

# Quantum Information Theory

oliverobrien111

July 2021

## 1 Lecture 1

### 1.1 Internal waves

#### 1.1.1 Minimal maths version

Imagine we have some sort of stratification that is given by density profile  $\hat{\rho}$ . We are going to require that this is smooth and as we will see later and it is only changing the gradient over a length scale that is large compared to the length scale of the waves. Consider a small volume  $V$  of fluid of density  $\rho_0$  and I will magically displaced it up by a distance of  $\zeta$  (assuming it retains volume, density and shape). There will clearly be a buoyancy force  $B$ , and for small displacements:

$$B = gV\zeta \frac{d\hat{\rho}}{dz}$$

Newton's second law gives

$$\rho_0 V \ddot{\zeta} = B = gV\zeta \frac{d\hat{\rho}}{dz}$$

$$\ddot{\zeta} + \left(0 \frac{g}{\rho_0} \frac{d\hat{\rho}}{dz}\right) \zeta = 0$$

This will clearly have solutions:

$$\zeta = A \cos Nt + B \sin Nt$$

where  $N$  is the buoyancy (or Brunt-Vaisala frequency)  $N = \sqrt{-\frac{g}{\rho_0} \frac{d\hat{\rho}}{dz}}$

There is a key ingredient that hasn't been taken into account in this treatment. There hasn't been anything about continuity e.g. how does the fluid get out the way when it falls back down. To highlight this if we displace a long slim slab of fluid instead of a sphere. If this long slim slab is vertical and we displace it upwards then we get exactly the same maths, but we can make it thin enough that we don't need to worry about continuity to the first order. Now what happens if we take a long thin slab of fluid at an angle and move it along itself. Would it fall vertically downwards or slide back along itself. It

is intuitively hard to fall downwards as this would cause a continuity issue of needing to compress all of the fluid below the line (in order to fall down a lot of fluid needs to be moved past the line which would make a big pressure difference resiting the motion). So the slab will slide back along itself. This will have only displaced each parcle of fluid by  $\zeta \cos \theta$  upwards and it will only experience a force of  $g \cos \theta$  back along its original path. Therefore this would give:

$$\ddot{\zeta} + N^2 \cos^2 \theta \zeta = 0$$

$$\ddot{\zeta} + \omega \zeta = 0$$

This gives the dispersion relation for internal gravity waves:

$$|\omega/N| = |\cos \theta|$$

This logic works if we stack slabs on top of each other and as long as we only move each on by a small amount. They can all oscillate up and down along themselves but have no need to be in the same phase, so you could send a wave though them all perpendicular to the slabs. This would mean each slab is a line of constant phase with energy being transmitted along the slab.

Now lets think about the  $\cos \theta$ . If  $\mathbf{k} = (k, l, m)$  is a vector perpendicular to the slabs, then we have:

$$\cos \theta = \frac{\sqrt{k^2 + l^2}}{\sqrt{k^2 + l^2 + m^2}}$$

### 1.1.2 More rigorous derivation

$$\nabla \cdot \mathbf{u} = 0$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \rho = \kappa \nabla^2 \rho$$

For now  $\kappa = 0$ :

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p - \rho g \hat{z} + \rho \nu \nabla^2 \mathbf{u}$$

For now  $\nu = 0$ . Take  $\rho = \rho_0 + \rho'$ . Linearise with  $\rho' \ll \rho_0$  and Boussinesq  $|\frac{\nabla \mathbf{u}}{\nabla t}| \ll g$ . The Boussinesq means that the  $\rho'$  contributes to the gravity term but not to the inertial term. We can rewrite the equations using reduced gravity to make this clear  $g' = g \frac{\rho - \rho_0}{\rho} = g \frac{\rho'}{\rho}$ :

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho_0} \nabla(p + \rho g z) - g' \hat{z}$$

To linearise we take  $\rho = \hat{\rho}(z) + \rho'(x, t)$  and  $\mathbf{u}' \sim \eta \omega \ll \frac{\omega}{|k|} \implies |k| \eta \ll 1$ . By combining this with our Boussinesq condition we have  $|\nabla \rho'| \ll |\frac{d\hat{\rho}}{dz}|$ . This gives the navier stokes equation:

$$\frac{\partial \rho'}{\partial t} + w \frac{d\hat{\rho}}{dz} = \frac{\partial \rho'}{\partial t} - w \frac{\rho_0}{g} N^2 = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0} \nabla(p_0 + \hat{p} - p') - \frac{\hat{\rho} + \rho'}{\rho_0} \mathbf{z} = -\frac{1}{\rho_0} \nabla(p_0 + \hat{p}) - g \frac{\hat{\rho}}{\rho_0} \hat{\mathbf{z}} - \frac{1}{\rho_0} \nabla p' - g \frac{\rho'}{\rho_0} \hat{\mathbf{z}}$$

Unperturbed state gives the first two terms as they are much larger than the other terms

$$0 = -\frac{1}{\rho_0} \nabla(p_0 + \hat{p}) - g \frac{\hat{\rho}}{\rho_0} \hat{\mathbf{z}}$$

therefore

$$p_0 + \hat{p} = - \int g \hat{\rho} dz$$

Let

$$b = -g \frac{\rho'}{\rho_0}$$

Therefore:

$$\frac{\partial b}{\partial t} = -w N^2 \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0} \nabla p' + b \hat{\mathbf{z}} \quad (2)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (3)$$

## 1.2 Vorticity

:  $\zeta = \nabla \times \mathbf{u}$  We are going to deal in 2D as it is easier. In 2-D voricity can be expressed interms of the streamfunction:

$$\zeta = -\nabla^2 \psi, \boldsymbol{\psi} = \psi \hat{\mathbf{y}}, \mathbf{u} = \nabla \times \boldsymbol{\psi}$$

Take curl of momentum equation to remove pressure:

$$\frac{\partial \zeta}{\partial t} = -\hat{\mathbf{z}} \times \nabla b$$

$$(\hat{\mathbf{z}} \times \nabla) \cdot (\hat{\mathbf{z}} \cdot \nabla w) = \nabla_H^2 w$$

so gives voricity equation:

$$(\nabla^2 \frac{\partial^2}{\partial t^2} + N^2 \nabla_H^2) w = 0 \quad (4)$$

Pose a plane wave type solution ansatz and see what happens. Let :

$$w(\mathbf{x}, t) = \text{Re}(\hat{w}(z) e^{i(kx + ly - \omega t)})$$

$$\frac{d^2 \hat{w}}{dz^2} + (k^2 + l^2) \left( \frac{N^2}{\omega^2} - 1 \right) \hat{w} = 0$$

So if we let  $m^2 = (k^2 + l^2) \left( \frac{N^2}{\omega^2} - 1 \right)$  then:

$$\hat{w} = \text{Re}(A e^{imz} + B e^{-imz})$$

If  $\omega > N$  then  $m$  is imaginary let  $\gamma = \sqrt{1 - \frac{N^2}{\omega^2}}$ :

$$w = (\hat{A}e^{-\gamma k_h z} + \hat{B}e^{\gamma k_h z})e^{i(kx+ly-\omega t)}$$

This is sort of showing how the velocity field changes with depth away from a disturbance on the surface. If we have  $N = 0$  then this is just a surface wave. If we have  $0 < N < \omega$  the  $\gamma$  is just giving a vertical rescaling of the behaviour beneath the surface wave. This means if we produce a sinusoidal disturbance with a frequency bigger than the buoyancy frequency then the disturbance looks like potential flow, as we increase the stratification of a fluid it will decrease the decay rate of that motion as we move away from that boundary. The limiting case is when we reach  $\omega = N$  the entire water depth is moving in phase and with the same magnitude as the surface.

In the case  $\omega < N$  we have  $m$  is real so we get sinusoidal variations in the vertical direction:

$$w = w_0 e^{ikx+ly+mz-\omega t} = w_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$$

$$\phi = \mathbf{k} \cdot \mathbf{x} - \omega t$$

so

$$w = w_0 e^{i\phi}$$

We want to get an idea of the relationships between the different parameters. To start with consider continuity:

$$\nabla \cdot \mathbf{u} = 0 \implies \frac{\partial u}{\partial x} + \frac{w}{\partial z} = 0$$

So

$$u = \int \frac{\partial w_0 e^{i\phi}}{z} dx = -\frac{m}{k} w_0 e^{i\phi} = -\frac{\tan \theta}{\cos \theta} w_0 e^{i\phi}$$

Considering surface variation  $\eta(x, t) = \tilde{\eta}(x, t)e^{i\phi}$  therefore by differentiating this and matching with  $u$  and  $w$  at the surface:

$$\mathbf{u} = \frac{\partial \eta}{\partial t} \implies w_0 = i\omega \cos \theta \tilde{\eta}$$

Now considering the relationship arising from the buoyancy equation:

$$\frac{b}{\partial t} = -wN^2 \implies i\omega \tilde{b} = -wN^2$$

$$b = -\eta \frac{\omega^2}{\cos \theta} e^{i\phi} = -\eta \omega N e^{i\phi}$$

Now consider the momentum equation:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p'}{\partial x} \implies \tilde{p} = i \frac{\omega N}{|\mathbf{k}|} \eta \sin \theta$$

### 1.3 Wave velocities

#### Phase velocity

$$\phi = \mathbf{k} \cdot \mathbf{x} - \omega t = k_i x_i - \omega t$$

The below identity is very obviously zero:

$$\frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x_i} = 0$$

$$k_i \frac{\partial \phi}{\partial t} + \omega \frac{\partial \phi}{\partial x_i} = 0$$

Divide across by  $k_i$ :

$$\frac{\partial \phi}{\partial t} + \frac{\omega}{|k|^2} k_i \frac{\partial \phi}{\partial x_i} = 0$$

Therefore,  $c_p = \frac{\omega}{|k|^2} \mathbf{k}$  as:

$$\frac{\partial \phi}{\partial t} + (\mathbf{c}_p \cdot \nabla) \phi = 0$$

## 2 Lecture 3

#### Group velocity:

$$\frac{\partial^2 \phi}{\partial x_i \partial t} - \frac{\partial^2 \phi}{\partial t \partial x_i} = 0$$

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0$$

As  $\omega = \omega(k)$  we have:

$$\frac{\partial \omega}{\partial x_i} = \frac{\partial \omega}{\partial k_j} \frac{\partial k_j}{\partial x_i}$$

$$\frac{\partial k_j}{\partial x_i} = \frac{\partial^2 \phi}{\partial x_j \partial x_i} = \frac{\partial k_i}{\partial x_j}$$

Therefore:

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial k_j} \frac{\partial k_i}{\partial x_j} = 0$$

Therefore,  $c_g = \frac{\partial \omega}{\partial k_i}$  as:

$$\frac{\partial k_i}{\partial t} + \mathbf{c}_g \cdot \nabla k_i = 0$$

So the wavenumber vector is being advected outwards with the group velocity.

**Surface waves:**  $\omega = gk$ ,  $c_g = \frac{\partial \omega}{\partial k} \sqrt{gh} = \frac{1}{2} c_p$ ,  $c_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}}$ .

## 2.1 Superposition

$$\eta = \cos((k + \delta k)x - (+\delta\omega)t) + \cos((k - \delta k)x - (\omega - \delta\omega)t)$$

$$\eta = 2 \cos(\delta kx - \delta\omega t) \cos(kx - \omega t)$$

As  $\delta\omega = \frac{\partial\omega}{\partial k} \delta k$  for  $|\delta k| \ll |k|$  then

$$\eta = 2 \cos((x - \frac{\partial\omega}{\partial k}t)\delta k) \cos(kx - \omega t)$$

This is a wave and envelope speed of  $\frac{\partial\omega}{\partial k}$ .

## 2.2 Internal wave velocities

As  $\frac{\omega^2}{N^2} = \frac{k^2 + l^2}{|k|^2} = \cos^2 \theta$

$$\mathbf{c}_p = \frac{\omega}{|k|^2} \mathbf{k} = \frac{N(k^2 + l^2)^{\frac{1}{2}}}{|k|^{\frac{3}{2}}} \mathbf{k} = \frac{N|\cos \theta|}{|k|^2} \mathbf{k}$$

This is the polar coordinate equation for two circles touching at the origin at every  $\phi$  so they sort of form a torus.

Now lets look at the group velocity:

$$\mathbf{c}_g = \frac{\partial\omega}{\partial k_i} = \frac{1}{2\omega} \frac{\partial\omega^2}{\partial k_i} = \frac{\omega}{|k|^2} \left( \frac{N^2}{\omega^2} (\mathbf{k} - k_z \hat{\mathbf{z}}) - \mathbf{k} \right) = \frac{N|\sin \theta|}{|k|} \begin{pmatrix} \cos \phi \sin \theta \\ \sin \phi \sin \theta \\ -\cos \theta \end{pmatrix}$$

$$|c_g| = \frac{N}{|k|} |\sin \theta|$$

This means that in the horizontal direction the phase velocity is always perpendicular to the group velocity. They form the same circle just one with  $\sin \theta$  and one with  $\cos \theta$  and as  $\sin \theta = \cos(\pi/2 - \theta)$ . As the angles on a semicircle subtend 90 degrees we can sum the two and we will get the opposite side of the circle always. Therefore,

$$\mathbf{c}_p + \mathbf{c}_g = \frac{N}{|\mathbf{k}|} \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}$$

$$|\mathbf{c}_p + \mathbf{c}_g| = \frac{N}{|\mathbf{k}|}, c_{p,z} = -c_{g,z}, \mathbf{c}_p \cdot \mathbf{c}_g = 0$$

## 2.3 Equipartition of energy

$$\mathbf{u} \cdot (\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \nabla p' + \rho' g \mathbf{z}) = 0$$

Recalling that  $\frac{\partial \rho'}{\partial t} - w \frac{\rho_0}{g} N^2 = 0 \implies w = \frac{g}{\rho_0 N^2} \frac{\partial \rho'}{\partial t}$ , and the incompressibility condition to get:

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho_0 |\mathbf{u}|^2 + \frac{1}{2} \frac{g^2}{\rho_0 N^2} p'^2 \right) + \nabla \cdot (p' \mathbf{u}) = 0$$

If we go back to the start and consider the displacement of a packet of fluid by  $\zeta$  we have change in potential energy of:

$$\Delta PE = \int_{z_0}^{z_0 + \zeta} g \frac{d\hat{\rho}}{dz} (z - z_0) dz = \frac{1}{2} \rho_0 N^2 \zeta^2$$

$$\rho' = -\frac{d\hat{\rho}}{dz} \zeta = \frac{\rho_0}{g} N^2 \zeta$$

$$PE = \frac{1}{2} N^2 \rho_0 \zeta^2 = \frac{1}{2} \rho_0 \frac{b^2}{N^2}$$

If we want to consider the total energy equation:

$$\int_V \frac{\partial}{\partial t} (KE + PE) dV + \int_S p' \mathbf{u} \cdot \mathbf{n} dS' = 0$$

$\mathbf{F}_E = p' \mathbf{u}$  is the flux of energy.

Lets consider 2D:

$$u = \eta \omega \sin \theta \sin \phi, w = -\eta \omega \cos \theta \sin \phi, b = \eta \frac{\omega^2}{\cos \theta} \cos \phi, p' = \eta \rho_0 \frac{\omega^2}{|k|} \tan \theta \sin \phi$$

Substitute into kinetic energy:

$$KE = \frac{1}{2} \rho_0 (u^2 + w^2) = \frac{1}{2} \rho_0 \omega^2 \eta^2 \sin^2 \phi$$

$$\bar{KE} = \frac{1}{4} \rho_0 \omega^2 \eta^2$$

$$PE = \frac{1}{2} \rho_0 \omega^2 \eta^2 \cos^2 \phi$$

$$\bar{PE} = \frac{1}{4} \rho_0 \omega^2 \eta^2$$

So you have equipartition of energy for linear waves  $\bar{KE} = \bar{PE}$ . We can also write down an expression for the flux of energy:

$$\mathbf{F}_E = p' \mathbf{u} = \rho_0 \omega^2 \eta^2 \sin^2 \phi \frac{N}{|k|} \sin \theta \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix} = \frac{1}{2} \rho_0 \omega^2 \eta^2 \mathbf{c}_g = \bar{E} \mathbf{c}_g$$

### 3 Lecture 4

#### 3.1 Oscillating cylinder

We are interested in the case where the oscillation of the cylinder  $a$  is much smaller than the diameter. You might think this is sufficient to make the waves linear, but it is not as we have these delta functions on the singularities on the tangent planes to the cylinder (so we will always have the amplitudes being large compared to the wave lengths here but we will ignore this). If everything is at rest to start with it will take a bit of time for the oscillations to propagate out into the whole space. As  $|c_g| = \frac{N}{|k|} \sin \theta$  the area of which is influenced by the oscillation will look like two causality envelopes (circles of increasing size touching at the centre). The waves form a st. andrews cross pattern with the waves being bi modal near the cylinder and unimodal further away this is due to viscosity

##### 3.1.1 Decay along a beam

We are going to take it being 2D and the buoyancy frequency  $N = 1$  and the mass dispersion  $\kappa = 0$  but include viscosity  $\nu \neq 0$ .

$$\begin{aligned}\nabla \cdot \mathbf{u} &= 0 \\ \frac{\partial u}{\partial t} + \frac{1}{\rho_0} \frac{\partial p}{\partial x} &= \nu \nabla^2 u \\ \frac{\partial w}{\partial t} + \frac{1}{\rho_0} \frac{\partial p}{\partial z} - b &= \nu \nabla^2 w \\ \frac{\partial b}{\partial t} + N^2 w &= 0\end{aligned}$$

To make our lives easier we are going to use a streamfunction:

$$\boldsymbol{\psi} = (0, \psi, 0), \mathbf{u} = (\nabla \times \boldsymbol{\psi}) e^{-i\omega t} = \begin{pmatrix} -\frac{\partial \psi}{\partial z} \\ 0 \\ \frac{\partial \psi}{\partial x} \end{pmatrix} e^{-i\omega t}$$

Take  $\zeta$  and  $\xi$  to be displacement in the wavevector  $k$  direction and then group velocity  $c_g$  direction. let  $\theta$  be the angle of the group velocity from the vertical which is the same as the angle of the cross. Therefore

$$\begin{aligned}\zeta &= x \cos \theta - z \sin \theta \\ \xi &= x \sin \theta + z \cos \theta\end{aligned}$$

Boynancy equation becomes:

$$\frac{\partial b}{\partial t} + \frac{\partial \psi}{\partial \zeta} \cos \theta + \frac{\partial \psi}{\partial \xi} \sin \theta = 0$$



**Vorticity:**  $\nabla \times \mathbf{u} = -\nabla^2 \psi e^{-i\omega t}$

$$-i\omega\left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \zeta^2}\right) - \frac{\partial b}{\partial \xi} \sin \theta - \frac{\partial b}{\partial \zeta} \cos \theta - \nu \nabla^2 \nabla^2 \psi = 0$$

Let  $b = (b_0 + \epsilon b_1 + \dots)e^{-i\omega t}$  and  $\psi = (\psi_0 + \epsilon \psi_1 + \dots)$ : We take small viscosity to make equations nice e.g. small  $\nu = 2\epsilon$  and  $\epsilon = \frac{1}{2}\nu$  as this is dimensional it is not clear what we mean by small parameter. As this is dimensional it means the dimensions of  $\psi_0$  and  $\psi_1$  are going to be different. This is not the ideal way of doing this but it is going to allow us to see more clearly what is happening.

$$\chi = \frac{\epsilon}{\sin \theta} \xi$$

Here  $\epsilon$  being small means we are interested in gradual changes in the direction of the group velocity but fast changes in the wave vector direction so we keep that sinusoidal behaviour.

$$\frac{\partial}{\partial \xi} = \frac{\partial \chi}{\partial \xi} \frac{\partial}{\partial \chi} = \frac{\epsilon}{\sin \theta} \frac{\partial}{\partial \chi}$$

Plug these into the buoyancy and vorticity equation and compare terms of the same order:

$$\begin{aligned} \epsilon_0 : \frac{\partial \psi_0}{\partial \zeta} &= -b_0, \quad \frac{\partial^2 \psi_0}{\partial \zeta^2} = i \frac{\partial b_0}{\partial \zeta} \\ \epsilon_1 : \omega \frac{\psi_1}{\partial \zeta} - i\omega b_1 &= -\frac{\partial \psi_0}{\partial \chi}, \quad i\omega \frac{\partial^2 \psi_1}{\partial \zeta^2} + \omega \frac{\partial b_1}{\partial \zeta} = i \frac{\partial^2 \psi_0}{\partial \zeta \partial \chi} - 2 \frac{\partial^4 \psi_0}{\partial \zeta^4} \end{aligned}$$

Can eliminate the LHS of both of these to give:

$$\begin{aligned} \frac{\partial^4 \psi_0}{\partial \zeta^4} &= i \frac{\partial^2 \psi_0}{\partial \zeta \partial \chi} \\ \frac{\partial^3 \psi_0}{\partial \zeta^3} &= i \frac{\partial \psi_0}{\partial \chi} + f(\chi) \end{aligned}$$

For a point sources as  $|\zeta| \rightarrow \infty$  then we expect  $\psi \rightarrow \psi_0 = F(\zeta)G(\chi) \frac{F''}{F} = i \frac{G'}{G} = -ik^3 G(\chi) = e^{-k^3 \chi} F(\zeta) = e^{ik\zeta} \psi \approx \psi_0 e^{-i\omega t} = A e^{-k^3 \chi} e^{i(k\zeta - \omega t)} \mathbf{k} = (k, 0)$  in  $(\zeta, \xi)$

$$\chi = \frac{\epsilon}{\sin \theta} \xi = \frac{\nu}{2 \sin \theta} \xi = \frac{\nu}{2N \sin \theta} \xi$$

for any  $N$ . If we take a whole spectrum of linear superposition of waves  $A(k)$ :

$$\psi = e^{-i\omega t} \int_{-\infty}^{\infty} A(k) \exp(ik\zeta - \frac{\nu k^3}{2N \sin \theta} \xi) dk \quad (5)$$

If we think about our cylinder with the two delta functions at the edge of the cylinder. As the fourier transform of the delta function is just a constant. So  $A(k)$  would just be a constant so the higher wavenumber modes would decay

more rapidly due to the  $k^3$  in the exponential. We want it to be clear that the length scale over which the day is happening is small compared to the length scale of the oscillations. Lets imagine we are going to scale  $\xi$ :

$$\frac{\nu k^2}{2N \sin \theta} k\xi$$

Recall that  $|c_g| = \frac{N}{|k|} \sin \theta$  so

$$\frac{\nu k^2}{2N \sin \theta} k\xi = \frac{\nu k}{2} \frac{1}{|c_g|} k\xi = -\pi Re^{-1} k\xi$$

With Reynolds number  $Re = \frac{\lambda |c_g|}{\nu}$  with  $\lambda = \frac{2\pi}{k}$ . We were requiring  $\epsilon$  to be small we were really requiring the Reynolds number to be large enough.

### 3.2 Mass diffusivity

$$(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla) \rho = \frac{D\rho}{Dt} = \kappa \nabla^2 \rho$$

Lets think about what could cause a difference in the density in the fluid e.g.  $S$  salt concentration, moisture content or temperature  $T$ . Carbon dioxide that we are breathing out is denser than the other air we are breathing out, which is more or less balanced by the humidity of the exhaled breath which is higher than the surroundings. Water vapour is less dense than air. The diffusivity of salt and the diffusivity of temperature are different so we can't write down an equation like above. What we can write down is equations of the diffusivity of these two:

$$\frac{DS}{Dt} = \kappa_s \nabla^2 S, \frac{DT}{Dt} = \kappa_T \nabla^2 T$$

For water  $\frac{\kappa_T}{\kappa_s} \approx 100$ . If you move hot salty water down into cold nonsalty water then it quickly becomes cold salty water so it rapidly becomes more dense than the fluid around it. Like wise if you took a parcel of cold fresh up into the hot area it will stay fresh but rapidly heat and so will be less dense than its surroundings and want to rise. This is called salt fingering. Equally if you have cold fresh water above hot salty water. As temperature diffuses relatively quickly the gradient will be much steeper for the salt concentration than the temperature, this creates convection. This is called double-diffusive convection.

Prandtl number:  $\frac{\nu}{\kappa_T}$

Schmidt number:  $\frac{\nu}{\kappa_S}$  with  $\kappa_S$  is the diffusivity of some thing like salt

### 3.3 Reflections of waves

With light the angle of incidence is equal to the angle of reflection.