In kinetic theory they would say there is a distribution function of velocities at every point and time, but in fluids we characterise every point with just one number describing the velocities.
If given L (size of the system) and λ (mean path of particles), then the fluid approach is valid when $\lambda << L$. Usual formula for λ is $\lambda = \frac{1}{n\sigma}$ with n number density and δ cross-section.
Neutrals: $\delta = 10^{-15} cm^2$ way you figure it out is by roughly taking the cross-sectional area of a hydrogen atom.
Ionized species: $\delta = 10^{-4}\left(\frac{k}{T}\right) cm^2$. Interaction is so strong between positive and negative charges they are scattering through 90 degrees. So the higher the temperature the faster the particles move and the small the distance between the particles has to be before they get deflected as much.
Consider in the case of the bow and termination shocks of the sun. The gas that has been flowing towards the sum has $n \approx 1cm^{-3}$ and $T \approx 10^4 K$, and the bow shock occurs about about $100 - 150AU$. For neutrals this gives $\lambda = 10^{15} cm = 70AU$ which is quite close to the size of the shock, so therefore the neutrals are weakly collisional so the fluid approach is not going to be very good at describing their behaviour. For ionized particles, this gives $\sigma = 10^{-12} cm$, so $\lambda = 10^{12} cm = 0.1AU$. Therefore, for ionized species $\lambda << L$ so the fluid approach is good.

0.1 Definitions

Characterise fluid by $\boldsymbol{u}$ (sometimes written $\boldsymbol{v}$), p(pressure) and ρ (density) at point $\hat{X}(\hat{r})$ and time t
Eulerian Derivative: Characterises the derivative at a given point in space

$$\frac{\partial}{\partial t}|_{\hat{X}}$$

Lagrangian time Derivative: Taken while moving with the fluid. $Df = f(x_0 + u\delta t, t + \delta t) - f(x_0, t) = (\frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla) f)\Delta t$

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla) f$$

0.2 Evolution of a line Element

$$\delta x(t + \Delta t) = \delta x(t) + \Delta t(\delta x(t) \cdot \nabla)\boldsymbol{u}$$

$$\frac{D\delta x}{Dt} = (\delta x \cdot \nabla)\boldsymbol{u}$$

0.3 Continuity Equation

Adopt Eulerian approach. Fixed $\hat{x}$, fixed volume element, and mass. $M = \int_V \rho dV$ - varies with time with:

$$\frac{\partial M}{t} = \frac{\partial}{\partial t}\int_V \rho dV = \int_V \frac{\partial \rho}{t} dV$$

$$\frac{\partial M}{t} = -\int_{\partial V} \rho \boldsymbol{u} \cdot d\boldsymbol{S} = -\int_V \nabla \cdot (\rho \boldsymbol{u}) dV$$

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V \nabla \cdot (\rho \boldsymbol{u}) dV$$

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u)\right) dV = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \tag{1}$$

0.4 Momentum Equation

Stick to Lagrangian approach. Follow mass element M as it moves and apply newtons law to this element. $M\frac{d\boldsymbol{u}}{dt} = \frac{D\boldsymbol{u}}{Dt} = F$. Possible forces: Pressure, gravity, body forces

Pressure - always acts on the surface so : $p = -\int_{\partial V} pd\boldsymbol{S} = -\int_V \nabla p dV$

Gravity - given by gravitational potential (ϕ) ($-\nabla \phi M = -\nabla \phi \int_V \rho dV = -\int_V \rho \nabla \phi dV$

Bodily forces - $f_e = \frac{F_e}{M}$

$$M\frac{D\boldsymbol{u}}{Dt} = \int_V \frac{D\boldsymbol{u}}{dt} \rho dV = -\int_V \nabla p dV - \int_V \rho \nabla \phi dV + \int_V f_e \rho dV$$

$$\rho \frac{Du}{Dt} = -\nabla p - \rho \nabla \phi + \rho f_e \tag{2}$$

$$\frac{\partial u}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{\nabla P}{\rho} - \nabla \phi + f_e \tag{3}$$

If we know $\phi(x)$ (Cowling Approximation) and $P = P(\rho)$ (barotopic approximation) then these equations are sufficient to describe the system.

0.5 Examples of barotopic fluids

Isothermal fluid: $P = c_s^2 \rho$, c_s is sound speed and $c_(x) =$ constant
Adiabatic fluid: has constant entropy everywhere, $S(x) =$ constant

As $S(P, \rho) = constant$ there must be a unique $P(\rho)$ relation.

For an ideal gas, $S = c_\sigma \ln \frac{P}{\gamma\rho}$ where c_σ is the specific heat at a constant volume and γ is the adiabatic index. γ is given by

$$\gamma = \frac{c_p}{c_s} = \frac{T\frac{\partial S}{\partial T}_p}{T\frac{\partial S}{\partial T}_v} = \frac{\frac{\partial S}{\partial T}_P}{\frac{\partial S}{\partial T}_\rho}$$

As S is constant $P = k\rho^\gamma$ for $k = e^{\frac{S}{c_v}}$ (the adiabatic equation of state). The value of γ depends on the type of gas and often depends on the polytropic index n by $\gamma = 1 + \frac{1}{n}$.

Monoatomic gas: $n = \frac{3}{2}$ and $\gamma = \frac{5}{3}$
Diatomic gas: $n = \frac{5}{2}$ and $\gamma = \frac{7}{5}$

What if $\phi(x)$ depends on the density distribution. You need to use the Poisson equation:

$$\nabla^2\phi = 4\pi G\rho \tag{4}$$

$$\phi(x) = -G\int_V \frac{\rho(\boldsymbol{x}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|} d\boldsymbol{x}' - G\int_{V_{ex}} \frac{\rho(\boldsymbol{x}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|} d\boldsymbol{x}'$$

0.6 Drop barotropic assumption

Still assume ideal fluid so no dissapative effects - no heat transport/radiation transport/conductivity. Then entropy will still be conserved for each fluid element

Entropy is a material property so it belongs to a particular particle or element, so entropy being conserved means that its material derivative is 0: $\frac{DS}{Dt} = 0$. No longer have $S(P, \rho) = constant$ but still have one to one relationship and ρ.

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial \rho}_S \frac{D\rho}{Dt} = \frac{P}{\rho} \frac{\frac{1}{p}\partial p}{\frac{1}{\rho}\partial\rho} \frac{D\rho}{Dt} = \frac{p}{\rho} (\frac{\partial \ln p}{\partial \ln \rho})_s \frac{D\rho}{Dt} = \frac{p}{\rho}\Gamma_1 \frac{D\rho}{Dt}$$

$$\Gamma_1 = (\frac{\partial \ln p}{\partial \ln \rho})_s$$

where Γ_1 is the first adiabatic exponent. For an ideal gas $\gamma = (\frac{\partial \ln p}{\partial \ln \rho})_s \implies$

$$\frac{Dp}{Dt} = \gamma \frac{p}{\rho} \frac{D\rho}{Dt}$$

From continuity: $\frac{\partial \rho}{\partial t} + (\boldsymbol{u} \cdot \nabla)\rho + \rho\nabla \cdot \boldsymbol{u} = 0$. So:

$$\frac{D\rho}{Dt} = -\rho\nabla \cdot \boldsymbol{u} \tag{5}$$

So

$$\frac{Dp}{Dt} = -\gamma p \nabla \cdot \boldsymbol{u}$$

Can rewrite as energy equation:

$$\frac{\partial p}{\partial t} + (\boldsymbol{u} \cdot \nabla)p + \gamma p \nabla \cdot \boldsymbol{u} = 0 \tag{6}$$

An ideal fluid is described by equations (1), (3), (4) and (6).

0.7 Departures from $p = \frac{n}{\mu} kT$

We used the above assumption to get the equation for γ for an ideal gas of ($\gamma = (\frac{\partial \ln p}{\partial \ln \rho})_s$). In general $\Gamma_1 = (\frac{\partial \ln p}{\partial \ln \rho})_T \gamma$, and for an ideal gas $(\frac{\partial \ln p}{\partial \ln \rho})_T = 1$.

Whenever T is very high, $p = \frac{\rho}{\mu} k_B T + aT^4$ where $a = \frac{4}{3}\frac{\sigma_{sb}}{c}$.

Radiation pressure is important in: Early Universe, Centres of stars, Inner parts of acretion disks around neutron stars and black holes

1 MHD

Maxwell's equations

$$\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} \tag{7}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{8}$$

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \tag{9}$$

$$\nabla \cdot \boldsymbol{E} = 4\pi \rho_g = 4\pi \Sigma_i q_i \tag{10}$$

Ideal MHD assumes that conductivity of the fluid is infinite, $\sigma \to \infty$.

Lets switch to a co-moving with a fluid frame (that is moving with velcoity u). In this frame $\boldsymbol{J'}$, $\boldsymbol{E'}$ are related by Ohm's Law: $\boldsymbol{J'} = \sigma \boldsymbol{E'}$. If $\sigma \to 0$ then $\boldsymbol{E'} = 0$ in the co-moving frame.
Lorentz transformation $\boldsymbol{E'} = \frac{\boldsymbol{E} + \frac{\boldsymbol{u} \times \boldsymbol{B}}{c}}{\sqrt{1 - \frac{u^2}{c^2}}}$ but we will be considering non-relativistic limit with $\frac{u}{c} << 1$. So

$$\boldsymbol{E'} = \boldsymbol{E} + \frac{\boldsymbol{u} \times \boldsymbol{B}}{c} + O(\frac{u^2}{c^2})$$

As $E' = 0$:

$$\boldsymbol{E} = -\frac{\boldsymbol{u} \times \boldsymbol{B}}{c} \tag{11}$$

Say L and T are typical length and time scales of the problem $L \sim uT$:

$$\nabla \times B \sim \frac{B}{L}$$

$$\frac{1}{c}\frac{\partial E}{\partial t} \sim \frac{E}{cT} \sim \frac{B}{cT}\frac{u}{c}$$

So $\frac{\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}}{\nabla \times \boldsymbol{B}} \sim \frac{u^2}{c^2} << 1$, so:

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} \tag{12}$$

Can use 7 and 11 to derive the induction equation:

$$\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \frac{\boldsymbol{u} \times \boldsymbol{B}}{c}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \tag{13}$$

If $\boldsymbol{u}$ is known then we can solve this equation for $\boldsymbol{B}$. The equation is linear if $\boldsymbol{u}$ is specified. Kinematic limit when the B-field doesn't affect $\boldsymbol{u}$ much.

Take divergence of induction equation gives: $\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = 0$ so if a system starts solenoidal then it stays that way.

1.0.1 Magnetic force

per unit volume

$$\boldsymbol{F} = \frac{1}{c}\sum_i f_i \boldsymbol{u}_i \times \boldsymbol{B} = \frac{\boldsymbol{J} \times \boldsymbol{B}}{c} = \frac{c}{4\pi}\frac{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}}{c} = \frac{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi}$$

as:

$$\frac{\partial u}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{\nabla P}{\rho} - \nabla\phi + f_e = -\frac{\nabla P}{\rho} - \nabla\phi + \frac{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi}$$

$$\frac{\partial u}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{\nabla P}{\rho} - \nabla\phi + \frac{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi}$$

(14)

This has no electrostatic term ($\rho_g E$) as in the non-relativistic limit it is negillibe compared to the magentic term.

$$F_{m,i} = \frac{1}{4\pi}{}_{ijk}{}^{jlm}\frac{\partial B_m}{\partial X_l}B_k = -\frac{1}{4\pi}{}_{ikj}{}^{jlm}\frac{\partial B_m}{\partial X_l}B_k = -\frac{1}{4\pi}(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl})\frac{\partial B_m}{\partial X_l}B_k$$

$$F_{m,i} = \frac{1}{4\pi}(\frac{\partial B_i}{\partial X_k}B_k - \frac{\partial B_k}{\partial X_i}B_k) = \frac{1}{4\pi}((\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \frac{1}{2}\nabla \cdot \boldsymbol{B}^2)_i$$

gives the **isotropic magnetic pressure**:

$$\boldsymbol{F} = \frac{(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}}{4\pi} - \nabla\cdot\frac{\boldsymbol{B}^2)}{8\pi} \tag{15}$$

$$(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} = B\frac{\partial \boldsymbol{B}}{\partial s} = B\frac{\partial B\boldsymbol{s}}{\partial s} = B\boldsymbol{s}\frac{\partial B}{\partial s} + B^2\frac{\partial \boldsymbol{s}}{\partial s}$$

To figure out change in $\boldsymbol{s}$ with respect to s we look at the difference between the vectors as they turn an arc with radius R_c. Therefore, $\delta\boldsymbol{s} = \boldsymbol{n}\phi$ with $\boldsymbol{n}$ being a normal vector to the arc of radius R_c, and $\phi = \frac{ds}{R_c}$ as the angle is the distance divided by the radius. Therefore, $\delta\boldsymbol{s} = \boldsymbol{n}\frac{dS}{R_c}$:

$$\frac{\partial\boldsymbol{s}}{s} = \frac{\boldsymbol{n}}{R_c}$$

Therefore:

$$\frac{(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}}{4\pi} = \boldsymbol{s}\frac{\partial}{\partial s}(\frac{B^2}{8\pi}) + \frac{B^2}{4\pi}\frac{\boldsymbol{n}}{R_c} = \nabla_{||}(\frac{B^2}{8\pi} + \frac{B^2}{4\pi}\frac{\boldsymbol{n}}{R_c}$$

$$\boldsymbol{F}_m = \nabla_{||}(\frac{B^2}{8\pi}) + \frac{B^2}{4\pi}\frac{\boldsymbol{n}}{R_c} - \nabla_{||}(\frac{B^2}{8\pi}) - \nabla_{\perp}(\frac{B^2}{8\pi})$$

$$\boldsymbol{F}_m = \frac{B^2}{4\pi}\frac{\boldsymbol{n}}{R_c} - \nabla_{\perp}(\frac{B^2}{8\pi})$$

(16)

As

$$\nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = \boldsymbol{u}(\nabla\cdot\boldsymbol{B}) + (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - \boldsymbol{B}(\nabla\cdot\boldsymbol{u})$$

We have $\nabla\cdot\boldsymbol{B} = 0$ and $\nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = \frac{\partial\boldsymbol{B}}{\partial t}$ so

$$\frac{D\boldsymbol{B}}{Dt} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} - \boldsymbol{B}(\nabla\cdot u)$$

From the continuity equation: $\nabla\cdot\boldsymbol{u} = -\frac{1}{\rho}\frac{D\rho}{Dt}$

$$\frac{D\boldsymbol{B}}{Dt} - \frac{\boldsymbol{B}}{\rho}\frac{D\rho}{Dt} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u}$$

This gives the motion of the field lines.

$$\frac{D}{Dt}(\frac{\boldsymbol{B}}{\rho}) = \frac{\boldsymbol{B}}{\rho}\cdot\nabla\boldsymbol{u} \tag{17}$$

This is equivalent to the motion of laine eleemnet from the start of the course so $\frac{\boldsymbol{B}}{\rho} \sim \delta\boldsymbol{x}$ scale proportionally.

Consider a cyclinder of length δx and cross section δS with magnetic field $\boldsymbol{B}$ so $\delta m = \rho \delta S \delta x = \boldsymbol{B} \delta S$. Therefore, as the mass is fixed so must the magentic flux element must stay constant. This also holds more generally, consider magnetic flux throguh a surface S

$$\phi = \int_S \boldsymbol{B} \cdot d\boldsymbol{S}$$

In a Lagrangian frame, contour gets advected with the fluid. Look at the change of the flux as the contour gets advected.

$$\frac{D\phi}{Dt} = \int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{S} + \int_C \boldsymbol{B} \cdot (\boldsymbol{u} \times d\boldsymbol{l})G$$

with first term is changing magnetic flux and the second is the changing surface (considered seperately).

$$\frac{D\phi}{Dt} = \int_s \nabla \times (\boldsymbol{u} \times B) \cdot d\boldsymbol{S} + \int_C \boldsymbol{B} \cdot (\boldsymbol{u} \times d\boldsymbol{l})$$

$$\frac{D\phi}{Dt} = \int_C (\boldsymbol{u} \times B) \cdot d\boldsymbol{l} + \int_C \boldsymbol{B} \cdot (\boldsymbol{u} \times d\boldsymbol{l}) = 0$$

This illustrates flux-freezing. In MHD the magnetic flux is conserved as the fluid moves.

Example: Star formation

Should be able to calculate the magnetic flux in the star from knowing the magnetic flux of the initial cloud. This would lead to an estimate of $10^8 G$ but the actual flux is $10^2 G$ and this is because the MHD assumption is not accurate.

Critical flux

$$E_m \sim \frac{\phi^2}{6R_c}$$

$$E_{grav} \sim \frac{GM^2}{R_c}$$

If

$$\frac{E_m}{E_{grav}} = \frac{\phi}{M_c}^2 \frac{1}{G} < 1$$

then collapse occurs so need $\phi < \phi_{crit} = M_c G^{\frac{1}{2}}$ in order for it collapse. For our sun the flux is too large for the collapse to have occured. The solution to this is the solar formation occured at very cold temperatures so the ionization is very low so the conductivity is not infinite so has other effects to consider such as the hall effect and resisitivity and mainly antipolar diffusion. Antipolar diffusion is when B field couples to charges and charges then couple to neutrals by collisions so the magnetic field does not affect the neturals and so the magnetic field just slips through the cloud of gas. So therefore magentic flux is not conserved and we expexpect to lose a lot of magnetic flux.

Example: Formation of neutron stars A neutron star is only 10 km radius whereas the core of the star that collapses is about 10^{11} cm.

$$\phi \sim B_* R_*^2 \sim B_{NS} R_{NS} \implies B_{NS} \sim B_* \frac{R_*}{R_{NS}} = 10^{12} G$$

This is actually very accurate so the model works well for the transition from a star to a neutron star.

1.1 Rotating, axisymmetric systems: induction equation

Use cyclindrical coordinates

$$\boldsymbol{u} = u_\psi \boldsymbol{e}_\psi = R\Omega(R,z)\boldsymbol{e}_\psi$$

$$\boldsymbol{B} = \boldsymbol{B}_p + B_\psi \boldsymbol{e}_\psi$$

$$\boldsymbol{B}_p = B_R \boldsymbol{e}_R + B_z \boldsymbol{e}_z$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} - \boldsymbol{B}(\nabla \cdot \boldsymbol{u})$$

Last term can be dropped as incompressible behaviour

$$\nabla = (\frac{\partial}{\partial R}, \frac{\partial}{\partial z}, \frac{1}{R}\frac{\partial}{\psi}$$

$$\frac{\partial \boldsymbol{e}_R}{\partial \psi} = \boldsymbol{e}_\psi, \frac{\partial \boldsymbol{e}_\psi}{\partial \psi} = -\boldsymbol{e}_R$$

$$(\boldsymbol{B} \cdot \nabla)\boldsymbol{u} = B_r \frac{\partial (u_\psi \boldsymbol{e}_\psi}{\partial R} + B_z \frac{\partial (u_\psi \boldsymbol{e}_\psi}{\partial z} + \frac{B_\psi}{R}\frac{\partial (u_\psi \boldsymbol{e}_\psi}{\partial \psi} = B_r \frac{\partial (R\Omega)}{\partial R}\boldsymbol{e}_\psi + B_z \frac{\partial (R\Omega)}{\partial z}\boldsymbol{e}_\psi + B_\psi \Omega(-\boldsymbol{e}_R)$$

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{B} = \frac{u_\psi}{R}\left(\frac{\partial}{\partial \psi} B_R \boldsymbol{e}_R + \frac{\partial}{\partial \psi} B_z \boldsymbol{e}_z + \frac{\partial}{\partial \psi}(B_\psi \boldsymbol{e}_\psi\right)$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{e}_\psi R(\boldsymbol{B}_p \cdot \nabla)\Omega`$$

$$\frac{\partial \boldsymbol{B}_p}{\partial t} = 0, \frac{\partial B_\psi}{\partial t} = R(\boldsymbol{B}_p \cdot \nabla)\Omega$$

In steady state have $\frac{\partial B_\psi}{\partial t} = 0 \implies (\boldsymbol{B}_p \cdot \nabla)\Omega = 0$ so the gradient of Ω is always orthogonal to $\boldsymbol{B}_p$ which means Ω is constant on magnetic surfaces. This is called the **Ferronos' isozotation law**

1.2 Magnetic Forces - force balance

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{\nabla P}{\rho} - \nabla\phi + \frac{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi\rho}$$

L, T are the length and time scales. LHS and first term comparison $\frac{u\rho L}{\rho c_s^2 T} = \frac{v}{c}^2$ if $u << c_s$ can neglect ineterial LHS compared to 1st term on RHS.
Now compare LHS and 3rd term. $\frac{4\pi u L\rho}{TB^2} \sim \frac{u^2}{\frac{B^2}{4\pi\rho}}$. $\frac{B}{\sqrt{4\pi\rho}}$ is the Alfeven velocity u_A. so this ratio is $\frac{u}{u_A}^2$. So if $u << u_A$ can neglect inertial term compared to magnetic force. Consider third term against first term: $\frac{(3)}{(1)} \sim \frac{B^2}{4\pi\rho L}\frac{\rho L}{\rho c_s^2} = \frac{u_A}{c_s}^2$

When $u << c_s$ neglect inertial vs pressure
When $u << u_A$ can neglect inertial vs magnetic stresses
When $c_s << u_A$ can neglect thermal pressure vs magnetic pressure

magnetostatic equilibrium: is a static solution when $\boldsymbol{u} = 0$

Plasma β
p - thermal pressure
$p_B = \frac{B^2}{8\pi}$ - magnetic pressure

$$\beta = \frac{p}{p_B} \sim \frac{c_s}{u_A}^2$$

So whenever $\beta << 1$ then thermal pressure is negligible, and if $\beta >> 1$ then magnetic pressure dominates.

1.3 Magnetic Buoyancy

If we have a magnetic structure in which the velocity is very small so we can set the magnetic structure to zero:

$$0 = -\nabla p - \rho\nabla\phi + \frac{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi} = -\nabla p - \rho\nabla\phi + \frac{1}{4\pi}B^2\frac{\boldsymbol{n}}{R_c} - \nabla_\perp p_B$$

Imagine a flux tube with considerably stronger flux density than the surrounding fluid which the magnetic field going in the $-x$ direction with graviational acceleration in $-z$ direction. Look z in a plane. Assume $R_c \to \infty$ as magnetic field lines straight and parallel, also as $\nabla\phi$ is in $-z$ direction. The only contributions in the x direction is:

$$0 = -\nabla_\perp(p + p_B) \implies p + p_B = p + \frac{B^2}{8\pi} = c(z)$$

Let B_2 be the much stronger magnetic flux inside the tube and B_1 be the flux outside.

$$p_1 + \frac{B_1^2}{8\pi} = p_2 + \frac{B_2^2}{8\pi} \implies p_1 > p_2$$

This means that $\rho_2 < \rho_1$ if we assume the temperature remains constant in the centre of a star as the thermal timescales are to quick and equillibrium will be found quickly. This generates an archimedian force that pushes the flux tube up hence the magnetic buoyancy. So strong areas of magnetic fields will float up to the surface of a star for instance.
Sunspot
At some point on the surface of the sun you have an area with a much higher magnetic field flux (B_2) than the surrounding area (B_1). Can again fix z constant we would find the same equations again:$p_1 + \frac{B_1^2}{8\pi} = p_2 + \frac{B_2^2}{8\pi} \implies p_1 > p_2$ at fixed z.Typically the temperature does vary a lot here rather than density, as it is easily possible to lose heat. As emissivity $\sim 6T^4$ we get a magnetically dominated areas being dark sun spots.

1.4 Force-free equilibrium

Assume that $\beta << 1$ meaning that $p_B >> p$, then we can neglect gravity, so the magnetic force dominates the force balance. Useful if gravity is very weak. This means $\boldsymbol{F}_k = 0$ which looks weird as it looks like the magnetic force is zero but it actually just means the magnetic terms are so much large than the rest. Remember the decompistion of the magnetic force:

$$\boldsymbol{F}_k = \frac{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi} = \frac{\boldsymbol{J} \times \boldsymbol{B}}{c} = 0$$

so when $\boldsymbol{J}||\boldsymbol{B}$ we are in the free force state. Therefore, $\boldsymbol{J} = \alpha(\boldsymbol{u})\boldsymbol{B}$ so $\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} = \frac{4\pi\alpha(\boldsymbol{u})}{c}\boldsymbol{B}$. Take the divergence of this equation:

$$0 = \frac{4\pi}{c}\left((\boldsymbol{B} \cdot \nabla)\alpha + \alpha(\nabla \cdot \boldsymbol{B})\right) \implies (\boldsymbol{B} \cdot \nabla)\alpha = 0$$

This looks very similar to the Ferrono's isozotation but here it means that $\alpha(\boldsymbol{u})$ is constant along the magnetic field lines so not just an arbitrary function. We have

$$\frac{4\pi}{c}\alpha(\boldsymbol{u})\boldsymbol{B} = \nabla \times \boldsymbol{B}$$

which is linear in $\boldsymbol{B}$. If know distribution of $\alpha(S)$ on a particular surface S then I can solve this whole system. The structure of the upper solar atmosphere is force free, same is true for magnetosphere of pulsars.

Lets consider the consquences of assuming α is constant. Then we would find that $\nabla \cdot \boldsymbol{B} = \frac{4\pi\alpha}{c}\boldsymbol{B}$ (take the curl):

$$\nabla \times (\nabla \cdot \boldsymbol{B}) = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2\boldsymbol{B} = \frac{4\pi\alpha}{c}\nabla \times \boldsymbol{B} = (\frac{4\pi\alpha}{c})^2\boldsymbol{B}$$

This gives a Helmholtz equation (this is a nice linear equation):

$$\nabla^2\boldsymbol{B} + (\frac{4\pi\alpha}{c})^2\boldsymbol{B} = 0$$

Cylindrical force free equilibrium
Assume α is constant and cylindrically symmetrical. $\boldsymbol{r} = (R, z, \psi)$, $\boldsymbol{B} = (B_R, B_z, B_\psi)$, $\frac{\partial}{\partial\psi} = 0, \frac{\partial}{\partial z} = 0$

$$\nabla \cdot \boldsymbol{B} = 0 \implies \frac{1}{R}\frac{\partial}{\partial R}(RB_R) = 0 \implies RB_R(R) = \text{constant}$$

This is a solution but it is a bad solution as it diverges at $R = 0$ and the only way to get a non-singular system is to set the constant to 0 so $B_R = 0$.

$$\nabla \times \boldsymbol{B} = -\frac{\partial B_z}{\partial R}\boldsymbol{e}_\psi + \frac{1}{R}\frac{\partial}{\partial R}(RB_\psi\boldsymbol{e}_z$$
$$\frac{4\pi\alpha}{c}\boldsymbol{B} = \frac{4\pi\alpha}{c}B_\psi\boldsymbol{e}_\psi + \frac{4\pi\alpha}{c}B_z\boldsymbol{e}_z$$
$$B_\psi = \frac{c}{4\pi\alpha}(-\frac{\partial B_z}{R}), \alpha B_z = \frac{c}{4\pi}\frac{1}{R}\frac{\partial}{\partial R}(RB_\psi)$$
$$\alpha B_z = \frac{c}{4\pi}\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{c}{4\pi\alpha}(-\frac{\partial B_z}{\partial R})\right)$$

2 Lecture 8

$$\frac{1}{R}\frac{\partial}{\partial R}(R\frac{\partial B_z}{\partial R}) + (\frac{4\pi\alpha}{c})^2 B_z = 0$$

This is a version of Helmholtz equation and remember that in cyclindrical geometry

$$\nabla^2 f = \frac{1}{R}\frac{\partial}{\partial R}(R\frac{\partial f}{\partial R}) + \frac{\partial^2 f}{\partial z^2} + \frac{1}{R^2}\frac{\partial^2 f}{\partial \phi^2}$$

The solutions to this equation have form:

$$B_z = H_a J_a(\tilde{\alpha}R)$$
$$B_\phi = B_a J_1(\tilde{\alpha}R)$$

where $\tilde{\alpha} = \frac{4\pi}{c}\alpha$, J_o, J_1 are Bessel functions. Which satisify:

$$(tJ_o'(t))' + tJ_0(t) = 0, J_1(t) = -J_0'(t)$$

This length scale α depends on the stength of the current running along the field lines. If the current is very high then the field variations will be on a shorter length scale.

2.1 Potential Field

This gets realised whenever $\boldsymbol{J} = 0$ so $\alpha = 0$ everywhere. We know that $\boldsymbol{J} \sim \nabla \times \boldsymbol{B}$ so $\nabla \times \boldsymbol{B} = 0$, so therefore $\boldsymbol{B} = \nabla\phi$ so is a potential field. As we still have $\nabla \cdot \boldsymbol{B} = 0$ we get $\nabla^2\phi = 0$. This is a familar equation that we know how to solve. e.g. Consider a magnetic star with boundary condition on surface of star if we know $\boldsymbol{B}$ at $r = R_*$, then $\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(a_{lm}r^l + b_{lm}r^{-(l+1)})P_l^m(\cos\theta)e^{im\phi}$.

2.2 Grad-Shafranov Equation

Lets consider a system with pressure and B-field in equilibrium. The equation of motion of this system is

$$-\nabla p + \frac{\boldsymbol{J} \times \boldsymbol{B}}{c} = 0$$

Take a scalar product with $\boldsymbol{B}$:

$$(\boldsymbol{B} \cdot \nabla)p = 0$$

So the gradient of magnetic pressure has to be perpendicular to magnetic pressure/magnetic field lines have to be perpendicular to surfaces where the pressure is constant. If you take scalar product with $\boldsymbol{J}$:

$$(\boldsymbol{J} \cdot \nabla)p = 0$$

so same result as for $\boldsymbol{B}$. If I construct surfaces of constant p: $p(x, y, z)pconstant$ then it follows that $\boldsymbol{B}, \boldsymbol{J}$ must lie on these surfaces.

Assume axisymmetry $\frac{\partial}{\partial \phi} = 0$:

$$\boldsymbol{B} = \boldsymbol{B}_p + B_\phi \boldsymbol{e}_\phi$$

Introduce magentic flux:

$$\psi(R, z) = \int_0^R B_z 2\pi R dR$$

Introduce current:

$$I(R, z) = \int_0^R J_z 2\pi R dR$$

Flux can be considered like a label of magnetic surfaces by the value of ψ that they are enclosing.

$$I = I(\psi), p = p(\psi)$$

(current and pressure are constant on surfaces of given ψ). From defintion of flux:

$$B_z = \frac{1}{2\pi R}\frac{\partial \psi}{\partial R}$$

from defintion of current:

$$J_z = \frac{1}{2\pi R}\frac{\partial I}{\partial R}$$

From $\nabla \cdot \boldsymbol{B} = 0$:

$$\frac{1}{R}\frac{\partial R}{R}(RB_R) + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{1}{2\pi R}\frac{\partial^2 \psi}{\partial z R} = 0$$

$$RB_R + \frac{1}{2\pi}\frac{\partial\psi}{\partial z} = c(z) \implies B_R = -\frac{1}{2\pi R}\frac{\partial\psi}{\partial z} + \frac{c(z)}{R}$$

As we want a solution that is non-singular so the last term must be zero

$$B_R = -\frac{1}{2\pi R}\frac{\partial\psi}{\partial z}$$

We now know B_R and B_z so

$$\boldsymbol{B}_p = -\frac{1}{2\pi R}(\frac{\partial\psi}{\partial z}, \frac{\partial\psi}{\partial R}) = \frac{\nabla\psi \times \boldsymbol{e}_\phi}{2\pi R}$$

Lets look at the current:

$$I = \int_S \boldsymbol{J}\cdot d\boldsymbol{S} = \int_S \frac{c}{4\pi}(\nabla\times\boldsymbol{B})\cdot d\boldsymbol{S} = \frac{c}{4\pi}\int_C \boldsymbol{B}\cdot d\boldsymbol{l} = \frac{c}{4\pi}B_\phi 2\pi R = \frac{cB_\phi R}{2}$$

$$B_\phi = \frac{2I(\psi)}{Rc}$$

$$\boldsymbol{B} = \frac{\nabla\psi\times\boldsymbol{e}_\phi}{2\pi R} + \frac{2I(\psi)}{Rc}\boldsymbol{e}_\phi$$

For $\boldsymbol{J} = \frac{c}{4\pi}\nabla\times\boldsymbol{B} \implies \nabla\cdot\boldsymbol{J} = 0$. We could do the same derivation the current and we will get exactly the same expression for the poliodal component of hte current:

$$\boldsymbol{J}_p = \frac{\nabla I(\psi)\times\boldsymbol{e}_\phi}{2\pi R}$$

For J_ϕ we can consider $J_\phi = \frac{c}{4\pi}(\nabla\times\boldsymbol{B})_\phi$ so

$$\boldsymbol{J} = \frac{\nabla I(\psi)\times\boldsymbol{e}_\phi}{2\pi R} + \frac{c}{4\pi}(\nabla\times\boldsymbol{B})_\phi\boldsymbol{e}_\phi$$

This expression has second derivatives of ψ in it as we differentiate B again.

3 Lecture 9

$$\nabla p = \frac{\partial p}{\partial\psi}\nabla\psi = \frac{\boldsymbol{J}\times\boldsymbol{B}}{c}$$

This is normal to the surface of constant pressure, and so it is also normal to the flux surfaces and the current surfaces. As the RHS is also normal to the flux and current surface this equation only has one component, therefore:

$$\frac{\boldsymbol{J}\times\boldsymbol{B}}{c} = G(\nabla\psi)$$

This force is non-linear (it is quadratic in ψ, $\boldsymbol{B}$ contains first derivatives in ψ and $\boldsymbol{J}$ contains second derivatives of ψ). He will not be deriving the expression for G, just giving the final result which is:

$$\frac{\partial p}{\partial\psi}\nabla\psi = G\nabla\psi \implies \nabla\psi(G - \frac{\partial p}{\partial\psi}) = 0$$

Grad-Shafrahov equation:

$$G - \frac{\partial p}{\partial \psi} = 0$$

Often called equation of cross-field balance or trans-field balance.
G in particular case: Cyclindrical coordinates (R, z)

$$\frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R}\frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial z^2} = -16\pi^3 R^2 \frac{\partial p}{\partial \psi} - \frac{8\pi^2}{c^2}\frac{\partial I^2(\psi)}{\partial \psi}$$

Don't have to memorise this lol i'm glad. The power of this approach is we can just solve this system and we can then get the behaviour of everything (current and pressure) just from the flux $\psi(R, z)$. LHS does not reduce very nicely unfortunately it is: $R^2 \nabla \cdot (\frac{1}{R^2}\psi)$. What is interesting is flux function was originally used as just a coordinate for labeling our pressure surfaces and it has become an independant function that we need to solve for. This equation does not have to be linear as $p(\psi)$ and $I(\psi)$ could be anything (normally specified at the beginning of the problem). We can very easily get force free by dropping pressure term.

Consider force-free cyclindrical equilibrium structure ($\frac{\partial}{\partial z} = 0, p = 0$). Assume that $I(\psi) = \kappa\psi$ so:

$$\frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R}\frac{\partial \psi}{\partial R} = -\frac{16\pi^2\kappa^2}{c^2}\psi$$

$$\frac{\partial}{\partial R}(R\frac{1}{2\pi R}\frac{\partial \psi}{\partial R}) - \frac{1}{2\pi R}\frac{\partial \psi}{\partial R} = -\frac{8\pi\kappa^2}{c^2}\psi$$

$$\frac{\partial}{\partial R}(RB_z) - B_z = R\frac{\partial B_z}{\partial R} = -\frac{8\pi\kappa^2}{c^2}\psi$$

$$\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial B_z}{\partial R} = -\frac{16\pi^2\kappa^2}{c^2}\frac{1}{2\pi R}\frac{\partial \psi}{\partial R} = -\frac{16\pi^2\kappa^2}{c^2}B_z$$

let $\alpha = \frac{4\pi\kappa}{c}$ and this reduces to the Helmholtz equation

$$\nabla^2 B_z + \alpha^2 B_z = 0$$

4 Conservation Laws

Some variable has a density (amount per volume) $q(\boldsymbol{x}, t)$ that satisfies an equation:

$$\frac{\partial q}{\partial t} + \nabla \cdot \boldsymbol{F} = 0$$

then we say that $\boldsymbol{F}$ is a flux of q. By analogy with continiuity equation: $q = \rho$, $\boldsymbol{F} = \rho\boldsymbol{u}$. Amount of q in volume V: $Q = \int_V q(\boldsymbol{x}, t)dV$.

$$\frac{\partial Q}{\partial t} = -\int_V \nabla \cdot \boldsymbol{F} dV = -\int_S \boldsymbol{F} \cdot d\boldsymbol{S} \text{ flux of } q \text{ across surface of V}$$

Define a material invariant as a scalar field $f(\boldsymbol{x}, t)$ for which $\frac{Df}{Dt} = 0$. e.g. Entropy is a material invariant. Then $\frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla) f = 0$, therefore $\rho \frac{\partial f}{\partial t} + (\rho \boldsymbol{u} \cdot \nabla) f = 0$ and as $f \frac{\partial \rho}{\partial t} + f \nabla \cdot (\rho \boldsymbol{u}) = 0$:

$$\frac{\partial (\rho f)}{\partial t} + \nabla \cdot (\rho f \boldsymbol{u}) = 0$$

Consider energy density:

$$\epsilon = \rho(\frac{u^2}{2} + \phi + e) + \frac{B^2}{8\pi}$$

We want to find $\boldsymbol{F}_\epsilon$ such that

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot \boldsymbol{F}_\epsilon = 0$$

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial \rho}{\partial t}(\frac{u^2}{2} + \phi + e) + \frac{1}{8\pi}\frac{\partial B^2}{\partial t} + \rho(\frac{1}{2}\frac{\partial u^2}{\partial t} + \frac{\partial \phi}{\partial t} + \frac{\partial e}{\partial t})$$

$$\frac{1}{8\pi}\frac{\partial B^2}{\partial t} = \frac{\boldsymbol{B}}{4\pi}\frac{\partial \boldsymbol{B}}{\partial t} =^{maxwells} -\frac{\boldsymbol{B}c}{4\pi}(\nabla \times \boldsymbol{E}) = -\frac{c}{4\pi}\boldsymbol{B} \cdot (\nabla \times \boldsymbol{E})$$

$$\rho \frac{1}{2}\frac{\partial u^2}{\partial t} = \rho \boldsymbol{u}\frac{\partial \boldsymbol{u}}{\partial t} = \rho \boldsymbol{u}(-(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \frac{\nabla p}{\rho} - \nabla \phi \rho + \frac{1}{4\pi\rho}((\nabla \times \boldsymbol{B}) \times \boldsymbol{B}))$$

$$\frac{De}{Dt} = T\frac{Ds}{Dt} - p\frac{dV}{Dt} = t\frac{Ds}{Dt} + \frac{p}{\rho^2}\frac{D\rho}{Dt}$$

$$\frac{\partial e}{\partial t} = T\frac{Ds}{Dt} - (\boldsymbol{u} \cdot \nabla)e - \frac{p}{\rho}(\nabla \cdot \boldsymbol{)}$$

5 Lecture 9

$$\frac{\partial \epsilon}{\partial t} = -\frac{\partial \rho}{\partial t}(\frac{u^2}{2}+\phi+e) - \rho\boldsymbol{u}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\phi + \frac{\boldsymbol{u} \cdot ((\nabla \times \boldsymbol{B}) \times \boldsymbol{B})}{4\pi} + \frac{\partial \phi}{\partial t} - (\rho\boldsymbol{u})\cdot\nabla e - p(\nabla\cdot\boldsymbol{u}) + \rho T\frac{Ds}{Dt}$$

$$\frac{\partial \epsilon}{\partial t} = -\frac{\partial \rho}{\partial t}(\frac{u^2}{2}+\phi+e) - \rho(\boldsymbol{\cdot}\nabla)\frac{u^2}{2} -)\rho\boldsymbol{u}\cdot\nabla)e - p(\nabla\cdot\boldsymbol{u}) - (\boldsymbol{u}\cdot\nabla)p - (\boldsymbol{u}\cdot\nabla)\phi - \frac{c}{4\pi}\boldsymbol{B}\cdot(\nabla\times\boldsymbol{E}) + \frac{c(\nabla)\boldsymbol{B} \times \frac{\boldsymbol{u}}{c}}{4\pi} + \rho\frac{\partial \phi}{\partial t} + \rho$$

$$\frac{\partial \epsilon}{\partial t} = -\frac{\partial \rho}{\partial t}(\frac{u^2}{2}+\phi+e) - (\rho\boldsymbol{u}\cdot\nabla)(\frac{u^2}{2}+e+\phi) - \nabla\cdot(p\boldsymbol{u}) - \frac{c}{4\pi}(\boldsymbol{B}\cdot(\nabla\times\boldsymbol{E}) - (\nabla\times\boldsymbol{B})\boldsymbol{E})$$

$$\frac{\partial \epsilon}{\partial t} = -\nabla \cdot (\rho\boldsymbol{u}(\frac{u^2}{2} + \phi + e) + p\boldsymbol{u}) - \frac{c}{4\pi}(\boldsymbol{B} \cdot (\nabla \times \boldsymbol{E}) - (\nabla \times \boldsymbol{B})\boldsymbol{E}) + \rho\frac{\partial \phi}{\partial t} + \rho T\frac{Ds}{Dt}$$

as

$$\nabla\cdot(\boldsymbol{M}\times\boldsymbol{N}) = \boldsymbol{N}\cdot(\nabla\times\boldsymbol{M}) - \boldsymbol{M}\cdot(\nabla\times\boldsymbol{N}) \implies \boldsymbol{B}\cdot(\nabla\times\boldsymbol{E}) - (\nabla\times\boldsymbol{B})\boldsymbol{E} = \nabla\cdot(\boldsymbol{E}\times\boldsymbol{B})$$

$$\frac{\partial \epsilon}{\partial t} = -\nabla\cdot(\rho\boldsymbol{u}(\frac{u^2}{2}+\phi+e)+p\boldsymbol{u}) - \nabla\cdot(\frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B}) + \rho\frac{\partial \phi}{\partial t} + \rho T\frac{Ds}{Dt} = -\nabla\cdot(\rho\boldsymbol{u}(\frac{u^2}{2}+\phi+e+\frac{P}{\rho}) + \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B}) + \rho\frac{\partial \phi}{\partial t} + \rho$$

so

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot \boldsymbol{F}_\epsilon = \rho\frac{\partial \phi}{\partial t} + \rho T\frac{Ds}{Dt} \quad (18)$$

for $F_{epsilon} = \rho\boldsymbol{u}(\frac{u^2}{2} + \phi + h) + \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{B}$, with $e + \frac{P}{\rho} = h$ (enthalpy)

5.1 Bernoulli invariant

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot \boldsymbol{F}_\epsilon = 0$$

In steady state $\frac{\partial \epsilon}{\partial t} = 0$ so $\nabla \cdot \boldsymbol{F}_\epsilon = 0$. Assume hydrodynamic limit, $\boldsymbol{B} \to 0$ which implis

$$\nabla \cdot (\rho \boldsymbol{u}(\frac{u^2}{2} + \phi + h) = 0$$

$$\nabla \cdot (\rho \boldsymbol{u})(\frac{u^2}{2} + \phi + h) + \rho(\boldsymbol{u} \cdot \nabla)(\frac{u^2}{2} + \phi + h)) = 0$$

so $\boldsymbol{u} \cdot \nabla)B_B = 0, and\ \mathrm{B}_B = \frac{u^2}{2} + \phi + h$ so B_B is constant along streamlines of the flow.

5.2 Magnetic helicity conservation

Vector potential $\boldsymbol{A}$, $\boldsymbol{B} = \nabla \times \boldsymbol{A} \implies \nabla \cdot \boldsymbol{B} = 0$. Define magnetic helicity:

$$H_m = \int_V \boldsymbol{A} \cdot \boldsymbol{B} dV$$

$$\frac{\partial}{\partial t} \boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{A} \cdot \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{B} \cdot \frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{A} \cdot (\nabla \times (\boldsymbol{u} \times \boldsymbol{B}))$$

Maxwell's equation

$$-\nabla \times \boldsymbol{E} = \frac{1}{c} \frac{\partial \boldsymbol{B}}{\partial t} = \frac{1}{c} \nabla \times \frac{\partial \boldsymbol{A}}{\partial t} \implies \nabla \times (\frac{1}{c} \frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{E}) = 0$$

$$\boldsymbol{E} + \frac{1}{c} \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla \phi_l$$

where ϕ_l is th electrostatic potential

$$\frac{\partial \boldsymbol{A}}{\partial t} = -c(\boldsymbol{E} + \nabla \phi_l)$$

so

$$\frac{\partial}{\partial t} \boldsymbol{A} \cdot \boldsymbol{B} = -c(\boldsymbol{B} \cdot \nabla \phi_l) - c(\boldsymbol{E} \cdot \boldsymbol{B}) + \boldsymbol{A} \cdot (\nabla \times (\boldsymbol{u} \times \boldsymbol{B}))$$

Second term vanishes as $E = -\frac{\boldsymbol{u} \times \boldsymbol{B}}{c}$

$$\nabla \cdot (\boldsymbol{A} \times (\boldsymbol{u} \times \boldsymbol{B})) = (\boldsymbol{u} \times \boldsymbol{B}) \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{u} \times \boldsymbol{B}) - \boldsymbol{A} \cdot \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

so

$$\boldsymbol{A} \cdot \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = -\nabla \cdot (\boldsymbol{A} \times (\boldsymbol{u} \times \boldsymbol{B}))$$

First term (use fact $\nabla \cdot \boldsymbol{B} = 0$):

$$(\boldsymbol{B} \cdot \nabla)\phi_e = (\boldsymbol{B} \cdot \nabla)\phi_e + \phi_e \nabla \cdot \boldsymbol{B} = \nabla \cdot (\boldsymbol{B} \phi_e)$$

so:

$$\frac{\partial}{\partial t}\boldsymbol{A}\cdot\boldsymbol{B} = -c\nabla\cdot(\boldsymbol{B}\phi_e) - \nabla\cdot(\boldsymbol{A}\times(\boldsymbol{u}\times\boldsymbol{B})) = -\nabla\times(c\boldsymbol{B}\phi_e + \boldsymbol{A}\times(\boldsymbol{u}\times\boldsymbol{B}))$$

$$\frac{\partial H_m}{\partial t} + \nabla\cdot\boldsymbol{F}_{H_m} = 0 \tag{19}$$

for $\boldsymbol{F}_{H_m} = c\boldsymbol{B}\phi_e + \boldsymbol{A}\times(\boldsymbol{u}\times\boldsymbol{B})$ (often have to derive conservation laws like this in exams so understand and learn technique)

6 Lecture 10

There is also kinetic heliocity:

$$H_k = \int_V \boldsymbol{u}\cdot\boldsymbol{\omega}dV, \omega = \nabla\times\boldsymbol{u}$$

this gives completely anologous results for vorticity and magnetic field so

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{\omega})$$

6.1 Momentum conservation law

Momentum density $\rho\boldsymbol{u}$
One can show that there exists a symmetric tensor $\Pi_{ij} = \Pi_{ji}$ s.t.

$$\frac{\partial(\rho\boldsymbol{u}}{\partial t} + \nabla\cdot\boldsymbol{\Pi} = 0$$

consdier ith component:

$$\frac{\partial\rho}{\partial t}u_i + \rho\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i}\Pi_{ij} = (-\nabla\cdot(\rho\boldsymbol{u}))u_i + \rho\frac{\partial u_i}{\partial t} + \frac{\partial\Pi_{ij}}{\partial x_j} = 0$$

$$-u_i\frac{\partial(\rho u_i)}{\partial x_j} - \rho u_j\frac{\partial u_i}{\partial x_j} + \rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} + \frac{\partial\Pi_{ij}}{\partial x_j} = 0$$

$$-\frac{\partial}{\partial x_j}\rho u_i u_j + \rho\frac{Du_i}{Dt} + \frac{\partial Pi_{ij}}{\partial x_j} = 0$$

$$\rho\frac{Du_i}{Dt} = \frac{\partial}{\partial x_j}(\rho u_i u_j - \Pi_{ij}) = \frac{\partial}{\partial x_j}T_{ij}$$

Stress tensor: $T_{ij} = \rho u_i u_j - \Pi_{ij}$

$$\frac{D\boldsymbol{u}}{Dt} = \nabla\cdot\boldsymbol{T} \tag{20}$$

The first term in the stress tensor we are removing the advection so are just looking at the stresses.

Find T_{ij}

$$\rho\frac{D\boldsymbol{u}}{Dt} = \nabla\cdot\boldsymbol{T} = -\nabla p - \rho\nabla\phi + \frac{1}{4\pi}((\nabla\times\boldsymbol{B})\times\boldsymbol{B})$$

Let $\boldsymbol{p}$ be an arbitary vector. Form a tensor $\boldsymbol{P} = P_{ij} = \boldsymbol{p}_i\boldsymbol{p}_j$. We can prove the following to be true:

$$\nabla\cdot(\boldsymbol{P} - \frac{1}{2}p^2\boldsymbol{I}) = (\nabla\cdot\boldsymbol{p})\boldsymbol{p}p + (\nabla\times\boldsymbol{p})\times\boldsymbol{p}$$

Recall that:

$$(\nabla\times\boldsymbol{p})\times\boldsymbol{p} = (\boldsymbol{p}\cdot\nabla)\boldsymbol{p} - \nabla(\frac{\boldsymbol{p}}{2})$$

So:

$$\frac{\partial}{\partial x_j}(p_ip_j - \frac{p^2}{2}I_{ij} = ((\nabla\cdot\boldsymbol{p})\boldsymbol{p} + (\nabla\times\boldsymbol{p})\times\boldsymbol{p})_i$$

Introduce Maxwell's stress tensor $\boldsymbol{M}$:

$$M_{ij} = \frac{1}{4\pi}(B_iB_j - \frac{B^2}{2}I_{ij}$$

implies an expresssion for the lorentz force:

$$\frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = \nabla\cdot\boldsymbol{M}$$

Consider pressure:

$$(\nabla p)_i = \frac{\partial p}{\partial x_j}I_{ij} = \frac{\partial}{\partial x_j}(pI_{ij}) = \nabla\cdot(p\boldsymbol{I})$$

Consider gravity:

$$4\pi\rho G = \nabla^2 G = -\nabla\cdot\boldsymbol{g}$$

$$-\rho\nabla\phi = (-\frac{\nabla\cdot\boldsymbol{g}}{4\pi G}(-\boldsymbol{g}) = -\frac{1}{4\pi G}(\nabla\cdot\boldsymbol{g})\boldsymbol{g}$$

$$(\nabla\cdot\boldsymbol{g})\boldsymbol{g} + (\nabla\times\boldsymbol{g})\times\boldsymbol{g} = \nabla\cdot(g_ig_j - \frac{g^2}{2}I_{ij}$$

Second term is zero so:

$$-\rho\nabla\phi = -\frac{1}{4\pi G}\nabla\cdot(g_ig_j - \frac{g^2}{2}I_{ij}$$

so we get a gravitional stress tesnor

$$G_{ij} = -\frac{1}{4\pi G}\nabla\cdot(g_ig_j - \frac{g^2}{2}I_{ij}$$

$$-\rho\nabla\phi = \nabla\times G$$

so combining it all together we get:

$$\nabla\boldsymbol{T} = \nabla\cdot(-p\boldsymbol{I} + \boldsymbol{G} + \boldsymbol{M})$$

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{G} + \boldsymbol{M}$$

$$T_{ij} = -pI_{ij} - \frac{1}{4\pi G}\nabla\cdot(g_ig_j - \frac{g^2}{2}I_{ij}) + \frac{1}{4\pi}(B_iB_j - \frac{B^2}{2}I_{ij}) \tag{21}$$

6.2 Virial relations

Stem from considering $I_{ij} = \int_V x_i x_j \rho dV$. Take second time derivative

$$\frac{1}{2}\frac{D^2 I_{ij}}{Dt^2} = \int_V (2K_{ij} - T_{ij})dV$$

for $K_{ij} = \rho u_i u_j$. This is a statement of the Tensor Virial Theorem. Take average over long timescales to give:

$$< . >= \frac{1}{T}\lim_{T\to\infty}\int_0^T dt$$

The LHS will go to zero so you will will ened up with the difference of two first derivatives inside the limit but these are finite and as $T \to \infty$ this will go to zero. So:

$$2 < K_{ij} >=< T_{ij} >$$

Scalar viral theorem is obtained by taking the trace of this expression. So you get the total kinetic energy of the system from the LHS, and the potential energy of the field from the RHS:

$$TrK_{ij} = Tr\frac{1}{2}\int \rho u_i u_j dV = \frac{1}{2}\rho u^2 dV = K, TrT_{ij} = 3(\gamma-1)E_{thermal} + E_{mag} + E_{grav}$$

Therefore:

$$2 < K > +3(\gamma - 1) < E_{th} > + < E_m > + < E_g >= 0$$

For $< E_{th} >=< E_m >= 0 \implies -2 < K >=< E_g >$. So the total energy: $< E >=< E_k > + < E_g >= \frac{E_g}{2}$.

7 Examples class 1

Standard identity it is worth learning:

$$(\nabla \times \boldsymbol{u}) \times \boldsymbol{u} = (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \frac{1}{2}\nabla(u^2)$$

YOU GET SO MANY FORMULAS AT THE FRONT OF THE EXAM BOOKLET!!

$$de = TdS - pdV \implies \nabla e = T\nabla S - P\nabla V \implies \nabla e = T\nabla S - p\nabla V \implies \nabla\times\nabla e = 0 = \nabla T\times\nabla S - \nabla p\times\nabla V$$

In ideal fluid:

$$\frac{DS}{Dt} = \frac{\partial S}{\partial t} + \boldsymbol{u}\cdot\nabla S = 0$$

Thermal energy equation:

$$\frac{DP}{Dt} = -\Gamma_1 P\nabla\cdot\boldsymbol{u}$$

Be careful when doing things like $A_{ij}c_i d_j = B_{ij}c_i d_j \implies A_{ij} = B_{ij}$ as this is not true generally if $c = d$ so the right tensor is symmetric as then it only has 6 independant variables. In this case (/in all cases) you could take the derivative with respect to c or d to remove them safely. As clearly ${}_{ijk}x_j x_k = 0$ but $\epsilon_{ijk} \neq 0$.

8 Lecture 11

8.1 MHD Waves

Start with basic static state, characterised by unperturbed variables: $\boldsymbol{B}_0(\boldsymbol{x})$ unperturbed $\boldsymbol{B}$ field, $\rho_0(\boldsymbol{x})$ unperturbed density, $p_0(x)$ unperturbed pressure, $\phi_0(\boldsymbol{x})$ unperturbed potential, $\boldsymbol{u}_0(\boldsymbol{x}) = \boldsymbol{0}$ unperturbed velocity. We then introduce a few perturbations e.g. $\delta\rho, \delta p, \delta\phi$.

Governing equations:

$$\frac{\partial \rho}{\partial t} = -(\boldsymbol{u} \cdot \nabla)\rho - \rho(\nabla \cdot \boldsymbol{u})$$

$$\frac{\partial p}{\partial t} = -(\boldsymbol{u} \cdot \nabla)p - \gamma p(\nabla \cdot \boldsymbol{u})$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

Now linearise thsese terms in perturbations:

$$\frac{\partial \delta\rho}{\partial t} = -(\delta\boldsymbol{u} \cdot \nabla)\rho_0 - \rho_0(\nabla \cdot \delta\boldsymbol{u})$$

$$\frac{\partial \delta p}{\partial t} = -(\delta\boldsymbol{u} \cdot \nabla)p_0 - \gamma p_0(\nabla \cdot \delta\boldsymbol{u})$$

$$\frac{\partial \delta\boldsymbol{B}}{\partial t} = \nabla \times (\delta\boldsymbol{u} \times \boldsymbol{B}_0)$$

Can simplify these equations by introducing a displacement vector ξ: $\delta\boldsymbol{u} = \frac{\partial \boldsymbol{\xi}}{\partial t}$. You don't need to know the physical meaning of this. The physical meaning is the rate of change of the vectors connecting the perturbed trajectory and the unpertubed trajectory.

$$\frac{\partial \delta\rho}{\partial t} = -(\frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \nabla)\rho_0 - \rho_0(\nabla \cdot \frac{\partial \boldsymbol{\xi}}{\partial t})$$

$$\frac{\partial \delta p}{\partial t} = -(\frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \nabla)p_0 - \gamma p_0(\nabla \cdot \frac{\partial \boldsymbol{\xi}}{\partial t})$$

$$\frac{\partial \delta\boldsymbol{B}}{\partial t} = \nabla \times (\frac{\partial \boldsymbol{\xi}}{\partial t} \times \boldsymbol{B}_0)$$

Can then integrate over time as only time dependance in $\boldsymbol{\xi}$:

$$\delta\rho = -(\boldsymbol{\xi} \cdot \nabla)\rho_0 - \rho_0(\nabla \cdot \boldsymbol{\xi}) \tag{22}$$

$$\delta p = -(\boldsymbol{\xi} \cdot \nabla)p_0 - p_0(\nabla \cdot \boldsymbol{\xi}) \tag{23}$$

$$\delta\boldsymbol{B} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}_0) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{\xi} - \boldsymbol{B}(\nabla \cdot \boldsymbol{\xi}) - (\boldsymbol{\xi} \cdot \nabla)\boldsymbol{B} \tag{24}$$

So knowing $\boldsymbol{\xi}$ gives you the full behaviour of the system. Use equation of motion to determine $\boldsymbol{\xi}$:

$$\rho\frac{\partial \boldsymbol{u}}{\partial t}+\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p-\rho\nabla\phi+\frac{\nabla\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi} = -\nabla p-\rho\nabla\phi+\frac{(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}-\nabla(\frac{B^2}{2})}{4\pi}$$

Linearise:

$$\rho_0\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\nabla(\delta p+\delta(\frac{B^2}{8\pi}))-\rho\nabla\delta\phi+\delta\rho\nabla\phi+\frac{1}{4\pi}((\delta\boldsymbol{B}\cdot\nabla)\boldsymbol{B}+(\boldsymbol{B}\cdot\nabla)\delta\boldsymbol{B})$$

We have poisson: $\nabla^2\phi = 4\pi G\rho \implies \nabla^2\delta\phi = 4\pi G\delta\rho$. There is a slight issue with this equation when we look at a homogenous system this comes as a result of Jeans swindle, as this does not have a well behaved solution. The above equations come about from assuming that the background state is steady and in equilibirum so $0 = -\nabla p_0 - \rho_0\nabla\phi_0 + \frac{(\nabla\times\boldsymbol{B}_0)\times\boldsymbol{B}_0}{4\pi}$.

Consider $\delta(p+\frac{B^2}{8\pi}) = \delta p = \boldsymbol{B}\cdot\frac{\delta\boldsymbol{B}}{4\pi} = -(\boldsymbol{\xi}\cdot\nabla)p-\gamma(\nabla\cdot\boldsymbol{\xi})+\frac{\boldsymbol{B}}{4\pi}((\boldsymbol{B}\cdot\nabla)\boldsymbol{\xi}-\boldsymbol{B}(\nabla\cdot\boldsymbol{\xi})-(\boldsymbol{\xi}\cdot\nabla)\boldsymbol{B})$ so:

$$\delta(p+\frac{B^2}{8\pi}) = -(\boldsymbol{\xi}\cdot\nabla)(p+\frac{B^2}{8\pi})-(\gamma p+\frac{B^2}{4\pi})(\nabla\cdot\boldsymbol{\xi})+\frac{1}{4\pi}\boldsymbol{B}\cdot(\boldsymbol{B}\cdot\nabla)\boldsymbol{\xi}$$

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Assume homogeneity and neglect gravity to simplify equaitons to:

$$\delta\rho = \rho(\nabla\cdot\boldsymbol{\xi})$$

$$\delta p = -\gamma p(\nabla\cdot\boldsymbol{\xi})$$

$$\delta\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{\xi}-\boldsymbol{B}(\nabla\cdot\boldsymbol{\xi})$$

$$\delta(p+\frac{B^2}{8\pi}) = -(\gamma P+\frac{B^2}{4\pi})+\frac{1}{4\pi}\boldsymbol{B}((\boldsymbol{B}\cdot\nabla)\boldsymbol{\xi})$$

$$\rho\frac{\partial\boldsymbol{\xi}}{\partial t^2} = -\nabla\delta(p+\frac{B^2}{8\pi}+\frac{1}{4\pi}\boldsymbol{B}\cdot\nabla((\boldsymbol{B}\cdot\nabla)\boldsymbol{\xi}-\boldsymbol{B}(\nabla\cdot\boldsymbol{\xi}))$$

Since we are interested in wave like equations we will plug in an ansatz. Assuming that $\boldsymbol{\xi} = \tilde{}$

$\xi e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$ (wave ansatz). This means a lot of simplifications:

$$\nabla f\to i\boldsymbol{k}f, \frac{\partial f}{\partial x}\to -i\omega f, \nabla\cdot\boldsymbol{f}\to i\boldsymbol{k}\cdot\boldsymbol{f}, (\boldsymbol{F}\cdot\nabla)f\to i(\boldsymbol{k}\cdot\boldsymbol{F})f$$

so

$$\delta(p+\frac{B^2}{4\pi})\to -(\gamma p+\frac{B^2}{4\pi})i\boldsymbol{k}\cdot\boldsymbol{\xi}+\frac{i}{4\pi}(\boldsymbol{B}\cdot\boldsymbol{K})(\boldsymbol{B}\cdot\boldsymbol{\xi})$$

This gives EOM:

$$\rho(-i\omega)^2\boldsymbol{x}i = i\boldsymbol{k}(-(\gamma p+\frac{B^2}{4\pi})i\boldsymbol{k}\cdot\boldsymbol{\xi}+\frac{i}{4\pi}(\boldsymbol{B}\cdot\boldsymbol{K})(\boldsymbol{B}\cdot\boldsymbol{\xi}))+\frac{1}{4\pi}(\boldsymbol{B}\cdot i\boldsymbol{k})(\boldsymbol{B}\cdot i\boldsymbol{k})\boldsymbol{\xi}-\boldsymbol{B}i(\boldsymbol{k}\cdot\boldsymbol{\xi})$$

$$\rho\omega^2\boldsymbol{\xi} = (\gamma p + \frac{B^2}{4\pi})(\boldsymbol{k}\cdot\boldsymbol{\xi})\boldsymbol{k} - (\boldsymbol{B}\cdot\boldsymbol{k})(\boldsymbol{B}\cdot\boldsymbol{\xi})\frac{\boldsymbol{k}}{4\pi} + \frac{1}{4\pi}(\boldsymbol{B}\cdot\boldsymbol{k})((\boldsymbol{B}\cdot\boldsymbol{k})\boldsymbol{\xi} - -\boldsymbol{B}(\boldsymbol{k}\cdot\boldsymbol{\xi}))$$

take dot with $\boldsymbol{k}$:

$$\rho\omega^2(\boldsymbol{k}\cdot\boldsymbol{\xi}) = (\gamma p + \frac{B^2}{4\pi}k^2(\boldsymbol{k}\cdot\boldsymbol{\xi}) - \frac{1}{4\pi}(\boldsymbol{B}\cdot\boldsymbol{k})(\boldsymbol{B}\cdot\boldsymbol{\xi})k^2 + \frac{1}{4\pi}(\boldsymbol{B}\cdot\boldsymbol{k})\times((\boldsymbol{B}\cdot\boldsymbol{k})(\boldsymbol{k}\cdot\boldsymbol{\xi}) - (\boldsymbol{B}\cdot\boldsymbol{k})(\boldsymbol{k}\cdot\boldsymbol{\xi}))$$

$$(\rho\omega^{2\text{‘}} - k^2(\gamma p + \frac{B^2}{4\pi}))(\boldsymbol{k}\cdot\boldsymbol{\xi}) + \frac{1}{4\pi}(\boldsymbol{B}\cdot\boldsymbol{k})(\boldsymbol{B}\cdot\boldsymbol{\xi})k^2 = 0 \tag{25}$$

Take the first equation with doted $\boldsymbol{B}$ this time:

$$\gamma p(\boldsymbol{B}\cdot\boldsymbol{k})(\boldsymbol{k}\cdot\boldsymbol{\xi}) - \rho\omega^2(\boldsymbol{B}\cdot\boldsymbol{\xi}) = 0 \tag{26}$$

$$\rho\omega^{2\text{‘}} - k^2(\gamma p + \frac{B^2}{4\pi})\frac{1}{4\pi}(\boldsymbol{B}\cdot\boldsymbol{k})\gamma p(\boldsymbol{B}\cdot\boldsymbol{k}) - \rho\omega^2(\boldsymbol{k}\cdot\boldsymbol{\xi})(\boldsymbol{B}\cdot\boldsymbol{\xi}) = 0$$

First possibility is $(\boldsymbol{k}\cdot\boldsymbol{\xi}) = (\boldsymbol{B}\cdot\boldsymbol{\xi}) = 0$ implying the displacement vector is orthongonal to both $\boldsymbol{k}$ and $\boldsymbol{B}$.If we plug this assumption back into the orginal equaiont we get:

$$\rho\omega^2\boldsymbol{\xi} = \frac{(\boldsymbol{B}\cdot\boldsymbol{k})^2}{4\pi}\boldsymbol{\xi} \implies \omega^2 = \frac{(\boldsymbol{B}\cdot\boldsymbol{k})^2}{4\pi\rho} = (\boldsymbol{k}\cdot\boldsymbol{u}_a)^2$$

where $\boldsymbol{u}_a = \frac{\boldsymbol{B}}{\sqrt{4\pi\rho}}$ is the alfven velcotiy and the dispersion relation of the Alfen waves is $\omega = \pm(\boldsymbol{k}\cdot\underline{\mathrm{u}}_a) - \boldsymbol{k}u_a\cos\theta$ with phase velocity $\boldsymbol{u}_{ph} = \frac{\omega}{k}\hat{\boldsymbol{k}} = \pm u_a\cdot\theta\hat{\boldsymbol{k}}$ and group velocity $\boldsymbol{u}_{gr} = \frac{\partial\omega}{\partial k} = \pm\boldsymbol{u}_a$. we have $\boldsymbol{u}_{ph}||\boldsymbol{k}$ and $\boldsymbol{u}_{gr}||\boldsymbol{B}$.

Magnetic tension is the restoring force for the Alfen waves, and we have

$$\delta\rho = 0, \delta p = 0$$

so Alfen waves are incompressible. So it is just oscillation of the magnetic field lines.

The second possibility is $(\boldsymbol{k}\cdot\boldsymbol{\xi}) \neq 0$ and $(\boldsymbol{B}\cdot\boldsymbol{\xi}) \neq 0$ so only get a solition when determinant is zero:

$$-\rho\omega^2(\rho\omega^2 - k^2(\gamma p + \frac{B^2}{4\pi})) - \frac{1}{4\pi}(\boldsymbol{B}\cdot\boldsymbol{k})(\gamma p(\boldsymbol{B}\cdot\boldsymbol{k}) = 0$$

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In the non-magnetic case ($\boldsymbol{B} = 0$):

$$\rho\omega^2(\rho\omega^2 - k^2\gamma p) = 0$$

So for propagating waves we need $\omega \neq 0$ so have $\omega^2 = \frac{\gamma p}{\rho}k^2$. If we have isotropic waves then $p = k\rho^\gamma$, $c_s^2 = \frac{p}{\rho} = \gamma p^{\gamma-1}k = \frac{\gamma P}{\rho}$ so $\omega^2 = (c_s k)^2$ so these are sound waves. For sound waves the phase velocity: $\boldsymbol{u}_p = \frac{\omega}{k}\hat{\boldsymbol{k}} = \pm c_s\hat{\boldsymbol{k}}$ and group velocty: $\boldsymbol{u}_{gr} = \pm c_s\hat{\boldsymbol{k}}$. So for sound waves phase velocity is the same as group velocity and are longitudinal as $\boldsymbol{\xi}||\boldsymbol{k}$.

In the magnetic case:

$$\omega^2(\omega^2 - k^2(\frac{\gamma p}{\rho} + \frac{B^2}{4\rho}) + \frac{\gamma P}{\rho}k^2(\frac{\boldsymbol{B}}{\sqrt{4\pi\rho}}\boldsymbol{k})^2 = 0$$

$$\omega^4 - \omega^2 k^2(c_s^2 + u_A^2) + k^2 c_s^2(\boldsymbol{u}_A \cdot \boldsymbol{k})^2 = 0$$

so the disperion relation is:

$$\omega^2 = \frac{1}{2}(k^2(c_s^2 + u_A^2) \pm \sqrt{k^4(c_s^2 + u_A^2)^2 - 4k^2 c_s^2(\boldsymbol{u}_A \cdot \boldsymbol{k})^2}$$

Let $u_p = \frac{\omega}{k}$:

$$u_p^2 = frac12((c_s^2 + u_A^2) \pm \sqrt{(c_s^2 + u_A^2)^2 - 4u_A^2 c_s^2 cos^2\theta}$$

This gives two modes with the fast wave given by the "+" ssign and the slow wave given by the "-" sign. These are called magentosonic waves as it contains both thermal and magnetic pressure act as restoring forces.

Consider the limit as $\theta = 90^o$ $(\boldsymbol{kB})$. For the fast wave we get $u_p = c^s + u_A^2$ and for the slow wave $u_p = 0$.
Consider $\theta = 0, 180^o$ $(\boldsymbol{k}||\boldsymbol{B})$ gives:

$$u_p^2 = \frac{1}{2}(c_s^2 + u_A^2 \pm |c_s^2 - u_A^2|)$$

For $c_s > u_A$ we have $u_p^2 = c_s^2$ (fast) and $u_p^2 = u_A^2$ (slow) and for $c_s < u_A$ we have $u_p^2 = u_A^2$ (fast) and $u_p^2 = c_s^2$ (slow). So for

$$u_p^2 = \begin{cases} max(c_s^2, u_A^2) & \text{fast} \\ min(c_s^2, u_A^2) & \text{slow} \end{cases}$$

Look up Friedrichs diagram as no way I can reproduce it here. u_p for fast wave is always between $\sqrt{c_s^2 + u_a^2}$ and $max(c_s, u_A)$ as can be seen from the diagram. But this is not very much of a deivateion from c_s so waves propogate quasi-isotropically. They are restored by total pressure $p + \frac{B^2}{8\pi}$. Slow waves are sound waves strongly channeled along the $\boldsymbol{B}$field when the field is strong.

10.1 Non-linear waves

Considering non-magnetic, no gravity, assume spatial uniformity of background, assume isentropic setup so $p = P(\rho)$ and $c_s = c_s(\rho) = c_s(p)$ and only consider 1D setup $u = (u, 0, 0)$.

Start by considering a general wave solution (**Simple wave**):

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Assume that $u = u(\rho)$ so $u = u(c_s)$:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial \rho}\frac{\partial \rho}{\partial x} = 0 \tag{27}$$

$$\frac{\partial u}{\partial t} + (u + \frac{1}{\rho}\frac{\partial p}{\partial u})\frac{\partial u}{\partial x} = 0 \tag{28}$$

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$$\rho(x,t) \implies d\rho = \frac{\partial \rho}{\partial t}dt + \frac{\partial \rho}{\partial x}dx$$

For fixed ρ ($d\rho = 0$):

$$\frac{\partial \rho}{\partial t} + \frac{\rho}{\partial x}(\frac{\partial x}{\partial t})_\rho \implies (\frac{\partial x}{\partial t})_\rho = -\frac{\frac{\partial \rho}{t}}{\frac{\partial \rho}{\partial x}}$$

Similarly consider $u = u(x,t)$ for fixed u:

$$(\frac{\partial x}{\partial t})_u = -\frac{\frac{\partial u}{\partial t}}{\frac{\partial u}{\partial x}}$$

$$(\frac{\partial x}{\partial t})_\rho = \frac{\frac{\partial(u\rho}{\partial \rho}\frac{\partial \rho}{\partial x}}{\frac{\partial \rho}{\partial x}} = u + \rho\frac{\partial u}{\partial \rho}$$

similarly and using 27‘:

$$(\frac{\partial x}{\partial t})_\rho = u + \frac{1}{\rho}\frac{\partial p}{\partial \rho}$$

Since $u = u(\rho)$, $(\frac{\partial x}{\partial t})_\rho = (\frac{\partial x}{\partial t})_u$:

$$u + \rho\frac{\partial u}{\partial \rho} = u + \frac{1}{\rho}\frac{\partial p}{\partial u}$$

$$\frac{\partial u}{\partial \rho} = \frac{1}{\rho^2}\frac{\partial p}{\partial u} = \frac{1}{\rho^2}\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial u} = \frac{c_s^2}{\rho^2}(\frac{\partial u}{\partial \rho})^{-1}$$

so

$$(\frac{\partial u}{\partial \rho})^2 = \frac{{c_s}^2}{\rho} \implies \frac{\partial u}{\partial \rho} = \pm\frac{c_s}{\rho}$$

So therefore:

$$u = \pm \int \frac{c_s}{\rho} d\rho = \pm \int \frac{1}{c_s \rho} dp = u(\rho) = u(p)$$

For Polytropic gas: $P = k\rho^\gamma$, $c_s^2 = \gamma k \rho^{\gamma-1}$, $c_s = (\gamma k)^{1/2} \rho^{\frac{\gamma-1}{2}}$.

$$u = \pm(\gamma k)^{\frac{1}{2}} \int \rho^{\frac{\gamma-3}{2}} d\rho = \pm \frac{2}{\gamma - 1} (\gamma k)^{\frac{1}{2}} (\rho^{\frac{\gamma-1}{2}} - \rho_0^{\frac{\gamma-1}{2}})$$

with ρ_0 is the unperturbed density i.e $\rho = \rho_0$ when $u = 0$. Can be rewritten:

$$u = \pm \frac{2}{\gamma - 1} (c_s - c_{s,0}) \implies c_s = c_{s,0} \pm \frac{\gamma - 1}{2} u$$

Need lots of practice with polytropic gas is going to be used everywhere YAY waves part II love that stuff.

Simple wave From $u = \pm \int \frac{dp}{\rho c_s}$:

$$\frac{\partial p}{\partial u} = \pm \rho c_s$$

But $(\frac{\partial x}{\partial t}_u = u + \frac{1}{\rho} \frac{\partial \rho}{\partial u} = u \pm c_s(u)$. Can integrate directly and find an algebraic relation between u, x and t (an implict solution for $u(x, t)$.

$$x = (u \pm c_s(u))t + f(u)$$

At $t = 0$, $x = f(u)$ so f is just the initial condition of some sort.

If at $t = 0$ we have velocity distribution in space s.t. $u(x, t = 0) = g(x)$ as $x = f(u)$ so $f = g^{-1}$. FOR THIS SECTION USE PART II WAVES NOTES IT WAS TAUGHT MUCH BETTER. Though here we seem to only think about the right propogating wave and also it is possible to have non-polytropic gas but it must have some relation $P = (\rho)$. We consider shock waves. They don't talk much about chartersitics and lecture seems keen on intutitive understanding of what shockwaves look like (eventually look like sawtooths would eventually tip over). Explanation given by shockwaves is that u cannot be multivalued in hydrodynamics(unlike in kinetic theory) so it just becomes a shock (discontinuity).

Riemann invariants

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial p} \frac{\partial p}{\partial t} = \frac{1}{c_s^2} \frac{p}{\partial t}, \frac{\partial \rho}{\partial x} = \frac{1}{c_s^2} \frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = \frac{1}{c_s^2} \frac{\partial p}{\partial t} + \rho \frac{\partial u}{\partial x} + \frac{u}{c_s^2} \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p}{\partial t} + \rho c_s^2 \frac{\partial u}{\partial x} + u \frac{\partial p}{\partial x} = 0$$

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Standard riemann derivation(my jam!). Silly astrophysicists don't use $Q = \int \frac{dp}{c_s \rho}$

$$\frac{\partial u}{\partial t} \pm \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x}(u \pm c_s) + \frac{\partial p}{\partial x}(\frac{1}{\rho} \pm \frac{u}{\rho c_s}) = 0$$

$$(\frac{\partial}{\partial t}(u \pm \int \frac{dp}{\rho c_s}) + (u \pm c_s)\frac{\partial}{\partial x}(u \pm \int \frac{dp}{\rho c_s}) = 0$$

Riemann invariants: $J_{pm} = u \pm \int \frac{dp}{\rho c_s}$. These are constant along lines of $\frac{\partial x}{\partial t}_{J_{\pm}} = u \pm c_s$. Did about twenty minutes of example of a simple wave ect. all done much better in waves notes so just re read them.

For a simple wave when does it break given some initial conditions. Denote $v = u \pm c_s(u) \implies \frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} \pm \frac{\partial c_s}{\partial y}\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t}(1 \pm \frac{\partial c_s}{\partial u})$ similarly $\frac{partialv}{\partial x} = \frac{\partial u}{\partial x}(1 \pm \frac{\partial c_s}{\partial u})$. As we know:

$$\frac{\partial u}{\partial t} + (u + \frac{1}{\rho}\frac{\partial p}{\partial u})\frac{\partial u}{\partial x} = 0, \frac{\partial p}{\partial u} = \pm \rho c_s$$

So,

$$\frac{\partial u}{\partial t} = -(u \pm c_s)\frac{\partial u}{\partial x}$$

$$\frac{\partial v}{\partial t} = -(u \pm c_s)\frac{\partial u}{\partial x}(1 \pm \frac{\partial c_s}{\partial y}) = -v\frac{\partial v}{\partial x}$$

So gives the inviscid burgers equation:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0 \tag{29}$$

It is has a very simple solution:

$$v(x,t) = F(x - vt)$$

F determined from initial conditions. At $t = 0$, $v(x,0) = F(x)$.

$$\frac{\partial v}{\partial x} = F'(1 - \frac{\partial v}{\partial x}t) \implies \frac{\partial v}{\partial x} + F't\frac{\partial v}{\partial x} = F'$$

$$\frac{\partial v}{\partial x} = \frac{F'}{1 + F't} = \frac{v_0'(x - vt)}{1 + v_0'(x - vt)}$$

So v is advected along charactersics $x = x_0 + vt$ and a shock forms when $\frac{\partial v}{\partial x} \to \infty$.

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To find where on wave it will first break is by finding the steepest part of the slope: $max(-v_0'(x_0))$ at $t = (max(-v_0'(x_0))^{-1}$. After the shock it continues to evolve after the shock, and it is called an "N-wave". It can be shown that the amplitude of the "N"-wave decays with $A(t)t^{-\frac{1}{2}}$ and $w(t)t^{\frac{1}{2}}$. As it is 3D as the wave propogrates it has to decay in energy so needs to decay in amplitude, as the energy flux is proporotional to the amplitude squared but there are also other effects such as damping (like heat conduction).

At the shock visocitity starts to play an important role.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{4}{3}\frac{\eta}{\rho}\frac{\partial^2 u}{\partial x^2}$$

The last term looks like $\frac{\eta}{\rho}(\nabla^2 \boldsymbol{u} + \frac{1}{3}\nabla(\nabla \cdot \boldsymbol{u}))$ where η is the bulk viscosity and $\frac{\eta}{\rho} = \nu$ is the kinematic viscosity. Can always be written as $\nu \sim c_s\lambda$ where λ is the mean free path of particles. When you are apporaching a shock your second derivative takes on a very large value and so becomes important. Estimate shock width, assume term 2 ($\frac{u^2}{\delta x}$) and term 4 ($\nu\frac{u}{\delta x^2}$) are of comparable size so $\delta x =\sim \frac{\nu}{\mu} \sim \frac{\nu}{c_s} \sim \lambda$. This is not surprising as this is the length scale of collisions occuring in the system. In astrophysics we sometimes have very large mean free paths so our shocks are often mediated not by collisions but by the electromagnetic effects. These collisionless shocks are very complicated entities.

13.0.1 Shock jump conditions

Planar shock propogating in x-direction with speed v_{sh} so $\frac{\partial}{\partial z} = \frac{\partial}{\partial y} = 0$. Switch to frame comoving with shock, so pre shock velocity is $v_1 = v_{pre} - v_{sh}$ and post shock velocity is $v_2 = v_{post} - v_{sh}$. The jump conditions are the relations between the collection $\rho_1, v_1, c_{s,1}, p_1$ and $\rho_2, v_2, c_{s,2}, p_2$. Turns out it is determined by Mach number $M = \frac{v_1}{c_{s,1}}$ which needs $M > 1$ for the shock to happen. We use the conservation laws.
Mass conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{F}_m = 0, \boldsymbol{F}_m = \rho \boldsymbol{v} \implies \rho_1 v_{x,1} = \rho_2 v_{x,2}$$

Momentum conservation:

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} + \nabla\cdot\boldsymbol{\Pi} = 0, \Pi_{ij} = \rho v_i v_j - T_{ij} = \rho v_i v_i + (p + \frac{B^2}{8\pi})\delta_{ij} - \frac{B_i}{B_j} \implies [\rho v_x v_z - \frac{B_x B_z}{4\pi}] = 0$$

Energy conservation:

$$\frac{\partial}{\partial t}(\rho(\frac{v^2}{2}+e)+\frac{B^2}{8\pi})+\nabla\cdot\boldsymbol{F}_E = 0, \boldsymbol{F}_E = \rho\boldsymbol{v}(\frac{v^2}{2}+h)+\frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B}) \implies [\rho v_x(\frac{v^2}{2}+h)+\frac{c}{4\pi}(E_y B_z - E_z B_y)] = 0$$

Electrodynamic constraints:
$\boldsymbol{B}_n$ (normal magnetic field) is continous so $[\boldsymbol{B}_n] = 0$ so

$$[B_x] = 0$$

$\boldsymbol{E}_t$ (tangental electric field) is continous so $[E_t] = 0 \implies [E_y] = [E_z] = 0$. As $\boldsymbol{E} = -\frac{\boldsymbol{v}\times\boldsymbol{B}}{c}$ so

$$[v_x B_z - v_z B_x] = 0$$

$$[v_x B_y - v_y B_x] = 0$$

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14.1 Non-magnetic case

Rankine-Hugoniot relations:

$$[\rho v_x] = 0$$

$$[\rho v_x + p] = 0 \implies [v_x] = -\frac{[p]}{\rho v_x}$$

$$[\rho v_x v_y] = 0 \implies [v_y] = 0$$

$$[\rho v_x v_z] = 0 \implies [v_z] = 0$$

$$[\rho v_x(\frac{v^2}{2} + h)] = 0$$

In the shock $[p] > 0$, $[\rho] > 0$ and $[T] > 0$. In the frame of the shock it is deccelerating the fluid however in the inertial frame it could be accelerating the fluid. In hydrodynamic fluids we see that the tangential velocity is preserved unlike in MHD where we have $\rho v_x[v_y] = [\frac{B_x B_y}{4\pi}] \implies [v_y] = \frac{B_x[B_y]}{4\pi\rho v_x}$. In MHD you can have three different types of shocks unlike here (you can have alfven shocks fast shocks or slow shocks).

Rewatch minutes 15 to 25 of 17/11/2021 as it is a dicussion about contact discontinuity and the reverse and forward shock around the sun (includes disccusion of the size of vorticity).

14.2 Super nova explosion

$E \sim 10^{51} eng$ $M_{ej} \sim 1 M_\odot$ Assume the fluid is cold as it has negillible impact. $\frac{1}{2}M_{ej}v_{ej}^2 \sim E \implies \mathrm{v}_{ej} = (\frac{2E}{M_{ej}})^{\frac{1}{2}} \sim 10^4 \frac{km}{s} E_{51}^{\frac{1}{2}} M_{ej,1}^{-\frac{1}{2}}$. THis is much higher than the speed of sound so mach number is $M \sim 10^2 - 10^3 >> 1$. As the ejecta is expanding the density will be going down, when it encounters the interstellar medium it would send an forward shock into the interstellar medium with some velocity v_{sh} (not the same as v_{ej})$.Intheframeoftheejectortheinterstellarmediumisrunningintoyousothat$

$\frac{4\pi}{3}m_p n_0 R_{tz}^3$ giving the transition radius of $R_{tz} \sim (\frac{3}{4\pi}\frac{M_{ej}}{m_p n_0})^{\frac{1}{3}} \sim 2pc(\frac{M_{ej,1}}{n_{0,1}}$. The time this takes is $t_{tz} \sim \frac{R_{tz}}{v_{ej}} \sim 200yrsE^{-\frac{1}{2}}M_{ej}^{\frac{5}{6}}n_{0,1}^{-\frac{1}{3}}$. For $t > t_{tz}$ we can forget about the initial conditions. So the two remaining parameters are E, n_0.The kinetic energy of the explosion is so much larger than the energy lost through luminscene so can consider this to be an adiabatic explosion. Also adopt a spherically symmetric set up. Assume that at some point in time there was an explision that released a lot of energy E and now we just have some shock propogating into the interstellar medium with density n_0 with some radius $R(t)$. $v_{sh} = \dot{R}(t)$. **The ratio across a strong shock is** $\rho = \rho_o\frac{\gamma+1}{\gamma-1}$. So for $\gamma = \frac{5}{3}$ $\rho = 4\rho_0$ so this is indepedant of the energy of the shock whereas the pressure does depend on speed et.c $p = \frac{2}{\gamma+1}\rho_0 v_{sh}^2$.

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In the shock frame:

$$[\rho u] = 0 \implies \rho(R)u\rho(R) = \rho u_{sh} \implies u(R) = \frac{\rho_0}{\rho}u_{sh} = \frac{\gamma-1}{\gamma+1}u_{sh}$$

In the inertial frame

$$u(R) = u_{sh} - \frac{\gamma-1}{\gamma+1}\frac{2}{\gamma+1}u_{sh} = \frac{2}{\gamma1}u_{sh}$$

. Continuity equation:

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho u) = 0$$

Momentum equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial\rho}{\partial r} = 0$$

Energy equation:

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2(\frac{1}{2}\rho u^2 + \frac{\gamma p}{\gamma-1})u) = 0$$

We have the boundary conditions and the equations of motion but we have no knowledge of where the boundary conitions should be applied we don't know where R should be. We generally solve this issue using dimensional anlysis. We have to charactersitics of the problem we ahve energy ($E = [ML^2T^{-2}]$) and density ($\rho_0 = [ML^{-3}]$). We need an additional variable of time in order to get a length scale so we add the time scale of the explosion into the mix. $\frac{E}{\rho_0} = [L^5T^{-2}]$ so say consider $\frac{E}{\rho_0}t^2 = [L^5]$ so a natural scale for the strong explosion is: $R(t) = (\alpha\frac{E}{\rho_0}t^2)^{1/5}$ with α some dimensionless constant. The dependance on R can come only in the form $\xi = \frac{r}{R(t)}$. This is a similarility variable and is typical of self-similar problems.

If I have unknown $f(r,t) = \phi(t)\tilde{f}\frac{r}{R(t)}$ (similarity anzatz). $\rho = \rho_0\tilde{\rho}(\xi)$, $u = \dot{R}(t)\tilde{u}(\xi)$, $p = \rho_0\dot{R}^2\tilde{p}(\xi)$ the tilde functions are always dimensionless (similarity solution you pull out all the dimensions into the prefactors).

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial t} = -\frac{r}{R}\dot{R}\frac{\partial f}{\partial \xi} = -\frac{\dot{R}}{R}\xi\frac{\partial f}{\partial \xi}$$

$$\frac{\partial f}{\partial r} = \frac{1}{R}\frac{\partial f}{\partial \xi}$$

$$\frac{\dot{R}}{R} = \frac{\partial \ln R}{\partial t} = \frac{\partial(\ln t^{\frac{3}{5}})}{\partial t} = \frac{2}{5}t^{-1}$$

$$\ddot{R} = -\frac{3}{5}Ct^{-\frac{8}{5}} = -\frac{3}{5}Ct^{-\frac{3}{5}}t^{-1} = -\frac{3}{5}\dot{R}\frac{5}{2}\frac{\dot{R}}{R} = -\frac{3}{2}\frac{\dot{R}^2}{R}$$

Continuity equation becomes:

$$-\frac{\dot{R}}{R}\xi\frac{\partial\tilde{\rho}}{\partial\xi}\rho_0 + \frac{1}{R^2\xi^2}\frac{1}{R}\frac{\partial}{\partial\xi}(R^2\xi^2\rho_0\tilde{\rho}\dot{R}\tilde{u}) = 0$$

$$-\xi\frac{\partial\tilde{\rho}}{\partial\xi} + \tilde{u}\frac{\partial\tilde{\rho}}{\partial\xi} + \frac{\tilde{\rho}}{\xi^2}\frac{\partial}{\partial\xi}(\xi^2\tilde{u}) = 0$$

$$(\tilde{u}-\xi)\frac{\partial\tilde{\rho}}{\partial\xi} + \frac{\tilde{\rho}}{\xi^2}\frac{\partial}{\partial\xi}(\xi^2\tilde{u}) = 0 \tag{30}$$

From momentum equation:

$$\frac{\partial\dot{R}}{\partial t}\tilde{u} + \dot{R}\frac{\partial\tilde{u}}{\partial t} + \dot{R}^2\frac{1}{R}\frac{\partial\tilde{u}}{\partial\xi} + \frac{1}{\rho_0\tilde{\rho}R}\frac{\partial}{\partial\xi}(\rho_0\dot{R}^2\tilde{\rho}) = 0$$

$$\frac{-3}{2}\frac{\dot{R}^2}{R}\tilde{u} + \frac{\dot{R}}{R}\xi\frac{\partial\tilde{u}}{\partial\xi} + \frac{\dot{R}^2}{R}\tilde{u}\frac{\partial\tilde{u}}{\partial\xi} + \frac{\dot{R}^2}{R}\frac{1}{\tilde{\rho}}\frac{\partial}{\partial\xi}\tilde{p} = 0$$

$$(\tilde{u}-\xi)\frac{\partial\tilde{u}}{\partial\xi} - \frac{3}{2}\tilde{u} + \frac{1}{\tilde{\rho}}\frac{\partial\tilde{p}}{\partial\xi} = 0 \tag{31}$$

Total energy is conserved so :

$$E = 4\pi\int_0^R r^2dr(\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1} = 4\pi R^3\int_0 1\xi^2d\xi(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\tilde{p}}{\gamma-1})\rho_0\dot{R}^2$$

using expressions for $\dot{R}$:

$$E = 4\pi R^3\rho_0\frac{4}{25}t^{-2}R^2\int_0^1\xi^2d\xi(\frac{1}{2}\tilde{\rho}\tilde{u}+\frac{\tilde{p}}{\gamma-1}) = \frac{16\pi}{25}\rho_0\alpha^5\frac{E}{\rho_0}t^2t^{-2}\int_0^1\xi^2d\xi(\frac{1}{2}\tilde{\rho}\tilde{u}+\frac{\tilde{p}}{\gamma-1})$$

This gives a condition on α, so once we have the full solution we can find α:

$$1 = \frac{16\pi}{25}\alpha^5\int_0^1\xi^2d\xi(\frac{1}{2}\tilde{\rho}\tilde{u} + \frac{\tilde{p}}{\gamma-1})$$

Rewrite energy equation in form:

$$\frac{\partial q}{\partial t} + \frac{\partial F}{\partial r} = 0, q = r^2(\frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1}), F = r^2(\frac{1}{2}\rho u^2 + \frac{\gamma p}{\gamma - 1})$$

get $q = \rho_0 \dot{R}^2 R^2 \xi^2 (\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\tilde{p}^2}{\gamma-1})$, $F = \rho_0 \dot{R}^2 R^2 (\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\gamma\tilde{p}}{\gamma-1})\xi^2\tilde{u}$. Define $\tilde{q}(\xi)$ and $\tilde{F}(\xi)$ as everything to the right of R^2 in the prevoious equations so the energy quation becomes

$$\rho_0 \frac{\partial}{\partial t}(\dot{R}^2 R^2)\tilde{q} + \rho_0 \dot{R}^2 R^2(-\xi\frac{\dot{R}}{R}\frac{\partial \tilde{q}}{\partial \xi}) + \rho_0 \dot{R}^3 R^2 \frac{1}{R}\frac{\partial \tilde{F}}{\partial \xi} = 0$$

16 Lecture 20

Excercise show that:

$$\frac{\partial}{\partial t}(\dot{R}^2 R^2) = -\dot{R}^3 R$$

So can write the relation as:

$$-\rho_0 \dot{R}^4 R\tilde{q} - \rho_0 \dot{R}^2 R^2 \xi \frac{\partial \tilde{q}}{\partial \xi}\frac{\dot{R}}{R} + \rho_0 \dot{R}^3 R \frac{\tilde{F}}{\partial \epsilon} = 0$$

All of the dimensions canncel to give:

$$-\tilde{q} - \xi\frac{\partial \tilde{q}}{\partial \xi} + \frac{\partial \tilde{F}}{\partial \xi} = 0$$

$$\frac{\partial F}{\partial \xi} = \frac{\partial}{\partial \xi}\xi\tilde{q}$$

No integration constant as in centre $\xi = 0$

$$\tilde{F} = \xi\tilde{q} \implies \xi^2\tilde{u}(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\gamma\tilde{p}}{\gamma - 1}) = \xi^3(\frac{1}{2}\tilde{\rho}\tilde{u}^2 + \frac{\tilde{p}^2}{\gamma - 1})$$

$$\tilde{p} = \frac{\gamma - 1}{2}\tilde{\rho}\tilde{u}^2 \frac{u^2 - \xi}{\xi - \gamma\tilde{u}} \tag{32}$$

Left with 2 equations for $\tilde{\rho}$ and $\tilde{u}$, can be solved by making subsitution $\tilde{u} = \xi v(\xi)$. We find a solution for $\xi = \xi(v), \tilde{\rho}(v), \tilde{p}(v), \tilde{u}(v)$.
For gamma = $\frac{5}{3}$, $\tilde{\rho}(1) = 4, \tilde{u} = \frac{3}{4}$:

$$\xi(v) = (\frac{4v}{3})^{-\frac{2}{5}}(\frac{20v}{3} - 4)^{\frac{2}{13}}(\frac{5}{2} - 2v)^{-\frac{82}{195}}$$

$$\tilde{\rho}(v) = 4(4 - 4v)^{-4}(\frac{20v}{3} - 4)^{\frac{7}{13}}(\frac{5}{2} - 2v)^{-\frac{82}{13}}$$

$$\tilde{p}(v) = \frac{3}{4}(\frac{4v}{3})^{\frac{6}{5}}(4 - 4v)^{-5}(\frac{5}{2} - 2v)^{-\frac{82}{15}}$$

$\xi \in (0,1)$ maps to $v \in (\frac{3}{5}, \frac{3}{4})$. Need to know the qualitative shape of the curves$p(\xi), \rho(\xi), u(\xi)$etc. Lecture 20 20th miniute. Derive these shapes by considering how they behave as ξ goes to 0.

1) Most mass is in a thin shell
2) Inside the shell pressure is roughly constant. As temperature is very high the speed of sound is very high so sound waves can smooth out deviations from the equilibirum very quickly.
3) Velocity structure is close to a Hubble Flow (so the velocity is increasing with distance).

This is called the Sedov-Taylor phase and gives a post shock temperature of:

$$R(t) = 5pcE_{51}^{\frac{1}{5}} t_3^{\frac{2}{5}} n_1^{-\frac{1}{5}}$$

$$v(t) = 2000\frac{km}{s} E_{51}^{\frac{1}{5}} t_3^{-35} n_1^{-\frac{1}{5}}$$

$$T(t, r = R) = 5 \times 10^7 K E^{\frac{2}{5}} t_3^{-\frac{6}{5}} n_1^{-\frac{2}{5}}$$

At some point cooling becomes important. The plasma will cool by free-free emission (Bremsttruhlung), when you have a sea of protons and a sea of fast moving electrons and when an electron is deflected it causes emmission of a photon. The free-free emissivity (power per unit volume) goes like $\dot{e}T^{\frac{1}{2}}n^2$ then the cooling time is $t_{cool} \sim \frac{e}{\dot{e}} \frac{nT}{T^{\frac{1}{2}}} = T^{\frac{1}{2}} n^{-1}$. As n is roughly constant, but T decreases with $Tt^{-\frac{6}{5}}$ so t_{cool} goes down and t goes up so at some point $t_{cool} \sim t$ which is the end of the Sedov-Taylor phase. ST stops at around 2000 years. First there was an ejector stage (first 200 years) then there was a Sedov-Taylor phase (till 2000 years) and then there is a slow plough solution.

Snow plough solution: As the shell cools down the inside of the shell doesn't as it is sooooo hot, so as the shell cools down its pressure falls and it gets compressed by the inside of the shell. SO then the inside high pressure and temperature is pushing along a cold very dense region that keeps picking up more and more matter.

Sometimes supernova can all explode in a cluster forming a superbubble this causes momentum feedback on the gas and might even expel it from the galaxy!

17 Lecture 21

17.1 Bondi Accretion

Same assumptions as before no B fields, sphereical symmtery, adiabacity but now include attractor.

$$\phi = -\frac{GM}{r}$$

Steady-state: $\frac{\partial \rho}{\partial t} = 0 = -\nabla \cdot (\rho \bar{u})$, $\bar{u} = (u_r, 0, 0) = (u, 0, 0)$ so

$$-\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho u) = 0$$

Thus, $r^2 \rho u$ is constant. We define a mass accretion rate $\dot{M} = 4\pi r^2 \rho u$ (sometimes this is defined with a minus sign).

$$\rho u \frac{du}{dr} = -\rho \frac{d\phi}{dr} - \frac{dp}{dr} \implies u\frac{du}{dr} = -\frac{d\phi}{dr} - \frac{1}{\rho}\frac{dp}{dr}$$

Integrate to get bernoulli constant: $B = \frac{1}{2}u^2 + \phi + h$ with $h = \int \frac{dp}{\rho}$.
Transform $-\frac{dp}{dr} = -\frac{dp}{d\rho}\frac{d\rho}{dr} = -c_s^2 \frac{d\rho}{dr}$ and use the constant $r^2 \rho u$ to get $\frac{d\rho}{dr} = -\frac{2\rho}{r} - \frac{\rho}{u}\frac{du}{dr}$. So $-\frac{dp}{dr} = c_s^2 \rho (\frac{2}{r} + \frac{1}{u}\frac{du}{dr})$ and therefore EOM becomes:

$$\rho u \frac{du}{dr} = -\rho \frac{d\phi}{dr} + \rho c_s^2 \frac{2}{r} + \frac{\rho c_s^2}{u}\frac{du}{dr} \implies u^2 \frac{du}{dr} = -u\frac{d\phi}{dr} + c_s^2 \frac{2u}{r} + c_s^2 \frac{du}{dr}$$

$$(u^2 - c_s^2)\frac{du}{dr} = u\left(\frac{2c_s^2}{r} - \frac{d\phi}{dr}\right) \tag{33}$$

This gives a constraint. At a point in the flow where $u = c_s$ smooth passage through this point means that we have $\frac{d\phi}{dr} = \frac{2c_s^2}{r}$ or $u = c_s = 0$. This point is called a sonic point (because the velocity of the fluid becomes equal to the sound speed. Denote all variables at the sonic point with subscript s: $\rho_s, r_s, ...$

$$\frac{2c_{s,s}^2}{r} = \frac{d\phi}{dr} = \frac{GM}{r_s^2}, c_{s,s}^2 = \frac{GM}{2r_s}$$

Bernoulli constant at the sonic point:

$$h = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} = \frac{c_s^2}{\gamma - 1}$$

$$B = \frac{c_{s,s}^2}{2} - \frac{GM}{r_s} + h = \frac{GM}{2r_s}\left(\frac{1}{2} + \frac{1}{\gamma - 1}\right) - \frac{GM}{r_s} = \frac{GM}{r_s}\left(\frac{\gamma + 1}{4(\gamma - 1)} - 1\right) = \frac{GM}{r_s}\frac{5 - 3\gamma}{4(\gamma - 1)}$$

$$r_s = \frac{GM}{B}\frac{5 - 3\gamma}{4(\gamma - 1)}, c_{s,s}^2 = B\frac{2(\gamma - 1)}{5 - 3\gamma}$$

So can get the value for the sonic radius from the Bernoulli constant. These constraints only occur if at some point in the flow it goes from subsonic to supersonic.

Look for transonic solutions, smoothly passing through the sonic point.
Wind: Start near central object with $u = 0$ but high c_s and go to $r \to \infty$, where you end up with $u = constant$, $\rho \to \frac{1}{r^2}$.
Accretion: Start with $r \to \infty$ with some $\rho_0, c_{s,0}$ and at rest $u = 0$. Then accerlerate towards the center passing through the sonic point.

Diagram of wind and accretion solutions at 30 min of 21 lecture.

For accretion at $\to \infty$ $B = \frac{c_{s,0}^2}{\gamma-1}$, so $r_s = \frac{GM}{c_{s,0}^2}\frac{5-3\gamma}{4}$ and $c_{s,s}^2 = c_{s,0}^2\frac{2}{5-3\gamma}$.

$$\dot{M} = 4\pi r_s^2 \rho_s u_s = 4\pi r_s^2 \rho_s c_{s,s}$$

$$P \sim \rho^\gamma, c_s^2 \rho^{\gamma-1} \implies (\frac{c_{s,s}}{c_{s,0}})^2 = (\frac{\rho_s}{\rho_0})^{\gamma-1} \implies \rho_s = \rho_0(\frac{2}{5-3\gamma}^{\frac{1}{\gamma-1}}$$

$$\dot{M} = 4\pi(\frac{GM}{c_{s,0}^2})^2(\frac{5-3\gamma}{2})^2\rho_0(\frac{2}{5-3\gamma})^{\frac{1}{\gamma-1}}c_{s,0}(\frac{2}{5-3\gamma}^{\frac{1}{2}} = \pi(\frac{GM}{c_{s,0}})^2\rho_0 c_{s,0} f(\gamma)$$

with $f(\gamma) = (\frac{2}{5-3\gamma})^{\frac{5-3\gamma}{2(\gamma-1)}}$. So to have a transonic solution you need $1 < \gamma < \frac{5}{3}$.
$f(\frac{5}{3}) = 1, f(1) = e^{\frac{3}{2}}$.

Bondi radius: $R_B = \frac{GM}{c_{s,0}^2}$ and $\dot{M} = \pi R_B^2 \rho_0 c_{s,0}$ so the accretion rate corresponds to the rate at which the fluid flows across the Bondi radius with the sound speed and density at infinity.

$$c_{s,0}^2 = \frac{GM}{R_B}$$

So the thermal energy of gas at ∞ is of order of the gravitional energy at R_B.

18 Lecture 22

The above situation of a critical surface arising when the flow speed is equal to the speed of wave propogration is a common motif in inflow and outflow problems of a variety of types.

Outside of r_s (the cirtical surface) the pressure is strong, assume that for $r < r_s$ assume that gravity wins and we can neglect the pressure term. So we can guess that the velovity of the flow is something liek $v = \sqrt{\frac{GM}{r}}$ this would imply that $\dot{M} = 4\pi r^2 u \rho_1 \implies \rho \sim \frac{1}{r^2 u} r^{-\frac{3}{2}}$. Need to check that this assumption is valid:

$$\phi \sim \frac{GM}{r}, \frac{p}{\rho} \sim \rho^{\gamma-1} \sim r^{-\frac{3}{2}(\gamma-1}$$

only γ appraoching $\frac{5}{3}$ is $\frac{P}{\rho} \sim r^{-1}$ so for any other γ this assumption is valid.

Accretion onto a moving object: Not going to solve it but will dicuss certain characteristics. As fluid passes by the object it will deflect the fluid until it forms a shock behind the object and then will be acreted (if it passes clsoe enough ot the object. Now in addition to thermal energy we also have kinteic energy of the gas going by. So the acretion radius is where the sum of the kinetic and thermal energies is of order $\frac{GM}{r_a}$ so $v^2 + c_{s,0}^2 \sim \frac{GM}{r_a}$, $r_a \sim \frac{GM}{v^2 + c_{s,0}^2}$.

$$\dot{M} \sim \pi r_a^2 \rho_a \sqrt{v^2 + c_{s,0}^s} = \pi \frac{(GM)^2}{(v^2 + c_{s,0}^2)}^{\frac{3}{2}} \rho_0$$

This is called Bondi-Littleton accretion rate.

For $v >> c_{s,0}$ (supersonic acretor we have $r_a \sim \frac{GM}{v^2} = \frac{GM}{c_{s,0}^2 M^2} = \frac{R_B}{M^2}$. As $M >> 1$ for supersonic speed, so for supersonic speeds the acretion radius and the accretion rate are massively decreased:

$$\dot{M}_{BL} \sim \pi \frac{(GM)^2}{v^3} \rho_0 \sim \dot{M}_B \frac{1}{M^3} << \dot{M}_B$$

SMBH in our galaxy with $M \sim 4 \times 10^6 M_\odot$, we have measure ρ_0 at R_B found that the actual acretion rate is much less than the calculated creation rate. A similar situation exists in the bulk of the galaxy with neutron stars and black holes floating around not being anywhere near as bright as they owuld be with a acretion rate of the calculated size.

18.1 Axisymmetric MHD outflows

$\boldsymbol{B} = \boldsymbol{B}_p + \boldsymbol{B}_\phi = \frac{\nabla\psi \times \boldsymbol{e}_\phi}{2\pi R} + B_\phi \boldsymbol{e}_\phi$ with $B_\phi = \frac{2I(\psi)}{CR}$. Since we arrived at these equations from just the induction equaiotn they remain valid for MHD with $\boldsymbol{u} \neq 0$.

Mass loading Equation
In steady state induction gives $\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = 0$ which means that $\boldsymbol{u} \times \boldsymbol{B} = \nabla\phi(r, z)$ (purely polidal).

$$\boldsymbol{u} \times \boldsymbol{B} = \boldsymbol{u}_p \times \boldsymbol{B}_p + \boldsymbol{u}_p \times \boldsymbol{e}_\phi B_\phi + u_\phi \boldsymbol{e}_\phi \times \boldsymbol{B}_p$$

But also is purely polidal $\boldsymbol{u}_p \times \boldsymbol{B}_p = 0$ so $\boldsymbol{u}_p || \boldsymbol{B}_p$ so $\boldsymbol{u}_p$ lies on magnetic surfaces:

$$\rho \boldsymbol{u}_p = k \boldsymbol{B}_p$$

Can now apply the continiuty equation. In steady stateL

$$0 = \nabla \cdot (\rho \boldsymbol{u}) = \nabla \cdot (\rho \boldsymbol{u}_p) = \nabla \cdot (k \boldsymbol{B}_p) = \boldsymbol{B}_p \cdot \nabla k + k(\nabla \cdot \boldsymbol{B}_p) = (\boldsymbol{B}_p \cdot \nabla) k = 0$$

This tells us that $\boldsymbol{B}_p k$ and so k is constant on magnetic surfaces, meaning that $k(r) = k(\psi)$. so $\boldsymbol{u}_p = \frac{k(\psi)\boldsymbol{B}_p}{\rho}$. Now look at the second part of the polidal equation:

$$\boldsymbol{u}\times\boldsymbol{B} = \boldsymbol{e}_\phi(u_\phi\boldsymbol{B}_p - B_\phi\boldsymbol{u}_p) = \boldsymbol{e}_\phi\times(u_\phi\boldsymbol{B}_p - \frac{B_\phi k(\psi)}{\rho}\boldsymbol{B}_p) = (u_\phi - \frac{B_\phi k(\psi)}{\rho})\boldsymbol{e}_\phi\times\boldsymbol{B}_p$$

As

$$\boldsymbol{e}_\phi \times \boldsymbol{B}_p = \frac{1}{2\pi R}\boldsymbol{e}_\phi \times (\nabla\psi \times \boldsymbol{e}_\phi) = \frac{\nabla\psi - \boldsymbol{e}_\phi(\nabla\psi \cdot \boldsymbol{e}_\phi)}{2\pi R} = \frac{\nabla\psi}{2\pi R}$$

$$0 = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = \frac{1}{2\pi}\nabla \times (\frac{u_\phi}{R} - \frac{B_\phi k}{\rho R}\nabla\psi) = \frac{1}{2\pi}\nabla(\frac{u_\phi}{R} - \frac{B_\phi k}{\rho R}) \times \nabla\psi = 0$$

so $\nabla(\frac{u_\phi}{R} - \frac{B_\phi k}{\rho R})||\nabla\psi$ therefore $w(\psi) = \frac{u_\phi}{R} - \frac{B_\phi k}{\rho R}$.

19 Lecture 23

$$u_\phi = R\omega(\psi) + \frac{k(\psi)B_\psi}{\rho}$$

so full velocity of flow is:

$$\boldsymbol{u} = \frac{k\boldsymbol{B}_p}{\rho} + \frac{kB_\phi\boldsymbol{e}_\phi}{\rho} + R\omega\boldsymbol{e}_\phi = \frac{k\boldsymbol{B}}{\rho} + \omega R\boldsymbol{e}_\phi$$

Let's switch to the frame rotating with angular vecloty $\omega(\psi)$

$$\boldsymbol{u}' = \boldsymbol{u} - \omega R\boldsymbol{e}_\phi = \frac{k\boldsymbol{B}}{\rho}$$

so $\boldsymbol{u}'||\boldsymbol{B}$ so fluid flows along the field in this grame like deads on a wire.

Entropy

$$\frac{Ds}{Dt} = 0 \implies (\boldsymbol{u}\cdot\nabla)s = 0 \implies (\boldsymbol{u}_p\cdot\nabla)s = 0 \implies (\boldsymbol{B}_p\cdot\nabla)S = 0$$

So $S = S(\psi)$

Angular momentum

EoM in MHD steady state:

$$\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla P - \nabla\phi\rho + \frac{1}{4\pi}((\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \nabla(\frac{B^2}{2}))$$

Take scalar product with $\boldsymbol{e}_\phi$:

$$\rho\boldsymbol{e}_\phi\cdot(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \frac{1}{4\pi}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B})\cdot\boldsymbol{e}_\phi$$

As LHS:

$$\rho\boldsymbol{e}_\phi\cdot(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \boldsymbol{e}_\phi\cdot(\boldsymbol{e}_\phi(\boldsymbol{u}_p\cdot\nabla)u_\phi + \frac{u_\phi u_R}{R}\boldsymbol{e}_\phi)$$

as the same thing is true fro the RHS:

$$\rho((\boldsymbol{u}_p \cdot \nabla)u_\phi + u_\phi u_R R) = \frac{1}{4\pi}(\boldsymbol{B}_p \cdot \nabla)B_\phi + \frac{B_\phi B_R}{R})$$

Look at

$$\frac{1}{R}(\boldsymbol{u}_p \cdot \nabla)(Ru_\phi) = \frac{1}{R}(u_z \frac{\partial}{z}(Ru_\phi) + u_R \frac{\partial}{\partial R}(Ru_\phi)) = (\boldsymbol{u}_p \cdot u_\phi + \frac{u_\phi u_R}{R}$$

So

$$\rho(\boldsymbol{u}_p \cdot \nabla)(Ru_\phi) = \frac{1}{4\pi}(\boldsymbol{B}_P \cdot \nabla)(RB_\phi)$$

$$k(\psi)\rho(\boldsymbol{B}_p \cdot \nabla)(Ru_\phi) = \frac{1}{4\pi}(\boldsymbol{B}_P \cdot \nabla)(RB_\phi)$$

So

$$(\boldsymbol{B}_p \cdot \nabla)(k(\psi)Ru_\phi - \frac{RB_\phi}{4\pi}) = 0$$

So angular momentum is $l(\psi) = k(\psi)Ru_\phi - \frac{RB_\phi}{4\pi}$ (first term is advected angular momentum and the second is the magnetic stress) is a function of ψ.

$$Ru_\phi = \frac{RB_\phi}{4\pi k(\psi)} + l(\psi)$$

Bernoulli intergral

$$\nabla \cdot (\rho \boldsymbol{u}(\frac{u^2}{2} + \phi + h) + c\frac{\boldsymbol{E} \times \boldsymbol{B}}{4\pi}) = 0$$

As

$$\boldsymbol{E} = -\frac{1}{c}\boldsymbol{u} \times \boldsymbol{B} = -\frac{1}{c}(\frac{k\boldsymbol{B}}{\rho} + \omega R\boldsymbol{e}_\phi) \times \boldsymbol{B} = -\frac{\omega R}{c}\boldsymbol{e}_\phi \times \boldsymbol{B} = -\frac{\omega R}{c}\boldsymbol{e}_\phi \times \boldsymbol{B}_p$$

so is purely polidal and so therfore:

$$(\boldsymbol{E} \times \boldsymbol{B})_p = -\frac{\omega R}{c}(\boldsymbol{e}_\phi \times B_P) \times (B_\phi \boldsymbol{e}_\phi) = -\frac{\omega R B_\phi}{c}\boldsymbol{B}_p$$

so energy equation becomes

$$\nabla \cdot (\boldsymbol{B}_p(k(\psi)(\frac{u^2}{2} + \phi + h) - \frac{\omega R B_\phi}{4\pi})) = 0$$

$$(\boldsymbol{B}_p \cdot \nabla)(k(\psi)(\frac{u^2}{2} + \phi + h) - \frac{\omega R B_\phi}{4\pi})) = 0$$

So therefore the following is a function of ψ:

$$\epsilon(\psi) = \frac{1}{2}u^2 + \phi + h - \frac{\omega R B_\phi}{4\pi k(\psi)}$$

To get $\psi(R, z)$ we need to look at the projection of EoM onto the poloidal plane such as the Grad-Shafanov equation

$$(EoM)_p = \nabla\psi(GSeq)$$

This will be a second order in ψ PDE wiht a number of integrals of motion in it. So it exhbitis a number of cirtical surfaces like in bondi accretion (sonic surface/point).

Define poloidal Alfven velocity

$$v_{o,p}^2 = \frac{B_p^2}{4\pi\rho}$$

define polidal alfven number

$$A = \frac{u_p}{v_{a,p}} \implies A^2 = \frac{(u_p\rho)^2}{B_p^2}\frac{4\pi}{\rho} = \frac{4\pi k^2\psi}{\rho}$$

20 Lecture 24

A goes from 0 to ∞ crossing $A = 1$ inbetween as at infinity $\rho = 0$ so A is infinite and at origin $\psi = 0$. We have

$$Ru_\phi = \frac{RB_\phi}{4\pi k} + l(\psi), \frac{u_\phi}{R} = \frac{kB_\phi}{R\rho} + \omega(\psi)$$

Eliminate B_ϕ and get:

$$(u_\phi - \frac{l}{R}) = \frac{\rho}{4\pi k^2}(u_\phi - \omega R)$$

$$u_\phi - \frac{R^2\omega(\psi) - A^2 l(\psi)}{R(1 - A^2)}$$

Therefore $A = 1$ is a special point with $R^2 = \frac{l(\psi)}{\omega(\psi)} \implies R_A = \sqrt{\frac{l(\psi)}{\omega(\psi)}}$ is the Alfven radius/surfce. For $A << 1$ this is sub-Alfvenic, and we have $u_\phi \approx \omega(\psi)R$ - solid body rotation in forced by magnetic stresses!

$$A^2 \sim \frac{\epsilon_{kin}}{\epsilon_{mag}}$$

For $A >> 1$ - super Alfvenic and we ahve $u_\phi \approx \frac{l(\psi)}{R}$ so now the fluid expands preserving its angular momentum so magnetic stresses are no longer important

Disk orbits at $\Omega_k = (\frac{GM}{R^3})^{\frac{1}{2}}$ in ideal MHD we have that $\omega(\psi) = \Omega_k(R(\psi))$ as the magnetic field lines are frozen in the disk. Initial AM of the fluid is $l_0 = \Omega_k(R_0)R_0^2$. At Alfven point $l_A \approx \Omega_k(R_0)R_A^2$. "Lever arm effect" leads to a much larger loss of angular momentum than the fluid would have when it was just part of the disk, the magnetic field is transfering some of the angular momentum into the outflowing wind. Angular momentum flow through the disk is propotiaonl to the angular moemntum flow in the wind $\dot{M}_d\omega_k(R_o)R_o^2 \sim \dot{M}_w\omega_k(R_0)R_A^2$ so $\frac{\dot{M}_d}{\dot{M}_\omega} = (\frac{R_A}{R_0})^2$. Sometimes called the slingshot effect - this is very important need to look into this. Is a very efficent way to lose angular momentum very a small amount of fluid if the flow is signitficantly mangetised and the alfven radius is large.

How do you launch the wind from the disk?
New integral of motion (Jacobi integral): $\epsilon'(\psi) = \epsilon(\psi) - \omega(\psi)l(\psi) = \frac{1}{2}u^2 + \phi + h -$
4 $\pi k - \omega(Ru_\phi - \frac{RB_\phi}{4\pi k} = \frac{1}{2}u_p^2 + \frac{1}{2} - \omega Ru_\phi + \frac{1}{2}(\omega R)^2 + h + \phi$

$$\epsilon'(\psi) = \frac{1}{2}u_p^2 + \frac{1}{2}(u_p - \omega R)^2 + h + \phi - \frac{1}{2}\omega^2 R^2$$

with $\phi_{cg} = \phi - \frac{1}{2}\omega^2 R^2$.

Consider a cold flow so $h = 0$, near the launching point, $A << 1$ so $u_\phi \approx \omega R$. So $\epsilon' \approx \frac{1}{2}u_p^2 + \phi_{cg}(R, z)$. To acclerate and launch fluid ϕ_{cg} needs to decrease along the flux surface.

$$\phi_{cg} = -\frac{GM}{\sqrt{R^2 + z^2}} - \frac{1}{2}\Omega_k^2(R_0)R^2$$

Look at diagram in minute 35 of lecture 24 to see what this potential looks like. We get a saddle point of this potential (on the graph of z against R at $z = 0$ $R = R_0$. So in order to launch to infinity we need to launch into the RHS section of the graph that has negative and we need to figure out the maximium angle at which is is possible (which is the samea s finding the angle of the equipotential hitting R_0.

$$\phi_{cg}(R, z) = \phi_{cg}(R_0, 0) + \frac{1}{2}\phi_{cg,RR}(R - R_0)^2 + \frac{1}{2}\phi_{cg,zz}z^2$$

$$\phi_{cg,R} = \frac{RGM}{(R^2 + z^2)^{\frac{3}{2}}} - \Omega_k^2(R_0)R$$

$$\phi_{cg,RR} = -3\frac{GM}{R_0}^3$$

$$\phi_{cg,z} = \frac{-1}{2}(-\frac{2GMz}{(R^2 + z^2}^{\frac{3}{2}} = \frac{GMz}{(R^2 + z^2)^{\frac{3}{2}}}$$

$$\phi_{cg,zz} = \frac{GM}{R_0^3}$$

So:

$$\phi_{cg}(R, Z) = \phi_{cg}(R_0, 0) - \frac{1}{2}\frac{GM}{R_0^3}(R - R_0)^2 + \frac{1}{2}\frac{GM}{R_0^3}(z - z_0)^2$$

At separatrix: $\phi_{cg} = \phi_{cg}(R_0, 0)$ so the angle is the tangent of the line:

$$3(R - R_0)^2 = z^2$$

which is $\tan\alpha = \frac{z}{R-R_0} = \sqrt{3}$ so the critical angle is 60 degrees. So the lines have to be bent more than 60 degrees and then you can launch a wind.

21 Example sheet 4

Specific enthalpy:

$$h = e + \frac{p}{\rho}$$

in an ideal gas:

$$e = \frac{p}{(\gamma - 1)\rho}$$