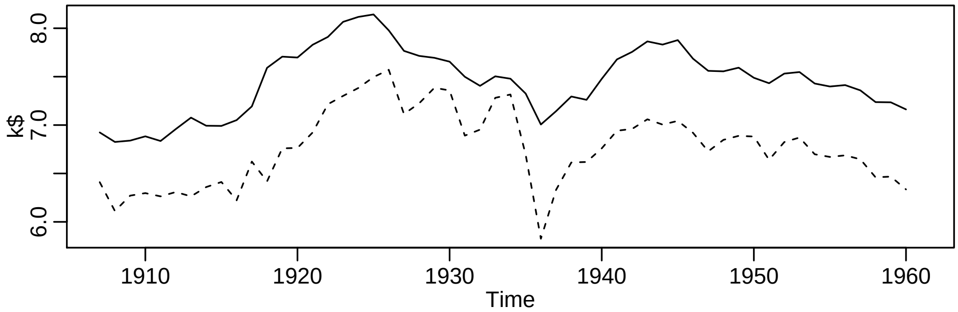
Solution to Series 6

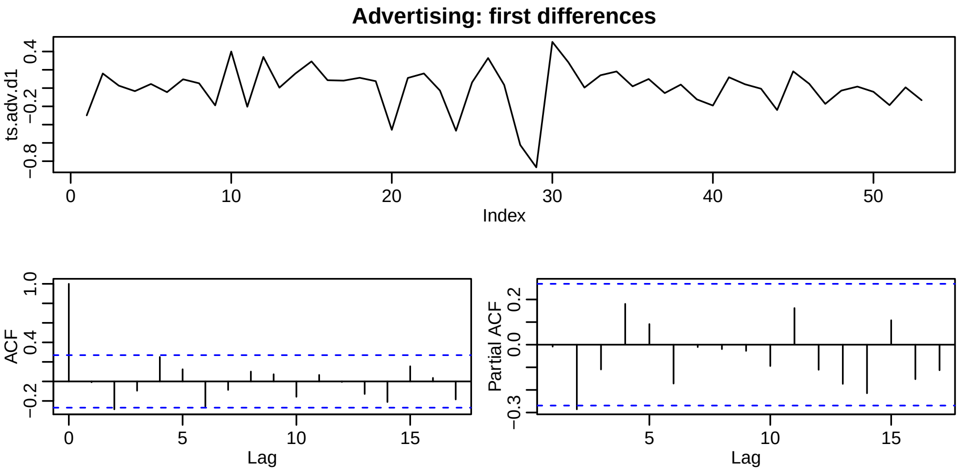
1. a) The plots clearly show that the time series are *not* stationary:

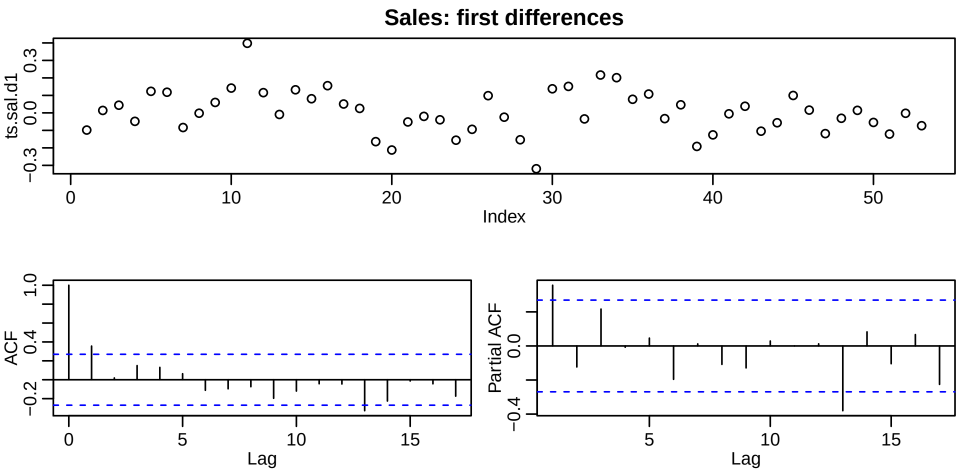


b) We first remove the missing values (last entry of the time series) and then calculate the first differences:

```
> ts.adv.d1 <- diff(ts.advert[!is.na(ts.advert)])
> ts.sal.d1 <- diff(ts.sales[!is.na(ts.sales)])
```

By differencing we can achieve stationarity as the following plots show (more or less):





c) The transfer function model

$$Y_{2,t} = \sum_{j=0}^{\infty} \nu_j Y_{1,t-j} + E_t$$

makes the assumption that a change in the advertising expenditures ($Y_{1,t}$) causes a change in the (future) sales ($Y_{2,t}$), but *not* vice versa.

d)

- From the correlogram of `d.adv.d1` we see that the input series $Y_{1,t} = X_{1,t} - X_{1,t-1}$ can be described as an $\mathrm{AR}(2)$ model. We fit it as follows:

```
> (r.fit.adv <- arima(ts.adv.d1, order = c(2, 0, 0)))
Call:
arima(x = ts.adv.d1, order = c(2, 0, 0))

Coefficients:
          ar1      ar2  intercept
      -0.0066  -0.2875    -0.0003
s.e.   0.1331   0.1314     0.0244

sigma^2 estimated as 0.05171:  log likelihood = 3.21,  aic = 1.59
```

Hence we get the model

$$Y_{1,t} = -0.0066 \cdot Y_{1,t-1} - 0.2875 \cdot Y_{1,t-2} + D_t \,,$$

where D_t is a white noise with variance $\widehat{\sigma}_D^2 = 0.052$ (see component `r.fit.adv$sigma2`). The mean of the time series can be regarded as zero (one gets an estimate of -0.0014).
Remark: One could also fit the $\mathrm{AR}(2)$ model of the first differences with the function `ar.burg()` or `ar.yw()`, resp. The estimates of the coefficients are quite similar, though.

- We apply the transformation as in the lecture:

```
> ts.D <- resid(r.fit.adv)
> ts.Z <- filter(ts.sal.d1, c(1, -r.fit.adv$model$phi), sides = 1)
```

In the transformed model

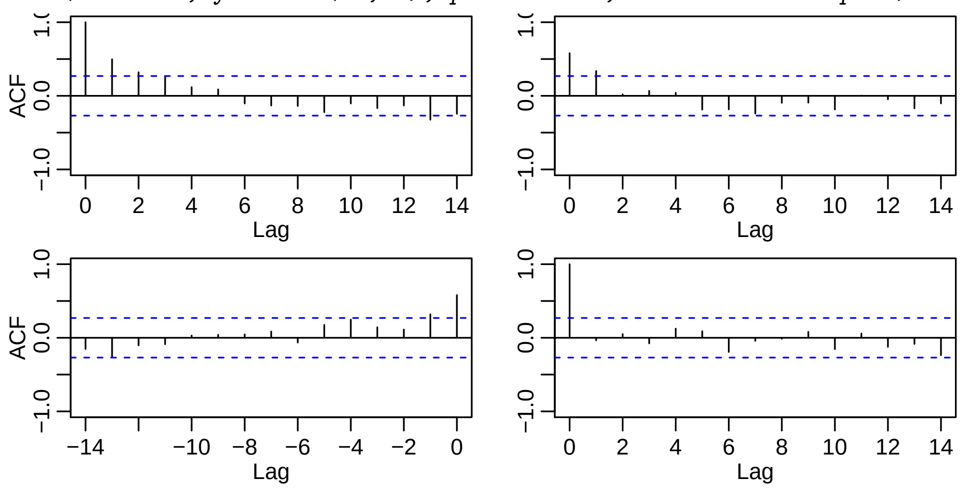
$$Z_t = \sum_{j=0}^{\infty} \nu_j D_{t-j} + U_t \ ,$$

the coefficients are the same as in the original transfer function model of part c). However, the time series D_t is *uncorrelated* here. Hence we can estimate the coefficients ν_j by

$$\widehat{\nu}_k = \frac{\widehat{\gamma}_{21}(k)}{\widehat{\sigma}_D^2}, \quad k \geq 0$$

where $\widehat{\rho}_{21}(k)$ denotes the empirical cross correlations of D_t and Z_t. The estimated coefficients $\widehat{\nu}_k$ are hence proportional to the empirical cross correlations $\widehat{\rho}_{21}(k)$ shown in the following plot.

```
> ts.trans <- ts.intersect(ts.Z, ts.D)
> acf(ts.trans, ylim = c(-1, 1), plot = TRUE, na.action = na.pass)
```



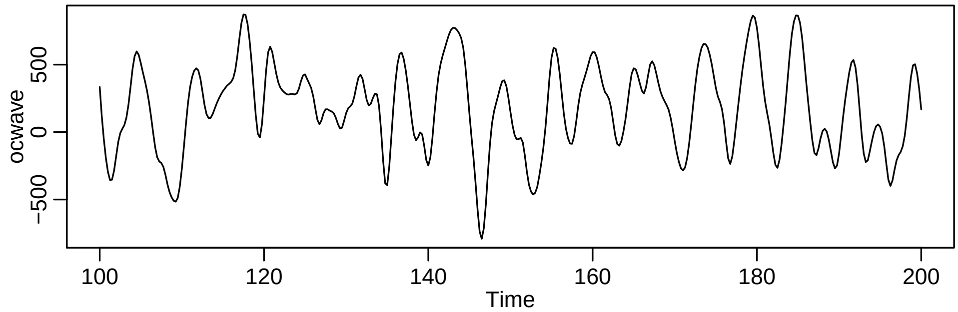
We see that $\widehat{\rho}_{21}(0)$ has the largest value. We find another large value at lag $k = -1$. This shows that, *contrary to our assumption* in part c), there is an influence of $Y_{2,t}$ on $Y_{1,t}$. Hence the modeling approach is not allowed since the prerequisites are not fulfilled. However, our analysis shows that there is a mutual influence between $Y_{2,t}$ and $Y_{1,t}$.
A change in the sales hence also causes a change in the advertising expenditures. This seems to be plausible in practice: the budget for advertising is usually established based on past sales, e.g. as a percentage of last year's sales.

- Estimation of the coefficients ν_j in R :

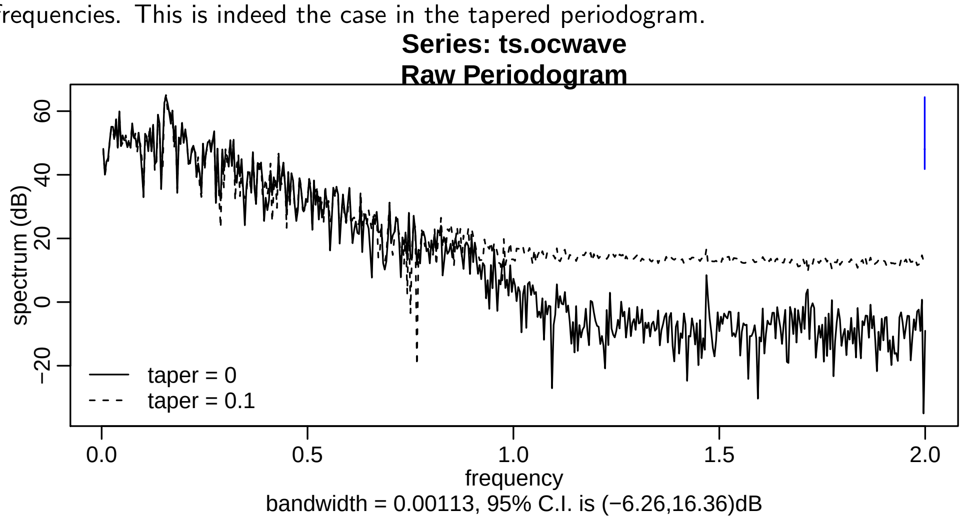
```
> gamma21 <- acf(ts.trans, plot = FALSE, type = "covariance",
    na.action = na.pass)$acf[, 1, 2]
> round(gamma21/r.fit.adv$sigma2, 2)[1:6]
[1]  0.33  0.20  0.01  0.04  0.02 -0.11
```

2. **a)** In the plot of the time series, oscillations with a period of approximately 6.5 s are apparent. To determine the period, it may be necessary to plot only a short region of the complete time series, see below.

```
> plot(window(ts.ocwave, start = 100, end = 200), ylab = "ocwave")
```

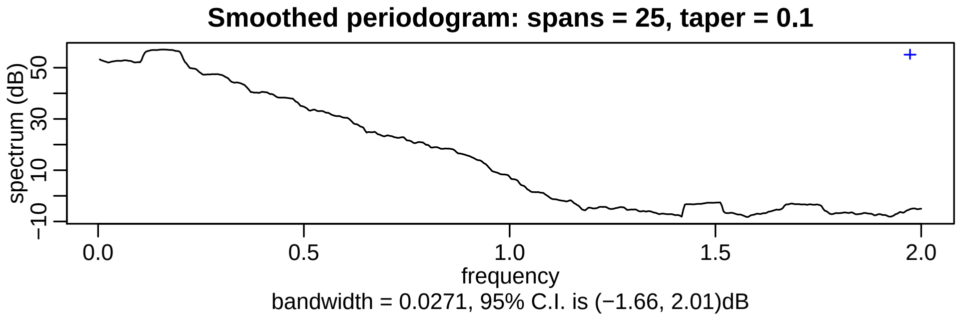


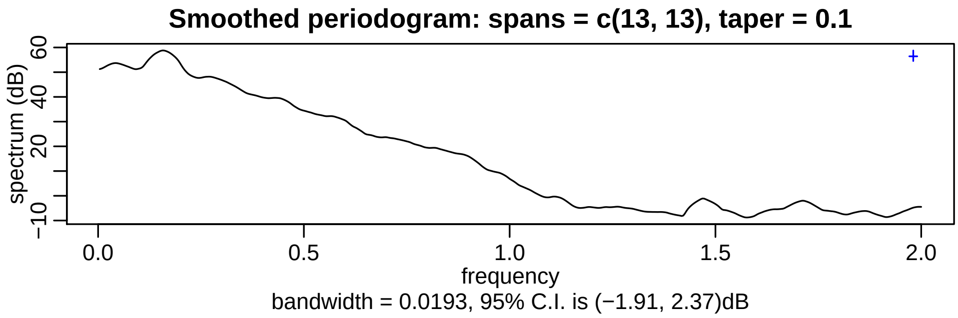
b) The raw periodogram (without tapering) overestimates the spectrum at high frequencies. The bias is reduced by tapering. The strongly reduced variance in the range $[1, 2]$ compared to the range $[0, 1]$ in the raw periodogram is a hint for the problems in estimating the spectrum without tapering: we know from theory that the estimation accuracy in logarithmic scale should be constant over varying frequencies. This is indeed the case in the tapered periodogram.



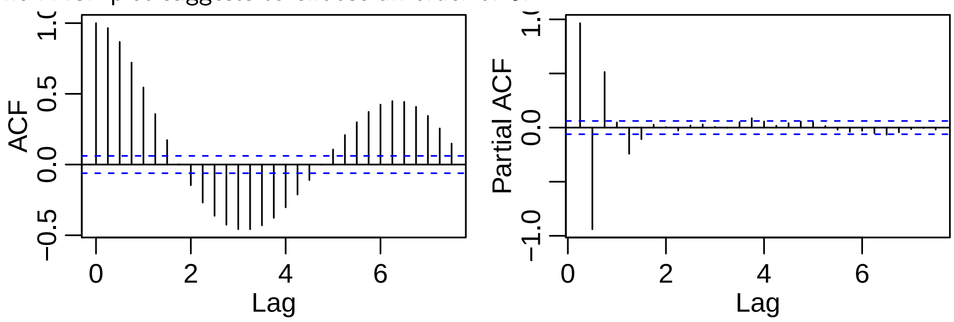
Note: the frequency range goes from 0 to 2, since the Nyquist frequency is $\frac{1}{2\Delta t}$. Hence the unit of the frequency scale is Hz.

c) The spectrum is maximal for the frequency $\nu \approx 0.16$ Hz, corresponding to a period of $T = \frac{1}{\nu} \approx 6.2$ s. This is the same period we found in the time series plot in task a).





d) The PACF plot suggests to choose an order of 6:



The Burg estimator for order 6 has a prediction variance of 149, the Yule-Walker estimator a much larger one of 797.

The respective spectra show clear differences. The method of Burg yields a better estimate of the spectrum as the method of Yule-Walker: as we have seen in part b), the spectrum should shrink to low values in the high frequency range.

```
> spec.ar(ocwave.burg, log = "dB")
> spec.ar(ocwave.yw, log = "dB", add = TRUE, lty = 2)
> legend("bottomleft", legend = c("Burg", "Yule-Walker"), lty = 1:2, bty = "n")
```