

# Quantum Information Theory

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## 1 Lecture 1

### 1.1 Internal waves

#### 1.1.1 Minimal maths version

Imagine we have some sort of stratification that is given by density profile  $\hat{\rho}$ . We are going to require that this is smooth and as we will see later and it is only changing the gradient over a length scale that is large compared to the length scale of the waves. Consider a small volume  $V$  of fluid of density  $\rho_0$  and I will magically displaced it up by a distance of  $\zeta$  (assuming it retains volume, density and shape). There will clearly be a buoyancy force  $B$ , and for small displacements:

$$B = gV\zeta \frac{d\hat{\rho}}{dz}$$

Newton's second law gives

$$\rho_0 V \ddot{\zeta} = B = gV\zeta \frac{d\hat{\rho}}{dz}$$

$$\ddot{\zeta} + \left(0 \frac{g}{\rho_0} \frac{d\hat{\rho}}{dz}\right) \zeta = 0$$

This will clearly have solutions:

$$\zeta = A \cos Nt + B \sin Nt$$

where  $N$  is the buoyancy (or Brunt-Vaisala frequency)  $N = \sqrt{-\frac{g}{\rho_0} \frac{d\hat{\rho}}{dz}}$

There is a key ingredient that hasn't been taken into account in this treatment. There hasn't been anything about continuity e.g. how does the fluid get out the way when it falls back down. To highlight this if we displace a long slim slab of fluid instead of a sphere. If this long slim slab is vertical and we displace it upwards then we get exactly the same maths, but we can make it thin enough that we don't need to worry about continuity to the first order. Now what happens if we take a long thin slab of fluid at an angle and move it along itself. Would it fall vertically downwards or slide back along itself. It

is intuitively hard to fall downwards as this would cause a continuity issue of needing to compress all of the fluid below the line (in order to fall down a lot of fluid needs to be moved past the line which would make a big pressure difference resiting the motion). So the slab will slide back along itself. This will have only displaced each parcle of fluid by  $\zeta \cos \theta$  upwards and it will only experience a force of  $g \cos \theta$  back along its original path. Therefore this would give:

$$\ddot{\zeta} + N^2 \cos^2 \theta \zeta = 0$$

$$\ddot{\zeta} + \omega \zeta = 0$$

This gives the dispersion relation for internal gravity waves:

$$|\omega/N| = |\cos \theta|$$

This logic works if we stack slabs on top of each other and as long as we only move each on by a small amount. They can all oscillate up and down along themselves but have no need to be in the same phase, so you could send a wave though them all perpendicular to the slabs. This would mean each slab is a line of constant phase with energy being transmitted along the slab.

Now lets think about the  $\cos \theta$ . If  $\mathbf{k} = (k, l, m)$  is a vector perpendicular to the slabs, then we have:

$$\cos \theta = \frac{\sqrt{k^2 + l^2}}{\sqrt{k^2 + l^2 + m^2}}$$

### 1.1.2 More rigorous derivation

$$\nabla \cdot \mathbf{u} = 0$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \rho = \kappa \nabla^2 \rho$$

For now  $\kappa = 0$ :

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p - \rho g \hat{z} + \rho \nu \nabla^2 \mathbf{u}$$

For now  $\nu = 0$ . Take  $\rho = \rho_0 + \rho'$ . Linearise with  $\rho' \ll \rho_0$  and Boussinesq  $|\frac{\nabla \mathbf{u}}{\nabla t}| \ll g$ . The Boussinesq means that the  $\rho'$  contributes to the gravity term but not to the inertial term. We can rewrite the equations using reduced gravity to make this clear  $g' = g \frac{\rho - \rho_0}{\rho} = g \frac{\rho'}{\rho}$ :

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho_0} \nabla(p + \rho g z) - g' \hat{z}$$

To linearise we take  $\rho = \hat{\rho}(z) + \rho'(x, t)$  and  $\mathbf{u}' \sim \eta \omega \ll \frac{\omega}{|k|} \implies |k| \eta \ll 1$ . By combining this with our Boussinesq condition we have  $|\nabla \rho'| \ll |\frac{d\hat{\rho}}{dz}|$ . This gives the navier stokes equation:

$$\frac{\partial \rho'}{\partial t} + w \frac{d\hat{\rho}}{dz} = \frac{\partial \rho'}{\partial t} - w \frac{\rho_0}{g} N^2 = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0} \nabla(p_0 + \hat{p} - p') - \frac{\hat{\rho} + \rho'}{\rho_0} \mathbf{z} = -\frac{1}{\rho_0} \nabla(p_0 + \hat{p}) - g \frac{\hat{\rho}}{\rho_0} \hat{\mathbf{z}} - \frac{1}{\rho_0} \nabla p' - g \frac{\rho'}{\rho_0} \hat{\mathbf{z}}$$

Unperturbed state gives the first two terms as they are much larger than the other terms

$$0 = -\frac{1}{\rho_0} \nabla(p_0 + \hat{p}) - g \frac{\hat{\rho}}{\rho_0} \hat{\mathbf{z}}$$

therefore

$$p_0 + \hat{p} = - \int g \hat{\rho} dz$$

Let

$$b = -g \frac{\rho'}{\rho_0}$$

Therefore:

$$\frac{\partial b}{\partial t} = -w N^2 \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0} \nabla p' + b \hat{\mathbf{z}} \quad (2)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (3)$$

## 1.2 Vorticity

:  $\zeta = \nabla \times \mathbf{u}$  We are going to deal in 2D as it is easier. In 2-D voricity can be expressed interms of the streamfunction:

$$\zeta = -\nabla^2 \psi, \boldsymbol{\psi} = \psi \hat{\mathbf{y}}, \mathbf{u} = \nabla \times \boldsymbol{\psi}$$

Take curl of momentum equation to remove pressure:

$$\frac{\partial \zeta}{\partial t} = -\hat{\mathbf{z}} \times \nabla b$$

$$(\hat{\mathbf{z}} \times \nabla) \cdot (\hat{\mathbf{z}} \cdot \nabla w) = \nabla_H^2 w$$

so gives voricity equation:

$$(\nabla^2 \frac{\partial^2}{\partial t^2} + N^2 \nabla_H^2) w = 0 \quad (4)$$

Pose a plane wave type solution ansatz and see what happens. Let :

$$w(\mathbf{x}, t) = \text{Re}(\hat{w}(z) e^{i(kx + ly - \omega t)})$$

$$\frac{d^2 \hat{w}}{dz^2} + (k^2 + l^2) \left( \frac{N^2}{\omega^2} - 1 \right) \hat{w} = 0$$

So if we let  $m^2 = (k^2 + l^2) \left( \frac{N^2}{\omega^2} - 1 \right)$  then:

$$\hat{w} = \text{Re}(A e^{imz} + B e^{-imz})$$

If  $\omega > N$  then  $m$  is imaginary let  $\gamma = \sqrt{1 - \frac{N^2}{\omega^2}}$ :

$$w = (\hat{A}e^{-\gamma k_h z} + \hat{B}e^{\gamma k_h z})e^{i(kx+ly-\omega t)}$$

This is sort of showing how the velocity field changes with depth away from a disturbance on the surface. If we have  $N = 0$  then this is just a surface wave. If we have  $0 < N < \omega$  the  $\gamma$  is just giving a vertical rescaling of the behaviour beneath the surface wave. This means if we produce a sinusoidal disturbance with a frequency bigger than the buoyancy frequency then the disturbance looks like potential flow, as we increase the stratification of a fluid it will decrease the decay rate of that motion as we move away from that boundary. The limiting case is when we reach  $\omega = N$  the entire water depth is moving in phase and with the same magnitude as the surface.

In the case  $\omega < N$  we have  $m$  is real so we get sinusoidal variations in the vertical direction:

$$w = w_0 e^{ikx+ly+mz-\omega t} = w_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$$

$$\phi = \mathbf{k} \cdot \mathbf{x} - \omega t$$

so

$$w = w_0 e^{i\phi}$$

We want to get an idea of the relationships between the different parameters. To start with consider continuity:

$$\nabla \cdot \mathbf{u} = 0 \implies \frac{\partial u}{\partial x} + \frac{w}{\partial z} = 0$$

So

$$u = \int \frac{\partial w_0 e^{i\phi}}{z} dx = -\frac{m}{k} w_0 e^{i\phi} = -\frac{\tan \theta}{\cos \theta} w_0 e^{i\phi}$$

Considering surface variation  $\eta(x, t) = \tilde{\eta}(x, t)e^{i\phi}$  therefore by differentiating this and matching with  $u$  and  $w$  at the surface:

$$\mathbf{u} = \frac{\partial \eta}{\partial t} \implies w_0 = i\omega \cos \theta \tilde{\eta}$$

Now considering the relationship arising from the buoyancy equation:

$$\frac{b}{\partial t} = -wN^2 \implies i\omega \tilde{b} = -wN^2$$

$$b = -\eta \frac{\omega^2}{\cos \theta} e^{i\phi} = -\eta \omega N e^{i\phi}$$

Now consider the momentum equation:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x} \implies \tilde{p} = i \frac{\omega N}{|\mathbf{k}|} \eta \sin \theta$$

### 1.3 Wave velocities

#### Phase velocity

$$\phi = \mathbf{k} \cdot \mathbf{x} - \omega t = k_i x_i - \omega t$$

The below identity is very obviously zero:

$$\frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x_i} = 0$$

$$k_i \frac{\partial \phi}{\partial t} + \omega \frac{\partial \phi}{\partial x_i} = 0$$

Divide across by  $k_i$ :

$$\frac{\partial \phi}{\partial t} + \frac{\omega}{|k|^2} k_i \frac{\partial \phi}{\partial x_i} = 0$$

Therefore,  $c_p = \frac{\omega}{|k|^2} \mathbf{k}$  as:

$$\frac{\partial \phi}{\partial t} + (\mathbf{c}_p \cdot \nabla) \phi = 0$$

## 2 Lecture 3

#### Group velocity:

$$\frac{\partial^2 \phi}{\partial x_i \partial t} - \frac{\partial^2 \phi}{\partial t \partial x_i} = 0$$

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0$$

As  $\omega = \omega(k)$  we have:

$$\frac{\partial \omega}{\partial x_i} = \frac{\partial \omega}{\partial k_j} \frac{\partial k_j}{\partial x_i}$$

$$\frac{\partial k_j}{\partial x_i} = \frac{\partial^2 \phi}{\partial x_j \partial x_i} = \frac{\partial k_i}{\partial x_j}$$

Therefore:

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial k_j} \frac{\partial k_i}{\partial x_j} = 0$$

Therefore,  $c_g = \frac{\partial \omega}{\partial k_i}$  as:

$$\frac{\partial k_i}{\partial t} + \mathbf{c}_g \cdot \nabla k_i = 0$$

So the wavenumber vector is being advected outwards with the group velocity.

**Surface waves:**  $\omega = gk$ ,  $c_g = \frac{\partial}{\partial k} \sqrt{gh} = \frac{1}{2} c_p$ ,  $c_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}}$ .

## 2.1 Superposition

$$\eta = \cos((k + \delta k)x - (+\delta\omega)t) + \cos((k - \delta k)x - (\omega - \delta\omega)t)$$

$$\eta = 2 \cos(\delta kx - \delta\omega t) \cos(kx - \omega t)$$

As  $\delta\omega = \frac{\partial\omega}{\partial k} \delta k$  for  $|\delta k| \ll |k|$  then

$$\eta = 2 \cos\left(\left(x - \frac{\partial\omega}{\partial k}t\right)\delta k\right) \cos(kx - \omega t)$$

This is a wave and envelope speed of  $\frac{\partial\omega}{\partial k}$ .

## 2.2 Internal wave velocities

As  $\frac{\omega^2}{N^2} = \frac{k^2 + l^2}{|k|^2} = \cos^2 \theta$

$$\mathbf{c}_p = \frac{\omega}{|k|^2} \mathbf{k} = \frac{N(k^2 + l^2)^{\frac{1}{2}}}{|k|^{\frac{3}{2}}} \mathbf{k} = \frac{N|\cos \theta|}{|k|^2} \mathbf{k}$$

This is the polar coordinate equation for two circles touching at the origin at every  $\phi$  so they sort of form a torus.

Now lets look at the group velocity:

$$\mathbf{c}_g = \frac{\partial\omega}{\partial k_i} = \frac{1}{2\omega} \frac{\partial\omega^2}{\partial k_i} = \frac{\omega}{|k|^2} \left( \frac{N^2}{\omega^2} (\mathbf{k} - k_z \hat{\mathbf{z}}) - \mathbf{k} \right) = \frac{N|\sin \theta|}{|k|} \begin{pmatrix} \cos \phi \sin \theta \\ \sin \phi \sin \theta \\ -\cos \theta \end{pmatrix}$$

$$|c_g| = \frac{N}{|k|} |\sin \theta|$$

This means that in the horizontal direction the phase velocity is always perpendicular to the group velocity. They form the same circle just one with  $\sin \theta$  and one with  $\cos \theta$  and as  $\sin \theta = \cos(\pi/2 - \theta)$ . As the angles on a semicircle subtend 90 degrees we can sum the two and we will get the opposite side of the circle always. Therefore,

$$\mathbf{c}_p + \mathbf{c}_g = \frac{N}{|\mathbf{k}|} \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}$$

$$|\mathbf{c}_p + \mathbf{c}_g| = \frac{N}{|\mathbf{k}|}, c_{p,z} = -c_{g,z}, \mathbf{c}_p \cdot \mathbf{c}_g = 0$$

## 2.3 Equipartition of energy

$$\mathbf{u} \cdot (\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \nabla p' + \rho' g \mathbf{z}) = 0$$

Recalling that  $\frac{\partial \rho'}{\partial t} - w \frac{\rho_0}{g} N^2 = 0 \implies w = \frac{g}{\rho_0 N^2} \frac{\partial \rho'}{\partial t}$ , and the incompressibility condition to get:

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho_0 |\mathbf{u}|^2 + \frac{1}{2} \frac{g^2}{\rho_0 N^2} p'^2 \right) + \nabla \cdot (p' \mathbf{u}) = 0$$

If we go back to the start and consider the displacement of a packet of fluid by  $\zeta$  we have change in potential energy of:

$$\Delta PE = \int_{z_0}^{z_0 + \zeta} g \frac{d\hat{\rho}}{dz} (z - z_0) dz = \frac{1}{2} \rho_0 N^2 \zeta^2$$

$$\rho' = -\frac{d\hat{\rho}}{dz} \zeta = \frac{\rho_0}{g} N^2 \zeta$$

$$PE = \frac{1}{2} N^2 \rho_0 \zeta^2 = \frac{1}{2} \rho_0 \frac{b^2}{N^2}$$

If we want to consider the total energy equation:

$$\int_V \frac{\partial}{\partial t} (KE + PE) dV + \int_S p' \mathbf{u} \cdot \mathbf{n} dS' = 0$$

$\mathbf{F}_E = p' \mathbf{u}$  is the flux of energy.

Lets consider 2D:

$$u = \eta \omega \sin \theta \sin \phi, w = -\eta \omega \cos \theta \sin \phi, b = \eta \frac{\omega^2}{\cos \theta} \cos \phi, p' = \eta \rho_0 \frac{\omega^2}{|k|} \tan \theta \sin \phi$$

Substitute into kinetic energy:

$$KE = \frac{1}{2} \rho_0 (u^2 + w^2) = \frac{1}{2} \rho_0 \omega^2 \eta^2 \sin^2 \phi$$

$$\bar{K}E = \frac{1}{4} \rho_0 \omega^2 \eta^2$$

$$PE = \frac{1}{2} \rho_0 \omega^2 \eta^2 \cos^2 \phi$$

$$\bar{P}E = \frac{1}{4} \rho_0 \omega^2 \eta^2$$

Also have:

$$PE = \frac{1}{2} \rho_0 \frac{b^2}{N^2}$$

So you have equipartition of energy for linear waves  $\bar{K}E = \bar{P}E$ . We can also write down an expression for the flux of energy:

$$\mathbf{F}_E = p' \mathbf{u} = \rho_0 \omega^2 \eta^2 \sin^2 \phi \frac{N}{|k|} \sin \theta \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix} = \frac{1}{2} \rho_0 \omega^2 \eta^2 \mathbf{c}_g = \bar{E} \mathbf{c}_g$$

### 3 Lecture 4

#### 3.1 Oscillating cylinder

We are interested in the case where the oscillation of the cylinder  $a$  is much smaller than the diameter. You might think this is sufficient to make the waves linear, but it is not as we have these delta functions on the singularities on the tangent planes to the cylinder (so we will always have the amplitudes being large compared to the wave lengths here but we will ignore this). If everything is at rest to start with it will take a bit of time for the oscillations to propagate out into the whole space. As  $|c_g| = \frac{N}{|k|} \sin \theta$  the area of which is influenced by the oscillation will look like two causality envelopes (circles of increasing size touching at the centre). The waves form a st. andrews cross pattern with the waves being bi modal near the cylinder and unimodal further away this is due to viscosity

##### 3.1.1 Decay along a beam

We are going to take it being 2D and the buoyancy frequency  $N = 1$  and the mass dispersion  $\kappa = 0$  but include viscosity  $\nu \neq 0$ .

$$\begin{aligned}\nabla \cdot \mathbf{u} &= 0 \\ \frac{\partial u}{\partial t} + \frac{1}{\rho_0} \frac{\partial p}{\partial x} &= \nu \nabla^2 u \\ \frac{\partial w}{\partial t} + \frac{1}{\rho_0} \frac{\partial p}{\partial z} - b &= \nu \nabla^2 w \\ \frac{\partial b}{\partial t} + N^2 w &= 0\end{aligned}$$

To make our lives easier we are going to use a streamfunction:

$$\boldsymbol{\psi} = (0, \psi, 0), \mathbf{u} = (\nabla \times \boldsymbol{\psi}) e^{-i\omega t} = \begin{pmatrix} -\frac{\partial \psi}{\partial z} \\ 0 \\ \frac{\partial \psi}{\partial x} \end{pmatrix} e^{-i\omega t}$$

Take  $\zeta$  and  $\xi$  to be displacement in the wavevector  $k$  direction and then group velocity  $c_g$  direction. let  $\theta$  be the angle of the group velocity from the vertical which is the same as the angle of the cross. Therefore

$$\begin{aligned}\zeta &= x \cos \theta - z \sin \theta \\ \xi &= x \sin \theta + z \cos \theta\end{aligned}$$

Boynancy equation becomes:

$$\frac{\partial b}{\partial t} + \frac{\partial \psi}{\partial \zeta} \cos \theta + \frac{\partial \psi}{\partial \xi} \sin \theta = 0$$



**Vorticity:**  $\nabla \times \mathbf{u} = -\nabla^2 \psi e^{-i\omega t}$

$$-i\omega\left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \zeta^2}\right) - \frac{\partial b}{\partial \xi} \sin \theta - \frac{\partial b}{\partial \zeta} \cos \theta - \nu \nabla^2 \nabla^2 \psi = 0$$

Let  $b = (b_0 + \epsilon b_1 + \dots)e^{-i\omega t}$  and  $\psi = (\psi_0 + \epsilon \psi_1 + \dots)$ : We take small viscosity to make equations nice e.g. small  $\nu = 2\epsilon$  and  $\epsilon = \frac{1}{2}\nu$  as this is dimensional it is not clear what we mean by small parameter. As this is dimensional it means the dimensions of  $\psi_0$  and  $\psi_1$  are going to be different. This is not the ideal way of doing this but it is going to allow us to see more clearly what is happening.

$$\chi = \frac{\epsilon}{\sin \theta} \xi$$

Here  $\epsilon$  being small means we are interested in gradual changes in the direction of the group velocity but fast changes in the wave vector direction so we keep that sinusoidal behaviour.

$$\frac{\partial}{\partial \xi} = \frac{\partial \chi}{\partial \xi} \frac{\partial}{\partial \chi} = \frac{\epsilon}{\sin \theta} \frac{\partial}{\partial \chi}$$

Plug these into the buoyancy and vorticity equation and compare terms of the same order:

$$\begin{aligned} \epsilon_0 : \frac{\partial \psi_0}{\partial \zeta} &= -b_0, \quad \frac{\partial^2 \psi_0}{\partial \zeta^2} = i \frac{\partial b_0}{\partial \zeta} \\ \epsilon_1 : \omega \frac{\psi_1}{\partial \zeta} - i\omega b_1 &= -\frac{\partial \psi_0}{\partial \chi}, \quad i\omega \frac{\partial^2 \psi_1}{\partial \zeta^2} + \omega \frac{\partial b_1}{\partial \zeta} = i \frac{\partial^2 \psi_0}{\partial \zeta \partial \chi} - 2 \frac{\partial^4 \psi_0}{\partial \zeta^4} \end{aligned}$$

Can eliminate the LHS of both of these to give:

$$\begin{aligned} \frac{\partial^4 \psi_0}{\partial \zeta^4} &= i \frac{\partial^2 \psi_0}{\partial \zeta \partial \chi} \\ \frac{\partial^3 \psi_0}{\partial \zeta^3} &= i \frac{\partial \psi_0}{\partial \chi} + f(\chi) \end{aligned}$$

For a point sources as  $|\zeta| \rightarrow \infty$  then we expect  $\psi \rightarrow \psi_0 = F(\zeta)G(\chi) \frac{F''}{F} = i \frac{G'}{G} = -ik^3 G(\chi) = e^{-k^3 \chi} F(\zeta) = e^{ik\zeta} \psi \approx \psi_0 e^{-i\omega t} = A e^{-k^3 \chi} e^{i(k\zeta - \omega t)} \mathbf{k} = (k, 0)$  in  $(\zeta, \xi)$

$$\chi = \frac{\epsilon}{\sin \theta} \xi = \frac{\nu}{2 \sin \theta} \xi = \frac{\nu}{2N \sin \theta} \xi$$

for any  $N$ . If we take a whole spectrum of linear superposition of waves  $A(k)$ :

$$\psi = e^{-i\omega t} \int_{-\infty}^{\infty} A(k) \exp(ik\zeta - \frac{\nu k^3}{2N \sin \theta} \xi) dk \quad (5)$$

If we think about our cylinder with the two delta functions at the edge of the cylinder. As the fourier transform of the delta function is just a constant. So  $A(k)$  would just be a constant so the higher wavenumber modes would decay

more rapidly due to the  $k^3$  in the exponential. We want it to be clear that the length scale over which the day is happening is small compared to the length scale of the oscillations. Lets imagine we are going to scale  $\xi$ :

$$\frac{\nu k^2}{2N \sin \theta} k\xi$$

Recall that  $|c_g| = \frac{N}{|k|} \sin \theta$  so

$$\frac{\nu k^2}{2N \sin \theta} k\xi = \frac{\nu k}{2} \frac{1}{|c_g|} k\xi = -\pi Re^{-1} k\xi$$

With Reynolds number  $Re = \frac{\lambda |c_g|}{\nu}$  with  $\lambda = \frac{2\pi}{k}$ . We were requiring  $\epsilon$  to be small we were really requiring the Reynolds number to be large enough.

### 3.2 Mass diffusivity

$$(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla) \rho = \frac{D\rho}{Dt} = \kappa \nabla^2 \rho$$

Lets think about what could cause a difference in the density in the fluid e.g.  $S$  salt concentration, moisture content or temperature  $T$ . Carbon dioxide that we are breathing out is denser than the other air we are breathing out, which is more or less balanced by the humidity of the exhaled breath which is higher than the surroundings. Water vapour is less dense than air. The diffusivity of salt and the diffusivity of temperature are different so we can't write down an equation like above. What we can write down is equations of the diffusivity of these two:

$$\frac{DS}{Dt} = \kappa_s \nabla^2 S, \frac{DT}{Dt} = \kappa_T \nabla^2 T$$

For water  $\frac{\kappa_T}{\kappa_s} \approx 100$ . If you move hot salty water down into cold nonsalty water then it quickly becomes cold salty water so it rapidly becomes more dense than the fluid around it. Like wise if you took a parcel of cold fresh up into the hot area it will stay fresh but rapidly heat and so will be less dense than its surroundings and want to rise. This is called salt fingering. Equally if you have cold fresh water above hot salty water. As temperature diffuses relatively quickly the gradient will be much steeper for the salt concentration than the temperature, this creates convection. This is called double-diffusive convection.

Prandtl number:  $\frac{\nu}{\kappa_T}$

Schmidt number:  $\frac{\kappa_T}{\kappa_S}$  with  $\kappa_S$  is the diffusivity of some thing like salt

### 3.3 Reflections of waves

With light the angle of incidence is equal to the angle of reflection.

## 4 Lecture 6

The thing being conserved on reflection is the wavelength (in the case of light) and therefore the colour. If the speed of light in the medium is constant then the frequency is constant, then the wavelength must also be constant. However, in our internal wave system our frequency  $\omega$  is not constant as  $|\frac{\omega}{N}| = |\cos \theta|$ . The angle to the vertical must be conserved not the angle of incidence and angle of reflection in order for the frequency to be the same at the wall. So only in the case of a horizontal wall is the angle of incidence equal to the angle of reflection. This difference will mean that the wavelength will not be conserved in general.

Let the displacement on the incident ray  $\eta_i$  and reflected rays  $\eta_r$ . Therefore in example sheet we show that a slope of angle  $\alpha$  to the vertical is:

$$|\eta_r| = \gamma |\eta_i|, \gamma = \left| \frac{\sin(\theta + \alpha)}{\sin(\theta - \alpha)} \right|$$

This comes about as the same volume of fluid must be displaced but the distance between neighbouring rays is smaller, so the displacement must be larger.

### 4.0.1 Energy density upon reflection

$$|\mathbf{k}_r| = \gamma |\mathbf{k}_i| \implies \lambda_r = \frac{1}{\gamma} \lambda_i$$

$$|\tilde{\mathbf{u}}_r| = \gamma |\tilde{\mathbf{u}}_i|$$

$$|\tilde{\boldsymbol{\eta}}_r| = \gamma |\tilde{\boldsymbol{\eta}}_i|$$

Recall  $|c_g| = \frac{N}{|\mathbf{k}|} \sin \theta$ :

$$|c_{g,r}| = \frac{1}{\gamma} |c_{g,i}|$$

Flux of energy per wavelength must be preserved on reflection. We have to be careful what we mean by flux of energy as we have two different wavelength so we can talk about flux of energy per wavelength or flux of energy per unit length:

Energy density per wavelength:

$$\begin{aligned} \tilde{E} &= \int_0^\lambda PE + KE d\zeta \\ \tilde{\mathbf{F}} &= \tilde{E} \mathbf{c}_g \end{aligned}$$

Therefore:

$$\tilde{E}_r = \gamma \tilde{E}_i$$

As  $\tilde{E}_r \sim \lambda(PE + KE) = \lambda(\tilde{\boldsymbol{\eta}}\tilde{\boldsymbol{\eta}}^* + \tilde{\mathbf{u}}\tilde{\mathbf{u}}^*) \sim \lambda(|\tilde{\boldsymbol{\eta}}|^2 + |\tilde{\mathbf{u}}|^2)$  and as  $\lambda \sim \frac{1}{\gamma}$  and  $|\boldsymbol{\eta}|^2 \sim \gamma^2$  so  $\tilde{E}_r \sim \gamma \tilde{E}_i$ .

Energy density per unit length:

$$\bar{E} = \frac{1}{\lambda} \int_0^\lambda PE + KE = \frac{1}{\lambda} \tilde{E}$$

$$|\tilde{F}| = \tilde{E}|c_g| = \lambda \bar{E}|\mathbf{c}_g|$$

Flux of energy per wavelength was conserved:

$$|\tilde{F}_r| = \lambda_r \bar{E}_r |c_{g_r}| = \frac{1}{\gamma} \lambda_i \bar{E}_r \frac{1}{\gamma} |c_{g,i}| = |\tilde{F}_i| = \lambda_i \bar{E}_i |c_{g,i}|$$

So

$$\bar{E}_r = \gamma^2 \bar{E}_i$$

Again consisten with  $\bar{E} \sim PE + KE \sim |\tilde{\eta}|^2 + |\tilde{u}|^2$ . The typeset notes have hats for energy desnity and flux per wavelength.

We also might want to think about the total energy of a whole broad region of a wave coming in and going out:

$$TE_r = \int_{-\frac{L_r}{2}}^{\frac{L_r}{2}} PE_r + KE_r d\zeta = \gamma^2 \int_{-\frac{1}{\gamma}\frac{L_i}{2}}^{\frac{1}{\gamma}\frac{L_i}{2}} PE_i + KE_i = \gamma TE_i$$

Spectral energy density  $S(k)$  if we have a whole lot of waves of different wavenumbers and want to consider how that spectrum will change:

$$S_r(k) = \gamma S_i\left(\frac{K}{\gamma}\right)$$

#### 4.0.2 Critical reflection

Under sub critical reflection the vertical direction of the propogation reverses upon reflection.

Under supercritical reflection the vertical direction of propogation is maintained. On the boundary between these modes is the critical reflection where the reflected wave is along the slope of the wall in both directions so we have  $\alpha = \theta$ . Shifting slightly one way would lead to the wave reflecting slightly above the upper portion of the wall and slightly the other would lead to the the wave reflection slightly above the lower poriton oft he wave.

As we approach critical reflection,  $\tilde{\eta} \rightarrow \infty$ ,  $k_r \rightarrow \infty$ , dissipation which scales like  $k^3$  also tends to infinty. Viscosity becomes important and we get non-linearities as  $k\tilde{\eta} \rightarrow \infty$  as linear waves require  $k\tilde{\eta} \ll 1$ . Viscosity also plays a role away from critical conditions due to no-slip boundary.

### 4.1 Ray tracing

$$(\nabla^2 \frac{\partial^2}{\partial t^2} + N^2 \nabla_H^2)w = 0$$

$$(\nabla^2 \frac{\partial^2}{\partial t^2} + N^2 \nabla_H^2)\psi = 0$$

In 2D:  $\psi = \tilde{\psi}(x, z)e^{-i\omega t} : (N^2 - \omega^2) \frac{\partial^2 \tilde{\psi}}{\partial x^2} - \omega^2 \frac{\partial^2 \tilde{\psi}}{\partial z^2} = 0$   $\Lambda^2 = \frac{\omega^2}{N^2 - \omega^2}$  givethisPoincarewaveequation( $\frac{\partial^2}{\partial x^2} - \Lambda^2 \frac{\partial^2}{\partial z^2}$ ) $\tilde{\psi} = 0$  If domain bounded with  $= 0$  on boundary this is an ill-posed problem and we will use ray tracing in st

## 5 Lecture 6

Lets consider a constant slope at the edge of the ocean with free surface with constant bounancy frequency. Imagine we have some waves entering the system in the body of the fluid 1 wavelength apart. These reflect off the free surface and here they remain exactly the same except for a change of vertical orientation. The reflection off the slope shortens the wavelength as we considered in the previous section. Therefore, the energy reflection goes up, group vocolity is going down, wavenumber goes down and the velocity gets steeper on each reflection off the slope. In this case the energy is trapped into the courner by a sequence of focusing reflections.

For large Reynolds number  $Re = \frac{|c_g|}{|k|\nu}$  nonlinearities will end up dominating leading to wave breaking, mixing and other frequencies and wave numbers coming out of the system.

Energy density increases:

$$\tilde{E}_{n+1} = \gamma \tilde{E}_n = \gamma^{n+1} \tilde{E}_0$$

Steepness increases

$$|k_{n+1}| \tilde{\eta}_{n+1} = \gamma^{n+1} k_0 \eta_0$$

### 5.1 Reflections from rough topography

How well do we actually know what is down at the bottom of the ocean?

Lets imagine we have some idealised rough topography, a sine wave of amplitude  $h_0$  and wavelength  $\lambda_T = \frac{2\pi}{k_T}$ . We can make use of our ray tracing.

Draw rays one wavelength apart and you can visually see that the reflected wavelength varies depending on which part of the sine wave they reflect off. This means the spectrum has changed so we can no longer just talk about the wavenumber vector. If we going to try and analyse this we can try some sort of linearisation by considering small amplitude variations in topography with  $k_T h_0 \ll 1$ .

Now lets zoom in and consider what is happening at a close scale (here we aren't actually assuming it is small), we have a wave coming in towards  $x_i$ , then reflects at  $x_0$  and then appears to be leaving from  $x_r$ . We define  $\delta x = x_0 - x_r = x_i - x_0$  as the angle of the reflected and incident wave from the vertical are identical. Take the height of the boundary to be  $z = h_0 \sin k_T x$ . Define  $\beta = \cot \theta$ . Now geometrically,  $h_0 \sin k_T x_0 = \beta \delta x$ . Now consider the amplitude of an incident ray

$$\eta_i(x, t) = \tilde{\eta}_i \sin(k_i x - \omega t), \eta_r(x, t) = -\tilde{\eta}_i \sin(k_i(x + \frac{2h_0}{\beta} \sin k_T x) - \omega t)$$

For small amplitude  $\frac{k_i h_0}{\beta} \ll 1$ :

$$\eta_r(x, t) = -\tilde{\eta}_i(\sin(k_i x - \omega t) + k_i(\frac{2h_0}{\beta}) \sin(k_T x) \cos(k_i x - \omega t)) = -\tilde{\eta}_i \sin(k_i x - \omega t) - \tilde{\eta}_i \frac{k_i k_0}{\beta} (-\sin((k_i - k_T)x - \omega t))$$

So we have three different wave numbers in the reflection:  $k_i = k_R, k_i - k_T = k_B, k_i + k_T = k_F$ .

If  $k_B = k_i - k_T < 0$  then we need to start being quite careful as in order to match the boundary condition it seems like the group velocity seems to be moving backwards from the right. This violates causality and the reason for this is we were making an assumption about the direction of the wavenumber vector associated with the reflection. What we actually have is two waves reflected in the forwards direction with group velocity  $c_{gR}, c_{gF}$  and a backwards reflected one with  $c_{gB}$ .

In general, for subcritical reflection we will end up with a backscatter from rough topography.

What happens if we have super critical reflection when the topography is steeper than the angle of the incident waves. This leads to a very complex spectrum as neighbouring rays could end up with very different end points as the waves can reflect multiple times off the topography before escaping. This problem is not very analytically tractable.

## 5.2 Wave attractors

Now let's look at smooth boundaries but make it a bit more complex by considering a bounded domain.

### 5.2.1 Rectangular basin

With width  $X$  and height  $Y$

If we have an eigenmode, then

$$\frac{X}{Y} \frac{n}{m} = \tan \theta$$

$m$  is the number of reflection from top boundary,  $n$  is the number of reflections of the left-hand boundary.

The simplest possible case is take  $n = m = X = Y = 1$  gives  $\tan \theta = 1 \implies \theta = \frac{\pi}{4}$  so you get a rectangle reflecting around the inside of the rectangle.

## 6 Lecture 7

One of the questions that came up in the background on the chat on wednesday is imagine we have this undular boundary to our domain and it is close to having critical slopes to our domain. A critical slope will lead to a massive increase in wavenumber and in steepness of the wave. If we have a near horizontal surface with near critical surfaces then we will end up filling the gaps between the undulations with denser fluid than above due to the mixing so a lot of the wave energy won't end up getting into the valleys as it will reflect off the changing stratification. To some approximation if the water can pool in it then we can ignore the fluctuations in the surface, however if the slope is at a gradient and so the water in the pools can flow away then they remain important.

Returning the immediate discussion, if we don't have an eigenmode ( $\frac{X}{Y} \frac{n}{m} \neq \tan \theta$ ) then reflections were space-filling. Remember we defined  $m = \frac{1}{2}$  number of subcritical reflections, and  $n = \frac{1}{2}$  number of supercritical reflections (reflection from a vertical rule will always be a supercritical reflection but if there is an angle it will depend on  $\theta$ )

**Trapezoidal basin:** Length of one side of 2, length of other side of 1 and width of 2. At 45 degrees, we converge to a corner to corner attractor. At 40 degrees we saw roughly diamond shaped stuff. (the trick of drawing attractors is to draw the attractor and then draw the boundary on, this method allows us to easily see the other possible domains that would give the same attractor.  $\theta = \tan^{-1} \frac{1}{2}$  gives the attractor between the other two corners. Any angles between 45 degrees and  $\theta = \tan^{-1} \frac{1}{2}$  will give nice diamond shapes, and outside this range we will get more complex shapes like figure of 8s. Eventually as we get higher and higher modes we have more different directions of waves in the same space so get lots more dissipation, we also start to see viscosity having an impact as the scale of the waves gets smaller.

Define a trapezium with bottom length  $X$  and height  $Z$  and define the bottom left to the the origin and the angle of the left hand side to be  $\alpha$  to the vertical. Define the starting point of the attractor as  $(x_0, z_0)$  and let every point of reflection thereafter be labeled  $(x_i, z_i)$ . Lets label  $a = \cot \alpha$  and  $b = \cot \theta$  to clean up algebra. First off we can see that  $z_0 = ax_0$ ,  $z_1 = Z$  and  $x_1 = \frac{Z+(b-a)x_0}{b}$ ,  $x_2 = 2$ ,  $z_2 = 2Z - bX - (b-a)x_0 \dots x_4 = \frac{2(bX-Z)+(a-b)x_0}{a+b}$ ,  $y_4 = ax_4$  simply from basic trigonometry. So we can now follow the packet of energy as it goes round once and we can repeat this as it goes around a second time until we find where it converges. Alternatively to find the attractor we just set  $x_4 = x_0$  which gives  $x_0 = X - \frac{Z}{b}$ . This looks a bit like a general iteration where you are trying to find the roots of  $f(x) = 0$ . One of these techniques is to rewrite this as  $x - g(x) = 0$  and then take  $x_n = g(x_{n-1})$  which is like what we have done here. This technique only converges under the right circumstances when  $|g'(x)| < 1$ . Here once we get close enough to the attractor this is equivalent to  $\frac{|a-b|}{|a+b|} < 1 \iff \frac{|\sin(\theta-\alpha)|}{|\sin(\theta+\alpha)|} = \frac{1}{\gamma} < 1 \iff \gamma > 1$ . Possible typo here check with

typed version.

One of the things we have seen here is that viscosity will play a role here as the wave number goes up and up as we go round and dissipation grows with the cube of the wavenumber. So we can start thinking about what the equilibrium energy spectrum looks like:

### 6.0.1 Energy spectrum for attractor

$$\gamma = \frac{\sin(\theta + \alpha)}{\sin(\theta - \alpha)}$$

Therefore with each reflection

$$k_n = \gamma k_{n-1}$$

$$\tilde{E}_n = \gamma \tilde{E}_{n-1} = \gamma^n \tilde{E}_0 = \frac{k_n}{k_0} \tilde{E}_0$$

provided there is no dissipation.

$$|\tilde{\eta}_n| \sim \frac{k_n}{k_0} |\tilde{\eta}_0|$$

The wave steepness is  $|k_n \tilde{\eta}_n| = \frac{k_n^2}{k_0} |\tilde{\eta}_n|$ .

Recall:

$$|\mathbf{u}| \sim e^{-\frac{k^2 \nu \zeta}{2N \sin \theta}, \tilde{E} \sim \Gamma}$$

$$\mathbf{u}^2 \sim e^{-k^3 \nu \zeta} N \sin \theta L - \text{length once around one circuit of the attractor}$$

$$\frac{\tilde{E}_n^{(end)}}{\tilde{E}_n^{(start)}} = e^{-\frac{k^3 \nu L}{N \sin \theta}} \tilde{E}_n^{(end)} = \frac{k_n}{k_0} e^{-\Gamma((\frac{k_n}{k_0})^3 - 1)} \tilde{E}_0 \text{ with } \Gamma = \frac{\nu k_0^3 L}{(\gamma^3 - 1) N \sin \theta}.$$

## 7 Examples Class 1

Does  $\nabla \cdot \mathbf{u} = 0$  always hold?

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x}$$

## 8 Lecture 8

### 8.1 Non-linear stratification

If we have a stratification that changes slowly compared to the wavelength, then we can treat the buoyancy frequency as being almost constant using the WKB approximation where we keep  $N$  constant locally and vary it along a ray:

$$\frac{\omega}{N} = \cos \theta \rightarrow \frac{\omega}{N(z)} = \cos \theta(z)$$



For example if:

$$N(z) = N_0 e^{-\frac{z}{H}}, \cos \theta(z) = \frac{\omega}{N_0} e^{z/H}$$

If we take some rays from a source at  $z = 0$  with  $x = X(z)$  then:

$$X(z) = \int_0^z \frac{dX}{dz} dz = \int_0^z \tan \theta dz$$

use substitution  $\cos \theta = \frac{\omega}{N_0} e^{z/H}$  to get

$$X(z) = H(\theta - \theta \tan \theta - (\theta_0 - \tan \theta_0)) = H(\cos^{-1}(e^{z/H} \cos \theta_0) - \frac{\sqrt{1 - e^{2z/H} \cos^2 \theta_0}}{e^{z/H} \cos \theta_0})$$

## 8.2 Lee waves

### 8.2.1 Extended range of hills

Imagine we have a sinusoidal topography with a wavelength of  $\lambda_T$  and amplitude  $\eta_0$  and we have a uniform wind of speed  $U$  moving past.

First we change the frame of reference so the fluid is stationary, which effectively means the mountain range is moving the opposite direction with speed  $U$ . We know  $\omega = k_T U = N \cos \theta \leq 1$  then we have waves. If  $\frac{k_T U}{N} > 1$  then we have no internal waves just forced oscillations with exponentially decaying disturbances. For  $\frac{k_T U}{N} \leq 1$  we will have internal waves.

Firstly, if these hills are making the waves then the vertical component of the waves must be upwards so  $c_{gz} > 0$ . A point of constant phase on a hill that moves to the left with speed  $-U$  so the phase velocity will also be moving to the left so  $c_{px} < 0$ . We also know that  $\mathbf{c}_g \cdot \mathbf{c}_p = 0$ . So we can combine these two facts to tell us which quadrant the group velocity must be in. So the phase velocity will point down to the left and the group velocity will point up to the left.

Now consider how this works with the frame with mountains at rest. It will look the same. We will still have lines of constant phase coming from the mountains but now the directions of the velocity will change. The change of direction of speed will change the direction of the horizontal component of the velocities. As the lines of constant phase do not move the phase velocity must be parallel to the line of constant phase, therefore the group velocity must be perpendicular to the lines of constant phase. we have transformed the velocities by adding  $U$  like:

$$\mathbf{c}_p = \mathbf{c}'_p - \mathbf{U}, \mathbf{c}_g = \mathbf{c}_g - \mathbf{U}$$

## 8.3 Kelvin ship waves

$$\phi = \mathbf{k} \cdot \mathbf{x} - \omega t$$

$$d\phi = \frac{\partial\phi}{\partial k_i} dk_i + \frac{\partial\phi}{\partial x_i} dx_i + \frac{\partial\phi}{\partial \omega} d\omega + \frac{\partial\phi}{\partial t} dt$$

Principle of stationary phase:  $d\phi = 0, d\mathbf{k} = 0, \omega = \omega(\mathbf{k}) \implies d\omega = 0$

$$\frac{\partial\phi}{\partial t} + (\mathbf{u} \cdot \nabla)\phi = 0$$

If  $\mathbf{u} = \mathbf{U}$  then  $\frac{\partial\phi}{\partial t} + (\mathbf{U} \cdot \nabla)\phi$

Kelvin ship waves: surface waves  $\mathbf{U} = (U, 0, 0)$

$$-\omega + U \frac{\partial\phi}{\partial x} = 0 = -\omega + kU = 0$$

We need  $\omega = kU$  for stationary phase:

Deep water waves  $\omega^2 = |\mathbf{k}|g$   $\mathbf{k} = (k, l, 0)$  and we also have  $|\mathbf{c}_g| = \frac{1}{2}|\mathbf{c}_p| = \frac{1}{2}\frac{\omega}{|\mathbf{k}|}$

but  $\omega = kW = U|\mathbf{k}| \cos \theta = \frac{1}{2}U \cos \theta$

Therefore for a ship moving on the surface of the water we get a angle of  $\tan \alpha = \frac{\cos \theta \sin \theta}{2 - \cos^2 \theta}$  between the point the wave has propagated from and the ship for waves transmitted with angle  $\theta$ .

## 9 Lecture 9

This is calculated by knowing that the group velocity is  $\frac{1}{2}U \cos \theta$  so the waves make a circle of radius  $\frac{1}{2}U \cos \theta$  with its centre  $UT$  behind the ship. We have a whole lot of these circles for waves produced at different times and we want to figure out the envelope that contains them all so we want to look for the maximum angle by differentiating this value to get  $\theta_{max} = \frac{1}{2} \cos^{-1} \frac{1}{3}$ .

Something very similar happens in the case of internal waves

### 9.1 Stationary phase for internal waves

Imagine you stand on top of an isolated mountain with wind going past at speed  $U$ . As before:

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right)\phi = 0$$

$$\omega = kU$$

This time  $\omega = N \cos \theta = N \frac{k}{|\mathbf{k}|}$  in 2D so we have

$$|\mathbf{k}| = \frac{N}{U}$$

for the phase to be stationary. We can also remember that we have

$$|\mathbf{c}_g| = \frac{N}{|\mathbf{k}|} \sin \theta, |\mathbf{c}_p| = \frac{N}{|\mathbf{k}|} \cos \theta$$

Won't be an exam question on 3D stationary waves.

## 9.2 Shear Flows

### 9.2.1 Sheared base state

$$\mathbf{u} = U\hat{\mathbf{x}} + \mathbf{u}'$$

Linearise about  $U\hat{\mathbf{x}}$ :

$$\begin{aligned}\frac{\partial b}{\partial t} + U \frac{\partial b}{\partial x} &= -w' N^2 \\ \frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + \frac{dU}{dz} w' &= -\frac{1}{\rho_0} \frac{\partial p'}{\partial x} \\ \nabla \cdot \mathbf{u}' &= 0\end{aligned}$$

Combining all this gives:

$$\left( \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \left( \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial z^2} \right) + N^2 \frac{\partial^2}{\partial x^2} - U'' \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \frac{\partial}{\partial x} \right) w' = 0$$

In the limit  $U = 0$ :

$$\left( \frac{\partial^2}{\partial t^2} \nabla^2 + N^2 \nabla_H^2 \right) w' = 0$$

as before.

If we choose a frame where the flow is steady with  $\frac{\partial}{\partial t} = 0$  then

$$\left( \frac{\partial^2}{\partial x^2} \nabla^2 + \left( \frac{N^2}{U^2} - \frac{U''}{U} \right) \frac{\partial^2}{\partial x^2} \right) w' = 0$$

Provided  $w'$  is bounded at infinity:

$$\left( \nabla^2 + \frac{N^2}{U^2} - \frac{U''}{U} \right) w' = 0$$

Consider the case:

$$w' = \tilde{w} e^{i(kx + mz)}$$

Consider easier case with no curvature  $U'' = 0$  then

$$\left( -(k^2 + m^2) + \frac{N^2}{U^2} \right) w' = 0$$

For non-trivial solution  $|\mathbf{k}| = \frac{N}{U}$

### 9.2.2 Critical layer reflection

Let's return to ray tracing to show how we can use this. In the simple case we have already looked at with a single mountain with a constant speed wind. We know that this gives quarter circles enclosed in semicircular causality envelopes. Lets consider a steady state so when the wind has been blowing forever. The phase velocity is along the lines of constant phase which is necessary for them

to be stationary as then you cannot see the phase move.

If we want to know how the position of one of these rays is moving with time we need to consider

$$\frac{dZ}{dx} = \frac{c'_{gz}}{c'_{gx}} = \frac{c_{gz}}{c_{gx} + U} = \frac{m}{k}$$

and as  $|\mathbf{k}|^2 = k^2 + m^2 = \frac{N^2}{U^2}$ .

We take the assumption that  $k$  is preserved as we move along a ray. However, if  $U \neq \text{const}$  or  $N \neq \text{const}$  then we must have  $m \neq \text{const}$ . Therefore, the orientation of  $\mathbf{k}$  changes and  $|\mathbf{k}|$  changes:

$$m^2 = \frac{N^2}{U^2} - k^2$$

Need to take negative square root as the phase velocity is perpendicular to the group velocity which has positive vertical motion. So

$$\frac{dZ}{dx} = \frac{m}{k} = \frac{(\frac{N^2}{U^2} - k^2)^{\frac{1}{2}}}{|\mathbf{k}|} = ((\frac{N}{kU})^2 - 1)^{\frac{1}{2}}$$

as  $k$  is also negative.

If  $\frac{N}{kU} = 1$  then the ray stops propagating vertically upwards and so cannot propagate through height at which  $\frac{N}{kU} = 1$ . Consider  $N = \text{const}$  and  $U$  increases with height. Clearly at the level of the mountain we produce waves as we can always find some  $k$  that works as  $\frac{N}{kU} > 0$ . So waves will always be produced for an isolated mountain. So the rays will slowly bend round until at a certain height  $z_c$  (where  $\frac{N}{kU} = 1$ ) they are horizontal.

## 10 Lecture 10

Near this critical height  $z_c$  we can do a Taylor series expansion to find out what happens if it isn't quite attained.

$$\frac{dZ}{dx} = ((Z - z_c) \frac{d}{dz} (\frac{N^2}{k^2 U^2}))^{\frac{1}{2}}$$

So:

$$Z = z_c + \frac{1}{4} \frac{d}{dz} (\frac{N^2}{k^2 U^2})|_{z=z_c} (x - x_c)^2$$

This will give us a quadratic behaviour where the waves rise up to  $z_c$  then they reflect back down in a parabolic curve. The lines of constant phase are perpendicular to this parabolic  $c_g$ . So the waves go from having almost horizontal phase lines and then completely vertical phase lines. We have been a bit

naughty here as we have made use of the WKB approximation but we need to ask if this is a valid approximation.

While  $U(z)$  may change over a length scale that is large compared with  $\lambda = \frac{2\pi}{|k|}$ , the vertical extent of the wave is going towards infinity as  $m \rightarrow 0$  as  $z \rightarrow z_c$ . So at the top we are saying the wave crests are vertical so at the top there must be some variation in velocity across the wave crest. Luckily whilst it is not clear that WKB is valid it works well in practice. Approximations often work well even when they have no right to do so.

Important to note that the height at which this reflection occurs depends on the wavenumber of the waves, so a spectrum of waves will reflect at differing heights.

### 10.0.1 Critical layer absorption

This is the opposite case to the above. What happens if  $\frac{U}{N}$  reduces with height and at some point reduces to zero. This time we can think about  $\frac{dz}{dx} = \frac{(\frac{N^2}{U^2} - k^2)^{\frac{1}{2}}}{|k|} = \frac{m}{k}$  so as  $U \rightarrow 0$ , we get  $m \rightarrow \infty$  and  $|k| \rightarrow \infty$ . This means that this time the ray is going to bend upwards and become vertical at  $U = 0$ , and also as we become more and more vertical the wave number increases and so the wave crests get closer and closer together (this is because  $k$  remains the same as  $m$  increases). As  $|k| \rightarrow \infty$ ,  $|c_g| = \frac{N}{|k|} \sin \theta \rightarrow 0$  and  $|c_p| = \frac{N}{|k|} \cos \theta \rightarrow 0$ . We also remember that the dissipation of the waves (the change of energy per wavelength) will be  $d\tilde{E}d\zeta = e^{-k^3\nu\zeta} N \sin \theta$  so the energy will be dissipating out very quickly which is referred to as critical layer absorption. Critical layer absorption occurs when  $U$  changes sign.

We aren't covering 3D stuff this year or columnar wave modes which were focused on in previous year. Part of the reason for this is it is more or less impossible to ask exam questions on as it is too algebraically messy.

## 10.1 Resonant triads

### 10.1.1 Weakly nonlinear internal waves

Q1 on Example Sheet 1 goes part of the way towards getting the background for this. Here we should note that the Linear wave solution satisfies the nonlinear equations but you cannot use linear superposition as addition of linear wave solutions does not solve non-linear equations.

$$\mathbf{u} = \tilde{\mathbf{u}}_1 e^{i\phi_1} + \tilde{\mathbf{u}}_2 e^{i\phi_2} + \tilde{\mathbf{u}}_3 e^{i\phi_3}$$

$$\phi_1 = \mathbf{k}_1 \cdot \mathbf{x} - \omega t, \dots$$

$$b = \tilde{b}_1 e^{i\phi_1} + \tilde{b}_2 e^{i\phi_2} + \tilde{b}_3 e^{i\phi_3}$$

Linear terms in equations, we are going to allow  $\tilde{\mathbf{u}}_i = \tilde{\mathbf{u}}_i(t)$  to be a slowly varying function allowing time dependance in phase and amplitude:

$$\frac{\partial}{\partial t}(\mathbf{u}) = (\dot{\tilde{\mathbf{u}}}_1 - i\omega_1 \tilde{\mathbf{u}}_1)e^{i\phi_1} + \dot{\tilde{\mathbf{u}}}_2 - i\omega_2 \tilde{\mathbf{u}}_2)e^{i\phi_2} + \dot{\tilde{\mathbf{u}}}_3 - i\omega_3 \tilde{\mathbf{u}}_3)e^{i\phi_3}$$

We are going to take the case with no viscosity and no mass diffusivity  $\nu = \kappa = 0$ . So in the linear equations this would end up with  $\dot{\tilde{\mathbf{u}}}$  would vanish but because we have the non-linear terms this time they might not.

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = i(\tilde{\mathbf{u}}_1 \cdot \mathbf{k}_1)\tilde{\mathbf{u}}_1 e^{2i\phi_1} + i(\tilde{\mathbf{u}}_2 \cdot \mathbf{k}_2)\tilde{\mathbf{u}}_2 e^{2i\phi_2} + i(\tilde{\mathbf{u}}_3 \cdot \mathbf{k}_3)\tilde{\mathbf{u}}_3 e^{2i\phi_3} + i((\tilde{\mathbf{u}}_1 \cdot \mathbf{k}_2)\tilde{\mathbf{u}}_2 + (\tilde{\mathbf{u}}_2 \cdot \mathbf{k}_1)\tilde{\mathbf{u}}_1)e^{i(\phi_1 \pm \phi_2)} + \dots e^{i(\phi_2 \pm \phi_1)} + \dots$$

The first self interactions just represent sort of local forcing but the cross over terms are much more interesting as they show how three different waves interact.

Triadic resonance condition to allow a sustained interaction between  $\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2, \tilde{\mathbf{u}}_3$  over sustained time and space. Order for this to happen the cross over terms must feedback into the part of the interaction not included in themselves so must have  $\pm\phi_3 = \pm\phi_1 \pm \phi_2$ . So if we put two waves into the system they will produce a third wave in addition to the forcing waves. As there are two waves you could produce a new wave we need to further consideration to discern which of the potential waves are sustained.

The two waves generated have:  $\pm\phi_3 = \pm\phi_1 \pm \phi_2$  so  $\pm\mathbf{k}_3 = \pm\mathbf{k}_1 \pm \mathbf{k}_2$  and  $\omega_3 = \pm\omega_1 \pm \omega_2$

For sustained interaction need:

$$\frac{\omega_1}{N} = \frac{k_1}{|\mathbf{k}_1|} = \cos \theta_1$$

$$\frac{\omega_2}{N} = \frac{k_2}{|\mathbf{k}_2|} = \cos \theta_2$$

$$\frac{\omega_3}{N} = \frac{k_3}{|\mathbf{k}_3|} = \cos \theta_3$$

This means if we are going to draw things on wavenumber diagram only part of the space is going to marry up.

## 11 Example sheet 1

Linerised equation of motion with non zero viscosity and mass diffusion:

$$(\frac{\partial}{\partial t} = \kappa \nabla^2)(\frac{\partial}{\partial t} - \nu \nabla^2)\nabla^2 \psi = -N^2 \frac{\partial^2 \psi}{\partial x^2}$$

which is derived by straight away combining:

$$\frac{\partial u}{\partial t} = \frac{\partial b}{\partial x} + \nu \nabla^2 u, \quad \frac{\partial b}{\partial t} = -N^2 w - \kappa \nabla^2 b$$

In general  $\frac{D}{Dt}$  and  $\frac{\partial}{\partial x}$  do not commute in general.

If it doesn't say derive the dispersion relation you can just write it down. So worth memorising some dispersion relationships like the deep water one  $\omega^2 = gh \tanh kH$ . I made mistake of thinking the dispersion relation did not change so need to practice deriving this dispersion relation.

Marks questions out of 40.

Define potential energy for deep water as:

$$\bar{P}E = PE_0 + \int_{-H}^{\eta} \rho g z dz$$

we can define reference state to be  $PE_0 = - \int_{-H}^0 \rho g z dz$  so

$$\bar{P}E = \int_0^{\eta} \rho g z dz = \frac{1}{4} \rho_0 g \eta_0^2$$

When determining the structure of a decaying wave we can take the integration constant to be zero, as we want solutions oscillating around zero?? Email the guys about this.

Won't set anything as algebraically messy as question 5 in the exam again. Try question 5 again i completely didn't get it at all!

He uses a completely different technique to me for figuring out the reflection of waves. His is a geometric approach where you consider the continuity at the boundary.

Question 8 is also bizarre, seems to be using techniques from fluids II to do with boundary layers. You are meant to assume the form  $\eta = \eta_0 e^{i\omega t} e^{-(1+i)\epsilon/\delta}$  try it again with this.

## 12 Lecture 11

We force with  $\phi_1$ ,  $\phi_2$  and nonlinearly generate  $\phi_3$ :

$$\phi_3 = \pm \phi_1 \pm \phi_2$$

$$\omega_3 = N \frac{k_3}{|\mathbf{k}_3|}$$

Triadic resonance transfer term Take  $\psi = \tilde{\psi}_1 e^{i\phi_1} + \tilde{\psi}_2 e^{i\phi_2} + \tilde{\psi}_3 e^{i\phi_3}$ . Also take  $\kappa = 0$  and  $\nu$  is small. Assume  $\tilde{\psi}_j$  are functions of  $t$ . Now let's just look at the non linear  $\mathbf{u} \cdot \nabla \mathbf{u}$  terms and relate them to the linear terms  $\frac{\partial \mathbf{u}}{\partial t}$  then we find:

$$\dot{\tilde{\psi}}_1 = I_1 \tilde{\psi}_2 \tilde{\psi}_3 - \frac{1}{2} \nu |\mathbf{k}_1|^2 \tilde{\psi}_1$$

$I_1$  is in the typeset version of the notes but he doesn't want us to memorise it just have an idea that it exists. We are going to take this body and use the euler equations for this body, this gives a mechanical analogy:

Euler equations for a rigid body:

$$J\dot{\omega} + \omega \times (J\omega) = M$$

with  $J$  is the Inertia tensor and  $M$  is the applied torques. Take

$$J = \begin{pmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{pmatrix}$$

with  $J_{11} < J_{22} < J_{33}$ .

If  $M = 0$  then conserve energy and we have:

$$\omega \cdot J\dot{\omega} + \omega \cdot (\omega \times (J\omega)) = 0$$

$$J_{11}\dot{\omega}_1 + (J_{33} - J_{22})\omega_2\omega_3 = 0$$

$$J_{22}\dot{\omega}_2 + (J_{11} - J_{33})\omega_3\omega_1 = 0$$

$$J_{33}\dot{\omega}_3 + (J_{22} - J_{11})\omega_1\omega_2 = 0$$

If  $|\omega_1| \gg |\omega_2|, |\omega_3|$  then  $J_{11}\dot{\omega}_1 = 0$  and  $J_{22}\dot{\omega}_2 + (J_{11} - J_{33})\omega_3\omega_1 = 0 \dots$   
Eliminate  $\omega_3$  to get:

$$J_{22}J_{33}\ddot{\omega}_2 - (J_{11} - J_{33})(J_{22} - J_{11})\omega_1^2\omega_2 = 0$$

There is a similar result for  $\ddot{\omega}_3$ , the RHS coefficient can be easily seen to be less than 0 so this gives harmonic oscillation so if perturbed it will oscillate slightly but be pretty stable. Similarly if  $|\omega_3| \gg |\omega_1|, |\omega_2|$  we get harmonic oscillation. However, for  $|\omega_2| \gg |\omega_1|, |\omega_3|$  we get:

$$J_{11}J_{33}\ddot{\omega}_1 - (J_{33} - J_{22})(J_{22} - J_{11})\omega_2^2\omega_1 = 0$$

so  $\omega_2$  will grow (and decay) exponentially if perturbed so this is an instability.

### 12.0.1 Triadic resonance instability

$$|\tilde{\psi}_1| \gg |\tilde{\psi}_2|, |\tilde{\psi}_3|$$

We have oscillatory behaviours in the central regions of the sustained graph of solutions, and exponentially growing solutions on the top and bottom. The TRI is a linear instability of internal waves when we consider weakly non-linear behaviour.

It would be too hard to write an exam question on this area, he tried to write



an exam question on this last year and it was too algebraically nasty to do any more than we just covered.

We want to figure out what the growth rate actually looks like for a plane wave on the exponentially growing portion of the curve. We can do this analytically to figure out:

$$\sigma_{\pm} = -\frac{1}{4}\nu(|\mathbf{k}_2|^2 + |\mathbf{k}_3|^2) \pm \sqrt{\frac{1}{16}\nu^2(|\mathbf{k}_2|^2 + |\mathbf{k}_3|^2 + I_2 I_3 |\tilde{\psi}_1|^2)}$$

we have  $\sigma_+ \geq 0$  if  $|\tilde{\psi}_1| > 0$ . We actually find that the maximum of the growth rate occurs when  $\omega_2 = -\omega_3 = \frac{1}{2}\omega_1$ . This has been known for a while and is known as the parametric subharmonic instability (PSI). In practice we don't tend to see this as in practice we are considering beams of waves rather than plane waves.

## 13 Lecture 12

### 13.1 Shallow water

There is a strong analogy between shallow water flow and flows in incompressible fluids. We can write down the rules for incompressible fluid flow:

$$\nabla \cdot \mathbf{u} = 0, \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

If we throw away a dimension we get the shallow water flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \neq 0$$

though in the shallow water flow the flow is still 3D.

In the examples sheet 1 we derived the dispersion of interfacial waves to be:

$$\omega^2 = (\rho_1 - \rho_2)gk \left( \frac{\rho_1}{\tanh kH_1} + \frac{\rho_2}{\tanh kH_2} \right)^{-1}$$

We are going to be interested in one specific limit of this dispersion relation but first let's consider a different limit. The short wave limit  $kH_1, kH_2 \gg 1$  so  $\tanh kH_i \rightarrow 1$  this gives  $\omega^2 = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} gk = Agk$ . Therefore in the limit  $A \rightarrow 1$  we recover the deep water dispersion relation  $\omega^2 = gk$ .

Introduce the 'reduced gravity'  $g' = \frac{\Delta\rho}{\rho} g = 2Ag$  so we get:

$$\omega^2 = \frac{1}{2}g'k, c_p = \sqrt{\frac{Ag}{k}} = \sqrt{\frac{g'}{2k}}, c_g = \frac{1}{2}c_p$$

We can clearly see from graphs of height against phase velocity and frequency that there are two clear limits. One for short waves (where the frequency depends strongly on the wavelength) and one for when one height is much bigger

than the other for long waves ( the long wave limit).

$kH_1, kH_2 \ll 1$  so  $\tanh kH_i \rightarrow kH_i$  so

$$c_p = c_g = \sqrt{\frac{H_1 H_2}{H_1 + H_2} g'}$$

can think of  $H_E = \frac{H_1 H_2}{H_1 + H_2}$  as an effective depth. We only need one of  $kH_1$  or  $kH_2$  to be small for waves to become non-dispersive. Let  $H_s$  be the depth of the shallow layer and  $H_d$  be the depth of the deep layer with  $\frac{H_s}{\rho_s} \ll \frac{H_d}{\rho_d}$  so we get  $c_p = c_g = \sqrt{H_s g'}$  with  $g' = \frac{\rho_1 - \rho_2}{\rho_s} g$ .

In a lot of what we are going to be doing we are going to be taking the Boussinesq limit of  $\Delta\rho \ll \bar{\rho}$  so  $g' = \frac{\Delta\rho}{\bar{\rho}} g$  therefore:

$$\omega'' = \frac{g' k}{\coth kH_1 + \coth kH_2}$$

with  $\coth kH_i \rightarrow \frac{1}{kH_i}$  for  $kH_i \ll 1$  and  $\coth kH_i \rightarrow 1$  for  $kH_i \gg 1$ . To start with mainly consider one shallow layer to make the mathematics easier.

### Equipartition of energy

$$\bar{E} = \bar{K}E + \bar{P}E, \bar{F}_E = c_g \bar{E}$$

#### 13.1.1 Shallow water equations

We have just considered linear waves but we need to think of non-linear waves to be useful so need to derive the shallow water equations.

Consider a shallow body of water with height  $h$  and horizontal scale  $X$  with  $h \ll X$ .

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\frac{1}{\rho} \nabla(p + \rho g z) + \nu \nabla^2 \mathbf{u} \end{aligned}$$

for homogenous fluid. Take scales  $U, W, X, Z, P, T$  with:

$$\frac{U}{X} + \frac{W}{Z} = 0$$

in shallow water  $\frac{X}{Z} \rightarrow \infty$  and so  $\frac{W}{U} \rightarrow 0$

$$\frac{W}{T} + \frac{UW}{X} + \frac{W^2}{Z} = \frac{1}{\rho} \frac{P}{Z} + g + \nu \left( \frac{W}{X^2} \frac{W}{Z^2} \right)$$

as the vertical pressure gradient is predominantly hydrostatic so the other terms become small and  $\frac{1}{\rho} \frac{P}{Z} \sim g$  so:

$$\frac{\partial p}{\partial x} \approx -\rho g z = p_0 - \rho g z$$

and

$$\frac{\partial w}{\partial z} \ll g$$

now look at horizontal momentum equation:

$$\frac{U}{T} + \frac{U^2}{X} + \frac{WU}{Z} = \frac{\rho g Z}{\rho X} + \nu \left( \frac{U}{X^2} + \frac{U}{Z^2} \right)$$

can neglect the viscosity term as we are dealing with high reynolds number. Physically insight allows us to neglect the  $\frac{WU}{Z}$  term as we can see that  $W$  should be small as it is a shallow flow so what would be generating a large velocity up or down. this gives:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial}{\partial x} (p + \rho g z) + \nu \frac{\partial^2 u}{\partial z^2}$$

### From first principles:

We now cover how to do this from first principles as often we will be asked to derive the shallow water equations for bizarre channels like triangular tubes or something and this gives us an insight into how to do that.

Take a flat bottom, unit width, slowly varying depth of water with  $u(x, t)$  under the hydrostatic limit and  $Re \gg 1$ .

The volume flux is given by  $Q = uh$  so  $2\delta x \frac{\delta h}{\delta t} = Q_{x-\delta x} - Q_{x+\delta x}$ . So as  $\delta t \rightarrow 0$  we get:

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\ \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} &= 0 \end{aligned}$$

If cross-section of channel is not rectangular, then this expression changes. Here the cross-sectional area is just  $h$  times unit width but if say a triangular channel then the cross-sectional area would be scaling with  $h^2$  instead.

We can also consider the momentum flux and the pressure forces. So the pressure force towards the right on the right hand face

$$\int_0^{h - \frac{\partial h}{\partial x} \delta x} \rho g \left( h - \frac{\partial h}{\partial x} \delta x - z \right) dz = \rho \left( \frac{h^2}{2} - h \frac{\partial h}{\partial x} \delta x \right) + O(\delta x^2)$$

we also get:

$$\frac{\partial}{\partial t} (uh) + \frac{\partial}{\partial x} \left( u^2 h + \frac{1}{2} g h^2 \right) = 0$$

with  $F_M = u(uh) = uM$  for 1D shallow water. Note the similarities with

$$\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2 + p) = 0$$

which is 1 D compressible. So you can see that  $h$  and  $\rho$  play similar roles.

In Example sheet 2, we vary  $b(x, z)$  as the width of the channel. This is still a 1D shallow water flow provided  $\frac{u^2}{b} \frac{d^2 b}{dx^2} \ll g$

**Boussinesq vs non-Boussinesq:**

$$g' = \frac{\rho_1 - \rho_2}{\frac{1}{2}(\rho_1 + \rho_2)} = \frac{\Delta\rho}{\bar{\rho}} g$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g' \frac{\partial h}{\partial x} = 0$$

For most things a single-layer shallow water flow ( Boussinesq flow) and a single-layer free surface flow, the equations are the same except for  $g' < - > g$ . This will not be the case where the shallow water assumptions are violated - we will look at some of these.

## 14 Lecture 13

One instance of this not working is if the lower layer is deep as then we could plausibly have much lower velocities in the lower layer than the upper layer. If we were to take the pressure being lower in the lower layer then we would end up having higher pressure where the upper layer is larger which would force the pressure to be funneled in the direction of the narrowest upper layer. If the upper layer has a free surface or a surface with say air then it can rise up slightly allowing the pressure to be uniform along the boundary with the deep layer below.

**Averaging**

$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$ ,  $\mathbf{u} = \mathbf{u}(x, y, z, t)$  with  $\bar{\mathbf{u}} = \bar{\mathbf{u}}(x, t)$  and  $h = h(x, t)$  with:

$$\bar{\mathbf{u}} = \frac{1}{A} \int_{-b/2}^{b/2} \int_0^h \mathbf{u} dz dy$$

with  $A = bh = A(x, t)$ . If we take the divergence equation:

$$\int_{-b/2}^{b/2} \nabla \cdot \mathbf{u} dz dy = 0$$

Consider:

$$\int_{-b/2}^{b/2} \frac{\partial w}{\partial z} dz dy = \int_{-b/2}^{b/2} w|_{z=h} - w|_{z=0} dy$$

using the kinematic boundary condition

$$w(z = h) = \frac{\partial h}{\partial t} + u|_{z=h} \frac{\partial h}{\partial x}$$

gives

$$b \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} \int_{-b/2}^{b/2} u|_{z=h} dy = 0$$

similar result holds for  $\frac{\partial v}{\partial y}$  so these together gives:

$$\int_{-b/2}^{b/2} \nabla \cdot \mathbf{u} dz dy = b h \frac{\partial \bar{u}}{\partial x} + \bar{u} (b \frac{\partial h}{\partial x} + h \frac{db}{dx}) + b \frac{\partial h}{\partial x} = 0$$

If we do the same for the momentum equation:

$$\int \int_A \frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla) u dA$$

which gives:

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} - \nu \frac{\partial^2 \bar{u}}{\partial x^2} = \nu \left[ \frac{1}{b} \left( \frac{\partial u'}{\partial y} \Big|_{y=\frac{b}{2}} - \frac{\partial u'}{\partial y} \Big|_{y=-\frac{b}{2}} \right) + \frac{1}{h} \left( \frac{\partial u'}{\partial z} \Big|_{z=h} - \frac{\partial u'}{\partial z} \Big|_{z=0} \right) \right] - (u' \frac{\partial \bar{u}'}{\partial x} + v' \frac{\partial \bar{u}'}{\partial y} + w' \frac{\partial \bar{u}'}{\partial z})$$

The equations for  $\bar{u}$  require a knowledge of the equations for  $u'^2$ . This is generally referred to as the closure problem.

At high Re, then  $\nu$  may be negligible but  $u'_i \bar{u}'_j$  may not be.

We can try to model the unknown terms on the RHS by considering the scaling of the different terms with  $\frac{\nu}{h} \frac{\partial u'}{\partial z} \sim \frac{\nu \bar{u}}{h^2}$  and  $\frac{\nu}{b} \frac{\partial u'}{\partial y} \sim \frac{2\nu \bar{u}}{b^2}$ . So could model the RHS as:

$$\nu \left[ \frac{1}{b} \left( \frac{\partial u'}{\partial y} \Big|_{y=\frac{b}{2}} - \frac{\partial u'}{\partial y} \Big|_{y=-\frac{b}{2}} \right) + \frac{1}{h} \left( \frac{\partial u'}{\partial z} \Big|_{z=h} - \frac{\partial u'}{\partial z} \Big|_{z=0} \right) \right] = C_L \nu \frac{\bar{u}}{h^2} (1 + 2 \frac{h^2}{b^2})$$

where  $C_L$  is an O(1) laminar drag coefficient. In the slow viscous flow course we could find that  $C_L = \frac{2}{3}$  for  $Re \ll 1$  and  $b \gg h$ . However for high Re,  $\nu$  is negligible and we can think about the other terms:

$$-(u' \frac{\partial \bar{u}'}{\partial x} + v' \frac{\partial \bar{u}'}{\partial y} + w' \frac{\partial \bar{u}'}{\partial z}) \approx C_t \frac{\bar{u} |\bar{u}|}{h} (1 + 2 \frac{h}{b})$$

The reason for this form instead of just  $\bar{u}^2$  is we want it to retrain the flow so we need to retain the sign. The simplest way we can model this is with  $C_T$  is an O(1) turbulent drag coefficient that dependson roughness of boundaries. For smooth boundaries it can be small  $C_T \approx 0.03$  for rough boundaries it is roughly 0.1. Often  $h \ll b$  so  $C_t \frac{\bar{u} |\bar{u}|}{h}$ . Dropping overbars.

$$\frac{\partial h}{\partial t} + b^{-1} \frac{\partial}{\partial x} (b h u) = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = -C_L \nu \frac{u}{h^2} - C_T \frac{u |u|}{h}$$

Do Q13 and 17 on ES2.

## 14.1 Hyperbolic system

These equations support wavelike solutions of the form:

$$F_i(x - c_i t)$$

where  $c_i$  is the wave speed and  $F(\cdot)$  expresses the evolution following a wave. The equations we are looking at are quasi linear.

**Recap linearity:**

Linear equations:  $a(x, t) \frac{\partial u}{\partial x} + b(x, t) \frac{\partial u}{\partial t} = c(x, t)u + d(x, t)$

Semi-linear equations:  $a(x, t) \frac{\partial u}{\partial x} + b(x, t) \frac{\partial u}{\partial t} = c(x, t, u)$

Quasi-linear equations:  $a(x, t, u) \frac{\partial u}{\partial x} + b(x, t, u) \frac{\partial u}{\partial t} = c(x, t, u)$

### 14.1.1 A model for traffic flow

We have a road with speed depending only on the local traffic density  $u = u(\rho)$ . So can easily write down continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(u\rho) = 0$$

Now lets just imagine we have got helicopters flying over the traffic proving traffic reports with some speed  $\lambda$  so:  $x = x_0 + \lambda s$ ,  $t = t_0 + s$ . The helicopter will see:

$$\frac{df}{ds} = \frac{dt}{ds} \frac{\partial f}{\partial t} + \frac{dx}{ds} \frac{\partial f}{\partial x}$$

as  $\frac{d}{ds} = \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x}$  and so  $\frac{d\rho}{ds} = \frac{\partial \rho}{\partial t} + \lambda \frac{\partial \rho}{\partial x}$ .

Imagine  $u(\rho) = 1 - \rho$  then we will get a quadratic flux. Also our continuity equation would give us:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}((1 - \rho)\rho) = 0 = \frac{\partial \rho}{\partial t} + (1 - 2\rho) \frac{\partial \rho}{\partial x} = 0$$

If the pilot flies at speed  $\lambda = 1 - 2\rho$  then  $\frac{d\rho}{ds} = 0$  and so  $\rho = \text{const}$ . So  $\lambda$  is the characteristic speed of the problem. We have reduced the pde to ode by choosing this moving view point. When characteristics intersect we get a shock. These two different traffic densities are producing the same flux as we still must have continuity so we have:

$$\rho_l(u_l - u_s) = \rho_r(u_r - u_s)$$

so in this particular case of constant traffic densities on each side of the shock we get:

$$u_s = 1 - (\rho_r + \rho_l) = \frac{1}{2}(\lambda_r + \lambda_l)$$

## 15 Lecture 14

### 15.1 Shallow water characteristic treatment

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = 0$$

$$x = x_0 + \lambda s, t = t_0 + s$$

with stopwatch time  $s$

$$x = x_0 + \zeta, t = t_0 + \zeta$$

with tape-measure distance  $\zeta$ . As we have  $\frac{\partial}{\partial t} = \frac{d}{ds} - \lambda \frac{\partial}{\partial x}$  so our equations transform to:

$$\frac{dh}{ds} + (u - \lambda) \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0$$

$$\frac{du}{ds} + (u - \lambda) \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = 0$$

multiply the first by  $u - \lambda$  and the second by  $h$  and add them together to get:

$$(u - \lambda) \frac{dh}{ds} - h \frac{du}{ds} + ((u - \lambda)^2 - gh) \frac{\partial h}{\partial x} = 0$$

To make into ODE set  $(u - \lambda)^2 - gh = 0$  which occurs when  $u \pm \sqrt{gh} = u \pm c$  as  $\sqrt{gh} = c_p = c_g = c$  for long waves on stationary fluid.

Now lets do the same manipulations a bit differently using matrix formulism:

$$\begin{pmatrix} u & h \\ g & u \end{pmatrix} \begin{pmatrix} h \\ u \end{pmatrix}_x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h \\ u \end{pmatrix}_t = 0$$

$$A \mathbf{v}_x + B \mathbf{v}_t = \mathbf{f}$$

Using transformation  $\frac{\partial}{\partial t} = \frac{d}{ds} - \lambda \frac{\partial}{\partial x}$ :

$$\begin{pmatrix} h \\ u \end{pmatrix}_s + \begin{pmatrix} u & h \\ g & u \end{pmatrix} \begin{pmatrix} h \\ u \end{pmatrix}_x = 0$$

$x - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h \\ u \end{pmatrix}_x = 0$  We then premultiply by  $\mathbf{q}^T$  and we want to chose  $\mathbf{q}$  so it eliminates the RHS quantity

$$\mathbf{q}^T (A - \lambda B) = \mathbf{0}^T$$

Therefore we can think about the **general approach**. In general we have:

$$A \mathbf{v}_x + B \mathbf{v}_t = \mathbf{f}$$

These are quasi linear as  $A = A(x, t, \mathbf{v})$ ,  $B = B(x, t, \mathbf{v})$ ,  $f = f(x, t, \mathbf{v})$ . Consider a linear combination of equations:

$$\mathbf{q}^T A \mathbf{v}_x + \mathbf{q}^T B \mathbf{v}_t = \mathbf{q}^T \mathbf{f}$$

The observer  $\mathbf{v}_s = \lambda \mathbf{v}_x + \mathbf{v}_t$ . Take a linear combination of these as well:

$$\mathbf{m}^T \mathbf{v}_s = \mathbf{m}^T (\lambda \mathbf{v}_x + \mathbf{v}_t) = ?$$

Compare these two equations it would make sense to equate the following:

$$\lambda \mathbf{m}^T = \mathbf{q}^T A, \mathbf{m}^T = \mathbf{q}^T B$$

Then we can eliminate  $\mathbf{m}$  from these coupled equations to give;

$$\mathbf{q}^T A = \mathbf{q}^T B \iff \mathbf{q}^T (A - B) = \mathbf{0}^T$$

This is our generalised left-hand eigenvalue problem. For a non-trivial solution we need:

$$|A - \lambda B| = 0$$

This is how we chose  $\lambda$ . We can now go back in the other direction as we also know that :

$$\mathbf{m}^T \mathbf{v}_s = \mathbf{q}^T \mathbf{f}$$

and

$$\mathbf{m}^T = \frac{1}{\lambda} \mathbf{q}^T A = \mathbf{q}^T B$$

so

$$\mathbf{q}^T (A \mathbf{v}_s - \lambda \mathbf{f}) = 0, \mathbf{q}^T (B \mathbf{v}_s - \mathbf{f}) = 0$$

If we very simply sketch what was happening to our simple shallow water, our eigenvalue problem is:

$$\mathbf{q}^T \begin{pmatrix} u - \lambda & h \\ g & u - \end{pmatrix} = \mathbf{0}^T$$

So need

$$|A - B| = (u - \lambda)^2 - gh = 0 \implies \lambda = u \pm c, c = \sqrt{gh}$$

We also have:

$$\begin{pmatrix} 1 & q \end{pmatrix} \begin{pmatrix} \pm c & h \\ g & \pm c \end{pmatrix} = \mathbf{0}^T$$

so  $\mathbf{q}^T = (1 \quad \pm \frac{c}{g})$

$$\mathbf{q}^T (A \mathbf{u}_s - \lambda \mathbf{f}) = (1 \pm \frac{c}{g})(A \mathbf{u}_s - \lambda \mathbf{0}) = 0$$

Use  $h = \frac{c^2}{g}$  and solving this gives:

$$0 = \begin{cases} (u + c) \frac{d}{ds}(2c + u) & \lambda = u + c \\ (u - c) \frac{d}{ds}(2c - u) & \lambda = u - c \end{cases}$$



$$u \pm 2c = \text{const on } \lambda = u \pm c$$

**Higher-order equations:**

$$\frac{\partial^2 \eta}{\partial t^2} + \gamma \frac{\partial^2 \eta}{\partial x^2} = 0$$

Let  $v = \frac{\partial \eta}{\partial x}$  and  $w = \frac{\partial \eta}{\partial t}$  then:

$$\frac{\partial v}{\partial t} - \frac{\partial w}{\partial x} = 0$$

$$\frac{\partial w}{\partial t} - \gamma \frac{\partial v}{\partial x} = 0$$

for  $\lambda = \pm \gamma^{\frac{1}{2}}$ .

### 15.1.1 Implications of being hyperbolic

What makes things hyperbolic? Well we required that  $|A - \lambda B| = 0$  and it is hopefully obvious that if  $\lambda$  is representing a trajectory in space time then we need real eigenvalues.

$$\mathbf{q}^T (B\mathbf{u}_s - \mathbf{f}) = 0$$

If we want to form the ODE along these characteristics we need to know the eigenvectors so we can form two ODEs. So we need a complete set of eigenvectors. There will be some occasions with repeated eigenvalues but still have a full set of eigenvectors. If we cannot find a full set of eigenvalues the problem is not hyperbolic. The eigen values (velocity) are telling us how the information is propagating. There is an analogy between shallow water and compressible gas: For compressible gas we have the Mach number:

$$M = \frac{\text{velocity}}{\text{speed of sound}}$$

In the case of shallow water the equivalent is the Froude number :

$$F_r = \frac{\text{fluid velocity}}{\text{wave speed}} = \frac{u}{c} = \frac{u}{\sqrt{gh}}$$

We are considering if the information can propagate in two directions or just one direction. If the fluid is moving fast enough then the waves will only every propagate in the direction of the flow.

If  $|F_r| < 1$  then waves can propagate in both directions (subcritical flow)

If  $|F_r| > 1$  the waves can only propagate in one direction ( supercritical flow)

If  $|F_r| = 1$  then flow is critical so either  $\lambda^+ = u + c$  or  $\lambda_- = u - c$  are zero. One ( or both) of the the waves do not propagate relative to the observer.

Imagine we know the initial speed and depth at two points then we can follow

the characteristics leaving these points and meeting at a point. We can find the new speed and depth by considering the conservation along these characteristics :

$$u_1 + 2c_1 = u_a + 2c_a, u_1 - 2c_1 = u_b - 2c_b$$

so

$$u_1 = \frac{1}{2}(u_a + u_b) + (c_a - c_b)$$

$$c_1 = \frac{1}{4}(u_a - u_b) + \frac{1}{2}(c_a + c_b)$$

Zone of dependance for  $u_1, c_1$  is everything between  $u_a, c_a$  and  $u_b, c_b$ . The zone of influence is future times between  $\lambda_+ = u + c$  and  $\lambda_- = u - c$  characteristics.

### 15.1.2 The Saint-Venant dam-break problem

This is an example of the Riemann problem. If we have a dam of depth  $H$  and length  $L$  if we move one of the walls away at an accelerating rate  $V$ , at what point does the water not keep up with the dam. On  $C^+$  we have  $\lambda_+ = u + c$  and at initial time  $u + 2c = 2c'$ . So as  $c = \sqrt{gH} > 0$  so the maximum value that  $u$  can have is  $2c'$  when we have  $c = 0$  and therefore  $h = 0$ . So when  $V = 2c'$  the water will no longer be able to keep up. Everywhere before that point the  $C^+$  characteristics will be able to reach the dam, so we will be able to figure out the depth of the water at the dam by using  $u = V$  and  $u + c = 2c'$ . Therefore, we can figure out the value of  $c$  at every point on the dam, and therefore as  $u + 2c$  is constant everywhere and  $u - 2c$  is constant along  $C^-$  we have  $u$  and  $c$  are constant along  $C^-$  so the line is straight. It is nice to think of  $C^-$  as  $C^+$  characteristics reflected off the dam.

If we shrink this accelerating movement down to nothing to model an instantaneous removal of the dam. Then the information about the removal of the dam will move backwards along characteristic  $\lambda_- = -c'$ . We will have squeezed every point on the expansion fan to originate from the same point and as they are all straight this means the characteristics are  $\lambda_- = \frac{x}{t}$ . We can start to build up the solution as along  $\lambda_- = u - c$  we have  $u - 2c = -2c'$  in the quiescent region and along  $\lambda_+ = u + c$  we have  $u + 2c = 2c'$  everywhere in the fluid. The fastest moving fluid is at the front with  $u = 2c'$  from limit  $h \rightarrow 0, c \rightarrow 0$  so on  $\lambda_+ = u + 2c = 2c'$  so  $u \rightarrow 2c'$ . In the expansion fan  $\lambda_- = \frac{x}{t}$  along which  $u - c = \lambda_- = \frac{x}{t}$ . So as  $u - c = \frac{x}{t}$  and  $u + 2c = 2c'$  we have  $3c = 2c' - \frac{x}{t} \implies h = \frac{1}{9g}(2c' - \frac{x}{t})^2$ .

But do we see this for shallow water. We predicted its maximum velocity at the thinnest point. We have also violated shallow water. At  $t = 0$  we have violated the assumption that  $h$  and  $u$  change only over length scales large compared to  $H$ . We have also ignored the density of the fluid above and ignored viscosity.

What would be the effect of drag on the boundary. We have been assuming that  $u \rightarrow 2c'$  as  $h \rightarrow 0$  with the st. venant solution. In any real solution this is

going to cause a problem, as soon as this causes a problem we are going to break shallowater as shallow ater requires tat the front goes smoothly to zero and  $w$  is asymptotically small. If we actually have the velocity going to zero because of any drag what so ever at the front we will end up with a stagnation point forming at the front. This means a discountuity in slope must be forming at the front then this requires vertical acceleration of the fluid and this violates the shallow ater assumptions. If we did an expirment of water into air you would find that you don't get an infintely thin film travelling on, though it is a thin front due to the large density difference.

### Moving dam problem

Lets have the speed of the dam being less than  $2c'$ . Now in this case when the characteristics make it to the dam they are still going to have a non-zero height so the reflected ones can propogate backwards relative to the dam, and as everythign is propogating at constant speed, then all of these will be reflected with the same speed and same depth. So we get three areas the undisturbed fluid, the rarefaction wave travelling back and a region of constant depth at the front.

Because of the constant depth region the Froude number of rthe dam is constant  $Fr_f = \frac{u_f}{c_f}$ . This can be used to characterise the problem.

What if we were using a finite volume of fluid rather than the fluid behind the dame extending forever? Then you have the reflected rarefaction wave coming back towards the dam.

## 15.2 Gravity Currents

Brief description: They have some common features, they have a well defined front with large density gradients and there is often mixing behind the front at  $Re \gg 1$ . The no-slip boundary condition causes the ambient fluid to be over run. The no-slip condition means the fluid has to roll over the ground so it ends up over running some lighter fluid which sets up a static instability. This means it ends up convecting and forming 'lobes and clefts'. So if we look from the front we start seeing wave like lobes and clefts. This is caused by the lighter fluid trying to rise up through the front. This part of the current is often refered to as the head, and in this region shallow water odes not strictly apply but it does work well in the 'tail'.

Frontogenesis - the formation of the front. Let's consider what would happen if we had a fluid that started dense on the left and as we moved to the right it gets less dense iwth  $\frac{\partial \rho}{\partial x}|_{t=0} \neq 0$ . If we make our lives simple and make it 2D then in the Bousseqs case the vroticity equation gives:

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla) \omega = - \frac{g}{\rho_0} \frac{\partial \rho}{\partial x}$$

## 16 Example sheet 12

Can just state the vorticity equation in 2D and bousinesq case:

$$\frac{\partial \zeta}{\partial t} + (\mathbf{u} \cdot \nabla) \zeta = -\frac{g}{\rho_0} \frac{\partial \rho}{\partial x} + \nu \nabla^2 \zeta$$

with  $\zeta = -\nabla^2 \psi$ . If we wasn't bousinesq we have to worry about how we are doing the derivatives as we are taking the curl of  $\rho \frac{\partial \mathbf{u}}{\partial t}$  and we will also get a  $\nabla \rho \times \nabla p$ . If it was 3D we would also have a  $(\zeta \cdot \nabla) \mathbf{u}$  term. It is worth actually doing this computation as it looks quite straight forward but unless you remember the write order it turns out to be a bit more of a can of worms.

$$KE = \int_{-H}^{\eta} \frac{1}{2} \rho (u^2 + v^2 + w^2)$$

$$PE = \int_{-H}^{\eta} \rho g z dz - \int_{-H}^0 \rho g z dz$$

$$\mathbf{F} = \int_{-H}^{\eta} u(p + \rho g z) dz$$

but what we really want is  $KE = \int_{-H}^0 \frac{1}{2} \rho (u^2 + v^2 + w^2) dz$  as the  $u, v, w$  already have  $\eta$  baked in so the contribution from the bit at the top is another order of  $\eta$  further down. No need to remember the results of these integrals as the book work parts of the question will almost always end in book work to make his marking easier.

$$(2 \frac{\partial}{\partial \chi} + i \frac{\partial^2}{\partial \xi^2}) \psi_0 = f(\chi)$$

The reason we can discard the  $f(\chi)$  with  $\zeta, \chi$  in direction of group velocity and  $\xi$  in the perpendicular direction. Is because we assume a plane solution in the  $\zeta$  direction and so the integrating up from  $(2 \frac{\partial}{\partial \chi} + i \frac{\partial^2}{\partial \xi^2}) \frac{\partial \psi_0}{\partial \xi} = 0$  would average to zero.

For reflection questions draw a diagram with two incoming waves, and draw on all the wavevectors, wavelengths and group velocity. Then resolve velocities so you have zero normal velocity at the surface. Express the distance between the points of impact in terms of both wavelengths to get the relationship between the two of them. This is some standard exam question book work so make sure you can do this. Can quote:

$$|\mathbf{F}| = \bar{E} |\mathbf{c}_g|$$

We also know that the flux of energy is conserved per wavelength.

When thinking about attractors we can often simplify the algebra by noticing a symmetry, also standard practice to define  $\beta = \frac{1}{\tan \theta}$  and therefore the line through a point has equation:  $z = z_0 + \beta(x - x_0)$ . In order to figure out the magnification factor either you figure out the slopes at the positions of the reflections and then use the  $\gamma$  expressions or you could use the iterative formula for getting back to the same position.

Make sure to always draw on wavevector to diagrams remembering it is perpendicular to the group velocity, and it is often tidier to write  $\mathbf{k} = |\mathbf{k}|(\cos \theta, -\sin \theta)$  than  $\mathbf{k} = (k, m)$ .

Refamiliarise with stokes boundary layer condition from Fluids II. Remember form of solution.

When finding expressions for  $|\mathbf{k}|$  to find envelopes remember that we need  $|\mathbf{k}|$  so there are some values of  $\theta$  that aren't possible. Equally remember to think about the direction of phase velocity of stationary waves, it must be in the same direction as the observer in order for them to exist.

## 17 Lecture 16

### 17.1 Early models for gravity currents

The starting point for a lot of this is dimensional analysis. At high Reynolds number we will have  $u_f^2 \sim gH$  and as we found with the St. Venant solution we found that  $u_f^2 = 2gH$  but this requires  $h \rightarrow 0$  smoothly. Generally we find this is not the case as at the boundary we have factors like drag or surface tension, so the region close to the boundary will be non-hydrostatic as we need to have a vertical velocity so we violate shallow water at the front (doesn't mean it can't be applied elsewhere). Experimentally it has been found that  $F_r = \frac{u_f}{\sqrt{gh_f}} = f\left(\frac{h}{H}\right)$ .

### 17.2 Cavity Flows

First understood in 1960s by Benjamin. He said let's imagine we have a duct with a height  $H$  and high Reynolds number. We assume that out at infinity we have velocity zero and we assume we have a cavity extending into the duct with height  $h$  and density of 0 and moving left with velocity  $U$ . Therefore, the volume flux of the fluid must be equal to the volume displaced by the cavity in the opposite direction. It is easier to think about this in frame of reference of the front of the cavity. We pick a control volume with boundaries far enough from the front in either direction that we can assume the velocity is parallel to the duct. By continuity of volume flux we must have  $u_2 = \frac{UH}{H-h}$  with  $u_2$  being the speed under the cavity. Let us assume it is inviscid, irrotational and energy

conserving and steady. This means we can use Bernoulli:

$$\frac{1}{2}|\mathbf{u}|^2 + \rho gz + p = \text{const}$$

consider pressure at the top of the fluid before the cavity and under the cavity on the same streamline.

$$p_1 = p_0 - \frac{1}{2}\rho U^2$$

$$p_2 = p_0 = p_0 + \rho gh - \frac{1}{2}\rho u_2^2$$

The first of these is the pressure of the cavity and the second is given by the Bernoulli constant

$$u_2^2 = 2gh$$

"Flow force" on the upstream side is:

$$\int_0^H p dz + \rho U^2 H = p_0 H + \frac{1}{2}\rho U^2 H + \frac{1}{2}\rho g H^2$$

"Flow force" on the downstream is:

$$2\rho gh(H - h) + p_0 H + \frac{1}{2}\rho g(H - h)^2$$

So these balancing gives:  $h^3 = \frac{1}{2}Hh^2$ . This clearly gives two energy-conserving solutions  $h = 0$  and  $h = \frac{1}{2}H$ .

If we allow some turbulence to develop then we can get an adverse pressure gradient at the back of the head. We want to draw our control volume far enough downstream of these turbulence that all the flow is parallel again. Now the height is  $h - \Delta$  which is effectively the loss of pressure caused by the energy loss. This adverse pressure gradient often leads to instabilities, potentially to rotational flow and dissipation of energy. Provided we make the box long enough we can ignore all that and just think of the head loss (loss of height of the cavity). Now

$$u_2^2 = 2g(h - \Delta)$$

This is our energy loss represented as a reduction in hydrostatic head. Unless there is an internal source of energy we must have  $\Delta \geq 0$ . Now work through the same maths as before to find:

$$\Delta = \frac{(H - 2h)h^2}{2(H - h)(H + h)}$$

Note  $h \rightarrow \frac{1}{2}H \implies \Delta \rightarrow 0$  and as  $\frac{h}{H} \rightarrow 0 \implies \Delta \rightarrow \frac{h}{2H} \rightarrow 0$

We can write the Frude number as:

$$F_f = \frac{u_f}{\sqrt{gh_f}} = \left( \frac{(1 - \frac{h}{H})(2 - \frac{h}{H})}{1 + \frac{h}{H}} \right)^{\frac{1}{2}}$$

### 17.3 Bonden and Meiburg (2013)

They said we don't need to worry about a lot of these things (e.g. things being hydrostatic) we just need to take a control volume and a current. In the current we have density  $\rho_0 + \rho'$  and in the ambient fluid we have density  $\rho_0$ . We have height  $h$  of our cavity and then a boundary layer of height  $\delta$  and then we will let  $\delta$  shrink to zero. We want to think about the circulation

$$\Gamma = \int_V \omega dV = \int_S \mathbf{u} \cdot d\mathbf{x}$$

This time we have a Baroclinic torque associated with the density difference across the boundary:

$$G_\omega = - \int_V \frac{g}{\rho_0} \frac{\partial \rho'}{\partial x} dV = g' h$$

The flux of vorticity:

$$F_\omega = \int_{h-\frac{\delta}{2}}^{h+\frac{\delta}{2}} \omega u dz$$

we have  $\omega \sim -\frac{u_2}{\delta}$  and  $\bar{u} \sim \frac{1}{2}u_2$  so therefore:

$$F_\omega = -\frac{1}{2}u_2^2 \text{as } \delta \rightarrow 0$$

We have continuity  $u_1 H = u_2 (H - h)$  which leads to a different Froude number:

$$F_H = \frac{u_f}{\sqrt{g'H}} = \left(2\frac{h}{H}\right)^{\frac{1}{2}} \left(1 - \frac{h}{H}\right)$$

$$F_f = \frac{u_f}{\sqrt{g'h}} = 2^{\frac{1}{2}} \left(1 - \frac{h}{H}\right)$$

In exam questions we can get a long way by saying the Froude number is some function of the final depth and this works really really well for predicting the overall behaviour of the gravity current.

### 17.4 Simplified models of gravity currents

#### 17.4.1 Integral model

This is simply a series of rectangles with conserved volumes. Sometimes we go a step further and pick another shape that more accurately approximates the shape. We will be using this plus a boundary condition on the front based on a constant Froude number.

So we will start off with a region  $L(t)h(t) = L_0 h_0$ . If we now apply a Froude number for the front:

$$u_f = F_f \sqrt{g'h} = \dot{L}$$

then

$$\dot{L} = F_f \sqrt{g' \frac{h_0 L_0}{L}} \sim t^{\frac{2}{3}}$$

Can do similar calculation for channel with width  $b(x)$ .

We will have the front moving linearly with time  $L \sim t$  until the initial rarefaction catches back up with the front after reflecting off the back wall when it will move with  $L \sim t^{\frac{2}{3}}$

#### 17.4.2 Entrainment into shallow water

What is all this mixing doing and how is it changing the shallow water equations. At high Re it is often turbulent which may be from the roughness of the lower boundary and/or shear instability at the interface.

Let us imagine we have a shallow water layer, a control volume and densities  $\rho_0$  and  $\rho_1$  and some process that is causing mixing across the interface. This process is entraining fluid with  $\omega_e$  into the lower layer and detraining fluid with  $\omega_d$  into the upper layer. We will assume the mixing within layers keeps density constant over the depth of the layer so  $\frac{\partial \rho}{\partial z} = 0$  and  $\rho = \rho(x)$  within a layer. We will assume the upper layer is very deep so  $\rho_2 = \text{const.}$

Momentum equation with no drag:

$$\frac{\partial}{\partial t}(\rho_1 h_1 u_1) + \frac{\partial}{\partial x}(u_1 \rho_1 h_1 u_1 + \frac{1}{2}(\rho_1 - \rho_2) g h_1^2) = \rho_2 u_2 - \omega_d \rho_1 u_1$$

We can ignore  $\rho_2 u_2$  as  $u_2 = 0$ .

Volume conservation gives:

$$\frac{\partial h_1}{\partial t} + \frac{\partial}{\partial x}(u_1 h_1) = \omega_e - \omega_d$$

If we assume 'linear mixing'

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Note the following

$$\begin{aligned} \frac{\partial}{\partial t}(\rho_1 h) &= \rho_1 \frac{\partial h}{\partial t} + h \frac{\partial \rho_1}{\partial t} \\ \frac{\partial}{\partial t}(\rho_1 h_1 u_1) &= \rho_1 u_1 \frac{\partial h}{\partial t} + \rho_1 h_1 \frac{\partial u_1}{\partial t} + h_1 u_1 \frac{\partial \rho_1}{\partial t} \end{aligned}$$

define:

$$B = \begin{pmatrix} h & \rho_1 & 0 \\ 0 & 1 & 0 \\ h_1 u_1 & \rho_1 u_1 & \rho_1 h_1 \end{pmatrix}$$



$$\mathbf{v} = (\rho_1 h_1 u_1)$$

so we can simplify:

$$\begin{aligned} \frac{\partial h_1}{\partial t} + u_1 \frac{\partial h_1}{\partial x} + h_1 \frac{\partial u_1}{\partial x} &= \omega_e - \omega_d \\ g' &= g'(x, t) = \frac{\rho_1 - \rho_2}{\bar{\rho}} g \\ \frac{\partial g'}{\partial t} + u_1 \frac{\partial g'}{\partial x} &= -g' \frac{\omega_e}{h_1} \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g' \frac{\partial h}{\partial x} + \frac{1}{2} h \frac{\partial g'}{\partial x} &= -\omega_e \frac{u_1 - u_2}{h} \\ \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_1 & h_1 \\ \frac{1}{2} h_1 & g' & u_1 \end{pmatrix} \begin{pmatrix} g' \\ h_1 \\ u_1 \end{pmatrix}_x + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} g' \\ h_1 \\ u_1 \end{pmatrix}_t &= \begin{pmatrix} -g' \frac{\omega_e}{h} \\ e - \omega_d \\ -\omega_e \frac{u_1}{h} \end{pmatrix} \end{aligned}$$

If we define a characteristic  $s = x - \lambda t$  this reduces to an ode when:

$$\left| \begin{pmatrix} u_1 - \lambda & 0 & 0 \\ 0 & u_1 - \lambda & h_1 \\ \frac{1}{2} h_1 & g' & u_1 - \lambda \end{pmatrix} \right| = (u_1 - \lambda)((u_1 - \lambda)^2 - g' h) = 0$$

So we get three characteristics  $\lambda_{\pm} = u_1 \pm c$  and  $\lambda_{g'} = u_1$  and  $c = \sqrt{g' h}$  with ODE

$$\frac{dg'}{ds} = -g' \frac{\omega_e}{h_1}$$

for charactersitics  $\lambda_{g'} = u_1$  (left as an exerceise for the rest).

It is very easy to incorporate this into integral models, one of the questions we might ask is what sets our entrainment velocity ( $\omega_e$  or  $\omega_d$ ) ( possible exam question). We are only going to worry about  $\omega_e$  and not  $\omega_d$  (this second one is left for exam questions).

Using the Buckingham Pie theorem. This quanity will be governed by some number of dimensionless parameters e.g. Froude number  $\frac{u}{\sqrt{g' h}}$ , Reynolds number  $\frac{u h}{\nu}$ , schmidt number  $\frac{\nu}{\kappa}$ , **Richardson number**  $\frac{g' l}{u'^2}$ . Make assumption that  $l \sim h$  (depth) and that  $u' \sim u$  not tha they are equal but that they scale with them - the exact ratio of these will be to do with things like how rough is the channel which is encapsulated by the drag coefficient  $C_T$ . We find that provided  $Re \gg 1$  it doesn't matter that we have replaced the Froude number with the drag coefficient and we can also hope that the schmidt number is not important (which is how we used to think but now we know it is more important) so we take the assumption that for sufficently large  $S_c$  (schmidt number) we can ignore it. So maybe:

$$\alpha = \frac{\omega_e}{u} = f(R_i, C_T)$$

Luckily we are only dealing with a small range of richardson numbers in a shallow flow. We find that for small richarson number it increases linearly and then it sort of asymptotes and it is in the asymptotic region that we have gravity currents. So we have an almost constant entraining coefficient. For the moment we will treat  $\omega_i = \alpha u$  with constant  $\alpha$ . It turns out mixing difficult or quite easy depending on the initial conditions (e.g. if you start with dense stuff on top it is easy and if it is on bottom then it is hard). We can think of mixing of stirring and diffusion. Stirring requires KE from somewhere, and diffusion is the result of chemical potential energy and maximising entropy. If we just do horizontal stirring there is no potential penalty for switching parcels of fluids. If we try pulling tendrils of fluid up each time we need potentially energy input, however some ways we can do this easily for instance waves which lift up fluid in a reversible way. So once we have uplifted the fluid mixing horizontally doesn't have any potential cost so if we have an instability type thing we can mix horizontally.

We could have our gravity current running down a slope which changes our momentum equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \cos \theta \frac{\partial h}{\partial x} + g \sin \theta$$

Imagine if we had a gravity current moving along that then collides with an inclined slope.

If we had a gravity current going down a slope then without entrainment it would make no sense the intergal model would get a taller and narrower rectangle clearly violating shallow water.

## 19 Lecture 18

### 19.1 Single-layer Hydraulics

Assume we have some surface with topography given by  $H(x)$  and a shallow water flow across it with height  $h(x)$ , this flow is through a channel with width  $b(x)$ . The velocity of the fluid is  $u(x)$ . Assume the pressure field is hydrostatic which is equivalent to saying the vertical accelerations are small.

We are going to make use of various physical concepts.

First and foremost continuity which equates conservation of volume flux  $Q = uhb = \text{const.}$  (in exam questions sometimes this hasn't been constant)

Momentum - this is too hard for us to make progress in these complex geometries as the pressure distribution on the boundaries will have a streamwise component.

Energy - assuming no dissipation (though has reared its head in exam questions)

Note: It is not uncommon for people to misunderstand the Bernoulli equation

which is a statement about energy and not about momentum the first thing to note is:

$$\nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) = \mathbf{u} \times \boldsymbol{\omega} + (\mathbf{u} \cdot \nabla)\mathbf{u}$$

so the momentum equation becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}|\mathbf{u}|^2 + \frac{P}{\rho} + gz\right) = \mathbf{u} \times \boldsymbol{\omega}$$

Energy:  $\mathbf{u} \cdot$  "momentum equations" gives:

$$\frac{1}{2} \frac{\partial |\mathbf{u}|^2}{\partial t} + (\mathbf{u} \cdot \nabla)\left(\frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + gz\right) = 0$$

So for a steady state we have the Bernoulli potential

$$B = \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + gz$$

which is advected along the fluid velocity.

The momentum equation can be written in terms of this:

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla B = \mathbf{u} \times \boldsymbol{\omega}$$

but this is isn't that helpful much more helpful in the energy equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|\mathbf{u}|^2\right) + (\mathbf{u} \cdot \nabla)B = 0$$

so for steady flow  $B(\psi)$  is conserved along streamlines.

If  $\boldsymbol{\omega} = 0$  we can return to the momentum equation to get:

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla B = 0 \implies \nabla\left(\frac{\partial \phi}{\partial t} + B\right) = 0$$

so  $\frac{\partial \phi}{\partial t} + B = \text{constant}$ . For hydraulics we will simply be considering steady flows with  $B = \text{constant}$ .

### 19.1.1 Specific energy

This is just a way of rewriting the Bernoulli potential a little bit differently.

$$E = \frac{1}{2} \frac{Q^2}{b^2 h^2 g} + H + h = \text{const}$$

Now consider how this changes as function of  $h$ . As  $h \rightarrow 0$  it asymptotes to infinity and as  $h \rightarrow \infty$  it tends towards being linear, so it must have a stationary point at some point.

For a channel of uniform width with  $b(x) = \text{const.}$  with varying depth. Now we have:

$$E - H(x) = \frac{Q^2}{b^2 h(x)^2 g} + h(x)$$

So we can just consider how  $H(x)$  varies along the channel, as  $E$  is constant. Now draw the  $E - H$  curve that we discussed above with its stationary point. Now start labelling the points on this graph and consider how the value of  $E - H$  changes as you move along the streamline. So if you move towards a bump  $H$  will increase so you would move down the curve to the left, when you move past the maximum of the bump you will start to move back up the curve till you reach your original point. For streamlines too close to the bump you cannot move low enough on the curve as  $E - H$  would be below the minimum of  $\frac{Q^2}{b^2 h(x)^2 g} + h(x)$ . There is a critical stationary streamline that will exactly hit the minimum of  $E - H$  and then has the option of following the left hand of the curve or the right hand, either corresponding to a symmetrical solution or one that rapidly decreases in depth after the bump.

### 19.1.2 Informatoin propagation

$$\frac{\partial E}{\partial h} = -\frac{Q^2}{b^2 h^3 g} + 1 = \frac{-u^2}{hg} + 1 = 1 - F_r^2$$

$$F_r = \frac{u}{c} = \frac{u}{\sqrt{gh}}$$

Note that  $\lambda_+ = u + c, \lambda_- = u - c$  so  $\frac{\lambda_+ \lambda_-}{c^2} = \frac{u^2}{c^2} - 1$  so:

$$F_r = \frac{\lambda_+ \lambda_-}{c^2} + 1$$

So we can see if we draw our specific energy curve then on the left of the minimum we have  $F_r > 1$  supercritical flow and to the right of the minimum we have  $F_r < 1$  subcritical flow and at the minimum we have  $F_r = 1$ .

Often these bumps are referred to as sills or weirs.

### 19.1.3 Shocks

If we have a flow that is subcritical before the bump and then have a supercritical flow after, and for instance there is a smaller bump further downstream we will need to have a hydraulic jump formed.

In the subcritical region we have  $\lambda_-$  characteristic pointing up stream and  $+$  points downstream. As we approach the top of the bump the  $-$  characteristic shrinks to nothing. In the supercritical region to the right of the bump we have  $-$  and  $\lambda_+$  pointing downstream. Before the second smaller bump we must have a subcritical region with a  $-$  characteristic pointing downstream. So we must

have characteristics of the same type intersecting giving a shock. This jump will dissipate some energy so we will be able to move onto a lower specific energy curve. MSo it jumps from the left hand side of the minimum to the right hand side of the minimum on a lower curve.

We have constant  $E$  so:

$$\frac{dE}{dx} = \frac{\partial E}{\partial x} + \frac{\partial E}{\partial \phi} \frac{dQ}{dx} + \frac{\partial E}{\partial b} \frac{db}{dx} + \frac{\partial E}{\partial H} \frac{dH}{dx} + \frac{\partial E}{\partial h} \frac{dh}{dx} = 0$$

As in this case we have assumed  $b$  is constant,  $Q$  and  $E$  are constant so:

$$\frac{\partial E}{\partial H} \frac{dH}{dx} + \frac{\partial E}{\partial h} \frac{dh}{dx} = 0$$

At  $\frac{dH}{dx} = 0$  either  $\frac{dh}{dx} = 0$  (giving a symmetric solution) or  $\frac{\partial E}{\partial h} = 0 \implies Fr = 1$ .

If we had a  $Fr > 1$  then as it climbs up the slope it is going to become thinner and slower moving. So there are cases where if it is not sufficiently supercritical to make it over the bump so there is a jump where it goes subcritical.

Across the jump shallow water is violated, momentum is conserved, volume flux is conserved and energy is dissipated.

#### **Differences between Boussinesq and non-Boussinesq**

Are the fluids miscible? and also need to be concerned about the pressure difference across the jump.

This is not too bad for a stationary hydraulic jump ( a moving one is often called a bore). We can write down stationary jump conditions:

$$u_l h_l = u_r h_r$$

$$u_l^2 h_l + \frac{1}{2} g h_l^2 = u_r^2 h_r + \frac{1}{2} g h_r^2$$

If jump is stationary the pressure distribution in the upper layer must be hydrostatic.

Typically takes one of two forms: If only a small amount of energy needs to be dissipated you get an "undular jump" with nonlinear waves that are stationary in the frame of the jump. If too much energy for nonlinear waves then you have a sharp turbulent jump.

For a "bore" ( a moving hydraulic jump) a Boussinesq fluid needs to accelerate ambient fluid out of the way leading to non-hydrostatic pressure across the jump.

## 20 Lecture 19

### 20.1 Jets, plumes and thermals

#### 20.1.1 Connections with shallow water

$$u(x, z, t) = \bar{u}(x, t) \phi\left(\frac{z}{h}\right)$$

$$\bar{u}(x, t) = \frac{1}{h} \int_0^\infty u(x, z, t) dz$$

with  $\eta = \frac{z}{h}$  so:

$$\int_0^\infty \phi\left(\frac{z}{h}\right) dz = h \int_0^\infty \phi(\eta) d\eta = h$$

$$\bar{\rho}(x, t) = \rho_0 + \frac{1}{h} \int_0^\infty (\rho(x, z, t) - \rho_0) dz = \rho_0 + (\bar{\rho}(x, t) - \rho_0) \psi\left(\frac{z}{h}\right)$$

$$\int_0^\infty \psi\left(\frac{z}{h}\right) dz = h, \int_0^\infty \psi(\eta) d\eta = 1$$

This is only a useful choice

#### Top-hat profile

$$\phi = \begin{cases} 1 & \eta \leq 1 \\ 0 & \eta > 1 \end{cases}$$

$$\psi = \begin{cases} 1 & \eta \leq 1 \\ 0 & \eta > 1 \end{cases}$$

Momentum flux is given by:

$$M(x, t) = \int_0^\infty \rho u^2 dz = \rho_0 \int_0^\infty u^2 dz = \chi \rho_0 \bar{u}^2 h$$

Here  $\chi$  is the shape factor and satisfies:

$$\chi = \int_0^\infty (\phi(\eta))^2 d\eta \geq \left( \int_0^\infty \phi(\eta) d\eta \right)^2 \geq 1$$

So for top-hat profile  $\chi = 1$  and for any other profile we have  $\chi > 1$  for any real flow  $\chi \geq 1$  even if only a little bit.

The buoyancy flux is given by:

$$F(x, t) = \int_0^\infty \frac{\rho - \rho_0}{\rho_0} g u dz = \gamma \frac{\bar{\rho} - \rho_0}{\rho_0} g \bar{u}$$

$$\gamma = \int_0^\infty \phi(\eta) \psi(\eta) d\eta$$

Schwarz inequality tells us:

$$|\gamma|^2 \leq \chi \int_0^\infty \psi^2 d\eta \geq 1 \implies \gamma \geq 1$$

**Mention in examples class this logic is off?**

Take  $h$  to be height of shallow water along a slope of angle  $\theta$ . Assume it is predominatly hydrostatic as for shallow water with:

$$\frac{\partial p}{\partial z} = \rho g \cos \theta$$

across the thickness of the layer. So we are saying the pressure within the layer at a given  $x$  is the same as the pressure outside the layer at a given  $x$  with  $x$  taken along the slope.

We may not have a well defined interface but entrainment can still increase the volume /decrease the density of the flow. We will still have  $\int_0^\infty \phi(\eta) d\eta = \int_0^\infty \psi(\eta) d\eta = 1$  by defintion, therefore the volume chang is represnted by an increase in  $h$ .

$$\omega_e = \alpha \bar{u}$$

$\alpha$  is the entrainment coefficent or the 'Batchelor entrainment'. As already discussed  $\alpha = \alpha(R_i, R_e, S_c, F_r, \dots, \theta)$  and for high enough  $Re$  and  $S_c$  we have  $\approx \alpha(R_i, \theta)$ . For typical flows we are looking at  $R_i \sim \frac{1}{F_r^2}$  and  $\cos \theta$  plays a role in  $F_r$  so maybe we can take the approximation  $\alpha \approx \alpha(\theta)$ , but actually if we let the richarson number go all the way to zero then the entrainment is different.

**Conservation of volume:**

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(\bar{u}h) = \omega_e = \alpha|\bar{u}|$$

**Conservation of mass:** Recall

$$\int_0^\infty \rho u dz = \int_0^\infty (\rho_0 + (\bar{\rho} - \rho_0)\psi)\bar{u}\phi dz = \rho_0\bar{u}h + \gamma(\bar{\rho} - \rho_0)\bar{u}h$$

let  $\Delta\rho = \bar{\rho} - \rho_0$  so:

$$\frac{\partial}{\partial t}(\bar{\rho}h) + \frac{\partial}{\partial x}(\rho_0\bar{u}h + \gamma\Delta\rho\bar{u}h) = \alpha\rho_0|\bar{u}|$$

Now expand and make use of the conservation of mass:

$$\frac{\partial \Delta\rho}{\partial t} + \frac{\gamma-1}{h}\Delta\rho \frac{\partial}{\partial x}(\bar{u}h) + \gamma\bar{u}\frac{\partial}{\partial x}\Delta\rho = -\alpha\frac{\Delta\rho}{h}|\bar{u}|$$

let  $g' = \frac{\Delta\rho}{\rho_0}g$  gives:

$$\frac{\partial g'}{\partial t} + \gamma\bar{u}\frac{\partial g'}{\partial x} + \frac{\gamma-1}{h}g'\frac{\partial}{\partial x}(\bar{u}h) = -\alpha\frac{g'}{h}|\bar{u}|$$

In the Boussinesq case the momentum equation becomes:

$$\frac{\partial}{\partial t}(\rho_0 \bar{u} h) + \frac{\partial}{\partial x}(\rho_0 \chi \bar{u}^2 h + \frac{1}{2} \bar{\rho} g' h^2 \cos \theta) = g' h \sin \theta$$

so this simplifies to

$$\frac{\partial \bar{u}}{\partial t} + \chi \bar{u} \frac{\partial \bar{u}}{\partial x} + g' \cos \theta \frac{\partial h}{\partial x} + \frac{1}{2} h \cos \theta \frac{\partial g'}{\partial x} = g' \sin \theta - \alpha \frac{\bar{u} |\bar{u}|}{h}$$

Therefore in matrix notation:

$$\begin{pmatrix} \bar{u} & h & 0 \\ g' \cos \theta & \chi \bar{u} & \frac{1}{2} h \cos \theta \\ (\gamma - 1) g' \frac{\bar{u}}{h} & (\gamma - 1) g' & \gamma \bar{u} \end{pmatrix} \begin{pmatrix} h \\ \bar{u} \\ g' \end{pmatrix}_x + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h \\ \bar{u} \\ g' \end{pmatrix}_t = \begin{pmatrix} \alpha \frac{\bar{u} |\bar{u}|}{h} \\ g' \sin \theta - \alpha \frac{\bar{u} |\bar{u}|}{h} \\ -\alpha g' \frac{|\bar{u}|}{h} \end{pmatrix}$$

If we expand across the bottom row:

$$(\gamma \bar{u} - \lambda)((\bar{u} - \lambda)(\chi \bar{u} - \lambda) - g' h \cos \theta) - (\gamma - 1) g' (\bar{u} - \lambda) \frac{1}{2} h \cos \theta + (\gamma - 1) g' \frac{\bar{u}}{h} \frac{1}{2} h^2 \cos \theta = 0$$

This is too much mess so let us look at the specific case with  $\gamma = 1$  then:

$$(\bar{u} - \lambda)((\bar{u} - \lambda)(\chi \bar{u} - \lambda) - g' h \cos \theta) = 0$$

So  $\lambda_1 = \hat{u} + \hat{c}$ ,  $\lambda_2 = \hat{u} - \hat{c}$ ,  $\lambda_3 = \bar{u}$  with  $\hat{u} = \frac{\chi+1}{2} \bar{u}$  and  $\hat{c} = \sqrt{g' h \cos \theta + (\frac{\chi-1}{2} \bar{u})^2}$ . So a different velocity is responsible for  $\lambda_1$  or  $\lambda_2$  than  $\lambda_3$  (which relates to just how the information is being carried on the flow of the fluid).

In the limit  $\chi = 1$  we get  $\hat{u} = \bar{u}$  and  $\hat{c} = c = \sqrt{g' h \cos \theta}$  which gives  $\lambda_1 = \bar{u} + c$ ,  $\lambda_2 = \bar{u} - c$ ,  $\lambda_3 = \bar{u}$ . Same as before but now we have the  $\cos \theta$  there. Now we need to start asking the question is the system still hyperbolic. The requirement for hyperbolic is that we have three real eigenvalues with a full set of eigenvectors. So to check we need to find the eigenvectors of these eigenvalues.

$$\lambda_1 : \bar{q}_1^T \begin{pmatrix} u - \lambda & h & 0 \\ g' \cos \theta & u - \lambda & \frac{1}{2} h \cos \theta \\ 0 & 0 & u - \lambda \end{pmatrix} = (q, 1, r_1) \begin{pmatrix} -c & h & 0 \\ g' \cos \theta & -c & \frac{h}{2} \cos \theta \\ 0 & 0 & -c \end{pmatrix} = \mathbf{0}$$

which gives:

$$\bar{q}_1^T = (\frac{g' \cos \theta}{c}, 1, \frac{h \cos \theta}{2c}) = ((\frac{g' \cos \theta}{h})^{\frac{1}{2}}, 1, (\frac{h \cos \theta}{4g'})^{\frac{1}{2}})$$

Can do the same for  $\bar{q}_2^T$  to get:

$$\bar{q}_2^T = (0, (\frac{g' \cos \theta}{h})^{\frac{1}{2}}, 1, -(\frac{h \cos \theta}{4g'})^{\frac{1}{2}})$$

and

$$\bar{q}_3^T = (0, 0, 1)$$



As  $\theta \rightarrow \frac{\pi}{2}$  we get:

$$\mathbf{q}_1^T = (0, 1, 0), \mathbf{q}_2^T = (0, 1, 0), \mathbf{q}_3^T = (0, 0, 1)$$

so no longer hyperbolic as we can only form two distinct ODES, so it is now parabolic.

In the limit  $\theta = \frac{\pi}{2}$  but with  $\chi > 1$  and  $\gamma > 1$  we get:

$$|A - \lambda B| = (\gamma \bar{u} -)((\bar{u} - \lambda)(\chi \bar{u} -)) + \dots$$

with  $\gamma$  sufficiently close to 1 that we can ignore the other bits. this will give us distinct eigenvalues:

$$\lambda_1 = \bar{u}, \lambda_2 = \chi \bar{u}, \lambda_3 = \gamma \bar{u}$$

so we have three distinct eigenvalues and will have a full set of eigenvectors that are still hyperbolic. So as soon as we go away from the top hat profile we get a hyperbolic system.

## 20.2 Jet

A source of momentum - typically with a very confined direction. We are looking for something that is producing a flow that is quite narrow so we can assume the pressure is uniform across it, very much the same as with the shallow water, though in this case there will be no hydrostatics. They can be considered as boundary layers. We will be looking at the high Reynolds number case and assume we have Batchelor entrainment, noting that  $R_i = 0$ , so we might expect  $\alpha_J > \alpha_{gravitycurrent}$ . Indeed it is! The question about how it compares to the entrainment coefficient to a plume (which has a density difference and so has a richardson number), in terms of the entrainment density plays a role but is not stabilising. For a point source of momentum  $M_0 > 0$  but no volume flux at point  $Q_0 = 0$ .

Point source of momentum has no length scale, so the only length scale in the problem is the distance from the point source. So we might expect either the width of the jet remains 0 or it is going to grow linearly with space. Lets pick a coordinate system with  $z$  in the direction we are pointing and  $r$  in the radial plane, and we might anticipate that things will be conical, at least if we look at a time or ensemble average to remove turbulent fluctuations.

We can think of this as having a self-similar shape with:

$$\bar{u} = U(E)\phi\left(\frac{r}{b}\right)$$

consider our  $\phi$  for a top hat profile:

$$\phi\left(\frac{r}{b}\right) = \phi(\eta) = \begin{cases} 1 & \eta \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

So effcientively we will have our jet expanding with this circular cross-section, so we can think about our momentum flux:

$$\pi M = \frac{1}{T} \int_0^T \int_{-\pi}^{\pi} \int_0^{\infty} w^2 r dr d\theta dt = \pi b^2 w^2 \chi$$

with  $\chi = \int_0^{\infty} (\phi(\eta))^2 d\eta$  for the top hat  $\chi = 1$  and  $M = b^2 w^2$ .

## 21 Lecture 20

Volume flux:

$$\pi Q = \frac{1}{T} \int_0^T \int_{-\pi}^{\pi} \int_0^{\infty} w r dr d\theta dt = \pi b^2 w$$

so  $Q = b^2 W$  if we define:

$$\int_{-\pi}^{\pi} \int_0^T w dt d\theta = 2\pi T W \phi\left(\frac{r}{b}\right)$$

The momentum flux is going to be constant so:

$$\frac{dM}{dz} = 0 \implies M = M_0$$

but the volume flux will be increasing due to entrainment so:

$$\frac{dQ}{dz} = \alpha W 2b$$

This comes from  $\frac{d}{dz}(\pi Q) = 2\pi b u_e = 2\pi b \alpha W$ . If we note we can write:  $bW = M^{\frac{1}{2}}$  then we can write:

$$\frac{dQ}{dz} = 2\alpha M_0^{\frac{1}{2}} = \text{const.}$$

so our volume flux increases linearly with distance from source:

$$Q = 2\alpha(M_0^{\frac{1}{2}} + z_v)$$

If  $Q = 0$  at  $z = 0$  then  $z_v = 0$ . we have:

$$b = \frac{Q}{M^{\frac{1}{2}}} = \frac{Q}{M_0^{\frac{1}{2}}} = 2\alpha(z + z_v)$$

so our jet is conical. We have assumed here the the jet is self-similar, but of course a real source is unlikely to coincide exactly with that. Imagine if we had a tube ending at  $z = 0$  then initially it will come out a smooth flow, but then as we progressively grow shear instabilities it will start to look like our turbulent jet. We could be tempted to put  $z_v$  at the edge of the tube, but that won't be accuract so we actually label a virtual origin as it will asymptotically approach the solution in the far field as though it is issuing from  $z_v$  wiht  $M_0$  and  $Q_0 = 0$ .

Going to motivate this in part by looking briefly at what a plume really looks like.

## 21.1 General equations for a plume

Steady plume: Morton, Taylor, Turner (1956). however most of the ideas here come from a paper by Batchelor (1954).

Have plume coming from point source, assume a self-similar solution, steady consider velocity profile across it  $w(r, z, t)$  which tends to 0 as  $r \rightarrow \infty$ . Take center lines suppor to be  $W$  and density profile  $\rho(r, z, t)$  with compact support and take width of plume  $b(z, t)$ .

In homogeneous environemnt, we need to consider the bouyancy flux  $F = F_0$  if the plume is steady, in general  $F(z = 0, t) = F_0(t)$ . We also need to consider the volume/mass flux by taking the volume flux at the point source to be zero so  $Q(z = 0, t) = 0$  and  $M(z = 0, t) = 0$ :

$$\phi\left(\frac{r}{b}\right) = \begin{cases} 1 & |r| \leq b \\ 0 & \text{otherwise} \end{cases}$$

In the example sheet we will take this to be a guassian.

We can simply write down what our mass flux is for our top hat profile:

$$\pi Q = \pi \rho b^2 W$$

the momentum flux will be:

$$\pi M = \pi \rho b^2 W^2$$

Note: if not top-hat would need  $\chi$  shape factor. This time we also have buoyancy:

$$\pi F = \pi(\rho_0 - \rho)gb^2W$$

this is  $> 0$  for a buoyant plume. Note: if not top-hat then  $\gamma \neq 1$ .

Now lets do some reverse engineering to write down an expression for  $W$  in terms of  $M, Q$  and  $F$ :

$$\begin{aligned} W &= \frac{M}{Q} \\ b &= \frac{Q}{(\rho M)^{\frac{1}{2}}} \\ g' &= \frac{F}{Q} \end{aligned}$$

As before with the jet we will use Batchelors entrainment hypothesis:

$$u_e = \alpha \left(\frac{\rho}{\rho_0}\right)^{\frac{1}{2}} W$$

If Boussinesq  $u_e = \alpha W$  so  $\rho_0 u_e^2 = \alpha \rho W^2$ . We can think about the  $\frac{\rho}{\rho_0})^{\frac{1}{2}}$  as some statement of energy.

Here we are interested in a plume version of  $\alpha$  rather than a jet version of  $\alpha$ . We can experimentally verify that for a top hat profile we would have  $\alpha$  lying roughly between 0.1 to 0.16 with typically  $\alpha = 0.13$ . For comparison in a jet typically  $\alpha 0.065 - 0.08$ . This is surprising as in the plume we have less dense fluid above a dense fluid so we would expect it to be more stable and therefore have less entrainment. However, the reason for this is when you introduce turbulent ins and outs at the edges you get lots of places with light fluid below dense fluid, so the presence of this static instability (related to Rayleigh-Taylor instability) leads to greatly increased mixing so  $\alpha_{plume} > \alpha_{jet}$ .