Vector Calculus

VII: The Integral Theorems of Vector Analysis

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1 Green's Theorem

1.1 Simple and Elementary Regions and Their Boundaries

A simple closed curve C that is the boundary of an elementary region has two orientations - counterclockwise (positive) and clockwise (negative). Denote C with the counterclockwise orientation as C^+, and with the clockwise orientation as C^-. The boundary C of a y-simple region can be decomposed into bottom and top portions C_1 and C_2, and (if applicable) left and right vertical portions, B_1 and B_2. Hence, we can write

$$C^+ = C_1^+ + B_2^+ + C_2^- + B_1^-, \tag{1}$$

Where the pluses denote curves oriented in the direction of left to right or bottom to top, and the minuses is the opposite.

A similar decomposition of the boundary of an x-simple region can also be made. Similarly, a simple region has two decompositions: one into upper and lower halves, the other into left and right halves.

1.2 Green's Theorem

Let D be a y-simple region and let C be its boundary. Suppose $P : D \to \mathbb{R}$ is of class C^1. Then

$$\int_{C^+} P\ dx = -\iint_D \frac{\delta P}{\delta y}\ dx\ dy. \tag{2}$$

Note that that the left-hand side denotes the line integral $\int_{C^+} P\ dx + Q\ dy$, where $Q = 0$.

Now let D be an x-simple region with boundary C. Then if $Q : D \to \mathbb{R}$ is C^1,

$$\int_{C^+} Q\ dy = \iint_D \frac{\delta Q}{\delta x}\ dx\ dy. \tag{3}$$

Note that that the left-hand side denotes the line integral $\int_{C^+} P\ dx + Q\ dy$, where $P = 0$.

Now intuitively, let D be a simple region and let C be its boundary. Suppose $P : D \to \mathbb{R}$ and $Q : D \to \mathbb{R}$ are of class C^1. Then

$$\int_{C^+} P\ dx + Q\ dy = \iint_D \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y} \right)\ dx\ dy \tag{4}$$

1.3 Generalizing Green's Theorem

Green's theorem actually applies to any "decent" region in $\mathbb{R}^2$. For example, Green's theorem applies to regions that are not simple, but that can be broken up into pieces, each of which are simple.

Green's theorem is useful because often the line integral is easier to calculate than the area integral or vice versa.

1.4 Area of a Region

If C is a simple closed curve that bounds a region to which Green's theorem applies, then the area of the region D bounded by $C = \delta D$ is

$$A = \frac{1}{2} \int_{\delta D} x\ dy - y\ dx. \tag{5}$$

1.5 Vector Form Using the Curl

Green's theorem can be rewritten in vector fields and generalized to $\mathbb{R}^3$.

Let $D \subset \mathbb{R}^2$ be a region to which Green's theorem applies, let δD be its (positively oriented) boundary, and let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a C^1 vector field on D. Then

$$\int_{\delta D} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k}\ dA = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\ dA. \tag{6}$$

1.6 Vector Form Using the Divergence

Another way to generalize Green's theorem to $\mathbb{R}^3$ is with divergence, known as the Divergence Theorem.

Let $D \subset \mathbb{R}^2$ be a region to which Green's theorem applies and let δD be its boundary. Let $\mathbf{n}$ denote the outward unit normal to δD. If $\mathbf{c} : [a, b] \to \mathbb{R}^2$, $t \mapsto \mathbf{c}(t) = (x(t), y(t))$ is a positively oriented parametrization of δD, $\mathbf{n}$ is given by

$$\mathbf{n} = \frac{(y'(t), -x'(t))}{\sqrt{[x'(t)]^2 + [y'(t)]^2}}. \tag{7}$$

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a C^1 vector field on D. Then

$$\int_{\delta D} \mathbf{F} \cdot \mathbf{n} \; ds = \iint_D \text{div } \mathbf{F} \; dA. \tag{8}$$

2 Stokes' Theorem

Stokes' theorem relates the line integral of a vector field around a simple closed curve C in $\mathbb{R}^3$ to an integral over a surface S for which C is the boundary. In other words, Green's theorem is the two-dimensional version of the more general Stokes' theorem.

2.1 Strokes' Theorem for Graphs

Consider a surface S that is the graph of a function $f(x, y)$, so that S is parametrized by

$$\begin{cases} x = u \\ y = v \\ z = f(u, v) = f(x, y) \end{cases} \tag{9}$$

For some (u, v) in some domain D in the plane. Recall that the integral of a vector function $\mathbf{F}$ over S is

$$\iint_F \mathbf{S} \cdot d\mathbf{S} = \iint_D \left[F_1 \left(-\frac{\delta z}{\delta x} \right) + F_2 \left(-\frac{\delta z}{\delta y} \right) + F_3 \right] \; dx \; dy, \tag{10}$$

Where $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$.

Let S be the oriented surface defined by a C^2 function $z = f(x, y)$, where $(x, y) \epsilon D$, a region to which Green's theorem applies, and let $\mathbf{F}$ be a C^1 vector field on S. Then if δS denotes the oriented boundary curve of S as just defined, we have

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\delta S} \mathbf{F} \cdot d\mathbf{s}. \tag{11}$$

2.2 Stokes' Theorem for Parametrized Surfaces

Suppose $\mathbf{\Phi} : D \to \mathbb{R}^3$ is a parametrization of a surface S and $\mathbf{c}(t) = (u(t), v(t))$ is a parametrization of δD. δS is not necessarily the boundary of S in a visualized geometric sense. For example, spheres have no boundaries. We get around this by assuming that $\mathbf{\Phi}$ is one-to-one on all of D. Then, the image of δD under $\mathbf{\Phi}$, or $\mathbf{\Phi}(\delta D)$, is the geometric boundary of $S = \mathbf{\Phi}(D)$. If $\mathbf{c}(t) = (u(t), v(t))$ is a parametrization of δD in the positive direction, we define δS to be the oriented closed curve that is the image of the mapping $\mathbf{p} : t \mapsto \mathbf{\Phi}(u(t), v(t))$, with the orientation of δS induced by $\mathbf{p}$.

Thus, we have Stokes' Theorem for parametrized surfaces. Let S be an oriented surface defined by a one-to-one parametrization $\mathbf{\Phi} : D \subset \mathbb{R}^2 \to S$, where D is a region to which Green's theorem applies. Let δS denote the oriented boundary of S and let $\mathbf{F}$ be a C^1 vector field on S. Then

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\delta S} \mathbf{F} \cdot d\mathbf{s}. \tag{12}$$

If S has no boundary, and this includes surfaces such as the sphere, then the integral on the left is zero.

3 Conservative Fields

3.1 When Are Vector Fields Gradients?

Let $\mathbf{F}$ be a C^1 vector field defined on $\mathbb{R}^3$, except possibly for a finite number of points. The following conditions on $\mathbf{F}$ are all equivalent:

1. For any oriented simple closed curve C, $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$
2. For any two oriented simple curves C_1 and C_2 that have the same endpoints,
$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}. \tag{13}$$
3. $\mathbf{F}$ is the gradient of some function f; that is, $\mathbf{F} = \nabla f$.
4. $\nabla \times \mathbf{F} = \mathbf{0}$.

A vector field satisfying one (and hence all) of the conditions above is called a conservative vector field.

4 Gauss' Theorem

4.1 Elementary Regions and Their Boundaries

Closed surfaces can be oriented in two ways. The outward orientation makes the normal point outward into space and the inward orientation makes the normal point into the bounded region.

Suppose S is a closed surface oriented in one of those two ways and $\mathbf{F}$ is a vector field on S. Then, we previously defined:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \sum_i \iint_{S_i} \mathbf{F} \cdot d\mathbf{s}, \tag{14}$$

Where S_i is the ith part of the surface S. If S is given an outward orientation, the integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ measures the total flux of $\mathbf{F}$ outward across S.

Recall that another common way of writing these surface integrals is by explicitly specifying the orientation of S. Let the orientation of S be given by a unit normal vector $\mathbf{n}(x, y, z)$ at each point S. Then we have the oriented integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n}) dS. \tag{15}$$

In this section, if S is a closed surface enclosing a region W, we adopt the default convention that $S = \delta W$ is given the outward orientation, with outward unit normal $\mathbf{n}(x, y, z)$ at each point $(x, y, z) \epsilon S$. Thus,

$$\iint_{\delta W} \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n}) dS = -\iint_S [\mathbf{F} \cdot (-\mathbf{n})] dS = -\iint_{\delta W_{\mathrm{opp}}} \mathbf{F} \cdot d\mathbf{S} \tag{16}$$

4.2 Gauss' Theorem

This theorem relates surface integrals to volume integrals. It states that if W is a region in $\mathbb{R}^3$, then the flux of a vector field $\mathbf{F}$ outward across the closed surface δW is equal to the integral of div $\mathbf{F}$ over W.

Let W be a symmetric elementary region in space. Denote by δW the oriented closed surface that bounds W. Let $\mathbf{F}$ be a smooth vector field defined on W. Then

$$\iiint_W (\nabla \cdot \mathbf{F})\ dV = \iint_{\delta W} \mathbf{F} \cdot d\mathbf{S} \tag{17}$$

Or

$$\iiint_W (\text{div } \mathbf{F})\ dV = \iint_{\delta W} (\mathbf{F} \cdot \mathbf{n})\ dS. \tag{18}$$

4.3 Generalizing Gauss' Theorem

We can extend the theorem to any region that can be broken up into symmetric elementary regions.

4.4 Gauss' Law

Let M be a symmetric elementary region in $\mathbb{R}^3$. Then if $(0,0,0) \not\epsilon\ \delta M$, we have

$$\iint_{\delta M} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3} dS = \begin{cases} 4\pi & \text{if } (0,0,0)\ \epsilon\ M \\ 0 & \text{if } (0,0,0) \not\epsilon\ M \end{cases} \tag{19}$$

Where

$$\mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{20}$$

And

$$r(x,y,z) = ||\mathbf{r}(x,y,z)|| = \sqrt{x^2 + y^2 + z^2}. \tag{21}$$