1 Dynamics and Geometry of the Universe

Homogeneous and isometric. The invariant line element is given by:

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu$$

In special relativity the metric is fixed at $\{-1, +1, +1, +1\}$ whereas in general relativity the metric is a function of time and position which is influenced by the content of the universe. because homogeneous and isomeric we can describe space-time via a time-ordered sequence of symmetric 3-spaces with 3-d line element:

$$dl^2 = \gamma_{ij}(t) dx^i dx^j$$

giving the general 4d metric:

$$ds^2 = dt^2 - \gamma_{ij} dx^i dx^j$$

Therefore to determine the metric of our universe just write down line elements dl^2 of all possible maximally symmetric 3-spaces. Therefore either flat, positive curvature or negative curvature.
Flat space:

$$dl^2 = \delta_{ij} dx^i dx^j$$

Positive curvature:

$$dl^2 = d\boldsymbol{x}^2 + du^2$$
$$\boldsymbol{x}^2 + u^2 = a^2$$

Negative curvature

$$dl^2 = d\boldsymbol{x}^2 - du^2$$
$$\boldsymbol{x}^2 - u^2 = -a^2$$

1.1 Unified Line element

$$dl^2 = a^2 \left[d\boldsymbol{x}^2 + k \frac{(\boldsymbol{x} \cdot d\boldsymbol{x})^2}{1 - k\boldsymbol{x}^2} \right] \tag{1}$$

$$d\boldsymbol{x}^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \boldsymbol{x} \cdot d\boldsymbol{x} = r dr$$

1.2 FRW metric

$$ds^2 = dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \tag{2}$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

entire dynamics of universe will be encoded in the scale factor $a(t)$. Can be scaled arbitrarily so we pick $a(t_0) = 1$
The coordinates r, θ and ϕ are unchanged as $a(t)$ evolves, and so are called the **co-moving coordinates**.

1.2.1 Physical coordinate

Position - $a(t)\boldsymbol{x}$
Velocity - $\frac{d}{dt}(a(t)\boldsymbol{x}) = \dot{a}(t)\boldsymbol{x} + a(t)\frac{d\boldsymbol{x}}{dt} = H\boldsymbol{x} + a(t)\boldsymbol{v}$
Hubble parameter - $H(t) = \frac{\dot{a}}{a}$

$$dl^2 = a^2(t)\left[dX^2 + S_k(X)d\Omega^2\right]$$

$$S_k(X) = X, \sin X, \sinh X$$

$$dX = \frac{dr}{1-kr^2}$$

If we introduce conformal time $d\tau = \frac{dt}{a}$:

$$ds^2 = a^2(\tau)\left[d\tau^2 - (dX^2 + +S_k(X)d\Omega^2)\right] \tag{3}$$

This is useful as photons have $ds = 0$, so radial trajectory has $\Delta\tau = \Delta X$.

1.2.2 Calculating motion in FRW

Velocity of a particle on trajectory $X^\mu(s)$ is $U^\mu(s) = \frac{dX^\mu}{ds}$.
Particles travel along geodesics which extremise the proper time δs along their path

$$\frac{dU^\mu}{ds} + \eta^\mu \alpha\beta U^\alpha U^\beta = 0$$

$$\eta^\mu \alpha\beta = \frac{1}{2}({}_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta})$$

$$\frac{dU^\mu}{ds} = \frac{dX^\alpha}{ds}\frac{dU^\mu}{dX^\alpha} = U^\alpha \frac{dU^\mu}{dX^\alpha}$$

$$\rho^\mu = mU^\mu$$

$$\rho^\alpha \frac{\partial \rho^\mu}{\partial X^\alpha} + \eta^\mu \alpha\beta \rho^\alpha \rho^\beta = 0 \tag{4}$$

valid equation of geodesic line for massive and massless paritcles $\mu = 0$ component for FRW:

$$E\frac{dE}{dt} = -\frac{\dot{a}}{a}p^2$$

with $E^2 - \rho^2 = m^2$ get $EdE = pdp$. Therefore,

$$\frac{\dot{p}}{p} = -\frac{\dot{a}}{a}$$

so $p\frac{1}{a}$.
For massless particles $E = p$, and since $E = \frac{\hbar}{\lambda}$, λa.
Therefore, $\lambda_0 = \frac{a(t_0)}{a(t_1)}\lambda_1$
Redshift: $z = \frac{\lambda_0 - \lambda_1}{\lambda_1}$ so $1 + z = \frac{1}{a}$ For near by sources we can taylor expand:
$a(t_1) = a(t_0 + (t_1 - t_0)) = a(t_0) + \dot{a}(t_1 - t_0)$
a(t_1) = $a(t_0)(1 + (t_1 - t_0)H_0 + ...)$
$z = H_0(t_0 - t_1)$ as it is a nearby source $z = H_0 d$ as $t_0 - t_1 = d/c$ (for near things)