Properties of Harmonic Functions

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1 Introduction

This project in mathematical analysis, studies harmonic functions on given a domain Ω, where Ω is an open subset in $\mathbb{R}^n$, for which $n \in \mathbb{N}$. The harmonic function u which equals to a function f on the boundary of the domain. Denote the boundary of Ω by $\partial\Omega$. We explore what sort of functions we can expect within Ω.

1.1 Laplace's Equations

The Laplace's equation are second order partial differential equations. For a function $u : \mathbb{R}^n \to \mathbb{R}$, where u is also twice differentiable. Functions which are continuous and twice differentiable we say are in C^2, thus $u \in C^2$. An n dimensional Laplace's equation is defined as

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \tag{1}$$

The Laplacian is an operator, which can be denoted by just Δ. Let f be a function in the domain. The non-homogeneous version of Laplace's equation is known as Poisson's equation

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = f. \tag{2}$$

1.2 Harmonic Functions

If a function $u : \mathbb{R}^n \to \mathbb{R}$ satisfies Laplace's equation $\Delta u = 0$ and is continuous with second derivatives that exist and are continuous too (in this case we say that $u \in C^2$), then it is called a harmonic function (Axler, Bourdon, & Wade, 2013). In more formal terms:

Definition 1.1. Let $\Omega \subseteq \mathbb{R}^n$ be an open and connected set where $n \in \mathbb{Z}^+$. A function $u \in C^2(\Omega)$ is called a harmonic function if $\Delta u = 0$ for all $x \in \Omega$.

Henceforth we let Ω denote an open and connected set subset of $\mathbb{R}^n$ throughout the rest of the paper. Now, here are three examples of harmonic functions in $\mathbb{R}^2$

1. $u = x + y$, where Ω is any open set on the plane

2. $u = x^2 - y^2$, where Ω is any open set on the plane

3. $u = ln(x^2 + y^2)$, where Ω is any open set on the plane that does include 0.

If in the third example Ω contained 0, then the function would not be harmonic, as u would not be in $C^2(\Omega)$. In particular u would not even be defined at 0. Therefore the second derivative would also not be continuous.

2 Mean Value Property

2.1 Mean Value Property in the plane

Harmonic functions have a variety of interesting properties. We shall now prove that all harmonic functions satisfy the First Mean Value Theorem for harmonic functions (Tisdell, 2018).

In the following theorem we introduce the path integral. The theorem essentially states that the value of u at any point in Ω is the same as the mean value of u over any circle around that point. In particular the expression in (4) below, in two dimensions, can be also written as

$$u(a, b) = \frac{1}{2\pi} \int_0^{2\pi} u(a + rcos(t),\ b + rsin(t))\ \ dt. \tag{3}$$

Theorem 2.1. Let u be a harmonic function in an open set $\Omega \subseteq \mathbb{R}^2$. We let $B_r(a, b)$ denote a ball with radius r centered at a point (a, b) in $\mathbb{R}^2$ with $\bar{B}_r(a, b) \subseteq \Omega$ then the First Mean Value Theorem states that:

$$u(a, b) = \frac{1}{2\pi r} \oint_{\partial B_r(a,b)} u(x, y) dS. \tag{4}$$

Proof. The proof for the first mean value theorem that is highlighted here requires the divergence theorem which is also known as Gauss's theorem. It says that if $\vec{F} = M(x, y)\vec{i} + N(x, y)\vec{j}$ then

$$\int\int_{B_r(a,b)} \nabla \cdot \vec{F} dxdy = \oint_{\partial B_r(a,b)} M dy - N dx.$$

Where $\vec{F}$ is a vector field in the place and both M and N are functions of x, y. So $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto \vec{F}(x, y)$, such that for each point $(x, y) \in \mathbb{R}^2$, $\vec{F}(x, y)$ is a vector depending on x, y. This theorem tells us that if you take the double integral of the divergence of vector field F then you can write it as a line integral over the boundary of the ball. As ∇u is also a vector field, we let $\nabla u = \vec{F}$ which means the components of $\vec{F}$ are the partial derivatives of u where $M = \frac{\partial u}{\partial x}$ and $N = \frac{\partial u}{\partial y}$. Then by the divergence theorem we get

$$\int\int_{B_R(a,b)} \nabla \cdot \nabla u \ dxdy = \oint_{\partial B_R(a,b)} \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx.$$

Notice here we are integrating over an arbitrary disc with radius R. As u is a harmonic function, the left hand side of the equation becomes 0 as $\Delta u = 0$ which gives us

$$0 = \oint_{\partial B_R(a,b)} \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx.$$

To evaluate a line integral the general idea is that we would need to parameterise the curve we are integrating over. Which would allow us to be able to reduce the line integral to a standard integral. Parameterising $\partial B_R(a,b)$ by using

$$\vec{r}(t) = (x(t),\ y(t)) = (a + R\cos(t),\ b + R\sin(t)) \qquad\qquad 0 \le t \le 2\pi$$

so that

$$\vec{r}\,'(t) = (x'(t), y'(t)) = (-R\sin(t), R\cos(t)) \qquad\qquad 0 \le t \le 2\pi.$$

$$dx = -R\sin(t)\ dt \qquad\qquad dy = R\cos(t)\ dt$$

Hence

$$0 = \oint_{\partial B_R(a,b)} \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx.$$

Now substituting in our values for dx and dy,

$$0 = \int_0^{2\pi} \frac{\partial u}{\partial x}(a + R\cos(t),\ b + R\sin(t)) R\cos(t) + \frac{\partial u}{\partial y}(a + R\cos(t),\ b + R\sin(t)) R\sin(t)\ \ dt.$$

By using the chain rule in reverse we get

$$0 = \int_0^{2\pi} \frac{\partial}{\partial R} u(a + R\cos(t),\ b + R\sin(t))\ \ dt$$

and now by Leibniz's rule we have

$$0 = \frac{d}{dR} \int_0^{2\pi} u(a + R\cos(t),\ b + R\sin(t))\ \ dt.$$

We are able the interchange the derivative and integral here by the Lebesgue's Dominated Convergence Theorem which we will discuss in a later chapter. Now to proceed we wish to exchange from R to r, to do this we integrate from $R = 0$ to $R = r$

$$0 = \int_0^r \frac{d}{dR} \int_0^{2\pi} u(a + R\cos(t),\ b + R\sin(t))\ \ dtdR.$$

By using the Fundamental Theorem of Calculus we compute

$$\begin{aligned} 0 &= \int_0^{2\pi} u(a + r\cos(t),\ b + r\sin(t))\ \ dt \ \ - \int_0^{2\pi} u(a,b)\ dt \\ &= \int_0^{2\pi} u(a + r\cos(t),\ b + r\sin(t))\ \ dt \ \ - 2\pi u(a,b) \end{aligned}$$

as the second integrand doesn't depend on t at all. This can then be easily rearranged to get us equation (3). $\square$

Proposition 2.1. If the First Mean Value Theorem holds then the function is harmonic.

Proof. This proof will be covered in the general sense in $\mathbb{R}^n$ in the following chapter. $\square$

come back to these bullet points, "i would make this much more precise. It is OK to be a bit repetitivie in order to claify matters. I suggest introducing a seperate definition environment." 26 minutes in to the recording

- A subharmonic function "$\Delta u \geq 0$" is created if an inequality is used rather than an equality. By following the proof in a similar manner we obtain the First Mean Value Theorem for subharmonic funcitons
$$u(a,b) \leq \frac{1}{2\pi r} \oint_{\partial B_r(a,b)} u(x,y)dS. \tag{5}$$
- A superharmonic function is obtained if we reverse the inequality "$\Delta u \leq 0$."

2.2 Mean Value Theorem in $\mathbb{R}^n$

In the previous subsection we stated and proved the Mean Value Theorem in the plane, i.e. in $\mathbb{R}^2$. Now that we have a better intuitive understanding of what the theorem is, we shall now define and prove it again but now extending the idea into n dimensions with the open set Ω now in $\mathbb{R}^n$. In this section we will introduce slightly different notation to the one we used in $\mathbb{R}^2$.

Definition 2.1. Let $B(x,r)$ be an open n^{th} dimensional ball in $\mathbb{R}^n$, centred at x with radius r it is defined as
$$B(x,r) = \{y \in \mathbb{R}^n | d(x,y) < r\}$$
where $d(x,y)$ is the Euclidean metric.

We now introduce a new integral notation, an integral with a dashed line across it, this represents an integral average which is the ordinary integral averaged out across the measure of the domain of integration. For example here the measure of the n^{th} dimensional ball B is the volume of B. The volume of B in $\mathbb{R}^n$ is equal to some constant c_n depending on dimension, multiplied by the radius of the ball to the power of n.

Theorem 2.2. Let $u \in C^2$ be a harmonic function in an open set $\Omega \subseteq \mathbb{R}^n$. The mean value theorem is defined as
$$u(x) = ⨍_{\partial B(x,r)} u \; dS = ⨍_{B(x,r)} u \; dV \tag{6}$$
for a ball centred on x, where x is a point in n dimensional space, with radius r in Ω, $B(x,r) \subset \Omega$.

Noting the above definitions for the integral average and the volume of the ball, eqaution (6) can be rewritten as such,
$$⨍_{B(x,r)} u \; dV = \frac{1}{c_n r^n} \int_{B(x,r)} u \; dV.$$
One can interpret c_n as the volume of the unit sphere in the n^{th} dimension. Looking at $\partial B(x,r)$ we can similarly conclude by working out the measure of the boundary of the ball, essentially calculating the surface area, we have
$$⨍_{\partial B(x,r)} u \; dS = \frac{1}{n c_n r^{n-1}} \int_{\partial B(x,r)} u \; dS.$$

The Mean Value Theorem here then states that the value of $u(x)$ is equal to the mean value of u over the surface of the sphere $\partial B(x, r)$ and $u(x)$ is equal to the mean value of u over the entire interior of the ball $B(x, r)$ as long as they are all in the open set Ω.

Proof. The following proof is based on (Evans, 2010, Section 2.2.2.) First we set

$$\phi(r) = \fint_{\partial B(x.r)} u(y)\ dS(y).$$

Which when written as an ordinary integral gives us

$$\phi(r) = \frac{1}{nc_n r^{n-1}} \int_{\partial B(x,r)} u(y)\ dS(y).$$

Next we substitute $y = x + rz$ with such values x and z so that we can map the boundaries of the ball to the unit ball centred at zero, while noting that $dS(y) = r^{n-1} dS(z)$, where r^{n-1} represents the Jacobian for the n dimensional change in variables. We get

$$\phi(r) = \frac{1}{nc_n r^{n-1}} \int_{\partial B(0,1)} u(x + rz)\ r^{n-1}\ dS(z),$$

cancelling out the r^{n-1} we are left with,

$$= \frac{1}{nc_n} \int_{\partial B(0,1)} u(x + rz)\ dS(z).$$

Note that nc_n works out the area of the boundary of a ball with dimension n when it has an unit radius which means by integral average notation we have

$$\phi(r) = \fint_{\partial B(0,1)} u(x + rz)\ dS(z).$$

What has been achieved by doing this, is that we have successfully removed the geometry from the integral, which has now been put it into the function instead, leaving the integral with just the unit ball.

Carrying on with the proof we now differentiate $\phi(r)$ with respect to r

$$\phi'(r) = \frac{\partial}{\partial r} \left[\fint_{\partial B(0,1)} u(x + rz)\ dS(z) \right].$$

Once again using the Lebesgue's Dominated Convergence Theorem covered in a later chapter, we differentiate this function with respect to r by using the multivariate chain rule, first differentiating the multi-variable function u with respect to all of its components $x_1, x_2, ..., x_n$ and then multiplying it with the differential of its $x_{j^{th}}$ component, where $1 \leq j \leq n$, with respect to r

$$\phi'(r) = \fint_{\partial B(0,1)} \left(\sum_{j=1}^{n} \left[\frac{\partial u}{\partial x_j}(x + rz) \frac{\partial}{\partial r}(x + rz)_j \right] \right) dS(z)$$

$$= \fint_{\partial B(0,1)} \left(\sum_{j=1}^{n} \left[\frac{\partial u}{\partial x_j}(x + rz)\ z_j \right] \right) dS(z).$$

Looking at the summation inside the integral we recognise it as the dot product of z and $\nabla u(x+rz)$, thus we have

$$\phi'(r) = \fint_{\partial B(0,1)} (\nabla u(x + rz) \cdot z)\, dS(z).$$

Now consequently, by putting the geometry back into the integral and mapping the unit ball back to $\partial B(x, r)$ by once again substituting $y = x + rz$ and using the inverse of the Jacobian from before, we can compute

$$\begin{aligned}
\phi'(r) &= \frac{1}{nc_n} \int_{\partial B(x,r)} \left(\nabla u(y) \cdot \frac{y - x}{r} \right) \frac{1}{r^{n-1}} dS(z) \\
&= \frac{1}{nc_n r^{n-1}} \int_{\partial B(x,r)} \left(\nabla u(y) \cdot \frac{y - x}{r} \right) dS(z) \\
&= \fint_{\partial B(x,r)} (\nabla u(y) \cdot \frac{y - x}{r}) dS(z).
\end{aligned}$$

This conveniently leads to

$$\phi'(r) = \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y)\ dS(z)$$

where ν represents the normal vector which is orthogonal to the tangent vector at point y on the surface of the ball B. We computed this last step by using the fact that if you wanted to know the derivative of u in some direction ν you take the full gradient of u and dot product it with the unit vector in that direction $\nabla u(y) \cdot \hat{\nu}(y)$. The vector $y - x$ points from the centre of the circle to the point on the boundary y. By the laws of simple euclidean geometry it is therefore the normal direction to the circle. As r is the radius of the circle this means

$$\hat{\nu}(y) = \frac{y - x}{r}$$

therefore

$$\frac{\partial u}{\partial \nu}(y) = \nabla u(y) \cdot \hat{\nu}(y) = \nabla u(y) \cdot \frac{y - x}{r}.$$

Add diagram here

Now using the Green's formulas (see Theorem 3 from [6])

THEOREM 3 (Green's formulas). *Let* $u, v \in C^2(\bar{U})$. *Then*

(i) $\int_U \Delta u\, dx = \int_{\partial U} \frac{\partial u}{\partial \nu}\, dS$,

(ii) $\int_U Dv \cdot Du\, dx = -\int_U u \Delta v\, dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u\, dS$,

(iii) $\int_U u\Delta v - v\Delta u\, dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu}\, dS$.

we can conclude the following,

$$\phi'(r) = \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y)\ dS(z) = \frac{1}{nc_n r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y)\ dS(z).$$

Using the first green's formula will turn the integral from averaging over the boundary of the ball to averaging over the solid ball

$$\phi'(r) = \frac{1}{nc_n r^{n-1}} \int_{B(x,r)} \Delta u(y)\ dy = \frac{r}{n} \fint_{B(x,r)} \Delta u(y)\ dy = 0.$$

Recall that $\Delta u = 0$ because it is a harmonic. It can be concluded now that ϕ is in fact constant as ϕ' is always equal to zero. We can use the fact that ϕ is constant to complete our proof since

$$\phi(r) = \lim_{t \to 0} \phi(t) = \lim_{t \to 0} \fint_{\partial B(x.r)} u(y)\ dS(y).$$

Finally by Lebesgue's Differentiation Theorem (Evans, 2010, Section E.4.), which holds for every x because u is continuous, (prove at the end if we have time) which states that all integral averages converge to the function value at the centre as we let the radius of the balls go to zero, we finally get

$$\phi(r) = \lim_{t \to 0} \phi(t) = \lim_{t \to 0} \fint_{\partial B(x.r)} u(y)\ dS(y) = u(x).$$

□

Theorem 2.3. If $u \in C^2(\Omega)$ satisfies the mean value property

$$u(x) = \fint_{\partial B(x,r)} u\ dS = \fint_{B(x,r)} u\ dV \tag{7}$$

then the function is harmonic

Proof. □

Thus concludes our proofs of the Mean-value formulas for Laplace's equation.

3 The Maximum Principle

3.1 The Maximum Principle in the plane

Harmonic functions satisfy the maximum principle which states that if u is a harmonic function, where u is on a bounded and connected set. If u attains its maximum value M within the set, then $u = M$ for all $x \in \mathbb{R}^2$. More formally:

Proposition 3.1. Let u be a harmonic function in Ω ($\Delta u = 0$), where $\Omega \subset \mathbb{R}^2$ is an open and connected set. If u achieves maximum M at a point $(x_0, y_0) \in \Omega$ then $u \equiv M$ for all $(x, y) \in \Omega$.

Rigorous definitions of open and connected sets can be found in Section 3.2. Proposition 3.1 can be changed into the maximum principle for subharmonic functions.

Proposition 3.2. Let u be a subharmonic function in Ω "$\Delta u \geq 0$", where $\Omega \subset \mathbb{R}^2$ is an open and connected set. If u achieves maximum M at a point $(x_0, y_0) \in \Omega$ then $u \equiv M$ for all $(x, y) \in \Omega$.

We shall focus here on proving 3.2, the proof of 3.1 is very similar.

Proof. Let $(x_0, y_0) \in \Omega$ where Ω is an open an connected set. There exists $r > 0$ such that $B_r(x_0, y_0) \subset \Omega$ and since that $u \leq M$ and $u(x_0.y_0) = M$ then it follows from mean value theorem for subharmonic functions (4) we have

$$M = u(x_0, y_0) \leq \frac{1}{2\pi r} \oint_{\partial B_r(a,b)} u(x, y) dS \leq M$$

Obviously if this was the proof for harmonic functions then we would use equation (3) instead. The only way to satisfy this mean value property would be to have $u(x_0, y_0) = u(x, y) = M$ for all $(x, y) \in B_r(x_0, y_0)$. Now let's assume there exists a point $(x_n, y_n) \in \Omega$ such that $u(x_n, y_n) < M$. The mean value property would also hold in the neighbourhood of (x_n, y_n). We can connect (x_n, y_n) and (x_0, y_0) with a continuous curve. Moving along the curve in the direction towards (x_n, y_n) from (x_0, y_0) there has to be a last point on the curve where $u = M$, as at our destination $(x_n, y_n) < M$. Let this point be (x_1, y_1). This point has all of the same properties as (x_0, y_0), namely there exists a ball at around that point where $u = M$. This is a contradiction as u can not be identical to M on any sufficiently small circle centred at (x_1, y_1) as we have established this point to be the last point where $u = M$. This therefore proves the result that there can not exist a point where u is less than the maximum. Thus $u = M$ for all $(x, y) \in \Omega$. □

Exchanging u with $-u$ we get the following corollary.

Corollary 3.1. Let u be a superharmonic function in Ω "$\Delta u \leq 0$", where $\Omega \subset \mathbb{R}^2$ is an open and connected set. If u achieves minimum m at a point $(x_0, y_0) \in \Omega$ then $u \equiv m$ for all $(x, y) \in \Omega$. This is the minimum principle.

Using the above results we can conclude that the only way for a subharmonic function u to be non-constant would be so that it does not achieve its max or min within $\bar{\Omega}$ but to attain it on the boundary $\partial\Omega$.

Proposition 3.3. Every non-constant subharmonic function on a bounded set Ω must achieve its maximum on $\partial\Omega$.

Proposition 3.4. Let u be a harmonic function in Ω. Then u must attain its maximum and minimum values on the boundary $\partial\Omega$.

3.2 Open, Closed and Connected Sets

Before we progress further let us first define some topological definitions that we have already been using in a more mathematically rigorous manner.

Definition 3.1 (Open and closed sets)**.** Let (X, d) be a metric space, where $U \subseteq X$.

1. Then U is open if for all points x in U, there exists a positive real number ϵ where ϵ is greater than zero such that there exists a ball around point x of radius ϵ, where the ball is also in U, that is the following holds (see Figure 1):

$$\forall x \in U \;\; \exists \epsilon > 0, \;\; s.t. \; B(x, \epsilon) \subseteq U$$

.

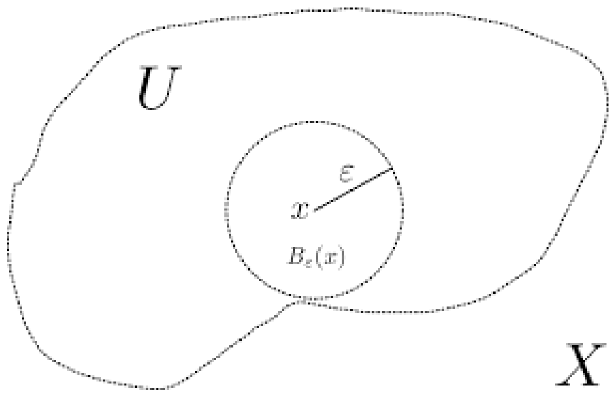

Figure 1: Ball centred at x with radius ϵ inside set U.

2. U is closed if $X - U = \{p \in X \ \ s.t. \ p \notin U\}$ is open.

Remark. U can be both open and closed (clopen) or neither.

Here is an example of a set that is both open and closed in $\mathbb{R}$ with the standard modulus metric $d(x, y)$. We can take U to be the entire set such that $U = \mathbb{R}$. Then $\mathbb{R} - U = \emptyset$. The empty set is open, trivially because there is nothing in the empty set. Therefore U is closed, but U is also open because if we consider $x \in U = \mathbb{R}$ then the ball $B(x, \epsilon) \subset \mathbb{R} = U$, where $\epsilon \in \mathbb{R}$ and is greater than 0. This then further implies that the empty set is also a clopen set. Besides the empty set and the entire set of $\mathbb{R}$, there are no other subsets of $\mathbb{R}$ that are clopen. This conclusion can be drawn due to the concept of connected spaces.

Definition 3.2 (Connected space)**.** Let (X, T) be a topological space in $\mathbb{R}^n$. We say that X is connected if it can not be expressed as a union of two open, non-empty and disjoint subsets of X.

Proposition 3.5. As a result of Definition 3.2 we can conclude the only clopen sets of a connected space X in $\mathbb{R}^n$ are X and $\emptyset$.

Proof. Let A be a clopen subset of X where X is a connected space in $\mathbb{R}^n$. Assume that A is non-empty. Then we can define $X = A \cup (X - A)$. When A is clopen, both A and $(X - A)$ are open sets. However this is a contradiction as X is connected so A and $(X - A)$ cannot be both open, unless they represent the sets X and $\emptyset$. $\square$

Definition 3.3 (Connected set)**.** A set $S \subseteq \mathbb{R}^n$ is called connected if there do not exist open sets U_1, $U_2 \subset \mathbb{R}^n$ such that $S \subseteq U_1 \cup U_2$ and

$$S \cap U_1 \neq \emptyset, \quad S \cap U_2 \neq \emptyset, \quad S \cap U_1 \cap U_2 \neq \emptyset.$$

Definitions above based on those found in (Munkres, 2000, Section 3.23).

3.3 The Maximum Principle for $\mathbb{R}^n$

In Section 3.1 we were able to gain a intuitive grasp of the maximum principle in the plane. Now, we shall state and prove it in the space of $\mathbb{R}^n$. In the following, Ω denotes an open subset of $\mathbb{R}^n$.

Definition 3.4 (Strongly harmonic)**.** We say that u is a strongly harmonic function in Ω if u is twice continuously differentiable in the open set Ω and continuously differentiable on the closure of Ω, so $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

In Definition 3.4, $\bar{\Omega}$ represents the closure of the set, we include this so that u is not a function that will tend to infinity as we approach the boundary. This is because of the Extreme Value Theorem which states that continuous functions on closed and bounded sets had closed and bounded range.

Theorem 3.1. Let u be a strongly harmonic fuction in Ω (where $u \in C^2(\Omega) \cap C(\bar{\Omega})$).

1. If u achieves its maximum value in $\bar{\Omega}$, then it must attain the maximum value somewhere on $\partial\Omega$.
2. Additionally, if Ω is a connected set and there is a point $x_0 \in \Omega$ where $u(x_0)$ is the maximum of u, then u is constant within all of Ω.

Similar to what we did with the maximum principle in the plane, exchanging u with $-u$ gives us an analogous minimum principle i.e. the above statements hold where each instance of maximum is replaced by minimum.

Proof. Let $x_0 \in \Omega$ be a point where the function u achieves its maximum $u(x_0) = M$. There exists $r > 0$ such that $B(x_0, r) \subset \Omega$ then it follows from Theorem 3.1

$$M = u(x_0) = \fint_{B(x_0,r)} u \, dV \leq \fint_{B(x_0,r)} M \, dV = M. \tag{8}$$

This last equality is true because M is a constant so it can be taken to the front of the integral average

$$\fint_{B(x_0,r)} M \, dV = M \fint_{B(x_0,r)} 1 \, dV = M \frac{1}{|B(x_0,r)|} \int 1 dV = M. \tag{9}$$

Equality 8 will only hold if $u \equiv M$ inside $B(x_0, r)$ why?. Therefore $u(y) = M$ for all $y \in B(x_0, r)$. Let $S = \{x \in \Omega \mid u(x) = M\}$. Thus the set S is an open set by Definition 3.1 as regardless of the point in S, a ball of radius r can always be placed around it as we have clearly shown using the mean value property. The set is also a closed set relative to Ω because of a property of continuous functions. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous if the pre-image of every closed set is closed, see (Munkres, 2000, Section 2.18). The set $\{M\}$, where continuous function $u(y) = M$, is certainly a closed set, therefore it's pre-image $u^{-1}(M) = \{y : u(y) = M\}$ is then also a closed set. Which means that the set $\{x \in \Omega \mid u(x) = M\}$ is clopen. Then following from the properties of clopen sets and connected spaces, the set equals Ω if Ω is connected. $\square$

4 The Perron Method

Now that we are armed with the Maximum Principle and the Mean Value Theorem we are able to illustrate the Perron method, the following is based on (John, 1991, Chapter 4 Section 4). This method aims to give a proof of the existence of solutions for the Dirichlet problem using subharmonic functions.

Definition 4.1 (Dirichlet problem)**.** Let Ω be a bounded, open and connected set. The aim is to find $u \in C^2(\Omega) \cap C(\overline{\Omega})$, such that for a given boundary data $f \in C(\partial\Omega)$ it holds that

$$\Delta u(\xi) = 0 \quad \forall \xi \in \Omega$$
$$u(\xi) = f(\xi) \quad \forall \xi \in \partial\Omega.$$

So the objective of the Perron method is to find functions that satisfy $u \in C^2(\Omega) \cap C(\overline{\Omega})$, where u is harmonic, given continuous boundary data. It is important to notice that in the definition of the Dirichlet problem the ξ represents something different on each line. In the first equation ξ is in Ω and in the second equation ξ is on the boundary of Ω. Assume we have a function w that is a solution to the Dirichlet problem. So w satisfies all our requirements from Definition 4.1

$$w \in C^2(\Omega) \cap (\overline{\Omega})$$

$$\Delta w = 0 \quad \text{in } \Omega$$
$$w = f \quad \text{on } \partial\Omega.$$

This is a big assumption that we can actually find such a w and that there exists a solution to the Dirichlet problem. Now let u be a subharmonic function in Ω. This is more likely to be true compared to our previous assumption as being subharmonic is a much less stringent restriction, that the laplacian is greater than or equal to 0 rather than being a strict equality. Suppose that there exists

$$u \in C^2(\Omega) \cap (\overline{\Omega})$$
$$\Delta u \geq 0 \quad \text{in } \Omega$$
$$u \leq f \quad \text{on } \partial\Omega.$$

Suppose such a u and w has been obtained. Then $u - w \leq 0$ on the boundary $\partial\Omega$ because

$$u \leq f = w \quad \text{on } \partial\Omega$$

$$u \leq w \quad \text{on } \partial\Omega$$
$$u - w \leq 0 \quad \text{on } \partial\Omega.$$

Hence, by the maximum principle when applied to $u - w$

$$u - w \leq 0 \quad \text{in } \Omega$$

as

$$\Delta(u - w) = \Delta u - \Delta w$$

$$= \Delta u - 0 = \Delta u \geq 0.$$

Therefore $u - w$ is subharmonic, so by the maximum principle for subharmonic functions does this need to explained further in the section above?

$$(u - w)(\xi) \leq \max_{\partial\Omega} (u - w) \quad \forall \xi \in \Omega.$$

The sum of two subharmonics functions is subhamonic by linearity property of derivatives and therefore we know that if on the boundary their difference is less than or equal to zero, then that result must extend to everywhere else on the domain. So by this result we can conclude that

$$u(\xi) \leq w(\xi) \quad \forall \xi \in \Omega$$

So the value of our solution w of the Dirichlet problem will be larger than or equal to the value of u at ξ at every single point inside the domain over all possible such sub solutions. As a result of this, a good ansatz (guess) for what w is, might be to take the supremum of u

$$w(\xi) := \sup\{u(\xi) : u(\xi) \leq w(\xi) \quad \forall \xi \in \Omega\}.$$

This extremal property tells us that if we fix an ξ in Ω, the solution, if it exists, will be bigger than or equal to u at every ξ which is a sub solution. Note that the aim is to find ξ in Ω, because the solution on the boundary is already known to be equal to f be definition.

Definition 4.2 (Perron function)**.** Suppose that f is continuous on the boundary of the domain. Define the Perron function $\omega_f : \Omega \to \mathbb{R}$ by $\omega_f(\xi) = \sup\{u(\xi) : u(\xi) \leq w(\xi) \quad \forall \xi \in \Omega\}$.

The motivation of the Perron method is to prove that the Perron function is also a harmonic function and satisfies

$$\Delta w = 0 \quad \text{in } \Omega.$$

As it is defined, the Perron function is the supremum of all values of u. And the subharmonic function u satisfies the equation but with an inequality. A theory which one might come up with right now is that if we were to take the supremum of u, it will push $u \leq w$ to an equality. This is the current motivation preceding forwards. The Perron method takes full advantage of the maximum principle and the solvability of the Dirichlet problem for a ball, which we will come back to in a later chapter. Look at this bit after you finished the solvability of the Dirichlet problem of a ball.

Now that we are familiar with the motivation we can work to begin constructing the solution to the Dirichlet problem by using the extremal property and a sequence of lemmas which we will prove and verify.

Denoting, as we have done similarly in the previous sections, $B(\xi, r)$ to be an open ball with centre ξ and radius r in $\mathbb{R}^n$. Then $\overline{B}(\xi, r)$ is the closure of this ball and $\partial B(\xi, r)$ is its boundary. For a continuous function u, we define the mean value of u on ∂B as

$$M_u(\xi, r) := \fint_{\partial B(\xi, r)} u \; dS.$$

Let Ω be an open, connected and bounded set in $\mathbb{R}^n$. In this section we will define subharmonic functions as such.

Definition 4.3 (Subharmonic function)**.** We call $u \in C^2(\Omega)$ such that $\Delta u \geq 0$ a subharmonic function. We denote the set of all subharmonic functions as $s(\Omega)$.

We now provide a corresponding definition for weakly subharmonic functions.

Definition 4.4 (Weakly subharmonic function)**.** If $u \in C^0(\Omega)$ and

$$u(\xi) \leq M_u(\xi, r) \ \ \forall \xi \in \Omega \ \ \forall r > 0, \tag{10}$$

and r is sufficiently small so that the ball does not cross the boundary of Ω at any point, that is the following holds

$$\forall r > 0 \text{ s.t. } B(\xi, r) \subseteq \Omega,$$

then u is a weakly subharmonic function. We denote the set of functions which are weakly subharmonic in Ω as $\sigma(\Omega)$.

Proposition 4.1. If a function u is subharmonic in the normal sense, then it is also weakly subharmonic.

Proof. If u satisfies Definition 4.3 then by the mean value theorem for subharmonic functions we know that

$$u(\xi) \leq M_u(\xi, r)$$

and if $u \in C^2(\Omega)$, that is, u continuously twice differentiable, then $u \in C^0$. Therefore we can conclude that $s(\Omega) \subseteq \sigma(\Omega)$. □

Lemma 4.1. Weakly subharmonic functions is an additive property. That is to say the sum of two weakly subharmonic functions is subharmonic.

Proof. Let u_1, $u_2 \in \sigma(\Omega)$. We have that

$$u_1 \leq M_{u_1}(\xi, r) = \fint_{\partial B(\xi,r)} u_1 \, dS,$$

$$u_2 \leq M_{u_2}(\xi, r) = \fint_{\partial B(\xi,r)} u_2 \, dS.$$

Adding together the above inequalities we have

$$u_1 + u_2 \leq \fint_{\partial B(\xi,r)} u_1 \, dS + \fint_{\partial B(\xi,r)} u_2 \, dS = \fint_{\partial B(\xi,r)} u_1 + u_2 \, dS = M_{u_1+u_2}(\xi, r).$$

By this we have proved that $u_1 + u_2 \in \sigma(\Omega)$. □

Lemma 4.2. Let u be a weakly subharmonic function such that $u \in \sigma(\Omega) \cap C^0(\overline{\Omega})$ then the maximum principle for subharmonic functions also holds, we have

$$\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u. \tag{11}$$

Proof. Let u be a weakly subharmonic function such that $u \in \sigma(\Omega) \cap C^0(\overline{\Omega})$. Weakly harmonic functions means that we are assuming the mean value inequality $u(\xi) \leq M_u(\xi, r)$. Once we have the mean value inequality you can prove the maximum principle in the exact same way we did before in Section 3.3. The solution vanishes when we recognise it is the algebraic property of Definition 4.4 that gives us the maximum principle for weakly subharmonic functions. □

To aid us in our search for the solution to the Dirichlet problem we now introduce a definition for what we will call $u_{x,r}$ but first, we will expand a bit more on the potential solutions for the Dirichlet problem. We say that the Dirichlet problem on Ω is **well-posed** if there exists a unique solution to the Dirichlet problem (on Ω) for each boundary data $f \in C(\partial\Omega)$.

Theorem 4.1. If $\Omega = B(\xi, r)$, then the Dirichlet problem is well-posed.

This means that Ω is a ball and that we can find a solution and this solution is unique. We have mentioned this before at the start of this chapter, the solvability of the Dirichlet problem for a ball remains a key ingredient to our construction of w. This will remain to be proved in a later section. Moving ahead for now, we will give a definition for $u_{\xi,r}$.

Definition 4.5. Given a function $u \in C^0(\Omega)$ and a ball $B(\xi, r)$, where $\overline{B}(\xi, r) \subset \Omega$. We define $u_{\xi,r}$ as a function that is continuous in Ω and that

$$u_{\xi,r}(x) = u(x) \text{ for } x \in \Omega - B(\xi, r) \tag{12}$$

$$\Delta u_{\xi,r}(x) = 0 \text{ for all } \quad x \in B(\xi, r) \tag{13}$$

Note that for $u_{\xi,r}$ inside the ball it is a solution to the Dirichlet problem in $B(\xi, r)$ according to Theorem 4.1, with the boundary data of $\partial B(\xi, r)$ equal to u as the ball given in equations 12 and 13 is an open ball. We have $u_{\xi,r}$ is a solution to the Dirichlet problem in $B(\xi, r)$ where the boundary data $f = u$ at $\partial B(\xi, r)$. Furthermore it can be concluded that the function $u_{\xi,r}$ is continuous everywhere in Ω. By definition, a solution of the Dirichlet problem is continuous everywhere in the closure of the set, so $u_{\xi,r}$ is continuous in the closed ball. Therefore it is continuous everywhere as on the boundary of the ball $u_{\xi,r}$ is equal to u.

Lemma 4.3. For a weakly subharmonic function $u \in \sigma(\Omega)$, and a closed ball $\overline{B}(\xi, r) \subset \Omega$ we have

$$u(x) \leq u_{\xi,r}(x) \quad \forall x \in \Omega \tag{14}$$

$$u_{\xi,r} \in \sigma(\Omega). \tag{15}$$

We have already established that $u_{\xi,r}$ is continuous everywhere in Ω, this lemma tells us that $u_{\xi,r}$ is also weakly subharmonic on the whole domain and that specifically inside the ball we are now saying that this function we have manufactured is in fact larger than or equal to u. Note that the first inequality is not saying much for values outside of the ball because by definition of $u_{\xi,r}$ it equals to u outside of the open ball. The proof of Lemma 4.3 is as follows:

Proof. To prove inequality 14 for the case where $x \in \Omega - B(\xi, r)$ by equation 12 from Definition 4.5 we have that $u_{\xi,r}(x) = u(x)$. Thus inequality 14 holds with an equality in this case. By definition of the lemma, it is already established that $u \in \sigma(\Omega)$ in Ω. If we were to restrict u to the ball

$B(\xi, r)$, inside this ball u is still a weakly subharmonic function. We note that if a function is subharmonic in Ω and then we restrict it to a smaller subset it is still subharmonic in that set. In Definition 4.4 we said that for **every** ξ in Ω we have the inequality $u(\xi) \leq M_u(\xi, r)$ for small enough r that it stays inside our ball. Therefore $u \in \sigma(B(\xi, r))$.

Furthermore, inside $B(\xi, r)$, by equation 13 from Definition 4.5 $u_{\xi,r}$, is a harmonic function. Then by elementary property of harmonic functions $-u_{\xi,r}$ is also a harmonic function inside the ball. It can be further concluded by Lemma 4.1 that $u - u_{\xi,r} \in \sigma(\Omega)$. From equation 12, we are told that on the boundary of the ball $u = u_{\xi,r}$, as a result we have $u - u_{\xi,r} = 0$ on the boundary. It follows from Lemma 4.2, the maximum principle for weakly harmonic functions, that $u - u_{\xi,r} \leq 0$ inside $B(\xi, r)$. The closed ball is controlled by the maximum on the boundary, but the maximum of the boundary is zero. This then fully proves inequality 14.

In order to prove 15, by using the definition for $\sigma(\Omega)$, we need to prove that

$$u_{\xi,r}(z) \leq M_{u_{\xi,r}}(z, t) \tag{16}$$

for sufficiently small t and z is an arbitrary point within Ω. Sufficiently small in this case, similar to before means that $t > 0$ and that the ball $B(z, t)$ stays inside the domain. Again we note that for $\xi \in \Omega - B(\xi, r)$ this is trivial as the left hand side of inequality 16 is equal to u which is weakly harmonic by definition, so inequality 16 holds outside of the ball $B(\xi, r)$.

Inside the ball by Definition 4.5, $u_{\xi,r}$ is a genuinely harmonic function which means that it is a subharmonic function, by Proposition 4.1 this means that it is also a weakly harmonic function. Then $u_{\xi,r} \in \sigma(\Omega)$ inside the ball.

To prove the last case where $z \in \partial B(\xi, r)$ we first recognise that because $u \leq u_{\xi,r}$ for all $\xi \in \Omega$,

$$M_u(z, t) := \fint_{\partial B(z,t)} u \, dS \leq \fint_{\partial B(z,t)} u_{\xi,r} \, dS$$

we have that $M_u(z, t) \leq M_{u_{\xi,r}}(z, t)$. Moreover, on the boundary of $B(\xi, r)$ we have that $u_{\xi,r} = u$. Putting these facts together we can see that $u_{\xi,r} \in \sigma(\Omega)$ on the boundary of $B(\xi, r)$ as

$$u_{\xi,r}(z) = u(z) \leq M_u(z, t) \leq M_{u_{\xi,r}}(z, t),$$

as a result fully completing our proof for 15 in the above Lemma.

add graphs ☐

Lemma 4.4. If u is a weakly harmonic function $u \in \sigma(\Omega)$ then the inequality from Definition 4.4 $u(\xi) \leq M_u(\xi, r)$ holds for closed balls whenever $\overline{B}(\xi, r) \subseteq \Omega$.

Proof. Let $\xi \in \Omega$ and $r > 0$ such that $\overline{B}(\xi, r) \subseteq \Omega$, by Lemma 4.3 we know that

$$u(\xi) \leq u_{\xi,r}(\xi)$$

where ξ is the centre of the ball, so it is certainly in Ω. Moreover, it is the centre of the closure of the ball with a positive radius, so even if the boundary of the closed ball touches the boundary of Ω, ξ would still be in Ω. Now, by (13) of Definition 4.5 we know that the function $u_{\xi,r}$ is strongly harmonic at its center i.e. $\Delta u_{\xi,r}(\xi) = 0$. Thus we can use the Mean Value Theorem (Theorem 2.2), which gives us an equality for harmonic functions

$$u_{\xi,r}(\xi) = M_{u_{\xi,r}}(\xi, r).$$

In the statement of the Mean Value Theorem, the function u does not need be harmonic on the closure of the ball, just on the open ball therefore we are safe in applying the Mean Value Theorem here. Further by (12) of Definition 4.5, we know that $u_{\xi,r}(x) = u(x)$ for all $x \in \partial B(\xi, r)$. Giving us

$$\fint_{\partial B(\xi,r)} u_{\xi,r} \, dS = M_{u_{\xi,r}}(\xi, r) = M_u(\xi, r) = \fint_{\partial B(\xi,r)} u \, dS.$$

Here, note that we are integrating around the boundary of the ball, therefore all of those points are not in the open ball but are in the closed ball, so they are in Ω. Thus when we put these three parts together it gives us

$$u(\xi) \leq u_{\xi,r}(\xi) = M_{u_{\xi,r}}(\xi, r) = M_u(\xi, r).$$

□

The key advancement we have made here is that we can now incorporate the closure of the ball into the definition of $u \in \sigma(\Omega)$. By Definition 4.4, which is if $u(\xi) \leq M_u(\xi, r)$ holds for each $\xi \in \Omega$ and $r > 0$ such that $B(\xi, r) \subseteq \Omega$, by this new lemma, Lemma 4.4 we can now replace $B(\xi, r) \subseteq \Omega$ with $\overline{B}(\xi, r) \subseteq \Omega$.

Lemma 4.5. A necessary and sufficient condition for u to be a harmonic function in Ω is that u and $-u$ are both weakly harmonic.

Proof. Assume u is harmonic in Ω. Then the Mean Value Theorem (Theorem 2.2) says that

$$M_u(\xi, r) = u(\xi) \;\; \forall \xi \in \Omega \text{ such that } B(\xi, r) \subseteq \Omega$$

We are trying to deduce that u and $-u \in \sigma(\Omega)$. We know u is harmonic, this implies

$$u(\xi) = M_u(\xi, r) \leq M_u(\xi, r),$$

therefore

$$u \in \sigma(\Omega).$$

Multiplying the Mean Value Theorem by -1 on both sides we compute

$$-u(\xi) = -M_u(\xi, r),$$

$$-u(\xi) = -\fint_{\partial B(\xi,r)} u \, dS,$$

$$-u(\xi) = \fint_{\partial B(\xi,r)} -u \, dS,$$

$$-u(\xi) = M_{-u}(\xi, r) \leq M_{-u}(\xi, r).$$

Therefore we have $-u \in \sigma(\Omega)$. Now we need to prove the converse, that is assuming $u \in \sigma(\Omega)$ and also $-u \in \sigma(\Omega)$ we need to show that u is harmonic. By Lemma 4.4 we have that

$$u(x) \leq u_{\xi,r}(x)$$

$$-u(x) \leq -u_{\xi,r}(x) \ \ \forall \ \ x \in \Omega.$$

Assuming the closed ball $\overline{B}(\xi, r) \subseteq \Omega$, we multiply the second inequality by -1 to get

$$u_{\xi,r}(x) \leq u(x) \leq u_{x,r},$$

therefore

$$u(x) = u_{\xi,r}(x) \ \forall x \in \Omega$$

Hence, by what we defined $u_{\xi,r}$ to be in Definition 4.5, we have that $u(x)$ is harmonic in $B(\xi, r)$ because $u_{\xi,r}$ is harmonic in $B(\xi, r)$.

□

Lemma 4.6. If u is continuous in Ω, and that for each $\xi \in \Omega$

$$u(\xi) = M_u(\xi, r)$$

for $r > 0$ such that $B(\xi, r)$, then u is a harmonic function in Ω

Proof. This implies that both u and $-u$ are weakly harmonic, thus by Lemma 4.5 this proves the above lemma. □

Definition 4.6. For $f \in C^0(\partial\Omega)$, let us define a set of functions

$$\sigma_f(\overline{\Omega}) = \{u | u \in C^0(\overline{\Omega}) \cap \sigma(\Omega), \ u \leq f \text{ on } \partial\Omega\}. \tag{17}$$

We further define a supremum to $u(x)$, similar to the Perron function we defined in our motivation, now we do it in a more rigorous manner

$$w_f(x) = \sup_{u \in \sigma_f(\overline{\Omega})} u(x) \text{ for } x \in \Omega. \tag{18}$$

To get a better intuition for what the function $w_f(x)$ is, think of x as being fixed in Ω, and observe the value of $u(x)$ for every single possible function u and take the supremum. We can write w_f in set notation

$$w_f(x) = \sup\{u(x) : u \in \sigma_f(\overline{\Omega})\} \text{ for } x \in \Omega. \tag{19}$$

We set

$$m = \inf\{f(x) : x \in \partial\Omega\}, \tag{20}$$

$$\mu = \sup\{f(x) : x \in \partial\Omega\}. \tag{21}$$

By the completeness axiom cite for the real line we know that any set that is bounded above has a supremum and any set that is bounded below has an infimum. Both m and μ are real numbers because f is continuous and it is closed and bounded so it satisfies the extreme value theorem cite. The extreme value theorem states that, a function that is continuous, closed and bounded has a maximum and a minimum and they are finite real numbers. So m is a real number, we could associate with that real number a function that is constant and equal to it, let

$$\tilde{m}(x) := m \ \ \forall x \in \overline{\Omega}. \tag{22}$$

The function $\tilde{m}$ belongs to $\sigma_f(\overline{\Omega})$, $\tilde{m}$ is clearly continuous on the closure as it is just a constant function, it is weakly harmonic because

$$\tilde{m}(\xi) = M_{\tilde{m}}(\xi, r).$$

Again due to the fact that $\tilde{m}$ is a constant function that equals to m everywhere. Furthermore $\tilde{m} \leq f$ on Ω by (20). It can be concluded from this that the set $\sigma_f(\Omega)$ is not empty.

Furthermore by Lemma 4.2 we have that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u = \mu.$$

Which means that

$$u(x) \leq \mu \text{ for } u \in \sigma_f(\overline{\Omega}), \quad x \in \overline{\Omega}. \tag{23}$$

We have just shown that w_f is well defined. In order for the supremum to exist the set needs to be non-empty and bounded above. We have verified that w_f is a non-empty set, bounded above by μ. So therefore by completeness axiom the supremum is a real number.

Lemma 4.7. Let $u_1, ..., u_k \in \sigma_f(\overline{\Omega})$ and $v = \max(u_1, ..., u_k)$. Then $v \in \sigma_f(\overline{\Omega})$.

Proof. By the definition of weakly subharmonic functions (Definition 4.4) for all $\xi \in \Omega$ and r that is sufficiently small we compute that

$$\begin{aligned} v(\xi) &= \max(u_1(\xi), ..., u_k(\xi)) \\ &\leq \max(M_{u_1}(\xi, r), ..., M_{u_k}(\xi, r)) \\ &= \max(\fint_{\partial B(\xi,r)} u_1 \, dS, ..., \fint_{\partial B(\xi,r)} u_k \, dS) \\ &\leq \fint_{\partial B(\xi,r)} \max(u_1, ..., u_k) \, dS \\ v(\xi) &= M_v(\xi, r). \end{aligned}$$

It remains to be proved that $v \in C^0(\overline{\Omega})$. We know that $u_1, ..., u_k \in C^0(\overline{\Omega})$ to prove that $v \in C^0(\overline{\Omega})$ we will need to verify that the maximum of two continuous functions is continuous, which can be iterated to get to k functions. Let $k = 2$, we have $v = \max(u_1, u_2)$. If $u_1 \leq u_2$ then we are done as $v = u_2$ and u_2 is continuous. Similarly if $u_2 \leq u_1$. If there is point where u_1 and u_2 cross, we need to prove that the function v is continuous at that point. Let there be a point $\xi_0 \in \overline{\Omega}$ such that $u_1(\xi_0)$=$u_2(\xi_0)$. Our aim is to show that v is continuous at ξ_0.

Let $\epsilon > 0$, there exists a $\delta_{u_1} > 0$ such that

$$|u_1(\xi) - u_1(\xi_0)| < \epsilon \text{ for } |\xi - \xi_0| < \delta_{u_1}.$$

Similarly for u_2 there exists a $\delta_{u_2} > 0$ such that

$$|u_2(\xi) - u_2(\xi_0)| < \epsilon \text{ for } |\xi - \xi_0| < \delta_{u_2}.$$

Furthermore at ξ_0,

$$v(\xi_0) = u_1(\xi_0) = u_2(\xi_0).$$

We now need to show that there exists a $\delta > 0$ such that $|v(\xi) - v(\xi_0)| < \epsilon$, whether $v(\xi) = u_1(\xi)$ or $v(\xi) = u_2(\xi)$ when $|\xi - \xi_0| < \delta$. First suppose we have the case that $v(\xi) = u_1(\xi) = \max(u_1(\xi), u_2(\xi))$

$$\begin{aligned} |u_1(\xi) - u_1(\xi_0)| &< \epsilon \text{ for } |\xi - \xi_0| < \delta_{u_1}, \\ |v(\xi) - u_1(\xi_0)| &< \epsilon, \end{aligned}$$

but $v(\xi_0) = u_1(\xi_0)$, so when $|\xi - \xi_0| < \delta_{u_1}$ we have

$$|v(\xi) - v(\xi_0)| < \epsilon.$$

Now suppose we have the case that $v(\xi) = u_2(\xi) = \max(u_1(\xi), u_2(\xi))$

$$\begin{aligned} |u_2(\xi) - u_2(\xi_0)| &< \epsilon \text{ for } |\xi - \xi_0| < \delta_{u_2}, \\ |v(\xi) - u_2(\xi_0)| &< \epsilon, \end{aligned}$$

but $v(\xi_0) = u_2(\xi_0)$, so when $|\xi - \xi_0| < \delta_{u_2}$ we have

$$|v(\xi) - v(\xi_0)| < \epsilon.$$

Let $\delta = \min(\delta_{u_1}, \delta_{u_2})$, so there exists a $\delta > 0$ such that $|v(\xi) - v(\xi_0)| < \epsilon$, we can conclude the function v is continuous at ξ_0. Thus we have verified that $v \in C^0(\overline{\Omega})$ and that $v(\xi) = M_v(\xi, r)$, therefore $v \in \sigma_f(\overline{\Omega})$.

□

Lemma 4.8. The function w_f is harmonic in Ω.

Proof. Let there be a closed ball in Ω centred as ξ with radius r. Using the same notation as before we write, $\overline{B}(\xi, r) \subseteq \Omega$. Let there be another smaller open ball centred at ξ, $B(\xi, r')$ where $r' < r$. We now take a sequence of points (any arbitrary sequence of points), labelling them $x^1, x^2 ...$, that are scattered inside the smaller ball $B(\xi, r')$

$$(x^k) \subseteq B(\xi, r') \text{ where } k \in \mathbb{N}.$$

There are an infinite number of these points, imagine them as dust covering the smaller ball. Note we are not saying these points converge to anything or are in any distinguishable pattern, simply they exist inside $B(\xi, r')$.

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Let us now fix a $k \in \mathbb{N}$, in other words we pick an arbitrary x^k which is now a fixed point. For each $j \in \mathbb{N}$ we claim there exists a sequence of functions $u_k^j \in \sigma_f(\overline{\Omega})$. Then by (19) of Definition 4.6 we have that

$$\lim_{j \to \infty} u_k^j(x^k) = w_f(x^k) = \sup\{u(x^k) : u \in \sigma_f(\overline{\Omega})\}. \tag{24}$$

The above equation is because w_f is the supremum of all functions in $\sigma_f(\overline{\Omega})$ for $x \in \Omega$, see (19). The supremum is the least upper bound, so there is a sequence of points that approach it that get arbitrarily close. Note that if we have a function $u \in \sigma_f(\overline{\Omega})$ where $u_k^j \leq u$ in Ω, relation (24) is preserved if we replace any u_k^j with a function u instead, since

$$u_k^j(x^k) \leq u(x^k) \leq w_f(x^k).$$

The first inequality in the above is because we defined the function u to be as such and the second inequality is because w_f is supremum of all functions $u \in \sigma_f(\overline{\Omega})$.

We now define a new function

$$u^j(x) := \max(u_1^j(x), u_2^j(x), ..., u_j^j(x)) \ \ \forall j \in \mathbb{N}, \ \forall x \in \Omega. \tag{25}$$

Then by Lemma 4.7, it can be seen that $u^j(x) \in \sigma_f(\overline{\Omega})$. Furthermore for $j \geq k$

$$u^j(x) \geq u_k^j(x) \tag{26}$$

If $j \geq k$ then u_k^j would be inside the set itself

$$u^j(x) := \max(u_1^j(x), u_2^j(x), ..., u_k^j(x), ..., u_j^j(x)).$$

Therefore as long as $j \geq k$ then $u^j(x) \geq u_k^j(x)$ for $x \in \Omega$. Suppose there is a function $v^j \in \sigma_f(\Omega)$ where $v^j(x) \geq u_k^j(x)$ for all $j \geq k$, then

$$\lim_{j \to \infty} v^j(x^k) = w_f(x^k) \text{ for all } k \in \mathbb{N}. \tag{27}$$

This is evident by noting that

$$u_k^j(x^k) \leq v^j(x^k) \leq w_f(x^k).$$

Then taking the limit of every term from j to infinity and using relation (24)

$$\lim_{j \to \infty} u_k^j(x^k) \leq \lim_{j \to \infty} v^j(x^k) \leq \lim_{j \to \infty} w_f(x^k),$$

$$w_f(x^k) \leq \lim_{j \to \infty} v^j(x^k) \leq w_f(x^k).$$

Finally the Sandwich Lemma citation needed proves relation (27). We now define

$$v^j(x) := \max(u_1^j(x), u_2^j(x), ..., u_j^j(x), \tilde{m}) \ \ \forall j \in \mathbb{N}, \ \forall x \in \Omega. \tag{28}$$

Where $\tilde{m}$ is function (22) defined in Definition 4.6. The constant function $\tilde{m}$ is in $\sigma_f(\overline{\Omega})$, thus Lemma 4.7 implies $v^j \in \sigma_f(\overline{\Omega})$. Also $v^j(x) \geq u_k^j(x)$ for all $j \geq k$ by (26) since adding $\tilde{m}$ into the set does not make the function smaller. Hence relation (27) holds for this v^j. Now we have

$$m \leq v^j(x) \leq \mu \text{ for } x \in \Omega \tag{29}$$

where μ is what was defined in (21) of Definition 4.6. In the above the first inequality is due to the definition of v^j and we have the second inequality by (23) of Definition 4.6.

Recall by (14) of Lemma 4.3 we can have a function

$$v^j_{\xi,r}(x) \geq v^j(x) \quad \forall x \in \Omega,$$

note that this function $v^j_{\xi,r} \geq m$. Moreover by (15) of Lemma 4.3

$$v^j_{\xi,r} \in \sigma(\Omega).$$

Also $v^j_{\xi,r} = v^j$ on $\overline{\Omega} \setminus B(\xi, r)$, so $v^j_{\xi,r} \in C^0(\overline{\Omega})$ with

$$v^j_{\xi,r}(x) = v^j(x) \quad \forall x \in \partial\Omega.$$

As a result we have that

$$v^j_{\xi,r}(x) \leq f(x) \quad \forall x \in \partial\Omega,$$

where f denotes the boundary data. Putting this together, we have that $v^j_{\xi,r} \in C^0(\overline{\Omega}) \cap \sigma(\Omega)$, $v^j_{\xi,r} \leq f$ on $\partial\Omega$ thus it is clear that $v^j_{\xi,r} \in \sigma_f(\overline{\Omega})$, hence

$$v^j_{\xi,r} \leq \mu \quad \forall x \in \Omega,$$

by (23) of Definition 4.6. Hence we have

$$m \leq v^j(x) \leq v^j_{\xi,r}(x) \leq \mu \quad \forall x \in \Omega,$$

$$v^j_{\xi,r}(x^k) \leq w_f(x^k) \quad \forall k \in \mathbb{N}.$$

Using relation (27) and the sandwich lemma we can conclude the following

$$m \leq v^j_{\xi,r} \leq \mu \quad \forall x \in \Omega, \tag{30}$$

$$\lim_{j\to\infty} v^j_{\xi,r}(x^k) = w_f(x^k) \quad \forall k \in \mathbb{N}. \tag{31}$$

By construction of $v^j_{\xi,r}$ we know that the function is harmonic in $B(\xi, r)$, that is $\Delta(v^j_{\xi,r}) = 0$ in $B(\xi, r)$ (see Definition 4.5). We now can take advantage of the compactness property of harmonic functions, see (John, 1991, Chapter 4 Section 3), which states that if there is exists a compact set $K \subseteq \Omega$, and we have a sequence of functions v_j, where $j \in \mathbb{N}$, that are bounded above and below and they are harmonic in Ω. Then there exists a sub-sequence v_{j_k} such that

$$\lim_{k\to\infty} v_{j_k}(x) = H(x) \quad \forall x \in K, \tag{32}$$

where $\Delta H = 0$. Thus if we let $\Omega_1 = B(\xi, r)$ and $K = \overline{B}(\xi, r')$, by the compactness property we acquire that

$$\lim_{j\to\infty} v^j_{\xi,r} = H(x), \tag{33}$$

where $H(x)$ is a harmonic function. We are able to do this because the functions $v^j_{\xi,r}$ are bounded above and below by relation (30) and they are harmonic in $B(\xi, r)$ by construction. Then it follows from (31) that

$$w_f(x^k) = H(x^k). \tag{34}$$

(Note that right now, H could be dependent on our choice for the sequence of x^k and the subsequence of the js.)

In this next stage of our proof we need to prove that w_f is continuous on $B(\xi, r)$. Let $x \in B(\xi, r')$. Choose any sequence of $(x^k)_{k=1,2,...} \in B(\xi, r')$ such that $\lim_{k\to\infty} x^k = x$ for all $k \in \mathbb{N}$. Then by (34) and the continuity of the harmonic function H we can compute that

$$\lim_{k\to\infty} w_f(x^k) = \lim_{k\to\infty} H(x^k) = H(x). \tag{35}$$

This shows that $\lim_{k\to\infty} w_k(x^k)$ exists. Next, to prove that $\lim_{k\to\infty} w_k(x^k) = w_f(x)$, choose $(x^k)_{k=1,2,...}$ as above except also with $x^1 = x$, so then by (35) and (34) we have that

$$\lim_{k\to\infty} w_k(x^k) = H(x) = H(x^1) = w_f(x^1) = w_f(x).$$

We conclude that

$$\lim_{k\to\infty} w_k(x^k) = w_f(x), \tag{36}$$

for any sequence $x^k \subseteq B(\xi, r')$ with $\lim_{k\to\infty}(x^k = x)$, hence w_f is continuous at x.

Finally, if we take another sequence $(x^k)_{k=1,2,...} \in B(\xi, r')$ that completely fills the ball $B(\xi, r')$. We say that this sequence is dense inside this ball. Now, by (35) we know that w_f agrees with the harmonic function H for all x^k. Therefore, as w_f is continuous then $w_f = H$ for all $x \in B(\xi, r')$, this means that w_f is a harmonic function in a neighbourhood of ξ and as a result this extends to all of Ω. Proving that w_f is harmonic in Ω.

$\square$

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