# Vector Calculus

## III: Vector-Valued Functions

Oliver Zhao

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# 1 Acceleration and Newton's Second Law

## 1.1 Differentiation of Paths

Recall that a path in $\mathbb{R}^n$ is a map $\mathbf{c}$ of $\mathbb{R}$. If the path is differentiable, its derivative at each time $t$ is a $n$ x 1 matrix, or

$$\mathbf{c}(t) = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \text{ and } \mathbf{c}'(t) = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ \dots \\ dx_n/dt \end{bmatrix} \tag{1}$$

Recall that $\mathbf{c}'(t)$ is the tangent vector to the path at point $\mathbf{c}(t)$. If $\mathbf{c}$ represents the path of a moving particle, then the velocity vector is

$$\mathbf{v} = \mathbf{c}'(t), \tag{2}$$

And its speed is

$$s = ||\mathbf{v}||. \tag{3}$$

Likewise, the corresponding acceleration vector is just

$$\mathbf{a} = \mathbf{v}' = \mathbf{c}''. \tag{4}$$

Note that a path is considered a regular path if the following property is satisfied

$$\mathbf{v}(t) = \mathbf{r}'(t) \neq 0. \tag{5}$$

## 1.2 Differentiation Rules for Paths

Let $\mathbf{b}(t)$ and $\mathbf{c}(t)$ be differentiable paths in $\mathbb{R}^3$ and $p(t)$ and $q(t)$ be differentiable scalar functions:

| | |
|---|---|
| Sum Rule: | $\frac{d}{dt}[\mathbf{b}(t) + \mathbf{c}(t)] = \mathbf{b}'(t) + \mathbf{c}'(t)$ |
| Scalar Multiplication Rule: | $\frac{d}{dt}[p(t)\mathbf{c}(t)] = p'(t)\mathbf{c}(t) + p(t)\mathbf{c}'(t)$ |
| Dot Product Rule: | $\frac{d}{dt}[\mathbf{b}(t) \cdot \mathbf{c}(t)] = \mathbf{b}'(t) \cdot \mathbf{c}(t) + \mathbf{b}(t) \cdot \mathbf{c}'(t)$ |
| Cross Product Rule: | $\frac{d}{dt}[\mathbf{b}(t) \times \mathbf{c}(t)] = \mathbf{b}'(t) \times \mathbf{c}(t) + \mathbf{b}(t) \times \mathbf{c}'(t)$ |
| Chain Rule: | $\frac{d}{dt}[\mathbf{c}(q(t))] = q'(t)\mathbf{c}'(q(t))$ |

## 1.3 Newton's Second Law

Newton's second law states that the acceleration of a path $\mathbf{c}(t)$ is

$$\mathbf{a}(t) = \mathbf{c}''(t), \tag{6}$$

And if $\mathbf{F}$ is the force acting and $m$ is the mass of the particle, then

$$\mathbf{F} = m\mathbf{a}. \tag{7}$$

**Circular Orbits**

Consider a particle of mass $m$ moving at a constant speed $s$ in a circular path of radius $r_0$. Supposing that it moves in the $xy$ plane, we can compress the third component and write its location as

$$\mathbf{r}(t) = \left( r_0 cos\left(\frac{st}{r_0}\right), r_0 sin\left(\frac{st}{r_0}\right)\right). \tag{8}$$

Note that this is a circle of radius $r_0$ and its speed is given by $||\mathbf{r}'(t)|| = s$. The quantity $s/r_0$ is the frequency, denoted as $\omega$. Thus,

$$\mathbf{r}(t) = (r_0 cos(\omega t), r_0 sin(\omega t)). \tag{9}$$

The acceleration is given by

$$\mathbf{a}(t) = \mathbf{r}''(t) = \left(-\frac{s^2}{r_0} cos\left(\frac{st}{r_0}\right), -\frac{s^2}{r_0} sin\left(\frac{st}{r_0}\right)\right) = -\frac{s^2}{r_0^2}\mathbf{r}(t) = -\omega^2 \mathbf{r}(t). \tag{10}$$

**Kepler's Law**

The square of the period is proportional to the cube of the radius in circular gravitational orbit, or

$$T^2 = r_0^3 \frac{(2\pi)^2}{GM}. \tag{11}$$

**Components of Acceleration Vector**

Before going into the acceleration vector, note the unit tangent vector $\mathbf{T}(t)$ and unit normal vector $\mathbf{N}(t)$. $\mathbf{T}(t)$ describes the instantaneous velocity and is in the same direction as the velocity. Meanwhile, $\mathbf{N}(t)$ is the unit tangent vector of the unit tangent vector function, used to describe the change in the velocity vector, point towards the "inside" of how a curve is curving.

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{||\mathbf{v}(t)||}, \tag{12}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}. \tag{13}$$

The acceleration vector can be decomposed into the tangential and normal components, through the following equations:

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{||\mathbf{v}||}, \tag{14}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{||\mathbf{v} \times \mathbf{a}||}{||\mathbf{v}||}, \tag{15}$$

Where

$$\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t). \tag{16}$$

To prove this, we show that

$$\begin{aligned}
\mathbf{v}(t) &= ||\mathbf{c}'(t)||\mathbf{T}(t) = v(t)\mathbf{T}(t)\\
\mathbf{a}(t) &= \mathbf{v}'(t)\\
&= v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)\\
&= v'(t)\mathbf{T}(t) + (v(t)||\mathbf{T}'(t)||)\ \mathbf{N}(t)\\
&= \text{Tangential Component} + \text{Normal Component}
\end{aligned} \tag{17}$$

Thus,

$$\begin{aligned}
a_{\mathbf{T}} &= v'(t) = \frac{d}{dt}||\mathbf{r}'(t)||\\
a_{\mathbf{N}} &= v(t)||\mathbf{T}'(t)||
\end{aligned} \tag{18}$$

## 1.4 Angular Motion

**Angular Momentum**
Angular momentum, or $\mathbf{L}$, is the cross product of linear position $\mathbf{r}$ and linear momentum $\mathbf{p}$, where momentum $\mathbf{p} = m\mathbf{v}$,

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v}. \tag{19}$$

**Torque**
When we take the derivative of angular momentum, we get the net torque, or

$$\begin{aligned}
\frac{d}{dt}(\mathbf{r}(t) \times m\mathbf{v}(t)) &= \mathbf{r}'(t) \times m\mathbf{v}(t) + \mathbf{r}(t) \times m\mathbf{v}'(t)\\
&= \mathbf{r}(t) \times m\mathbf{v}'(t)\\
&= \mathbf{r}(t) \times \mathbf{F}(t)
\end{aligned} \tag{20}$$

Note that this is true because the $\mathbf{r}'(t) \times m\mathbf{v}(t) = 0$, since the two vectors are in the same direction.

**Central Force**
The central force $\mathbf{F}(t)$ on an object is the force $f(t)$ that is directed along the line $\mathbf{r}(t)$ joining the object and the origin, or

$$\mathbf{F}(t) = f(t)\mathbf{r}(t). \tag{21}$$

Note that the torque of the central force is zero, because

$$\begin{aligned}
\text{Torque} &= \mathbf{F}(t) \times \mathbf{r}(t)\\
&= f(t)\mathbf{r}(t) \times \mathbf{r}(t)\\
&= 0
\end{aligned} \tag{22}$$

Note that the angular momentum for a central force is constant. We show prove this by definition

$$\begin{aligned}\text{Angular Momentum} &= \mathbf{r}(t) \times m\mathbf{v}(t) \\ &= m(r(t)\hat{\mathbf{r}}(t)) \times (r'(t)\hat{\mathbf{r}}(t) + r(t)\theta'(t)\hat{\theta}) \\ &= m(r(t)\hat{\mathbf{r}}(t) \times r(t)\theta'(t)\hat{\theta}) \\ &= mr^2(t)\theta'(t)\hat{z},\end{aligned} \tag{23}$$

Where $r^2(t)\theta'(t)$ is constant.

## 1.5 Velocity and Acceleration in Polar Coordinates

**Basic Polar Conversions**
The position vector $\mathbf{r}$ can be expressed in polar coordinates as

$$\mathbf{r}(t) = r(t)\hat{\mathbf{r}}, \tag{24}$$

Where

$$\hat{\mathbf{r}}(t) = (cos(\theta(t)), sin(\theta(t))) = cos(\theta)\hat{\imath} + sin(\theta)\hat{\jmath}. \tag{25}$$

Along with the unit vector $\hat{\mathbf{r}}$, the unit vector $\hat{\theta}$ is

$$\hat{\theta} = (-sin(\theta(t)), cos(\theta(t))) = -sin(\theta(t))\hat{\imath} + cos(\theta(t))\hat{\jmath}. \tag{26}$$

Consequently, $\hat{\imath}$ and $\hat{\jmath}$ can be expressed as

$$\begin{aligned}\hat{\imath} &= cos(\theta(t))\hat{\mathbf{r}} - sin(\theta(t))\hat{\theta} \\ \hat{\jmath} &= sin(\theta(t))\hat{\mathbf{r}} + cos(\theta(t))\hat{\theta}\end{aligned} \tag{27}$$

**Velocity**
Recall that $\mathbf{v}(t) = \mathbf{r}'(t)$. Thus,

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) \\ &= (r(t)\hat{\mathbf{r}}(t))' \\ &= r'(t)\hat{\mathbf{r}}(t) + r(t)\hat{\mathbf{r}}'(t) \\ &= r'(t)\hat{\mathbf{r}}(t) + r(t)\theta'(t)\hat{\theta}(t).\end{aligned} \tag{28}$$

Additionally, note that

$$\begin{aligned}\frac{d\hat{\theta}(t)}{dt} &= (-\theta'(t)cos(\theta(t)), -\theta'(t)sin(\theta(t))) \\ &= -\theta'(t)\hat{\mathbf{r}}.\end{aligned} \tag{29}$$

**Acceleration**
Recall that $\mathbf{a}(t)\mathbf{v}'(t)$. Thus,

$$\begin{aligned}\mathbf{a}(t) &= \mathbf{v}'(t) \\ &= r''(t)\hat{\mathbf{r}}(t) + r'(t)\theta'(t)\hat{\theta}(t) \\ &+ (r'(t)\theta'(t) + r(t)\theta''(t))\hat{\theta}(t) \\ &+ r(t)\theta'(t)(-\theta'(t)\hat{\mathbf{r}}(t)) \\ &= (r''(t) - r(t)(\theta'(t))^2)\hat{\mathbf{r}}(t) \\ &+ (r'(t)\theta'(t) + r'(t)\theta'(t) + r(t)\theta''(t))\hat{\theta}(t)\end{aligned} \tag{30}$$

## 1.6 Path Curvature

**Parameterized Curves**
The curvature of a smooth curve describes how have a curve is changing directions as a given point. The formal definition of curvature is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|, \tag{31}$$

While two easier alternate formulas are

$$\kappa = \frac{||\mathbf{T}(t)||}{||\mathbf{r}'(t)||} = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}. \tag{32}$$

Recall that the unit tangent $\mathbf{T}$ is just

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}. \tag{33}$$

**Regular Cartesian Functions**
In the case of functions in the form of $y = f(x)$, the curvature is calculated as

$$\kappa = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{\frac{3}{2}}}. \tag{34}$$

Note that if $y = f(x)$ is parameterized as $x(t) = t$ and $y(t) = f(x(t))$, then the formula for curvature is simplified into

$$\kappa = \frac{y''}{(1 + y'^2)^{\frac{3}{2}}}. \tag{35}$$

## 1.7 Integral of Parameterized Curves

Recall that the integral of a function $f(x)$ is computed as

$$\text{Area} = \int_{x_1}^{x_2} f(x)dx = \int_{x_1}^{x_2} ydx. \tag{36}$$

For a parameterized curve, $x(t) = t$ and $y(t) = f(x(t))$. Thus,

$$\text{Area} = \int_{t_0}^{t_1} y(t)dx(t). \tag{37}$$

For example, suppose that $x(t) = cosh(t)$ and $y(t) = sinh(t)$, then

$$\text{Area} = \int_{t_0}^{t_1} sinh(t)sinh(t)dx, \tag{38}$$

Because

$$\frac{dx}{dt} = \frac{d}{dt}cosh(t) = sinh(t) \tag{39}$$

Or

$$dx = sinh(t)dt. \tag{40}$$

# 2 Arc Length

## 2.1 Definition of Arc Length

The length of a path $\mathbf{c}(t)$ is the integral of the speed $||\mathbf{c}'(t)||$, or

$$L(\mathbf{c}) = \int_{t_0}^{t_1} ||\mathbf{c}'(t)||dt. \tag{41}$$

For example, the length of the path $\mathbf{c}(t) = (x(t), y(t), z(t))$ for $t_0 \leq t \leq t_1$, is

$$L(\mathbf{c}) = \int_{t_0}^{t_1} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt. \tag{42}$$

Note that if a curve is made up of a finite number of pieces, each of which is $C^1$ (with bounded derivative), we compute the arc length by adding the lengths of the component pieces. Such curves are called piecewise $C^1$, or sometimes "piecewise smooth".

To understand the derivation of the equation above, consider a situation where we have dsicrete points of a path. to approximate the distance between $\mathbf{c}(t_i)$ and $\mathbf{c}(t_{i+1})$, the distance is

$$\begin{aligned} ||\mathbf{c}(t_{i+1}) - \mathbf{c}(t_i)|| &= \sqrt{(x(t_{i+1}) - x(t_i))^2 + (y(t_{i+1}) - y(t_i))^2 + (z(t_{i+1}) - z(t_i))^2} \\ &= \sqrt{(x'(t_i)(t_{i+1} - t_i))^2 + (y'(t_i)(t_{i+1} - t_i))^2 + (z'(t_i)(t_{i+1} - t_i))^2} \\ &= (t_{i+1} - t_i)\sqrt{x'(t_i)^2 + y'(t_i)^2 + z'(t_i)^2} \end{aligned} \tag{43}$$

We then arrive at

$$\sum_{i=1}^{N}\left((t_{i+1}-t_i)\sqrt{x'(t_i)^2+y'(t_i)^2+z'(t_i)^2}\right) \tag{44}$$

As we take the limit $N \to \infty$, the sum converted to the integral

$$\text{Arc Length} = \int_{t=a}^{t=b} \sqrt{x'(t)^2+y'(t)^2+z'(t)^2}dt. \tag{45}$$

## 2.2 Differential of Arc Length

An infiniteisimal displacement of a particle following a path $\mathbf{c}(t) = x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$ is

$$d\mathbf{s} = dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k} = \left(\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}+\frac{dz}{dt}\mathbf{k}\right)dt, \tag{46}$$

Where its length is the differential of the arc length, or

$$ds = \sqrt{dx^2+dy^2+dz^2} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}dt. \tag{47}$$

The formula above helps us remember the arc-length formula as

$$\text{Arc Length} = \int_{t_0}^{t_1} ds. \tag{48}$$

The formula can be easy generalized to $n$-dimensions. Consider a piecewise $C^1$ path $\mathbf{c}(t) : [t_0, t_1] \to \mathbb{R}^n$, its length is defined as

$$L(\mathbf{c}) = \int_{t_0}^{t_1} ||\mathbf{c}'(t)||dt = \int_{t_0}^{t_1} \sqrt{(x_1'(t))^2+(x_2'(t))^2+\cdots+(x_n'(t))^2}\, dt. \tag{49}$$

## 2.3 Arc Length of $y = f(x)$

When we have the function $y = f(x)$ for $a \leq x \leq b$, we can parameterize the function as

$$\begin{aligned}\mathbf{c}(x) &= (x, f(x))\\ \mathbf{c}'(x) &= (1, f'(x))\end{aligned} \tag{50}$$

Hence, the arc length is simply

$$\begin{aligned}L &= \int_a^b ||\mathbf{c}'(x)||dx\\ &= \int_a^b \sqrt{1^2+(f'(x))^2}dx\end{aligned} \tag{51}$$

### 2.4 Reparameterization of a Curve

When reparameterizing a curve, we essentially want to express arc length $L$ as $s(t)$. where we reexpress the original arc length $L$

$$L(t) = \int_a^b ||\mathbf{r}'(t)||dt \qquad \text{as} \qquad s(t) = \int_a^t ||\mathbf{r}'(\tau)||d\tau. \tag{52}$$

Let $\alpha : [a, b] \to \mathbb{R}$, where it is strictly increasing and is of class $C^1$. Let the path $\mathbf{c} : [a, b] \to \mathbb{R}^3$. We reparameterize as follows. If a value $s$ is in $[\alpha(a), \alpha(b)]$, then there is a $t$ such that $\alpha(t) = s$. We define the reparameterized as d, or

$$\begin{aligned} \mathbf{d}(s) &= \mathbf{c}(t) \\ \mathbf{d}(\alpha(t)) &= \mathbf{c}(t) \end{aligned} \tag{53}$$

Thus,

$$\mathbf{c}'(t) = \mathbf{d}'(\alpha(t))\alpha'(t). \tag{54}$$

Arc length reparameterization

$$\alpha(t) = \int_a^t ||\mathbf{c}'(\tau)||d\tau. \tag{55}$$

# 3 Vector Fields

### 3.1 Concept of Vector Field

A vector field in $\mathbb{R}^n$ is a map $\mathbf{F} : A \subset \mathbb{R}^n \to \mathbb{R}^n$ that assigns to each point $\mathbf{x}$ in its domain $A$ a vector $\mathbf{F}(\mathbf{x})$. If $n = 2$ and $n = 3$, $\mathbf{F}$ is a vector field in the plane and in the space, respectively.

A vector field on $\mathbf{R}^n$ has $n$ scalar component fields $F_1, F_2, \ldots, F_n$. For example, a vector field $\mathbf{F}(x, y, z)$ on $\mathbf{R}^3$ has three component scalar fields $F_1$, $F_2$, and $F_3$, such that

$$\mathbf{F}(x, y, z)) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)). \tag{56}$$

### 3.2 Gradient Vector Fields

A gradient vector field is often denoted as $\nabla f$, which describes the direction and magnitude of a gradient. For example, a three-dimensional gradient vector field in space is defined as

$$\nabla f(x, y, z) = \frac{\delta f}{\delta x}(x, y, z)\mathbf{i} + \frac{\delta f}{\delta y}(x, y, z)\mathbf{j} + \frac{\delta f}{\delta z}(x, y, z)\mathbf{k}. \tag{57}$$

### 3.3 Flow Lines

If $\mathbf{F}$ is a vector field, a flow line for $\mathbf{F}$ is a path $\mathbf{c}(t)$ such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)). \tag{58}$$

In other words, it can be visualized as the pathway a particle takes along a vector field. As the equation above suggests, a flow line can be viewed as a system of differential equations. For example, in a 3D space we can write $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$ as

$$\begin{aligned} x'(t) &= P(x(t), y(t), z(t)), \\ y'(t) &= Q(x(t), y(t), z(t)), \\ z'(t) &= R(x(t), y(t), z(t)), \end{aligned} \tag{59}$$

Where $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}. \tag{60}$$

# 4 Divergence and Curl

### 4.1 Divergence

We define the divergence of a vector field $\mathbf{F}$ by taking the dot product $\nabla$ with $\mathbf{F}$. For example, for a 3D field vector,

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}. \tag{61}$$

For a velocity field of a fluid, then div $\mathbf{F}$ represents the rate of expansion per unit volume under the flow of the fluid. For example, if div $\mathbf{F} < 0$, the gas is compression. Likewise, in a 2D velocity field, the divergence measures the rate of expansion of area.

### 4.2 Curl

We define the curl of a vector field $\mathbf{F}$ by taking the cross product of $\nabla$ with $\mathbf{F}$. For example, for a 3D field vector $\mathbf{F}$,

$$\begin{aligned}\text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\delta F_3}{\delta y} - \frac{\delta F_2}{\delta z}\right)\mathbf{i} + \left(\frac{\delta F_1}{\delta z} - \frac{\delta F_3}{\delta x}\right)\mathbf{j} + \left(\frac{\delta F_2}{\delta x} - \frac{\delta F_1}{\delta y}\right)\mathbf{k}\end{aligned} \tag{62}$$

Unlike divergence, which can be defined in $\mathbf{R}^n$ for any $n$, we define the curl only in 3D space (or for planar vector fields, regarding their third component as zero). The curl is associated with rotational motion for solid objects and fluids.

## 4.3 Properties of Divergences and Curls

The curl of a gradient for any $C^2$ function $f$ is zero. In other words, gradients are curl free, or

$$\nabla \times (\nabla f) = \mathbf{0}. \tag{63}$$

If $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a vector field in the plane, it can also be regarded as a vector field in space where $\mathbf{k}$ is zero and the other two components are independent of $z$. The curl of $\mathbf{F}$ then reduces to

$$\nabla \times \mathbf{F} = \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right)\mathbf{k}. \tag{64}$$

So the curl of $\mathbf{F}$ always points in the $\mathbf{k}$ direction and the value of $\mathbf{k}$ is called the scalar curl of $\mathbf{F}$.

As you may notice already, the Laplace operator $\nabla^2$ that operates of functions $f$ is the divergence of the gradient, or

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} + \frac{\delta^2 f}{\delta z^2} \tag{65}$$

Now let use note that basic identities of vector analysis:

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(cf) = c\nabla f$, for a constant $c$
3. $\nabla(fg) = f\nabla g + g\nabla f$

4. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$, at points $\mathbf{x}$ where $g(\mathbf{x} \neq 0)$
5. div $(\mathbf{F} + \mathbf{G}) =$ div $\mathbf{F}$ + div $\mathbf{G}$
6. curl $(\mathbf{F} + \mathbf{G}) =$ curl $\mathbf{F}$ + curl $\mathbf{G}$
7. div $(f\mathbf{F}) = f$ div $\mathbf{F} + \mathbf{F} \cdot \nabla f$
8. div $(\mathbf{F} \times \mathbf{G}) = \mathbf{G}\cdot$ curl $\mathbf{F} - \mathbf{F}\cdot$ curl $\mathbf{G}$
9. div curl $\mathbf{F} = 0$
10. curl $(f\mathbf{F}) = f$ curl $\mathbf{F} + \nabla f \times \mathbf{F}$
11. curl $\nabla f = \mathbf{0}$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f \cdot \nabla g)$
13. div $(\nabla f \times \nabla g) = 0$
14. div $(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$