Lecture 12.

Laws

• Functor Laws.

```
class Functor f where
    fmap :: (a -> b) -> f a -> f b

instance Functor Maybe where
    fmap f Nothing = Nothing.
    fmap f (Just x) = Just (f x)
```

Law 1. identity.

Given
```
id :: a -> a
id x = x
```

$$\underbrace{\text{fmap } \underbrace{\text{id}}_{a \to a}}_{f a \to f a} = \underbrace{\text{id}}_{fa \to fa}$$

Functor composition.

$$\text{fmap } (f \cdot g) = \text{fmap } f \cdot \text{fmap } g$$

---

class Monad m where
return :: a → m a
(>>=) :: m a → (a → m b) → m b

Monad Laws.

1. Left identity.

$$\underbrace{\underbrace{\text{return } \overset{a}{x}}_{m\ a} \gg= \underbrace{f}_{a \to m b}}_{m\ b} = \underbrace{f x}_{m\ b}$$

2. Right identity

$$\underbrace{\underbrace{mx}_{m\ a} \gg= \underbrace{\text{return}}_{a \to m a}}_{m\ a} = \underbrace{mx}_{m\ a}$$

- Associative law.

$$\underbrace{\underbrace{(\underbrace{mx}_{m\,a} \gg\!= \underbrace{f}_{a \to m\,b})}_{m\,b} \gg\!= \underbrace{g}_{b \to m\,c}}_{m\,c} =$$

$$\underbrace{\underbrace{mx}_{m\,a} \gg\!= \underbrace{(\lambda \underbrace{x}_{a} \to \underbrace{f\,x}_{m\,b} \gg\!= \underbrace{g}_{b \to m\,c})}_{a \to m\,c}}_{m\,c}$$

$$\text{foldr} :: (a \to b \to b) \to b \to [a] \to b$$
$$\text{foldr}\ f\ k\ [\,] = k$$
$$\text{foldr}\ f\ k\ (x:xs) = f\ x\ (\text{foldr}\ f\ k\ xs)$$

Fold Fusion.

$$f \cdot \text{foldr}\ g\ a = \text{foldr}\ h\ b$$

Base case: [ ]

$$(f \cdot \text{foldr}\ g\ a)\,[\,] = \text{foldr}\ h\ b\ [\,]$$
$$\Leftrightarrow \{\text{def} \cdot\}$$
$$f\,(\text{foldr}\ g\ a\ [\,]) = \text{foldr}\ h\ b\ [\,]$$
$$\Leftrightarrow \{\text{def foldr}\}$$
$$f\ a = b$$

Other case: (x:xs)

$$(f \cdot \text{foldr}\ g\ a)(x:xs) = \text{foldr}\ h\ b\ (x:xs)$$
$$\Leftrightarrow \{\text{def} \cdot\}$$
$$f\,(\text{foldr}\ g\ a\ (x:xs)) = \text{foldr}\ h\ b\ (x:xs)$$
$$\Leftrightarrow \{\text{def foldr}\}$$
$$f\,(g\ x\ (\text{foldr}\ g\ a\ xs)) = h\ x\ (\text{foldr}\ h\ b\ xs)$$
$$\Leftrightarrow \{\text{induction hypothesis}\}$$
$$f\,(g\ x\ (\underline{\text{foldr}\ g\ a\ xs})) = h\ x\ (f\ (\underline{\text{foldr}\ g\ a\ xs}))$$
$$\Leftarrow \{\text{abstraction}\}$$
$$f\,(g\ x\ y) = h\ x\ (f\ y)$$