Key: Definition | Theorem | Lemma | Example

Norms, Metrics & Topologies Summary

Exam is examples based so will only do some proofs, not all

Chapter 1 - Introduction

① **Countable:** A set A is countable if it is in bijection with a subset of $\mathbb{N}$. Equivalently, if there's an injection from A to $\mathbb{N}$. Could be finite or infinite. $\emptyset$ countable.

Chapter 2 - Normed Spaces

② **Norm on a vector space:** A norm on a vector space X is a map $\|\cdot\|: X \longrightarrow \mathbb{R}^+$ s.t.

no symmetry condition like a metric

(i) $\|x\| = 0 \iff x = 0$ [non-degeneracy]

(ii) $\|\lambda x\| = |\lambda| \|x\| \quad \forall \lambda \in \mathbb{R}$ (or $\mathbb{C}$), $x \in X$ [homogeneity]

(iii) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$ [triangle inequality]

③ **Normed space:** The pair $(X, \|\cdot\|)$, if standard norm, could just say the normed space X.

③ **Convex subset:** X vector space, $K \subset X$ convex if $\forall x, y \in K$, $0 \leq \lambda \leq 1$, the line segment joining x, y lies in K, $\lambda x + (1-\lambda) y \in K$.

closed unit ball

④ **Closed unit ball convex in a normed space:** In any $(X, \|\cdot\|)$, $\overline{B_X}$ convex

Pf: $x, y \in \overline{B_X} \Rightarrow \|x\| \leq 1$, $\|y\| \leq 1$ so $\|\lambda x + (1-\lambda) y\| \leq |\lambda| \|x\| + |1-\lambda| \|y\| \leq \lambda + 1 - \lambda = 1$

⑤ **Equivalent check for norm:** If $N: X \longrightarrow \mathbb{R}^+$ satisfies (i), (ii) in norm defn, plus $B = \{x \in X : \|x\| \leq 1\}$ convex, then N satisfies (iii) so a norm.

Pf: WTS triangle inequality. $\frac{x}{N(x)}, \frac{y}{N(y)} \in B$ so use convexity w/ $\lambda = \frac{N(x)}{N(x)+N(y)}$

⑥ **Examples**

- $X = \mathbb{R}^n$ $\quad \|x\|_{\ell^p} = \left(\sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}} \quad 1 \leq p < \infty \quad \|x\|_{\ell^\infty} = \max_{j \in \{1,\dots,n\}} |x_j|$

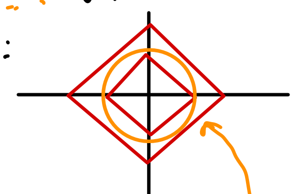
pth power summable sequences

- $X = \ell^p$ "The sequence space" $\ell^p = \{ (x_j)_{j=1}^{\infty} : \sum_{j=1}^{\infty} |x_j|^p < \infty \}$, $\|x\|_{\ell^p} = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{\frac{1}{p}}$

Basis is $\{e^{(j)}\}_{j=1}^{\infty}$ $\quad e^{(j)} = (0, 0, \dots, 1, \dots, 0, \dots)$ so infinite dimensional...

- $X = C([a,b])$ is the space of $\mathbb{R}$-cts functions on $[a,b]$, two norms:

$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$, $\quad \|f\|_{L^p} = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}$



norms must have spherically symmetric unit balls...

⑦ **Equivalent norms:** Norms $\|\cdot\|_1$, $\|\cdot\|_2$ are equivalent if $\exists$ constants $0 < c_1 \leq c_2$ s.t. $c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|_1 \quad \forall x \in X$ $\iff$ you can fit scaled copies of unit balls inside

⑧ **Minkowski Inequalities:** $\|x+y\|_{\ell^p} \leq \|x\|_{\ell^p} + \|y\|_{\ell^p}$ [works for $\mathbb{R}^n$ & ℓ^p]

Chapter 3 - Metric Spaces

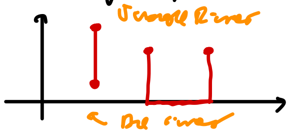
(X, d) is a metric space

⑨ **Metric on a set:** A metric d on a set X is a map $d: X \times X \longrightarrow \mathbb{R}^+$ s.t.

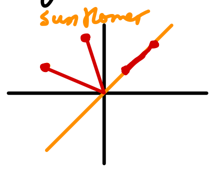
(i) $d(x,y) = 0 \iff x = y$, (ii) $d(x,y) = d(y,x) \quad \forall x, y \in X$, (iii) $d(x,z) \leq d(x,y) + d(y,z)$

(10) **Norm ⇒ metric:** X a vector space, $\|\cdot\|: X \to \mathbb{R}$ a norm, then $d(x,y) = \|x-y\|$ is a metric

(11) **Examples of norms:**



good for counter-examples

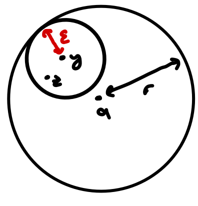


- discrete metric: $d(x,x)=0$, $d(x,y)=1$ if $x \neq y$.
- Hamming distance: # different pairs between genes. E.g. $d(AAAC, AAAA) = 1$
- combinatorial metric: Graph G, d = minimum # edges to join a graph G.
- Sunflower metric on $\mathbb{R}^2$: $d(x,y) = \begin{cases} \|x-y\| & \text{if } (x,y) \text{ lie on same line through } 0 \\ \|x\|+\|y\| & \text{otherwise} \end{cases}$
- Jungle River metric on $\mathbb{R}^2$: $d((x_1,y_1),(x_2,y_2)) = \begin{cases} |y_1-y_2| & \text{if } x_1 = x_2 \\ |y_1|+|x_1-x_2|+|y_2| & \text{if } x_1 \neq x_2 \end{cases}$

(12) **Metrics on Subspaces & Products:** (X,d) metric space, $A \subset X$, then $(A, d|_A)$ is a metric space and can do products too...

(13) **Open & Closed sets:** A list of some crucial definitions. (X,d) metric space.

- **Open ball** centered $a \in X$, radius r: $B(a,r) = \{x \in X : d(x,a) < r\}$
- **closed** " " " ": $\overline{B}(a,r) = \{x \in X : d(x,a) \leq r\}$
- Subset S of (X,d) **bounded** if $\exists a \in X$, $r > 0$ s.t. $S \subset B(a,r)$
- Subset $U \subset X$ is **open** if $\forall x \in U$, $\exists \varepsilon > 0$ s.t. $B(x,\varepsilon) \subset U$
- Subset $F \subset X$ is **closed** (in (X,d)) if $X \setminus F$ is open.



(14) **Fundamental Results:**

- Open balls are open. Pf: Pick $\varepsilon = r - d(y,a)$, then $B(y,\varepsilon) \subset B(a,r)$ $\because$ $d(z,a) \leq \dots \leq r$
- Finite intersections of open sets are open. $U_1, \dots, U_n$ open $\Rightarrow \bigcap_{i=1}^n U_i$ open

min is well defined... Pf: Pick $x \in \bigcap_{i=1}^n U_i$, pick $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$, then $B(x,\varepsilon) \subset B(x,\varepsilon_i) \subset U_i$ so $B(x,\varepsilon) \subset \bigcap_{i=1}^n U_i$. If countable, not true! $\bigcap_{i=1}^\infty \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ not open.

- Finite unions of closed sets are closed. $F_1, \dots, F_n$ closed $\Rightarrow \bigcup_{i=1}^n F_i$ closed.

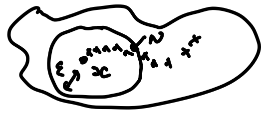
Pf: De-Morgan. $X \setminus \bigcup_{i=1}^n F_i = \bigcap_{i=1}^n (X \setminus F_i)$ so this open hence ■

- Any union of open sets is open. If $\{U_i : i \in I\}$ open $\Rightarrow U = \bigcup_{i \in I} U_i$ open

Pf: $x \in U \Rightarrow x \in U_i$ for $i \in I$, U_i open so this ε_i works for U

- Any intersection of closed sets is closed.

Pf: de morgan again.



(15) **Convergence of Sequences:**

- A sequence $x_n \to x$ in (X,d) if $\lim_{n\to\infty} d(x_n, x) = 0 \Leftrightarrow \forall \varepsilon > 0\ \exists N \geq 1$ s.t. $x_n \in B(x,\varepsilon)\ \forall n > N$
- Sequence in a metric space has at most one limit.

Pf: suppose not & $0 \leq d(x,y) \leq d(x,x_N) + d(x_N,y) \to 0 \Rightarrow x = y$ ■

- Open set convergence: $x_n \to x \Leftrightarrow \forall$ open sets U w/ $x \in U$, $\exists N \geq 1$ s.t. $x_n \in U\ \forall n \geq N$.

Pf: ...

- closed set convergence: $F \subset X$ closed $\Leftrightarrow x_n \in F$ w/ $x_n \to x \in X \Rightarrow x \in F$

(A set is closed $\Leftrightarrow$ it contains all of its limit points) Pf: contrapositive...

Chapter 4 – Continuity

(16) **Continuity in metric spaces:** Let (X, d_X), (Y, d_Y) be two metric spaces $f: X \to Y$

- $\lim_{x \to p} f(x) = y \in Y$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $0 < d_X(x,p) < \delta \Rightarrow d_Y(f(x), y) < \varepsilon$
- f continuous at $p \in X$ if $\lim_{x \to p} f(x) = f(p)$. use ε-δ above
- f Lipschitz cts if $\exists C > 0$ s.t. $d_Y(f(x), f(y)) \leq C\, d_X(x,y)$ $\forall x, y \in X$
- $f: X \to Y$ cts $\Leftrightarrow \forall x_n \in X$ s.t. $x_n \to x$, $f(x_n) \to f(x)$
- Still have usual addition & quotient laws. f/g cts $\forall x \in X$ s.t. $g(x) \neq 0$
- $f: X \to Y$ cts $\Leftrightarrow \forall$ open sets $U \subset Y$, $f^{-1}(U)$ open in X.

Proof: '$\Rightarrow$' ① Take open set $U \subset Y$, $x \in f^{-1}(U) \Rightarrow f(x) \in U$ open

so $\exists \varepsilon > 0$ s.t. $B_Y(f(x), \varepsilon) \subset U \Leftrightarrow$ if $d_Y(f(x), y) < \varepsilon \Rightarrow y \in U$

② f continuous: $\exists \delta > 0$ s.t. $d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$

③ So, if $x' \in B_X(x, \delta) \Rightarrow f(x') \in B_Y(f(x), \varepsilon) \subset U$

④ That is, $B_X(x, \delta) \subset f^{-1}(U) \Rightarrow f^{-1}(U)$ open

'$\Leftarrow$' Suppose $U \subset Y$ open $\Rightarrow f^{-1}(U)$ open. Take $x \in X$, $\varepsilon > 0$.

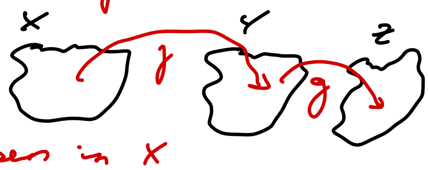
① Note that $B_Y(f(x), \varepsilon)$ open in $Y \Rightarrow f^{-1}(B_Y(f(x), \varepsilon))$ open in X

centering ball

② $x \in f^{-1}(B_Y(f(x), \varepsilon)) \xRightarrow{\text{by openness}} \exists \delta > 0$ s.t. $B_X(x, \delta) \subset f^{-1}(B_Y(f(x), \varepsilon))$

③ But this means $d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon \Rightarrow f$ cts ■

- **Composition cts:** $(X, d_X), (Y, d_Y), (Z, d_Z)$ metric spaces, Then $g \circ f : X \longrightarrow Z$ cts.



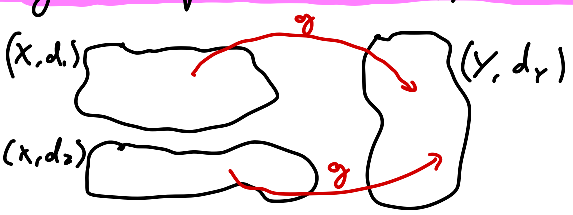
Pf: U open in $Z \Rightarrow g^{-1}(U)$ open in $Y \Rightarrow f^{-1}(g^{-1}(U))$ open in X

But $f^{-1}(g^{-1}(x)) = (g \circ f)^{-1}(x)$ so $(g \circ f)^{-1}(U)$ open in $X \Rightarrow g \circ f$ cts. ■

- Share open sets $\Rightarrow$ cts functions: If d_1, d_2 are two metrics on X, then

(i) Every set open in (X, d_1) is open in (X, d_2)

(ii) If



For any metric space (Y, d_Y), $g : X \longrightarrow Y$ cts from $(X, d_2) \to (Y, d_Y)$ $\Rightarrow g$ cts from (X, d_1) to (Y, d_Y)

[& vice versa]

- **Topologically equivalent if open sets coincide.**
(Lipschitz equivalent is same as equivalent norms)

(16)

Homeomorphism

If $f : X \to Y$ is a bijection and f, f^{-1} cts Then f is a homeomorphism

U open in $X \Leftrightarrow f(U)$ open in Y

Isometry

$f : X \longrightarrow Y$ bijective s.t. $d_Y(f(x), f(y)) = d_X(x, y)$ $\forall x, y \in X$ Then f isometry.

distance preserving

Chapter 5 – Topological Spaces

(17) **A topology:** τ on a set X, is a collection of subsets of X which we agree to call the open sets s.t.

(T1) T, ϕ open

(T2) intersection of finitely many open sets open $\bigcap_{i=1}^{n} U_i \in \tau$

(T2) arbitrary unions of open sets open $\bigcup_{i \in \tau} U_i$ open

The pair (X, τ) is a topological space.

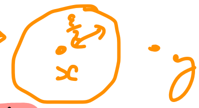
[Theres a closed set equivalent]

Examples

- topology induced by metric
- discrete topology: all subsets open
- indiscrete topology: only $\phi, X \in \tau$
- cofinite / co countable: set open if its ϕ, X or complement finite/countable

(18) (X, τ) is **metrisable** if there's a metric that gives rise to the same open sets.

open! and not $\emptyset$ or X



(19) **If $|X| \geq 2$, then the indiscrete topology is not metrisable.**

Pf: Suppose τ induced by a metric d on X. Pick $x, y \in X$ $x \neq y$, then $d(x,y) = \varepsilon > 0$. $B(x, \frac{\varepsilon}{2})$ open subset of (X, d), $x \in B(x, \frac{\varepsilon}{2})$ but $y \notin B(x, \frac{\varepsilon}{2})$ so found an open set that isn't $\emptyset$ or X ⨳

(20) **coarser / finer**: If τ_1, τ_2 are two topologies on X

- τ_1 **coarser** than τ_2 if $\tau_1 \subset \tau_2$ [τ_1 has fewer open sets]
- τ_2 **finer** than τ_2 if $\tau_1 \subset \tau_2$ [τ_2 has more open sets]

(21) **A basis for a topology τ on X**: is a collection $\beta \subset \tau$ s.t. every set in τ is a union of sets from β. $\forall U \in \tau$ $\exists C_U \subset \beta$ s.t. $U = \bigcup_{B \in C_U} B$

(22) **Subspace topology**: If (X, τ) top. space, $S \subset X$, then subspace topology on S is

$\tau_S = \{ U \cap S : U \in \tau \}$ ← topologised subspace of X

$$\begin{tikzcd} d_X \arrow[r, squiggly] \arrow[d, squiggly] & \tau_X \arrow[d, squiggly, "?"] \\ d_Y \arrow[r, squiggly, "?"'] \arrow[ur, phantom, "? \; ="] & \tau_Y \end{tikzcd}$$

(23) **Subspace topology matches metric subspace:**

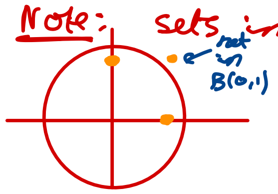
Pf: The key is that $B_X(a, \varepsilon) \cap Y = B_Y(a, \varepsilon)$

(24) **Product topology**: $(X, \tau_X), (Y, \tau_Y)$ two topological spaces, then product topology on $X \times Y$ is topology τ with basis:

$$\beta = \{ U_1 \times U_2 : U_1 \in \tau_X, U_2 \in \tau_Y \}$$

another check w/ metrics omitted.

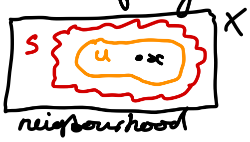
Note: sets in product topology τ are not all of the form $U_1 \times U_2$ for $U_i \in \tau_i$.



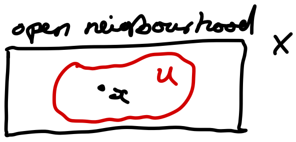
Take $B(0,1) \subset \mathbb{R}^2$. open in product topology.
Suppose $B = U_1 \times U_2$, then $(\frac{3}{4}, 0), (0, \frac{3}{4}) \in B(0,1)$
so $\frac{3}{4} \in U_1$, $\frac{3}{4} \in U_2$ but $U_1 \times U_2 \ni (\frac{3}{4}, \frac{3}{4}) \notin B(0,1)$

(25) **Closure, interior, boundary**: let (X, τ) be a topological space.

- A **neighbourhood of $x \in X$** is a subset S s.t. $\exists U \in \tau$ open with $x \in U \subset S$

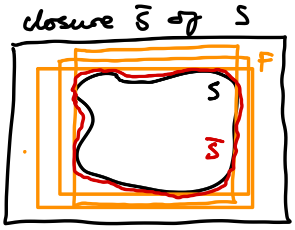


- An **open neighbourhood of $x \in X$** is a open set $U \in \tau$ s.t. $x \in U$



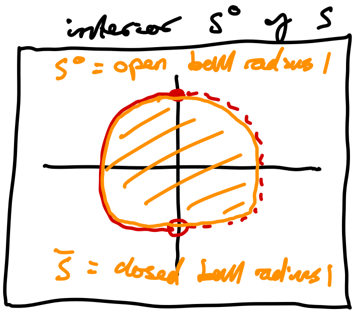
- The **closure of $S \subset X$, $\bar{S}$** is the intersection of all the closed sets that contain S.

$$\boxed{\bar{S} = \bigcap_{\substack{S \subset F \\ F \text{ closed}}} F}$$



- The **interior of S, S°** is the biggest open set contained in S, the best approximation to S from the inside.

$$\boxed{S^\circ = \bigcup_{\substack{U \subset S \\ U \text{ open}}} U}$$



- The **boundary ∂H of H** is the set of x s.t. every neighbourhood of x meets both H and its complement.

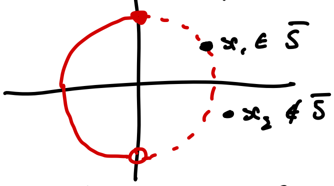
(always closed as $\partial H = \bar{H} \cap \overline{X \setminus H}$)

$\partial H = \{ x \in X : \text{if } x \in U \text{ open, then } U \cap H \neq \emptyset \text{ and } U \cap (X \setminus H) \neq \emptyset \}$

$$\boxed{\partial H = \bar{H} \setminus H^\circ}$$

(26) Results relating closure, interior, boundary:

- $\overline{S} = \bigcap_{\substack{S \subset F \\ F \text{ closed}}} F = \{ x \in X \mid \text{every open neighbourhood of } x \text{ intersects } S \} = \{ \text{limits of convergent sequences in } S \}$

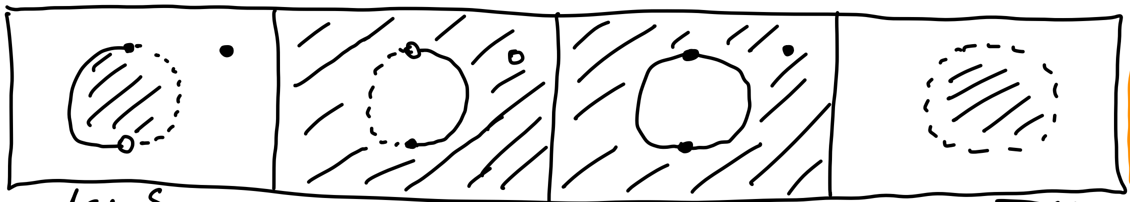


"look a little around x and must find S"

"set of things you can approximate by elements in your set"

- $S^\circ = \bigcup_{\substack{U \subset S \\ U \text{ open}}} U = \{ x \in X \mid S \text{ is a neighbourhood of } x \} = X \setminus \overline{(X \setminus S)}$

Similar picture for $\overline{S} = X \setminus (X \setminus S)^\circ$



Useful:
$\overline{A} = A \cup \partial A$
$A^\circ = A \setminus \partial A$

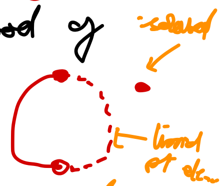
Other results:

- $S \subset \overline{S}$
- $\overline{S}$ is closed
- F closed, $S \subset F \Rightarrow \overline{S} \subset F$
- S closed $\Rightarrow \overline{S} = S$
- If $S_1 \subset S_2 \Rightarrow \overline{S_1} \subset \overline{S_2}$
- $\overline{S_1 \cup S_2} = \overline{S_1} \cup \overline{S_2}$

- S° open
- If U open $U \subset S$, then
  - S° open
  - $S^\circ \subset S$
  - $U \subset S^\circ \subset S$

(27) A limit point of a set: is a point $x \in X$ s.t. every neighbourhood of x intersects $S / \{x\}$ [doesn't count if you only see yourself]

An isolated point is a point in X that's not a limit point

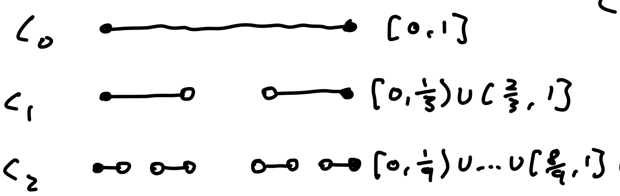


E.g. $X = \mathbb{R}$, $S = \{ \frac{1}{n} \mid n \in \mathbb{N} \} \cup \{0\}$. 0 is only limit pt, $\frac{1}{n}$ isolated.

(28) A subset $S \subset X$ is

can't be both (sliding scale!)
- dense in X if $\overline{S} = X$ — $\mathbb{Q}$ is dense in $\mathbb{R}$, sees all of $\mathbb{R}$ while $\mathbb{Q} \subset \mathbb{R}$
- nowhere dense if $(\overline{S})^\circ = \phi$ — $\mathbb{Z}$ nowhere dense in $\mathbb{R}$, $\overline{\mathbb{Z}} = \mathbb{Z}$, $(\overline{\mathbb{Z}})^\circ = \phi$

(29) Cantor's set: At every step, delete the middle 3rd. Fractal!



$C_0 \subset C_1 \subset \dots \subset C_N$ and we say

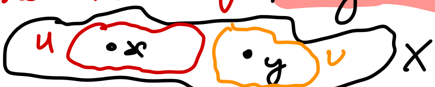
$$C = \bigcap_{n=0}^{\infty} C_n$$

① $C \neq \phi$ Pf: contains interval endpoints.
② C closed Pf: arbitrary intersection of closed sets
③ $\overline{C} = C$ ④ $C^\circ = \phi$
④ $\partial C = C$

C is nowhere dense
contains no isolated points.

(30) (X, τ) metrisable: if $\exists$ metric d on X s.t. $\tau = \{ \text{open sets in } (X,d) \}$

Note: To show not metrisable, show (X,τ) not Hausdorff [any metrisable topological space is Hausdorff]



(31) Topological space is Hausdorff: if $\forall x \neq y \in X$, $\exists$ disjoint $U, V \in \tau$ s.t. $x \in U$, $y \in V$

limits unique in this!

(32) Continuity: use open set versions from metric spaces as definitions!

(33) **Homeomorphisms:** (same as metric) (X, τ_X) and (Y, τ_Y) topological spaces. Then a bijection $f: X \to Y$ is homeomorphism if any hold
(i) f and f^{-1} continuous
(ii) V is open in $Y \iff f^{-1}(V)$ open in X
(iii) U is open in $X \iff f(U)$ open in Y

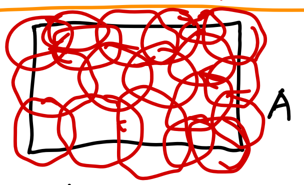
If this exists then X and Y are homeomorphic

(34) **Topological Invariant:** A property is — if it is preserved by homeomorphisms.
E.g. T finite, T Hausdorff, T metrisible, # connected components.

Chapter 6 - Compactness

This is a topological property.

$$\boxed{A \subset \bigcup_{u \in \mathcal{U}} u}$$



(35) **Compactness:**

- A **cover** of a set A is a collection $\mathcal{U}$ of sets whose union is A
- A **subcover** of a cover $\mathcal{U}$ is a subset of $\mathcal{U}$ that still covers A.
- A cover is **open** if all its elements are open.
- A topological space (X, τ) is **compact** if every open cover of X has a finite subcover. Same thing for subspaces.

Example: $X = (0,1)$ is not compact. Take $\mathcal{U} = \{(0,x): x \in X\}$. Finite subcover is $\{(0,x_1), \dots, (0,x_n)\}$. Finitely many so set $x = \max x_i$. Then $\bigcup_{i=1}^{n} (0, x_i) = (0,x)$. Pick $b \in (x, 1)$ $b \notin \bigcup_{i=1}^{n} (0, x_i)$ so no finite subcover.

This goes to $\mathbb{R}^n$

(36) **Heine Borel Theorem:** **Any closed interval $[a,b]$ is a compact subset of $\mathbb{R}$.**
Given an arbitrary open cover want to find a finite subcover.

Pf: Suppose $\mathcal{U}$ is any open cover of $[a,b]$. Let S be
$$S = \{x \in [a,b] \mid [a,x] \text{ has a finite cover } \mathcal{U}' \subset \mathcal{U}\}$$
Set of places that we have a finite cover

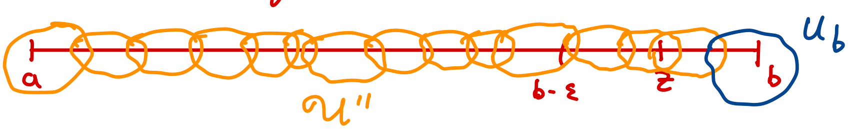
① Let $x = a$, $[a,a] = \{a\}$, since $\mathcal{U}$ covers $[a,b]$, $\exists U_a \in \mathcal{U}$ s.t. $a \in U_a$ so there exists an open set that a lies in $\Rightarrow S \neq \emptyset$

② $S \subset [a,b] \Rightarrow S$ bounded above by b. Least upper bound principle says S has a supremum $\sup(S) = c$, $c \in [a,b]$ [$S \neq \emptyset$ & bdd above]

③ WTS $c = b$, $b \in S$. For contradiction suppose $c < b$. As $c \in [a,b]$ & $\mathcal{U}$ is a cover of $[a,b]$, $\exists U_c \in \mathcal{U}$ s.t. $c \in U_c$. U_c open $\Rightarrow \exists \delta > 0$ s.t. $(c-\delta, c+\delta) \subset U_c$.

④ c is a least upper bd so $\exists x \in (c-\delta, c]$ s.t. $x \in S$. This means we have a finite subcover $\mathcal{U}' \subset \mathcal{U}$ of $[a,x]$. Then pick y s.t. $c < y < c+\delta \leq b$
$\mathcal{U}' \cup \{U_c\}$ is a finite subcover of $[a,y]$. $\Rightarrow y \in S$ but $y > c$ ✗.

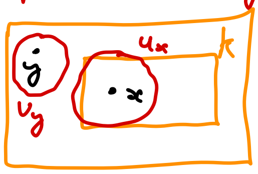
⑤ So $c = b$. Note: supremum's don't have to lie in the set so WTS $b \in S$.
As $\mathcal{U}$ covers $[a,b]$, $\exists U_b \in \mathcal{U}$ s.t. $b \in U_b$. U_b open $\Rightarrow \exists \varepsilon > 0$ s.t. $(b-\varepsilon, b] \subset U_b$. b supremum $\Rightarrow \exists z \in S$ s.t. $z \in (b-\varepsilon, b]$. So $\exists \mathcal{U}'' \subset \mathcal{U}$ s.t. $\mathcal{U}''$ finite & covers $[a,z]$. $\mathcal{U}'' \cup \{U_b\}$ finite subcover of $\mathcal{U}$ covering $[a,b] \Rightarrow [a,b]$ compact. ∎



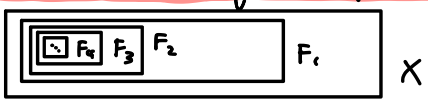
Banach space = complete normed space

(37) **Compactness Results** [should know proofs but omitted to save time]

- Any closed subset S of a compact space X is compact
  Pf: Let $\mathcal{U}$ cover S, $\mathcal{U} \cup \{X \setminus S\}$ covers X so finite subcover $\Rightarrow$ finite subcover of S.
- Any compact subset K of a Hausdorff space X is closed.



  Pf: Pick $y \in X \setminus K$, $\forall x \in K$ $\exists$ disjoint open sets $U_x \ni x$ and $V_y \ni y$. $\{U_x : x \in K\}$ open cover of K so $\exists$ finite subcover $U_{x_1}, \dots, U_{x_n}$ of K. Hence $V = \bigcap_{i=1}^{n} V_{y_i}$ is an open set s.t. $y \in V$ and $V \cap K = \emptyset$. Hence $X \setminus K$ open $\Rightarrow$ K closed.
- Any compact subset K of a metric space (X,d) is bounded.
  Pf: Cover with open balls, then finite subcover & pick the largest.
- A subset of $\mathbb{R}$ is compact $\Leftrightarrow$ it's closed & bounded.
  Pf: '$\Rightarrow$' Suppose K compact
  ① $\mathbb{R}$ metric : compact $\Rightarrow$ bbd
  ② $\mathbb{R}$ Hausdorff : compact $\Rightarrow$ closed
  '$\Leftarrow$' Suppose K closed & bounded
  ③ Since K bbd $\exists a,b$ s.t. $K \subset [a,b]$
  ④ By Heine Borel : $[a,b]$ compact
  ⑤ K closed in $[a,b]$ so closed in compact $\Rightarrow$ K compact.
- Let $F_1 \supset F_2 \supset \dots$ be non empty closed subsets of a compact space X, then
  $$\bigcap_{i=1}^{\infty} F_i \neq \emptyset$$



Proof: ① Assume $\bigcap_{i=1}^{\infty} F_i = \emptyset$. Then set $U_i := X \setminus F_i$, U_i open as F_i closed.
② As we have $\underbrace{}$ $\Rightarrow$ $\bigcup_{i=1}^{\infty} U_i = X$. X compact so $\exists U_1, \dots, U_n$ s.t.
③ $\bigcup_{i=1}^{n} U_i = X = U_n$ ← take largest n, $F_n = X \setminus U_n = X \setminus X = \emptyset$ but assumed $F_i \neq \emptyset$ ※

- If (X, τ_X), (Y, τ_X) compact, τ product topology, then $(X \times Y, \tau)$ compact.
  Pf: long and subtle.
- Subset of $\mathbb{R}^n$ compact $\Leftrightarrow$ closed and bounded
- A continuous image of a compact set is compact
  Pf: let $\mathcal{U}$ be an open cover of $f(X)$, then $f^{-1}(U_j)$ open & cover X so finite # & translate to $y = f(x)$ so $U_1, \dots, U_n$ cover $f(X)$.
- A continuous bijection of a compact space T onto a Hausdorff space S is a homeomorphism.
  Pf: show $f^{-1} : S \longrightarrow T$ cts. Take $K \subset T$ closed. Show $f(K)$ closed.
- If $X \neq \emptyset$ & compact, then $f : T \longrightarrow \mathbb{R}$ cts is bbd & attains its bounds.
  Pf: T compact, f cts so $\mathbb{R}$ compact $\Leftrightarrow$ closed & bbd $\overset{F}{\Longrightarrow}$ contains a supremum.
  So $\forall n \in \mathbb{N}$ $\exists x_n$ s.t. $\sup F - \frac{1}{n} \leq x_n \leq \sup F$. F closed so $x_n \longrightarrow \sup F \in F$
- All norms on $\mathbb{R}^n$ are equivalent.
  Pf: ugh...

open sets say big...

(38) **Lebesgue number**: Let $\mathcal{U}$ be an open cover of a metric space (X,d). A number $\delta > 0$ is Lebesgue # for $\mathcal{U}$ if $\forall x \in X$, $\exists U \in \mathcal{U}$ s.t. $B(x,\delta) \subset U$

Note: Every open cover $\mathcal{U}$ of a compact metric space has a lebesgue #
Proof: $\forall x \in X$ $\exists r(x)$ s.t. $B(x, r(x)) \subset U(x)$ $\forall U(x) \subset \mathcal{U}$.
$\{B(x, \frac{r(x)}{2}) : x \in X\}$ open cover of X $\Rightarrow$ finite subcover.
and $\delta = \min_i \left(\frac{r(x_i)}{2}\right)$ is our δ. $B(x,\delta) \subset \dots \subset U(x_j)$

Recall: $f : (X, d_X) \longrightarrow (Y, d_Y)$ uniformly cts if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.
$d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$ $\forall x, y \in X$

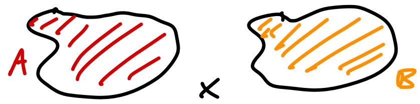
Apply: cts map from compact metric space to metric space is uniformly cts.
Pf: For $z \in X$ set $U_z = f^{-1}(B_Y(f(z), \varepsilon/2))$ open cover. use δ from lebesgue #

(39) **Sequential Compactness:** Subset K of metric space (X,d) is **sequentially compact** if every sequence in K has a convergent subsequence whose limit lies in K. $\forall x_n \in K \ \exists x_{n_k}$ s.t. $x_{n_k} \longrightarrow x \in K$.

↳ **Thm:** For a subset of a metric space **{compact} ⟺ {sequentially compact}**
Pf: Huge so omitted. ⇓ applies to normed spaces.

Note: closed unit ball in ℓ^p is not compact. Take $(e^{(j)})_{j=1}^{\infty}$. No convergent subsequence but $\|e^{(j)} - e^{(k)}\|_{\ell^p} = 2^{\frac{1}{p}}$ if p finite.

Chapter 7 — connectedness



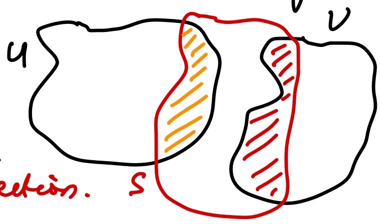
(40) **Connectedness definitions:** (A,B) **partition** X topological space, if $T = A \cup B$ and $A \cap B = \phi$. A & B partition X. X is **connected** if the only partitions of X into open sets are (X,ϕ) and (ϕ, X). X **disconnected** if not connected.

(41) **Equivalent disconnectedness criteria:** All below equivalent.
(i) X disconnected
(ii) X has a partition into two non-empty open / closed sets.
(iii) X has a subset that is both open & closed and neither ϕ or X.
(iv) There is a cts surjective function $f: X \longrightarrow \{0,1\}$ w/ discrete topology.

Pf: (ii) ⇒ (iv). Write $X = A \cup B$, A, B open, $A \cap B = \phi$. Set $f(x) = 0$ if $x \in A$, and $f(x) = 0$ if $x \in B$. $f^{-1}(0) = A$, $f^{-1}(1) = B$, $f^{-1}(\{0,1\}) = X$ so preimage of all open sets open so f cts.
(iv) ⇒ (ii) Assume $f: X \longrightarrow \{0,1\}$ cts surjection. Set $A = f^{-1}(0)$, $B = f^{-1}(1)$, both open as f cts. Non-empty as f surjective. $A \cup B = X$, $A \cap B = \phi$

Note: Can use (iv) to show set connected by showing any cts function constant.

(42) **Separated iff disconnected:** A set $S \subset X$ separated by subsets $U, V \subset X$ if $S \subset U \cup V$, $U \cap V \cap S = \phi$, $U \cap S \neq \phi$, $V \cap S \neq \phi$
A subset is disconnected ⟺ separated by some open subsets.

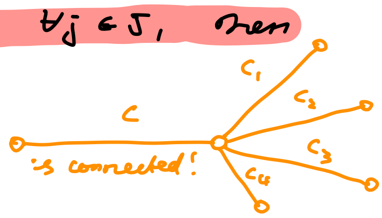


(43) **Connected subsets of $\mathbb{R}$:** Subset of $\mathbb{R}$ connected ⟺ interval.
Proof: ⟸ suppose I not connected. Then $\exists f: I \longrightarrow \mathbb{R}$ cts surjection. Set up $f: I \longrightarrow \mathbb{R}$ cts. to contradict IVT.

$\mathbb{R}^2$ not homeo to $\mathbb{R}$.

connectedness is a topological property if T, S homeomorphic, T connected ⇒ S connected

(44) **Operations on connected sets:**
- If C_j, $j \in J$ connected subsets of X, $C_i \cap C_j \neq \phi$ $\forall i,j$, then $K = \bigcup_{j \in J} C_j$ connected.
  Pf: suppose not, then f must be constant on K so not onto. ✗
- If C_1, C_2 connected, $\overline{C_1} \cap C_2 \neq \phi$, then $C_1 \cup C_2$ connected.
  Pf: again show f constant so not onto.
- If C and $\{C_j : j \in J\}$ connected, $C_j \cap \overline{C} \neq \phi$ $\forall j \in J$, then
  $$K = C \cup \bigcup_{j \in J} C_j \quad \text{connected.}$$



- Continuous image of a connected set is connected
  Pf: Suppose not, then $g \circ f : X \longrightarrow \{0,1\}$ cts & surjective so X disconnected ✗
- Product of two connected spaces connected.

homeomorphic.

Note: $X \setminus \{x\}$ is connected $\forall x \in X$ is a topological property. $\mathbb{R}^2 \neq \mathbb{R}$

(45) **Connected components:** $x \sim y$ if $x, y \in C$ where C is connected. The equivalence classes of $\sim$ are the **connected components** of X

– connected components are connected & closed & the maximal connected subsets of X

Note: Number of connected components is a topological property. ← invariant under homeomorphism

$f: \mathbb{R}^2 \to \mathbb{R}$. Pick $c \in \mathbb{R}$, then $\exists x$ s.t. $f(x) = c$. $\mathbb{R}^2 \setminus \{c\}$ connected, $\mathbb{R} \setminus \{x\}$ disconnected ✗.
$\therefore f(\mathbb{R}^2 \setminus \{c\}) = \mathbb{R} \setminus \{x\}$ & cts image of connected is connected.

(46) **Path connected:** If $u, v \in X$, a **path** from u to v is a cts map $\psi: [0,1] \longrightarrow X$ s.t. $\psi(0) = u$, $\psi(1) = v$. X is **path connected** if any two points in X can be joined by a path in X.

Note: path connected $\Rightarrow$ connected [not conversely]

Pf: Fix $u \in X$, for $v \in X$, $\exists$ path from u to v. $C_v = \text{im}(\phi)$ connected. Then $X = \{u\} \cup \bigcup_{v \in X} C_v$ & each C_v contains u so connected.

Note: connected open subsets of $\mathbb{R}^n$ are path connected.

Chapter 8 – Completeness in metric spaces

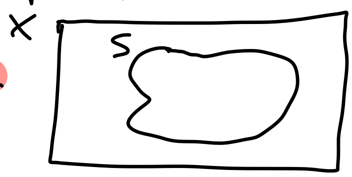
Proofs non-examinable here...

(47) **Completeness:** A metric space (X, d) is **complete** if any cauchy sequence in X converges

Note: $(0,1)$ not complete. $(1 - \frac{1}{n})_{\in (0,1)} \longrightarrow 1_{\notin (0,1)}$. $\mathbb{Q}$, not complete $1 + \frac{1}{2!} + \frac{1}{3!} + \dots \longrightarrow e \notin \mathbb{Q}$.
$\mathbb{R}$ is complete & $(0,1)$ & $\mathbb{R}$ homeomorphic $\Rightarrow$ completeness is not a topological property.

(48) **Relationships between complete / closed / compact:** (X,d) metric space, $S \subset X$.

1. If S complete, then S closed in X
2. If X complete & S closed in X, then S also complete.



(49) **Every compact metric space is complete.**

Note: complete $\not\Rightarrow$ compact [say $\mathbb{R}$. complete but not compact]

(50) **Examples of complete spaces**

- $\mathbb{R}^d$ is complete
- ℓ^p is complete (sequence space)
- For any set $X \neq \phi$, $B(X)$ – the space of bounded real-valued functions $f: X \to \mathbb{R}$ w/ sup norm $\|f\| = \sup_{x \in X} |f(x)|$ is complete.
- $\underbrace{C_b(X) \subset B(X)}_{\text{closed subspace of } B(X)} \Rightarrow C_b(X)$ complete. $[C_b(X) = \{ f: X \to \mathbb{R} \text{ bdd \& cts}\}]$
- If X compact, then $C(X)$ is complete with max norm $\|f\|_\infty = \max_{x \in X} |f(x)|$

Pf: $f \in C(X)$ & X compact $\Rightarrow f$ bdd $\Rightarrow C(X) = C_b(X) \Rightarrow f$ attains bounds so $\sup_{x \in X} |f(x)| = \max_{x \in X} |f(x)|$

maximum must lie in set
supremum doesn't have to

(51) **Completions:** How do you 'complete' a space A by adding missing limits?
Method 1: Find a complete metric space that contains A s.t. $X = \bar{A}$ [$\mathbb{R}$ is completion of $\mathbb{Q}$]

A List of topological Properties

1. Compactness   e.g. Cantor set C uncountable, $\mathbb{Q}$ is countable. not homeomorphic
2. countable / uncountable
3. Connectedness
4. number of connected components
5. number of connected components AFTER removing a point.

consider $[0,1) \cup [2,3)$ and $(0,1) \cup (2,3)$

remove $\{0, 2\}$ and still have two connected components

remove any point & will have 3 connected components

Review lecture

- continuous image of a connected space is connected.

- Q: $X=(0,1)$, X closed, X bounded, X not compact how?

closure where?
interior where?

closed in itself
open in itself
always relative to something.

confirming this in $B(\frac{1}{2}, \frac{1}{2})$ so bounded (metric)

topology matters.

$\mathcal{U} = \{(0,x) : x \in (0,1)\}$

$\mathcal{U} = \{(\frac{1}{n}, 1) : n \in \mathbb{N}\}$

Take finite so n largest

$(0, \frac{1}{n_{max}}) \subset (0,1)$

but $(0, \frac{1}{n_{max}}) \not\subset \mathcal{U}$

closed in $\mathbb{R}^n$ bdd $\Leftrightarrow$ compact

$(0,1)$ not closed in $\mathbb{R}$ so not compact.
(is closed in $(0,1)$)

- Compact $\Leftrightarrow$ closed & bounded on $\mathbb{R}^n$ [can show homeomorphic]

- open sets vs open neighbourhoods

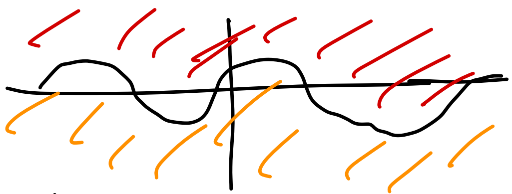
not clear where

neighbourhood of this point $x \in U$ and U open

- 2021 Exam Part 2 is too hard in an exam

Q3c $f : \mathbb{R} \longrightarrow [-1,1]$

$G = \{(x, f(x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2$



(a) Assume f cts. Show $\mathbb{R}^2 \setminus G$ disconnected.

$U = \{(x,y) \in \mathbb{R}^2 : y > f(x)\} \subset \mathbb{R}^2 \setminus G$ [upper part]

$L = \{(x,y) \in \mathbb{R}^2 : y < f(x)\} \subset \mathbb{R}^2 \setminus G$ [lower part]

$U \cap L = \emptyset$, $U \cup L = \mathbb{R}^2 \setminus G$ so partition $\mathbb{R}^2 \setminus G$

To show disconnected, $U, V \neq \emptyset$ and U, V open.

Take $x=0$, $f(x) \in \mathbb{R}$, pick $\underbrace{f(x)+1}_{y_1}$, $\underbrace{f(x)-1}_{y_2}$ $(x, y_1) \in U$, $(x, y_2) \in L$

so $U, L \neq \emptyset$

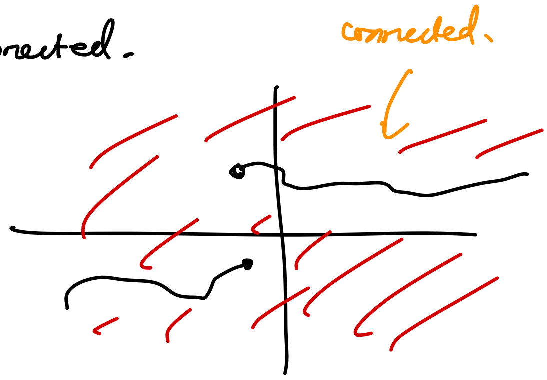
Now show open. why?

Define $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $h(x,y) = y - f(x)$ (cts) and $h^{-1}((0,\infty))$ open

$h^{-1}((-\infty, 0))$ open [as h cts & $(0,\infty), (-\infty, 0)$ open]

$\mathbb{R}^2 \setminus G$ is union of non-empty disjoint open sets. $\Rightarrow$ disconnected.
[do show non-empty]

(b) If f is discontinuous $\Rightarrow$ $\mathbb{R}^2 / G$ is connected.



Proof: Path connected $\Rightarrow$ connected.

BUT not nice jump discontinuity.

f disconnected $\Leftrightarrow$ $\exists f : X \to \{0,1\}$ surjective cts.

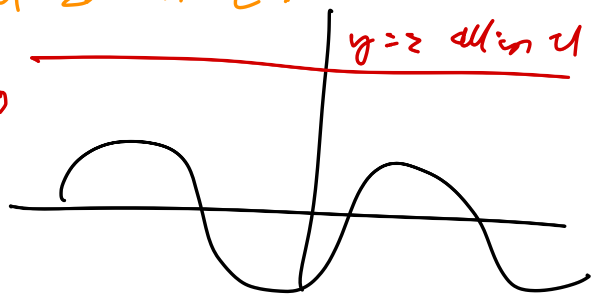
Prove every cts f on $\mathbb{R}^2 \setminus G$ only takes one value.

Consider U & L

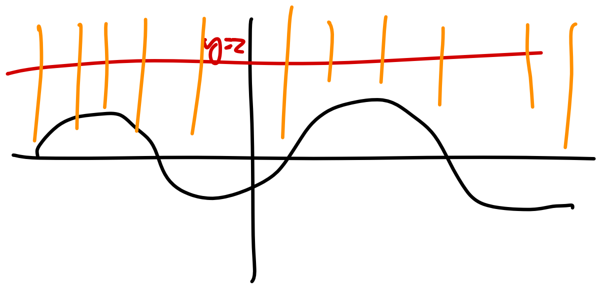
Suppose $h : \mathbb{R}^2 \setminus G \longrightarrow \{0,1\}$ cts.

Claim 1: Takes constant values on U & on L.

$h|_U$ constant.



So cts f can't change value on this line so all zero.



vertical lines are connected & meet horizontal line so f is all zero on all of U.

Similarly h constant on L. When changes?

on $L = U$

- Ideas in proofs are examinable. Spend more time on examples. What is the Lebesgue number? Is one a Lebesgue # for this open covers. Apply theorems.
- Past exam papers can maybe be ignored. 3 compulsory Qs.

Hard Questions

Exercise 2. (‡) Show that if (X, d) is a metric space then the metric $d_1(x, y) := \min(1, d(x, y))$ (introduced in Problem Sheet 2) is topologically equivalent to d.