Vector Calculus

IV: Double and Triple Integrals

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1 Introductory Concepts

1.1 Double Integrals as Volumes

Consider a continuous function of two variables $f : R \subset \mathbf{R}^2 \to \mathbf{R}$ whose domain R is a rectangle with sides parallel to the coordinate axes. The rectangle R can be described in terms of two closed intervals $[a, b]$ and $[c, d]$ on the x and y axes, respectively. Assume that $f(x, y) \geq 0$ on R. If we take the double integral of f over R, we are effectively finding the volume or

$$\text{Volume} = \int\int_R f(x,y)dA = \int\int_R f(x,y)dx\ dy. \tag{1}$$

1.2 Cavalieri's Principle

Suppose we have a solid body and we let $A(x)$ denote its cross-sectional area in a plane P_x measured at a distance x from a reference plane. According to Cavalieri's principle, the volume of a body is given by

$$\text{Volume} = \int_b^a A(x)dx, \tag{2}$$

Where a and b are the maximum distances from the reference plane. If we partition $[a, b]$ into n-equal parts where $a = x_0 < x_1 < \cdots < x_n = b$, then if $\Delta x = x_{i+1} - x_i$, the preceding integral is

$$\sum_{i=0}^{n-1} A(c_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} A(c_i)\Delta x. \tag{3}$$

1.3 Reduction to Iterated Integrals

Consider a solid region under a graph $z = f(x, y)$ defined on region $[a, b] \times [c, d]$, where f is continuous and greater than zero. Cavalieri's principle can be used to show that volume V is

$$V = \int_a^b A(x)dx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx. \tag{4}$$

This preceding integral is called an iterated integral because it is obtained by integrating along different axes sequentially.

With certain constraints that are usually satisfied in real applications, Fubini's theorem states that

$$\int_a^b \left[\int_c^d f(x,y)dy\right] dx = \int_c^d \left[\int_a^b f(x,y)dx\right] dy. \tag{5}$$

2 Double Integral Over a Rectangle

2.1 Definition of Integral

Consider a closed rectangle $R \subset \mathbf{R}^2$; that is, R is a Cartesian product of two intervals $R = [a,b] \times [c,d]$. By regular partition of R of order n, we mean the two ordered collections of $n+1$ equally spaced points $\{x_j\}_{j=0}^n$ and $\{y_k\}_{k=0}^n$ is satisfying the two conditions:

$$a = x_0 < x_1 < \cdots < x_n = b \qquad c = y_0 < y_1 < \cdots < y_n = d \tag{6}$$

$$x_{j+1} - x_j = \frac{b-a}{n} \qquad y_{k+1} - y_k = \frac{d-c}{n} \tag{7}$$

And

$$\Delta A = \Delta x \ \Delta y. \tag{8}$$

If the sequence $\{S_n\}$ converges to a limit S as $n \to \infty$ and if the limit S is the same for any choice of points $\mathbf{c}_{jk}$ in the rectangles R_{jk}, then we say that f is integrable over R and write the following for the limit S:

$$\int\int_R f(x,y)dA = \int\int_R f(x,y)dx \ dy \ \int\int_R f \ dx \ dy \tag{9}$$

2.2 Fubini's Theorem

This theorem states that if f is continuous, then

$$\int_a^b f(x)dx = F(b) - F(a), \tag{10}$$

Where F is the antiderivative of f, or $F' = f$. This theorem will not work for functions of multiple variables such as $f(x, y)$, but we can often reduce a double integral over a rectangle to iterated single integrals, where

$$\int\int_R f(x,y)dA = \int_a^b \left[\int_c^d f(x,y)dy\right] dx = \int_c^d \left[\int_a^b f(x,y)dx\right] dy \tag{11}$$

3 The Double Integral Over More General Regions

3.1 Elementary Regions

Suppose we have two continuous real-valued functions $\phi_1 : [a, b] \to \mathbf{R}$ and $\phi_2 : [a, b] \to \mathbf{R}$ that satisfy $\phi_1(x) \leq \phi_2(x)$ for all $x \in [a, b]$. Let D be the set of all points (x, y) such that $x \in [a, b]$ and $\phi_1(x) \leq y \leq \phi_2(x)$. This region D is said to be y-simple. The curves and straight-line segments that bound the region together constitute the boundary of D, denoted as δD.

We say that region D is x-simple if there are continuous functions ψ_1 and ψ_2 defined as $[c, d]$ such that D is the set of all points (x, y) satisfying

$$y \in [c,d] \text{ and } \psi_1(y) \leq x \leq \psi_2(y), \tag{12}$$

Where $\psi_1(y) \leq \psi_2(y)$ for all $y \in [c, d]$. Again, the curves that bound region D constitute its boundary δD. A simple region is one that is both a x-simple region and a y-simple region. Sometimes this simple regions are referred to as elementary regions.

3.2 The Integral over an Elementary Region

If D is an elementary region in the plane, choose a rectangle R that contains D. Given $f : D \to \mathbb{R}$, where f is continuous, define $\int\int_D f(x, y)dA$, the integral of f over the set D, as follows:

$$f^*(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \text{ and } (x,y) \in R \end{cases} \tag{13}$$

Then, we can define

$$\int\int_D f(x,y)dA = \int\int_R f^*(x,y)dA. \tag{14}$$

3.3 Reduction to Iterated Integrals

If $R = [a,b] \times [c,d]$ is a rectangle containing D, we can use the iterated integrals to obtain

$$\int\int_D f(x,y)dA = \int\int_R f^*(x,y)dA = \int_a^b \int_c^d f^*(x,y)dy\ dx. \tag{15}$$

Assume that D is a y-simple region determined by functions $\phi_1 : [a,b] \to \mathbf{R}$ and $\phi_2 : [a,b] \to \mathbf{R}$. Consider the iterated integral

$$\int_a^b \int_c^d f^*(x,y)dy\ dx, \tag{16}$$

And in particular, the inner integral $\int_c^d f^*(x,y)dy$ for some fixed x. By definition, $f^*(x,y) = 0$ if $y < \phi_1(x)$ or $y > \phi_2(x)$, giving

$$\int_c^d f^*(x,y)dy = \int_{\phi_1(x)}^{\phi_2(x)} f^*(x,y)dy = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy. \tag{17}$$

Consequently, the iterated integral for y-simple regions is

$$\int\int_D f(x,y)dA = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy\right] dx. \tag{18}$$

Similarly, the iterated integral for x-simple regions is

$$\int\int_D f(x,y)dA = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx\right] dy. \tag{19}$$

4 Changing the Order of Integration

4.1 Introduction

By changing which iterated integral we compute first, we can evaluate the easier integral first, which can be very useful.

4.2 Mean-Value Inequality

Suppose there are numbers m and M such that for all $(x, y) \epsilon D$ and $m \leq f(x, y) \leq M$, then integrating over D we get

$$m \cdot A(D) \leq \int\int_D f(x, y) dA \leq M \cdot A(D), \tag{20}$$

Where $A(D)$ is the area of region D. It can help us estimate integrals that we cannot evaluate exactly.

4.3 Mean-Value Equality

The mean-value inequality can be turned into an quality when f is continuous. Suppose $f : D \to \mathbf{R}$ is continuous and D is an elementary region. Then for some point (x_0, y_0) in D we have

$$\int\int_D f(x, y) dA = f(x_0, y_0) A(D), \tag{21}$$

Where $A(D)$ denotes the area D.

5 The Triple Integral

5.1 Definition of the Triple Integral

Our new objective is to define the triple integral of a function $f(x, y, z)$ over a rectangular parallelepiped $B = [a, b] \times [c, d] \times [p, q]$. Analogous to double integrals, we partition the three sides of B into n equal parts and form the sum

$$S_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(\mathbf{c}_{ijk}) \Delta V, \tag{22}$$

Where $\mathbf{c}_{ijk}$ is a point in B_{ijk}, the ijkth rectangular parallelepiped in the partition of B, and ΔV is the volume of B_{ijk}.

Let f be a bounded function of three variables defined on B. If $\text{limit}_{n\to\infty} S_n = S$ exists and is independent of any choice of $\mathbf{c}_{ijk}$, we call f integrable and call S the triple integral of f over B and denote it by

$$\int\int\int_B f\ dV = \int\int\int_B f(x,y,z)\ dV = \int\int\int_B f(x,y,z)\ dx\ dy\ dz. \tag{23}$$

5.2 Properties of Triple Integrals

Continuous functions defined on B are integrable and can be reduced into iterated integrals the same way that double integrals are reduced into iterated integrals.

Let $f(x,y,z)$ be integrable on the box $B = [a,b] \times [c,d] \times [p,q]$. Then any iterated integral that exists is equal to the triple integral, or

$$\begin{aligned} \int\int\int_B f(x,y,z)\ dx\ dy\ dz &= \int_p^q \int_c^d \int_a^b f(x,y,z)\ dx\ dy\ dz \\ &= \int_p^q \int_a^b \int_c^d f(x,y,z)\ dy\ dx\ dz \\ &= \int_a^b \int_p^q \int_c^d f(x,y,z)\ dz\ dy\ dx \end{aligned} \tag{24}$$

5.3 Elementary Regions

An elementary region in 3D space is defined by restricting one of the variables to be between two functions of the two remaining variables, the domains of these functions being an elementary region in the plane. For example, if D is an elementary region in the xy plane and if $\gamma_1(x,y)$ and $\gamma_2(x,y)$ are two functions with $\gamma_2(x,y) \geq \gamma_1(x,y)$, an elementary region consists of all (x,y,z) such that (x,y) lies in D and $\gamma_1(x,y) \leq z \leq \gamma_2(x,y)$.

5.4 Integrals over Elementary Regions

As with integrals in the plane, any function of three variables that is continuous over an elementary region is integrable on that region. Suppose that W is an elementary region described by bounding z between two functions x and y. Then either is true

$$\begin{aligned}
\int\int\int_W f(x,y,z)\ dx\ dy\ dz &= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\ dz\ dy\ dx \\
\int\int\int_W f(x,y,z)\ dx\ dy\ dz &= \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\ dz\ dx\ dy
\end{aligned} \tag{25}$$

Note if $f = 1$, we get the integral $\iiint_W dx\ dy\ dz$, or the volume of region W.