Exercises 3A

1) let $u, v \in R^3$ then $T(u+v) = (2(u_1+w_1) - 4(u_2+w_2) + 3(u_3+w_3) + b, 6(u_1+w_1) + c(u_1+w_1)(u_2+w_2)(u_3+w_3))$. So $b = 0$. Let $k \in R$, then $T(ku) = (k(2u_1 - 4u_2 + 3u_3) + b, 6ku_1 + ck^3u_1u_2u_3)$, so $c = 0$.

2) skipped

3) Let $e_1, ..., e_n$ be the column unit vectors in F^n, then we can write

$$X = x_1e_1 + x_2e_2 + ..., x_ne_n$$

. And

$$T(X) = T(x_1e_1) + ... + T(x_ne_n) = x_1T(e_1) + ... + x_nT(e_n)$$

. We now denote the unit vectos in F^m to be $d_1, ..., d_m$. Then we can write the linear transformation

$$T(e_1) = a_{11}d_1 + a_{12}d_2 + ... + a_{1m}d_m$$
$$\vdots$$
$$\vdots$$
$$T(e_n) = a_{n1}d_1 + a_{n2}d_2 + ... + a_{nm}d_m$$

Hence we have

$$T(e_1) = \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1m} \end{pmatrix}, ..., T(e_n) = \begin{pmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nm} \end{pmatrix}$$

Hence, the transpose of A is the matrix asked for in the question.

4) If $v_1, ..., v_m$ are linearly dependent, then there is a non-zero solution to

$$c_1v_1 + ... + c_mv_m = 0$$

. By linearity, we can map the non-zero solution to W with the mapping T,

$$T(c_1v_1 + ... + c_mv_m) = T(0) = 0$$

, which contradicts the fact that $Tv_1, ..., Tv_m$ are linearly independent.

5) skipped, so much work

6) skipped, so much work

7) skipped

8) $\varphi(v) = \sqrt{(v_1v_2)}$

9) skipped
10) skipped
11) skipped
12) skipped
13) Write $V = U \oplus U'$ and use the basis vectors to construct a linear transformation T such that all the basis vectors from U' and U are map into W. Then we can take S as the part of T that transforms the basis vectors from U'.
14) skipped
15) Let $w_1, ..., w_m$ be linearly independent. Since $v_1, ..., v_m$ are linearly dependent, there exists a vector $v_i = \sum_{-i} x_j v_j$, so $T(v_i) = T\left(\sum_{-i} x_j v_j\right)$. But the equality is impossible since w_i's are linearly independent.
16) skipped
17) skipped

Exercises 3B

1) Let $T : F^5 \to F^2 = (x_1, x_2, 0, 0, 0)$.
2) Let $x \in V$, we know that $S(T(x)) \in \text{null } T$. So $T(S(T(x))) = 0 \to (ST)^2(x) = 0$.
3)