Geometry & Motion - MA134
summary sheet

29/5/22

① Curves

$\underline{r}$ is vector-valued if it maps $t \in \mathbb{R}$ to a vector $\underline{r}(t) \in \mathbb{R}^n$

- smooth $\Leftrightarrow$ each component infinitely differentiable
- oriented $\Leftrightarrow$ curve is traced out in a direction ↗
- embedded $\Leftrightarrow$ [simple] does not intersect itself [injective]
- closed $\Leftrightarrow$ $\underline{r}(t)$ is parameterisation $t \in [a,b]$. $\underline{r}(a) = \underline{r}(b) \Rightarrow$ closed

function $\underline{r} : I \to \mathbb{R}^n$. $I \subseteq \mathbb{R}$ the set of image points

$$C = \{ \underline{r}(t) : t \in I \}$$

is a <u>curve</u> in $\mathbb{R}^n$. $\underline{r}$ is a parametrisation of the curve C.

— Rules: —

- $\frac{d}{dt}(\underline{u} + \lambda \underline{v}) = \underline{u}' + \lambda \underline{v}'$
- $\frac{d}{dt}(f \underline{u}) = f' \underline{u} + f \underline{u}'$
- $\frac{d}{dt}(\underline{u} \cdot \underline{v}) = \underline{u}' \cdot \underline{v} + \underline{u} \cdot \underline{v}'$
- $\frac{d}{dt}(\underline{u} \times \underline{v}) = \underline{u}' \times \underline{v} + \underline{u} \times \underline{v}'$
- $\frac{d}{dt}(\underline{u}(f(t))) = u'(f(t)) f'(t)$

Pf: write out in component form

[illegible]

Calculus of vector functions & curves ②

$\underline{r}'(c)$ is ~~the~~ a tangent vector to the curve at $t = c$

curve C parametrised by $\underline{r}(t)$, $t \in I$. $\underline{r}$ is a $\boxed{\text{regular parametrisation}}$ if $\underline{r}'(t) \neq 0 \Leftrightarrow |r'(t)| \neq 0$ at all points on the curve. A curve is regular if it has a regular parametrisation

$|r(t)|^2 = \underline{r} \cdot \underline{r} \Rightarrow f(t) = |\underline{r}(t)| \Rightarrow (f(t))^2 = \underline{r} \cdot \underline{r}$

$\Rightarrow 2 f(t) \cdot f'(t) = 2 \underline{r} \cdot \underline{r}'$

$\Rightarrow f'(t) = \dfrac{\underline{r} \cdot \underline{r}'}{|\underline{r}(t)|}$

$|\underline{r}(t)| =$ constant

$\Rightarrow \underline{r}(t)$ and $\underline{r}'(t)$ orthogonal

$\Downarrow$

circles!

Arc length

$$\frac{ds}{dt} = |r'(t)| \Rightarrow s = \int_a^b |\underline{r}'(t)| \, dt \qquad \left|\frac{dr}{dt}\right| = \frac{ds}{dt}$$

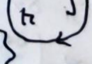
$$s(t) = \int_{t_0}^{t} |r'(u)| \, du$$ arc length parametrisation is $\underline{r}(s)$

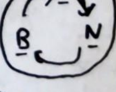
Also, $\left|\frac{dr}{ds}\right| = 1 \Rightarrow$ unit speed parametrisation

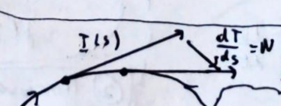
Differential Geometry of Curves ③

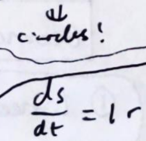
let $\underline{r}(s)$ be a unit speed curve

- curvature: $\kappa(s) = |\underline{r}''(s)|$
- radius of C: $1/\kappa(s)$ [osculating circle]
- tangent : $\underline{T}(s) = \underline{r}'(s)$
- Principle Normal : $\underline{N}(s)$ | $\underline{r}''(s) = \underbrace{\kappa(s)}_{}\underbrace{\underline{N}(s)}_{}$
  magnitude direction
- Binormal : $\kappa \neq 0$, $\underline{B} = \underline{T} \times \underline{N}$
- Torsion : $\underline{B}' = -\tau \underline{N}$, $\tau = -\underline{B}' \cdot \underline{N}$









Frennet - serret Equations

$$\begin{pmatrix} \underline{T}' \\ \underline{N}' \\ \underline{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \underline{T} \\ \underline{N} \\ \underline{B} \end{pmatrix}$$

$a \cdot b = |a||b| \cos\theta$, $|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}| \sin\theta$

Flowchart

Is $\underline{r}$ unit speed?

Yes:

- $\underline{T} = \underline{r}'$
- $\kappa = |\underline{T}'|$
- $\underline{N} = \dfrac{\underline{T}'}{|\underline{T}'|}$
- $\underline{B} = \underline{T} \times \underline{N}$
- $\tau = -\underline{B}' \cdot \underline{N}$

(functions of s)

NO: can we find arc length parametrisation easily?

YES → (functions of s)

NO:

- $\underline{T}(t) = \dfrac{\underline{r}'(t)}{|\underline{r}'(t)|}$
- $\kappa(t) = \left|\dfrac{dT}{ds}\right| = \left|\dfrac{dT}{dt} \cdot \dfrac{dt}{ds}\right| = \left|\dfrac{\underline{T}'(t)}{\underline{r}'(t)}\right|$
- $\underline{N}(t) = \dfrac{\underline{T}'(t)}{|\underline{T}'(t)|}$
- $\underline{B}(t) = \underline{T}(t) \times \underline{N}(t)$
- $\tau(t) = -\dfrac{dB}{dt} \cdot \dfrac{dt}{ds} \cdot \underline{N}(t)$
  $= \dfrac{-\underline{B}'(t) \cdot \underline{N}(t)}{|r'(t)|}$

$$\kappa(t) = \frac{|\underline{r}'(t) \times \underline{r}''(t)|}{|\underline{r}'(t)|^3}, \quad \tau(t) = \frac{r'(t) \times r''(t) \cdot r'''(t)}{|r'(t) \times r''(t)|^2}$$

etc

Differentiating functions of several variable ④

let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$. The graph of f is the set of points in $\mathbb{R}^3$ s.t. $z = f(x, y)$

Partial Derivatives of $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ at pt (a, b) are

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h} \quad \text{etc.}$$

The level sets of a function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ are the set of pts

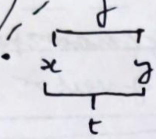
$$L_k = \{ \underline{x} \in \mathbb{R}^n \mid f(\underline{x}) = k \}$$

$n = 2 \implies$ contours

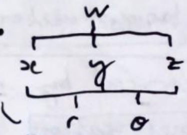
$n = 3 \implies$ isosurfaces

Product rule still holds for partial derivaties

Chain Rule

- 

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

- 

$$\frac{dw}{dr} = \frac{\partial w}{\partial x}\frac{\partial x}{dr} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$. The gradient of f, ∇f, grad f

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

$\nabla f : \mathbb{R} \to \mathbb{R}^n$

opperator too.
scalar → vector

$f : U \subseteq \mathbb{R}^3 \to \mathbb{R}$. $\hat{\underline{u}}$ is a unit vector. The Directional Derivative of f in the direction of u, $D_{\underline{u}} f$ defined as

$$D_u f(\underline{x}) = \lim_{h \to 0} \frac{f(\underline{x} + h\underline{u}) - f(\underline{x})}{h}$$

$$\boxed{D_{\underline{u}} f(x) = \nabla f \cdot \underline{u}}$$

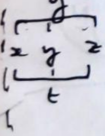
Pf: $D_{\underline{u}} f(\underline{x}) = \lim_{t \to 0} \frac{f(\underline{x} + t\underline{u}) - f(\underline{x})}{t}$

let $g(t) = f(\underline{r}(t))$ where, $\underline{r}(t) = \underline{x} + t\underline{u}$

$\underline{r}(0) = \underline{x}$ $\quad \underline{r}'(t) = \underline{u}$

$\therefore D_u f(\underline{x}) = \lim_{t \to 0} \frac{g(t) - g(0)}{t} = g'(0)$

but



$$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = \nabla g \cdot \underline{r}'(t)$$

$\therefore D_u f(\underline{x}) = g'(0) = \nabla f \cdot \underline{r}'(0) = \nabla f \cdot \hat{\underline{u}}$.

Geometry & Applications ⑤

① Linear Approximations

$$f(x, y) \approx f(a,b) + \left.\frac{\partial f}{\partial x}\right|_{(a,b)}(x - a) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)}(y - b)$$

linear approx of a function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(\underline{x})$ near $\underline{x}_0$ given by

$$f(\underline{x}) \approx f(\underline{x}_0) + \nabla f|_{\underline{x}_0}(\underline{x} - \underline{x}_0)$$

② Normal to a surface

∇f is normal to $f = $ constant

③ Critical Points & classification

(a, b) is a critical point of $f : \mathbb{R}^2 \to \mathbb{R}$ if $f_x = 0$ and $f_y = 0$ $[\nabla f = 0]$

$\Downarrow$

suppose $f(x, y)$ has a critical point at (a, b)

let $D = \det(H)$ where

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$$D = f_{xx}f_{yy} - f_{xy}^2$$

Then

(a) if $D > 0$, $f_{xx} > 0$ at (a, b) $\implies$ local minimum

(b) if $D > 0$, $f_{xx} < 0$ at (a, b) $\implies$ local maximum

(c) if $D < 0$ $\implies$ saddle point

(d) if $D = 0$ $\implies$ inconclusive test

Steepest acent/decent

$D_{\underline{\hat{u}}} f = \nabla f \cdot \underline{\hat{u}} = |\nabla f| \cos\theta$

where θ is the angle between ∇f and $\underline{\hat{u}}$, $\theta \in [0, \pi]$

$\Rightarrow$ max for $\theta = 0$ so f increasing the fastest in direction

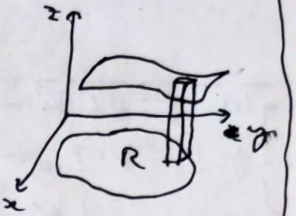
$$\underline{\hat{u}} = \frac{\nabla f}{|\nabla f|} \quad \text{[steepest acent]}$$

$\Rightarrow$ min for $\theta = \frac{\pi}{2}$ so f decreases the fastest for

$$\underline{\hat{u}} = -\frac{\nabla f}{|\nabla f|} \quad \text{[steepest decent]}$$

Integration I - cartesian coordinates ⑥

If you can't do an integral, try switching the order of integration!



$\iint_R f(x,y)\,dx\,dy$ represents the volume under the surface $z = f(x,y)$ above the region R

solid in $V \subseteq \mathbb{R}^3$ has density $\rho(x,y,z)$

$$M = \int dm = \iiint \rho(x,y,z)\,dV$$

centre of mass $\bar{x} = \frac{1}{m} \iiint \rho x \, dV$

Integration II - Special coordinates ⑦

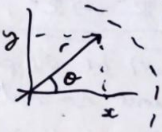
Polar coordinates:

$dA = r\,dr\,d\theta$

$x = r\cos\theta$,

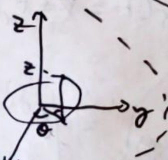
$y = r\sin\theta$

$x^2 + y^2 = r^2$, $\tan\theta = \frac{y}{x}$



cylindrical coordinates:

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ dV &= r\,dr\,d\theta\,dz \end{aligned}}$$

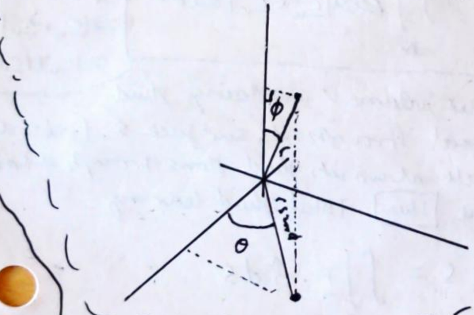


$I = \int_{-\infty}^{\infty} e^{-x^2} dx \Rightarrow I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dx\,dy$ and switch to polar coordinates

$\Rightarrow I^2 = \int_0^{2\pi}\int_0^{\infty} r e^{-r^2} dr\,d\theta$

$\Rightarrow I = \sqrt{\pi}$

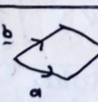
Spherical coordinates



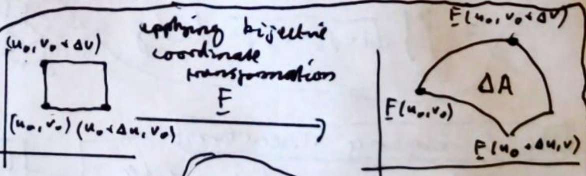
$$\boxed{\begin{aligned} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi \\ dV &= r^2 \sin\phi\,dr\,d\theta\,d\phi \end{aligned}}$$

Calculus of Multivariable Functions ⑧

note: $\Delta A \approx |\underline{a} \times \underline{b}| \approx$



$\underline{F}_u = \frac{\partial F}{\partial u}$, $\underline{F}_v = \frac{\partial F}{\partial v}$



$\underline{a} = \underline{F}_u \Delta u$ [evaluated at u_0, v_0]

$\underline{b} = \underline{F}_v \Delta v$ [evaluated at u_0, v_0] $\Rightarrow \Delta A = |\underline{F}_u \times \underline{F}_v| \Delta u \Delta v$

[derived from linear approximation form] $\Rightarrow \boxed{dA = |\underline{F}_u \times \underline{F}_v|\,du\,dv}$

$dA = dx\,dy = |\det \underline{DF}(u,v)|\,du\,dv$

$dV = |\det \underline{DF}(u,v,w)|\,du\,dv\,dw$

let $\underline{F} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\underline{F}(u,v) = (x,y)$ be a bijection [represents a coordinate transformation $x = x(u,v)$, $y = y(u,v)$] then the integral of $f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ can be:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(u,v)\,|\det DF(u,v)|\,du\,dv$$

where $S = F^{-1}(R)$ is the corresponding region in the u-v plane.

$$\underline{DF} = \begin{pmatrix} X_u & X_v \\ Y_u & Y_v \end{pmatrix} = \begin{pmatrix} \nabla x \\ \nabla y \end{pmatrix}$$

$\underline{F}(r,\theta) = (r\cos\theta, r\sin\theta)$

$$\underline{DF}(r,\theta) = \begin{pmatrix} X_r & X_\theta \\ Y_r & Y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$\underline{F}(r,\theta,\phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$

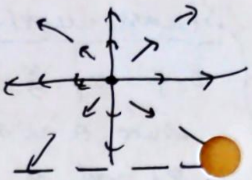
$\underline{F} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$

$\underline{F}$ is a vector field

A vector field which can be expressed as ∇f is called a <u>conservative</u> vector field.

$F(x,y) = x, y$

$f = \frac{x^2+y^2}{2}$



let $\underline{F} : U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$. The Divergence of $\underline{F}$, $\nabla \cdot \underline{F}$ or div $\underline{F}$ is

$$\operatorname{div} F = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

$\nabla\cdot$: vector → scalar

let $\underline{F} : U \subseteq \mathbb{R}^3 \to \mathbb{R}^3$. The <u>curl</u> of $\underline{F}$ $\nabla \times \underline{F}$ or curl F is

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$\nabla\times$: vector → vector.

If $\underline{F}(x,y,z)$ is a conservative field then curl $F = 0$ $[F = \nabla f]$

<u>Pf:</u> $\underline{F} = \nabla f = (f_x, f_y, f_z)$

$$\therefore \operatorname{curl} F = \nabla \times \nabla f = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix}$$

$$= \begin{pmatrix} f_{yz} - f_{zy} \\ f_{xz} - f_{zx} \\ f_{xx} - f_{yx} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If $z = f(x,y)$

parametrisation:

$\underline{r}(x,y) = (x, y, f(x,y))$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}$$

$$\therefore SA = \iint_R |\underline{r}_u \times \underline{r}_v|\, du\, dv$$

$$= \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

Surface Integrals + Divergence Theorem ⑨

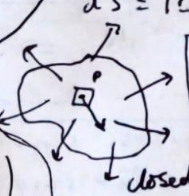
We can parametrise a surface S in $\mathbb{R}^3$ by $\underline{r} : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, $\underline{r}(u,v) = (x(u,v), y(u,v), z(u,v))$

Let $\underline{r}(u,v)$ be the parametrisation of a surface S. The unit normal at the point P on S given by $\hat{\underline{n}} = \pm \frac{\underline{r}_u \times \underline{r}_v}{|\underline{r}_u \times \underline{r}_v|}$ evaluated at pt. P.

$dS = |\underline{r}_u \times \underline{r}_v|\, du\, dv$ so surface area given by

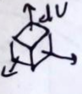
$$\boxed{SA = \iint_\Omega |\underline{r}_u \times \underline{r}_v|\, du\, dv}$$

Note: If $\underline{r}_u \cdot \underline{r}_v = 0$ $\Rightarrow |\underline{r}_u \times \underline{r}_v| = |\underline{r}_u||\underline{r}_v|$



closed container, volume V containing fluid that flows outward through its surface S. [velocity $\underline{v}$] $\underline{F} = \rho\underline{v}$ is the rate at which fluid flows through a small area dS. $\underline{F}$ is the $\boxed{\text{flux}}$. Total fluid leaving:

$$\text{Total flux across } S = \iint_S \underline{F} \cdot \hat{\underline{n}}\, dS$$



rate at which fluid flows out of volume element: $\nabla \cdot F\, dV$

$$\text{rate of outflow from } V = \iiint_V \nabla \cdot F\, dV$$

The Divergence Theorem

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable vector field. V volume. S is surface

$$\iiint_V \nabla \cdot \underline{F}\, dV = \iint_S \underline{F} \cdot \hat{\underline{n}}\, dS$$

where $\hat{\underline{n}}$ is outward pting unit n.

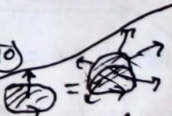
RHS $= \iint_S \underline{F} \cdot d\underline{S}$ where

$d\underline{S} = \hat{\underline{n}}\, dS$ ← nice geometry

$= \hat{\underline{n}}\, |\underline{r}_u \times \underline{r}_v|\, du\, dv$

$= \pm(\underline{r}_u \times \underline{r}_v)\, du\, dv$ ↖ not nice geometry

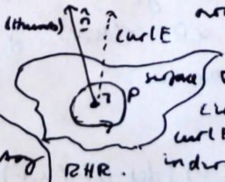
Line Integrals & Stokes Theorem ⑩



$\int_C F \cdot d\underline{r} = \int_a^b \underline{F} \cdot \frac{dr}{dt}\, dt$ $\underline{F} : \mathbb{R}^n \to \mathbb{R}^n$ C is a curve parametrised by $\underline{r}(t)$

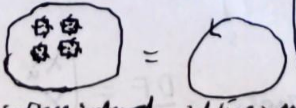
If $\underline{F}$ is a conservative field, $\int_C \underline{F} \cdot d\underline{r}$ independent of path.

How much fluid circulates around P?



Pointwise: $\nabla \times \underline{F} \cdot \hat{\underline{n}}$

circulation = curl F projected in direction of $\hat{n}$



Stokes Theorem

Let $F(x,y,z)$ be a vector field. Let S be a surface w/ unit normal $\hat{n}$ and boundary curve C, oriented w/ RHR.

$$\iint \nabla \times F \cdot \hat{\underline{n}}\, dS = \int_C \underline{F} \cdot d\underline{r}$$

[net circulation] [work done around C]