

# Ch 10 End of Linear Algebra MA10G

2/5/22

Def: **Diagonalisable:** [two versions - linear maps & matrices]

A linear map  $T: V \rightarrow V$  is diagonalisable if there is a basis for  $V$  such that the matrix  $A$  for  $T$  wrt this basis is diagonal. [ $a_{ij} = 0$  if  $i \neq j$ ]

An  $n \times n$  matrix  $A$  over  $\mathbb{k}$  is diagonalisable if there is a  $\mathbb{k}$ -basis for  $\mathbb{k}^n$  such that the linear map  $T: \mathbb{k}^n \rightarrow \mathbb{k}^n$  given by  $T(\underline{x}) = A\underline{x}$  is diagonal.

**Relationship between diagonal matrices & eigenvectors**

Let  $T: V \rightarrow V$  be a linear map. The matrix for  $T$  is diagonal wrt some basis of  $V$

 $\Leftrightarrow$ 

$V$  has a basis consisting of eigenvectors of  $T$

Let  $A$  be an  $n \times n$  matrix over  $\mathbb{k}$ . Then  $A$  diagonalisable

 $\Leftrightarrow$ 

$\mathbb{k}^n$  has a basis of eigenvectors of  $A$

Proof: [two statements equivalent from correspondence between linear maps and matrices and the definitions of eigenvalues & eigenvectors].

$\Rightarrow$  Suppose the matrix  $A = (a_{ij})$  of  $T$  is diagonal wrt the basis  $e_1, \dots, e_n$  of  $V$ . We know the image of the  $i$ th basis vector is  $e_1, \dots, e_n$  of  $V$ . We know the image of the  $i$ th basis vector is represented by the  $i$ th column of  $A$ , but since  $A$  is diagonal, this column has the single non-zero entry  $a_{ii}$ . So

$T(e_i) = a_{ii}e_i \Leftrightarrow Ae_i = a_{ii}e_i \Leftrightarrow$  each basis vector  $e_i$  is an eigenvector of  $A$ .

$\Leftarrow$  Suppose that  $e_1, \dots, e_n$  is a basis of  $V$  consisting entirely of eigenvectors of  $T$ . Then for each  $i$  we have  $T(e_i) = \lambda_i e_i$  for some  $\lambda_i \in \mathbb{k}$ . [But as the image of the  $i$ th basis vector is the  $i$ th column of the matrix  $A$  for  $T$  wrt that basis.] So the matrix  $A$  wrt this basis is the diagonal matrix  $A = (a_{ij})$  with  $a_{ii} = \lambda_i \forall i$ .  $\square$

Can't diagonalise every matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \det(A - xI) = \begin{pmatrix} 1-x & 1 \\ 0 & -x \end{pmatrix} = 0 \Rightarrow (1-x)^2 = 0$$

$$\text{so } \lambda = 1, \ker(A - I) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

can't diagonalise matrix  $A$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ so } \text{span}(\ker(A - I)) = \text{span} \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

so can't find two linearly independent eigenvectors. so no basis of  $\mathbb{k}^2$  of eigenvectors

distinct eigenvalues  $\Rightarrow$  corresponding eigenvectors linearly independent

Thm

Proof: Let's use induction on r. Suppose let  $\lambda_1, \dots, \lambda_r$  be distinct eigenvalues of  $T$ , let  $v_1, \dots, v_r$  be corresponding eigenvectors. Then  $v_1, \dots, v_r$  are linearly independent.

Proof: Idea is induction on r.

- $r=1$  .  $\lambda_1$  has eigenvector  $v_1$ , which is clearly linearly independent as  $v_1 \neq 0$ .
- Now for  $r > 1$  suppose  $\exists \alpha_i \in \mathbb{K}$  s.t.

$$\alpha_1 v_1 + \dots + \alpha_r v_r = 0 \quad (*)$$

Applying  $T$  yields

$$\alpha_1 T(v_1) + \dots + \alpha_r T(v_r) = 0$$

and as  $v_1, \dots, v_r$  are eigenvectors, we have

$$\alpha_1 \lambda_1 v_1 + \dots + \alpha_r \lambda_r v_r = 0 \quad (**)$$

we have two pieces of information about the  $\{\alpha_i v_i\}$  so let's combine.

Taking  $(**) - \lambda_1 (*)$  yields

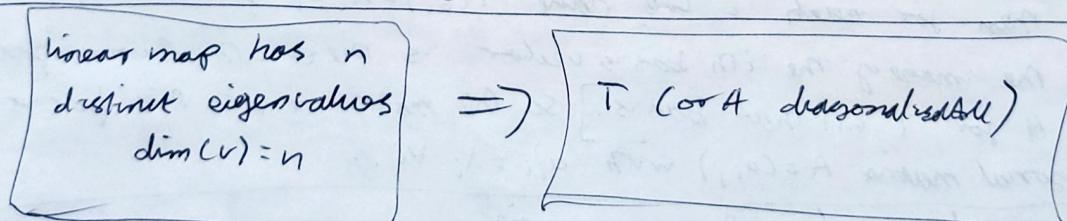
$$\alpha_2 (\lambda_2 - \lambda_1) v_2 + \dots + \alpha_r (\lambda_r - \lambda_1) v_r = 0$$

as we have 1 less  $\alpha$   
more for 1 more  $\lambda$

[ $r-1$  assumption]

By the induction hypothesis, the  $v_2, \dots, v_r$  are linearly independent so  $\alpha_i (\lambda_i - \lambda_1) = 0$ . we know  $\lambda_i \neq \lambda_1$ ,  $\forall i$  as the  $\lambda_i$  are distinct so  $\alpha_i = 0$

$\Rightarrow v_1, \dots, v_r$  linearly independent.



Pf:  $v_1, \dots, v_n$  linearly independent by above and  $\dim(V) = n$

so  $v_1, \dots, v_n$  must form a basis of  $V$  so

we have a basis of eigenvectors

$\Rightarrow T$  diagonalisable.

Note: can have ~~less~~  $n$  non-distinct eigenvalues but matrix would still be diagonalisable if we end up with ~~more~~ distinct eigenvectors. Matrices with ~~less~~  $n$   $\neq$  linearly independent.

# Change of basis & Equivalence

(21/5/22)      chapter 14

Let  $V$  be a vector space  $\dim(V) = n$ .

$e_1, \dots, e_n$   
basis  
of  $V$

$e'_1, \dots, e'_n$   
2nd basis  
of  $V$

The matrix  $P$  of the identity map  $I_v: V \rightarrow V$  using basis  $e_1, \dots, e_n$  in domain and using basis  $e'_1, \dots, e'_n$  in the range of  $\oplus I_v$ .  
 P is the change of basis matrix from the matrix  $\oplus e_i$ 's to the basis  $e'_i$ 's

Take  $P = (P_{ij})$  [matrix]. As  $I_v$  takes a vector written in  $e_1, \dots, e_n$  to a vector written in  $e'_1, \dots, e'_n$ , we can say combination of

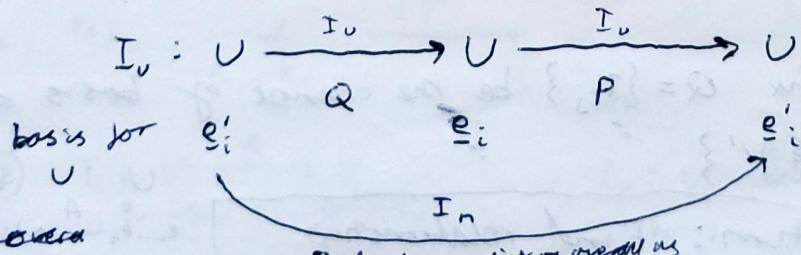
$$I_v(e_j) = e_j = \sum_{i=1}^n P_{ij} e'_i \quad \text{for } 1 \leq j \leq n \quad (*)$$

so the columns of the change of basis matrix of  $P$  are the coordinates of the old basis vectors  $e_i$  wrt the new bases vectors  $e'_i$ 's.

The change of base matrix is invertible

e.g. If  $P$  is the change of base matrix from the basis  $e_1, \dots, e_n$  to  $e'_1, \dots, e'_n$  then and  $Q$  is the change of base matrix from the  $e'_i$ 's to the basis of  $e_i$ 's, then  $P = Q^{-1}$

Proof: Take the composition of linear maps



[identity matrix overall as some domain basis for domain and range]

$$\Rightarrow \text{so } I_n = PQ \Leftrightarrow I_n = QP \Leftrightarrow P = Q^{-1}$$

The change of base matrix  
turns vectors coordinates wrt old basis  
into the same vectors coordinates  
wrt the new basis

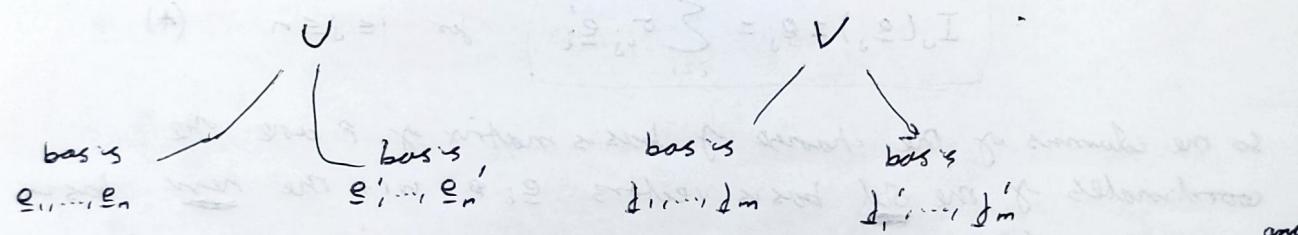
E.g.

- Let  $v \in V$ . Let  $\underline{v}_i$  be column vector of  $v$  wrt basis  $e_1, \dots, e_n$   
Let  $\underline{v}'_i$  be column vector of  $v$  wrt basis  $e'_1, \dots, e'_n$

$$\Rightarrow P \underline{v}_i = \underline{v}'_i$$

### Effect of change of bases on Linear Maps

- Let  $T: U \rightarrow V$  be a linear map.  $\dim(U) = n, \dim(V) = m$ .



- Let  $A = (a_{ij})$  be the  $m \times n$  matrix of  $T$  wrt bases  $\{e_i\}$  and  $\{e'_j\}$  of  $U$  and  $V$ .

$$\text{so } T(e_j) = \sum_{i=1}^n a_{ij} e_i \quad \text{for } 1 \leq j \leq m$$

- Let  $B = (b_{ij})$  be the  $m \times n$  matrix of  $T$  wrt. bases  $\{e'_i\}$  and  $\{e_j\}$  of  $U$  and  $V$ .

$$\text{so } T(e'_j) = \sum_{i=1}^m b_{ij} e_i \quad \text{for } 1 \leq j \leq n$$

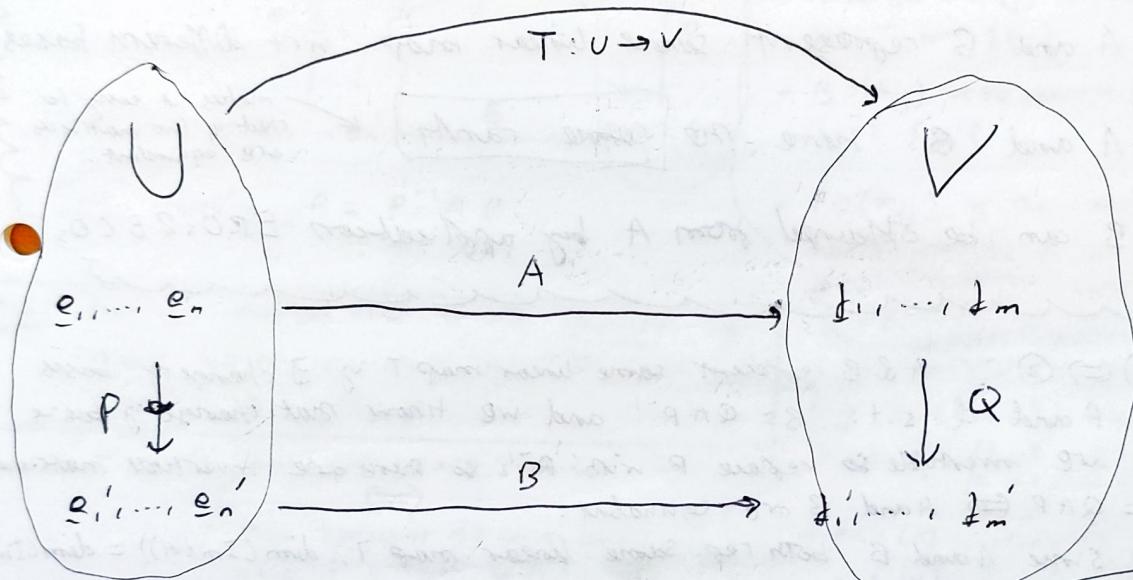
- Let  $P = (P_{ij})$  be the  $n \times n$  matrix be the change of basis matrix from  $\{e_i\}$  to  $\{e'_i\}$

- Let the  $m \times m$  matrix  $Q = \{Q_{ij}\}$  be the change of basis matrix from  $\{e_j\}$  to  $\{e'_j\}$

Aim: Find relationship  
between  $A$  and  $B$  wrt. the  
change of bases matrices  
 $P$  and  $Q$

$$\begin{array}{ccc} U & & V \\ e_1, \dots, e_n & \xrightarrow{A} & e'_1, \dots, e'_m \\ P \downarrow & & Q \downarrow \\ e'_1, \dots, e'_n & \xrightarrow{B} & e''_1, \dots, e''_m \\ B = QAP^{-1} \end{array}$$

rough sketch of chase  
relationships between  
changes of bases matrices  
and matrices for linear  
transformations w.r.t different  
bases



Theorem:  $B P = Q A \Leftrightarrow B = Q A P^{-1}$

$$\text{Proof: } ① I_U : U \xrightarrow{P} U$$

$$e_1, \dots, e_n \quad e'_1, \dots, e'_n$$

$$B T : U \xrightarrow{B} V$$

$$e'_1, \dots, e'_n \quad f'_1, \dots, f'_m$$

$$\therefore T \circ I_U : U \xrightarrow{BP} V$$

$$e_1, \dots, e_n \quad f_1, \dots, f_m$$

$$② T : U \xrightarrow{A} V$$

$$e_1, \dots, e_n \quad f_1, \dots, f_m$$

$$I_V : V \xrightarrow{Q} V$$

$$f_1, \dots, f_m \quad f'_1, \dots, f'_m$$

$$\therefore I_V \circ T : U \xrightarrow{QA} V$$

$$e_1, \dots, e_n \quad f'_1, \dots, f'_m$$

so combining ① and ②

$$\Rightarrow BP = QA$$

two min matrices A and  
B represent mesame linear  
map from an n-dimensional  
vector space to an m-d  
vector space

( $\Leftarrow$ )  
there exist invertible matrices  
nxn and mnm matrices  
P and Q with  
 $B = Q A P$

## Equivalent matrices

Def: Two  $m \times n$  matrices  $A$  and  $B$  are equivalent if there exist invertible  $P$  and  $Q$  with  $B = QAP$

(equivalent conditions)

- ①  $A$  and  $B$  are equivalent
- ②  $A$  and  $B$  represent same linear map wrt different bases
- ③  $A$  and  $B$  have the same rank makes it easy to check if two matrices are equivalent.
- ④  $B$  can be obtained from  $A$  by applications EROs & ECos?

Proof: ①  $\Rightarrow$  ②  $\because A$  &  $B$  represent same linear map  $T$  if  $\exists$  change of basis matrices  $P$  and  $Q$  s.t.  $B = QAP^{-1}$  and we know that change of basis matrices are invertible so replace  $P$  with  $P^{-1}$  so there are invertible matrices with  $B = QAP \Leftrightarrow A$  and  $B$  are equivalent.

②  $\Rightarrow$  ③ since  $A$  and  $B$  both rep. same linear map  $T$ ,  $\dim(\text{Im}(A)) = \dim(\text{Im}(B)) = \dim(\text{Im}(T))$  otherwise  $\exists i$  so  $\text{rank}(A) \neq \text{rank}(B)$ . non zero

③  $\Rightarrow$  ④  $\because$  as  $A$  and  $B$  both have ranks, can be brought into block form

$$E_s = \left( \begin{array}{c|c} I_s & 0_{s,n-s} \\ \hline 0_{m-s,s} & 0_{m-s,n-s} \end{array} \right)$$

operations operations

by EROs and ECos. As EROs & ECos invertible,  $A \xrightarrow{\text{EROs}} E_s \xrightarrow{\text{ECos}} B$

④  $\Rightarrow$  ① can write these transformations as a product of elementary row & column operations so  $\exists$  elementary row matrices  $R_1, \dots, R_r$  and EC matrices  $C_1, \dots, C_s$  s.t.  $B = R_1 \dots R_r A C_1 \dots C_s$  so take the product & it is invertible so  $\exists$  invertible  $P$  and  $Q$  s.t.  $B = QAP$ . □

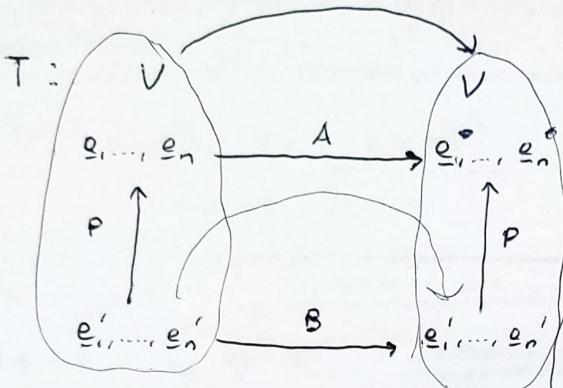
Note: any  $m \times n$  matrix  $A$  is equivalent to the matrix  $E_s$  above where  $s = \text{rank}(A)$ . The form  $E_s$  is a canonical form for  $m \times n$  matrices under equivalence.

Similar  $\Rightarrow$  equivalent

equivalent  $\not\Rightarrow$  similar

# Similar Matrices

$V$  vector space over field  $\mathbb{F}$ .  $\dim(V) = n$ .  $T: V \rightarrow V$  is the linear map



Theorem: clearly,  $B = P^{-1}AP$

Def: two  $n \times n$  matrices over  $\mathbb{F}$  are similar if there exists an  $n \times n$  invertible matrix  $P$  with  $B = P^{-1}AP$

In other words: two matrices are similar  
 $\Leftrightarrow$

They represent the same linear map  $T: V \rightarrow V$  w.r.t. different bases of  $V$

Similar matrices have the same characteristic polynomial & hence the same eigenvalues

Proof: Let  $A$  and  $B$  be similar matrices. Then  $\exists P$  [invertible] with  $B = P^{-1}AP$  so

$$\begin{aligned} \det(B - \lambda I_n) &= \det(P^{-1}AP - \lambda I_n) \\ &= \det(P^{-1}(A - \lambda I_n)P) \\ &= \det(P^{-1}) \det(A - \lambda I_n) \det(P) \\ &= \frac{\det P}{\det(P)} \det(A - \lambda I_n) \\ &= \det(A - \lambda I_n) \end{aligned}$$

so  $A$  and  $B$  have the same characteristic polynomial

$\Rightarrow$  same eigenvectors

## Setup:

- $e_1, \dots, e_n$  and  $e'_1, \dots, e'_n$  are two bases of  $V$ .
  - $A = (a_{ij})$  is the matrix for  $T$  wrt  $\{e_i\}$
  - $B = (b_{ij})$  is the matrix for  $T$  wrt  $\{e'_i\}$
  - $P = (p_{ij})$  is the change of basis matrix from  $\{e'_i\}$  to  $\{e_i\}$
- Opposite to before

E.g.  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  where

$$T(x) = A_2 = \begin{pmatrix} 1 & 2 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} x$$

in basis (of eigenvectors)

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

for  $\mathbb{R}^3 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$  and computing change of basis matrix from new field

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$P^{-1}A_2P = B \text{ as required}$$

so  $A_2$  &  $B$  similar.

Note: The different matrices corresponding to a linear map  $T$  are all similar so they all have the same characteristic equation so can refer to it as the characteristic polynomial of  $T$ .

hard to find a  $E_s$  like Row column reduced echelon form like we do for equivalent matrices.

## Similar vs Equivalent matrices

$T: U \rightarrow V$

Two  $m \times n$  matrices are equivalent if  $\exists$  invertible  $P, Q$  &  $R$  with  
 $B = QAP$

$T: V \rightarrow V$

Two  $n \times n$  matrices over  $\mathbb{K}$  are similar if  $\exists$   $n \times n$  invertible matrix  $P$  with  
 $B = P^{-1}AP$

Def:  $\underline{v} = (v_1, \dots, v_n), \underline{w} = (w_1, \dots, w_n), \underline{v}, \underline{w} \in \mathbb{R}^n$ .

$$\underline{v} \cdot \underline{w} = \sum_{i=1}^n v_i w_i \quad [\text{scalar product}]$$

$A$  &  $B$  have same characteristic polynomial & same eigenvalues.

orthonormal basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

A basis  $b_1, \dots, b_n$  of  $\mathbb{R}^n$  is orthonormal if

$$\textcircled{1} \quad b_i \cdot b_i = 1 \quad \text{for } 1 \leq i \leq n$$

$$\textcircled{2} \quad b_i \cdot b_j = 0 \quad \text{for } 1 \leq i, j \leq n \text{ and } i \neq j$$

↑ Standard basis for  $\mathbb{R}^3$ 's orthonormal

Symmetric matrices

$n \times n$  matrix  $A$  symmetric if

$$A^T = A$$

orthogonal matrices

$n \times n$  matrix  $A$  is orthogonal if

$$A^T = A^{-1}$$

$$\Leftrightarrow AA^T = A^T A = I_n$$

A orthonormal matrix

$$A \in M_{2 \times 2}(\mathbb{R})$$

rows  $e_1, \dots, e_n$  of  $A$

form an orthonormal basis of  $\mathbb{R}^n$

columns  $e_1, \dots, e_n$  of  $A$

form an orthonormal basis of  $\mathbb{R}^n$

A  $\Leftrightarrow$  is a real symmetric matrix



A has only real eigenvalues

if:  $\det(A - \lambda I_n)$  has a root  $\lambda \in \mathbb{C}$  which is an eigenvalue of  $A$

$$Av = \lambda v \text{ and transpose it } v^T A = \lambda v^T \quad (3)$$

$$A^T = \bar{\lambda} \bar{v} \quad (4) \quad \text{and transpose}$$

multiply (3) by  $\bar{v}^T$  on the right.

multiply (4) by  $v^T$  on the left. ] compare:

$$\bar{\lambda} = \lambda$$

A is a real symmetric matrix.  $\lambda_1, \lambda_2$  are two distinct eigenvalues of  $A$  [corresponding  $v_1, v_2$ ]



$$v_1 \cdot v_2 = 0$$

Let  $A$  be a real symmetric matrix ( $n \times n$ ). Then there exists a real orthogonal matrix  $P$  with  $P^{-1}AP = P^TAP$  ↓ diagonal.

change of basis matrix of normalized eigenvectors.

↓ get an orthonormal basis of  $\mathbb{R}^n$  [eigenvectors]

$$\text{so } P^{-1}AP \text{ diagonal Entries } (P^{-1}AP)_{ii} = \lambda_i$$