Quantum Information Theory

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1 Lecture 1

1.1 Internal waves

1.1.1 Minimal maths version

Imagine we have some sort of stratification that is given by density profile $\hat{\rho}$. We are going to require that this is smooth and as we will see later and it is only changing the gradient over a length scale that is large compared to the length scale of the waves. Consider a small volume V of fluid of density ρ_0 and I will magically displaced it up by a distance of ζ (assuming it retains volume, density and shape). There will clearly be a buoyancy force B, and for small displacements:

$$B = gV\zeta\frac{d\hat{\rho}}{dz}$$

Newton's second law gives

$$\rho_0 V\ddot{\zeta} = B = gV\zeta\frac{d\hat{\rho}}{dz}$$

$$\ddot{\zeta} + (0\frac{g}{\rho_0}\frac{d\hat{\rho}}{dz})\zeta = 0$$

This will clearly have solutions:

$$\zeta = A\cos Nt + B\sin Nt$$

where N is the buoyancy (or Brunt-Vaisala frequency) $N = \sqrt{-\frac{g}{\rho_0}\frac{d\hat{\rho}}{dz}}$

There is a key ingredient that hasn't been taken into account in this treatment. There hasn't been anything about continuity e.g. how does the fluid get out the way when it falls back down. To highlight this if we displace a long slim slab of fluid instead of a sphere. If this long slim slab is vertical and we displace it upwards then we get exactly the same maths, but we can make it thin enough that we don't need to worry about continuity to the first order. Now what happens if we take a long thin slab of fluid at an angle and move it along itself. Would it fall vertically downwards or slide back along itself. It is intuitively hard to fall downwards as this would cause a continuity issue of needing to compress all of the fluid below the line (in order to fall down a lot of fluid needs to be moved past the line which would make a big pressure difference resiting the motion). So the slab will slide back along itself. This will have only displaced each parcle of fluid by $\zeta \cos\theta$ upwards and it will only experience a force of $g \cos\theta$ back along its original path. Therefore this would give:

$$\ddot{\zeta} + N^2 \cos^2\theta\zeta = 0$$

$$\ddot{\zeta} + \omega\zeta = 0$$

This gives the dispersion relation for internal gravity waves:

$$|\omega/N| = |\cos\theta|$$

This logic works if we stack slabs on top of each other and as long as we only move each on by a small amount. They can all oscillate up and down along themselves but have no need to be in the same phase, so you could send a wave though them all perpendicular to the slabs. This would mean each slab is a line of constant phase with energy being transmitted along the slab.

Now lets think about the $\cos\theta$. If $\boldsymbol{k} = (k, l, m)$ is a vector perpendicular to the slabs, then we have:

$$\cos\theta = \frac{\sqrt{k^2 + l^2}}{\sqrt{k^2 + l^2 + m^2}}$$

1.1.2 More rigorous derivation

$$\nabla \cdot \boldsymbol{u} = 0$$

$$(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla)\rho = \kappa\nabla^2\rho$$

For now $\kappa = 0$:

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p - \rho g\hat{z} + \rho\nu\nabla^2\boldsymbol{u}$$

For now $\nu = 0$. Take $\rho = \rho_0 + \rho'$. Linearise with $\rho' << \rho_0$ and Boussinesq $|\frac{\nabla \boldsymbol{u}}{\nabla t}| << g$. The Boussieneq means that the ρ' contrbutes to the gravity term but not to the interial term. We can rewrite the equations using reduced gravity to make this clear $g' = g\frac{\rho - \rho_0}{\rho} = g\frac{\rho'}{\rho}$:

$$frac\partial\boldsymbol{u}\partial t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho_0}\nabla(p + \rho g z) - g'\hat{\boldsymbol{z}}$$

To linearise we take$\rho = \hat{\rho}(z) + \rho'(x, t)$ and $\boldsymbol{u}' \sim \eta\omega << \frac{\omega}{|k|} \implies |k|\eta << 1$. By combining this with our Boussineqs condition we have $|\nabla\rho'| << |\frac{d\hat{\rho}}{dz}|$. This gives the navier stokes equation:

$$\frac{\partial \rho'}{\partial t} + w\frac{d\hat{\rho}}{dz} = \frac{\partial \rho'}{\partial t} - w\frac{\rho_0}{g}N^2 = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{\rho_0}\nabla(p_0+\hat{p}-p') - \frac{\hat{\rho}+\rho'}{\rho_0}\boldsymbol{z} = -\frac{1}{\rho_0}\nabla(p_0+\hat{p}) - g\frac{\hat{\rho}}{\rho_0}\hat{\boldsymbol{z}} - \frac{1}{\rho_0}\nabla p' - g\frac{\rho'}{\rho_0}\hat{\boldsymbol{z}}$$

Unperturbed state gives the first two terms as they are much larger than the other terms

$$0 = -\frac{1}{\rho_0}\nabla(p_0+\hat{p}) - g\frac{\hat{\rho}}{\rho_0}\hat{\boldsymbol{z}}$$

therefore

$$p_0+\hat{p} = -\int g\hat{\rho}dz$$

Let

$$b = -g\frac{\rho'}{\rho_0}$$

Therefore:

$$\frac{\partial b}{\partial t} = -wN^2 \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p' + b\hat{\boldsymbol{z}} \tag{2}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{3}$$

1.2 Vorticity

: $\boldsymbol{\zeta} = \nabla \times \boldsymbol{u}$ We are going to deal in 2D as it is easier. In 2-D vorictity can be expressed interms of the streamfunction:

$$\zeta = -\nabla^2\psi, \boldsymbol{\psi} = \psi\hat{\boldsymbol{y}}, \boldsymbol{u} = \nabla \times \boldsymbol{\psi}$$

Take curl of momentum equation to remove pressure:

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} = -\hat{\boldsymbol{z}} \times \nabla b$$

$$(\hat{\boldsymbol{z}} \times \nabla) \cdot (\hat{\boldsymbol{z}} \cdot \nabla w) = \nabla_H^2 w$$

so gives voriticity equation:

$$(\nabla^2\frac{\partial^2}{\partial t^2} + N^2\nabla_H^2)w = 0 \tag{4}$$

Pose a plane wave type solution ansatz and see what happens. Let :

$$w(\boldsymbol{x},t) = Re(\hat{w}(z)e^{i(kx+ly-\omega t)})$$

$$\frac{d^2\hat{w}}{dz^2} + (k^2+l^2)(\frac{N^2}{\omega^2}-1)\hat{w} = 0$$

So if we let $m^2 = (k^2+l^2)(\frac{N^2}{\omega^2}-1)$ then:

$$\hat{w} = Re(Ae^{imz} + Be^{-imz}$$

If $\omega > N$ then m is imaginary let $\gamma = \sqrt{1 - \frac{N^2}{\omega^2}}$:

$$w = (\hat{A}e^{-\gamma k_h z + \hat{B}e^{\gamma k_h z}})e^{i(kx+ly-\omega t)}$$

This is sort of showing how the velocity field changes with depth away from a distribance on the surface. If we have $N = 0$ then this is just a surface wave. If we have $0 < N < \omega$ the γ is just giving a vertical rescaling of the behaviour beneath the surface wave. This means if we produce a sinosidal distributance with a frequency bigger than the bouyancy frequency then the distrubance looks like potential flow, as we increase the stratification of a fluid it will decrease the decay rate of that motion as we move away from that boundary. The limiting case is when we reach $\omega = N$ the entire water depth is moving in phase and with the same magnitude as the surface.

In the case $\omega < N$ we have m is real so we get sinsoidal variations in the vertical direction:

$$w = w_0 e^{ikx+ly+mz-\omega t} = w_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$$

$$\phi = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t$$

so

$$w = w_0 e^{i\phi}$$

We want to get an idea of the relationships between the different parameters. To start with consider continuity:

$$\nabla\cdot\boldsymbol{u} = 0 \implies \frac{\partial u}{\partial x} + \frac{w}{\partial z} = 0$$

So

$$u = \int \frac{\partial w_0 e^{i\phi}}{z} dx = -\frac{m}{k} w_0 e^{i\phi} = -\frac{\tan\theta}{\cos\theta} w_0 e^{i\phi}$$

Considering surface variation $\eta(x,t) = \tilde{\eta}(x,t)e^{i\phi}$ therefore by differentiing this and matching with u and w at the surface:

$$\boldsymbol{u} = \frac{\partial\boldsymbol{\eta}}{\partial t} \implies w_0 = i\omega\cos\theta\tilde{\eta}$$

Now considering the relationship arising form the bouyancy equation:

$$\frac{b}{\partial t} = -wN^2 \implies i\omega\tilde{b} = -wN^2$$

$$b = -\eta\frac{\omega^2}{\cos\theta}e^{i\phi} = -\eta\omega N e^{i\phi}$$

Now consider the momentum equation:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \implies \tilde{p} = i\frac{\omega N}{|\boldsymbol{k}|}\eta\sin\theta$$

1.3 Wave velocities

Phase velocity

$$\phi = \boldsymbol{k} \cdot \boldsymbol{x} - \omega t = k_i x_i - \omega t$$

The below identity is very obviously zero:

$$\frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial t}\frac{\partial \phi}{\partial x_i} = 0$$

$$k_i \frac{\partial \phi}{\partial t} + \omega \frac{\partial \phi}{\partial x_i} = 0$$

Divide across by k_i:

$$\frac{\partial \phi}{\partial t} + \frac{\omega}{|k|^2} k_i \frac{\partial \phi}{\partial x_i} = 0$$

Therefore, $c_p = \frac{\omega}{|\boldsymbol{k}|^2}\boldsymbol{k}$ as:

$$\frac{\partial \phi}{\partial t} + (\boldsymbol{c}_p \cdot \nabla)\phi = 0$$

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Group velocity:

$$\frac{\partial^2 \phi}{\partial x_i \partial t} - \frac{\partial^2 \phi}{\partial t \partial x_i} = 0$$

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0$$

As $\omega = \omega(k)$ we have:

$$\frac{\partial \omega}{\partial x_i} = \frac{\partial \omega}{\partial k_j}\frac{\partial k_j}{\partial x_i}$$

$$\frac{\partial k_j}{\partial x_i} = \frac{\partial^2 \phi}{\partial x_j \partial x_j} = \frac{\partial k_i}{\partial x_j}$$

Therefore:

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial k_j}\frac{\partial k_i}{\partial x_j} = 0$$

Therefore, $c_g = \frac{\partial \omega}{\partial k_i}$ as:

$$\frac{\partial k_i}{\partial t} + \boldsymbol{c}_g \cdot \nabla k_i = 0$$

So the wavenumber vector is being advected outwards with the group velocity.
Surface waves: $\omega = gk$, $c_g = \frac{\partial}{\partial k}\sqrt{gh} = \frac{1}{2}c_p$, $c_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}}$.

2.1 Superposition

$$\eta = \cos((k+\delta k)x - (+\delta\omega)t) + \cos((k-\delta k)x - (\omega - \delta\omega)t)$$

$$\eta = 2\cos(\delta kx - \delta\omega t)\cos(kx - \omega t)$$

As $\delta\omega = \frac{\partial\omega}{\partial k}\delta k$ for $|\delta k| << |k|$ then

$$\eta = 2\cos((x - \frac{\partial\omega}{\partial k}t)\delta k)\cos(kx - \omega t)$$

This is a wave and envelope speed of $\frac{\partial\omega}{\partial k}$.

2.2 Internal wave velocities

As $\frac{\omega^2}{N^2} = \frac{k^2+l^2}{|k|^2} = \cos^2\theta$

$$\boldsymbol{c}_p = \frac{\omega}{|k|^2}\boldsymbol{k} = \frac{N(k^2+l^2)^{\frac{1}{2}}}{|k|^{\frac{3}{2}}}\boldsymbol{k} = \frac{N|\cos\theta|}{|k|^2}\boldsymbol{k}$$

This is the polar coordinate equation for two circles touching at the origin at every ϕ so they sort of form a torus.

Now lets look at the group velocity:

$$c_g = \frac{\partial\omega}{\partial k_i} = \frac{1}{2\omega}\frac{\partial\omega^2}{\partial k_i} = \frac{\omega}{|k|^2}(\frac{N^2}{\omega^2}(\boldsymbol{k} - k_z\hat{\boldsymbol{z}}) - \boldsymbol{k}) = \frac{N|\sin\theta|}{|k|}\begin{pmatrix}\cos\phi\sin\theta \\ \sin\phi\sin\theta \\ -\cos\theta\end{pmatrix}$$

$$|c_g| = \frac{N}{|k|}|\sin\theta|$$

This means that in the horizontal direction the phase velocity is always perpendicular to the group velocity. They form the same circle just one with $\sin\theta$ and one with $\cos\theta$ and as $\sin\theta = \cos(\pi/2 - \theta)$. As the angles on a semicircle subtend 90 degrees we can sum the two and we will get the opposite side of the circle always. Therefore,

$$\boldsymbol{c}_p + \boldsymbol{c}_g = \frac{N}{|\boldsymbol{k}|}\begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix}$$

$$|\boldsymbol{c}_p + \boldsymbol{c}_g| = \frac{N}{|\boldsymbol{k}|}, c_{p,z} = -c_{g,z}, \boldsymbol{c}_p \cdot \boldsymbol{c}_g = 0$$

2.3 Equipartition of energy

$$\boldsymbol{u} \cdot (\rho_0\frac{\partial\boldsymbol{u}}{\partial t} + \nabla p' + \rho' g\boldsymbol{z}) = 0$$

Recalling that $\frac{\partial \rho'}{\partial t} - w\frac{\rho_0}{g}N^2 = 0 \implies w = \frac{g}{\rho_0 N^2}\frac{\partial \rho'}{\partial t}$, and the incompressibility condition to get:

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho_0|\boldsymbol{u}|^2 + \frac{1}{2}\frac{g^2}{\rho_0 N^2}p'^2) + \nabla\cdot(p'\boldsymbol{u}) = 0$$

If we go back to the start and consider the dispalcement of a packet of fluid by ζ we have change in potential energy of:

$$\Delta PE = \int_{z_0}^{z_0+\zeta} g\frac{d\hat{\rho}}{dz}(z-z_0)dz = \frac{1}{2}\rho_0 N^2\zeta^2$$

$$\rho' = -\frac{d\hat{\rho}}{dz}\zeta = \frac{\rho_0}{g}N^2\zeta$$

$$PE = \frac{1}{2}N^2\rho_0\zeta^2 = \frac{1}{2}\rho_0\frac{b^2}{N^2}$$

If we want to consider the total energy equation:

$$\int_V \frac{\partial}{\partial t}(KE+PE)dV + \int_S p'\boldsymbol{u}\cdot\boldsymbol{n}dS' = 0$$

$\boldsymbol{F}_E = p'\boldsymbol{u}$ is the flux of energy.

Lets consider 2D:

$$u = \eta\omega\sin\theta\sin\phi, w = -\eta\omega\cos\theta\sin\phi, b = \eta\frac{\omega^2}{\cos\theta}\cos\phi, p' = \eta\rho_0\frac{\omega^2}{|k|}\tan\theta\sin\phi$$

Subsitute into kinetic energy:

$$KE = \frac{1}{2}\rho_0(u^2+w^2) = \frac{1}{2}\rho_0\omega\eta^2\sin^2\phi$$

$$\bar{KE} = \frac{1}{4}\rho_0\omega^2\eta^2$$

$$PE = \frac{1}{2}\rho_0\omega^2\eta^2\cos^2\phi$$

$$\bar{PE} = \frac{1}{4}\rho_0\omega^2\eta^2$$

Also have:

$$PE = \frac{1}{2}\rho_0\frac{b^2}{N^2}$$

So you have equiparition of energy for linear waves $\bar{KE} = \bar{PE}$. We can also write down an expression for the flux of energy:

$$\boldsymbol{F}_E = p'\boldsymbol{u} = \rho_0\omega^2\eta^2\sin^2\phi\frac{N}{|k|}\sin\theta\begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix} = \frac{1}{2}\rho_0\omega^2\eta^2\boldsymbol{c}_g = \bar{E}\boldsymbol{c}_g$$

3 Lecture 4

3.1 Oscillating cyclinder

We are interested in the case where the oscillation of the cylinder a is much smaller than the diameter. You might think this is sufficent to make the waves linear, but it is not as we ahve these delta functions on the singularities on the tangent planes to the cylinder (so we will always have the amplitudes being large compared to the wave lengths here but we will ignore this). If everything is at rest to start with it will take a bit of time for the oscillations to propogate out into the whole space. As $|c_g| = \frac{N}{|\boldsymbol{k}|}\sin\theta$ the area of which is influenced by the oscillation will look like two causality envelopes (circles of increasing size touching at the centre). The waves form a st. andrews cross pattern with the waves being bi modal near the cyclinder and unimodal further way this is due to visocity

3.1.1 Decay along a beam

We are going to take it being 2D and the bouunancy frequency $N = 1$ and the mass dispersion $\kappa = 0$ but include visocity $\nu \neq 0$.

$$\nabla \cdot \boldsymbol{u} = 0$$
$$\frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = \nu\nabla^2 u$$
$$\frac{\partial w}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial z} - b = \nu\nabla^2 w$$
$$\frac{\partial b}{\partial t} + N^2 w = 0$$

To make our lives easier we are going to use a streamfunction:

$$\boldsymbol{\psi} = (0, \psi, 0), \boldsymbol{u} = (\nabla \times \psi)e^{-i\omega t} = \begin{pmatrix} -\frac{\partial\psi}{\partial z} \\ 0 \\ \frac{\partial\psi}{\partial x} \end{pmatrix} e^{-i\omega t}$$

Take ζ and ξ to be displacment in the wavevector k direction and then group velocity c_g direction. let θ be the angle of the group velocity from the vertical which is the asame as the angle of the cross. ThereforeL

$$\zeta = x\cos\theta - z\sin\theta$$

$$\xi = x\sin\theta + z\cos\theta$$

Boynancy equation becomes:

$$\frac{\partial b}{\partial t} + \frac{\partial\psi}{\partial\zeta}\cos\theta + \frac{\partial\psi}{\partial\xi}\sin\theta = 0$$

Vorticity: $\nabla \times \boldsymbol{u} = -\nabla^2 \psi e^{-i\omega t}$

$$-i\omega(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \zeta^2}) - \frac{\partial b}{\partial \xi}\sin\theta - \frac{\partial b}{\partial \zeta}\cos\theta - \nu\nabla^2\nabla^2\psi = 0$$

Let $b = (b_0 + \epsilon b_1 + ...)e^{-i\omega t}$ and $\psi = (\psi_0 + \epsilon\psi_1 + ...)$: We take small viscoity to make equations nice e.g. small $\nu = 2\epsilon$ and $\epsilon = \frac{1}{2}\nu$ as this is dimensional it is not clear what we mean by small parameter. As this is dimensional it means the dimensions of ψ_0 and ψ_1 are going to be different. This is not the ideal way of doing this but it is going to allow us to see more clearly what is happening.

$$\chi = \frac{\epsilon}{\sin\theta}\xi$$

Here ϵ being small means we are interested in gradual changes in the direction of the group velcoity but fast changes in teh wave vector direction so we keep that sinosdiual behaviour.

$$\frac{\partial}{\partial \xi} = \frac{\partial \chi}{\partial \xi}\frac{\partial}{\partial \chi} = \frac{\epsilon}{\sin\theta}\frac{\partial}{\partial \chi}$$

Plug these into the bouyancy and vorticity equation and compare terms of the same order:

$$\epsilon_0 : \frac{\partial \psi_0}{\partial \zeta} = -b_0, \frac{\partial^2 \psi_0}{\partial \zeta^2} = i\frac{\partial b_0}{\partial \zeta}$$

$$\epsilon_1 : \omega\frac{\psi_1}{\partial \zeta} - i\omega b_1 = -\frac{\partial \psi_0}{\partial \chi}, i\omega\frac{\partial^2 \psi_1}{\partial \zeta^2} + \omega\frac{\partial b_1}{\partial \zeta} = i\frac{\partial^2 \psi_0}{\partial \zeta \partial \chi} - 2\frac{\partial^4 \psi_0}{\partial \zeta^4}$$

Can eliminate the LHS of both of these to give:

$$\frac{\partial^4 \psi_0}{\zeta^4} = i\frac{\partial^2 \psi_0}{\partial \zeta \partial \chi}$$

$$\frac{\partial^3 \psi_0}{\partial \zeta^3} = i\frac{\partial \psi_0}{\partial \chi} + f(\chi)$$

For a point sources as $|\zeta| \to \infty$ then we expect $\psi \to \psi_0 = F(\zeta)G(\chi)\frac{F''}{F} = i\frac{G'}{G} = -ik^3G(\chi) = e^{-k^3\chi}F(\zeta) = e^{ik\zeta}\psi \approx \psi_0 e^{-i\omega t} = Ae^{-k^3\chi}e^{i(k\zeta-\omega t)}\boldsymbol{k} = (k, 0)$ in (ζ, ξ)

$$\chi = \frac{\epsilon}{\sin\theta}\xi = \frac{\nu}{2\sin\theta}\xi = \frac{\nu}{2N\sin\theta}\xi$$

for any N. If we take a whole spectrum of linear superposition of waves $A(k)$:

$$\psi = e^{-i\omega t}\int_{-\infty}^{\infty} A(k)\exp(ik\zeta - \frac{\nu k^3}{2N\sin\theta}\xi)dk \tag{5}$$

If we think about our cyclinder with the two delta functions at the edge of the cyclinder. As the fourier transform of the delta function is just a constant. So $A(k)$ would just be a constant so the higher wavenumber modes would decay more rapidly due to the k^3 in the exponential. We want it to be clear that the length scale over which the day is happening is small compared to the length scale of the oscillations. Lets imagine we are going to scale ξ:

$$\frac{\nu k^2}{2N\sin\theta}k\xi$$

Recall that $|c_g| = \frac{N}{|k|}\sin\theta$ so

$$\frac{\nu k^2}{2N\sin\theta}k\xi = \frac{\nu k}{2}\frac{1}{|c_g|}k\xi = -\pi Re^{-1}k\xi$$

With Reynolds number $Re = \frac{\lambda|c_g|}{\nu}$ with $\lambda = \frac{2\pi}{k}$. We we were requiring ϵ to be small we were really requiring the reynolds number to be large enough.

3.2 Mass difussivity

$$(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla)\rho = \frac{D\rho}{Dt} = \kappa\nabla^2\rho$$

Lets think about what could cause a difference in the density in the fluid e.g. S salt concentration, moisture content or temperature T. Carbon dioxide that we are breathing out is denser than the other air we are breathing out, which is more or less balanced by the humidity of the exhaled breath which is higher than the surroundings. Water vapour is less dense than air. The diffusivity of salt and the diffusivity of temperature are different so we can't write down an equation like above. What we can write down is equations of the diffusivity of these two:

$$\frac{DS}{Dt} = \kappa_s\nabla^2 S, \frac{DT}{Dt} = \kappa_T\nabla^2 T$$

For water $\frac{\kappa_T}{\kappa_s} \approx 100$. If you move hot salty water down into cold nonsalty water then it quickly becomes cold salty water so it rapidly becomes more dense than the fluid around it. Like wise if you took a parcel of cold fresh up into the hot area it will stay fresh but rapidly heat and so will be less dense than its surroundings and want to rise. THis is called salt fingering. Equally if you have cold fresh water above hot salty water. As tempearture diffuses relativitly quickly the gradient will be much steeper for the salt concentration than the temperature, this creates convection. This is called double-diffusive convection.

Prandtl number: $\frac{\nu}{\kappa_T}$
Schmidt number: $\frac{\nu}{\kappa_S}$ with κ_S is the diffusivity of some thing like salt

3.3 Reflections of waves

With light the angle of incidence is equal to the angle of reflection.

4 Lecture 6

The thing being conserved on relfection is the wavelength (in the case of light) and therefore the colour. If the speed of light in the medium is constant then the frequency is constant, then the wavelength must also be constant. However, in our internal wave system our frequency ω is not constant as $|\frac{\omega}{N}| = |\cos\theta|$. The angle to the vertical must be conserved not the angle of incidence and angle of reflection in order for the frequency to be the same at the wall. So only in the case of a horizontal wall is the angle of incidence equal to the angle of reflection. This difference will mean that the wavelength will not be conserved in general.

Let the displacement on the incident ray η_i and reflected rays η_r. Therefore in example sheet we show that a slope of angle α to the vertical is:

$$|\eta_r| = \gamma|\eta_i|, \gamma = |\frac{\sin(\theta+\alpha)}{\sin(\theta-\alpha)}$$

This comes about as the same volume of fluid must be displaced but the distance between neighbouring rays is smaller, so the dispalcement must be larger.

4.0.1 Energy density upon reflection

$$|\boldsymbol{k}_r| = \gamma|\boldsymbol{k}_i| \implies \lambda_r = \frac{1}{\gamma}\lambda_i$$
$$|\tilde{\boldsymbol{u}}_r| = \gamma|\tilde{\boldsymbol{u}}_i|$$
$$|\tilde{\boldsymbol{\eta}}_r| = \gamma|\tilde{\boldsymbol{\eta}}_i|$$

Recall $|c_g| = \frac{N}{|\boldsymbol{k}|}\sin\theta$:

$$|c_{g,r}| = \frac{1}{\gamma}|c_{g,i}|$$

Flux of energy per wavelength must be preserved on reflection. We have to be careful what we mean by flux of energy as we have two different wavelength so we can talk about flux of energy per wavelength or flux of energy per unit length:

Energy density per wavelength:

$$\tilde{E} = \int_0 PE + KEd\zeta$$
$$\tilde{\boldsymbol{F}} = \tilde{E}\boldsymbol{c}_g$$

Therefore:

$$\tilde{E}_r = \gamma\tilde{E}_i$$

As $\tilde{E}_r \sim \lambda(PE+KE) = \lambda(\tilde{\eta}\tilde{\eta^*} + \tilde{u}\tilde{u}^*) \sim \lambda(|\tilde{\eta}|^2 + |\tilde{u}|^2)$ and as $\lambda \sim \frac{1}{\gamma}$ and $|\eta|^2 \sim \gamma^2$ so $\tilde{E}_r \sim \gamma\tilde{E}_i$.

Energy density per unit length:

$$\bar{E} = \frac{1}{\lambda}\int_0^\lambda PE + KE = \frac{1}{\lambda}\tilde{E}$$

$$|\tilde{F}| = \tilde{E}|c_g| = \lambda \bar{E}|\boldsymbol{c}_g|$$

Flux of energy per wavelength was conserved:

$$|\tilde{F}_r| = \lambda_r \bar{E}_r |c_{g_r}| = \frac{1}{\gamma}\lambda_i \bar{E}_r \frac{1}{\gamma}|c_{g,i}| = |\tilde{F}_i| = \lambda_i \bar{E}_i |c_{g,i}|$$

So

$$\bar{E}_r = \gamma^2 \bar{E}_i$$

Again consisten with $\bar{E} \sim PE + KE \sim |\tilde{\eta}|^2 + |\tilde{u}|^2$. The typeset notes have hats for energy desnity and flux per wavelength.

We also might want to think about the total energy of a whole broad region of a wave coming in and going out:

$$TE_r = \int_{-\frac{L_r}{2}}^{\frac{L_r}{2}} PE_r + KE_r d\zeta = \gamma^2 \int_{-\frac{1}{\gamma}\frac{L_i}{2}}^{\frac{1}{\gamma}\frac{L_i}{2}} PE_i + KE_i = \gamma TE_i$$

Spectral energy density $S(k)$ if we have a whole lot of waves of different wavenumbers and want to consider how that spectrum will change:

$$S_r(k) = \gamma S_i(\frac{K}{\gamma})$$

4.0.2 Critical reflection

Under sub crictical reflection the vertical direction of the propogation reverses upon reflection.
Under supercitical reflection the vertical direction of propogation is maintained. On the boundary between these modes is the critical reflection where the reflected wave is along the slope of the wall in both directions so we have $\alpha = \theta$. Shifting slightly one way would lead to the wave reflecting slightly above the upper portion of the wall and slightly the other would lead to the the wave reflection slightly above the lower poriton oft he wave.

As we approach critical reflection, $\tilde{\eta} \to \infty$, $k_r \to \infty$, dissipation which scales like k^3 also tends to infinty. Viscosity becomes important and we get non-linearisties as $k\tilde{\eta} \to \infty$ as linear waves require $k\tilde{\eta} << 1$. Viscosity also plays a role away from critical conditions due to no-slip boundary.

4.1 Ray tracing

$$(\nabla^2 \frac{\partial^2}{\partial t^2} + N^2 \nabla_H^2)w = 0$$

$$(\nabla^2 \frac{\partial^2}{\partial t^2} + N^2 \nabla_H^2)\psi = 0$$

In 2D: $\psi = \tilde{\psi}(x,z)e^{-i\omega t}$:$(N^2 - \omega^2)\frac{\partial^2 \tilde{\psi}}{\partial x^2} - \omega^2 \frac{\partial^2}{\partial z^2}\tilde{\psi} = 0 \Lambda^2 = \frac{\omega^2}{N^2-\omega^2} give this Poincare wave equation (\frac{\partial^2}{\partial x^2} - \Lambda^2 \frac{\partial^2}{\partial z^2})\tilde{\psi} = 0 If domain bounded with = 0 on boundary this is an ill-posed problem and we will use ray tracing inst$

5 Lecture 6

Lets consider a constant slope at the edge of the ocean with free surface with constant bounancy frequncy. Imagine we have some waves entering the systemm in the body of the fluid 1 wavelength apart. These reflect off the free surface and here they remain exactly the same except for a change of vertical orientation. The reflection off the slope shorterns the wavelength as we considered in the previous section. Therefore, the energy reflection goes up, group vecolity is going down, wavenumber goes down and the velocity gets steeper on each reflection off the slope. In this case the energy is trapped into the courner by a sequence of focusing reflections.

For large Reynolds number $Re = \frac{|c_g|}{|k|\nu}$ nonlinearities will end up dominating leading to wave breaking, mixing and other frequencies and wave numbers coming out of the system.

Energy density increases:

$$\tilde{E}_{n+1} = \gamma \tilde{E}_n = \gamma^{n+1} \tilde{E}_0$$

Steepness increases

$$|k_{n+1}|\tilde{\eta}_{n+1} = \gamma^{n+1} k_0 \eta_0$$

5.1 Reflections from rough topography

How well do we actually know what is down at the bottom of the ocean?

Lets imagine we have some idealised rough topography, a sine wave of amplitude h_0 and wavelength $\lambda_T = \frac{2\pi}{k_T}$. We can make use of our ray tracing.

Draw rays one wavelength apart and you can visually see that the reflected wavelength varies depending on which part of the sine wave they reflect off. This means the spectrum has changed so we can no longer just talk about the wavenumber vector. If we going to try and analyse this we can try some sort of linearisation by considering small amplitude variations in topography with $k_T h_0 << 1$.

Now lets zoom in and consider what is happening at a close scale (here we aren't actually assuming it is small), we have a wave coming in towards x_i, then reflects at x_0 and then appears to be leaving from x_r. We define $\delta x = x_0 - x_r = x_i - x_0$ as the angle of the reflected and incident wave from the vertical are identical. Take the height of the boundary to be $z = h_0 \sin k_T x$. Define $\beta = \cot\theta$. Now geometrically, $h_0 \sin k_T x_0 = \beta \delta x$. Now consider the amplitude of an incident ray

$$\eta_i(x,t) = \tilde{\eta}_i \sin(k_i x - \omega t), \eta_r(x,t) = -\tilde{\eta}_i \sin(k_i(x + \frac{2h_0}{\beta}\sin k_T x) - \omega t)$$

For small amplitude $\frac{k_i h_0}{\beta} << 1$:

$$\eta_r(x,t) = -\tilde{\eta}_i(\sin(k_i x - \omega) + k_i(\frac{2h_0}{\beta})\sin(k_T x)\cos(k_i x - \omega t)) = -\tilde{\eta}_i \sin(k_i x - \omega) - \tilde{\eta}_i \frac{k_i k_0}{\beta}(-\sin((k_i - k_T)x - \omega t)$$

So we have three different wave numbers in the reflection:$k_i = k_R, k_i - k_T = k_B, k_i + k_T = k_F$.

If $k_B = k_i - k_T < 0$ then we need to start being quite careful as in order to match the boundary condition it seems like the group velocity seems to be moving backwards from the right. This violates causality and the reason for this is we were making an assumption about the direction of the wavenumber vector assoicated with the reflection. What we actually have is two waves reflected in the forwards direction with group velcoity c_{gR}, c_{gF} and a backwards refelcted one with c_{gB}.

In general, for subcritical reflectino we will end up with a backscatter from rough topography.

What happens if we have super critical reflection when the topography is steeper than the angle of the incident waves. This leads to a very complex spectrum as neighbouring rays could end up with very different end points as the waves can reflect multiple times off the topography before escaping. This problem is not very analytically tractable.

5.2 Wave attractors

Now lets look at smooth boundaries but make it a bit more complex by considering a bounded domain.

5.2.1 Rectangular basin

With width X and height Y

If we have an eigenmode, then

$$\frac{X}{Y}\frac{n}{m} = \tan\theta$$

m is the number of reflection from top boundary, n is the number of reflections of the left-hand boundary.

The simplest possible case is take $n = m = X = Y = 1$ gives $\tan\theta = 1 \implies \theta = \frac{\pi}{4}$ so you get a rectangle reflecting around the inside of the rectangle.

6 Lecture 7

One of the questions that came up in the background on the chat on wednesday is imagine we have this undular boundary to our domain and it is close to having critical slopes to our domain. A critical slope will lead to a massive increase in wavenumber and in steepnesss of the wave. If we have a near horizontal surface with near critical surfaces then we will end up filling the gaps between the undulations with denser fluid than above due to the mixing so a lot of the wave energy won't end up geting into the valleys as it will reflect off the changing stratificaiton. To some approximaiton if the water can pool in it then we can ignore the fluxuations in the surface, however if the slope is at a gradient and so the water in the pools can flow away then they remain important.

Returning the immediate discussion, if we don't have an eigenmode ($\frac{X}{Y}\frac{n}{m} \neq \tan\theta$ then reflections were space-filling. Remember we defined $m = \frac{1}{2}$ number of subcritical reflections, and $n = \frac{1}{2}$ number of supercritical reflections (reflection from a vertical rule will always be a supercritical refleciton but if there is an angle it will depend on θ

Trapezoidal basin: Length of one side of 2, length of other side of 1 and width of 2. At 45 degrees, we converge to a corner to corner attractor. At 40 degrees we saw roughly diamonded shaped stuff. (the trick of drawing attractors is to draw the attractor and then draw the boundary on, this method allows us to easily see the other possible domains that would give the same attractor. $\theta = \tan^{-1}\frac{1}{2}$ gives the attractor between the other two corners. Any angles between 45 degrees and $\theta = \tan^{-1}\frac{1}{2}$ will give nice diamond shapes, and outside this range we will get more complex shapes like figure of 8s. Eventually as we get higher and higher modes we have more different directions of waves in the same space so get lots more disspation, we also start to see viscoity having an impact as the scale of the waves gets smaller.

Define a trapezium with bottom length X and height Z and define the bottom left to the the origin and the angle of the left hand side to be α to the vertical. Define the starting point of the attractor as (x_0, z_0) and let every point of reflection thereafter be labeled (x_i, z_i). Lets label $a = \cot\alpha$ and $b = \cot\theta$ to clean up algebra. First off we can see that $z_0 = ax_0$, $z_1 = Z$ and $x_1 = \frac{Z+(b-a)x_0}{b}$, $x_2 = 2$, $z_2 = 2Z - bX - (b-a)x_0$ ... $x_4 = \frac{2(bX-Z)+(a-b)x_0}{a+b}$, $y_4 = ax_4$ simply from basis trigonometry. So we can now follow the packet of energy as it goes round once and we can repeat this as it goes around a second time until we find where it converges. Alternatively to find the attractor we just set $x_4 = x_0$ which gives $x_0 = X - \frac{Z}{b}$. This looks a bit like a general iteration where you are trying to find the roots of $f(x) = 0$. One of these techniques is to rewrite this as $x - g(x) = 0$ and then take $x_n = g(x_{n-1})$ which is like what we have done here. This technique only converges under the right circumstances when $|g'(x)| < 1$. Here once we get close enough to the attracotr this is equivalent to $\frac{|a-b|}{|a+b|} < 1 \iff |\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)} = \frac{1}{\gamma} < 1 \iff \gamma > 1$. Possible typo here check with typed version.

One of the things we have seen here is that viscoity will play a role here as the wave number goes up and up as we go round and dissaption grows with the cube of the wavenumber. So we can start thinking about what the equilibrium energy spectrum looks like:

6.0.1 Energy spectrum for attractor

$$\gamma = \frac{\sin(\theta + \alpha)}{\sin(\theta - \alpha)}$$

Therefore with each reflection

$$k_n = \gamma k_{n-1}$$

$$\tilde{E}_n = \gamma \tilde{E}_{n-1} = \gamma^n \tilde{E}_0 = \frac{k_n}{k_0} \tilde{E}_0$$

provided there is no dissipation.

$$|\tilde{\eta}_n| \sim \frac{k_n}{k_0} |\tilde{\eta}_0|$$

The wave steepness is $|k_n \tilde{\eta}_n| = \frac{k_n^2}{k_0} |\tilde{\eta}_n|$.

Recall:

$$|\boldsymbol{u}| \sim e^{-\frac{k^2 \nu \zeta}{2N \sin\theta}, \tilde{E} \sim \lceil}$$

u—$^2 \sim e^{-k^3 \nu \zeta} N \sin\theta$L$-lengthoncearoundonecircuitoftheattractor$
$\frac{\tilde{E}_n^{(end)}}{\tilde{E}_n^{(start)}} = e^{-\frac{k^3 \nu L}{N \sin\theta}} \tilde{E}_n^{(end)} = \frac{k_n}{k_0} e^{-\Gamma((\frac{k_n}{k_0})^3 - 1)} \tilde{E}_0 with \Gamma = \frac{\nu k_0^3 L}{(\gamma^3 - 1) N \sin\theta}$.

7 Examples Class 1

Does $\nabla \cdot \boldsymbol{u} = 0$ always hold?

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x}$$

8 Lecture 8

8.1 Non-linear stratification

If we have a stratification that changes slowly compared to the wavelength, then we can treat the bouyancy frequency as being almost constant using the WKB approximation were we keep N constant locally and vary it along a ray:

$$\frac{\omega}{N} = \cos\theta \rightarrow \frac{\omega}{N(z)} = \cos\theta(z)$$

For example if:

$$N(z) = N_0 e^{-\frac{z}{H}}, \cos\theta(z) = \frac{\omega}{N_0} e^{z/H}$$

If we take some rays form a source at $z = 0$ wtih $x = X(z)$ then:

$$X(z) = \int_0^z \frac{dX}{dz} dz = \int_0^z \tan\theta dz$$

use substitution $\cos\theta = \frac{\omega}{N_0} e^{z/H}$ to get

$$X(z) = H(\theta - \theta\tan\theta - (\theta_0 - \tan\theta_0)) = H(\cos^{-1}(e^{z/H}\cos\theta_0) - \frac{\sqrt{1 - e^{2z/H}\cos^2\theta}}{e^{z/H}\cos\theta_0})$$

8.2 Lee waves

8.2.1 Extended range of hills

Imagine we have a sinsuodial topography with a wavelength of λ_T and amplitude η_0 and we have a uniform wind of speed U moving past.

First we change the frame of reference so the fluid is stationary, which effectively means the mountain range is moving the opposite direction with speed U. We know $\omega = k_T U = N\cos\theta \leq 1$ then we have waves. If $\frac{k_T U}{N} > 1$ then we have no internal waves just forced oscillations with exponentially decaying distubrances. For $\frac{k_T U}{N} \leq 1$ we will have internal waves.

Firstly, if these hills are making the waves then the vertical component of the waves must be upwards so $c_{gz} > 0$. A point of constant phase on a hill that moves to the left with speed $-U$ so the phase velocity will also be moving to the left so $c_{px} < 0$. We also know that $\boldsymbol{c}_g \cdot \boldsymbol{c}_p = 0$. So we can combine these two facts to tell us which quadrant the group velocity must be in. So the phase velocity will point down to the left and and the group velocity will point up to the left.

Now consider how this works with the frame with mountains at rest. It will look the same. We will still have lines of constant phase coming from the mountains but now the directions of the velocity will change. The change of direction of speed will change the direction of the horizontal component of the velocities. As the lines of constant phase do not move the phase velocity must be parallel to the line of constant phase, therefore the group velocity must be perpendicular to the lines of constant phase. we have transformed the velocities by adding U like:

$$\boldsymbol{c}_p = \boldsymbol{c}'_p - \boldsymbol{U}, \boldsymbol{c}_g = \boldsymbol{c}_g - \boldsymbol{U}$$

8.3 Kelvin ship waves

$$\phi = \boldsymbol{k} \cdot \boldsymbol{x} - \omega t$$

$$d\phi = \frac{\partial \phi}{\partial k_i} dk_i + \frac{\partial \phi}{\partial x_i} dx_i + \frac{\partial \phi}{\partial \omega} d\omega + \frac{\partial \phi}{\partial t} dt$$

Principle of stationary phase: $d\phi = 0, d\boldsymbol{k} = 0, \omega = \omega(\boldsymbol{k}) \implies d\omega = 0$

$$\frac{\partial \phi}{\partial t} + (\boldsymbol{u} \cdot \nabla)\phi = 0$$

If $\boldsymbol{u} = \boldsymbol{U}$ then $\frac{\partial \phi}{\partial t} + (\boldsymbol{U} \cdot \nabla)\phi$
Kelvin ship waves: surface waves $\boldsymbol{U} = (U, 0, 0)$

$$-\omega + U\frac{\partial \phi}{\partial x} = 0 = -\omega + kU = 0$$

We need $\omega = kU$ for stationary phase:
Deep water waves $\omega^2 = |\boldsymbol{k}|g$ $\boldsymbol{k} = (k, l, 0)$ and we also have $|\boldsymbol{c}_g| = \frac{1}{2}|\boldsymbol{c}_p| = \frac{1}{2}\frac{\omega}{|\boldsymbol{k}|}$
but $\omega = kW = U|\boldsymbol{k}|\cos\theta = \frac{1}{2}U\cos\theta$

Therefore for a ship moving on the surface of the water we get a angle of $\tan\alpha = \frac{\cos\theta\sin\theta}{2-\cos^2\theta}$ between the point the wave has propogated from and the ship for waves transmitted with angle θ.

9 Lecture 9

This is calculated by knowing that the group velocity is $\frac{1}{2}U\cos\theta$ so the waves make a circle of radius $\frac{1}{2}U\cos\theta$ with its centre UT behind the ship. We have a whole lot of these circles for waves produced at different times and we want to figure out the envolope that contains them all so we want to look for the maximium angle by differentiaing this value to get $\theta_{max} = \frac{1}{2}\cos^{-1}\frac{1}{3}$.

Something very similar happens in the case of internal waves

9.1 Stationary phase for internal waves

Imagine you stand on top of an isolated mountain with wind going past at speed U. As before:

$$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\phi = 0$$
$$\omega = kU$$

This time $\omega = N\cos\theta = N\frac{k}{|\boldsymbol{k}|}$ in 2D so we have

$$|\boldsymbol{k}| = \frac{N}{U}$$

for the phase to be stationary. We can also remember that we have

$$|\boldsymbol{c}_g| = \frac{N}{|\boldsymbol{k}|}\sin\theta, |\boldsymbol{c}_p| = \frac{N}{|\boldsymbol{k}|}\cos\theta$$

Won't be an exam question on 3D stationary waves.

9.2 Shear Flows

9.2.1 Sheared base state

$$\boldsymbol{u} = U\hat{\boldsymbol{x}} + \boldsymbol{u}'$$

Linearise about $U\hat{\boldsymbol{x}}$:

$$\frac{\partial b}{\partial t} + U\frac{\partial b}{\partial x} = -w' N^2$$

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + \frac{dU}{dz}w' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\nabla \cdot \boldsymbol{u}' = 0$$

Combining all this gives:

$$\left((\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})(\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial z^2}) + N^2\frac{\partial^2}{\partial x^2} - U''(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\frac{\partial}{\partial x} \right) w' = 0$$

In the limit $U = 0$:

$$(\frac{\partial^2}{\partial t^2}\nabla^2 + N^2\nabla_H^2)w' = 0$$

as before.
If we choose a frame where the flow is steady with $\frac{\partial}{\partial t} = 0$ then

$$(\frac{\partial^2}{\partial x^2}\nabla^2 + (\frac{N^2}{U^2} - \frac{U''}{U})\frac{\partial^2}{\partial x^2})w' = 0$$

Provided w' is bounded at infinity:

$$(\nabla^2 + \frac{N^2}{U^2} - \frac{U''}{U})w' = 0$$

Consider the case:

$$w' = \tilde{w}e^{i(kx+mz)}$$

Consider easier case with no curvature $U'' = 0$ then

$$(-(k^2 + m^2) + \frac{N^2}{U^2})w' = 0$$

For non-trivial solution $|\boldsymbol{k}| = \frac{N}{U}$

9.2.2 Critical layer reflection

Let's return to ray tracing to show how we can use this. In the simple case we have already looked at with a single mountain with a constant speed wind. We know that this gives quarter circles enclosed in semicircular casuality envolopes. Lets consider a steady state so when the wind has been blowing forever. The phase velocity is along the lines of constant phase which is necessary for them to be stationary as then you cannot see the phase move.

If we want to know how the position of one of these rays is moving with time we need to consider
$$\frac{dZ}{dx} = \frac{c'_{gz}}{c'_{gx}} = \frac{c_{gz}}{c_{gx} + U} = \frac{m}{k}$$
and as $|\boldsymbol{k}|^2 = k^2 + m^2 = \frac{N^2}{U^2}$.
We take the assumption that k is preserved as we move alon a ray. However, if $U \neq$constant or $N \neq$const then we must have $m \neq$ const. Therefore, the orientation of $\boldsymbol{k}$ changes and $|\boldsymbol{k}|$ changes:

$$m^2 = \frac{N^2}{U^2} - k^2$$

Need to take negative square root as the phase velocity is perpendicular to the group velocity which has positive vertical motion. So

$$\frac{dZ}{dx} = \frac{m}{k} = \frac{(\frac{N^2}{U^2} - k^2)^{\frac{1}{2}}}{|\boldsymbol{k}|} = ((\frac{N}{kU})^2 - 1)^{\frac{1}{2}}$$

as k is also negative.

If $\frac{N}{kU} = 1$ then the ray stops propogating vertically upwards and so cannot propogate through height at which $\frac{N}{kU} = 1$. Consider $N =$ const and U increases with height. Clearly at the level of the mountain we produce waves as we can always find some k that works as $\frac{N}{kU} > 0$. So waves will always be produced for an isolated mountain. So the rays will slowly bend round until at a certain height z_c (where $\frac{N}{kU} = 1$) they are horizontal.

10 Lecture 10

Near this critical height z_c we can do a taylor series expansion to find out what happens if it isn't quite attained.

$$\frac{dZ}{dx} = ((Z - z_c)\frac{d}{dz}(\frac{N^2}{k^2U^2}))^{\frac{1}{2}}$$

So:
$$Z = z_c + \frac{1}{4}\frac{d}{dz}(\frac{N^2}{k^2U^2})|_{z=z_c}(x - x_c)^2$$

This will give us a quadratic behaviour where the waves rise up to z_c then the reflect back down in a parabolic curve. The lines of constant phase are prependicular to this parabolic c_g. So the waves go from having almost horizontal phase lines and then completely vertical phase lines. We have been a bit naughty here as we have made use of the WKB approximation but we need to ask if this is a valid approximation.

While $U(z)$ may change over a length scale that is large comapred with $\lambda = \frac{2\pi}{|k|}$, the vertical extent of the wave is going towards infinity as $m \to 0$ as $z \to z_c$. So at the top we are saying the wave crests are vertical so at the top there must be some variation in velocity across the wave crest. Luckily whilst it is not clear that WKB is valid it works well in practice. Approximations often work well even when they have no right to do so.

Important to note that the height at which this reflection occurs depends on the wavenumber of the waves, so a spectrum of waves will reflect at differing heights.

10.0.1 Critical layer absorption

This is the opposite case to the above. What happens if $\frac{U}{N}$ reduces with height and at some point reduces to zero. This time we can think about $\frac{dz}{dx} = \frac{(\frac{N^2}{U^2} - k^2)^{\frac{1}{2}}}{|\boldsymbol{k}|} = \frac{m}{k}$ so as $U \to 0$, we get $m \to \infty$ and $|\boldsymbol{k}| \to \infty$. This means that this time the ray is going to bend upwards and become vertical at $U = 0$, and also as we become more and more vertical the wave number increases and so the wave crests get closer and closer together (this is because k remains the same as m increases). As $|\boldsymbol{k}| \to \infty$, $|c_g| = \frac{N}{|\boldsymbol{k}|} \sin\theta \to 0$ and $|c_p| = \frac{N}{|\boldsymbol{k}|} \cos\theta \to 0$. We also remember that the dissapation of the waves (the change of energy per wavelength) will be $d\tilde{E}d\zeta = e^{-k^3 \nu \zeta} N \sin\theta$ so the energy will be dissapating out very quickly which is refered to as critical layer absorption. Critical layer absorption occurs when U changes sign.

We aren't covering 3D stuff this year or columnar wave modes which were focused on in previous year. Part of the reason for this is it is more or less impossible to ask exam questions on as it is too algebraically messy.

10.1 Resonant triads

10.1.1 Weakly nonlinear internal waves

Q1 on Example Sheet 1 goes part of the wave towards getting the background for this. Here we should that the Linear wave solution satisfies the nonlinear equations but you cannot use linear superposition as addition of linear wave solutions does not solve non-linear equations.

$$\boldsymbol{u} = \tilde{\boldsymbol{u}}_1 e^{i\phi_1} + \tilde{\boldsymbol{u}}_2 e^{i\phi_2} + \tilde{\boldsymbol{u}}_3 e^{i\phi_3}$$

$$\phi_1 = \boldsymbol{k}_1 \cdot \boldsymbol{x} - \omega t, ...$$

$$b = \tilde{b}_1 e^{i\phi_1} + \tilde{b}_2 e^{i\phi_2} + \tilde{b}_3 e^{i\phi_3}$$

Linear terms in equations, we are going to allow $\tilde{\boldsymbol{u}}_i = \tilde{\boldsymbol{u}}_i(t)$ to be a slowly varying function allowing time dependance in phase and amplitude:

$$\frac{\partial}{\partial t}(\boldsymbol{u}) = (\dot{\tilde{\boldsymbol{u}}}_1 - i\omega_1\tilde{\boldsymbol{u}}_1)e^{i\phi_1} + \dot{\tilde{\boldsymbol{u}}}_2 - i\omega_2\tilde{\boldsymbol{u}}_2)e^{i\phi_2} + \dot{\tilde{\boldsymbol{u}}}_3 - i\omega_3\tilde{\boldsymbol{u}}_3)e^{i\phi_3}$$

We are going to take the case with no viscosity and no mass diffusivity $\nu = \kappa = 0$. So in the linear equations this would end up with $\dot{\tilde{\boldsymbol{u}}}$ would vanish but because we have the non-linear terms this time they might not.

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = i(\tilde{\boldsymbol{u}}_1\cdot\boldsymbol{k}_1)\tilde{\boldsymbol{u}}_1 e^{2i\phi_1} + i(\tilde{\boldsymbol{u}}_2\cdot\boldsymbol{k}_2)\tilde{\boldsymbol{u}}_2 e^{2i\phi_2} + i(\tilde{\boldsymbol{u}}_3\cdot\boldsymbol{k}_3)\tilde{\boldsymbol{u}}_3 e^{2i\phi_3} + i((\tilde{\boldsymbol{u}}_1\cdot\boldsymbol{k}_2)\tilde{\boldsymbol{u}}_2 + (\tilde{\boldsymbol{u}}_2\cdot\boldsymbol{k}_1)\tilde{\boldsymbol{u}}_2)e^{i(\phi_1\pm\phi_2)} + ... e^{i(\phi_2\pm\phi$$

The first self interactions just reperesent sort of local forcing but the cross over terms are much more interesting as they show how three different waves interact.

Triadic resonance condition to allow a sustatined interaction between $\tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_2, \tilde{\boldsymbol{u}}_3$ over sustained time and space. Order for this to happen the cross over terms must feedback into the part of the interaction not included in themselves so must have $\pm\phi_4 = \pm\phi_1 \pm \phi_2$. So if we put two waves into the system they will produce a third waves in additoin to the forcing waves. As there are two waves you could produce a new wave we need to further consideration to descern which of the potential waves are sustained.

The two waves generated have: $\pm\phi_3 = \pm\phi_1 \pm \phi_2$ so $\pm\boldsymbol{k}_3 = \pm\boldsymbol{k}_1 \pm \boldsymbol{k}_2$ and $\omega_3 = \pm\omega_1 \pm \omega_2$
For sustained interaction need:

$$\frac{\omega_1}{N} = \frac{k_1}{|\boldsymbol{k}_1|} = \cos\theta_1$$

$$\frac{\omega_2}{N} = \frac{k_2}{|\boldsymbol{k}_2|} = \cos\theta_2$$

$$\frac{\omega_3}{N} = \frac{k_3}{|\boldsymbol{k}_3|} = \cos\theta_3$$

This means if we are going to draw things on wavenumber diagram only part of the space is going to marry up.

11 Example sheet 1

Linerised equation of motion with non zero visocity and mass diffusion:

$$(\frac{\partial}{\partial t} = \kappa\nabla^2)(\frac{\partial}{\partial t} - \nu\nabla^2)\nabla^2\psi = -N^2\frac{\partial^2\psi}{\partial x^2}$$

which is derived by straight away combining:

$$\frac{\partial u}{\partial t} = \frac{\partial b}{\partial x} + \nu\nabla^2 u, \frac{\partial b}{\partial t} = -N^2 w - \kappa\nabla^2 b$$

In general $\frac{D}{Dt}$ and $\frac{\partial}{\partial x}$ do not commute in general.

If it doesn't say derive the dispersion relation you can just write it down. So worth memorising some dispersion relationships like the deep water one $\omega^2 = gh \tanh kH$. I made mistake of thinking the dipersion relation did not change so need to practice deriving this dispersion relation.

Marks questions out of 40.

Define potential energy for deep water as:

$$\bar{PE} = PE_0 + \int_{-H}^{\eta} \rho g z dz$$

we can define reference state to be $PE_0 = -\int_{-H}^{0} \rho g z dz$ so

$$\bar{PE} = \int_{0}^{\eta} \rho g z dz = \frac{1}{4}\rho_0 g \eta_0^2$$

When determining the structure of a decaying wave we can take the integration constant to be zero, as we want solutions oscillating around zero?? Email the guys about this.
Won't set anything as algebraically messy as question 5 in the exam again. Try quesiton 5 again i completely didn't get it at all!

He uses a completely different technique to me for figuring out the reflection of waves. His is a geometric appoarch where you consider the continuity at the boundary.

Question 8 is also bizarre, seems to be using techniques from fluids II to do with boundary layers. You are meant to assume the form $\eta = \eta_0 e^{i\omega t} e^{-(1+i)\epsilon/\delta}$ try it again with this.

12 Lecture 11

We force wtih ϕ_1, ϕ_2 and nonlinearly generate ϕ_3:

$$\phi_3 = \pm\phi_1 \pm \phi_2$$

$$\omega_3 = N\frac{k_3}{|\boldsymbol{k}_3|}$$

Triadic resoncance transfer term Take $\psi = \tilde{\psi}_1 e^{i\phi_1} + \tilde{\psi}_2 e^{i\phi_2} + \tilde{\psi}_3 e^{i\phi_3}$. Also take $\kappa = 0$ and ν is small. Assume $\tilde{\psi}_j$ are functions of t. Now lets just look at the non linear $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ terms and relate them to the linear terms $\frac{\partial \boldsymbol{u}}{\partial t}$ then we find:

$$\dot{\tilde{\psi}}_1 = I_1 \tilde{\psi}_2 \tilde{\psi}_3 - \frac{1}{2}\nu |\boldsymbol{k}_1|^2 \tilde{\psi}_1$$

I_1 is in the typeset version of the notes but he doesn't want us to memorise it just have an idea that it exists. We are going to take this body and use the euler equations for this body, this gives a mecahnical analogy:

Euler equations for a rigid body:

$$J\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (J\boldsymbol{\omega}) = \boldsymbol{M}$$

wtih $\boldsymbol{J}$ is the Inertia tensor and $\boldsymbol{M}$ is the applied torques. Take

$$J = \begin{pmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_{33} \end{pmatrix}$$

with $J_{11} < J_{22} < J_3$.

If $\boldsymbol{M} = 0$ then conserve energy and we have:

$$\boldsymbol{\omega} \cdot J\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \cdot (\boldsymbol{\omega} \times (J\boldsymbol{\omega})) = 0$$

$$J_{11}\dot{\omega_1} + (J_{33} - J_{22})\omega_2\omega_3 = 0$$
$$J_{22}\dot{\omega_2} + (J_{11} - J_{33})\omega_3\omega_1 = 0$$
$$J_{33}\dot{\omega_3} + (J_{22} - J_{11})\omega_1\omega_2 = 0$$

If $|\omega_1| >> |\omega_2|, |\omega_3|$ then $J_{11}\dot{\omega_1} = 0$ and $J_{22}\dot{\omega_2} + (J_{11} - J_{33})\omega_3\omega_1 = 0$ ...
Elminate ω_3 to get:

$$J_{22}J_{33}\ddot{\omega_2} - (J_{11} - J_{33})(J_{22} - J_{11})\omega_1^2\omega_2 = 0$$

There is a similar result for $\ddot{\omega_3}$, the RHS coefficent can be easily seen to be less than 0 so this gives harmonic oscillation so if pertubred it will oscillate slightly but be pretty stable. Similarly if $|\omega_3| >> |\omega_1|, |\omega_2|$ we get harmonic osciallation. However, for $|\omega_2| >> |\omega_1||\omega_3|$ we get:

$$J_{11}J_{33}\ddot{\omega_1} - (J_{33} - J_{22})(J_{22} - J_{11})\omega_2^2\omega_1 = 0$$

so ω_2 will grow (and decay) exponentially if perturbed so this is an instability.

12.0.1 Triadic resonance instability

$$|\tilde{\psi_1}| >> |\tilde{\psi_2}||\tilde{\psi_3}|$$

We have oscillatory behaviours in the central regions of the sustained graph of solutions, and exponentially growing solutions on the top and bottom. The TRI is a linear instability of internal waves when we consider weakly non-linear behaviour.

It would be too hard to write an exam question on this area, he tried to write an exam question on this last year and it was too algebraically nasty to do any more than we just covered.

We want to figure out what the growth rate actually looks like for a plane wave on the expoentially growing portion of the curve. We can do this analytically to figure out:

$$\sigma_{\pm} = -\frac{1}{4}\nu(|\boldsymbol{k}_2|^2 + |\boldsymbol{k}_3|^2) \pm \sqrt{\frac{1}{16}\nu^2(|\boldsymbol{k}_2|^2 + |\boldsymbol{k}_3|^2 + I_2 I_3|\tilde{\psi}_1|^2}$$

we have $\sigma_+ \geq 0$ if $|\tilde{\psi}_1| > 0$. We actually find that the maximium of the growth rate occurs when $\omega_2 = -\omega_3 = \frac{1}{2}\omega_1$. THis has been known for a while and is known as the parametric subharmonic instability (PSI). In practice we don't tend to see this as in practice we are consdiering beams of waves rather than plane waves.

13 Lecture 12

13.1 Shallow water

There is a strong analogy between shallow water flow and flows in incompressible fluids. We can write down the rules for incompressible fluid flow:

$$\nabla \cdot \boldsymbol{u} = 0, \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

If we throw away a dimension we get the shallow water flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \neq 0$$

though in the shallow water flow the flow is still 3D.
In the examples sheet 1 we derived the dispersion of interfacial waves to be:

$$\omega^2 = (\rho_1 - \rho_2)gk(\frac{\rho_1}{\tanh kH_1} + \frac{\rho_2}{\tanh kH_2})^{-1}$$

We are going to be interested in one specific limit of this dispersion relation but first lets consdier a different limit. The short wave limit $kH_1, kH_2 >> 1$ so $\tanh kH_i \to 1$ this gives $\omega^2 = \frac{\rho_1-\rho_2}{\rho_1+\rho_2}gk = Agk$. Therefore in the limit $A \to 1$ we recover the deep water dispersion relation $\omega^2 = gk$.

Introduce the 'reduced gravity' $g' = \frac{\Delta\rho}{\bar{\rho}}g = 2Ag$ so we get:

$$\omega^2 = \frac{1}{2}g'k, c_p = \sqrt{\frac{Ag}{k}} = \sqrt{\frac{g'}{2k}}, c_g = \frac{1}{2}c_p$$

We can clearly see from graphs of height against phase velocity and frequency that there are two clear limits. One for short waves (where the frequency depends strongly on the wavelength) and one for when one hiehgt is much bigger than the other for long waves (the long wave limit).

$kH_1, kH_2 << 1$ so $\tanh kH_i \to kH_i$ so

$$c_p = c_g = \sqrt{\frac{H_1 H_2}{H_1 + H_2} g'}$$

can think of $H_E = \frac{H_1 H_2}{H_1 + H_2}$ as an effective depth. We only need one of kH_1 or kH_2 to be small for waves to become non-dispersive. Let H_s be the depth of the shallow layer and H_d be the depth of the deep layer with $\frac{H_s}{\rho_s} << \frac{H_d}{\rho_d}$ so we get $c_p = c_g = \sqrt{H_s g'}$ with $g' = \frac{\rho_1 - \rho_2}{\rho_s} g$.

In a lot of what we are going to be doing we are going to be taking the BOussinesq limit of $\Delta\rho << \bar{\rho}$ so $g' = \frac{\Delta\rho}{\bar{\rho}} g$ therefore:

$$\omega" = \frac{g' k}{\coth kH_1 + \coth kH_2}$$

with $\coth kH_i \to \frac{1}{kH_i}$ for $kH_i << 1$ and $\coth kH_i \to 1$ for $kH_i >> 1$. To start with mainly consider one shallow layer to make the mathematics easier.

Equipartion of energy

$$\bar{E} = \bar{KE} + \bar{PE}, \bar{F_E} = c_g \bar{E}$$

13.1.1 Shallow water equations

We have just considered linear waves but we need to think of non-linear waves to be useful so need to derive the shallow ater equations.

Consider a shallow body of water with height h and horizontal scale X with $h << X$.

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla(p + \rho g z) + \nu \nabla^2 \boldsymbol{u}$$

for homogenous fluid. Take scales U, W, X, Z, P, T with:

$$\frac{U}{X} + \frac{W}{Z} = 0$$

in shallow water $\frac{X}{Z} \to \infty$ and so $\frac{W}{U} \to 0$

$$\frac{W}{T} + \frac{UW}{X} + \frac{W^2}{Z} = \frac{1}{\rho}\frac{P}{Z} + g + \nu(\frac{W}{X^2}\frac{W}{Z^2})$$

as the vertical pressure gradient is predominantly hydrostatic so the other terms become small and $\frac{1}{\rho}\frac{P}{Z} \sim g$ so:

$$\frac{\partial p}{\partial x} \approx -\rho g p = p_0 - \rho g z$$

and

$$\frac{\partial w}{\partial z} << g$$

now look at horizontal momentum equation:

$$\frac{U}{T} + \frac{U^2}{X} + \frac{WU}{Z} = \frac{\rho g Z}{\rho X} + \nu(\frac{U}{X^2} + \frac{U}{Z^2})$$

can neglect the viscoity term as we are dealing with high reyonlods number. Physcailly insight allows us to neglect the $\frac{WU}{Z}$ term as we can see that W should be small as it is a shallow flow so what would be generating a large velocity up or down. this gives:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial}{\partial x}(p + \rho g z) + \nu\frac{\partial^2 u}{\partial z^2}$$

From first prinicples:
We now cover how to do this from first principles as often we will be asked to derive the shallow water equations for bizarre channels like triangluar tubes or something and this gives us an insight into how to do that.

Take a flat bottom, unit width, slowly varying depth of water with $u(x,t)$ under the hydrostatic limit and $Re >> 1$.

The volume flux is given by $Q = uh$ so $2\delta x\frac{\delta h}{\delta t} = Q_{x-\delta x} - Q_{x+\delta x}$. So as $\delta t \to 0$ we get:

$$\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0$$

If cross-section of channel is not rectangular, then this expression changes.Here the cross-sectional area is just h times unit width but if say a triangular channel then the cross-sectional area would be scaling with h^2 instead.

We can also consider the momentum flux and the pressure forces. So the pressure force towards the right on the right hand face

$$\int_0^{h-\frac{\partial h}{\partial x}\delta x} \rho g(h - \frac{\partial h}{\partial x}\delta x - z)dz = \rho(\frac{h^2}{2} - h\frac{\partial h}{\partial x}\delta x) + O(\delta x^2)$$

we also get:

$$\frac{\partial}{\partial t}(uh) + \frac{\partial}{\partial x}(u^2 h + \frac{1}{2}gh^2) = 0$$

with $F_M = u(uh) = uM$ for 1D shallow water. Note the simularityes with

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) = 0$$

which is 1 D compressible. So you can sse that h and ρ play similar roles.

In Example sheet 2, we vary $b(x, z)$ as the wdith of the channel. THis is still a 1D shallow awter flow provided $\frac{u^2}{b}\frac{d^2b}{dx^2} << g$

Boussineq vs non-Boussinesq:

$$g' = \frac{\rho_1 - \rho_2}{\frac{1}{2}(\rho_1 + \rho_2)} = \frac{\Delta\rho}{\bar{\rho}}g$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g'\frac{\partial h}{\partial x} = 0$$

For most things a single-layer shallow ater flow (Boussinesq flow) and a single-layer free surface flow, the equations are the same except for $g' <-> g$. This will not be the case where the shallow water assumptions are violated - we will look at some of these.

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One instance of this not working is if the lower layer is deep as then we could plausibly have much lower velocities in the lower layer than the upper layer. If we were to take the pressure being lower in the lower layer then we would end up having lhigher pressure where the upper layer is larger which would force the pressure to be funnneled in the direction of the narrowest upper layer. If the upper layer has a free surface or a surface with say air then it can rise up slightly allowing the pressure to be unifrom along the boundary with the deep layer below.

Averaging
$\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{u}', \boldsymbol{u} = \boldsymbol{u}(x, y, z, t)$ with $\bar{\boldsymbol{u}} = \bar{\boldsymbol{u}})(x, t)$ and $h = h(x, t)$ with:

$$\bar{\boldsymbol{u}} = \frac{1}{A}\int_{-b/2}^{b/2}\int_0^h \boldsymbol{u}dzdy$$

wtih $A = bh = A(x, t)$. If we take the diveregence equation:

$$\int_{-b/2}^{b/2} \nabla \cdot \boldsymbol{u}dzdy = 0$$

Consider:

$$\int_{-b/2}^{b/2} \frac{\partial w}{\partial z}dzdy = \int_{-b/2}^{b/2} w|_{z=h} - w_{z=0}dy$$

using the kinematic boundary condition

$$w(z = h) = \frac{\partial h}{\partial t} + u|_{z=h}\frac{\partial h}{\partial x}$$

gives

$$b\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\int_{-b/2}^{b/2} u|_{z=h}dy = 0$$

similar result holds for $\frac{\partial v}{\partial y}$ so these together gives:

$$\int_{-b/2}^{b/2} \nabla \cdot \boldsymbol{u} dzdy = bh\frac{\partial \bar{u}}{\partial x} + \bar{u}(b\frac{\partial h}{\partial x} + h\frac{db}{dx}) + b\frac{\partial h}{\partial x} = 0$$

If we do the same for the momentum equation:

$$\int\int_A \frac{\partial u}{\partial t} + (\boldsymbol{u}\cdot\nabla)udA$$

which gives:

$$\frac{\partial \bar{b}}{\partial t}+\bar{u}\frac{\partial u}{\partial x}+g\frac{\partial h}{\partial x}-\nu\frac{\partial^2 \bar{u}}{\partial x^2} = \nu[\frac{1}{b}(\overline{\frac{\partial u'}{\partial y}}|_{y=\frac{b}{2}}-\overline{\frac{\partial u'}{\partial y}}|_{y=-\frac{b}{2}})+\frac{1}{h}(\overline{\frac{\partial u'}{\partial z}}|_{z=h}-\overline{\frac{\partial u'}{\partial z}}|_{z=0})]-(u'\frac{\bar{\bar{\partial}} u'}{\partial x}+v'\frac{\bar{\bar{\partial}} u'}{\partial y}+w'\frac{\bar{\bar{\partial}} u'}{\partial z})$$

The equations for $\bar{u}$ requre a knowledge of the equations for $\bar{u'^2}$. This is generally referred to as the closure problem.

At high Re, then ν may be negligible but $\bar{\bar{u_i'u_j'}}$ may not be.
We can try to model the unknown terms on the RHS by consdiering the scaling of the different terms with $\frac{\nu}{h}\frac{\partial u'}{\partial z} \sim \frac{\nu\bar{u}}{h^2}$ and $\frac{\nu}{b}\partial u'\partial y \sim \frac{2\nu\bar{u}}{b^2}$. So could model the RHS as:

$$\nu[\frac{1}{b}(\overline{\frac{\partial u'}{\partial y}}|_{y=\frac{b}{2}} - \overline{\frac{\partial u'}{\partial y}}|_{y=-\frac{b}{2}}) + \frac{1}{h}(\overline{\frac{\partial u'}{\partial z}}|_{z=h} - \overline{\frac{\partial u'}{\partial z}}|_{z=0})] = C_L\nu\frac{\bar{u}}{h^2}(1+2\frac{h^2}{b^2})$$

where C_L is an O(1) laminar drag coefficent. In the slow viscous flow course we could find that $C_L = \frac{2}{3}$ for Re ¡¡1 and $b >> h$. However for high Re, ν is neglible and we can think about the other terms:

$$-(u'\frac{\bar{\bar{\partial}} u'}{\partial x} + v'\frac{\bar{\bar{\partial}} u'}{\partial y} + w'\frac{\bar{\bar{\partial}} u'}{\partial z}) \approx C_t\frac{\bar{u}|\bar{u}|}{h}(1+2\frac{h}{b})$$

The reason for this form instead of just $\bar{u}^2$ is we want it to retrarding the flow so we need to retain the sign. The simplest way we can model this is is with C_T is an O(1) turbulent drag coefficent that dependson roughness of boundaries. For smooth boundaries it can be small $C_T \approx 0.03$ for rough boundaries it is roughly 0.1. Often $h << b$ so $C_t\frac{\bar{u}|\bar{u}|}{h}$. Dropping overbars.

$$\frac{\partial h}{\partial t} + b^{-1}\frac{\partial}{\partial x}(bhu) = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = -C_L\nu\frac{u}{h^2} - C_T\frac{u|u|}{h}$$

Do Q13 and 17 on ES2.

14.1 Hyperbolic system

These equations support wavelike solutions of the form:

$$F_i(x - c_i t)$$

where c_i is the wave speed and $F(..)$ expresses the evolution following a wave. The equations we are looking at are quasi linear.
Recap linearity:
LInear equations: $a(x,t)\frac{\partial u}{\partial x} + b(x,t)\frac{\partial u}{\partial t} = c(x,t)u + d(x,t)$
Semi-linear equations: $a(x,t)\frac{\partial u}{\partial x} + b(x,t)\frac{\partial u}{\partial t} = c(x,t,u)$
Quasi-linear equations: $a(x,t,u)\frac{\partial u}{\partial x} + b(x,t,y)\frac{\partial u}{\partial t} = c(x,t,u)$

14.1.1 A model for traffic flow

We have a road with speed depending only on the local traffic density $u = u(\rho)$. So can easily write down continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(u\rho) = 0$$

Now lets just imagine we have got helicopters flying over the traffic proving traffic reports with some speed λ so: $x = x_0 + \lambda s$, $t = t_0 + s$. The helicopter will see:

$$\frac{df}{ds} = \frac{dt}{ds}\frac{\partial f}{\partial t} + \frac{dx}{ds}\frac{\partial f}{\partial x}$$

as $\frac{d}{ds} = \frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial x}$ and so $\frac{d\rho}{ds} = \frac{\partial \rho}{\partial t} + \lambda\frac{\partial \rho}{\partial x}$.

Imagine $u(\rho) = 1 - \rho$ then we will get a quadratic flux. Also our continuity equation would give us:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}((1-\rho)\rho) = 0 = \frac{\partial \rho}{\partial t} + (1 - 2\rho)\frac{\partial \rho}{\partial x} = 0$$

If the pilot flies at speed $\lambda = 1 - 2\rho$ then $\frac{d\rho}{ds} = 0$ and so $\rho = \text{const}$. So λ is the characteristic speed of the problem. We have reduced the pde to ode by choosing this moving view point. When characteristics intersect we get a shock. These two different traffic densities are producing the same flux as we still must have continuity so we have:

$$\rho_l(u_l - u_s) = \rho_r(u_r - u_s)$$

so in this particular case of constant traffic desnities on each side of the shock we get:

$$u_s = 1 - (\rho_r + \rho_l) = \frac{1}{2}(\lambda_r + \lambda_l)$$

15 Lecture 14

15.1 Shallow water characteristic treatment

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{x} + h\frac{\partial u}{\partial x} = 0$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$
$$x = x_0 + \lambda s, t = t_0 + s$$

with stopwatch time s

$$x = x_0 + \zeta, t = t_0 +^{-1} \zeta$$

with tape-measure distance ζ. As we have $\frac{\partial}{\partial t} = \frac{d}{ds} - \lambda\frac{\partial}{\partial x}$ so our equations transform to:

$$\frac{dh}{ds} + (u-\lambda)\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0$$
$$\frac{du}{ds} + (u-)\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

multiply the first by $u-$ and the second by h and add them together to get:

$$(u-\lambda)\frac{dh}{ds} - h\frac{du}{ds} + ((u-\lambda)^2 - gh)\frac{\partial h}{\partial x} = 0$$

To make into ODE set $(u-\lambda)^2 - gh = 0$ which occurs when $= u \pm \sqrt{gh} = u \pm c$ as $\sqrt{gh} = c_p = c_g = c$ for long waves on stationary fluid.

Now lets do the same manipulations a bit differently using matrix formulism:

$$\begin{pmatrix} u & h \\ g & u \end{pmatrix}\begin{pmatrix} h \\ u \end{pmatrix}_x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} h \\ u \end{pmatrix}_t = 0$$

$$A\boldsymbol{v}_x + B\boldsymbol{v}_t = \boldsymbol{f}$$

Using transformation $\frac{\partial}{\partial t} = \frac{d}{ds} - \lambda\frac{\partial}{\partial x}$:

$$\begin{pmatrix} h \\ u \end{pmatrix}_s + \begin{pmatrix} u & h \\ g & u \end{pmatrix} pmatrixhu$$

$_x - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} h \\ u \end{pmatrix}_x = 0 Wethenpremultiplyby\boldsymbol{q}^T$ and we want to chose $\boldsymbol{q}$ so it eliminates the RHS quantitiy

$$\boldsymbol{q}^T(A - \lambda B) = \boldsymbol{0}^T$$

Therefore we can think about the **general approach**. In general we have:

$$A\boldsymbol{v}_x + B\boldsymbol{v}_t = \boldsymbol{f}$$

These are quasi linear as $A = A(x, t, \boldsymbol{v}), B = B(x, t, \boldsymbol{v}), f = f(x, t\boldsymbol{v})$. Consider a linear combination of equations:

$$\boldsymbol{q}^T A \boldsymbol{v}_x + \boldsymbol{q}^T B \boldsymbol{v}_t = \boldsymbol{q}^T \boldsymbol{f}$$

The observer $\boldsymbol{v}_s = \lambda \boldsymbol{v}_x + \boldsymbol{v}_t$. Take a linear combination of these as well:

$$\boldsymbol{m}^T \boldsymbol{v}_s = \boldsymbol{m}^T (\lambda \boldsymbol{v}_x + \boldsymbol{v}_t) =?$$

Compare these two equations it would make sense to equate the following:

$$\lambda \boldsymbol{m}^T = \boldsymbol{q}^T A, \boldsymbol{m}^T = \boldsymbol{q}^T B$$

Then we can eliminate $\boldsymbol{m}$ from these coupled equations to give;

$$\boldsymbol{q}^T A = \boldsymbol{q}^T B \iff \boldsymbol{q}^T (A - B) = \boldsymbol{0}^T$$

This is our generalised left-hand eigenvalue problem. For a non-trial solution we need:

$$|A - \lambda B| = 0$$

This is how we chose λ. We can now go back in the other direction as we also know that :

$$\boldsymbol{m}^T \boldsymbol{v}_s = \boldsymbol{q}^T \boldsymbol{f}$$

and

$$\boldsymbol{m}^T = \frac{1}{\lambda} \boldsymbol{q}^T A = \boldsymbol{q}^T B$$

so

$$\boldsymbol{q}^T (A \boldsymbol{v}_s - \lambda f) = 0, \boldsymbol{q}^T (B \boldsymbol{v}_s - \boldsymbol{f}) = 0$$

If we very simply sketch what was happening to our simple shallow water, our eigenvalue problem is:

$$\boldsymbol{q}^T \begin{pmatrix} u - \lambda & h \\ g & u - \end{pmatrix} = \boldsymbol{0}^T$$

So need

$$|A - B| = (u - \lambda)^2 - gh = 0 \implies \lambda = u \pm c, c = \sqrt{gh}$$

We also have:

$$\begin{pmatrix} 1 & q \end{pmatrix} \begin{pmatrix} \pm c & h \\ g & \pm c \end{pmatrix} = \boldsymbol{0}^T$$

so $\boldsymbol{q}^T = \begin{pmatrix} 1 & \pm \frac{c}{g} \end{pmatrix}$

$$\boldsymbol{q}^T (A \boldsymbol{u}_s - \lambda \boldsymbol{f}) = (1 \pm \frac{c}{g})(A \boldsymbol{u}_s - \lambda \boldsymbol{0}) = 0$$

Use $h = \frac{c^2}{g}$ and solving this gives:

$$0 = \begin{cases} (u + c) \frac{d}{ds}(2c + u) & \lambda = u + c \\ (u - c) \frac{d}{ds}(2c - i) & = u - c \end{cases}$$

$u \pm 2c = \text{const on } \lambda = u \pm c$

Higher-order equations:

$$\frac{\partial^2 \eta}{\partial t^2} + \gamma \frac{\partial^2 \eta}{\partial x^2} = 0$$

Let $v = \frac{\partial \eta}{\partial x}$ and $w = \frac{\partial \eta}{\partial t}$ then:

$$\frac{\partial v}{\partial t} - \frac{\partial w}{\partial x} = 0$$

$$\frac{\partial w}{\partial t} - \gamma \frac{\partial v}{\partial x} = 0$$

for $\lambda = \pm \gamma^{\frac{1}{2}}$.

15.1.1 Implications of being hyperbolic

What makes things hyperbolic? Well we required that $|A - \lambda B| = 0$ and it is hopefully obvious that if λ is representing a trajectory in space time then we need real eigenvalues.

$$\boldsymbol{q}^T (B \boldsymbol{u}_s - \boldsymbol{f}) = 0$$

If we want to form the ODE along these characteristics we need to know the eigenvectors so we can form two ODEs. So we need a complete set of eigenvectors. There will be some ocassions with repeated eigenvalues but still have a full set of eigenvectors. If we cannot find a full set of eigenvalues the problem is not hyperbolic. The eigen values (velocity) are telling us how the information is propogating. There is an anaolgy between shallow water and compressible gas: For compressible gas we ahve the Mach number:

$$M = \frac{\text{velocity}}{\text{speed of sound}}$$

In the case of shallow water the equivalent is the Froude number :

$$F_r = \frac{\text{fluid velocity}}{\text{wave speed}} = \frac{u}{c} = \frac{u}{\sqrt{gh}}$$

We are cosndiering if the information can propogate in two directions or just one direction. If the fluid is moving fast enough then the waves will only every propogate in the direction of the flow.

If $|F_r| < 1$ then waves can propogate in both directions (subcritical flow)
If $|F_r| > 1$ the waves can only propogate in one direction (supercritical flow)
If $|F_r| = 1$ then flow is critical so either $\lambda^+ = u + c$ or $\lambda_= u - c$ are zero. One (or both) of the the waves do not propogate relative to the observer.

Imagine we know the initial speed and depth at two points then we can follow the characteristics leaving these points and meeting at a point. We can find the new speed and depth by considering the conservation along these characteristics :

$$u_1 + 2c_1 = u_a + 2c_a, u_1 - 2c_1 = u_b - 2c_b$$

so

$$u_1 = \frac{1}{2}(u_a + u_b) + (c_a - c_b)$$

$$c_1 = \frac{1}{4}(u_a - u)b + \frac{1}{2}(c_a + c_b)$$

Zone of dependance fo u_1, c_1 is everything between u_a, c_a and u_b, c_b. The zone of influce is future times between $\lambda_+ = u + c$ and $\lambda_- = u - c$ characteristics.

15.1.2 The Saint-Verant dam-break problem

This is an example of the Riemann problem.