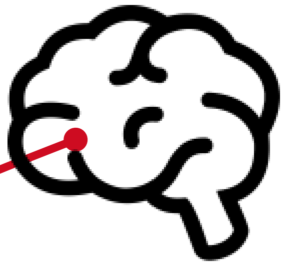

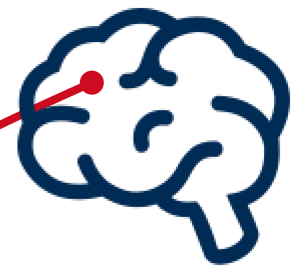

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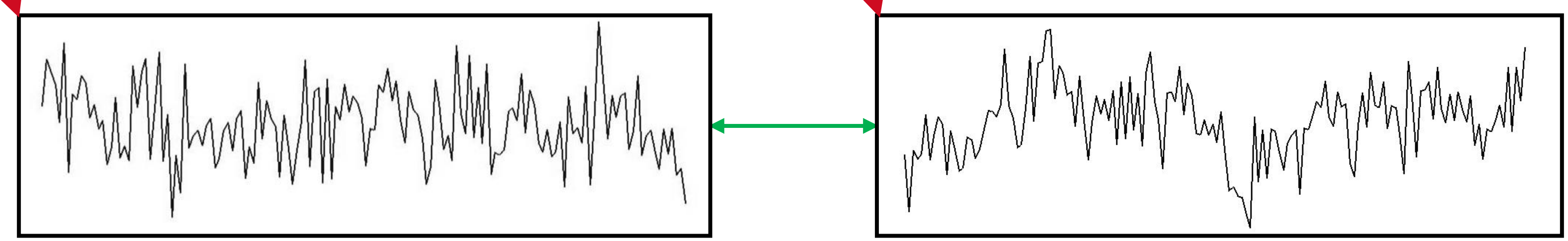
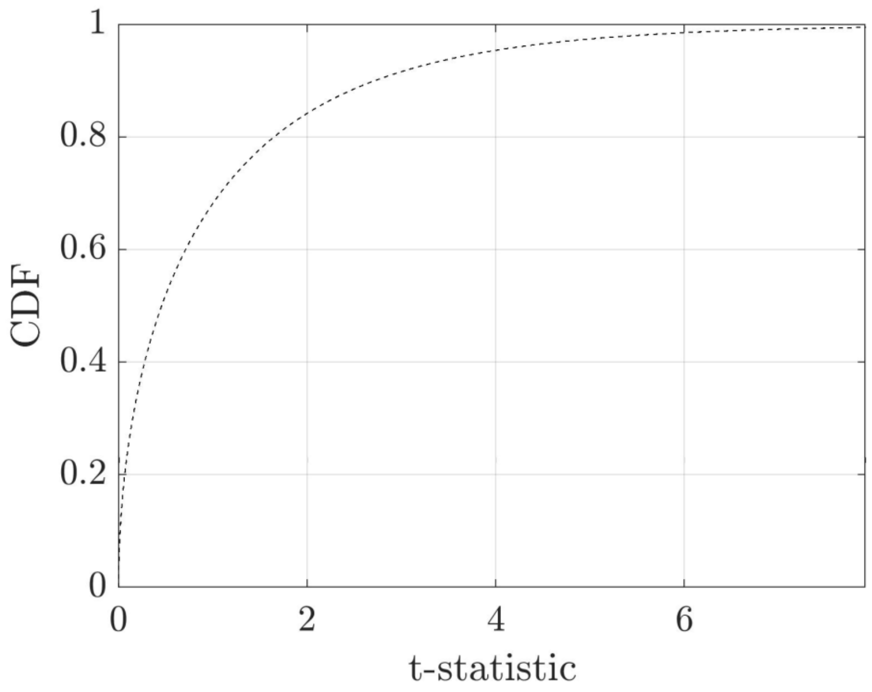
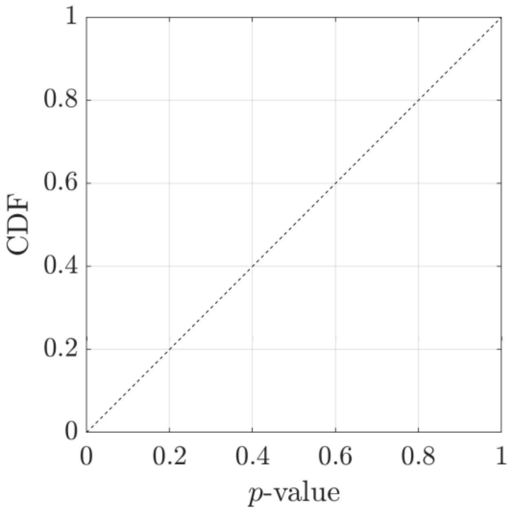


correlation, mutual information, Granger causality, etc.

Neuroimaging FPR: Pearson correlation

› Test for bivariate correlation
  - Assumes normally distributed **i.i.d variables**
  - Null case follows Student's t-distribution





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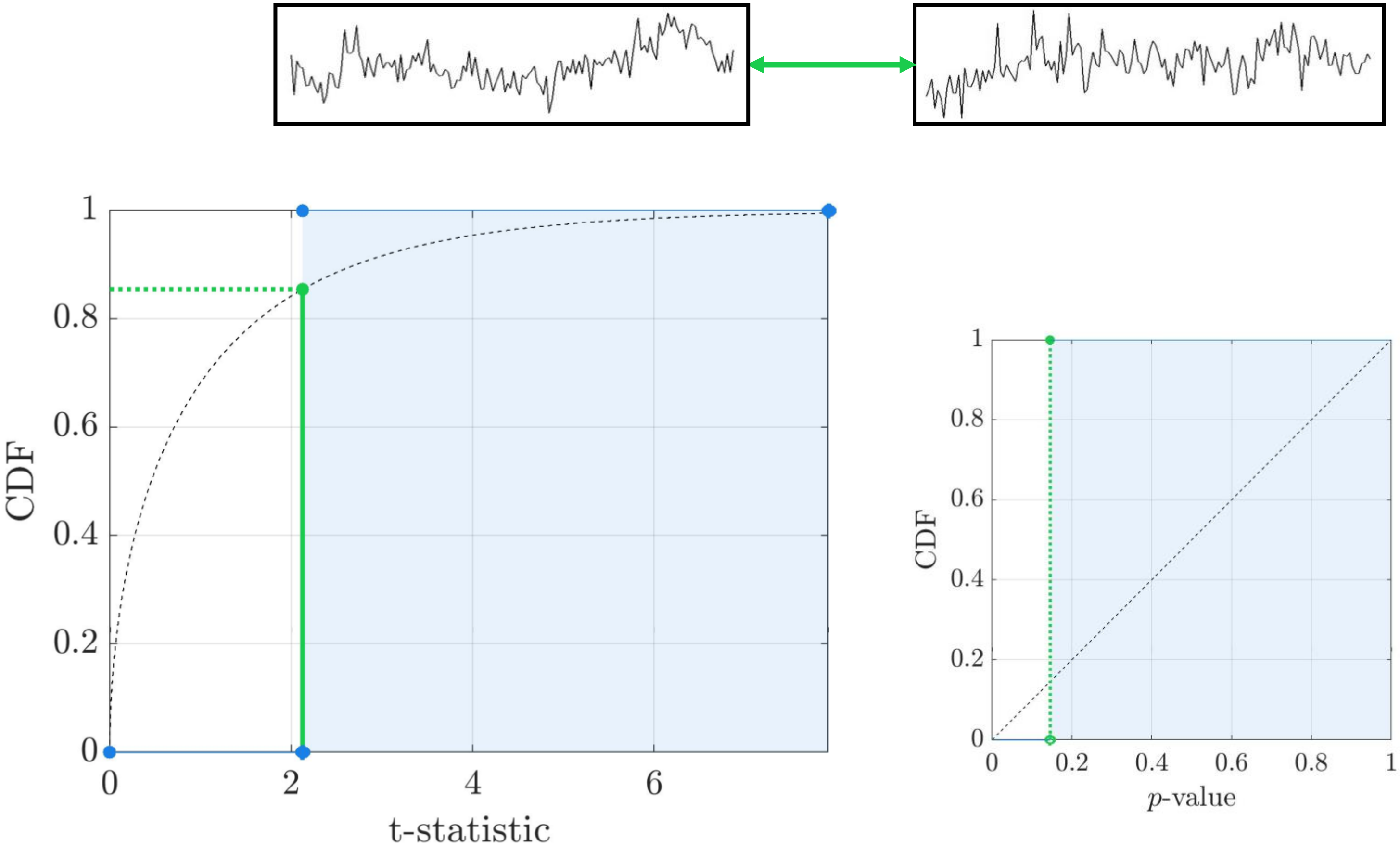

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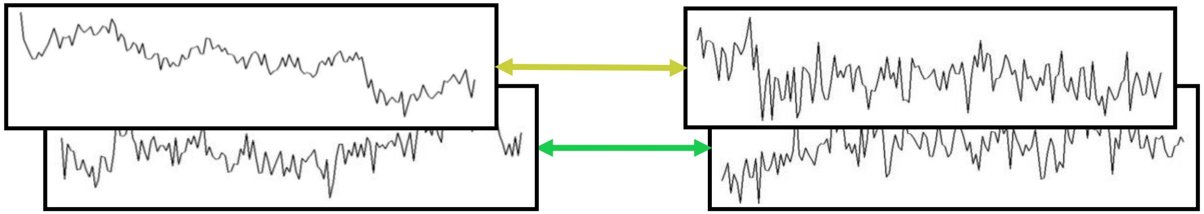

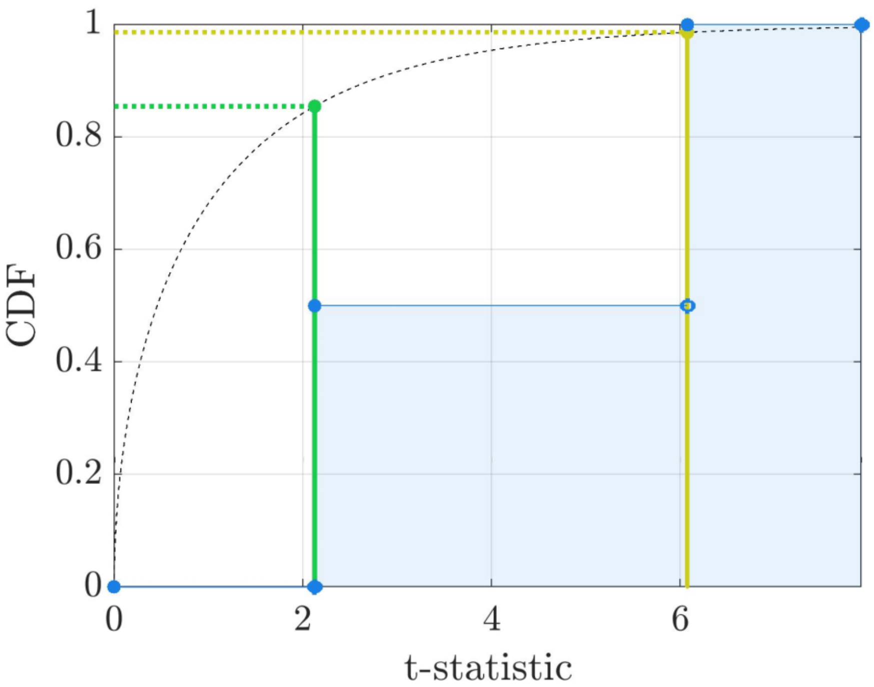

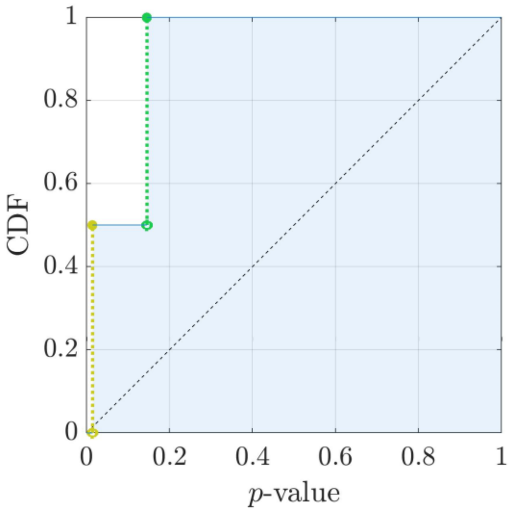

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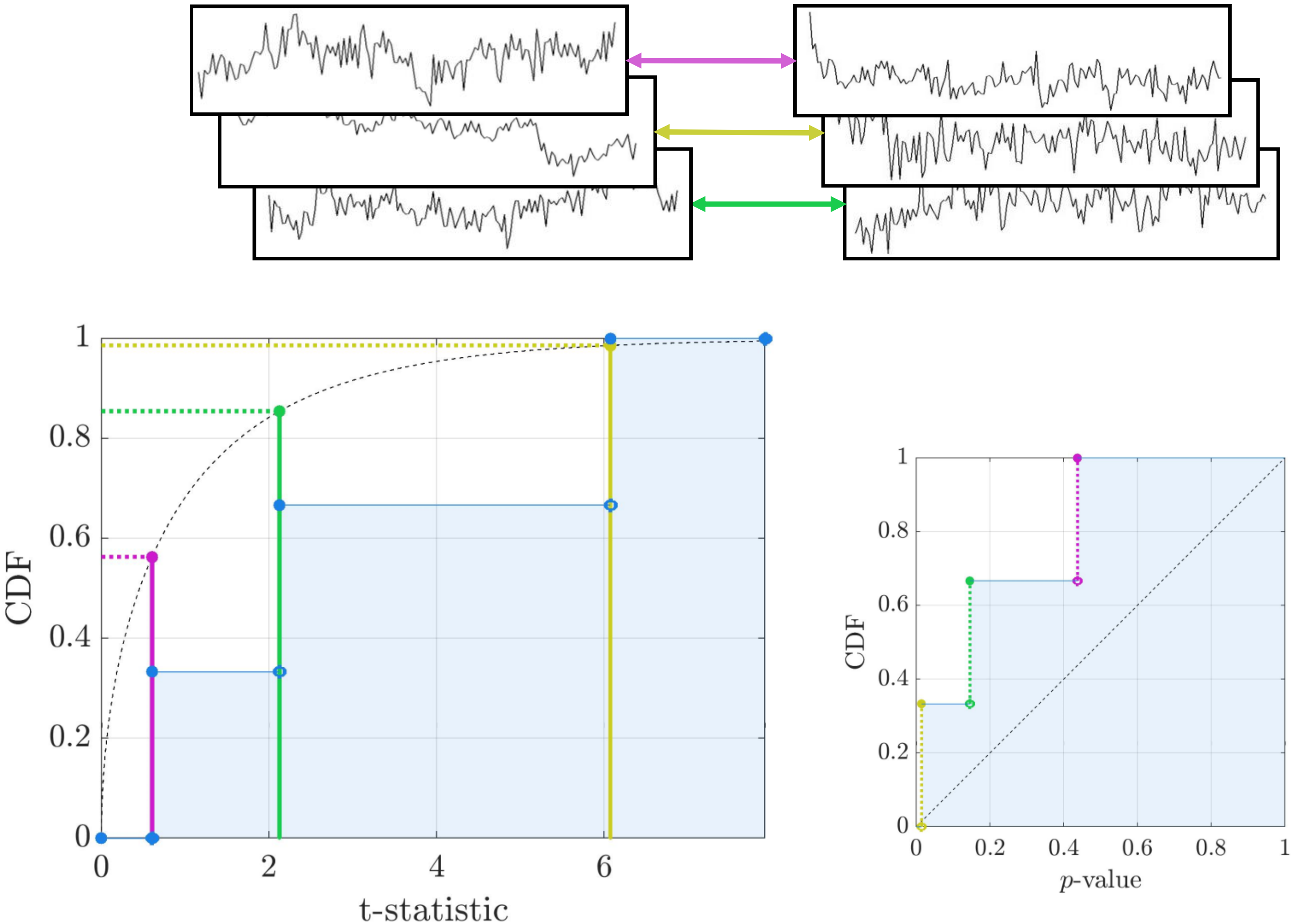

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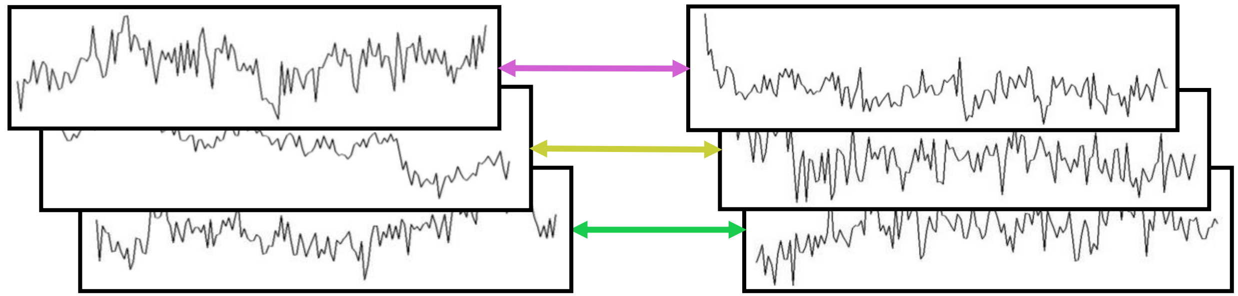

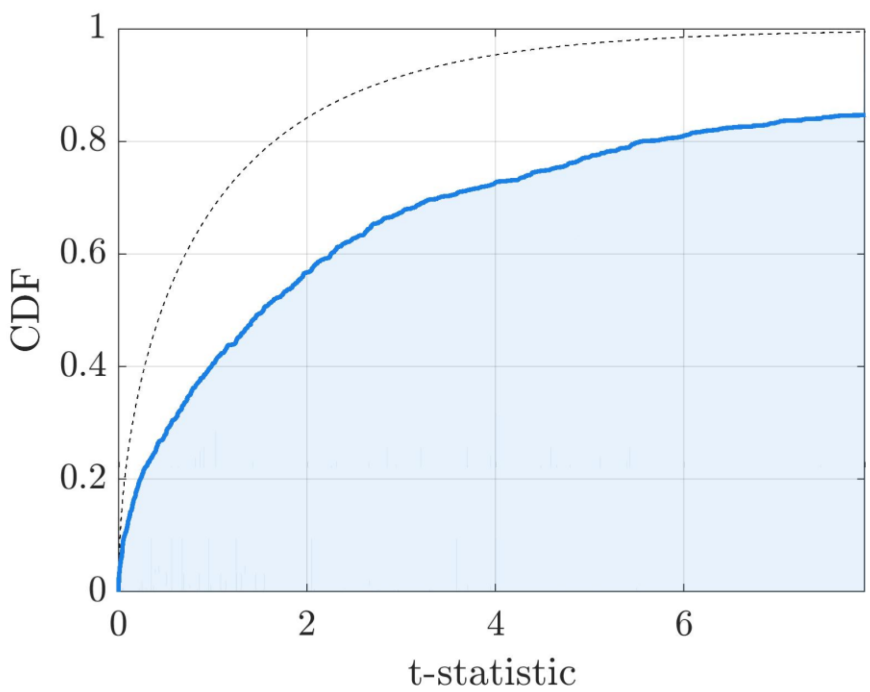

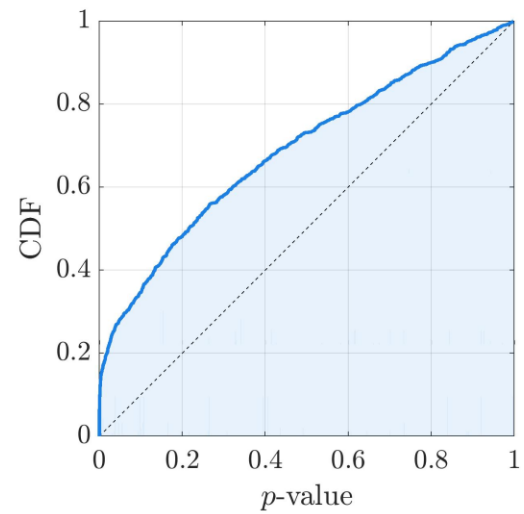

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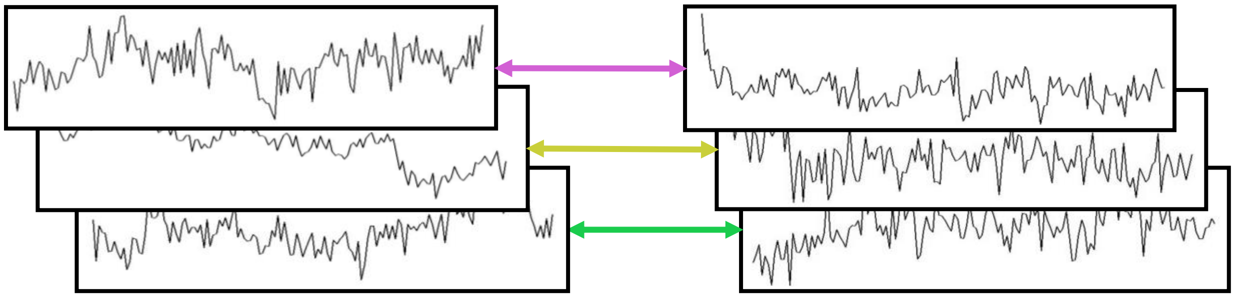
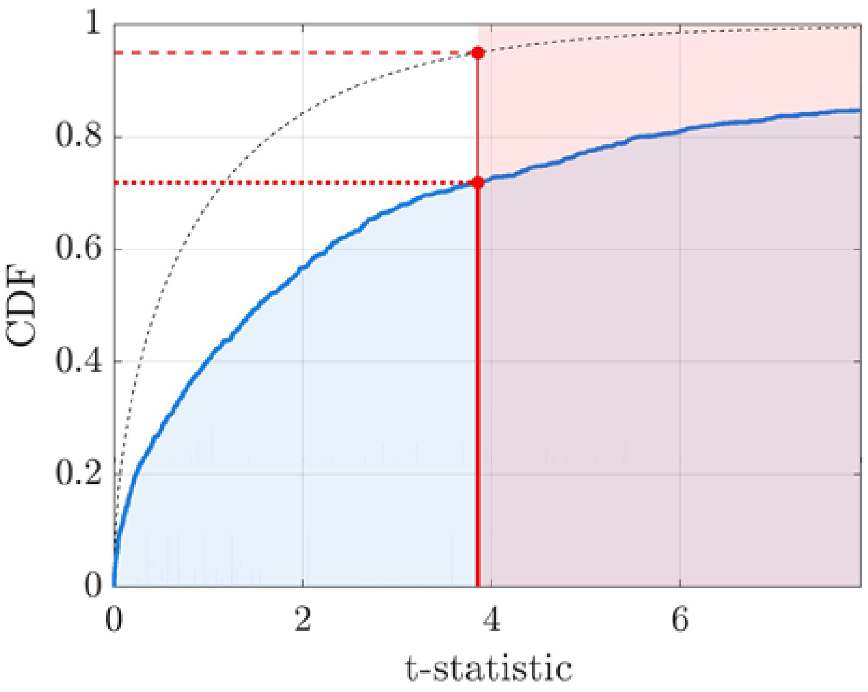
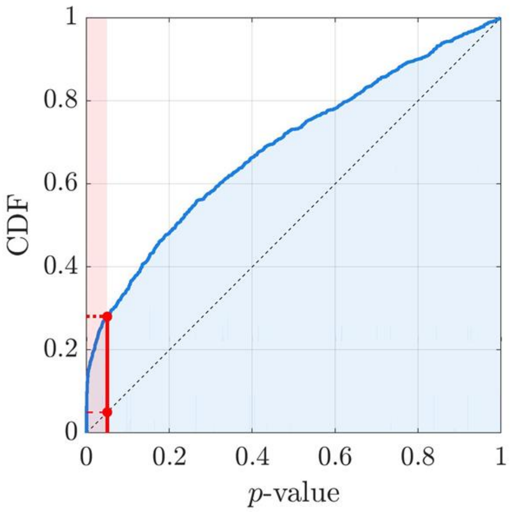




34% FPR (for a nominal 5%)



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Why do these false positives occur?

- This doesn't just happen with fMRI data but lots of stationary time series.

- Stationary time series are defined by two things:
  - Mean
  - Autocovariance

- Pearson correlation (and other linear-dependence measures) are scale-invariant, so the mean has no effect.

- Let's investigate the autocovariance/autocorrelation…

Linear dependence under autocorrelation

- Higher autocorrelation increases the FPR of sample correlation
  - This has been known since Yule/Pearson's work in the early 20th century

- How do we fix it?
  - **Bartlett's formula** [Bartlett (1935)]
    - Correct the sample size based on autocorrelation of univariate signals
    - Does the process pass the t-test with effective sample size?
  - **Granger causality** [Granger (1969)]
    - Define process through autocovariance
    - Does adding in another process improve predictability?

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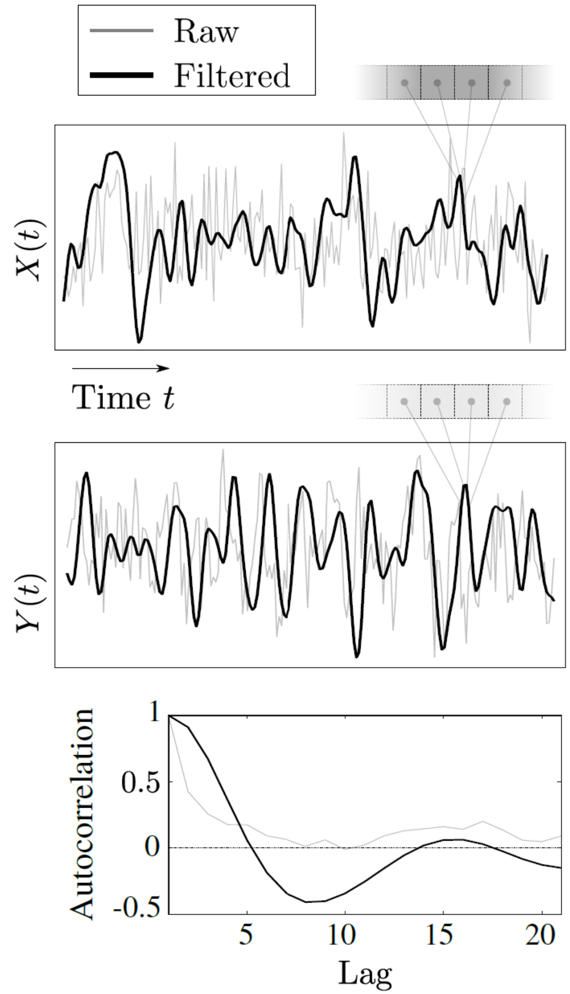
Bartlett's formula

- Compute the *effective sample size*:

$$\eta(x,y) = T\left[1 + 2\sum_{u=1}^{T-1} r_{xx}(u) r_{yy}(u)\right]^{-1}$$

- Rarely equal to sample size for time series.
  - **ESS < sample size** when:
    - both x and y are positively autocorrelated
  - **ESS > sample size** when:
    - One is positively autocorrelated and one is negatively autocorrelated



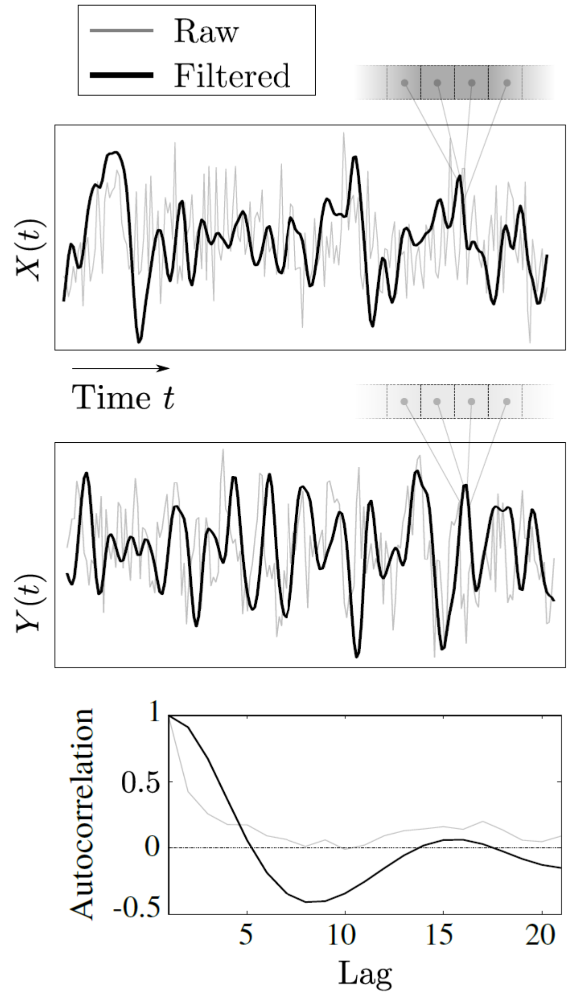
Bartlett's formula

› Test statistic against *t*-distribution with ESS:

$$r_{xy}\sqrt{\frac{\eta(x,y)-2}{1-r_{xy}^2}} \sim t(\eta(x,y)-2)$$

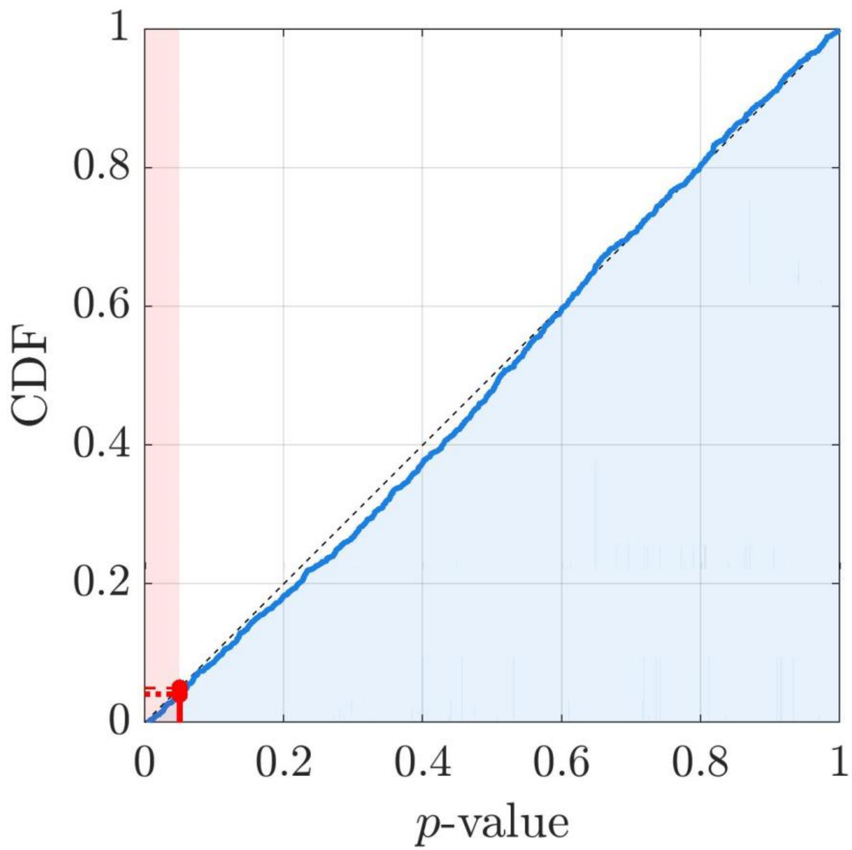
› Standard t-test Type I & II errors.

- False **positives** occur when**:**
  - both x and y are positively autocorrelated
- False **negatives** occur when**:**
  - One is positively autocorrelated and one is negatively autocorrelated



Exact Pearson correlation tests

5% FPR (for a nominal 5%)



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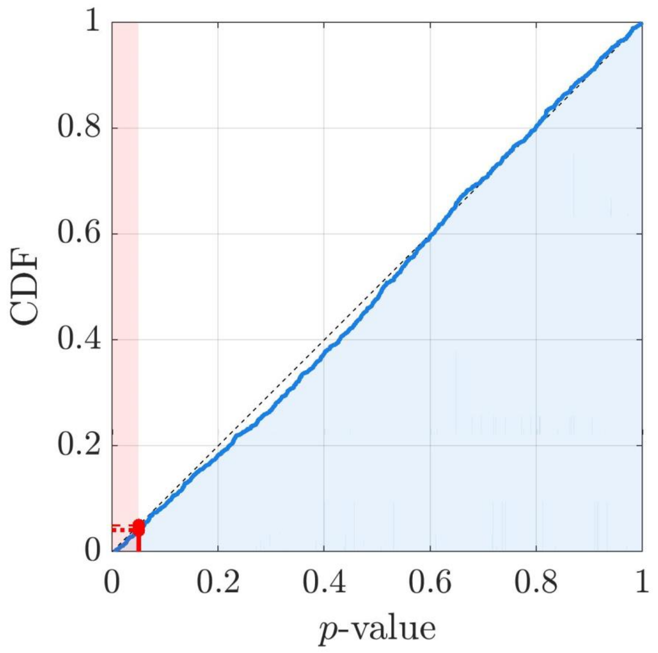

- Bartlett's formula corrects the false-positive bias in Pearson correlation
- This is but one linear-dependence measure, others can be directed and multivariate:
  - Granger causality
  - Mutual information
  - Canonical correlation analysis

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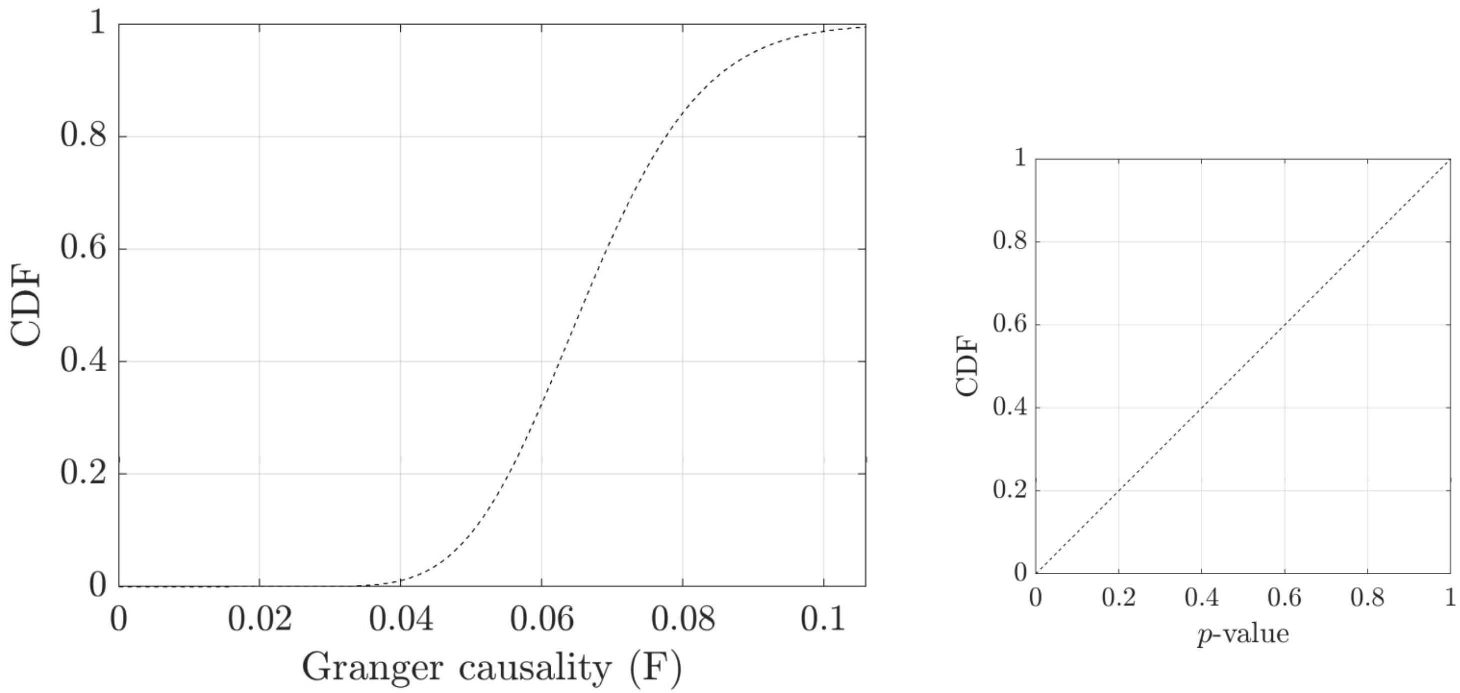
Neuroimaging FPR: Granger causality

- Univariate test for directed predictability
  - Assumes Markov chain (here we use order 50)
  - Null distribution is **asymptotically** chi-square



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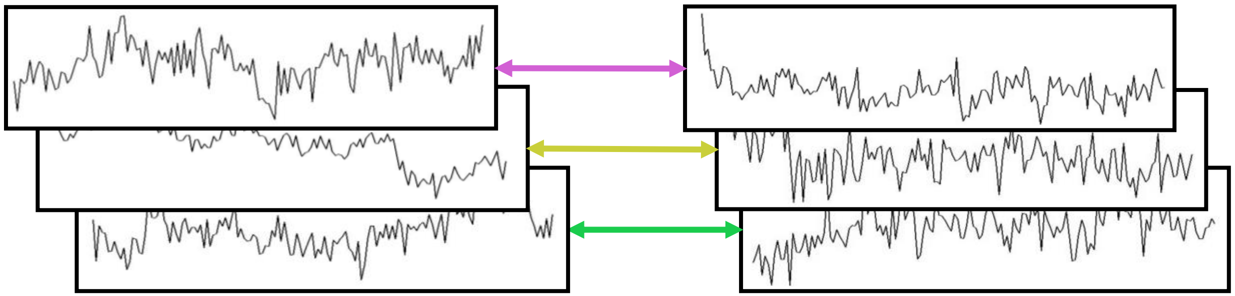

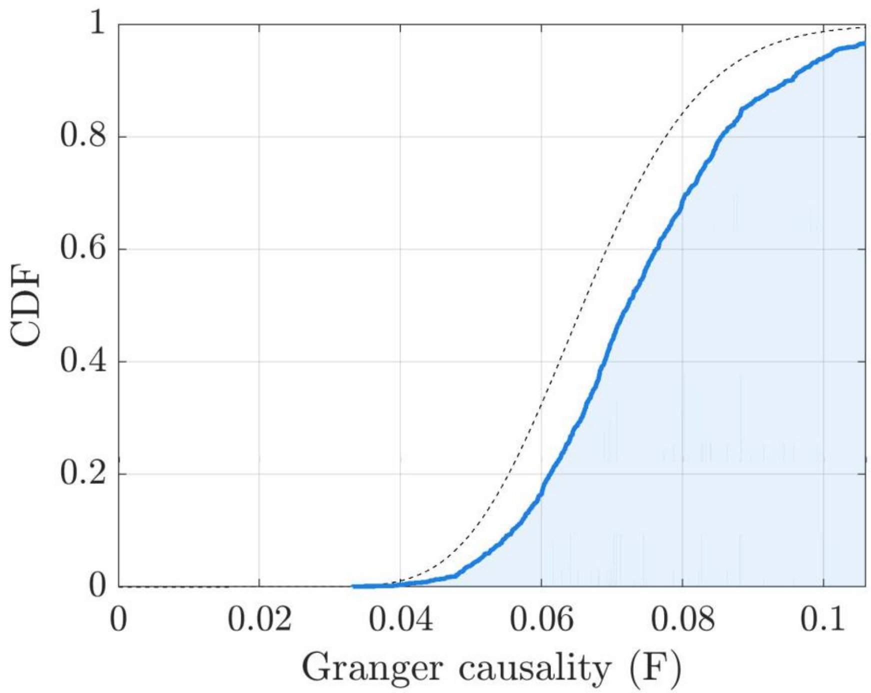

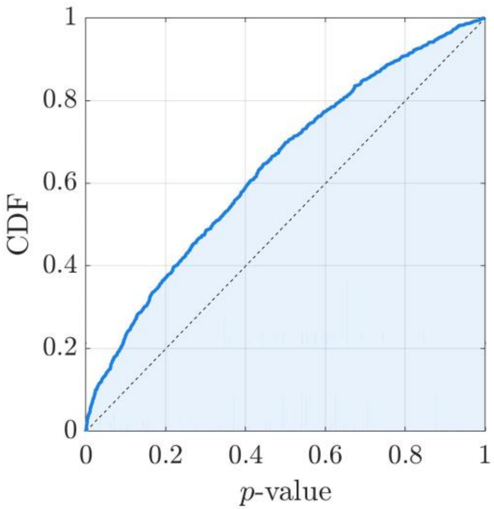

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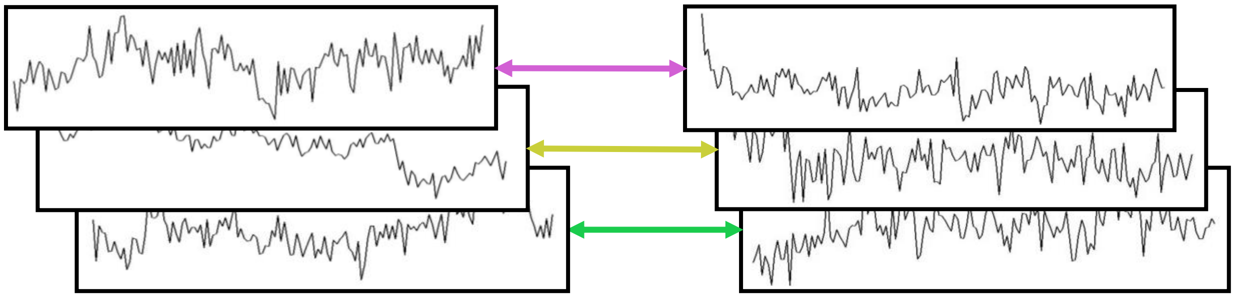
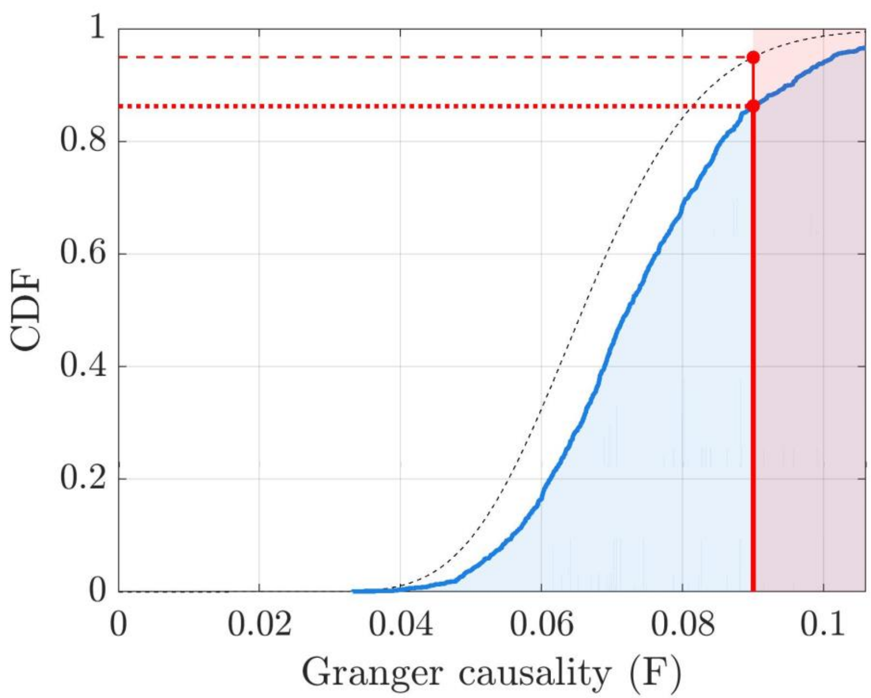
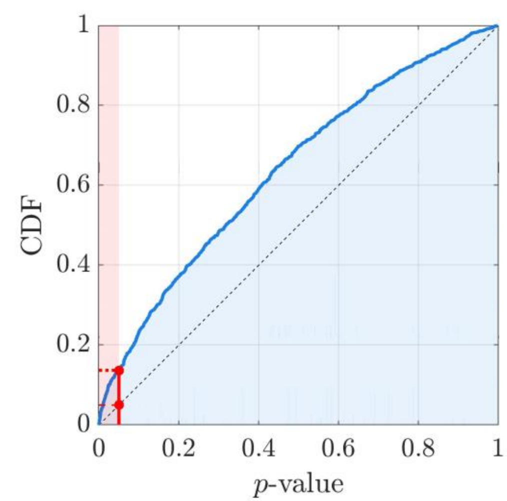




14% FPR (for a nominal 5%)



Unifying Bartlett’s formula and multivariate measures

- Bivariate correlation
  - Undirected and univariate time series
  - Exact t-tests given due to Bartlett (1935)

- Mutual information
  - Undirected and multivariate time series
  - Asymptotic chi-square tests due to Wilks (1938)
  - Exact F-tests for iid variables

- Granger causality
  - Directed and multivariate time series
  - Asymptotic chi-square tests due to Wilks (1938)

Sampling distribution of multivariate measures

1. **Partial Correlation**
   1. Derive one-tailed test from Bartlett's formula
   2. Exact two-tailed test by squaring
2. **Conditional mutual information (between two time series)**
   1. Express as a squared partial correlation
   2. Exact test from two-tailed PC test
3. **Conditional mutual information (between multiple time series)**
   1. Use chain rule to decompose as a sum of CMI terms
   2. Exact test by summing each two-tailed PC test
4. **Granger causality** (between two/multiple time series)
   1. Equivalent to conditional mutual information (Barnett et al., 2009)
   2. Exact test from CMI (above)

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Tests for partial correlation

› Take residuals of both processes:

$$e_{x|\boldsymbol{w}} = x - \hat{x}(\boldsymbol{w})$$
$$e_{y|\boldsymbol{w}} = y - \hat{y}(\boldsymbol{w})$$

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$$e_{x|\boldsymbol{w}} = x - \hat{x}(\boldsymbol{w})$$
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› Compute the correlation between those residuals:

$$r_{xy\cdot\boldsymbol{w}} = \frac{\sum_t \left[e_{x|\boldsymbol{w}}(t)\, e_{y|\boldsymbol{w}}(t)\right]^2}{\sum_t e^2_{x|\boldsymbol{w}}(t) \sum_t e^2_{y|\boldsymbol{w}}(t)} = \frac{S_{xy|\boldsymbol{w}}}{S_{x|\boldsymbol{w}}\, S_{y|\boldsymbol{w}}}$$

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Sum of squares

Tests for partial correlation

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› For **i.i.d. variables**, test against Student's t-distribution: $t(\nu)$

$$\nu = \boxed{T} - \boxed{c} - 2$$

Sample size

Dimension of conditional (c-variate)

Exact one-tailed tests for partial correlation

› Take residuals of both processes:

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› For **autocorrelated processes**, test against Student's t-distribution with ESS of residuals:

$$r_{xy\cdot\boldsymbol{w}} \sqrt{\frac{\boxed{\eta\left(e_{x|\boldsymbol{w}}, e_{y|\boldsymbol{w}}\right)} - \boxed{c} - 2}{1 - r^2_{xy\cdot\boldsymbol{w}}}} \sim t\left(\boxed{\eta\left(e_{x|\boldsymbol{w}}, e_{y|\boldsymbol{w}}\right)} - \boxed{c} - 2\right)$$

Effective sample size

Dimension of conditional

Exact two-tailed test for partial correlation

› Effective degrees of freedom:

$$N_{xy|\boldsymbol{w}} = \eta(e_{x|\boldsymbol{w}}, e_{y|\boldsymbol{w}}) - c - 2$$

› Now, squaring the statistic gives a two-tailed test:

$$r_{xy\cdot\boldsymbol{w}}\sqrt{\frac{\eta\big(e_{x|\boldsymbol{w}}, e_{y|\boldsymbol{w}}\big) - c - 2}{1 - r^2_{xy\cdot\boldsymbol{w}}}} \sim t(\eta\big(e_{x|\boldsymbol{w}}, e_{y|\boldsymbol{w}}\big) - c - 2)$$

$$\downarrow$$

$$N_{xy|\boldsymbol{w}}\,\frac{r^2_{xy\cdot\boldsymbol{w}}}{1 - r^2_{xy\cdot\boldsymbol{w}}} \sim F(1, N_{xy|\boldsymbol{w}})$$

The $\mathcal{L}$-distribution

› Let's take our (squared) statistic…

$$N_{xy|\boldsymbol{w}} \frac{r^2_{xy\cdot\boldsymbol{w}}}{1 - r^2_{xy\cdot\boldsymbol{w}}} \sim F(1, N_{xy|\boldsymbol{w}})$$

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- …and transform it:

$$\log\left(\frac{N_{xy|\boldsymbol{w}}}{N_{xy|\boldsymbol{w}}} \frac{r^2_{xy\cdot\boldsymbol{w}}}{1 - r^2_{xy\cdot\boldsymbol{w}}} + 1\right) = -\log\left(1 - r^2_{xy\cdot\boldsymbol{w}}\right)$$

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Equiv. to CMI

The $\mathcal{L}$-distribution

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$$N_{xy|\boldsymbol{w}} \frac{r^2_{xy\cdot\boldsymbol{w}}}{1 - r^2_{xy\cdot\boldsymbol{w}}} \sim F(1, N_{xy|\boldsymbol{w}})$$

- …and transform it:

$$\log\left(\frac{N_{xy|\boldsymbol{w}}}{N_{xy|\boldsymbol{w}}} \frac{r^2_{xy\cdot\boldsymbol{w}}}{1 - r^2_{xy\cdot\boldsymbol{w}}} + 1\right) = -\log\left(1 - r^2_{xy\cdot\boldsymbol{w}}\right)$$

- We label this statistic's null distribution *the $\mathcal{L}$-distribution*:

$$-\log\left(1 - r^2_{xy\cdot\boldsymbol{w}}\right) \sim \mathcal{L}(N_{xy|\boldsymbol{w}})$$

The $\mathcal{L}$-distribution

- This could be defined analytically from an F-distribution:

$$-\log\left(1 - r^2_{xy\cdot\boldsymbol{w}}\right) \sim \mathcal{L}(N_{xy|\boldsymbol{w}})$$

- But for our purposes it's best to sample $F \sim F(1, N)$

$$\log\left(\frac{F}{N} + 1\right) \sim \mathcal{L}(N)$$

- Because higher-dimensional measures (MI and Granger) are intractable.

$$\sum_{i=1}^{n} L_i = \sum_{i=1}^{n} \log\left(\frac{F_i}{N_i} + 1\right) \sim \mathcal{L}(\boldsymbol{N})$$

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$$\sum_{i=1}^{n} L_i = \sum_{i=1}^{n} \log\left(\boxed{\frac{F_i}{N_i}} + 1\right) \sim \mathcal{L}(\boldsymbol{N})$$

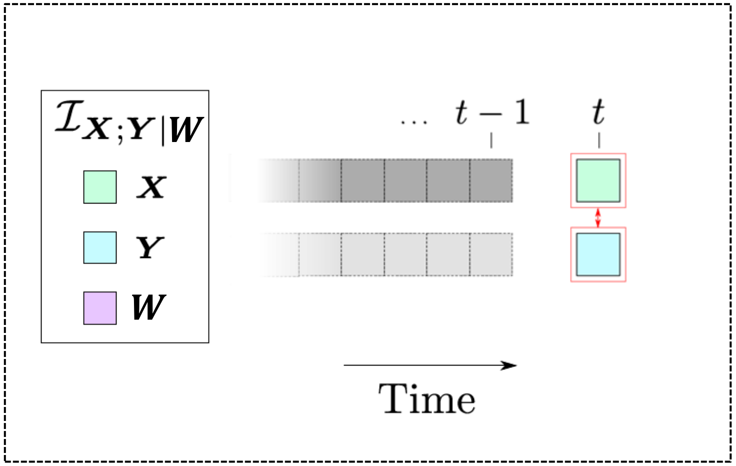
Potentially different for each L-term

Sampling distribution of multivariate measures

1. **Partial Correlation**
   1. Derive one-tailed test from Bartlett's formula
   2. Exact two-tailed test by squaring
2. **Conditional mutual information (between two time series)**
   1. Express as a squared partial correlation
   2. Exact test from two-tailed PC test
3. **Conditional mutual information (between multiple time series)**
   1. Use chain rule to decompose as a sum of CMI terms
   2. Exact test by summing each two-tailed PC test
4. **Granger causality** (between two/multiple time series)
   1. Equivalent to conditional mutual information (Barnett et al., 2009)
   2. Exact test from CMI (above)

(Conditional) mutual information

Between univariate Gaussians



- Equivalent to squared partial correlation [Davey (2013)]:

$$2\,\hat{\mathcal{I}}_{x;y|\boldsymbol{w}} = -\log(1 - r^2_{xy\cdot\boldsymbol{w}})$$

- Null distribution is asymptotically chi-square [Wilks (1938)]:

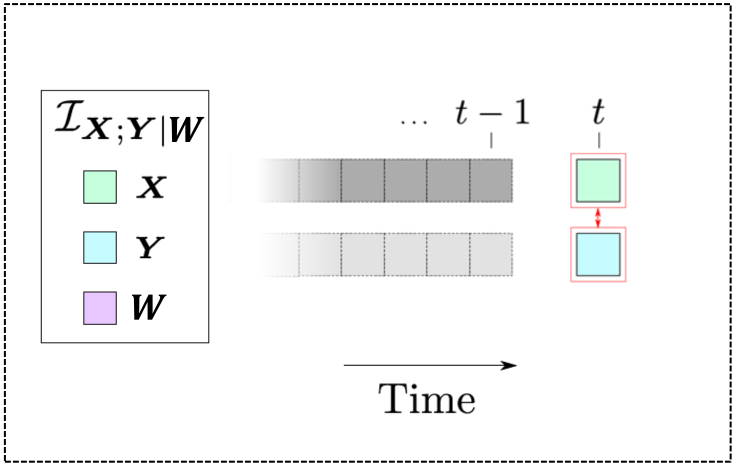
$$2T\,\hat{\mathcal{I}}_{\boldsymbol{x};\boldsymbol{y}|\boldsymbol{w}} \sim \chi^2(kl)$$

$$\begin{aligned} \dim(\boldsymbol{X}) &= k \\ \dim(\boldsymbol{Y}) &= l \\ \dim(\boldsymbol{W}) &= c \end{aligned}$$

- Exact null distribution for **i.i.d. variables** is F-distributed

$$\frac{T-(l+c+1)}{l}\left[\exp\left(2\hat{\mathcal{I}}_{x;\boldsymbol{y}|\boldsymbol{w}}\right)-1\right] \sim F(l, T-(l+c+1))$$

Between univariate Gaussians



› Equivalent to squared partial correlation [Davey (2013)]:

$$2\,\hat{\mathcal{I}}_{x;y|\boldsymbol{w}} = -\log(1 - r^2_{xy\cdot\boldsymbol{w}})$$

› Null distribution is asymptotically chi-square [Wilks (1938)]:

$$2T\,\hat{\mathcal{I}}_{\boldsymbol{x};\boldsymbol{y}|\boldsymbol{w}} \sim \boxed{\chi^2(kl)}$$

No conditional

$$\begin{aligned} \dim(\boldsymbol{X}) &= k \\ \dim(\boldsymbol{Y}) &= l \\ \dim(\boldsymbol{W}) &= c \end{aligned}$$

› Exact null distribution for **i.i.d. variables** is F-distributed

$$\frac{T-(l+\boxed{c}+1)}{l}\left[\exp\left(2\hat{\mathcal{I}}_{x;\boldsymbol{y}|\boldsymbol{w}}\right)-1\right] \sim F(l, T-(l+\boxed{c}+1))$$

Conditional

Exact (conditional) mutual information tests

Between univariate Gaussians

› Equivalent to squared partial correlation

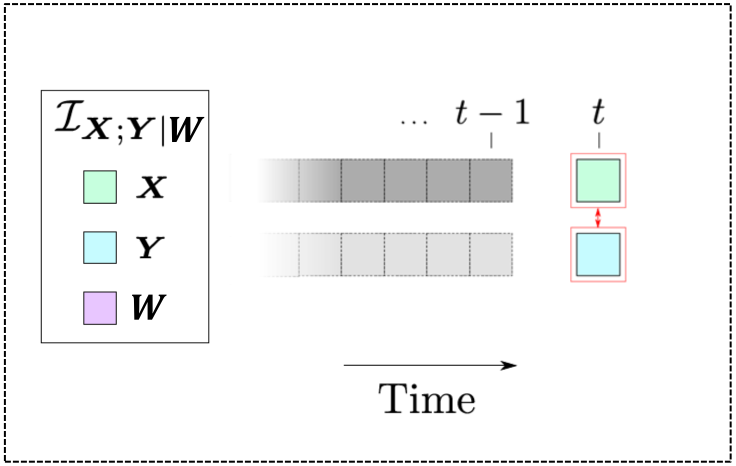
[Davey (2013)]

$$2\,\hat{\mathcal{I}}_{x;y|\boldsymbol{w}} = -\log(1 - r^2_{xy\cdot\boldsymbol{w}})$$

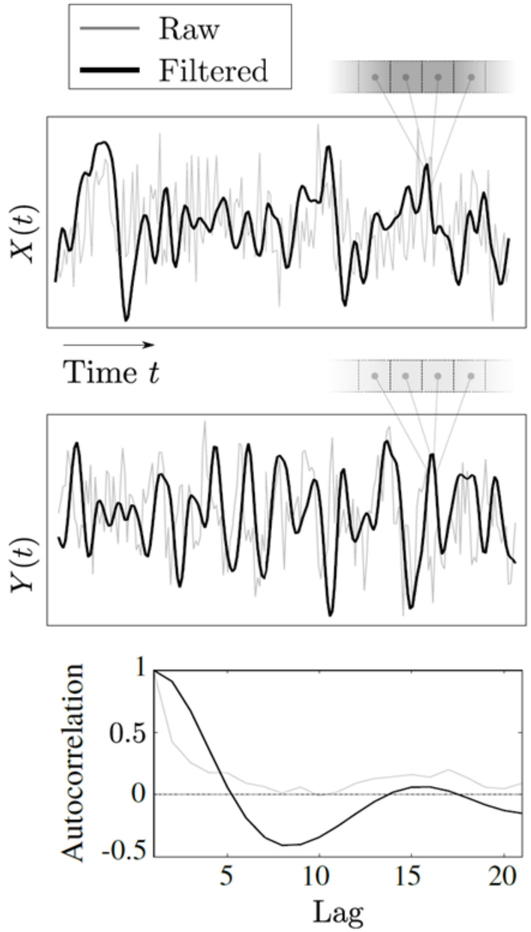
› So, we can use the $\mathcal{L}$-distribution:

$$-\log\left(1 - r^2_{xy\cdot\boldsymbol{w}}\right) \sim \mathcal{L}(N_{xy|\boldsymbol{w}})$$

$$2\,\hat{\mathcal{I}}_{x;y|\boldsymbol{w}} \sim \mathcal{L}(N_{xy|\boldsymbol{w}})$$

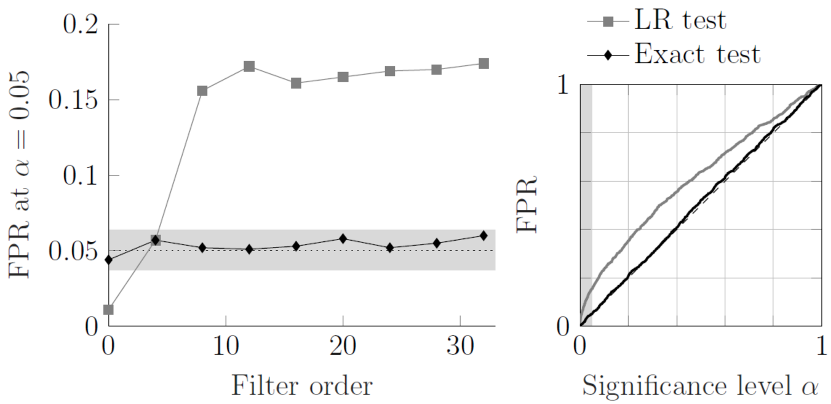


Numerical simulations

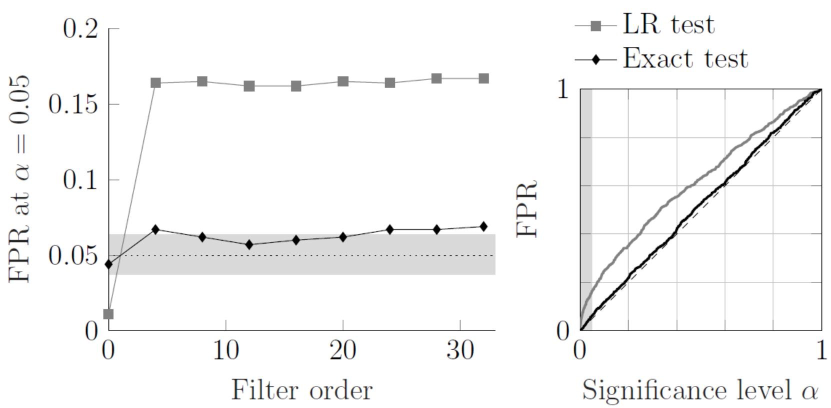


- Generate independent first-order autoregressive processes $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{W}$
  - 512 samples
  - $\dim(\boldsymbol{X}) = k,\ \dim(\boldsymbol{Y}) = l,\ \dim(\boldsymbol{W}) = c$
- Digitally filter with IIR/FIR filter
  - Increase filter order to increase AC
  - Common preprocessing step in neuro
- Perform 1000 trials of each configuration

Numerical simulations: MI for two time series



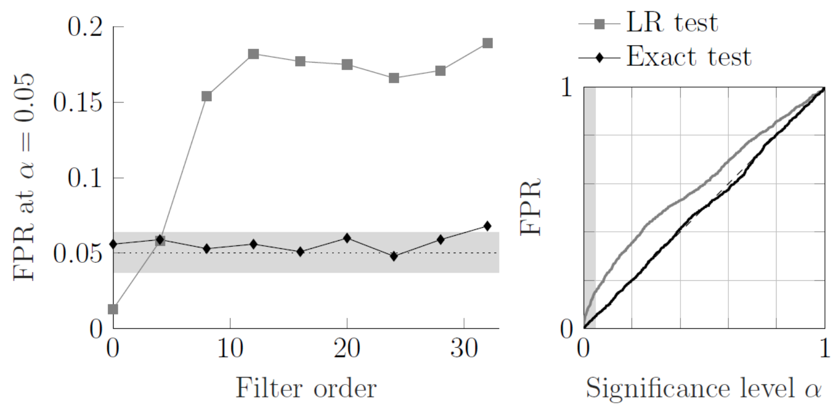
(a) FIR filter



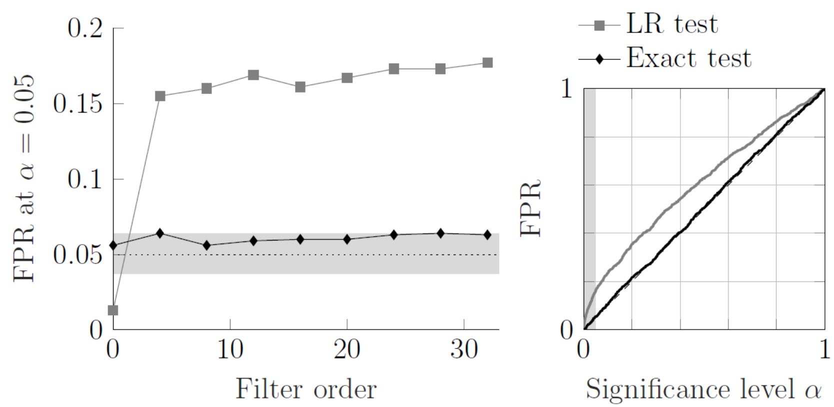
(b) IIR filter.

- Simulation:
  - X, Y are independent univariate processes ($k = l = 1$)
  - no conditional process W ($c = 0$)
  - Variable filter order
- Perform two tests:
  - **LR tests** are asymptotically valid
  - **Exact tests** are ours
- Both FIR and IIR increase the FPR dramatically to > 15% (3-times nominal value)
- Exact tests stay within (binominal) confidence intervals

Numerical simulations: CMI for two time series



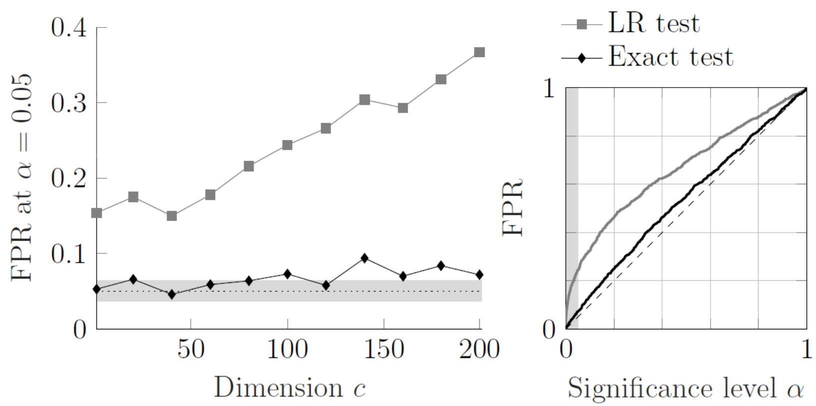
(a) FIR filter with univariate conditional.



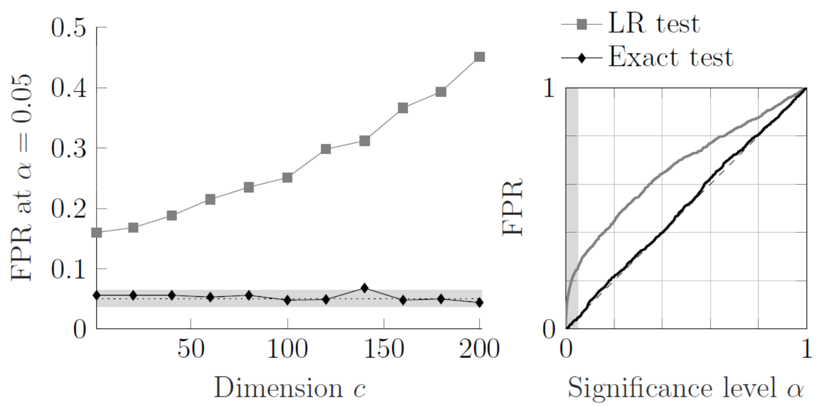
(b) IIR filter with univariate conditional.

- Simulation:
  - X, Y are independent univariate processes ($k = l = 1$)
  - Conditional W is univariate ($c = 1$)
  - Variable filter order
- Slight increase in LR test due to conditional not being taken into account

Numerical simulations: CMI for two time series



(a) FIR filter with c-variate conditional.



(b) IIR filter with c-variate conditional.

- Simulation:
  - X, Y are independent univariate processes ($k = l = 1$)
  - $\boldsymbol{W}$ is a higher-order conditional ($c = [1,200]$)
  - Fixed 8th order filter
- Approx. linear increase in FPR of LR test due to conditional
  - We stopped it at > 40% FPR (c=200)

Sampling distribution of multivariate measures

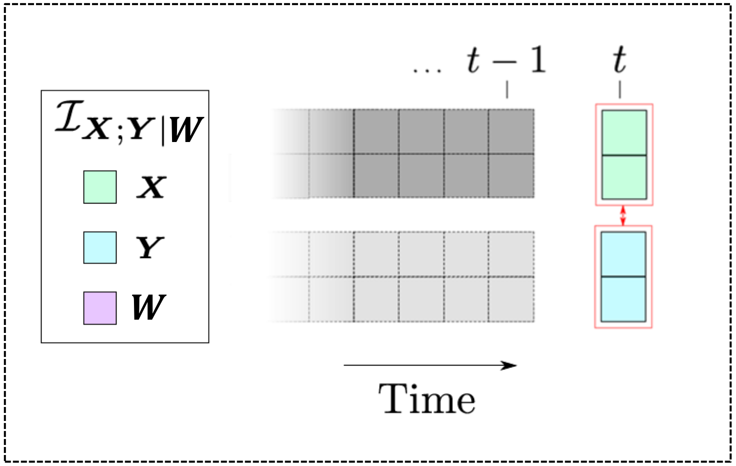
1. **Partial Correlation**
   1. Derive one-tailed test from Bartlett's formula
   2. Exact two-tailed test by squaring
2. **Conditional mutual information (between two time series)**
   1. Express as a squared partial correlation
   2. Exact test from two-tailed PC test
3. **Conditional mutual information (between multiple time series)**
   1. Use chain rule to decompose as a sum of CMI terms
   2. Exact test by summing each two-tailed PC test
4. **Granger causality** (between two/multiple time series)
   1. Equivalent to conditional mutual information (Barnett et al., 2009)
   2. Exact test from CMI (above)

(Conditional) mutual information

Between multivariate Gaussians

› Expressed as a log-likelihood of sum-of-squared residuals

$$\hat{\mathcal{I}}_{\boldsymbol{x};\boldsymbol{y}|\boldsymbol{w}} = -\frac{1}{2}\log\left(\frac{|\boldsymbol{S}_{\boldsymbol{x}\boldsymbol{y}|\boldsymbol{w}}|}{|\boldsymbol{S}_{\boldsymbol{x}|\boldsymbol{w}}||\boldsymbol{S}_{\boldsymbol{y}|\boldsymbol{w}}|}\right)$$



› Tested against the chi-square distribution:

$$2T\,\hat{\mathcal{I}}_{\boldsymbol{x};\boldsymbol{y}|\boldsymbol{w}} \sim \chi^2(kl)$$

Exact mutual information test

Multiple time series

$$\boldsymbol{v}^{\{gh\}}_{\boldsymbol{xy}|\boldsymbol{w}} = \begin{bmatrix} \boldsymbol{x}_{1:g-1} \\ \boldsymbol{y}_{1:h-1} \\ \boldsymbol{w} \end{bmatrix}$$

› Decompose mutual information by the chain rule:

$$\hat{\mathcal{I}}_{\boldsymbol{x};\boldsymbol{y}|\boldsymbol{w}} = \sum_{g=1}^{k}\sum_{h=1}^{l} \hat{\mathcal{I}}_{x_g;y_h|\boldsymbol{v}^{\{gh\}}_{\boldsymbol{xy}|\boldsymbol{w}}}$$

› Gives expression as squared partial correlations:

$$2\hat{\mathcal{I}}_{\boldsymbol{x};\boldsymbol{y}|\boldsymbol{w}} = -\sum_{g=1}^{k}\sum_{h=1}^{l} \log\left(1 - r^2_{x_g y_h \cdot \boldsymbol{v}^{\{gh\}}_{\boldsymbol{xy}|\boldsymbol{w}}}\right)$$

› Which is $\mathcal{L}$-distributed (under the null):

$$2\hat{\mathcal{I}}_{\boldsymbol{x};\boldsymbol{y}|\boldsymbol{w}} \sim \mathcal{L}(\boldsymbol{N}_{\boldsymbol{xy}|\boldsymbol{w}})$$

Exact mutual information test

Multiple time series

› Which is $\mathcal{L}$-distributed (under the null):

$$2\,\hat{\mathcal{I}}_{\boldsymbol{x};\boldsymbol{y}|\boldsymbol{w}} \sim \mathcal{L}(\boldsymbol{N}_{\boldsymbol{x}\boldsymbol{y}|\boldsymbol{w}})$$

$$\boldsymbol{v}^{\{gh\}}_{\boldsymbol{x}\boldsymbol{y}|\boldsymbol{w}} = \begin{bmatrix} \boldsymbol{x}_{1:g-1} \\ \boldsymbol{y}_{1:h-1} \\ \boldsymbol{w} \end{bmatrix}$$

› (Each term could have a different ESS)

$$N^{\{gh\}}_{\boldsymbol{x}\boldsymbol{y}|\boldsymbol{w}} = \boxed{\eta\left(e^{\{gh\}}_{\boldsymbol{x}|\boldsymbol{v}}, e^{\{gh\}}_{\boldsymbol{y}|\boldsymbol{v}}\right)} - \boxed{\dim\left(\boldsymbol{v}^{\{gh\}}_{\boldsymbol{x}\boldsymbol{y}|\boldsymbol{w}}(t)\right)} - 2$$

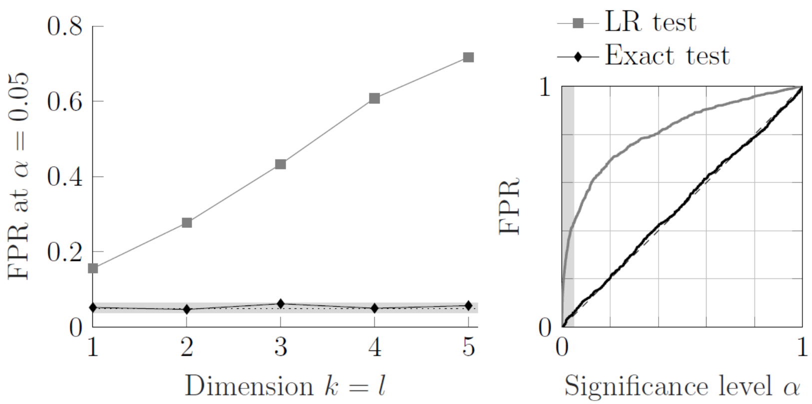
$$e^{\{gh\}}_{\boldsymbol{x}|\boldsymbol{v}} = x_g - \hat{x}_g\left(\boldsymbol{v}^{\{gh\}}_{\boldsymbol{x}\boldsymbol{y}|\boldsymbol{w}}\right)$$

$$e^{\{gh\}}_{\boldsymbol{y}|\boldsymbol{v}} = y_h - \hat{y}_h\left(\boldsymbol{v}^{\{gh\}}_{\boldsymbol{x}\boldsymbol{y}|\boldsymbol{w}}\right)$$

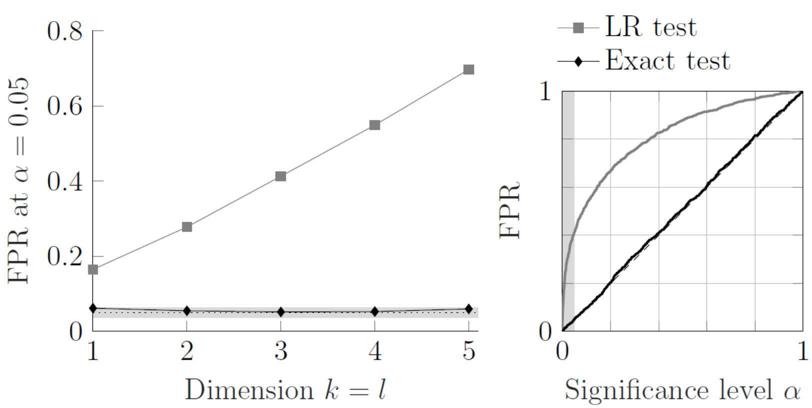
Effective sample size

Dimension of conditional

Numerical simulations: MI for multiple time series



(a) FIR filter.



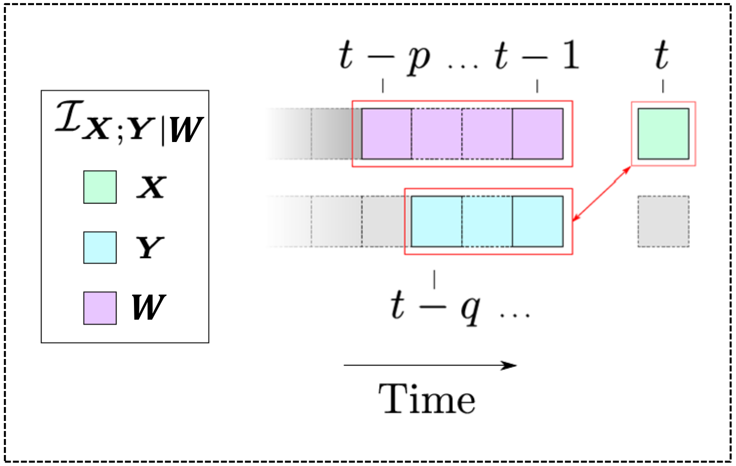
(b) IIR filter.

- Simulation:
  - X, Y are independent multivariate processes ($k = l \geq 1$)
  - No conditional process $\boldsymbol{W}$ ($c = 0$)
  - Fixed 8$^{\text{th}}$ order filter
- Approx. linear increase in FPR of LR test
  - Each term's Bartlett-correction and conditional not taken into account
  - Approaches 100% FPR

Sampling distribution of multivariate measures

1. **Partial Correlation**
    1. Derive one-tailed test from Bartlett's formula
    2. Exact two-tailed test by squaring
2. **Conditional mutual information (between two time series)**
    1. Express as a squared partial correlation
    2. Exact test from two-tailed PC test
3. **Conditional mutual information (between multiple time series)**
    1. Use chain rule to decompose as a sum of CMI terms
    2. Exact test by summing each two-tailed PC test
4. **Granger causality** (between two/multiple time series)
    1. Equivalent to conditional mutual information (Barnett et al., 2009)
    2. Exact test from CMI (above)

Granger causality



- Define relevant history of processes:
  - AIC, BIC, first minimum partial AC, etc.

$$\boldsymbol{X}^{(p)}(t) = \begin{bmatrix} \boldsymbol{X}(t-1) \\ \vdots \\ \boldsymbol{X}(t-p) \end{bmatrix}, \quad \boldsymbol{Y}^{(q)}(t) = \begin{bmatrix} \boldsymbol{Y}(t-1) \\ \vdots \\ \boldsymbol{Y}(t-q) \end{bmatrix}$$

- Expressed as a mutual information…

$$\mathcal{F}_{\boldsymbol{Y}\to\boldsymbol{X}|\boldsymbol{W}}(p,q) = 2\,\mathcal{I}_{\boldsymbol{X};\boldsymbol{Y}^{(q)}|\boldsymbol{X}^{(p)}\boldsymbol{W}}$$

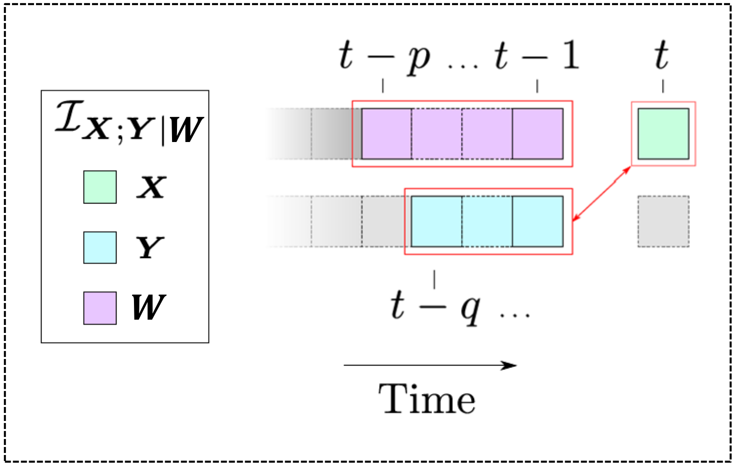
- …or equiv. as a log-ratio of the sum-of-squares

$$\hat{\mathcal{F}}_{\boldsymbol{y}\to\boldsymbol{x}|\boldsymbol{w}}(p,q) = \log\left(\frac{\left|\boldsymbol{S}_{\boldsymbol{x}|\boldsymbol{x}^{(p)}\boldsymbol{w}}\right|}{\left|\boldsymbol{S}_{\boldsymbol{x}|\boldsymbol{x}^{(p)}\boldsymbol{y}^{(q)}\boldsymbol{w}}\right|}\right)$$

Granger causality

› Asymptotic chi-square test:

$$T\,\hat{\mathcal{F}}_{\boldsymbol{y}\to\boldsymbol{x}|\boldsymbol{w}}(p,q) \sim \chi^2(klq)$$



› Exact test for **i.i.d. variables** with a univariate predictee/target:

$$\frac{T-(p+lq+c+1)}{lq}\left[\exp\left(\hat{\mathcal{F}}_{\boldsymbol{y}\to x|\boldsymbol{w}}(p,q)\right)-1\right]$$

$$\sim F(lq, T-(p+lq+c+1))$$

Exact Granger causality tests

Two time series

› Decomposes into q CMI terms via the chain rule:

$$\hat{\mathcal{F}}_{y \to x | \boldsymbol{w}}(p, q) = 2 \sum_{j=1}^{q} \hat{\mathcal{I}}_{x; y^j | \boldsymbol{v}^{\{j\}}_{y \to x | \boldsymbol{w}}}$$

$$\boldsymbol{v}^{\{j\}}_{y \to x} = \begin{bmatrix} \boldsymbol{x}^{(p)} \\ \boldsymbol{y}^{(j-1)} \\ \boldsymbol{w} \end{bmatrix}$$

› Gives the $\mathcal{L}$-distribution (under the null):

$$\hat{\mathcal{F}}_{y \to x | \boldsymbol{w}}(p, q) \sim \mathcal{L}(\boldsymbol{N}_{y \to x | \boldsymbol{w}})$$

Exact Granger causality tests

Two time series

$$\boldsymbol{v}^{\{j\}}_{y\to x} = \begin{bmatrix} \boldsymbol{x}^{(p)} \\ \boldsymbol{y}^{(j-1)} \\ \boldsymbol{w} \end{bmatrix}$$

› Decomposes into *q* CMI terms via the chain rule:

$$\hat{\mathcal{F}}_{y\to x|\boldsymbol{w}}(p,q) = 2\sum_{j=1}^{q} \hat{\mathcal{I}}_{x;y^j|\boldsymbol{v}^{\{j\}}_{y\to x|\boldsymbol{w}}}$$

› Gives the $\mathcal{L}$-distribution (under the null):

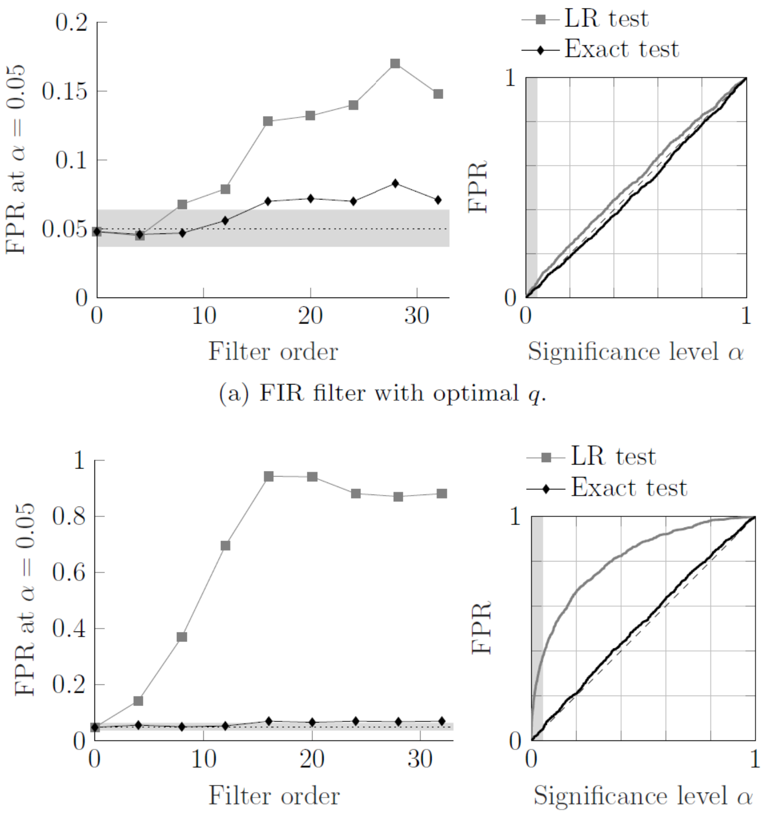
$$\hat{\mathcal{F}}_{y\to x|\boldsymbol{w}}(p,q) \sim \mathcal{L}(\boldsymbol{N}_{y\to x|\boldsymbol{w}})$$

› (Each term could have a different ESS)

$$N^{\{j\}}_{y\to x|\boldsymbol{w}} = \eta\left(e^{\{j\}}_{x|\boldsymbol{v}_{y\to x|\boldsymbol{w}}}, e^{\{j\}}_{y|\boldsymbol{v}_{y\to x|\boldsymbol{w}}}\right) - \dim\left(\boldsymbol{v}^{\{j\}}_{y\to x|\boldsymbol{w}}(t)\right) - 2$$

$$e^{\{j\}}_{x|\boldsymbol{v}_{y\to x|\boldsymbol{w}}} = x - \hat{x}\left(\boldsymbol{v}^{\{j\}}_{y\to x|\boldsymbol{w}}\right)$$
$$e^{\{j\}}_{y|\boldsymbol{v}_{y\to x|\boldsymbol{w}}} = y^j - \hat{y}^j\left(\boldsymbol{v}^{\{j\}}_{y\to x|\boldsymbol{w}}\right)$$

Numerical simulations: GC for two time series



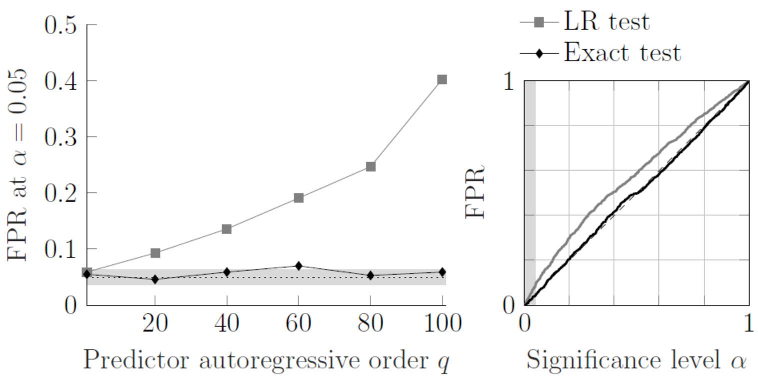
(a) FIR filter with optimal q.

(b) IIR filter with optimal q.

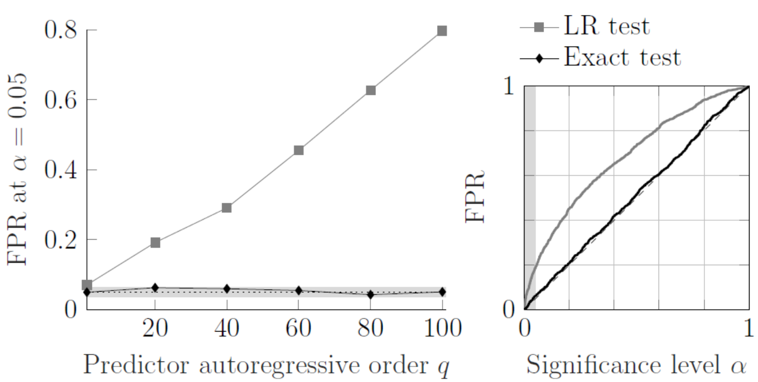
› Simulation:

- X, Y are independent univariate processes ($k = l = 1$)
- No conditional process $\boldsymbol{W}$ ($c = 0$)
- Variable filter order
- Optimal history lengths (p, q) chosen from partial ACF

Numerical simulations: GC for two time series



(a) FIR filter with varying q.



(b) IIR filter with varying q.

- Simulation:
  - X, Y are independent univariate processes ($k = l = 1$)
  - No conditional process $\boldsymbol{W}$ ($c = 0$)
  - Fixed 8th order filters
  - Variable predictor history length (q)
- LR test FPR increases approx. linearly with q due to no conditional
- Exact tests maintains correct FPR

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Exact Granger causality tests

Multiple time series

$$\boldsymbol{v}^{\{ghj\}}_{\boldsymbol{y}\to\boldsymbol{x}|\boldsymbol{w}} = \begin{bmatrix} \boldsymbol{x}^{(p)}_{1:g} \\ \boldsymbol{y}^{(q)}_{1:h-1} \\ \boldsymbol{y}^{(j-1)}_{h} \\ \boldsymbol{w} \end{bmatrix}$$

- Decomposes into *q x k x l* CMI terms

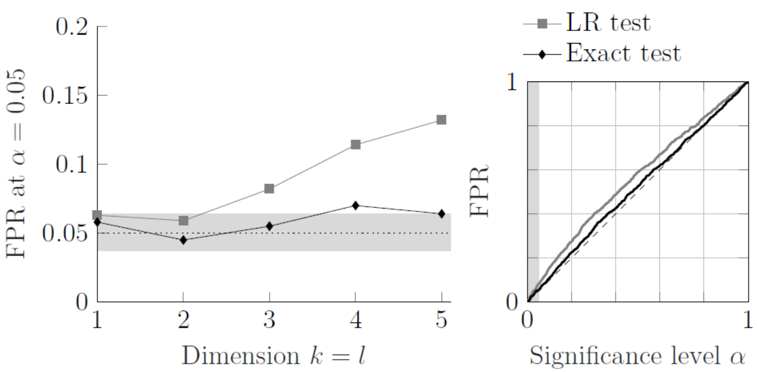
$$\hat{\mathcal{F}}_{\boldsymbol{x}\to\boldsymbol{y}|\boldsymbol{w}}(p,q) = 2\sum_{g=1}^{k}\sum_{h=1}^{l}\sum_{j=1}^{q}\hat{\mathcal{I}}_{x_g y_h^j | \boldsymbol{v}^{\{ghj\}}_{\boldsymbol{y}\to\boldsymbol{x}|\boldsymbol{w}}}$$

- Gives the $\mathcal{L}$-distribution (under the null):

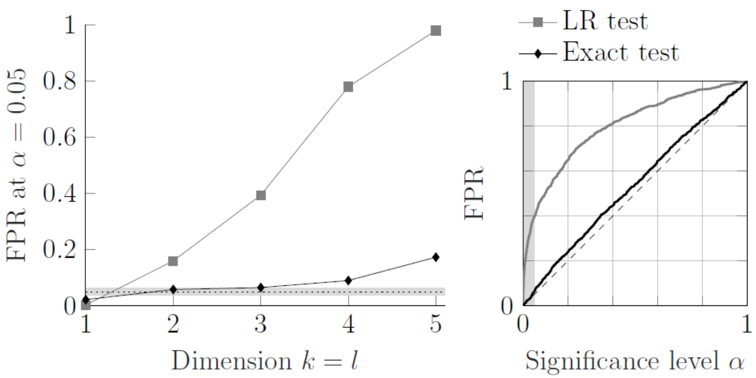
$$\hat{\mathcal{F}}_{\boldsymbol{y}\to\boldsymbol{x}|\boldsymbol{w}}(p,q) \sim \mathcal{L}(\boldsymbol{N}_{\boldsymbol{y}\to\boldsymbol{x}|\boldsymbol{w}})$$

- (Each term could have a different ESS)

Numerical simulations: GC for multiple time series



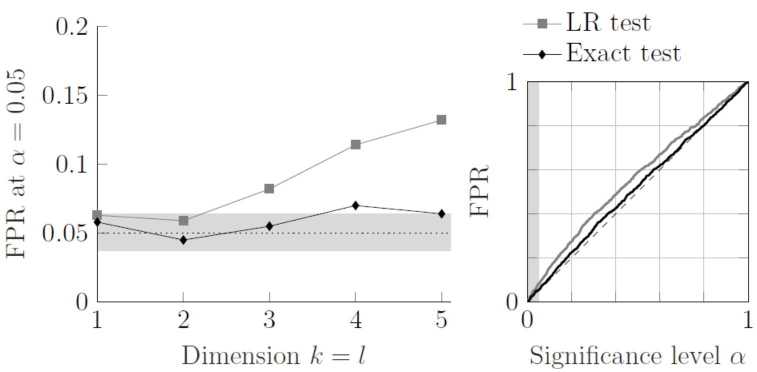
(a) FIR filter.



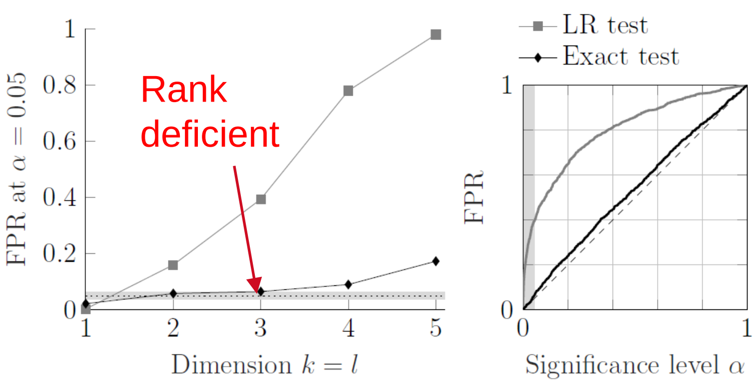
(b) IIR filter.

- Simulation:
  - $\boldsymbol{X}, \boldsymbol{Y}$ are independent multivariate processes ($k = l \geq 1$)
  - No conditional process $\boldsymbol{W}$ ($c = 0$)
  - Fixed 8th order filters
  - Optimal p but fixed $q = 1$
- LR test increases towards 100% FPR for IIR filter

Numerical simulations: GC for multiple time series



(a) FIR filter.



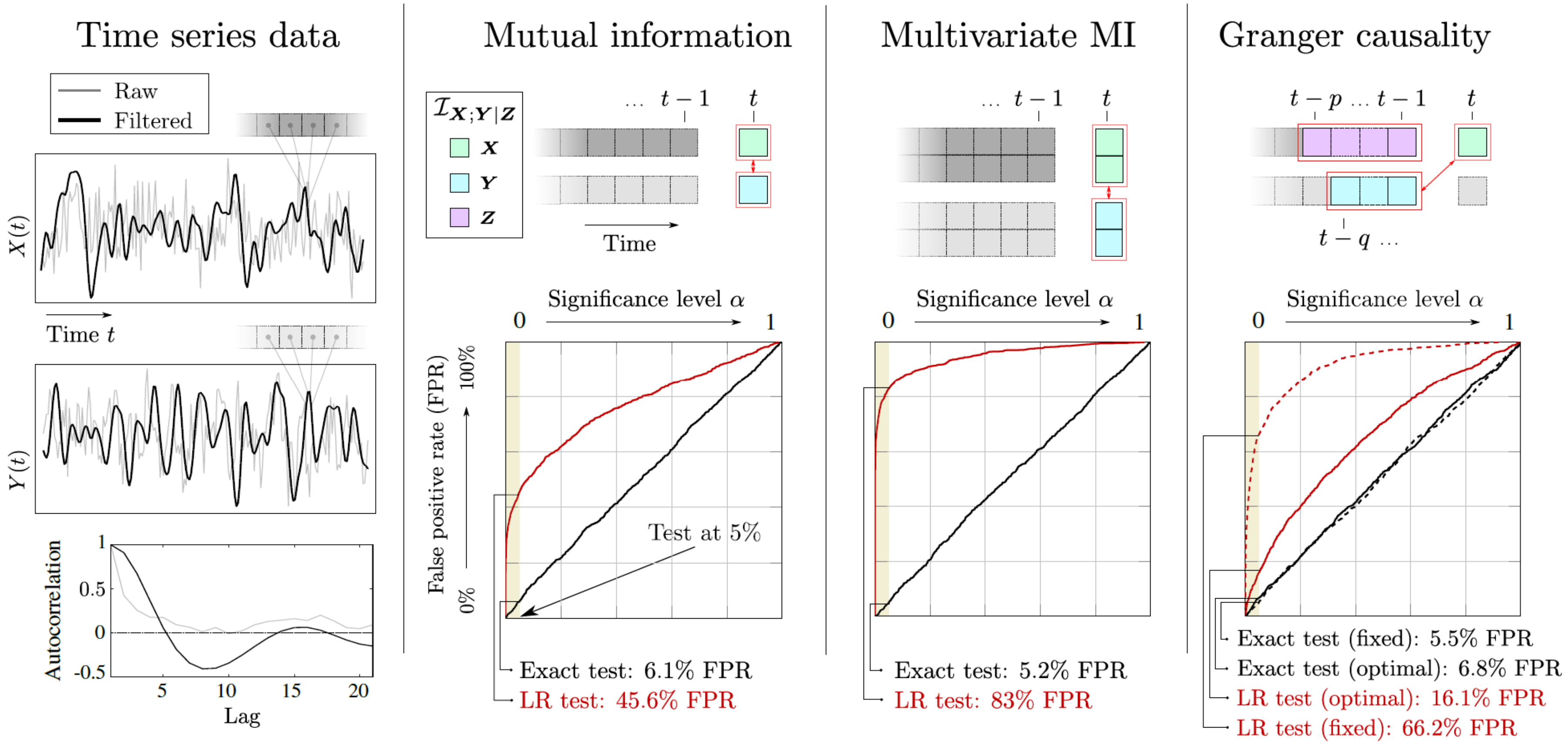
(b) IIR filter.

› Simulation:
  - $\boldsymbol{X}, \boldsymbol{Y}$ are independent multivariate processes ($k = l \geq 1$)
  - No conditional process $\boldsymbol{W}$ ($c = 0$)
  - Fixed 8th order filters
  - Optimal p but fixed $q = 1$

› LR test increases towards 100% FPR for IIR filter

› Regressions become rank deficient, so exact test begins to fail

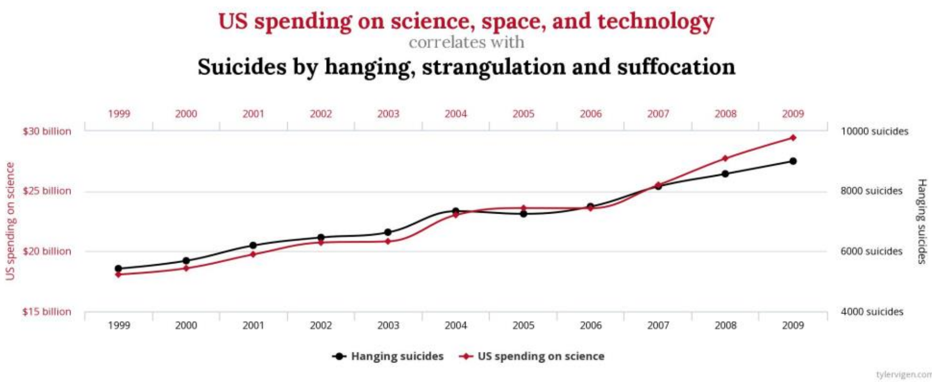
Case study: Human connectome project

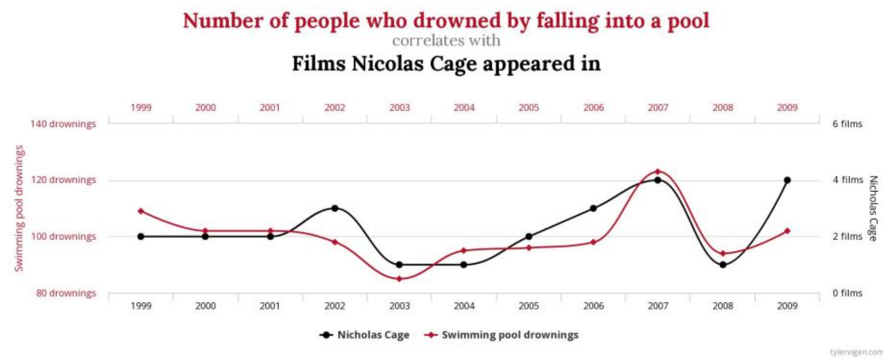


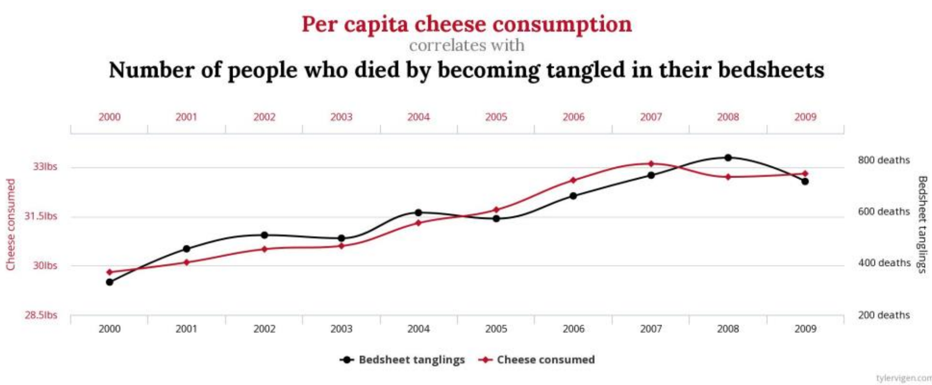
Take home messages

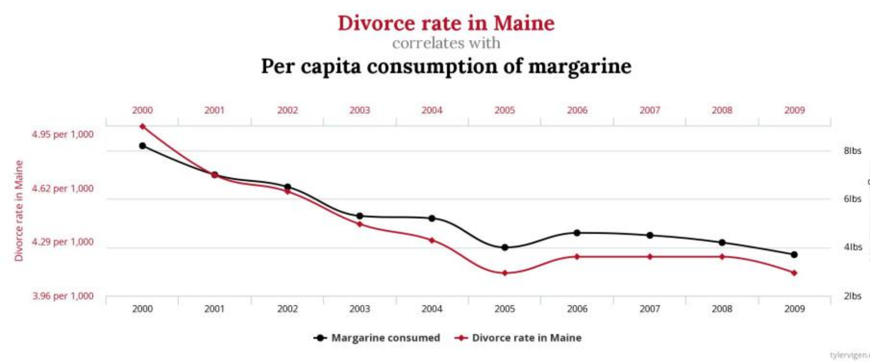
- Commonly-used linear-dependence measures exhibit bias for autocorrelated time series
- These measures can be represented as sums of squared partial correlations
- This representation allows us to derive their exact sampling distribution
- Before our work, these distributions were only valid asymptotically

Thank you!









Correlation image source: http://www.tylervigen.com/spurious-correlations