Vector Calculus

V: Change of Variables Formula and Applications of Integration

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1 Geometry of Maps from $\mathbf{R}^2$ to $\mathbf{R}^2$

1.1 Geomatric Mapping

Let D^* be a subset of $\mathbb{R}^2$, suppose we have a continuously differentiable map $T: \ D^* \to \mathbb{R}^2$, so T takes points in D^* to points in $\mathbb{R}^2$. We denote the set of image points by D or by $T(D^*)$. Hence, $D = T(D^*)$ is the set of all points $(x, y) \,\epsilon\, \mathbb{R}^2$ such that

$$(x, y) = T(x^*, y^*) \text{ for some } (x^*, y^*) \,\epsilon\, D^*. \tag{1}$$

1.2 Images of Maps

Let A be a 2×2 matrix with $\det A \neq 0$ and let T be the linear mapping of $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $T(\mathbf{x}) = A\mathbf{x}$. Then T transforms parallelograms into parallelograms and the vertices into vertices. Moreover, if $T(D^*)$ is a parallelogram, D^* must be a parallelogram.

1.3 One-to-One Maps

A mapping T is one-to-one on D^* if for (u, v) and $(u', v') \,\epsilon\, D^*$, $T(u, v) = T(u', v')$ implies that $u = u'$ and $v = v'$. This statement implies that two different points of D^* are never sent to the same point of D by T.

1.4 Onto Maps

The mapping T is onto D if for every point $(x, y) \,\epsilon\, D$ there exists at least one point (u, v) in the domain of T such that $T(u, v) = (x, y)$. Thus, if T is onto, we can solve the equation $T(u, v) = (x, y)$ for (u, v), given $(x, y) \,\epsilon\, D$.

A linear transformation of $\mathbb{R}^n$ to $\mathbb{R}^n$ given by mutiplication by a matrix A is one-to-one and onto when and only when $\det A \neq 0$.

2 The Change of Variables Theorem

Given two regions D and D^* in $\mathbb{R}^2$, a differentiable map T on D^* with image D, and ny real-valued integrable function $f : D \to \mathbb{R}$, we would like to express $\iint_D f(x, y)dA$ as an integral over D^* of the composite function $f \circ T$.

To do this, we assume that D^* is a region in the uv plane and that D is a region in the xy plane. The map T is given by two coordinate functions:

$$T(u,v) = (x(u,v), y(u,v)) \text{ for } (u,v) \;\epsilon\; D^*. \tag{2}$$

We may initially conjecture that

$$\iint_D f(x,y)\; dx\; dy = \int_{D^*} f(x(u,v), y(u,v)\; du\; dv, \tag{3}$$

Where $f \circ T(u,v) = f(x(u,v), y(u,v))$ is the composite function defined on D^*. But if we consider the function $f : D \to \mathbb{R}^2$ where $f(x,y) = 1$, then the equation above would imply

$$A(D) = \iint_D dx\; dy \stackrel{?}{=} \iint_{D^*} du\; dv = A(D^*). \tag{4}$$

The equation above only holds for special cases, and not for a general map T.

2.1 Jacobian Determinants

To find $A(D^*)$, we need to measure how a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ distorts the area of a region. This is given by the Jacobian determinant.

Let $T : \;\; D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 transformation given by $x = x(u,v)$ and $y = y(u,v)$. The Jacobian determinant of T, written $\delta(x,y)/\delta(u,v)$, is the determinant of the derivative matrix $\mathbf{D}T(u,v)$ of T:

$$\frac{\delta(x,y)}{\delta(u,v)} = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix} \tag{5}$$

Under certain restrictions on T, we can show that the area of $D = T(D^*)$ is obtained by integrating the absolute value of the Jacobian $\delta(x,y)/\delta(u,v)$ over D^*, or

$$A(D) = \iint_D dx\; dy = \iint_{D^*} \left| \frac{\delta(x,y)}{\delta(u,v)} \right| du\; dv. \tag{6}$$

2.2 Change of Variables Formula

Recall the method of substitution

$$\int_a^b f(x(u))\frac{dx}{du}du = \int_{x(a)}^{x(b)} f(x)\ dx, \tag{7}$$

Where f is continuous and $u \mapsto x(u)$ is continuously differentiable on $[a, b]$.

Suppose we have a C^1 function $u \mapsto x(u)$ that is one-to-one on $[a, b]$. Thus, w must have either $dx/du \geq 0$ on $[a, b]$ or $dx/du \leq 0$ on $[a, b]$. Let I^* denote the interval $[a, b]$, and let I denote the closed interval with endpoints $x(a)$ and $x(b)$. With these conventions, we can rewrite Equation (7) as

$$\int_{I^*} f(x(u)) \left|\frac{dx}{du}\right| du = \int_I f(x)dx. \tag{8}$$

Let D and D^* be elementary regions in the plane and let $T :\rightarrow D^* \rightarrow D$ be of class C^1. Suppose that T is one-to-one on D^*. Furtherfore, suppose that $D = T(D^*)$. Then for any integrable function $f : D \rightarrow \mathbb{R}$, we have

$$\iint_D f(x, y)dx\ dy = \iint_{D^*} f(x(u, v), y(u, v)) \left|\frac{\delta(x, y)}{\delta(u, v)}\right| du\ dv. \tag{9}$$

2.3 Polar Coordinates

One of the purposes of the change of variables theorem is to provide a method by which some double integrals can be simplified. For example, we can use it on polar coordinates where

$$\iint_D f(x, y)dx\ dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)r\ dr\ d\theta. \tag{10}$$

2.4 Change of Variables Formula for Triple Integrals

The Jacobian of a transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ (and beyond) is just a simple extension of the two-variable case. Let $T : W \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a C^1 function defined by the equations $x = x(u, v, w), y = y(u, v, w), z = z(u, v, w)$. Then the Jacobian of T, which is denoted $\delta(x, y, z)/\delta(u, v, w)$, is the determinant

$$\begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} & \frac{\delta x}{\delta w} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} & \frac{\delta y}{\delta w} \\ \frac{\delta z}{\delta u} & \frac{\delta z}{\delta v} & \frac{\delta z}{\delta w} \end{vmatrix}. \tag{11}$$

Just as in the two-variable case, the Jacobian measures how the transformation T distorts the volume of its domain. Hence, for triple (volume) integrals, the change of variables formula is

$$\begin{aligned} &\iiint_W f(x,y,z)\ dx\ dy\ dz \\ &\quad = \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\delta(x,y,z)}{\delta(u,v,w)}\right|\ du\ dv\ dw, \end{aligned} \tag{12}$$

Where W^* is an elementary region in the uvw space corresponding to W in the xyz space, under a mapping $T : (u,v,w) \mapsto (w(u,v,w), y(u,v,w), z(u,v,w))$, provided that T is of class C^1 and is one-to-one.

2.5 Cylindrical Coordinates

An example of a triple integral change of coordinates is converting Cartesian coordinates to cylindrical coordinates. Recall that

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z, \tag{13}$$

Where

$$\frac{\delta(x,y,z)}{\delta(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r. \tag{14}$$

Thus,

$$\iiint_W f(x,y,z)\ dx\ dy\ dz = \iiint_W f(r\cos\theta, r\sin\theta, z)\ r\ dr\ d\theta\ dz. \tag{15}$$

2.6 Spherical Coordinates

Meanwhile, for spherical coordinates recall that

$$x = \rho\text{sin}\phi\ \text{cos}\theta, \quad y = \rho\text{sin}\phi\ \text{sin}\theta, \quad z = \rho\text{cos}\phi, \tag{16}$$

Where

$$\frac{\delta(x,y,z)}{\delta(\rho,\theta,\phi)} = \begin{vmatrix} \text{sin}\phi\ \text{cos}\theta & -\rho\text{sin}\phi\ \text{sin}\theta & \rho\text{cos}\phi\ \text{cos}\theta \\ \text{sin}\phi\ \text{sin}\theta & \rho\text{sin}\phi\ \text{cos}\theta & \rho\text{cos}\phi\ \text{sin}\theta \\ \text{cos}\phi & 0 & -\rho\text{sin}\phi \end{vmatrix} \tag{17}$$

If we expand the formula, we get

$$\begin{aligned} &\iiint_W f(x,y,z)\ dx\ dy\ dz \\ &\qquad = \iiint_{W^*} f(\rho\text{sin}\phi\ \text{cos}\theta, \rho\text{sin}\phi\ \text{sin}\theta, \rho\text{cos}\phi)\rho^2\text{sin}\phi\ d\rho\ d\theta\ d\phi. \end{aligned} \tag{18}$$