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Department of Electrical Engineering

EEE4093F – Process Control and Instrumentation Project 2016 Report

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Executive Summary

This report details the design of three digital controllers for the position control of a DC motor. The controllers were designed using state feedback with internal model control, the QFT design method and the Ziegler-Nichols PID tuning method. Prior to the design procedure, the system was modelled and tested using ramp and step tests in order to build a mathematical model of the system. The controllers needed to be robust to changes in the plant (gain uncertainty) and needed to meet the following requirements:

- Zero steady state error in tracking a fixed set point, with a worst-case error of 5%.
- Steady state disturbance and noise response should also be within 5% of set point.
- Good disturbance rejection and noise attenuation.
- Less than 20% peak over/undershoot in transient tracking and disturbance responses.
- No oscillations in steady state
- 2% Settling time of less than 3 seconds
- The closed loop may not be slower than half of the plant.

The chosen controller was ultimately successful in controlling the positional of the motor in the real implemtaion and it met all the specifications.

Declaration

1. I know that plagiarism is wrong. Plagiarism is to use another's work and pretend that it is one's own.
2. I have used the IEEE convention for citation and referencing. Each contribution to, and quotation in, this final year project report from the work(s) of other people, has been attributed and has been cited and referenced.
3. This final year project report is my own work.
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1 Introduction and Project Plan

1.1 Introduction

The design and implementation of a control system is often an iterative process. In general, there are many possible valid solutions to a design problem, the key is to find the best solution and this is done by trying different design techniques and synthesis methods. To compare different controllers, each must be thoroughly tested and analysed to check if it stable, rejects input or output disturbances, what the control effort is etc. These quantifiable measures can be used to compare controllers such that one may be selected.

These controllers will be used in a negative feedback loop. The reason this is done is because negative feedback mitigates plant uncertainty, allowing for robust tracking of set points even when the model of the plant in not completely accurate or the plant changes. Negative feedback can also stabilise an unstable plant (in open loop) as well as diminish input and output disturbances [1].

1.2 Project Aim

In short, the aim of this project was to design a digital controller to control the position of a DC motor.

1.3 User requirements

The user requirements of the system were:

- Zero steady state error in tracking a fixed set point, with a worst-case error of 5%.
- Steady state disturbance and noise response should also be within 5% of set point.
- Good disturbance rejection and noise attenuation.
- Less than 20% peak over/undershoot in transient tracking and disturbance responses.
- No oscillations in steady state
- 2% Settling time of less than 3 seconds
- The closed loop may not be slower than half of the plant.

1.4 Project Plan

The following approach was used in the design of the controller:

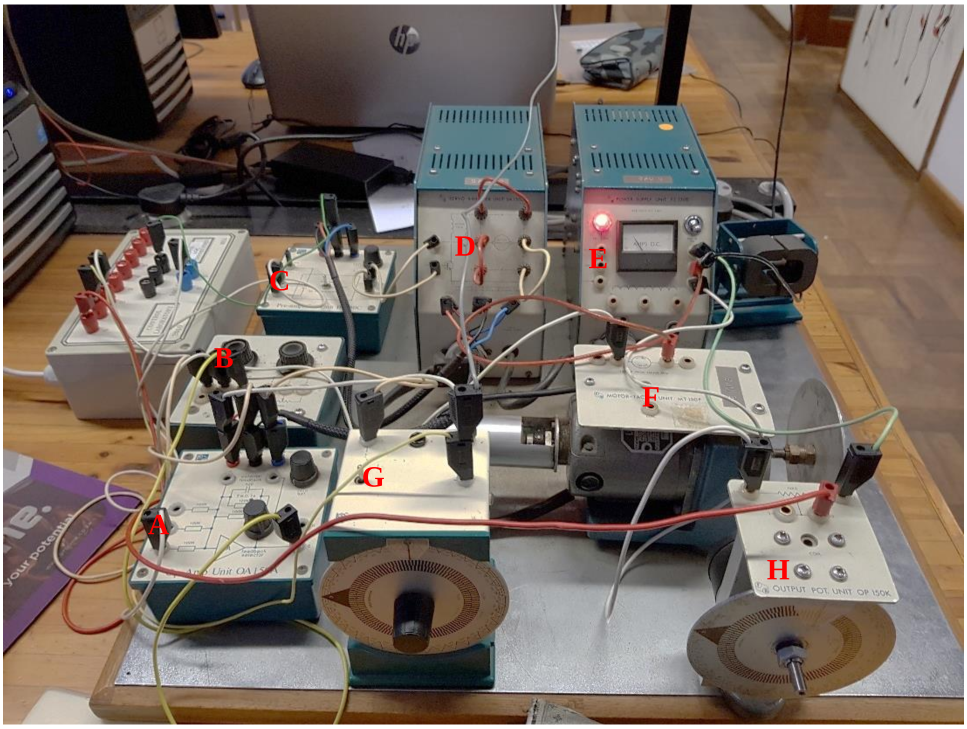
1. The physical system was first analysed
2. From this analysis, the system could be broken down into it component blocks and mathematically modelled
3. From this model the problem was directly formulated
4. The data captured from testing the system was used to derive the plant model
5. Using the problem formulation and the plant mode, three controller design techniques were used to design three controllers
6. Each controller was tested and compared using the criterion sated below
7. Finally, the chosen controller was implemented in C# on a computer in the lab

1.5 Criteria for choosing a controller

The best controller was considered to be the one that required the least control effort to achieve the required specifications, whilst stabilising the system and rejecting disturbances. Also, taken into consideration was the fact that the controller had to be simple and robust enough to mitigate the changes students may have made to the equipment in the lab, as well still working with external disturbances from other system that are being tested.

2 System Structure

Shown below is an overview of the whole system:



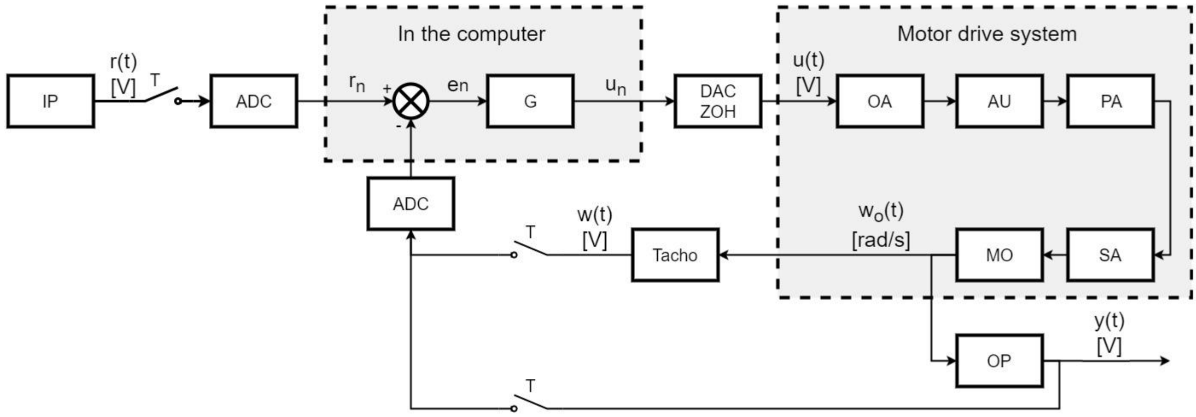
Each component is described in section 3.2.

3 Problem Formulation

In this section, an analysis of each component of the system is done, in order to better understand the underlying dynamics of the system and thus get a complete understanding of the problem at hand.

3.1 System Block Diagram

Shown below is a block diagram of the interconnections of the system:



This block diagram will be used to derive a final transfer function from the reference signal (r(t)) to the output (y(t)). G is the controller to be designed. It will receive sampled values of the set point reference signal as well the motor speed and current position, as measured in volts. Although it may look like this system has negative feedback and is in a closed loop, if u(t) is set to r(t) there is no control action, the feedback is ignored and the system can be considered to be in open loop. The system was initially tested by doing this.

3.2 Component Blocks

<table>
<tr><td>A - Op-Amp Feedback Unit [OA]</td><td>B - Attenuator Unit [AU]</td></tr>
<tr><td>

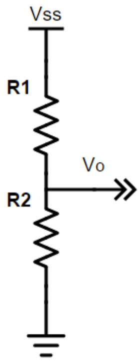
This competent is made from an op-amp (linear device) and applies an inverting (negative) gain to the transfer of the control signal:

$$T_{OA} = -1 = K_{OA}\left[\frac{V}{V}\right]$$

</td><td>

The attenuator applies a fractional gain (attenuating) to the signal. It is a resistor divider network (a potentiometer), thus it can be modelled as a linear competent:

$$T_{AU} = \frac{R_2}{R_1 + R_2} = K_{AU}\left[\frac{V}{V}\right]$$



</td></tr>
<tr><td>C - Pre-Amp Unit [PA]</td><td>D - Servo Amplifier Unit [SA]</td></tr>
<tr><td>

The pre-amp is from an ip-amp in a summing configuration, which internally applies a linear, inverting gain to the signal.

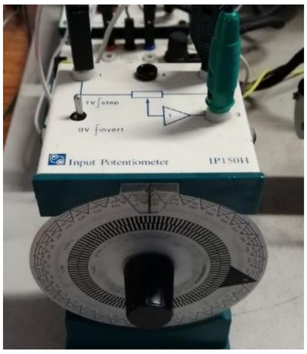
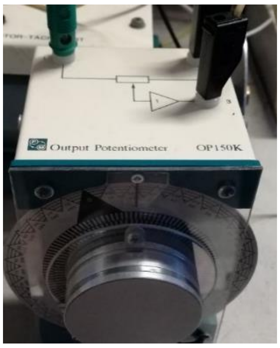
$$T_{PA} = -1 = K_{PA}\left[\frac{V}{V}\right]$$

</td><td>

The servo amplifier turns the control signal into a power signal that is inputted into the motor. The internal transistor are operated in their linear (active) region.

$$T_{SA} = K_{SA}\left[\frac{V}{V}\right]$$

</td></tr>
</table>

E - Power Supply Unit (PSU) [PS]	F - DC Motor and Tacho Unit [MO + Tacho]
This unit supplies power to the system and to the motor through the servo amplifier unit. However, it does not contribute to the overall transfer function of the system.  	The motor contains a dead band due to the static friction present and other parasitic properties. It is modelled as a linear device with a dead band. The motor additionality has tachometer attached to it, which measures the angular velocity in Volts, with the gain: $$T_{TA} = K_{TA}\left[\frac{V}{\text{rad/s}}\right]$$ 
G - External Input Potentiometer [IP]	**H - External Output Potentiometer [OP]**
The input potentiometer supplies the reference set point signal that the system must track when a controller has been complemented. It converts an angle to a voltage by dividing its supply by a factor that is proportion to the angle. $$T_{IP} = V_{SS}\frac{K_{in}}{R_t} = K_{IP}\left[\frac{V}{\text{rad}}\right]$$  	The potentiometer is used as a sensor to measure the angular position of the motor. The variable resistance is proportional to the integral of the angular velocity of the motor: $$\therefore T_{OP} = GR\frac{V_{SS}}{R_t} * \frac{1}{s} = K_{OP} * \frac{1}{s}\left[\frac{V}{\text{rad/s}}\right]$$ 

3.3 Block Analyses

Shown below is the derivation for the model for the motor:

Using Kirchhoff's voltage law:

$$V_a - V_{Ra} - V_{La} - V_c = 0$$

The voltages are given as:

$$V_{Ra} = i_a R_a, \qquad V_{La} = L_a \frac{d}{dt} i_a, \qquad V_c = k_v \omega_m$$

$$\therefore V_a - i_a R_a - L_a \frac{d}{dt} i_a - k_v \omega_m = 0$$

Because the motor does not experience a net torque acting on it, a torque balancing equation can be used:

$$T_m - T_{dyn} - T_{frict} - T_L = 0$$

Each component torque is given by:

$$T_m = k_t i_a, \qquad T_{dyn} = J\frac{d}{dt}\omega_a, \qquad T_{frict} = B\omega_m$$

Where: J is the rotational inertia of the motor and B is the damping coefficient

$$\therefore\ k_t i_a - J\frac{d}{dt}\omega_a - B\omega_m = 0$$

Rearranging both equations:

$$\frac{d}{dt}i_a = \frac{R_a}{L_a}i_a + \frac{k_v}{L_a}\omega_m - \frac{1}{L_a}V_a$$

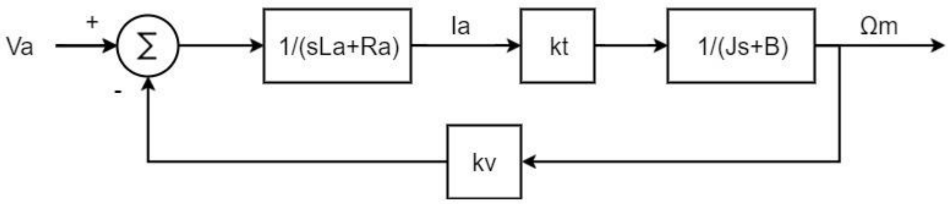
$$\frac{d}{dt}\omega_a = \frac{k_t}{J}i_a - \frac{B}{J}\omega_m$$

Taking the Laplace transform, assuming the initial conditions are 0:

$$I_a(s) = \frac{-k_v}{L_a s + R_a}\Omega_m + \frac{1}{L_a s + R_a}V_a = \frac{1}{L_a s + R_a}(V_a - k_v\Omega_m) = \frac{\frac{1}{R_a}}{\frac{L_a}{R_a}s + 1}(V_a - k_v\Omega_m)$$

$$\Omega_m(s) = \frac{k_t}{Js + B}I_a = \frac{\frac{k_t}{B}}{\frac{J}{B}s + 1}I_a$$

These two transfer functions can be used to create a new block diagram for the motor:



The derivation of the transfer function from Va to Ω_m is determined to be:

$$T_{\frac{\Omega_m}{V_a}} = \frac{\frac{k_t}{BR_a}}{\left(\frac{J}{B}s + 1\right)\left(\frac{L_a}{R_a}s + 1\right)} = MO(s)$$

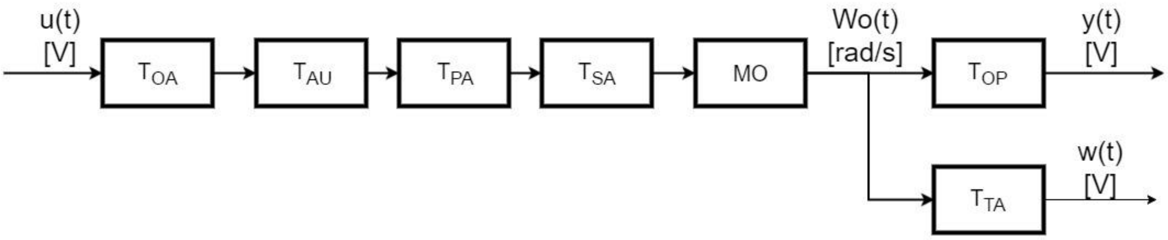
From this transfer function, the following characteristics may be derived:

- Motor gain $K = \frac{k_t}{BR_a}$
- Mechanical time constant $\tau_m = \frac{J}{B}$
- Field coil time constant $\tau_f = \frac{L_a}{R_a}$

In general, the mechanical time constant is much less than the field coil time constant, thus the $\left(\frac{J}{B}s + 1\right)$ pole is much more dominant than the $\left(\frac{L_a}{R_a}s + 1\right)$ pole, so MO(s) can be simplified to a first order transfer function:

$$\therefore\ MO(s) = \frac{K_m}{(T_m s + 1)}$$

The overall transfer function for the system can now be derived for the two outputs:



$$G_\omega(s) = T_{\frac{W}{U}} = T_{OA}T_{AU}T_{SA}MO * T_{TA} = K_{OA}K_{AU}K_{SA}\frac{K_m}{(T_m s + 1)}K_{TA} = \frac{K_S}{(T_m s + 1)}$$

$$G_p(s) = T_{\frac{Y}{U}} = T_{OA}T_{AU}T_{SA}MO * T_{OP} = K_{OA}K_{AU}K_{SA}\frac{K_m}{(T_m s + 1)}K_{OP} * \frac{1}{s} = \frac{K_P}{s(T_m s + 1)}$$

G_ω describes the system that was tested, G_p describes the model of the system that will be controlled.

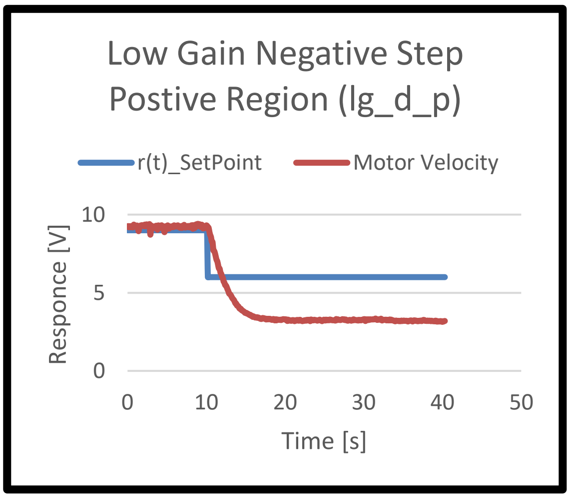
4 Mathematical Model

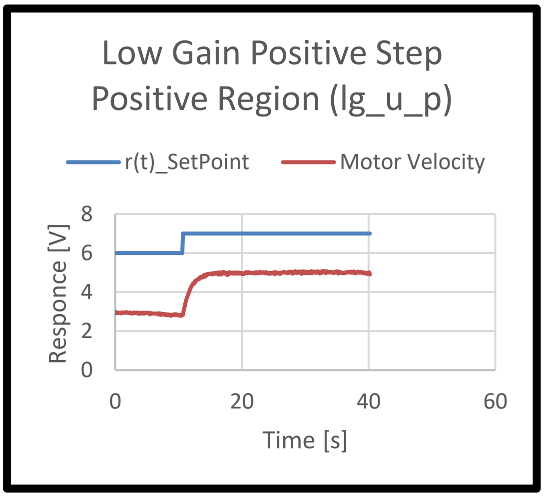
4.1 System Identification

4.1.1 Step tests

Various step inputs were used to test the response of the system. These steps were done at high plant gain (hg) and low plant gain (lg). This gain was controller by setting the attenuator unit in the system. Negative steps (d) as well as positive steps (u) were performed in the negative (n) and positive (p) range of inputs. 8 steps in total were performed.

Shown below are two example steps:

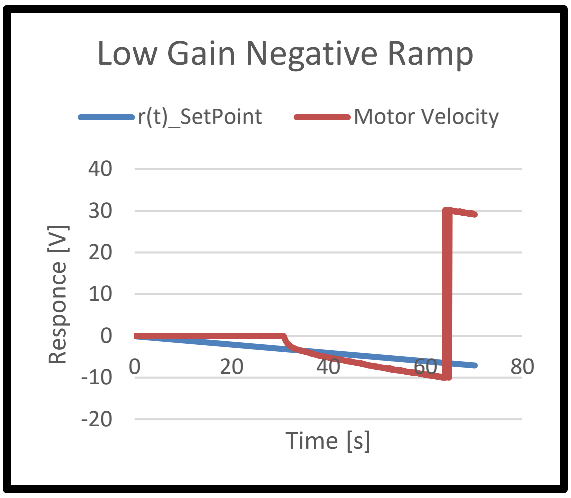


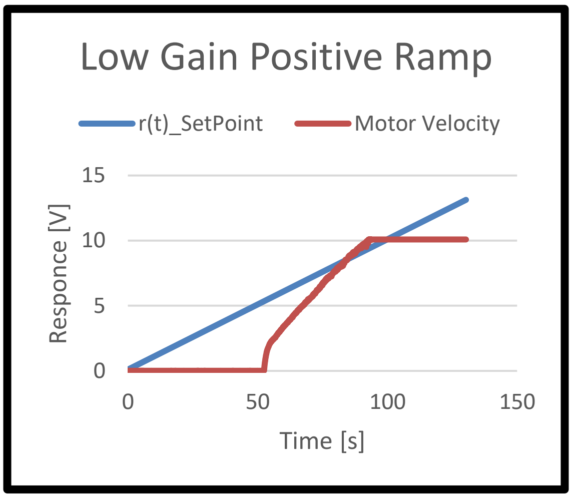


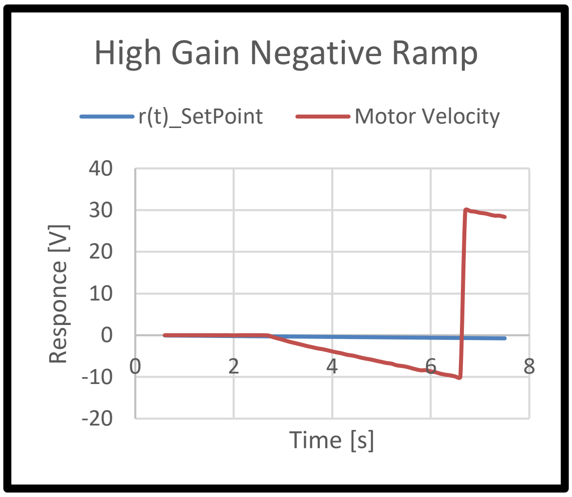
The steps show a fist order response. This fits the model derived for the DC motor above, if a step input is supplied. Thus, the derivation can be deemed to be valid.

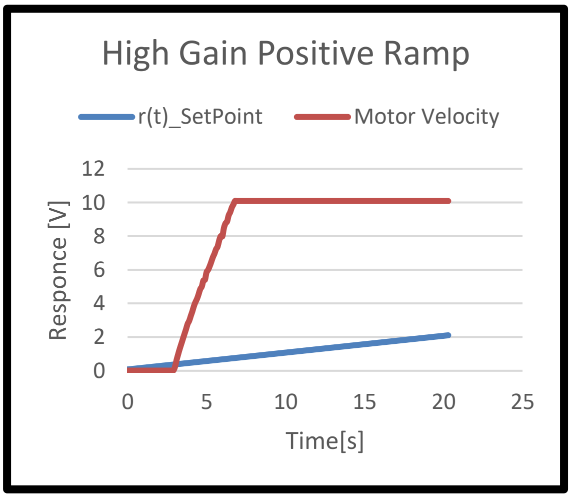
4.1.2 Ramp tests

The system also possessed some dead band. The following ramp tests of the system show the dead band and how it changes at different plant gains.









From these tests, the dead band occurs between the following r(t) values:

- At low plant gain: $[-3.16, 5.35]$ V
- At high plant gain: [-0.27, 3] V

The dead band for the plant with high gain is significantly smaller than the low gain one.

4.2 Dynamic Modelling

4.2.1 First order response

The dynamics of G(s) may be found using the following general method:

A first order system as the following transfer function:

$$G_{\omega}(s) = \frac{A}{\tau S + 1}$$

Steps of magnitude B are given by:

$$U(s) = \frac{B}{s}$$

$$\therefore Y(s) = \frac{A * B}{(\tau S + 1)s}$$

Taking the Inverse Laplace transform, the time domain function can be found:

$$y(t) = A * B\left(1 - e^{-\frac{t}{\tau}}\right)$$

The final values is thus given by (the e term goes to 0):

$$y(\infty) = A * B$$

$$\therefore A = \frac{y(\infty)}{B} = \frac{y_{final} - y_{inital}}{u_{final} - u_{inital}}$$

When t=τ, y is given by:

$$y(\tau) = A * B(1 - e^{-1}) = 0.632 * y(\infty) = 0632 * \left(y_{final} - y_{inital}\right) + y_{inital}$$

τ can be found by finding the time at which y(t) = $y(\tau)$, if the step occurred at t=0s.

4.2.2 Determining plant values

The following A and τ values were calculated for each test:

Exp	SI	SF	IV	FV	A	τ	AVG A	AVG τ
hg_u_p	0.40	0.58	4.21	9.45	29.11	1.40	29.58	1.60
	0.50	0.58	7.14	9.54	30.05	1.80		
hg_d_p	0.55	0.50	8.78	7.35	28.59	2.00	28.98	2.15
	0.58	0.40	9.63	4.34	29.38	2.30		
hg_u_n	-0.45	-0.30	-9.09	-4.34	31.69	2.50	29.85	2.38
	-0.45	-0.35	-8.81	-6.00	28.02	2.25		
hg_d_n	-0.30	-0.45	-4.10	-8.84	31.62	1.70	30.36	1.60
	-0.35	-0.40	-5.95	-7.40	29.10	1.50		
lg_u_p	6.00	7.00	2.92	5.00	2.08	1.40	2.08	1.40
	6.00	9.00	2.86	9.08	2.07	1.40		
lg_d_p	9.00	8.00	9.17	7.27	1.90	2.10	1.95	2.25
	9.00	6.00	9.18	3.20	1.99	2.40		
lg_u_n	-5.00	-4.00	-6.86	-4.77	2.09	1.80	2.08	1.40
	-6.00	-4.00	-8.84	-4.72	2.06	2.40		
lg_d_n	-4.00	-5.00	-4.99	-6.96	1.97	1.60	2.00	1.70
	-4.00	-6.00	-4.75	-8.78	2.02	1.80		

Using this data, the minimum and maximum A and τ values can be found:

MIN A	MAX A	AVG A	MIN τ	MAX τ	AVG τ
1.9	30.4	15.9	1.4	2.3	1.9

These values give the bounds on the uncertainties in the plant and must be accounted for when designing a controller.

4.2.3 Dealing with dead band

The dead band of the system may be dealt with by latching the controller output (u(t)) to the maximum or minimum bounds of the dead band. However, this requires *a priori* knowledge of the bounds of the dead band, which change for different plant gain values. Thus, a better approach is make sure that an integrator is present in the controller that can integrate the error sufficiently well such that when the controller needs to control the plant within the dead, the error will give it enough gain (eventually) to get out of the dead band.

5 Design of Controllers

5.1 Controller Requirements

From the user specification, the following requirements that the controllers must adhere to are derived:

- Zero steady state error in tracking a fixed set point, with a worst-case error of 5%:

$$0.95 < \left|\frac{L}{1+L}\right| < 1.05 \text{ as } \omega \to 0 \text{ (low } \omega\text{)}$$

- Steady state disturbance and noise response should also be within 5% of set point:

$$\left|\frac{1}{1+L}\right| < 0.05 \ (-26 \text{ dB}) \text{ as } \omega \to 0 \text{ (low } \omega\text{)}$$

- Good disturbance rejection and noise attenuation:

$$\left|\frac{1}{1+L}\right| < 3dB \ \forall \ \omega$$

- Less than 20% peak over/undershoot in transient tracking and disturbance responses:

$$M_p = \frac{y_p - y_\infty}{y_\infty} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.2$$

- No oscillations in steady state

$$\zeta > 0.707$$

- 2% Settling time of less than 3 seconds

$$t_{2\%} \cong \frac{4}{\omega_n \zeta} < 3$$

- The closed loop may not be slower than $\frac{1}{2}$ of the plant.

The dominate closed loop pole(s) must have an $\omega_n > \frac{1}{2}T$, where T is the time constant of the plant.

5.1.1 Requirements Analysis

The controller requirements stated above need to be analysed to determine the parameters of the bounds for the closed loop transfer characteristics.

From the 20% peak over/undershoot requirement, ζ can be found:

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.2$$

Solving for ζ:

$$\zeta > \pm\sqrt{\frac{\ln(0.2)^2}{(\pi^2 + \ln(0.2)^2}} = 0.46$$

$$\therefore \zeta > 0.46$$

For mechanical systems, ζ should be greater than 0.707 to ensure no oscillations, which will satisfy the condition derived above.

$$\therefore \zeta > 0.707$$

The system must not be over damped:

$$\zeta < 1$$

From the 2% settling time requirement, a bound on ω_n can be found:

$$\omega_n > \frac{4}{3\zeta}$$

This will give the largest values at $\zeta = 0.707$.

Therefore:

$$\omega_n > \frac{4}{3 * 0.707} = 1.89 \cong 1.9$$

Because the system cannot be expected to have infinite bandwidth, a reasonable upper bound on ω_n is chosen to be 4 rad/s, thus:

$$\omega_n < 4$$

Let K be the gain of the closed loop system at steady state, that is when $\omega \to 0$:

$$0.95 < K < 1.05$$

These parameters relate only to the dominate poles of the system.

In summary:

$$\mathbf{0.95 < K < 1.05}$$

$$\mathbf{1.9 < \omega_n < 4}$$

$$\mathbf{0.707 < \zeta < 1}$$

5.2 State Feedback Controller Design

For this approach, the various states of the system are directly controlled. This effectively allows one to choose the locations of the closed loop poles

5.2.1 State Space Model

The state-space model of the plant can be found using direct programming. The state space model will be done for the plant with the lowest A and T values.

Define the state vector of the system:

$$\hat{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \text{position } y \\ \text{velocity } v \end{pmatrix}$$

The state-space model of the plant is given by the following differential equation

$$\frac{d}{dt}\hat{x}(t) = \boldsymbol{A}\hat{x}(t) + \boldsymbol{B}u(t)$$
$$y(t) = \boldsymbol{C^T}\hat{x}(t) + \boldsymbol{D}u(t)$$

A general plant of the following form is used:

$$P(s) = \frac{A}{s * (Ts + 1)}$$

The state space model of the transfer function can be found by direct programming rules:

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & -\frac{\mathbf{1}}{\boldsymbol{T}} \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \mathbf{0} \\ \frac{\boldsymbol{A}}{\boldsymbol{T}} \end{pmatrix}, \quad \boldsymbol{C^T} = (\mathbf{1} \quad \mathbf{0}), \quad \boldsymbol{D} = \mathbf{0}$$

The discrete state space model is given by:

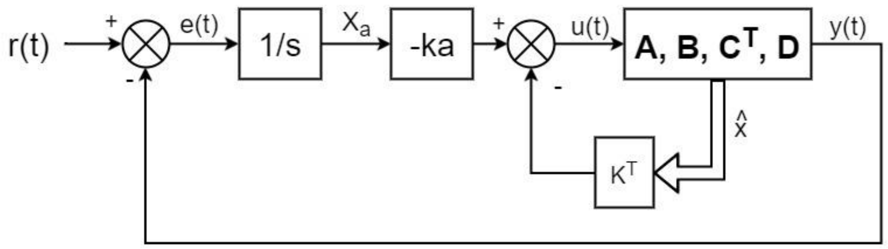
$$\frac{\hat{x}(n+1)-\hat{x}(n)}{T_s} = \boldsymbol{A}\hat{x}(n) + \boldsymbol{B}u(n) \rightarrow \hat{x}(n+1) = (T_s\boldsymbol{A} + \boldsymbol{I})\hat{x}(n) + T_s\boldsymbol{B}u(n)$$

$$y(n) = \boldsymbol{C^T}\hat{x}(n) + \boldsymbol{D}u(n)$$

$$\therefore \boldsymbol{A_z} = \begin{pmatrix} \boldsymbol{1} & T_s \\ \boldsymbol{0} & \boldsymbol{1} - \dfrac{T_s}{\boldsymbol{T}} \end{pmatrix}, \qquad \boldsymbol{B_z} = \begin{pmatrix} \boldsymbol{0} \\ \dfrac{T_s\boldsymbol{A}}{\boldsymbol{T}} \end{pmatrix}, \qquad \boldsymbol{C_z^T} = (\boldsymbol{1} \quad \boldsymbol{0}), \qquad \boldsymbol{D_z} = \boldsymbol{0}$$

5.2.2 Controller Model - State Feedback with Internal Model Control

State feedback will be employed with internal model control (IMC), which is needed in order to diminish steady state errors in the system as well as reject input and output disturbances. Although the plant is a type 1 system and contains an internal integrator, the position feedback gain will not necessarily be 1, resulting in a steady state error as the actual and desired position positions will not be directly compared, thus the necessity of the adding the integrator.



The input signal u(t) into the plant is given by:

$$u(t) = -k_aX_a - \boldsymbol{K^T}\hat{x}(t) = -(\boldsymbol{K^T}\ k_a)\bar{x}(t) = -\boldsymbol{\bar{K}}^T\bar{x}(t)$$

Where:

$$\bar{x} = \begin{bmatrix} \hat{x} \\ X_a \end{bmatrix}$$

This is known as the **control law** for the controller.

It follows that:

$$\frac{d}{dt}X_a(t) = e(t) = r - y = r - \boldsymbol{C^T}\hat{x}(t) - \boldsymbol{D}u(t)$$

The new state-space model of the whole system, from set point to output, is now given as (assuming D=0):

$$\frac{d}{dt}\bar{x} = \begin{pmatrix} \boldsymbol{A} & 0 \\ -\boldsymbol{C^T} & 0 \end{pmatrix}\bar{x} - \begin{pmatrix} \boldsymbol{B} \\ -\boldsymbol{D} \end{pmatrix}(\boldsymbol{K^T}\ k_a)\bar{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r(t) = \boldsymbol{\bar{A}}\bar{x} - \boldsymbol{\bar{B}\bar{K}^T}\bar{x} + \boldsymbol{q}r(t)$$

$$y = (1 \quad 0)\bar{x} - \boldsymbol{D\bar{K}}^T\bar{x}(t) = (\boldsymbol{\bar{C}^T} - \boldsymbol{D\bar{K}}^T)\bar{x}(t)$$

By taking the Laplace transform of the system and substituting $\bar{x}$ into the Y equation, the following is found:

$$Y(s) = \frac{(\boldsymbol{\bar{C}^T} - \boldsymbol{D\bar{K}}^T)\mathrm{Adj}(s\boldsymbol{I} - \boldsymbol{\bar{A}} + \boldsymbol{\bar{B}\bar{K}^T})\boldsymbol{q}}{|s\boldsymbol{I} - \boldsymbol{\bar{A}} + \boldsymbol{\bar{B}\bar{K}^T}|}R(s)$$

Thus, the characteristic equation is given by:

$$\therefore \phi_C = |s\boldsymbol{I} - \boldsymbol{\bar{A}} + \boldsymbol{\bar{B}\bar{K}^T}|$$

5.2.3 Controller Derivation

A nominal plant is chosen:

$$P(s) = \frac{6}{s * (1.9s + 1)}$$

Subsisting values into the state space model:

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 0 & -0.5263 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 \\ 3.158 \end{pmatrix}, \quad \boldsymbol{C^T} = (1 \quad 0), \quad \boldsymbol{D = 0}$$

$$\overline{\boldsymbol{K}}^{\boldsymbol{T}} = (k_1 \quad k_2 \quad k_a)$$

Converting to a discreet state space representation (sampling period of 2ms):

$$\therefore \boldsymbol{A_z} = \begin{pmatrix} 1 & 0.02 \\ 0 & 0.9895 \end{pmatrix}, \quad \boldsymbol{B_z} = \begin{pmatrix} 0 \\ 0.06283 \end{pmatrix}, \quad \boldsymbol{C_z^T} = (1 \quad 0), \quad \boldsymbol{D_z} = 0$$

Because the controller model was derived for the s-domain, but a digital controller is being designed, a good approximate of a ‘continuous’ z-domain is found using the Bilinear Transform, converting to the fictions w-domain, a continuous corollary to the z-domain:

$$w = \frac{2}{T_s}\frac{1 - z^{-1}}{1 + z^{-1}}$$

The state space model is now given as:

$$\boldsymbol{A_w} = \begin{pmatrix} 0 & 1 \\ 0 & -0.5263 \end{pmatrix}, \quad \boldsymbol{B_w} = \begin{pmatrix} 5.54\text{e-}05 \\ 3.158 \end{pmatrix}, \quad \boldsymbol{C_w^T} = (1 \quad -0.01), \quad \boldsymbol{D_w} = -5.54\text{e-}07$$

Substituting these into the matrices derived above, the characteristic equation can be found:

$$\overline{\boldsymbol{A_w}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -0.5263 & 0 \\ -1 & 0.01 & 0 \end{pmatrix}, \quad \overline{\boldsymbol{B_w}} = \begin{pmatrix} 5.54\text{e-}05 \\ 3.158 \\ 5.54\text{e-}07 \end{pmatrix}$$

$$\therefore \phi_C = \left| \begin{pmatrix} w & -1 & 0 \\ 0 & w + 0.5263 & 0 \\ -1 & -0.01 & w \end{pmatrix} + \begin{pmatrix} 5.54\text{e-}05 \\ 3.158 \\ 5.54\text{e-}07 \end{pmatrix} (k_1 \quad k_2 \quad k_a) \right|$$
$$= w^3 + (3.158k_2 + 0.5263)w^2 + (3.158k_1 + 0.03152k_a) * w - 3.158k_a$$

(some coefficients were approximated to 0)

From the controller requirements in section (5.1), the transfer function from set point to output must have the following parameters:

$$\boldsymbol{0.95 < K < 1.05}$$

$$\boldsymbol{1.9 < \omega_n < 4}$$

$$\boldsymbol{0.707 < \zeta < 1}$$

Therefore, $w_n = 2, \zeta = 0.9$ is a good choice for the pole locations. Using the characteristic equation of a second order system $s^2 + 2\zeta\omega_n s + \omega_n^2$, the desired poles locations is given by:

$$\phi_D = (w + 2)(w^2 + 3.6w + 4) = w^3 + 5.6w^2 + 11.2w + 8$$

Now, equating coefficients of w from ϕ_D and ϕ_c it is found:

$$\boldsymbol{k_1 = 3.571}$$

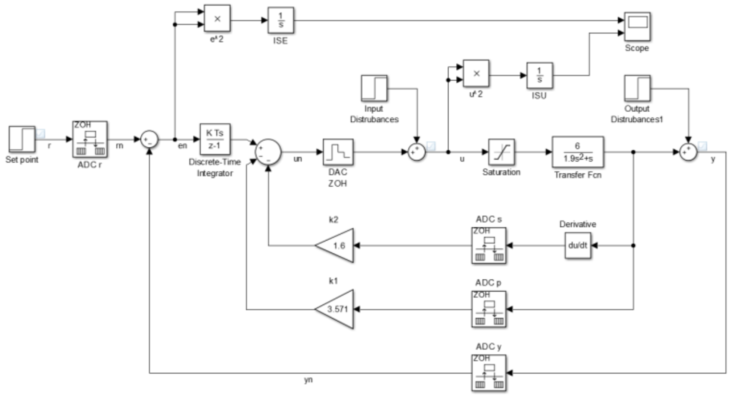
$$\boldsymbol{k_2 = 1.6}$$

$$\boldsymbol{k_a = -2.533}$$

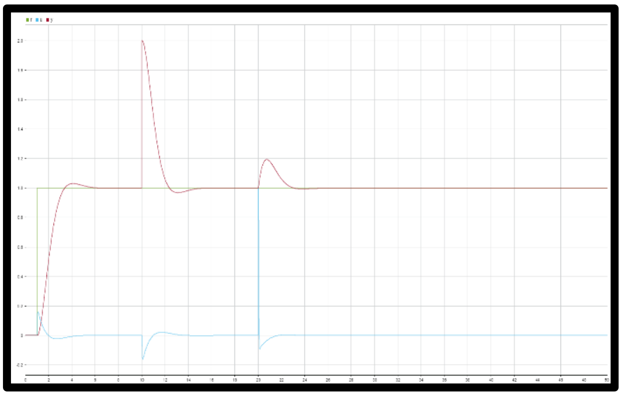
These gains seem reasonable. The negative k_a gain results in a positive in the loop because it was modelled as being negative.

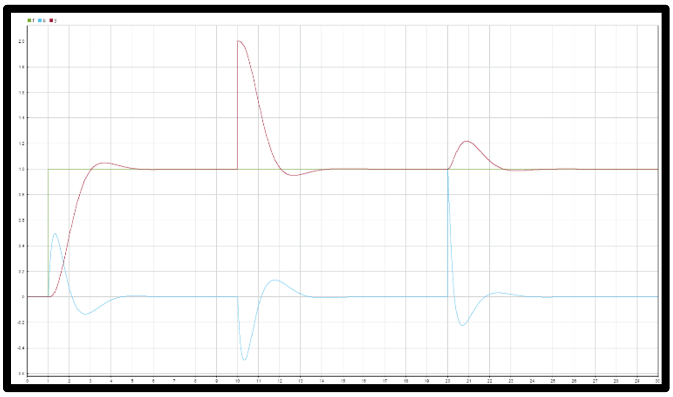
5.2.4 Controller Simulation

The controller was simulate using MATLAB Simulink, with the following diagram:



The response of the system to a step in the reference at t=1s, a step at the output disturbance at t=10s and a step at the input disturbance at t=20s is shown below. The plot on the left shows the nominal model response and the plot on the right shows the high gain, high time constant plant [A=30 T=2.3].



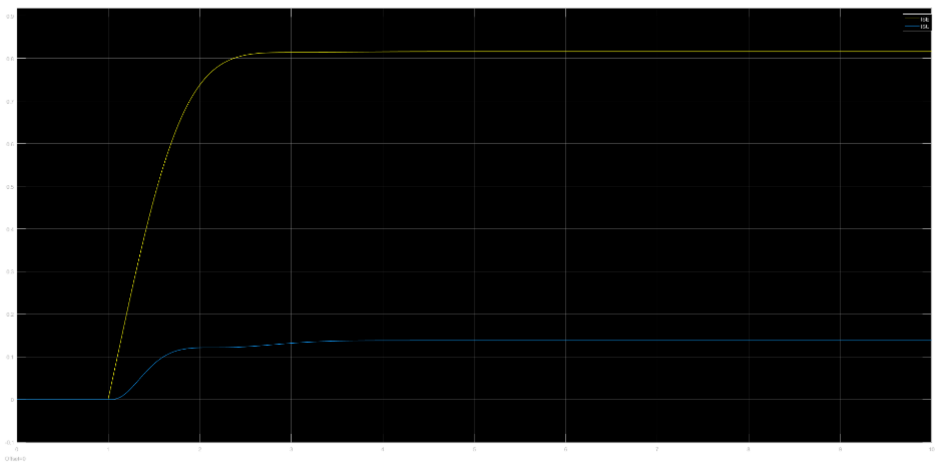


[Green: Set point r(t), Red: Output y(t), Blue: Plant Input u(t)]

The integrator gain was slightly increased to **3.5** in order to meet the rise time spec. For the high gain case, the system was stable and could still reject input and output disturbances. For the nominal case, the $t_{2\%}$ time was 2.9s, overshoot 5% and it completely rejected input and output disturbances. The controller met the specifications.

5.2.5 Controller Effort Analysis

The following shows the ISU and ISE for the nominal model of the system (for a step at the reference only):



[Yellow: ISE, Blue: ISU]

At steady state, the ISU and ISE values were relatively constant and approximated to be:

$$ISU = 0.14$$

$$ISE = 0.81$$

5.3 QFT Controller Design

Quantitative Feedback Theory provides a method that allows one to design a robust controller for a given plant, taking into account the uncertainties inherent in the plant. It allows the designer to visualize the current system model and the requirements it must meet using Bode plots and Nichols charts.

The response of the system in open loop is first plotted on a Nichols chart. The constraints and bounds that the system must conform to are mapped onto the Nichols chart. Using the knowledge of how different transfer functions in the controller will affect the open loop response (for example: adding a lag controller will introduce phase lag and pull the open loop response to the left in the Nichols chart), one can design the closed loop response to meet the requirements.

5.3.1 Controller Specification Bounds

The ideal response for the closed loop system is specified in terms of a second order system:

$$TF = \frac{K * \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K}{(\frac{s}{\omega_n})^2 + \frac{2\zeta}{\omega_n}s + 1}$$

It follows that in steady state ($s \to 0$), TF=K.

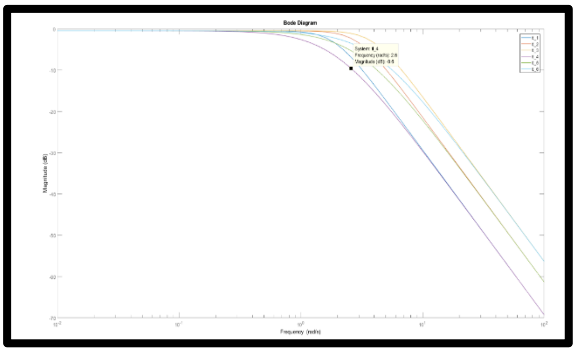
MATLAB was used to iterate through all bounds that the system must conform to in order to find the maximum and minimum possible bound.

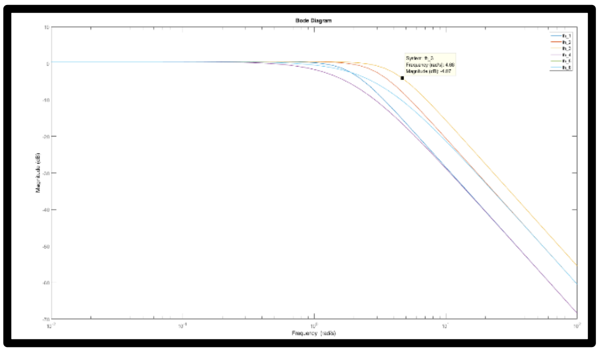
The bounds being (from section 5.1):

$$\mathbf{0.95 < K < 1.05}$$

$$\mathbf{1.9 < \omega_n < 4}$$

$$\mathbf{0.707 < \zeta < 1}$$





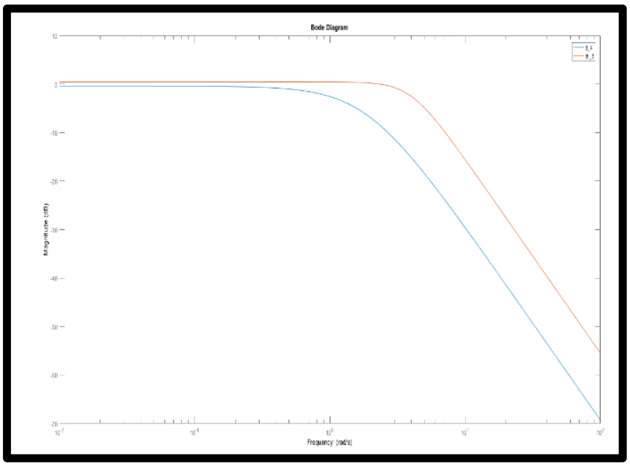
Minimum bound:

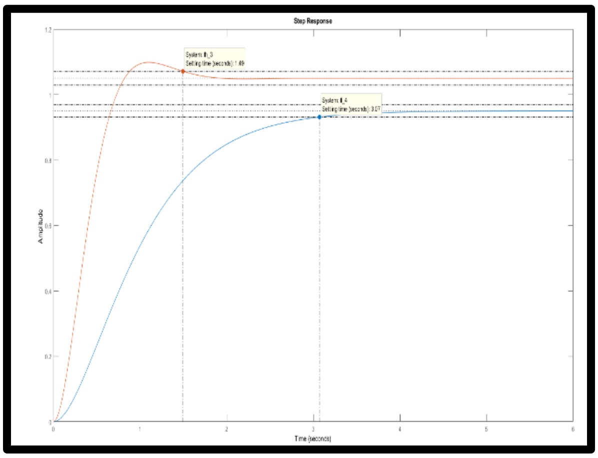
$$\mathbf{K = 0.95}$$
$$\mathbf{\zeta = 1}$$
$$\mathbf{\omega_n = 1.9}$$

Maximum bound:

$$\mathbf{K = 1.05}$$
$$\mathbf{\zeta = 0.707}$$
$$\mathbf{\omega_n = 4}$$

The magnitude and time response of the bounds is shown below:





[Orange: Maximum bound, Blue: Minimum bound]

These two bounds give the maximum and minimum responses that the closed loop system can have, when tracking a set point. That is, the closed loop system is bounded by the following constraints:

$$\left|\frac{\mathbf{0.95}}{\left(\frac{s}{1.9}\right)^2+\frac{2*1}{1.9}s+1}\right| < \left|\frac{L}{1+L}\right| < \left|\frac{\mathbf{1.05}}{\left(\frac{s}{4}\right)^2+\frac{2*0.707}{4}s+1}\right| \;\forall\omega$$

$$\left|\frac{1}{1+L}\right| < 0.05\ (-26\text{ dB})\text{ for low }\omega$$

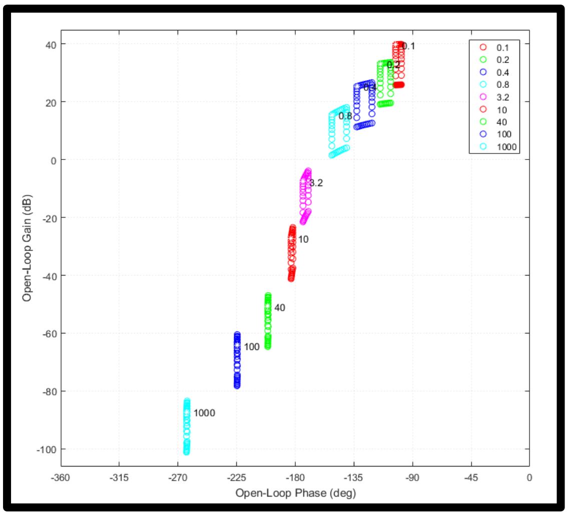
$$\left|\frac{1}{1+L}\right| < 3dB\ \forall\ \omega$$

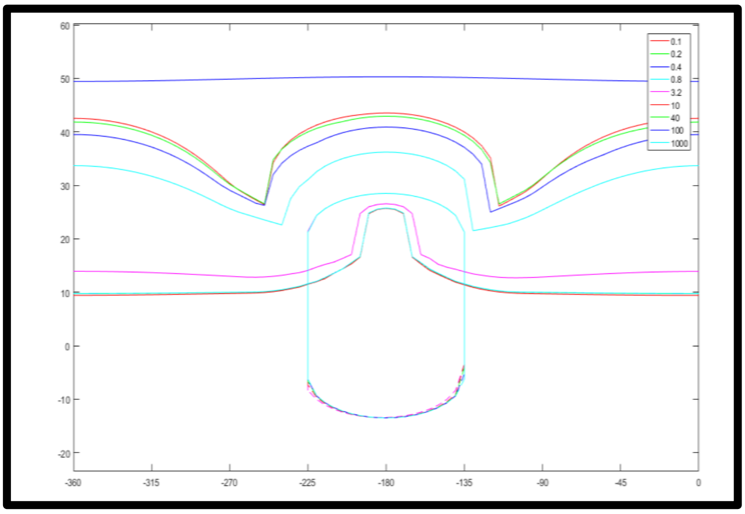
The plant is discretised and then taken to the w-domain so that the QFT design methods can be applied to it, while taking into account the phase lag that is introduced when sampling.

A nominal plant of A=6 and T = 1.9 was chosen:

$$P(s)=\frac{6}{1.9s^2+s}\rightarrow P(w)=\frac{-0.0315w+3.158}{w^2+0.5263w}$$

The plant template at different frequencies and the robust stability boundaries are shown graphically:

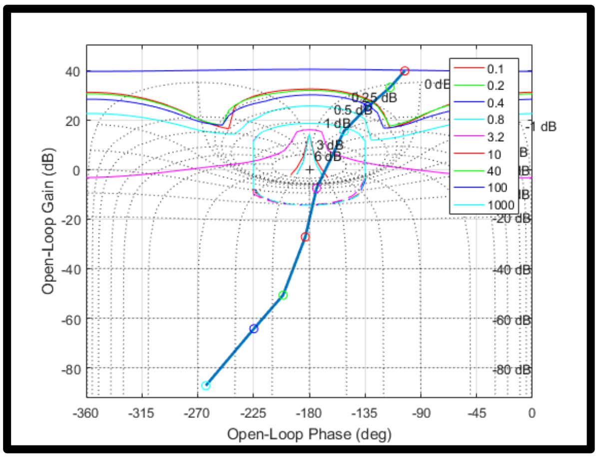




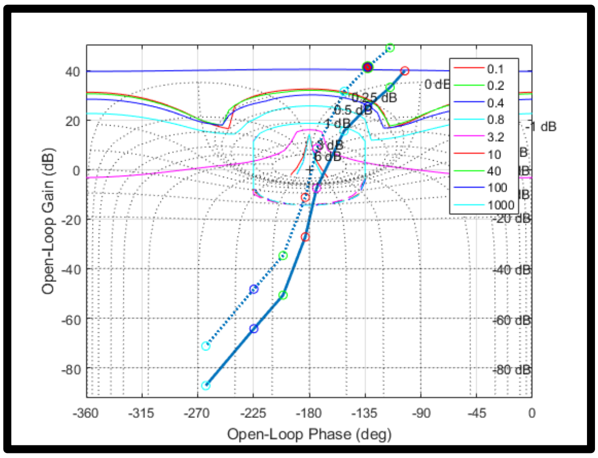
5.3.2 Controller Design

The following four pictures shows the design procedure from the initial system response to the finial response (that meets all the specification requirements).

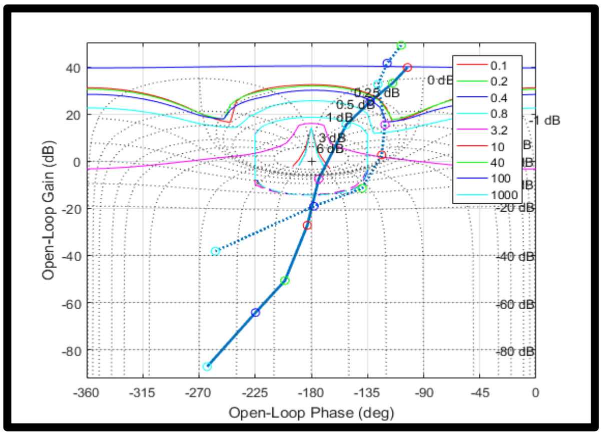
A



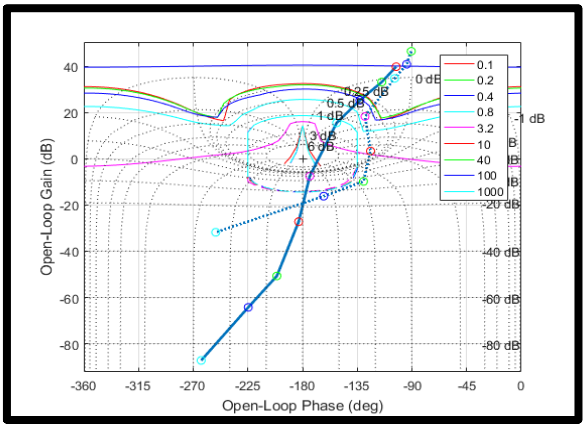
B



C



D



An explanation of the reasoning behind the design is given below:

A	B
The plant in closed loop does not meet the specification requirements. It goes through the 3dB circle (high frequencies are not robust enough) and the low frequencies require more gain. Thus, compensation is needed.	Gain is need to move the low frequencies points above the 40dB line (the blue line) to ensure no steady state error. Approximately 15dB of gain is added.
C	**D**
Two lead-lab factors are added to the controller to move the response to the right while maintaining a causal controller.	The gain and exact positions of the lead-lag factors were adjusted to 'just' meet the specification requirements, so as to not be over-engineered.

The finial controller came to be:

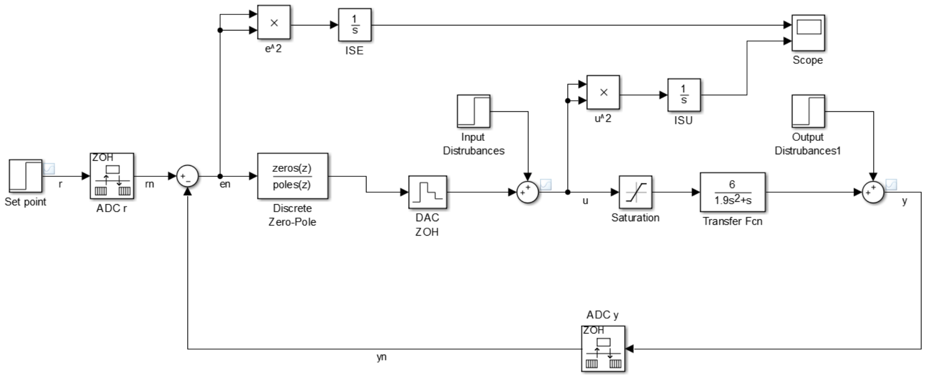
$$G(w) = \frac{602(s + 7.33)(w + 0.3588)}{(s + 218.2)(w + 1.784)}$$

Taking the bilinear transform:

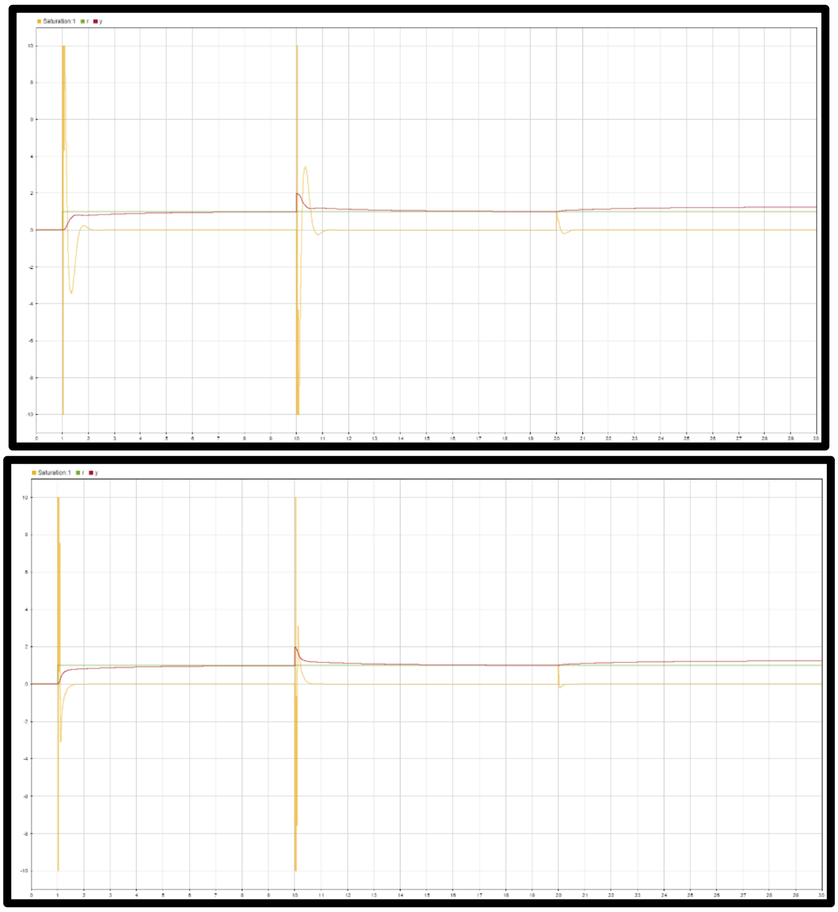
$$\boldsymbol{G(z) = \frac{200.47(z - 0.8634)(z - 0.99)}{(z + 0.3714)(z - 0.965)}}$$

5.3.3 Simulation

The following MATLAB Simulink model was used to simulated the system:



The set point was stepped at t =1s, the output disturbance at t=30s and the input disturbance at t=60s. The system had the following response for the nominal plant case (left) and high gain, high time constant case(right):

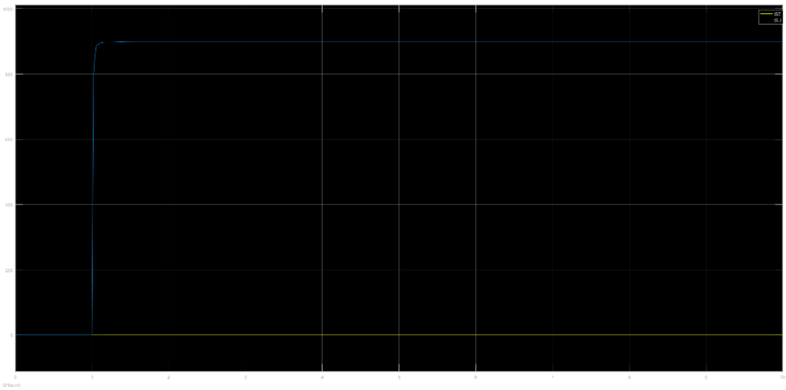


[Green: Set point, Red: Output y(t), Yellow: Plant input u(t)]

For the nominal plant case, the system's performance did not quite meet the specifications. The $t_{2\%}$ time was about 4s, although there was no overshot (0%). The controller rejected output disturbances quickly enough, however, input disturbances resulted in a steady state error. This is due to there being no integrator in the controller circuit. Due the very high gain of the controller, the plant inputs saturated, which would cause problems in a real implementation. For the high gain, high time constant plant case, the system was stable and performed similarly to the nominal plant case. The controller can thus be considered to be robust.

5.3.4 Controller Effort Analysis

The following shows the ISU and ISE of the nominal plant system (for only a step at the reference only):



[Blue: ISU, Yellow: ISE]

At steady state, the ISU and ISE values were constant and found to be:

$$ISU = 900$$

$$ISE = 3.98$$

The incredibly high ISU comes from the high gain of the controller.

5.4 Ziegler–Nichols PID Tuning Method

The Zielger-Nichols tuning method was developed by J.G. ZIEGLER and N. B. NICHOLS, presented in a paper they wrote called "Optimum Settings for Automatic Controllers" [2]. The method involves perturbing a system with a step input and changing a proportional feedback K_p until the output begins to oscillate with a period of T_u. At this point, the gain K_p has reached the 'ultimate gain' K_u. The method involves using a heuristic with the K_u gain and T_u period to give explicate values to the gains of a PID controller.

5.4.1 PID controller and the Tuning Heuristic

The transfer function of a discrete PID controller is given as:

$$PID(z) = K_p\left(1 + \frac{1}{T_i} * \frac{T_s}{z-1} + \frac{T_d}{\frac{T_d}{N} + \frac{T_s}{z-1}}\right)$$

Where:

- N is derivative filter divisor
- T_i is the integrator time constant
- T_d is the derivative time constant
- K_p is the proportional gain
- T_s is the sampling period

Depending on the type of controller to be implemented, there are different heuristics that are used:

Controller Type	K_p	T_i	T_d
P	$0.5K_u$	-	-
PI	$0.45K_u$	$\frac{T_u}{1.2}$	-
PID	$0.6K_u$	$\frac{T_u}{2}$	$\frac{T_u}{8}$
PID with some overshoot	$0.33K_u$	$\frac{T_u}{2}$	$\frac{T_u}{3}$

The nominal plant model the PID controller will be designed for is:

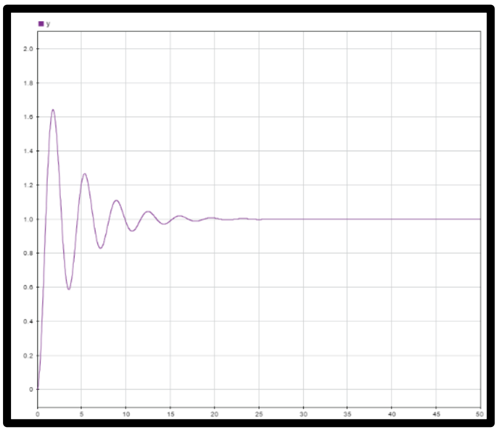
$$P(s) = \frac{6}{1.9s^2 + s}$$

The 'PID with some overshoot' heuristic will be used for the controller as it provides an improved $t_{2\%}$ time while maintaining an acceptable overshoot percentage.

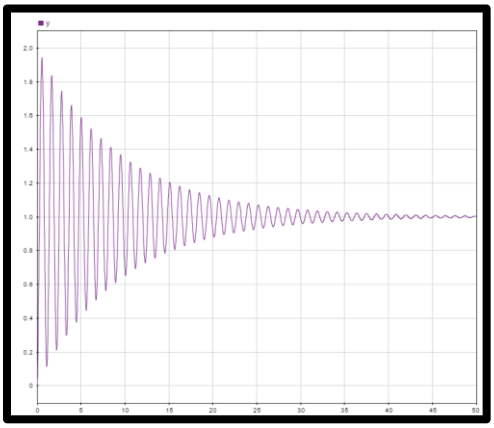
5.4.2 Find the tuning parameters

The plant was put into a closed loop with only a proportional gain. The follow are the results of vary the gain until the 'ultimate gain' was found:

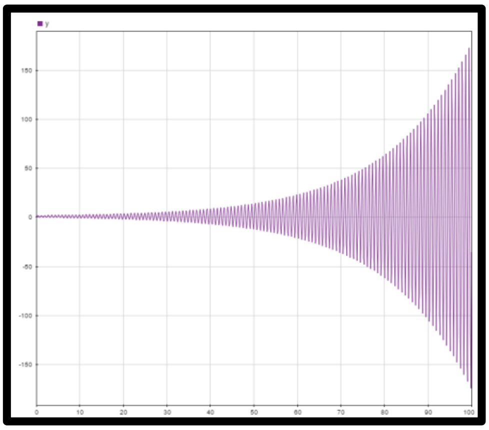
$K_p = 1$



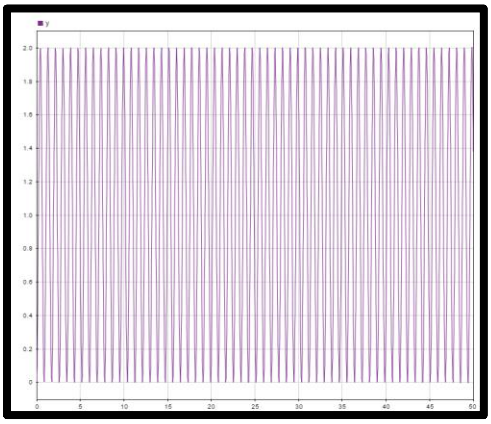
$K_p = 10$ more oscillations but they diminish



$K_p = 20$ Unstable



$K_p = 16.7 = K_u$ and T_u was found to be 0.85s



Now, the PID factors can be computed:

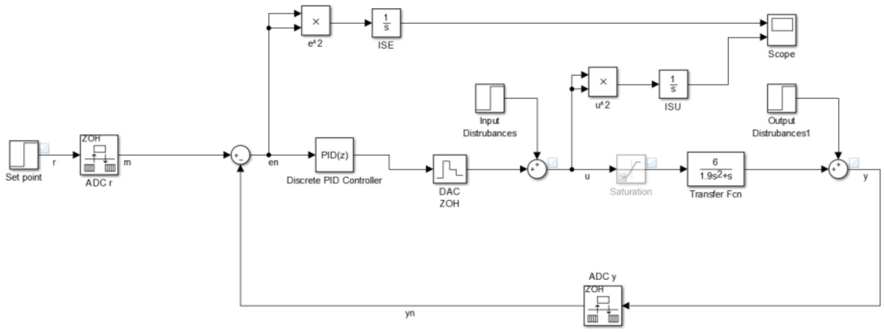
$$K_p = 5.51$$

$$T_i = 0.425$$

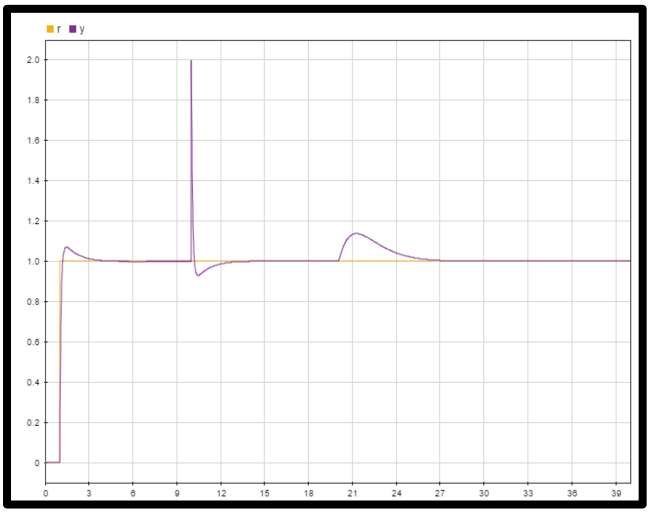
$$T_d = 0.283$$

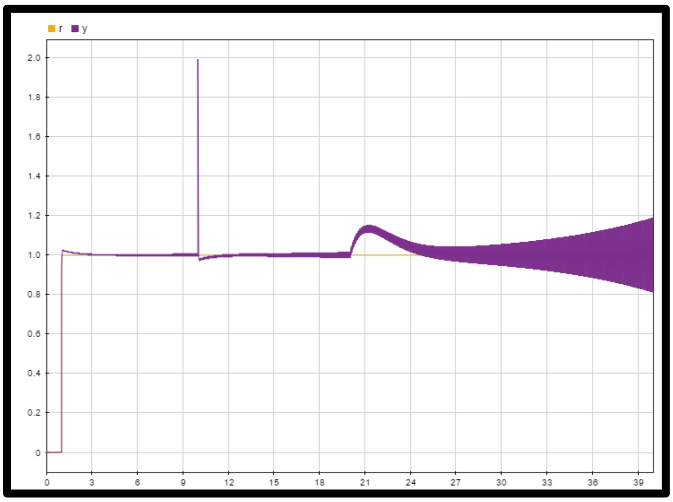
5.4.3 Simulation

The controller was simulated in MATLAB Simulink with the following model:



The response of the system to a step in the set point at t=1s, at the output disturbance at t=10s and at the input disturbance at t=20s. The plot on the left shows the response of the nonmail plant case and the right to the high gain, high time constant plant model.



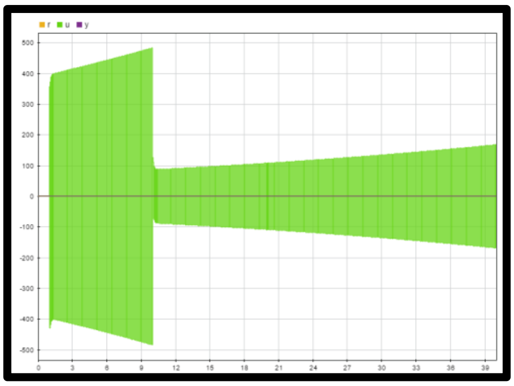


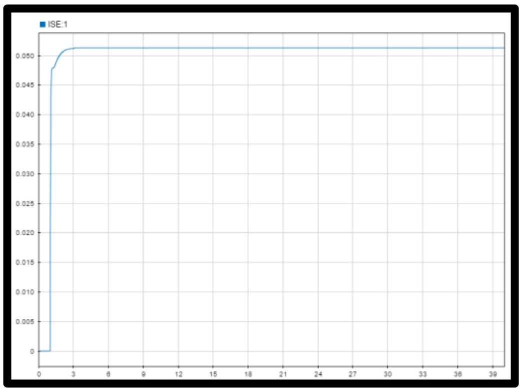
[Orange: Set point r(t), Purpale: Output y(t)]

The response of the system for the nominal plant case meets the specifications with a $t_{2\%}$ of 3 seconds, 8% overshoot with no oscillations and the controller effectivity mitigates input and output disturbances. However, the controller output (see section 5.4.4) was not stable and the controller was not robust to changes in the plant model, resulting in instability for the high gain, high time constant plant case, which does not meet the specifications.

5.4.4 Controller Effort Analysis

The plot on the left shows the controller output signal (the plant input). It is not stable and thus there is no ISU measure. The ISE signal was bounded and came to 0.515 in steady state for a step in the referance only for the nominal plant case.





6 Comparison of Designs and Final Selection

Three controllers were designed and simulated. In this section one must be selected for implementation. First the performance measures will be looked at, following that a comparative discussion on the three controllers will take place and finally one will be selected.

6.1 ISU and ISE for the controllers

Controller	ISU	ISE
State Feedback with IMC	0.14	0.81
QFT designed	900	3.98
Zielger-Nichols PID	Inf.	0.515

The state feedback controller had the lowest ISU and the Zielger-Nichols PID had the lowest ISE, however it also had an unmeasurable ISU. The QFT designed controller had very high gain and thus had a large ISU. The controller with the best overall ISU and ISE is the state feedback controller with IMC.

6.2 Comparative analysis

The QFT controller had a $t_{2\%}$ of around 4 seconds which does not meet the specifications, however it had a very ‘good’ response in that it got to the reference value with no overshoot and it was robust to changes in the plant model, but due to the high gain present in the controller, it saturated the plant input signal, which in a real implementation would cause unwanted behaviour in the system. Therefore, the QFT controller will not be selected. The Zielger-Nichols PID controller had good performance for the nominal model case, but when the plant model was changed, instability was introduced in the feedback loop. Additionality, the PID controller produced an unstable plant input signal which oscillated violently. In a real implementation, the controller would not be able to perform as it did in the simulation, especially considering it was not robust to changes in the plant model, therefore it will not be selected. That leaves the state feedback controller. It met the specifications after a slight modification to the integrator gain. It proved to be resilient to significant changes in the plant model, which was helped by the dedicated integrator present in the feedback loop, which integrated the error signal. The ISU and ISE measures for it are acceptable and thus I think it is a good choice.

6.3 Chosen controller

The state feedback controller with IMC was deemed to be the best controller to control the position of the DC motor.

7 Implementation of Selected Controller

7.1 Finding the differences equation of the relevant blocks

The selected controller only had gains and an integrator block. The integrator block is give by:

$$Int(z) = \frac{k_{int}T_s}{z-1} = \frac{u(z)}{e(s)} = \frac{0.02 * 3.5}{z-1}$$

$$\therefore u(z-1) = e(0.07)$$

$$\rightarrow u - uz^{-1} = 0.07ez^{-1}$$

$$\rightarrow u = 0.07ez^{-1} + uz^{-1}$$

Taking the inverse z-transform:

$$u_n = 0.07e_{n-1} + u_{n-1}$$

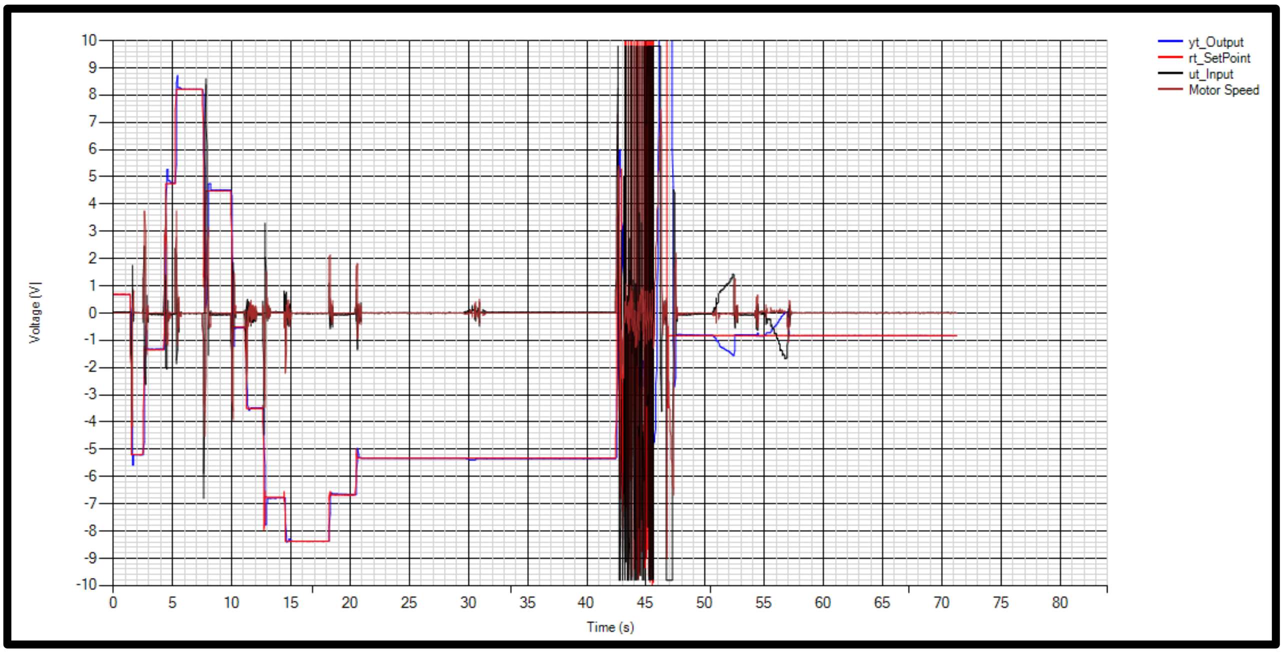
7.2 Implementation in C#

```
if (rt_SetPoint < -15)
{
    rt_SetPoint = -13;
}
if (rt_SetPoint > 15)
{
    rt_SetPoint = 13;
}


//calc error
en = yt_PlantOutput - rt_SetPoint;

ut_PlantInput = rt_SetPoint - (0.07*en1 + un1 + k1 * yt_PlantOutput + k2 * Velocity);
if (ut_PlantInput >= 9.8) ut_PlantInput = 9.8;
if (ut_PlantInput <= -9.8) ut_PlantInput = -9.8;

en1 = en;
un1 = ut_PlantInput;

// Send output to DAQ
string ControlLoop = comboBoxControlLoop.Text;
if (ControlLoop == "OPEN")
{ ut_PlantInput = rt_SetPoint; }

DAQ.Output(0, ut_PlantInput);
```

The initial values of en1 and un1 were 0. $k_1 = 3.5$, $k_2 = 1.6$. The IF states were implemented to prevent saturating the outputs.

7.3 Implemtaion Results

The state feedback implemtaion was ultimately successful in controlling the position of the motor. It tracked the reference within the specified time, rejected disturbances very well and handled changes in the plant gain gracefully. Below is a picture of the results:



8 Summary of Design

Three digital controllers were designed using state feedback, QFT design method and the Ziegler-Nichols PID tuning method. Each controller was tested in MATLAB Simulink, simulating the continuous and digital domains in the same simulation. The state feedback controller proved to be the most effective controller, it was robust to changes in the plant and met the specifications while mitigating input and output disturbances. The QFT design had too much gain, resulting in the saturation of the control signal and the PID controller was not robust to changes in the plant model and had an unstable control signal.

To achieve the controller designs, the system to be controlled was first analysed in order to understand the underlying dynamics. Step and ramp tests provided the data for the plant model. The plant had a simulated gain uncertainty in it (physically controlled by a potentiometer), thus the designed controllers had to be robust to changes in the plant model.

The implemented controller was successful in tracking the set point reference within the specified time, it rejected disturbances and performed well under all gain conditions.

References

[1] Boje, E, 2017. – Quantitative Feedback Theory (QFT) design.

[2] Ziegler, J.G. Nichols, N.B., 1942. Optimum settings for automatic controllers. *trans. ASME*, *64*(11).

[3] B. M, Control Engineering 1, Cape Town: South Africa, 1994.