17. All trees with more than one vertex have the same chromatic number. What is it, and why?

30. Find a planar drawing of the graph $\mathbf{K}_5 - e$, by which we mean the graph formed from the complete graph on 5 vertices by deleting any edge.

32. Show that every planar graph has a vertex that is incident to at most five edges.

33. Let $\mathbf{G} = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$. Its *degree sequence* is the list of the degrees of its vertices, arranged in nonincreasing order. That is, the degree sequence of $\mathbf{G}$ is $(\deg_{\mathbf{G}}(v_1), \deg_{\mathbf{G}}(v_2), \ldots, \deg_{\mathbf{G}}(v_n))$ with the vertices arranged such that $\deg_{\mathbf{G}}(v_1) \geq \deg_{\mathbf{G}}(v_2) \geq \cdots \geq \deg_{\mathbf{G}}(v_n)$. Below are five sequences of integers (along with n, the number of integers in the sequence). Identify

- the *one* sequence that ***cannot be the degree sequence of any graph***;
- the *two* sequences that could be the degree sequence of a ***planar*** graph;
- the *one* sequence that could be the degree sequence of a ***tree***;
- the *one* sequence that is the degree sequence of an ***eulerian*** graph; and
- the *one* sequence that is the degree sequence of a graph that must be ***hamiltonian***.

Explain your answers. (Note that one sequence will get two labels from above.)

a. $n = 10$: $(4, 4, 2, 2, 1, 1, 1, 1, 1, 1)$

b. $n = 9$: $(8, 8, 8, 6, 4, 4, 4, 4, 4)$

c. $n = 7$: $(5, 4, 4, 3, 2, 1, 0)$

d. $n = 10$: $(7, 7, 6, 6, 6, 6, 5, 5, 5, 5)$

e. $n = 6$: $(5, 4, 3, 2, 2, 2)$

34. Below are three sequences of length 10. One of the sequences cannot be the degree sequence (see Exercise 5.9.33) of any graph. Identify it and say why. For each of the other two, say *why* (if you have enough information) a *connected* graph with that degree sequence

- is definitely hamiltonian/cannot be hamiltonian;
- is definitely eulerian/cannot be eulerian;
- is definitely a tree/cannot be a tree; and
- is definitely planar/cannot be planar.

(If you do not have enough information to make a determination for a sequence without having specific graph(s) with that degree sequence, write “not enough information” for that property.)

a. $(6, 6, 4, 4, 4, 4, 2, 2, 2, 2)$

b. $(7, 7, 7, 7, 6, 6, 6, 2, 1, 1)$

c. $(8, 6, 4, 4, 4, 3, 2, 2, 1, 1)$

For the next Olympic Winter Games, the organizers wish to expand the number of teams competing in the sport of curling. Their goal is to divide 14 teams into two groups each. The organizers would like that in the first round, each team from a group will play a total of seven games against distinct opponents: five from their own group and two from the other group.

They're having trouble setting up such a schedule, so they've come to you. By using an appropriate graph-theoretic model, either argue that they cannot use their current plan or devise a way for them to do so.

Show that the average degree of a planar graph is less than 6, i.e. $\dfrac{\sum_{v \in V} \deg v}{|V|} < 6.$