Quantum Information Theory

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July 2021

1 Lecture 1

1.1 Path Integrals in Quantum Mechanics

Goal is to reformulate Schrodingers equation as a path integral.

$$\hat{H}(\hat{x},\hat{p}) \text{ with } [\hat{x},\hat{p}] = i\hbar$$

Assuming $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$. The schrodinger picture is:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

so

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(t)\rangle$$

Wavefunction: $\Psi(x,t) = \langle x| |\psi(t)\rangle$. We want to solve schordingers equation for this wavefucniton in a way that introduces the path integral:

$$\Psi(x,t) = \langle x| |\psi(t)\rangle = \langle x| e^{-\hat{H}t/\hbar} |\psi(0)\rangle$$

$$\Psi(x,t) = \int_{-\infty}^{\infty} K(x,x_0;t)\Psi(x_0,0)$$

where

$$K(x,x_0;t) = |x\rangle e^{-i\hat{H}t/\hbar} |x_0\rangle$$

is what we want a path integral expression for

Lets consider n intermediate times/positions. Let $0 = t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} = T$:

$$e^{-i\hat{H}T/\hbar} = e^{-i\hat{H}(t_{n+1}-t_n)/\hbar - i\hat{H}(t_1-t_0)/\hbar}$$

Insert identity: $I = \int dx_r |x_r\rangle \langle x_r|$:

$$K(x,x_0;t) = \int_{-\infty}^{\infty} \left(\prod_{r=1}^{n} dx_r \langle x_{r+1}| e^{-i\hat{H}(t_{r+1}-t_r)/\hbar} |x_r\rangle \right) \langle x_1| e^{-i\hat{H}(t_1-t_0)/\hbar} |x_0\rangle$$

Consider fixed $V(\hat{x}) = 0$. $K_0(x, x'; t) = \langle x| e^{-i\frac{\hat{p}^2}{2m\hbar}t} |x'\rangle$. Insert the identity: $I = \int \frac{dp}{2\pi\hbar} |p\rangle \langle p|$.

$$K_0(x, x'; t) = e^{\frac{im(x-x')^2}{2\hbar t}} \sqrt{\frac{m}{2\pi i\hbar t}}$$

For $V(\hat{x}) \neq 0$, we need very small time steps. Separate kinetic and potential parts (Suzuki-Trotter decomposition). Take $t_{r+1} - t_r = \delta t$ to be small and n large so $n\delta t = T$ (constant).

$$e^{-i\hat{H}\delta t/\hbar} = \exp(-\frac{i\hat{p}^2\delta t}{2\pi\hbar}) \exp(-\frac{iV(\hat{x})\delta t}{\hbar})(1 + O((\delta t)^2))$$

The last term here is vanishingly small under the Broker-Campbell-Henshorff thingy. Between any 2 position eigenstates:

$$\langle x_{r+1}| e^{-i\hat{H}\delta t/\hbar} |x_r\rangle = e^{-iV(x_r)\delta t/\hbar} K_0(x_{r+1}, x_r; \delta t)$$

Putting all these pieces together:

$$K(x, x_0; T) = \int [\prod_r dx_r](\frac{m}{2\pi i\hbar\delta t})^{\frac{n+1}{2}} \exp(i\sum_{r=0}^{n}[\frac{m}{2\hbar}(\frac{x_{r+1} - x_r}{\delta t})^2 - \frac{1}{\hbar}V(x)]\delta t) \tag{1}$$

In the limit $n \to a, \delta t \to 0$ the exponent becomes $\frac{1}{\hbar}\int_0^T dt[\frac{1}{2}m\dot{x}^2 - V(x)] = \int_0^T dt\mathfrak{L}(x, \dot{x})$. So this is classical action in the limit.

We have no found a path integral (functional integral)

$$K(x, x_0; T) = \langle x| e^{-i\hat{H}T/\hbar} |x\rangle = \int \mathcal{D}x e^{iS/\hbar}$$

where

$$\mathcal{D}x = \lim_{\delta t \to 0, n\delta t = T} \sqrt{\frac{m}{2\pi i\hbar\delta t}} \prod_{r=1}^{n} (\sqrt{\frac{m}{2\pi i\hbar\delta t}} dx_r)$$

One way of considering the classic limit is taking $\hbar \to 0$. For $e^{iS/\hbar}$ thisincreases the phases/frequencies. The Riemann-Lebesque lemma implies tha thte smallest frequncy (i.e. the path whihc minimises S dominates the integral). As smallest S is the hamiltonians principle of least action so this is equivalent to the classical treatment.

Another way is for $\hbar \neq 0$ the QM amplitude is the sum of all paths each weighted by phase $e^{iS/\hbar}$. This gives the interfence patterns we see in double sli, which is just represented by some classical line with no further diagrams.etc.

One trick we are going to play is dealing in imaginary time. You can analytically continue to imaginary time. Let $\tau = it$, then $\langle x| e^{-\hat{H}t/\hbar} |x_0\rangle = \int \mathcal{D}x e^{-S/\hbar}$. Here the $\hbar \to 0$ argument is much more clear as the smallest value of S will dominate in the limit. Analogy with statistical mechanics where $e^{-S/\hbar}$ is a botlzmann factor, and $\int \mathcal{D}x$ is the sum over microstates. These (with real exponentials) converge. Not all quantum questions can be answer in imaginary time. e.g. if there is a causality relationship between the initial and final space as we have convereted from mincovski to euclian.

2 Lecture 2

We showed that in quantum mechanics a path integral over positions weighted by the classical action:

$$\int \mathcal{D}x e^{iS[x]/\pi}$$

this came from non-relativisitic quantum mechanics whre the position is an operator. As we saw in quantum field theory this mixed treatment of space as an operator and time as a label is not appropriate for satisfying lorentz invariance, so we demote x to be a label so that space and time are treated the same. So we work with the appropriate fields. For much of this course we will work with scalar fields and then will generalise to fermionic fields and gauge fields. QM is 0+1 dimensional field theory.

2.1 Integrals and their diagram attic expansions

Goal of next couple lectures is to show mathematics and show that they generate the same diagrams as in QFT (have to take it on a little bit of faith will become clear towards the end of the course). We supress the interesting relationships between space and tiem for the this chapter and just treat them as labels.

0-dimentsional field: $p : \{\text{point}\} \to \mathbb{R}$ q real variable

Path integrals as if in imaginary time (which makes the integrals better behaved as we get expoentially decaying factors rather than complex integrands):

$$Z = \int_{\mathbb{R}} d\phi e^{-S(\phi)/\hbar}$$

For the purposes here just assume it is well-behaved. So assume it is an even polynomial so that as $\phi \to \pm\infty$ we have $S[\phi] \to \infty$. Also look at expectation values:

$$< f(\phi) >= \frac{1}{Z} \int d\phi f(\phi) e^{-S(\phi)/\hbar}$$

This are sometimes refeerred to as correlation functions. Assume that f does not grow so much it overwhelms the expoential so this is well behaved.

Now write down action corresponding to the free field theory which we can write down exactly and then we will do one that needs pertubation theory expansion.

2.1.1 Free theory

say we have N real scalar fields (variables). Let $a, b \in [1, N]$:

$$S[\phi] = \frac{1}{2} m_{ab} \phi_a \phi_b = \frac{1}{2} \phi^T m \phi$$

with m symmetric and positive definite ($detm > 0$). If m is diagonal it would obviously be a mass term, but it could also couple nearest neighbours and so could contain a discrete approximation to a derivate (so could represent difference operators on a discrte lattice but this isn't important today).

We can diagonalise m with orthogonal matrices P as m is symmetric and positive definite:

$$m = P \Lambda P^T$$

Λ is diagonal with elements λ_c with $c \in [1, N]$. Let $\chi = P^T Q$, then:

$$Z_0 = \int d^N \phi \exp(-\frac{1}{2\pi} \phi^T m \phi) = \prod_c \sqrt{\frac{2\pi\hbar}{\lambda_c}} = \sqrt{\frac{(2\pi\hbar)^N}{detM}}$$

We will need to play some tricks when we do fermionic fields as they are antisymmetric not symmetric so you end up with the detminant in the numerator rahter than the denominator.

To go from the partition function Z_0 to correlation function $f^{\boldsymbol{n}}$ we introduce an external source J (with N compononents) and replace the action $S_0(\phi)$ with $S_0(\phi) - J^T \phi$.

$$Z_0(J) = \int d^N \phi \exp(-\frac{1}{2\hbar} \phi^T m \phi + \frac{1}{\hbar} J^T \phi)$$

Let $\tilde{\phi} = \phi - m^{-1} J$:

$$Z_0(J) = \int d^N \phi \exp(-\frac{1}{2\hbar} \tilde{\phi}^T m \tilde{\phi}) \exp(\frac{1}{2\hbar} J^T m^{-1} J)$$

This is called the generating function or generating functional as it generates the correlation functions.

Example: Generate '2-point' function:

$$< \phi_a \phi_b >= \frac{1}{Z_0(0)} \int d^N \phi \phi_a \phi_b \exp(-\frac{1}{2\hbar} \phi^T m \phi + \frac{1}{\hbar} J^T \phi)|_{J=0} = \frac{1}{Z_0(0)} \int d^N \phi (\hbar \frac{\partial}{\partial J_a})(\hbar \frac{\partial}{\partial J_b}) \exp(-\frac{1}{2\hbar} \phi^T m \phi -$$

Now the ϕ depedance is only in the exponential so can bring out derivatives:

$$< \phi_a \phi_b >= \hbar^2 \frac{\partial}{\partial J_a} \frac{\partial}{\partial J_b} Z_0(J)|_{J=0} = \hbar^2 \frac{\partial}{\partial J_a} \frac{\partial}{\partial J_b} Z_0(0) \exp(\frac{1}{2\hbar} J^T m^{-1} J)|_{J=0} = \hbar (m^{-1})_{ab}$$

We represent this as a line between two points a and b also called a propagator.
Example: '4-point' function

$$< \phi_b\phi_c\phi_d\phi_f >= \hbar^2[(m^{-1})_{bc}(m^{-1})_{bc} + (m^{-1})_{bd}(m^{-1})_{cf} + (m^{-1})_{bf}(m^{-1})_{cd}]$$

These represent the three ways of linking up 4 points with 2 lines. For $2k$ field in ¡¿ then there should be $\frac{(2k)!}{2^k k!} = \frac{\text{permutate all 2k points}}{\text{(permutte all points inside pairs) (permute pairs)}}$ diagrams.

2.1.2 Interacting theory

We just want to go beyond the action we have written down to something a bit more complicated. In cases where exact intergration is not possible. So we seek an expansion about a classical point, with small $\hbar$. Integrals don't end up being convergent they are asymptotic. e.g.

$$\int d^N\phi f(\phi)e^{-S/\hbar}$$

wont have a Taylor expansion about $\hbar = 0$. All is not lost we can still make progress. The expansion we are going to look at in many cases is asymptotic which means that the various terms in the series get to be better and better approximations to the full results at least up to a point.

$$I(\hbar) \sim \sum_{n=0}^{\infty} c_n\hbar^n$$

iff

$$\lim_{\hbar\to 0^+} \frac{1}{\hbar^N}|I(\hbar) - \sum_{n=0}^{N} c_n\hbar^n| = 0$$

Series missed out some terms $e^{-\frac{1}{\hbar^2}}$ so there are non-perturbative effects. We won't cover it in this course but these terms do contribute effects in some gauge theories. In some weakly coupled thoeries like QED this is a very good expansions, as shown by how we can very accurate measure magnetic moment of an electron to 10^{-10} accuracy.

3 Lecture 3

Last time we looked at

$$S(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 = S_0(\phi) + \frac{\lambda}{4!}\phi^4, m^2 > 0, \lambda > 0$$

Partition f^1 (generating function with respect to $J = 0$):

$$Z = \int d\phi e^{-S/\hbar}$$

Seperate this action into the free part and the interaction part and then expand around classical fields

$$Z = \int d\phi e^{-S_0/\hbar} \sum_{v=0}^{\infty} \frac{1}{v!} [(-\frac{\lambda}{4!\hbar})\phi^4]^v$$

Truncate, swap order of sum and integral to give an asymptotic expansion.

In 0-dimensions we can integrate exactly. Let $x = \frac{1}{2\hbar} m^2 \phi^2$:

$$Z = \frac{\sqrt{2\hbar}}{m} \sum_{v=0}^{N} \frac{\hbar}{V!} (-\frac{\hbar\lambda}{4!m^4})^v 2^v \int_0^{\infty} e^{-x} x^{2v+\frac{1}{2}-1} = \frac{\sqrt{2\hbar}}{m} \sum_{v=0}^{N} \frac{\hbar}{V!} (-\frac{\hbar\lambda}{4!m^4})^v 2^v \Gamma(2v+\frac{1}{2})$$

$$Z = \frac{\sqrt{2\hbar}}{m} \sum_{v=0}^{N} \frac{\hbar}{V!} (-\frac{\hbar\lambda}{m^4})^v \frac{1}{(4!)^v v!} \frac{(4v)!}{2^{2v}(2v)!}$$

Use stirlings approximation:

$$V! \sim e^{V \log V}$$

So this factorial growth is very fast and so will make this an asympotic expansion as it means the terms will not converge. We can see that $\frac{1}{(4!)^v v!}$ comes from expanding $\exp(-S_I/\hbar)$, whereas $\frac{(4v)!}{2^{2v}(2v)!}$ comes from the combinatoric ways of pairing the $4v$ fields.

3.0.1 Generating function

$$Z(J) = \int d\phi \exp(-\frac{1}{\hbar}[S_0(\phi) + S_I(\phi) - J\phi])$$

Taylor expand $e^{-S_I/\hbar}$ then replace the ϕ with $\hbar\frac{\partial}{\partial J}$ then pull it out and sum that infinite series

$$Z(J) = \exp(-\frac{1}{\hbar} S_I(\hbar\frac{\partial}{\partial_J})) \int d\phi \exp(-\frac{1}{\hbar}[S_0(\phi) - J\phi])$$

Drop multiplicative factor $\exp(-\frac{\lambda}{4!\hbar}(\hbar\frac{\partial}{\partial J})^4)\exp(\frac{1}{2\hbar}J^T M^{-1} J)$. Here J is just a signle variable $M = m^2$.

$$Z(J) = \sum_{v=0}^{N} \frac{1}{V!} [-\frac{\lambda}{4!\hbar}(\hbar\frac{\partial}{\partial J})^4]^v \sum_{p=0} \frac{1}{p!} (\frac{1}{2\hbar} J^T M^{-1} J)^p \quad (2)$$

We represent this double series by diagrams of propogators and vertices. A propogrator is a line connecting to field. The propogator is M^{-1} which for the moment is just a boring m^{-2}. At each vertex we have: $-\frac{\lambda}{\hbar}(\hbar\frac{\partial}{\partial J})^4$.

Check $Z(0)$. In order for a term to survive to be nonzero we have to match up the derivatives with the Js. When $J = 0$, need number of derivates to be equal to the number of sources. Generally, lets call these external sources. e.g. in the case of this field $E = 4v - 2p$ and we want $E = 0$. First nontrivial terms are $(v, p) = (1, 2)$ and $(2, 4)$. So:

$$Z(0) = 1 + \text{figure eight} + ...$$

Product rule in differentiation turns into symmetry factors or combinatorial factors associated with each diagram. Think about the "pre-diagram" with the half edges (corresponding to derivatives) and floating propogators (with ends corresponding to sources) and then we label the sources so there are 4! ways of assigning 4 derivates (or half edges) to 4 sources (at a, a', b, b').

The numerator A is cancelled by denominator F.

$$F = (v!)(4!)^v(P!)2^p$$

F accounts for all the permuations of all vertices $V!$, and each vertex's legs and all propogators $P!$ and both ends of the propogators 2!. After dividing A by F we get the symmetry factor. This is important to remove double counting. As some of the ways of assigning derivates to sources are equivalent. Looking for the number of distinct ways of mapping from the half edges to themselves whilst preserving the graph but creating a distigusible set of half edge assignations. Number of ways of changing the labels but keeping the same graph.

4 Lecture 4

Now we want to look at diagrams with external terms e.g. $E = 2$ terms in the idagrammatic expansion. In the generating function:

$$Z(J) > | + |o + |\infty + |^o_o + |oo...$$

We get both the vaccuum bubbles and the connected diagrams, so can factor out the vacuum bubbles:

$$Z(J) > (| + 8 + ...)(| + |o + ...)$$

When we go to calculate expectation values:

$$< \phi^2 >= \frac{\hbar^2}{Z(0)})\frac{\partial}{\partial J)^2} Z(J)|_{J=0} = \frac{1}{Z(0)}(| + |o + |8 + ...) = (| + |0 + |oo)$$

The generalization is straightforward

$$< \phi^4 >= || + X + |oo| + ...$$

4.1 Effective actions

In this section we show that we only need to consider the connected vaccum bubbles as the action only deals in the sums of connected bubbles rather than Z which is the sum of all vaccumm bubbles.

We will show that $W = -\frac{1}{\hbar}\log Z$ Wilson effective action is sum of connected vacuum diagrams. Any diagram D as products of connected diagrams $D = \frac{1}{S_D}\sum_I (c_I)^{n_I}$

$$\frac{Z}{Z_0} = \sum_{\{n_I\}} D = \prod_I \sum_{n_I} (c_I)^{n_I} = \exp(\sum_I c_I) = e^{-(W-W_0)/\hbar}$$

$$W = W_0 - \hbar \sum_I C_I$$

Introduce external sources

$$-\frac{1}{\hbar}W(J) = \log Z(J)$$

$$-\frac{1}{\hbar}\frac{\partial^2}{\partial J"}W|_{J=0} = \frac{1}{Z(0)}\frac{\partial^2 Z}{\partial J^2}|_{J=0} - \frac{1}{(Z(0)^2}(\frac{\partial Z}{\partial J}_{J=0})^2 = \frac{1}{\hbar}(<\phi^2> - <\phi>^2)$$

In our theory for even actions the second term is zero. Above you can see we are finding the two point functions and then subtracting off the disconnected one point functions.

Why is W "effective":
Consider a theory with 2 scalars ϕ and χ:

$$S(\phi, \chi) = \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4}\phi^2\chi^2$$

Feynmann rules:
Propogrators of $\frac{\hbar}{m^2}$ and $\frac{\hbar}{M^2}$, vertex rules of $-\frac{\lambda}{\hbar}$. Therefore, $-\frac{W}{\hbar}$ = sum of connected diagrams with two dashed lines and two solid lines in coming out of each vertex.

$$<\phi^2> = | + |0 + (|) + ...$$

Look in typed notes to see how this works it is too difficult to latex the diagrams. If we want to remove the χ field then. Define $W(\phi)$

$$e^{-W(\phi)/\hbar} = \int d\chi e^{-S(\phi,\chi)/\hbar}$$

Treat $\chi^2\phi^2$ term as a source for χ^2 ($J = -\chi^2$) so the correlation funciton of the ϕ fields:

$$<f(\phi)> = \frac{1}{Z}\int d\phi d\chi f(\phi) e^{-S(\phi,\chi)} = \frac{1}{Z}\int d\phi f(\phi) e^{-W(\phi)/\hbar}$$

As this is simple theory we can use exact integration:

$$\int d\chi e^{-S(\phi,\chi)/\hbar} = e^{-m^2\phi^2/2\hbar}\sqrt{\frac{2\pi\hbar}{M^2+\frac{\lambda\phi^2}{2}}}$$

$$W(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\hbar}{2}\log(1+\frac{\lambda}{2M^2}\phi^2) + \frac{\hbar}{2}\log\frac{M^2}{2\pi\hbar}$$

$$W(\phi) = (\frac{m^2}{2}+\frac{\hbar\lambda}{4M^2})\phi^2 - \frac{\hbar\lambda^2}{16m^4}\phi^4 + \frac{\hbar\lambda^3}{48M^6}\phi^6 + ...$$

$$W(\phi) = \frac{m_{eff}^2}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + .. + \frac{\lambda^6}{6!}\phi^6$$

5 Lecture 5

Now we do it perturbatively using diagrams. Treat $\frac{\lambda}{4}\phi^2\chi^2$ as a source term:

$$Z = \int d\phi e^{-m^2\phi^2/2\hbar}\int d\chi \exp(-\frac{1}{\hbar}(\frac{M^2}{2}\chi^2 - J\chi^2))$$

$J = -\frac{\lambda}{4}\phi^2$ This leads to the Feynman rules with the propogator of $\frac{\hbar}{M^2}$ and we have self interaction terms with $-\frac{\lambda\phi^2 2\hbar}{}$ from the χ^2 source term. Now we can find the efffective action (given by the sum of the connected diagrams):

$$W(\phi) = -\hbar(....)$$

(above is all the connected diagrams with more and more vertices each with two legs so effectively looks like the increasing roots of unity)

$$W(\phi) = \frac{m^2}{\phi^2}2 - \frac{1}{2}\frac{\hbar\lambda}{2M^2}\phi^2 - \frac{1}{4}\frac{\hbar\lambda^2}{4M^4}\phi^4 + \frac{1}{3!}\frac{\hbar\lambda^2}{8M^6}\phi^6 + .. = \frac{m^2}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + ...$$

This is the same result as before. Now we have a theory of how the ϕ interacts. Now lets do the calculation with the full theory where we keep the ϕ explictly. Using $W(\phi)$:

$$<\phi^2>= \frac{1}{Z}\int d\phi\phi^2 e^{-W(\phi)/\hbar} = | + |o + ... = \frac{\hbar}{m_{eff}^2} - \frac{\lambda_4\hbar^2}{2m_{eff}^6} + ...$$

This agrees with the full caculation from earlier. We have shown that given a theory of two fields we can intergrate out one of them to get a theory in just one field, this changes the degrees of freedom which therefore changes the coefficent.

Lets continue on with effective actions, now we want to introduce the quantum effective action

5.0.1 Quantum effective action

Define average field in the presence of an external source $<\phi>=\Phi=-\frac{\partial W}{\partial J}=\frac{\hbar}{Z(J)}\frac{\partial}{\partial J}\int d\phi e^{-(S-J\phi)/\hbar}$. Legendre transform is a transformation from treating the source J as the independant variable to treating the mean field Φ as the independant variable

$$\Gamma(\Phi)=W(J_\Phi)+\Phi J_\Phi$$

J_Φ is the J which gives the correct expression $\frac{\partial W}{\partial J}|_{J_\Phi}=-\Phi$. Lets find the derivative:

$$\frac{\partial\Gamma}{\partial\Phi}=\frac{\partial W}{\partial\Phi}+J_\Phi+\Phi\frac{\partial J_\Phi}{\partial\Phi}=-\frac{\partial W}{\partial J}_{J_\Phi}\frac{\partial J}{\partial\Phi}+J_\Phi+\Phi\frac{J_\Phi}{\partial\Phi}$$

$$\frac{\partial\Gamma}{\partial\Phi}=J_\Phi$$

5.1 $\Gamma(\Phi)$ and Feynmann diagrams

External lines have one free end, whereas internal lines have no free ends. A bridge is any internal line of a connected graph which if cut would make the graph disconnected. A connected graph is said to be one-particle irreducible (1PI) iff it has no bridges. We are interested in these single particle irreducible graphs. Statement that we want to prove is that $\Gamma(\Phi)$ sums the 1PI graphs of the theory. We might expand about $\Phi=\Phi_0(J=0)$. Let $\varphi=\Phi$:

$$\Gamma(\Phi)=\Gamma^{(0)}+\frac{1}{2}\Gamma^{(2)}\varphi^2+...+\frac{1}{n!}\Gamma^{(n)}\varphi^n$$

Treat Φ as we did ϕ earlier. The quantum path integral for Φ:

$$e^{-W_\Gamma(J)/g}\int d\Phi e^{-(\Gamma(\Phi)-J\Phi)/g}$$

g is fictious planck constant.

$$W_\Gamma(J)=\text{ sum of connected diagrams }=\sum_{l=0}g^lW_\Gamma^{(l)}(J)$$

In $g0$ limit $W_\Gamma(J)=W_\Gamma(J)=\Gamma(\Phi)-J\Phi$ "classical" action of the path integral ($W(J)$) which is the effective action of the original theory. $W(J)=-\hbar\log\int d\phi e^{-S(\phi)/\hbar}$

$$W(J)=-\hbar\log(\int d\phi e^{-(S(\phi)-J\phi)/\hbar)}$$

The sum of connected diagrams with rules derived from action $S(\phi)-J\phi$ can be obtained as the sum of tree diagrams (no loops only bridges) using $W_\Gamma(J)$ rules derived from action $\Gamma(\Phi)-J\Phi$. Therefore the diagrams in the original theory that don't contain any bridges have to be absorbed into the coefficents of the tree diagrams of $W_\Gamma(J)$.

In a theory with several scalar fields: $\phi_a \; a = 1, ..., N$:

$$< \phi_a \phi_b >_J^{conn} = < \phi_a \phi_b >_J - < \phi_a >< \phi_b > = -\hbar \frac{\partial^2 W}{\partial J_a \partial J_b} = -\hbar \frac{\partial}{\partial J_a}(\frac{\partial W}{\partial J_b}) = \hbar \frac{\partial}{\partial J_a} \phi_B = \hbar (\frac{\partial J_a}{\partial \Phi_b})^{-1} = \hbar(\frac{\partial}{\partial \Phi}$$

The point is that the two point function which we know in the original thepry working with action $S(\phi)$ we have to sum the diagrams with two external legs so it starts of as just a single propogator plus the loop contributions. This tells us that if we know the quantum effective action then we can just read off the two point function, which is just represented by some classical line with no further diagrams.

What is basically happening here is we can split every diagram into irreducible parts and each of them can be expressed as a single vertex with the correct them of external lines and will represent all the possible way it can be done like with loads of loops and stuff etc. so basically we can hugely simplify the number of diagrams. We have to figure out what the vertex is.

6 Lecture 6

Given

$$\Gamma(\Phi) = \Gamma^{(0)} + \frac{1}{2}\Gamma^{(2)}(\Phi - \Phi_0)^2 + ... + \frac{1}{n!}\Gamma^{(n)}(\Phi - \Phi_0)^n + ...$$

we have

$$< \phi_a \phi_b >_J^{conn} = \hbar (\frac{\partial^2 \Gamma}{\partial \Phi_a \partial \Phi_b})^{-1} = \hbar (\Gamma^{(0)})^{-1}$$

in ϕ^4 theory:

$$< \phi_a \phi_b >_J^{conn} = - - + -o- + -o-o- + -8- + -\theta- + ..$$

$$= - - + -IPI- + -IPI-IPI- + -IPI-IPI-IPI- = \frac{1}{1 - IPI}$$

intuitively the last step comes from recognising the geometric series. On the example sheet we will do the same for three point function:

$$< \phi_a \phi_b \phi_c >_J^{conn} = -\frac{1}{\hbar}(\frac{\partial^3 \Gamma}{\partial \Phi_d \Phi_e \partial_f})^{-1} < \Phi_a \Phi_d >< \Phi_b \Phi_e >< \Phi_c \Phi_f >$$

$$(\frac{\partial^3 \Gamma}{\partial \Phi_d \Phi_e \partial_f})^{-1} = \Gamma^{(3)} = - < \Phi_d \Phi_e \Phi_F >^{IPI}$$

This is still trivial as you cant imagine having a non trivial bridge so we go to one higher level . If we go to a four point function we could have a bridge within the internal interactions e.g. it could be two 2 particle interactions rather than a single 4 particle interaction. Remember every external line has a two point interaction 1PI on it as there is a particle coming in a one coming out. This section is not in the written notes. It is slightly discussed later on page 41.

6.1 Fermions

Now have anticommutation relations rather than commutation relations. We introduce the abstract conscept of Grassmann numbers which are anti-commuting variables.
n numbers $\{\theta_a\} a = 1, ..., n$ and they obey:

$$\theta_a \theta_b = -\theta_b \theta_a \implies \theta_a^2 = 0$$

for any scalar $\phi_b \in \mathbb{C}$

$$\theta_a \phi_b = \phi_b \theta_a$$

Functions can be expressed as finite sums:

$$f(\theta) = f + \rho_a \theta_a + \frac{1}{2!} g_{ab} \theta_a \theta_b + ... + \frac{1}{n!} h_{a_1 a_2 ... a_n} \theta_{a_1} \theta_{a_2} ... \theta_{a_n}$$

where $g_{ab}, ..., h_{a_1...a_n}$ etc. are anti-symmetric in their indicies. This series is finite as adding any more θs would give a $\theta_a^2 = 0$ and vanish the term. Note

$$(\theta_a \theta_b)(\theta_c \theta_d) = (\theta_c \theta_d)(\theta_a \theta_b)$$

Differentiation is defined to anti commute:

$$\frac{\partial}{\partial \theta_a} \theta_b + \theta_b \frac{\partial}{\partial \theta_a} = \delta_{ab}, \frac{\partial}{\partial \theta_a} (\theta_b F(\theta)) = \delta_{ab} F(\theta) - \theta_b \frac{\partial F}{\partial \theta_a}$$

Integration: For a single grassamann θ

$$F(\theta) = f + \rho \theta$$

Define $\int d\theta$ and $\int \theta d\theta$. We want to require translational invariance and consider what it means to intergate over the whole range of θ. So impose requirement that

$$\int d\theta (\theta + \eta) = \int \theta d\theta$$

for constant grossman variable η. This implies that we want $\int d\theta = 0$ and we want $\int \theta d\theta = 1$ (this includes a choice of normalization). These are called the "Berezin rules".

Integration by parts is simplified:

$$\int d\theta \frac{\partial}{\partial \theta} F(\theta) = 0$$

So we have intoduced the rules for one variable now lets consider n Grassmann variables θ_a. In this case the only non-vanishing integral have to have one and only one θ:

$$\int d^n \theta \theta_1 \theta_2 ... \theta_n = 1 \iff \int d\theta_n d\theta_{n-1} ... d\theta_1 \theta_1 \theta_2 ... \theta_n = 1$$

A key point is in general if we start commuting these indicies we are going to start picking up signs so

$$\int d^n\theta\theta_{a_1}\theta_{a_2}...\theta_{a_n} = \epsilon_{a_1 a_2 \ldots a_n}$$

Now let use consider a change of variables $\theta' = X_{ab}\theta_b$ then:

$$\theta'_a = X_{ab}\theta_b, X_{ab} \in \mathbb{C}$$

$$\int d^n\theta'\theta_{a_1}...\theta_{a_n} = X_{a_1b_1}...X_{a_nb_n}\int d^n\theta\theta_{b_1}...\theta_{b_n} = X_{a_1b_1}...X_{a_nb_n}\epsilon^{b_1\ldots b_n} = \det X\epsilon^{a_1\ldots a_n} = \det X\int d^n\theta\theta_{a_1}...\theta$$

therefore $d^n\theta = detXd^n\theta'$ which compared to scalars where we have $\phi' = Y\phi \implies d^n\phi = \frac{1}{\det Y}d^n\phi'$. So fermions give the converse relationship here to what you would expect from scalars.

6.2 Free fermion field theory

$d = 0$, 2 fields θ_1, θ_2. Need a scalar action, only non-constant action

$$S(\theta) = \frac{1}{2}A\theta_1\theta_2, A \in \mathbb{R}$$

$$Z_0 =\in d^2\theta e^{-S(\theta)/\hbar} = \int d^2\theta(1 - \frac{A}{2\hbar}\theta_1\theta_2) = -\frac{A}{2\hbar}$$

$n = 2m$ fields

$$S = \frac{1}{2}A_{ab}\theta_a\theta_b$$

A_{ab} is antisymmetric matrix

$$Z_0 = \int d\theta^{2m}e^{-S/\hbar} = \int d^{2m}\theta\sum_{j=0}^{m}\frac{(-1)^j}{(2\hbar)^j j!}(A_{ab}\theta_a\theta_b)^j$$

$$Z_0 = \frac{(-1)^m}{(2\hbar)^m m!}\epsilon^{a_1\ldots a_n}A_{a_1a_2}...A_{a_{2m-1}a_{2m}} = \frac{(-1)^m}{\hbar}Pf(A) = \pm\sqrt{\frac{\det A}{\hbar^n}}$$

$Pf(A)$ is the Pfaffian.

6.2.1 External sources

Need to be grassman valued η:

$$S(\theta, \eta) = \frac{1}{2}A_{ab}\theta_a\theta_b - \eta_a\theta_a$$

Complete the square, using translation invariance to get the following integral that we can then do:

$$Z_0(\eta) = \exp(-\frac{1}{2\hbar}\eta^T A^{-1}\eta)Z_0(0)$$

Propogator

$$< \theta_a\theta_b >= \frac{\hbar^2}{Z_0(0)}\frac{\partial^2 Z_0(\eta)}{\partial\eta_a\partial\eta_b}|_{\eta=0} = \hbar(A^{-1})_{ab}$$

7 Lecture 7

7.1 LSZ reduction formula

This discussion is quite general and not very connected to the earlier section.

We are going to work through 2-2 scatting of ϕ particles. From last term:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E_b}(a(\boldsymbol{k})e^{-ik\cdot x} + a^\dagger(\boldsymbol{k})e^{ik\cdot x})$$

We are using the Minkowski metric $(+,-,-,-)$ so $k \cdot x = E_0 t - \boldsymbol{k}\cdot\boldsymbol{x}$. There is also a convention for the normalisation. We are taking the realtivistic normalisation for $a(\boldsymbol{k})$. It depends on whether you want the inner product of two functions to be the delta funcitno or the dealt function times E.

We invert to find $a(\boldsymbol{k})$:

$$\int d^3x e^{ik\cdot x}\phi(x) = \frac{1}{2E}a(\boldsymbol{k}) + \frac{1}{2E}e^{2iEt}a^\dagger(-\boldsymbol{k})$$

$$\int d^3x e^{ik\cdot x}\partial_a\phi(x) = \frac{-i}{2}a(\boldsymbol{k}) + \frac{i}{2}e^{2iEt}a^\dagger(-\boldsymbol{k})$$

These imply:

$$a(\boldsymbol{k}) = \int d^3x e^{ik\cdot x}(i\partial_a\phi(x) + E\phi(x))$$

$$a^\dagger(\boldsymbol{k}) = \int d^3x e^{-ik\cdot x}(-i\partial_a\phi(x) + E\phi(x))$$

In free theory: the 1 particle state is given by

$$|k\rangle = a^\dagger(\boldsymbol{k})|0\rangle, \langle 0||0\rangle = 1, a(\boldsymbol{k})|0\rangle = 0\forall \boldsymbol{k}k$$

Norm:

$$\langle k||k'\rangle = (2\pi)^3(2E)\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')$$
$$E^2 = \boldsymbol{k}^2 + m^2$$

We are going to assume that the interacting theory is close to the free theory.

Introduce a gaussian wave packet:

$$a_1^\dagger = \int d^3k f_1(\boldsymbol{k})a^\dagger(k)$$

with $f_1(k) \sim \exp(-\frac{(\boldsymbol{k}-\boldsymbol{k}_1)^2}{4\sigma^2})$ and similary

$$a_2^\dagger = \int d^3k f_2(\boldsymbol{k})a^\dagger(k)$$

with $\boldsymbol{k}_2 \neq \boldsymbol{k}_1$.

Now we evolve the gaussians backward in time until a time where the particles had no overlap in space and can be consider free. THere is a complication as due to the interaction $a_1^\dagger(t)$ and $a_2^\dagger(t)$ depend on t. However, the point is that as $t \to \pm\infty$ $a_1^\dagger$ and $a_2^\dagger$ coincide with free theory expressions.

Consdiring 2-2 scattering the initial state is $|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t)a_2^\dagger(t)|\Omega\rangle$ and the final state is $|f\rangle = \lim_{t\to\infty} a_1^\dagger(t)a_2^\dagger(t)|\Omega\rangle$. We also have $\langle i||i\rangle = \langle f||f\rangle = 1$ and $\boldsymbol{k}_1 \neq \boldsymbol{k}_2, \boldsymbol{k}_1' \neq \boldsymbol{k}_2'$.

We want to calculate the scattering amplitude $\langle f||i\rangle$. First note that

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = \int_\infty^\infty \partial_0 a_1^\dagger(t) = \int d^3 f_1(\boldsymbol{k}) \int d^4x \partial_0 (e^{-k\cdot x}(-i\partial_0\phi + c\phi)$$

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = i\int d^3k F_1(\boldsymbol{k}) \int d^4x e^{-ik\cdot x}(\partial_0^2 + E^2)\phi = -i\int d^3k f_1(\boldsymbol{k}) \int d^4x e^{-ik\cdot x}(\partial^2 + m^2)\phi$$

In free theory we have $(\partial^2 + m^2)\phi = 0$ (klein gordon equation) so the creation operator doesn't change.

$$\langle f||i\rangle = \langle\Omega| T a_{1'}(\infty)a_{2'}(\infty)a_1^\dagger(-\infty)a_2^\dagger(\infty)|\Omega\rangle$$

We can relate

$$a_j^\dagger(-\infty) = a_j^\dagger(\infty + i\int d^3k f_j \int d^4x e^{-ikx}(\partial^2 + m^2)\phi$$

$$a_{j'}^\dagger(\infty) = a_{j'}^\dagger(-\infty + i\int d^3k f_{j'} \int d^4x e^{ikx}(\partial^2 + m^2)\phi$$

Time ordering moves $a_j(-\infty)$ right and $a_j^\dagger(\infty)$ left. Therefore the t only nonzero term is the **LSZ reduction formula**:

$$\langle f||i\rangle = (i)^4 \int d^4x_1 d^4x_2 d^4x_1' d^4x_2' e^{-ik_1\cdot x_1} e^{-ik_2\cdot x_2}$$

$\mathrm{e}^{ik_3'\cdot x_3'}\mathrm{e}^{-ik_4'\cdot x_4'}(\partial_1^2 + m^2)(\partial_2^2 + m^2)(\partial_{1'}^2 + m^2)(\partial_{2'}^2 + m^2)\langle\Omega| T\phi(x_1)\phi(x_2)\phi(x_1')\phi(x_2')|\Omega\rangle having taken \sigma \to$ 0 limit of $f_j(k) \to \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}_j)$.

Generalisatoin to $m \to n$ scattering is straightforward. You performa a fourier transform to get:

$$\phi(y) = \int \frac{d^4q}{(2\pi)^4}\tilde{\phi}(q)e^{-iq\cdot y}$$

so

$$\langle f||i\rangle = \langle k_1'...k_n'||k_1...k_n\rangle = (i)^{m+n}\prod_{j=1}^m(-k_j^2 + m^2)\prod_{j=1}^n(-k_j'^2 + m^2)\langle\Omega| T\tilde{\phi}(k_1)...\tilde{\phi}(k_m)\tilde{\phi}(k_1')...\tilde{\phi}(k_n')|\Omega\rangle$$

momentum satisfy $k^2 = m^2$ propogators $\frac{k}{k^2+m^2}$. The $(-k^2+m^2)$ factors cancel the poles form the external propogators.

LSZ in momentum space:

$$\langle k_1'..k_n'| \langle k_1...k_n| = \langle \Omega| T\tilde{\phi}(k_1)...\tilde{(}k_n') |\Omega\rangle_{amputated}$$

Usually we are interested in all momentum being unequal, so all the particles are involved in the scattering. This implies we want connected diagrams.

8 Example sheet 1

When calculating symmetry factors you are genuinely looking for symmetries. Rather than trying to think about automorphisms first count the number of symmetries, then consider if any of them are already acounted for by exchange of particles and ignore them. If you get confused remember how it works with a single particle we don't say how many ways of picking which two to connect up is 6 but rather say we can slip those loops or exchange those loops (think a bit like polymod though in some instances this does not hold e.g. if the diagram is symmetric under exchange of two propogators which end on different particles this is a symmetry just like if they ended on the same particle). Try question 3 on ES1 it is a very good test of if you have got it cracked. It might be worth just memorising the symmetry factors of everything less than 3.

In order to show that

$$-\hbar^2 \frac{\partial^3 W}{\partial J_a \partial J_b \partial J_c}|_{J=0} =< \phi_a\phi_b\phi_c >^{conn}$$

we need to express in terms of Z and take each derivative at a time and only take J =0 at the end. This will give extra terms because the 1/Z will get differentiated as well as the $Z/\partial J$

If you are asked to find the feynmann rules, remember that the propogator is $< \phi_1\phi_2 >$ and the vertex rule you can get by adding the source term and then extracting the interaction term as derivatives and the vertex term is what connects the derivatives. You need to extract the interaction term first leaving the derivatives in the exponential outside the integral, then inside the intgeral you sometimes need to complete the square and then perform a translation.

For fermionic case remembder that A_{ij} and λ_{abcd} are antisymmetric and that for antisymmetric tensors $_{abcd} = \lambda\epsilon_{abcd}$ and $\epsilon_{abc...n}\epsilon_{abc...n} = n!$.

For grassman variables feynmann diagrams you cannot have more half edges than variables as $\theta_i^2 = 0$. Remember forumula for $Pf(A)$

9 Lecture 8

$$\langle k_1'...k_n'| |k_1...k_n\rangle = \langle\Omega| \tilde{\phi}(k_1')...\tilde{\phi}(k_n')\tilde{\phi}(-k_1)...\tilde{\phi}(-k_n) |\Omega\rangle$$

All positive signs means all particles coming out of the interaction so above we flipped the signs of the incoming particles so they are actually incoming.

In fact only weaker assumptions are needed for an LSZ formula

Unique ground state $|\Omega\rangle$ and the 1st excited state is a single particle

We want $\phi|\Omega\rangle$ to be a single-particle state (Generally ϕ could represent a composite operator) - i.e. $\langle\Omega|\phi|\Omega\rangle = 0$ (If $\langle\Omega|\phi|\Omega\rangle = v \neq 0$ then let $\tilde{\phi} = \phi - v$.

We whant ϕ to be properly normalised so that when it is far away it is nroamlised like a plane wave: $\langle k|\phi|\Omega\rangle = e^{ik\cdot x}$ as in the free case

Interactions mean we will probably need to rescale the *phi* to some scaled field $Z_\phi^{\frac{1}{2}}\phi$.
With these assumptions + a few more -¿ LSZ formula still applies. We may need to "renormalise" the field from

$$\mathfrak{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

to

$$\mathfrak{L} = \frac{1}{2}Z_\phi\partial^\mu\phi\partial_\mu\phi - \frac{Z_\phi}{2}m^2\phi^2 - \frac{\lambda Z_\phi^2}{4!}\phi^4$$

9.1 Scalar Field Theory

9.1.1 Wick rotation

We will would in Minkowski space-time with signature (+—)

$$\mathfrak{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$$

$$Z = \int \mathfrak{D}\phi e^{i\int dx^n L}, L = \int d^3x\mathfrak{L}$$

Propogator:

$$\frac{i}{k^2 - m^2 + i\epsilon} = \frac{i}{(k^0)^2 - |\boldsymbol{k}|^2 - m^2 + i\epsilon}$$

Let $ix^0 = x_4$ metric (++++), and arrange signs s.t.:

$$\mathfrak{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)$$

$$Z = \int \mathfrak{D}\phi e^{-\int d_{x_4} L}$$

Propogator:

$$\frac{1}{k^2 + m^2} = \frac{1}{k_4^2 + |\boldsymbol{k}|^2 + m^2}$$

This change to euclidean space time can be thought of as a rotation from the real axis to the imaginary axis and is called Wicks rotation. There are some situtaitons where we can't do this rotation but we are interested in times when we can.

9.1.2 Feynmann Rules

In the free case:

$$S_0(\phi, J) = \int d^4x(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2}m^2\phi^2 - J(x)\phi(x)$$

Fourier transform $\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}\tilde{\phi}(k)$

$$S_0[\tilde{\phi}, \tilde{J}] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left(\tilde{\phi}(-k)(k^2+m^2)\tilde{\phi}(k) - \tilde{J}(-k)\tilde{\phi}(k) - \tilde{J}(k)\tilde{\phi}(-k)\right)$$

$$S_0[\tilde{\phi}, \tilde{J}] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left(\tilde{\chi}(-k)(k^2+m^2)\chi(x) - \frac{\tilde{J}(-k)\tilde{J}(k)}{k^2+m^2}\right)$$

where $\tilde{\chi} = \tilde{\phi} - \frac{\tilde{J}}{k^2+m^2}$ so:

$$Z_0[J] = Z_0[0]\exp(\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{J}(-k)\tilde{J}(k)}{k^2+m^2}$$

We can ignore the premultiplier constant.
The free propogator is:

$$\tilde{\Delta}_0(q) = \frac{\delta^2 Z_0[\tilde{J}]}{\delta\tilde{J}(-qq)\delta\tilde{J}(q)} = \frac{1}{q^2+m^2}$$

Functional derivatives:

$$\frac{\delta}{\delta f(x_1)}f(x_2) = \delta^{(4)}(x_1 - x_2)$$

$$\frac{\delta}{\delta \tilde{f}(k_1)}\tilde{f}(k_2) = (2\pi)^4\delta^{(4)}(k_1 - k_2)$$

Now lets fourier transform back:

$$\Delta_0(x - x') = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-x')}}{k^2+m^2}$$

So we can write the partition funciton as:

$$Z_c[J] = \exp\left(\frac{1}{2}\int d^4xd^4x' J(x)\Delta(x-x')J(x')\right)$$

Interactions come about as before:

$$\mathfrak{L} = \mathfrak{L}_0 + \mathfrak{L}_I$$

As in $D = 0$

$$Z[J] = \int \mathfrak{D}\phi \exp(-\int d^4x(\mathfrak{L}_0+\mathfrak{L}_I-J\phi)) = \exp(-\int d^4y\mathfrak{L}_1(\frac{\delta}{\delta J(y)}))\exp\left(\frac{1}{2}\int d^4xd^4x' J(x)\Delta(x-x')J(x')\right)$$

$$Z[J] \sim \sum_{v=0}^{N}\frac{1}{v!}\left(-\int d^4y\mathfrak{L}_1(\frac{\delta}{\delta J(y)})\right)^v \sum_{p=0}\frac{1}{p!}\left(\frac{1}{2}\int d^4xd^4x' J(x)\Delta(x-x')J(x')\right)^p$$

For each term in $Z[J]$ there is a graph made up of a Propogator $\Delta_0(x-x')$ and verticles with n lines from $\phi^n > \mathfrak{L}_I$ and at each vertex we add $-\mathfrak{L}_I(\frac{\delta}{\delta J(y)})$. Then we integrate over internal positions and apply symmetry factors.

10 Examples Class 1

In general in order to find the n-th order perturbative expression for the partition function add a source term and then remove the derivatives and then set the source to zero. Ocassionally you don't need the source tterm and can compute it exactly.

Pay attention to the notation sometiems $Z_J(\lambda)$ so $Z_0(0)$ means $Z_{J=0}(\lambda = 0)$ where λ is the coupling constant and J is the invented source term. but sometimes $Z_\lambda(J)$. Nice way of thinking about which terms survive from expansion is that you need terms with the same number of $\frac{\partial}{\partial J}$ and J.

A nice way of writing effective actions are:

$$<\phi_a\phi_b> = \hbar^2 e^{W/\hbar}\frac{\partial}{\partial J_a}\frac{\partial}{\partial J_b}e^{-W/\hbar} = -\hbar\frac{\partial^2 W}{\partial J_a \partial J_b} + \frac{\partial W}{\partial J_a}\frac{\partial W}{\partial J_b} = <\phi_a\phi_b>^{conn} + <\phi_a>^{conn}<\phi_b>^{conn}$$

as $Z(J) = e^{-W/\hbar}$

$$\frac{\partial}{\partial\phi_d}(\frac{\partial^2\Gamma}{\partial\phi_b\partial\phi_c})^{-1} = (\frac{\partial^2\Gamma}{\partial\phi_b\partial\phi_e})^{-1}\frac{\partial^3\Gamma}{\partial\phi_d\partial\phi_e\partial\phi_f}(\frac{\partial^2\Gamma}{\partial\phi_d\partial\phi_c})^{-1}$$

$Thereasonfortheactionofgrassmannvariablesalwaysbeingantisymmetricprefactorsisthatthesymmetric$

-$\lambda\epsilon_{abcd}$ what this means for the diagram is that if we switch the order of two half edges then the sign needs to change. He is going to check exactly how this works and get back to us.

11 Lecture 9

Let's begin by looking at some examples, starting with the 1-loop function.

$$< \phi(x_2)\phi(x_1) >= -(\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_1)}W[J]$$

where $W[J] = -\log Z[J]$. Take $\mathcal{L}_1 = \frac{\lambda}{3!}\phi^3$:

$$< \phi(x_2)\phi(x_1) >= x_2 - x_1 + x_2 - y_2 o y_1 - x_1 + ...$$

Let's consider the first one loop diagram D here in more detail.

$$D = \frac{(-\lambda)^2}{2}\int d^4y_1 \int d^4y_2 \Delta_0(x_2 - y_2)\Delta_0(y_1 - x_1)(\Delta_0(y_2 - y_1))^2$$

Now take fourier transform

$$< \tilde{\phi}(p_2)\phi(p_1) >= \int d^4x_1 \int d^4x_2 e^{-i(p_1x_1+p_2x_2)} < \phi(x_2)\phi(x_1) >= -^{p_1}O-^{p_2}$$

Focus on $\tilde{D}$:

$$\tilde{D} = \frac{\lambda^2}{2}\int d^4x_1 d^4x_2 e^{-i(p_1x_1+p_2x_2)}\int d^4y_1 \int d^4y_2 \int (\prod_{j=1}^{4}\frac{d^4k_j}{(2\pi)^4})e^{ik_2(x_2-y_1)}e^{ik_1(y_1-x_2)}e^{i(k_3+k_4)-(y_2-y_1)}\tilde{\Delta}_0(k_1)$$

The integrals over x_1 and x_2 give: $(2\pi)^4\delta^{(4)}(p_1+k_1)$ and $(2\pi)^4\delta^{(4)}(p_2-k_2)$

$$\tilde{D}\int d^4y_1 \int d^4y_2 \int \frac{d^4k_3d^4k_4}{(2\pi)^8}e^{-i(p_1+k_3+k_4)-y_1}e^{-i(p_2-k_3-k_4)y_2}\tilde{\Delta}_0(-p_1)\tilde{\Delta}_0(p_2)\tilde{\Delta}_0(k_3)\tilde{\Delta}_0(k_4)$$

$$\tilde{D}\int \frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p_1+p_2)\tilde{\Delta}_0(-p_1)\Delta_0(p_2)\tilde{\Delta}_0(-(k-p_2))\tilde{\Delta}_0(k)$$

So the loop has momentum p coming in and coming out and then we have loop momentum k with one side of the loop having $k-p$ momentum and the other having k.

Another example in $\frac{\lambda}{3!}\phi^3$:

$$< \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) >^{conn}= \frac{\partial^4 W}{\partial J(x_1)\partial J(x_2)\partial J(x_3)\partial J(x_4)}|_{J=0}$$

This can be represented by three diagrams in the typed up lecture notes.

Using the LSZ in Euclidean spacetime to consider 2-2 scattering:

$$\langle f||i\rangle = \int d^4x_1d^4x_2d^4x_3d^4x_4e^{ik_1x_1}e^{ik_2x_2}e^{-ik_3x_3}e^{-ik_4x_4}(-\partial_1^2+m^2)...(-\partial_4^2+m^2) < \phi(x_1)\phi(x_2\phi(x_3)\phi(x_4) >^{co}$$

Klein-Gordon $(-\partial_i^2 + m^2)\Delta_0(x_i - y) = \delta^{(4)}(x_i - y)$
Integrals over x_i's collapse:

$$\langle f|\,|i\rangle = \lambda^2 \int d^4y \int d^4z \int \frac{d^4q}{(2pi)^4} \frac{e^{iq(y-z)}}{q^2+m^2}(e^{ik_y y}e^{ik_2 y}e^{-ik_3 z}e^{-ik_4 z} + ...)$$

Could intergate y and z to get delta functions and in this case there are no loops so there shouldn't be any loops left over so we are left with the following:

$$\langle f|\,|i\rangle^2 (2\pi)^4\delta^{(4)}(k_1+k_2-k_3-k_4)(\frac{1}{(k_1+k_2)(^2+m^2} + \frac{1}{(k_1-k_3)^2+m^2} + \frac{1}{(k_1+k_4)^2+m^2})$$

11.0.1 Momentum space Feynmann rules

Draw external lines for each incoming/outgoing particle

Leave one end of external line free other connected to a vector from $\mathfrak{L}_I$. Invlude internal lines to do this. Include topologically distinct diagrams

Incoming lines means you draw the momentum cominginto the vertex, whereas outocmming lines have momentum away from the vertex

Conserve momentum at each vertex

External lines get a factor of 1

Internal lines get propogation of $\frac{1}{q^2+m^2}$

For each vertex add the interaction coupling term $-\lambda_i$ for $\mathfrak{L}_I = \frac{\lambda_n}{n!}\phi^n$

A diagram with L loops will have L momenta l_i not fixed by momentum conservation so we integrate over these $\int \frac{d^4 l_i}{(2\pi)^4}$ (these can lead to divergences that we will come to soon.

Divide by symmetry factors to account for different ways of arranging internal propogators

12 Lecture 10

12.1 Quantum Effective Action and Vertex Functions

Same as before but now we have functionals rather than functions:

$$W[J] = -\log Z[J]$$

$$\Gamma[\Phi] = W[J] + \int d^4x J(x)\Phi(x)$$

The same steps as we have carried out before led us to conclude that

$$\frac{\delta W[J]}{\delta J(x)} = -\Phi(x), \frac{\delta \Gamma}{\delta \Phi(x)} = J(x)$$

$$Again when we are working with$$

W$wethinkofthesoruce$J$astheindependantfield, thenwenwemoveto$$\Gamma$ we think of Φ has the independant field. So we have:
$J(x)$ is independant and then correseponds to $\Phi(x)$ mean field by $< \phi(x) >_J$ whereas after the Lagrange transform we get $\Phi(x)$ is independant with a corresponding source $J(x)$ obtained by $< \phi(x) >_J = \Phi(x)$.
In momentum space, consider the sets of amputated 1PI diagrams with n external legswhere $n \geq 2$.
For $n = 2$ the summ of all such IP1 diagrams is called the self-energy which will be labelled as:

$$\Pi(k^2) = -_k(1PI)-_k$$

lorentz invariance and momentum conservation give us that k^2 is the parameter.
For $n \geq 3$ these vertex functions are lablled as:

$$V^{(n)}(k_1, ..., k_n) = \text{mass of n legs coming into 1PI}$$

Now lets think further in the momentum space field with $\tilde{\Phi} = \int d^4xe^{-ikx}\Phi(x)$

$$\Gamma[\tilde{\Phi}] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{\Phi}(-k)[k^2+m^2-\Pi(k')]\tilde{\Phi}(k) - \sum_{n=3} \frac{1}{n!}[\int \hat{\prod}_{i,j=1}^{n} \frac{d^4k_j}{(2\pi)^j}](2\pi)^4\delta(k_1+k_2+...+k_n)V^{(n)}(k_1, ..., k_n$$

This minus sign before the higher order terms is because want to think of this vertex as starting out as the classic vertex e.g. if we had $\mathfrak{L}_I = \frac{\lambda}{n!}\phi^n \implies V_0^{(n)} = -\lambda$ (corresponding to tree-level).

It is unusal that here the momentum has an impact on the interaction, as up until now we haven't been considering interaction terms with derivates that would bring in momentum dependance.
Note if $v^{(n)}$ has non-trivial momentum dependance then derivatives acting on Φ in coordinate space. Lets fourier tansform back

$$\Gamma[\Phi] = \int d^4x\{U[\Phi(x)] + Q[\Phi(x)]\partial_\mu\Phi\partial^\mu\Phi + ...\}$$

The first term here U is often called the effective potential.

12.1.1 (Quantum) Effective Potential

Generalising the earlier derivation we want to define:

$$< \phi(x_1)...\phi(x_n) >^{conn} = G^{(n)}(x_1, ...x_n) = -\prod_{i=1}^{n} \frac{\delta}{\delta J(x_i)} W[J]|_{J=0}$$

And we have:

$$\Gamma^{(n)} = \prod_{i=1}^{n} \frac{\delta}{\delta \Phi(x_i)} \Gamma[\Phi]|_{\Phi=\Phi_0}$$

We know that the quadratic parts are inverses of each other but now we need to integrate over :

$$\int d^4y G^{(2)}(x,y)\Gamma^{(2)}(y,z) = \delta^{(4)}(x-z)$$

12.2 Renormalisation

We will begin by picking a particular aciton and doing our first calculations in quantum higher dimensions. Lets chose the classical action:

$$S[\phi_0] = \int d^4x(\frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2}m_0\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4)$$

this example is not great as we don't have to change the field but we do have to in the ϕ^3 theory on the example sheet. Subscripts hint that these "original" or "bare" fields and parameters will need to be "renomarlised".
Consider quantum effects at 1-loop level. Take the 2-point function

$$\tilde{G}^{(2)}(p) = -0- = -+-1PI-+-1PI-1PI-+... = \frac{1}{p^2+m_0^2} + \frac{1}{p^2+m_0^2}\Pi(p^2)\frac{1}{p^2+m_0^2} + ...$$

$$\tilde{G}^{(2)}(p) = \frac{1}{p^2+m_0^2-\Pi(p^2)}$$

So

$$\tilde{\Gamma}^{(2)}(p) = [\tilde{G}^{(2)}(p)]^{-1} = p^2 + m_0^2 - \Pi(p^2)$$

Now we need to calculate the $\Pi(p^2)$ function.
In ϕ^4 theory we can calculate the one-loop contribution. In this theory there is only one diagram:

$$\Pi_1(p^2) = -_p^o-_p$$

we have developed the feynmann rules so this gives:

$$\Pi_1(p^2) = -\frac{\lambda_0}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m_0^2}$$

This will diverge for large k which we can show by introducing a cut off $|k| < \Lambda$. As the integrand only depends on the magnitude of k so

$$\Pi_1(p^2) = -\frac{\lambda_0}{2}\int_0^\Lambda \frac{dk}{(2\pi)^4}k^3 dk d\Omega_3 \frac{1}{k^2+m_0^2} = -\frac{\lambda_0 S_3}{2(2\pi)^4}\int_0 \frac{k^3}{k^2_{m0}{}^2}dk, S_3 = 2\pi^2$$

So let $n = \frac{k^2}{m_0^2}$

$$\Pi_1(p^2) = -\frac{\lambda_0}{32\pi^2}(\Lambda^2 - m_0^2\log(1+\frac{\Lambda^2}{m_0^2}))$$

This diverges quadaracticall as Λ goes to infinity.

Now lets look at the four point function f^n at 1-loop:

$$V_0^{(n)} = -\lambda_0$$

$$V_1^{(n)}(p_1,p_2,p_3,p_4) => o < + = o = +$$

basically you need three different diagrams to allow for every combination of the external lines meeting so first is 1-2, 3-4 then 1-3, 2-4 and the 1-4 2-3 and every diagram has two internal momenta between thsee verteces. Using the feynmann ruels this gives:

$$V_1^{(n)} = \frac{\lambda_0^2}{2}\int \frac{dk^4}{(2\pi)^4}\frac{1}{k^2+m_0^2}\sum_{p=\{p_1+p_2,p_1-p_3,p_1-p_4\}}\frac{1}{(p+k)^2+m_0^2}$$

This is a mess to integrate as you have external momentum dependance. However, there is really just one source of diverengence, as k gets large whatever these external momentum are they will be fixed so we only really have the one divergence which is important as we only have one parameter with which to fix the divergence. Now lets examine this divergence.
As $k \to \infty$, the external momenta become negligible. Examine

$$V_1^{(4)}(0,0,0,0) = \frac{3\lambda_0^2}{2}\int^\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+m_0^2)^2} = \frac{3}{\lambda_0^2}32\pi^2(\log(1+\frac{\Lambda^2}{m_0^2}) - \frac{\Lambda^2}{\Lambda^2+m_0^2})$$

so we have a logarithmic divergence.

13 Lecture 11

First we will find a shortcut for figuring out if a loop integral will diverge.

More generally in d dimensions we can write down the form of the integrals we were writing before (without factors of 2π as he was being schematic?)

$$(\int^\Lambda d^dk)^l(\frac{1}{k^2+\Delta})^I$$

Δ may contain external parameters such as mass and external momenta. For the purposes of investigating a large k we can drop the Δs are they are fixed.

$$\int^{\Lambda} \frac{(d^d k)^l}{k^{2I}}$$

The "superficial degree of divergence is $D = dL - 2I$. If $D > 0$ then the integral diverges like $\sim \Lambda^D$. If $D = 0$ then it diverges but logarithmically $\sim \log \Lambda$. If $D < 0$ then it is possibly finite but we cannot guarantee that. In this course we are only looking at one loop diagrams so if $D < 0$ then they will be finite.

13.0.1 Lohmann-Kollen propogator

$$\tilde{G}^{(2)}(p) = \int d^4x d^4y e^{-ip(x-y)} < \phi_0(x)\phi_0(y) >= \sum_n \frac{|\langle\Omega|\,\phi_0(0)\,|x\rangle\,|^2}{p^2 + m_n^2}$$

with $|\Omega\rangle$ is the vaccum state of the full theory, and $|n\rangle$ eigenstate of Hamiltonian with rest mass m_n. Look at Peskin and Schorder.

Focus on lowest energy, single particle excitation

$$\tilde{G}^{(2)}(p)\frac{|\langle\Omega|\,\phi_0(0)\,|1\rangle\,|^2}{p^2 + m_{phys}^2} + [\text{finte at } p^2 = -m_{phys}^2$$

So basically saying there is a pole at $p^2 = -m_{phys}^2$ with residue given above.

Last time we found for ϕ^4 theory that:

$$\tilde{G}^{(2)}(p) = \frac{1}{p^2 - m_0^2 - \Pi(p^2)}$$

with $\Pi(p^2) = -\frac{\lambda_0}{32\pi^2}(\Lambda^2 - m_0^2 \log(1 + \frac{\Lambda}{m_0^2}))$ This is constant in p^2 but in general is it not. e.g. in ϕ^3 theory the free energy depends on the momenta.

To deal with divergences we do the following:
We have the original Lagrangian: $\mathfrak{L}_0 = \frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4$
Rescale $\phi_0 = Z_\phi^{\frac{1}{2}}\phi$ where $Z_p^{\frac{1}{2}}$ determined s.t. $\tilde{G}^{(2)}(p)$ has unit residue at pole.
Separate out 2 sets of terms, write $\mathfrak{L}_0$ as follows:

$$\mathfrak{L}_0 = \mathfrak{L}_{\mathtt{rln}} + \mathfrak{L}_{\mathtt{ct}} = (\frac{1}{2}(\partial phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4)(\frac{\delta Z_\phi}{2}(\partial\phi)^2 + \frac{\delta m^2}{2}\phi^2 + \frac{\delta\lambda}{4!}\phi^4)$$

equate coefficents to get: $\delta Z_\phi = Z_\phi - 1$ and $\delta m^2 = Z_\phi m_0^2 - m^2$ and $\delta\lambda = Z_\phi^2\lambda_0 - \lambda$.
We now use this way of writing the lagrangian in perturbative calcualtions so the Feynmann rules need to be alterated a bit. The rules for $\mathfrak{L}_{rln}$ are the same as for $\mathfrak{L}_0$. Additionally, for $\mathfrak{L}_{ct}$ we have new rules given by two new interaction terms:

$$-\Box^{-p^2\Omega}-, -x^{-\delta m^2}-, X^{-\delta}$$

The diagramatics of these aren't imporatnt but it is useful to think of these terms as additional interaction terms at the one loop level. The tree diagrams containing $\mathfrak{L}_{ct}$ vertices are the same order as 1-loop diagrams containing $\mathfrak{L}_{rlm}$ vertices. Lets revisit:

$$\tilde{\Gamma}^{(2)}_{rlm}(p) = [\tilde{G}_{rlm}]^{-1} = p^2 + m^2 - \Pi_{rlm}(p^2)$$

From $\mathfrak{L}_{rlm}$, $\hat{\Pi}_1(p^2)$ is the same equation as for $\Pi_1(p^2)$ with m_0,λ_0, $\phi_0 \to m, \lambda, \phi$. From $\mathfrak{L}_{ct} : \Pi_{1,ct} = -x - + - \Box - = -\delta m^2 - \delta Z_\phi p^2$ these are additoinal diagrams that contribute to the two point function. Result for $\Pi_{rlm}(p^2) = \hat{\Pi}_1(p^2) + \Pi_{1,ct}(p^2)$. We know there is a divergence in $\hat{\Pi}(p^2)$ but we can make it finite by chosing specific δm^2 and $\delta\lambda^2$:

$$\delta m^2 = -\frac{\lambda}{32\pi^2}(^2 - m^2 \log(1 + \frac{\Lambda^2}{m^2}) + \text{finite}, \delta Z_\phi = 0$$

There is freedom in how we chose the finite part which goes by the naem of the "renormaliszation scheme" or "condition". This is a choice. e.g. we are using the "on-shell" scheme by imposing the following conditions:

$$\Pi_{rlm}(-m^2_{phys}) = m^2 - m^2_{phys}$$

Additionally, choose $m^2 - m^2_{phys} = 0$ this sets the renoramlised mass $m = m_{phys}$. The other "on shell" condition we could use is:

$$\frac{\partial \Pi_{rln}}{\partial p^2}|_{p^2 = -m^2_{phys}} = 0$$

These two conditions together give the propogator:

$$\tilde{G}^{(2)}(p) = \frac{1}{p^2 + m^2 - \Pi_{rlm}(p^2)} = \frac{1}{p^2 + m^2_{phys}}$$

To finish up we need to look at $\delta\lambda$, now lets look at corrections to the four point correlation function:

$$V^{(4)}_{rln}(0,0,0,0) = -\lambda + \hat{V}^{(4)}(0,0,0,0) + V^{(4)}_{1,ct}$$

$$V^{(4)}_{1,ct} = -\delta\lambda$$

Choosing $\delta\lambda = \frac{3^2}{32\pi^2}(\log\frac{\Lambda}{m^2} - 1)$ gives a finite vertex term as the divergence is cancelled by the second term

$$V^{(4)}_{rln}(0,0,0,0) = -\lambda + \frac{32\lambda^2}{32\pi}(\log(1 + \frac{\Lambda^2}{m^2}) + \frac{m^2}{m^2 + \Lambda^2}) = -\lambda_{eff}$$

where λ_{eff} is coefficient of $\frac{1}{4!}\Phi^4$ in $\Gamma[\Phi]$. Note we did make a decision about the finite piece which is that $\lambda_{eff} \to \lambda$ as $\to \infty$ which is another renormalisation choice or condition that we have decided to impose.

13.1 Dimensional regularization

This idea of momentum cutoff is fine for a scalar field theory but is no good for guage theory. In the context of perturbation theory we can regulate divergences by moving away from integer dimensions. Here $d = 4 - \epsilon$ for $0 < \epsilon << 1. in the case of \phi^3$ we are often interested in close to 6 dimensions. Take $S = \int d^4x(\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4)$

Dimensional analysis $[.]$ = mass dimension $\hbar = c = 1$. Given $[S] = 0$, $[\partial] = [m] = -[x] = 1$

$$[m^2\phi^2] = 2[m] + 2[\phi] = d \implies [\phi] = \frac{d}{2} - 1$$

$$[\lambda\phi^4] = d \implies [\lambda] = 4 - d = \epsilon$$

This means that away from 4 dimensions the formerly dimensionless coupling now has dimensions ϵ. This introduces a new mass scale μ s.t. $\lambda = \mu^\epsilon g(\mu)$ with a new dimensionless coupling g.

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We analytically continue things away from 4 dimensions to $d = 4-$, so the previously dimensionless coupling gains a dimension $\lambda = \mu^\epsilon g(\mu)$ with g dimensionless.

Now lets return to the one-loop diagrams that we looked at before. The one-loop self energy becomes:

$$\hat{\Pi}_1 = -\frac{1}{2}g(\mu)\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2 + m^2}$$

As before we can see that the integrand can be written as:

$$\hat{\Pi}_1 = -\frac{1}{2}g(\mu)\mu^\epsilon \frac{S_{d-1}}{(2\pi)^d}\int_0^\infty \frac{(k^2)^{d/2-1}}{2(k^2 + m^2)}dk^2$$

This is done in more general sense in the appendix of the written notes (including Feynmann's trick with Parametrization).
Can show that for $d \in \mathbb{Z}^+$, we have $S_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$. Analaytically continue this to $d \in \mathbb{R}^+$. We will also find the Schwinger trick useful which is two rewrite the denominator:

$$\frac{1}{A^n} = \frac{1}{A^n}\frac{1}{\Gamma(n)}\int_0^\infty dx e^{-x}x^{n-1}$$

Let $s = \frac{x}{A}$ then

$$\frac{1}{A^n} = \frac{1}{\Gamma(n)}\int_0^\infty ds e^{-As}s^{n-1}$$

So

$$\hat{\Pi} = -\frac{1}{2}g(\mu)\mu^\epsilon \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty dk^2 (k^2)^{d/2-1} \int_0^\infty dse^{-(k^2+m^2)s}$$

Let $u = sk^2$:

$$\hat{\Pi}_1 = -\frac{1}{2}g(\mu)\mu^\epsilon \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty dse^{-m^2 s} s^{-d/2} \int_0^\infty duu^{\frac{d}{2}-1}e^{-u}$$

We can identify this two terms:

$$\hat{\Pi}_1 = -\frac{1}{2}g(\mu)\mu^\epsilon \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \Gamma(1-\frac{d}{2})m^{d-2}\Gamma(d/2)$$

This is definitelly better just recognise Euler Beta fucntions B(s,t) like in notes):

$$B(s,t) = \int_0^1 duu^{s-1}(1-u)^{t-1} = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$$

Continying the maths:

$$\hat{\Pi}_1 = -\frac{g(\mu)m^2}{2(4\pi)^2}(\frac{4\pi\mu^2}{m^2})^{\epsilon/2}\Gamma(\frac{\epsilon}{2}-1)$$

We want to consider $\epsilon \to 0$. so need to expand about small ϵ. First term is straight forawrd:

$$(\frac{4\pi\mu^2}{m^2})^{\epsilon/2} = 1 + \epsilon/2\log(\frac{4\pi\mu^2}{m^2}) + O(\epsilon^2)$$

Expanding the gamma function needs to be analytically continued to think about what it means for negative arguments using $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$. There must be a pole at $\alpha = 0, -1, -2$ as $0\Gamma(0) = \Gamma(1)$ so near these poles there is a Laurent series with $\Gamma(\alpha) = \frac{1}{\alpha} - \gamma + O(\alpha)$ where $\gamma \approx 0.577216$.. so:

$$\Gamma(\frac{\epsilon}{2}-1) = -\frac{1}{1-\frac{\epsilon}{2}}\Gamma(\epsilon/2) = -\frac{2}{\epsilon} + \gamma - 1 + O(\epsilon)$$

therfore:

$$\hat{\Pi}_1 = \frac{g(\mu)m^2}{32\pi^2}(\frac{2}{\epsilon} - \gamma + 1 + \log(\frac{4\pi\mu^2}{m^2})) + O(\epsilon)$$

We have the same divergence with $\Lambda \to$ equivalent to $\frac{1}{\epsilon} = \frac{1}{4-d}$ pole. Add a counterterm $\frac{1}{2}\delta m^2\phi^2$.

Choice of scheme (finite term):
- On-shell such that $m^2 + \delta m^2 = m^2_{phys}$
- Minimal subtraction (MS) where the counter term just subtracts the divergence and adds no finite term.

- Modified minimal subtraction scheme ($\bar{MS}$) when you subtract the pole and also any constants that come along from the expansion e.g.

$$\delta^2 m = -\frac{g(\mu)m^2}{32\pi^2}(\frac{2}{\epsilon} - \gamma + \log(4\pi))$$

Then in $\bar{MS}$ we have:

$$\Pi^{\bar{MS}}_{rln} = \frac{g(\mu)m^2}{32\pi^2}(\log\frac{\mu^2}{m^2} - 1)$$

Now lets look at the one-loop corrections to the four-point function.

$$\hat{V}^{(4)}(0,0,0,0) = \frac{3g^2\mu^{2\epsilon}}{2}\int\frac{d^dk}{(2\pi)^4}\frac{1}{(k^2+m^2)^2} = 3g^2\mu^\epsilon\frac{1}{(4\pi)^{d/2}}(\frac{\mu}{m})^\epsilon\Gamma(2-\frac{d}{2})$$

$$\hat{V}^{(4)}(0,0,0,0) = 3g^2\mu^\epsilon 32\pi^2(\frac{2}{\epsilon} - \gamma + \log(\frac{4\pi\mu^2}{m^2}) + O(\epsilon)$$

Introduce a counter-term $-\delta g$ which is a four point vertex with $\delta g = \frac{3g^2}{32\pi^2}\frac{2}{\epsilon}$ in MS and $\delta g = \frac{3g^2}{32\pi^2}(\frac{2}{\epsilon} + \log(\frac{4\pi\mu^2}{m^2}))$ in $\bar{MS}$.

Explore the consequences of the new scale μ

Old fashioned approach (the next chapter will do the new approach).

Look at

$$\mathfrak{L}_0 = \frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2}m_0\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4$$

and rescale with $\phi_0 = Z_\phi^{1/2}\phi$ to get:

$$mathfrakL_0 = \frac{Z_\phi}{2}(\partial\phi_0)^2 + \frac{Z_\phi}{2}m_0\phi_0^2 + \frac{\lambda_0 Z_\phi^2}{4!}\phi_0^4 = \mathfrak{L}_{rln} + \mathfrak{L}_{ct}$$

$$\mathfrak{L}_{rln} + \mathfrak{L}_{ct} = \frac{(1+\delta Z_\phi)}{2}(\partial\phi)^2 + \frac{m^2+\delta m^2}{2}\phi^2 + \frac{(g+\delta g)\mu^\epsilon}{4!}\phi^4$$

Now we need to equate teh coeeficents between these two to give:

$$Z_\phi = 1 + \delta Z_\phi, m_0^2 = Z_\phi^{-1}(m^2 + \delta m^2), \lambda_0 = Z_\phi^{-2}(g+\delta g)\mu^\epsilon$$

Recall each loop brings in an additonal $\hbar$. For the vertex funciton we started off with the term:

$$V^{(4)}_{rln}(0,0,0,0) = \frac{1}{\hbar}(-\lambda + O(\hbar) - \delta g \sim O(\hbar)$$

$Basically as we have been ignoring the \hbar$s we need to remember that the counter-wieght term needs an extra $\hbar$ so it cancels with the loop terms.

Have $\lambda_0 = Z_\phi^{-2}(g+\delta g)\mu^\epsilon$ though in ϕ^4 $Z_\phi = 1$ so ignore it.
We define $\beta(g) = \frac{d}{d\log\mu} g(\mu) = \mu\frac{d}{d\mu}(g(\mu))$. This is an old fashioned attitude as there was nothing special about λ_0 (though it definitely doesn't depend on μ as it was introduced before we introduced μ):

$$0 = \frac{d}{d\log u}\lambda = \frac{d}{d\log u}((g+\delta g)\mu^\epsilon)$$

as $\delta g = \frac{3g^2\hbar}{16\pi^2\epsilon}$:

$$0 = \beta(g)(1+\frac{3g\hbar}{8\phi^2\epsilon}) + g(1+\frac{4g\hbar}{16\pi^2\epsilon})$$

we find that:

$$\beta(g) = (-\epsilon g - \frac{3g^2\hbar}{16\pi^2}(1+\frac{3g\hbar}{8\pi^2\epsilon})^{-1} = -\epsilon g + frac3g^2\hbar 8\pi^2 - \frac{3g^2\hbar}{16\pi^2 + O(\hbar^2}$$

There is a $O(\frac{1}{\epsilon}$ here but it is multipled by $O(\hbar^2)$ but that would be canceled by a 2 loop or three loop divergence so we ignore it as it is a higher order in pertrubation theory. so we get

$$\beta(g) = -\epsilon g + \frac{3g^2\hbar}{16\pi^2} \to \frac{3g^2\hbar}{16\pi^2} = \mu\frac{dg}{d\mu}$$

$$\int_{g(\mu)}^{g(\mu')} \frac{dg}{g^2} = \frac{3\hbar}{16\pi^2}\int_\mu^{\mu'}\frac{d\tilde{\mu}}{\tilde{\mu}} \implies \frac{1}{g(\mu')} = \frac{1}{g(\mu)} - \frac{3\hbar}{16\pi^2}\log\frac{u'}{u}$$

so

$$g(u') = \frac{g(\mu)}{1-\frac{3g\hbar}{16\pi^2}\log\frac{\mu'}{\mu}} \sim g(\mu)(1-\frac{3g(\mu)\hbar}{16\pi^2}\log\frac{\mu'}{\mu})$$

For $\mu' > \mu \implies g(\mu') > g(\mu)$ so the coupling "runs" to larger values as μ increases.

We see that is maybe possible that the denomiator may vanish so we coul dhave a landau pole. A Landau pole occurs if $\mu' \to \Lambda_{\phi^4}$ where

$$\frac{3g\hbar}{16\pi^2}\log\frac{\Lambda_{\phi^4}}{\mu} = 1$$

(at 1-loop order) as $\mu' \to \Lambda_{\phi^4}$ we have $g(\mu') \to \infty$. So Λ_{ϕ^4} can be a referecne scale:

$$g(\mu) = \frac{16\pi^2}{3\hbar^2}\frac{1}{\log(\frac{\Lambda_{\phi^4}}{\mu}}$$

So now we are comparing our running coupling to our reference scale. So we have gone from talking about dimensionless couplings to having a reference scale or mass scale ("dimensionfull") this is sometimes called a "Dimensional transmutation". This is not useful in theories with couplings that run like this that get weaker at higher energy scales like QED, but it is very important for strongly coupled theories.