# Vector Calculus
I: Differentiation

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## Contents

# 1 Geometry of Real-Valued Functions

## 1.1 Functions and Mapping

**Scalar-Valued and Vector-Valued Functions**
Scalar-valued functions have the form $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, for $m = 1$.
Vector-valued functions have the form $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, for $m \geq 2$.

**Graphs** Consider a function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$. A graph of $f$ is a set of points in $\mathbb{R}^{n+1}$, where $\{(x_1, \ldots, x_n, f(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \,\epsilon\, A\}$.

## 1.2 Level Sets: Level Curves and Level Surfaces

Consider a function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$. A level of $f$ is a subset of A where $f$ is a fixed constant.

**Level Set**: $\{\mathbf{x} \,\epsilon\, U | f(\mathbf{x}) = c\} \subset \mathbb{R}^n$.
**Level Curve**: $\{\mathbf{x} \,\epsilon\, U | f(\mathbf{x}) = c\} \subset \mathbb{R}^1$.
**Level Surface**: $\{\mathbf{x} \,\epsilon\, U | f(\mathbf{x}) = c\} \subset \mathbb{R}^2$.

## 1.3 Common 2D and 3D Shapes

**2D Shapes**
Circle: $(x - h)^2 + (y - k)^2 = r^2$. Center is at $(h, k)$ and radius is $r$.
Ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. Center is $(h, k)$. x width is $2a$ and y width is $2b$.
Parabola: $y = a(x - h)^2 + k$. Vertex is at $(h, k)$. Steepness is $a$.
Hyperbola: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$. Center is $(h, k)$. Vertices are $(h - a, k)$ and $(h + a, k)$.

**3D Shapes**
Circular Cylinder: $x^2 + y^2 = r^2$ for all $z$.
Parabolic Cylinder: $y = a(x - h)^2 + k$ for all $z$.
Sphere: $(x - h)^2 + (y - k)^2 + (z - n)^2 = r^2$.
Paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$.
Hyperbolic Paraboloid: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$.
Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$.
Hyperboloid (1 Sheet): $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.
Hyperboloid (2 Sheet): $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

# 2 Limits and Continuity

## 2.1 Definition and Properties of Limits

**Definition**
For a given limit $\lim_{x\to x_0} f(x) = b$, $f(x)$ is close to $b$ whenever $x$ is close to $x_0$, but not equal to $x_0$. Note that this does not necessarily mean $\lim_{x\to x_0} f(x) = f(x_0)$.

**Properties**
Scalar Property
If $\lim_{x\to x_0} f(x) = b$, then $\lim_{x\to x_0} cf(x) = cb$.

Additive Property
If $\lim_{x\to x_0} f(x) = b_1$ and $\lim_{x\to x_0} g(x) = b_2$, then $\lim_{x\to x_0}(f+g)(x) = b_1 + b_2$.

Multiplicative Property
If $\lim_{x\to x_0} f(x) = b_1$ and $\lim_{x\to x_0} g(x) = b_2$, then $\lim_{x\to x_0}(fg)(x) = b_1 b_2$.

Reciprocal Property
If $\lim_{x\to x_0} f(x) = b \neq 0$ and $f(x) \neq 0$, then $\lim_{x\to x_0} 1/f(x) = 1/b$.

## 2.2 Finding the Value of a Limit That Exists

**Plug in Values**
In the simplest case, to find the value of a limit, just plug in the values of the variables. For example, the value of the following limit

$$\lim_{(x,y)\to(5,-3)} \left(x^5 + 4x^3y - 6xy^2\right) = 5^5 + 4(5)^3(-3) - 6(5)(-3)^2 = 1355. \tag{1}$$

**Squeeze Theorem**
If the first method is not possible, we can use the squeeze theorem. For example, consider the value of the following limit

$$\lim_{(x,y)\to(0,0)} \frac{5xy^2}{x^2+y^2}. \tag{2}$$

We can show that

$$\begin{aligned} 0 \le \frac{5xy^2}{x^2+y^2} \le 5x, \quad x > 0 \\ 5x \le \frac{5xy^2}{x^2+y^2} \le 0, \quad x < 0. \end{aligned} \tag{3}$$

Hence, the limit of the function above is equal to zero.

**Rationalize**
We can also rationalize the limit. For example, consider the following limit

$$\lim_{(x,y)\to(0,0)} \frac{4(x^2+y^2)}{\sqrt{x^2+y^2+16}-4}. \tag{4}$$

We can rationalize it the following way

$$\begin{aligned} \frac{4(x^2+y^2)}{\sqrt{x^2+y^2+16}-4}\left(\frac{\sqrt{x^2+y^2+4^2}+4}{\sqrt{x^2+y^2+4^2}+4}\right) &= \frac{4(x^2+y^2)\sqrt{x^2+y^2+4^2}+4}{(x^2+y^2)} \\ &= 4\sqrt{x^2+y^2+4^2}+4. \end{aligned} \tag{5}$$

Clearly, the value of the limit now is

$$\lim_{(x,y)\to(0,0)} 4\sqrt{x^2+y^2+4^2}+4 = 4\sqrt{4^2}+4 = 20. \tag{6}$$

**Convert to Polar Equation**
Still, some limits are still not solvable. This is when converting the equation to polar form may be helpful. For example, consider the following limit converted into polar form

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \lim_{r\to0} \frac{(rcos\theta)^2(rsin\theta)}{r^2} = \lim_{r\to0} rcos\theta sin\theta = 0. \tag{7}$$

## 2.3 Showing That a Limit Does Not Exist

In order for a limit to exist, the value of the supposed limit should be the same regardless of the path (including nonlinear paths) that arrive at the given coordinates. Consequently, showing that the limit does not exist, often requires some creativity in finding a contradictory (and often nonlinear) path.

For example, consider the following limit

$$\lim_{(x,y)\to(0,0)} \frac{5xy^4}{x^2+y^8}. \tag{8}$$

It is standard to first find the value of the supposed limit along the x-axis and y-axis paths. Then, usually a third path is necessary to find a contradictory value of the supposed limit, as shown below

$$\begin{aligned} \lim_{(x,0)\to(0,0)} \frac{5xy^4}{x^2+y^8} &= \lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0, \\ \lim_{(0,y)\to(0,0)} \frac{5xy^4}{x^2+y^8} &= \lim_{(0,y)\to(0,0)} \frac{0}{y^8} = 0, \\ \lim_{(y^4,y)\to(0,0)} \frac{5xy^4}{x^2+y^8} &= \lim_{(y^4,y)\to(0,0)} \frac{5y^8}{2y^8} = \frac{5}{2}. \end{aligned} \tag{9}$$

## 2.4 Determining Set of Points in a Continuous Function

Determining the set of points at which a function is continuous is relatively simple, if the definition of a continuous function is well understood. Common points for discontinuity are:

- Values that would make the denominator in a fraction equal to zero.
- Values that are outside the range of a trigonometric function.
- Values that would make the square root an imaginary number.
- Values found at the boundaries of step-wise functions.

# 3 Differentiation

## 3.1 Definition of Partial Derivative

A differentiable function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ has partial derivatives at $\mathbf{x_0}$, where

$$\begin{aligned} \frac{\delta f_i}{\delta x_j} &= \lim_{h \to 0} \frac{f_i(x_1, x_2, \ldots, x_j + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h} \\ &= \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{e_j}) - f(\mathbf{x})}{h}. \end{aligned} \tag{10}$$

## 3.2 Affine Approximation of Derivatives

Consider the equation of the plane tangent to the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto f(x, y)$ at $(x_0, y_0)$

$$z = ax + by + c, \tag{11}$$

Where $a = \frac{\delta f}{\delta x}$ and $b = \frac{\delta f}{\delta y}$. Constant $c$ is determined by the fact that $z = f(x_0, y_0)$, so

$$z = f(x_0, y_0) + \left[\frac{\delta f}{\delta x}(x_0, y_0)\right](x - x_0) + \left[\frac{\delta f}{\delta y}(x_0, y_0)\right](y - y_0). \tag{12}$$

## 3.3 Differentiability of Functions

**Functions of Two Variables**
A function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{x_0}$ if the partial derivatives of $f$

exist at $\mathbf{x_0}$ and the linear approximation is a good approximation. For example, let $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$, we say $f$ is differentiable if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \left[\frac{\delta f}{\delta x}(x-x_0)\right] - \left[\frac{\delta f}{\delta y}(y-y_0)\right]}{||(x,y)-(x_0,y_0)||} = 0, \qquad (13)$$

**Functions of Multiple Variables**
For the more general case of a function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, the function is differentiable at $\mathbf{x_0}$ if the partial derivatives of $f$ exist at $\mathbf{x_0}$ and if

$$\lim_{\mathbf{x}\to\mathbf{x_0}} \frac{||f(\mathbf{x}) - f(\mathbf{x_0}) - \mathbf{T}(\mathbf{x}-\mathbf{x_0})||}{||\mathbf{x}-\mathbf{x_0}||} = 0. \qquad (14)$$

Where $\mathbf{T} = \mathbf{D}f(\mathbf{x_0})$ is a $m \times n$ with matrix elements $\frac{\delta f_i}{\delta x_j}$ evaluated at $\mathbf{x_0}$ and $\mathbf{T}(\mathbf{x}-\mathbf{x_0})$ means the product of $\mathbf{T}$ with $\mathbf{x}-\mathbf{x_0}$. We call $\mathbf{T}$ the **derivative** of $f$ at $\mathbf{x_0}$.

**Matrix of Partial Derivatives** In the more generalized case, where $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$, the matrix $\mathbf{D}f(\mathbf{x_0})$, or the matrix of partial derivatives of $f$ at $\mathbf{x_0}$, is defined as

$$\begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \dots & \frac{\delta f_1}{\delta x_n} \\ \dots & \dots & \dots \\ \frac{\delta f_m}{\delta x_1} & \dots & \frac{\delta f_m}{\delta x_n}. \end{bmatrix} \qquad (15)$$

**Gradients**
In the specific case of $f : U \subset \mathbb{R}^n \to \mathbb{R}$, $\mathbf{D}f(\mathbf{x})$ is a $1 \times n$ matrix, where

$$\mathbf{D}f(\mathbf{x_0}) = \begin{bmatrix} \frac{\delta f}{\delta x_1} & \dots & \frac{\delta f}{\delta x_n} \end{bmatrix}. \qquad (16)$$

The corresponding vector $(\delta f/\delta x_1, \dots, \delta f/\delta x_n)$ below is called the gradient, denoted by $\nabla f$. Consequently, given a vectorized step size of $h$,

$$\mathbf{D}f(\mathbf{x})(\mathbf{h}) = \nabla f(\mathbf{x}) \cdot \mathbf{h}. \qquad (17)$$

**Theorem Relating Differentiability and Continuity**
1. If $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^m$ is differentiable, then $f$ is continuous.
2. Suppose $f$ has continuous partial derivatives in some ball of radius $\epsilon$ about $\mathbf{x_0}$, or $B_\epsilon(\mathbf{x_0}) = \{\mathbf{x} \,\epsilon\, \mathbb{R}^n : ||\mathbf{x}-\mathbf{x_0}|| < \epsilon\}$. Then $f$ is differentiable at $\mathbf{x_0}$.

## 3.4 Leibniz Integral Rule

The Leibniz integral rule states that

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = \frac{db(x)}{dx} f(x,b(x)) - \frac{da(x)}{dx} f(x,a(x)) + \int_{a(x)}^{b(x)} \frac{\delta}{\delta x} f(x,t)dt. \qquad (18)$$

For example,

$$\frac{d}{dy}\left(\int_y^x cos(t^4)dt\right) = -cos(y^4). \tag{19}$$

# 4 Introduction to Paths and Curves

## 4.1 Tangent Line of Polar Equation

Recall that

$$\begin{aligned} x &= rcos\theta, \\ y &= rsin\theta. \end{aligned} \tag{20}$$

Consequently, we know that the following derivatives $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ are

$$\begin{aligned} \frac{dx}{d\theta} &= f'(\theta)cos\theta - f(\theta)sin\theta = \frac{dr}{d\theta}cos\theta - rsin\theta, \\ \frac{dy}{d\theta} &= f'(\theta)sin\theta + f(\theta)cos\theta = \frac{dr}{d\theta}sin\theta + rcos\theta. \end{aligned} \tag{21}$$

Thus, the derivative $\frac{dy}{dx}$ is

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}sin\theta + rcos\theta}{\frac{dr}{d\theta}cos\theta - rsin\theta}. \tag{22}$$

To calculate the equation of the tangent line, we must also solve for $x_0 = rcos\theta$ and $y_0 = rsin\theta$, to get the following solution

$$y - y_0 = \frac{dx}{dy}(x - x_0). \tag{23}$$

## 4.2 Velocity and Tangents to Paths

**Definition of a Path and Curve**
A path in $\mathbb{R}^n$ is a map "$[a,b] \to \mathbb{R}^n$. A collection $C$ of points $\mathbf{c}(t)$ as $t$ varies in the interval $[a,b]$ is called a curve. The path $\mathbf{c}$ is said to parametrize the curve C. For example, is $\mathbf{c}$ is a path in $\mathbb{R}^3$, we can write $\mathbf{c}(t) = (x(t), y(t), z(t))$.

**Velocity and Tangents** If $\mathbf{c}(t)$ is the position of a particle, then $\mathbf{c}'(t)$ is the velocity of the particle and $\mathbf{I}(t) = \mathbf{c}(t_0) + (t - t_0)\mathbf{c}'(t_0)$ is the tangent line to the path at $\mathbf{c}(t_0)$.

### 4.3 Parameterization of Common Shapes

**Linear Line**

$$\mathbf{c}(t) = \mathbf{x_0} + t\mathbf{v} \tag{24}$$

$$\mathbf{c}'(t) = \mathbf{v} \tag{25}$$

**Parabola**

$$\mathbf{c}(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix} \tag{26}$$

$$\mathbf{c}'(t) = \begin{bmatrix} 1 \\ 2t \end{bmatrix} \tag{27}$$

**Circle**

$$\mathbf{c}(t) = \begin{bmatrix} cos(t) \\ sin(t) \end{bmatrix} \tag{28}$$

**Helix**

$$\mathbf{c}(t) = \begin{bmatrix} cos(t) \\ sin(t) \\ t \end{bmatrix} \tag{29}$$

$$\mathbf{c}'(t) = \begin{bmatrix} -sin(t) \\ cos(t) \\ 1 \end{bmatrix} \tag{30}$$

**Ellipse**

$$\mathbf{c}(t) = \begin{bmatrix} sin(t) \\ 2cos(t) \end{bmatrix} \tag{31}$$

**Cycloid**

$$\mathbf{c}(t) = \begin{bmatrix} t - tsin(t) \\ 1 - cos(t) \end{bmatrix} \tag{32}$$

# 5 Properties of the Derivative

### 5.1 Sums, Products, Quotients

**Constant Multiple Rule**
Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\mathbf{x_0}$ and let $c$ be a real number. Then $h(\mathbf{x}) = cf(\mathbf{x})$ is differenitable at $\mathbf{x_0}$ and

$$\mathbf{D}h(\mathbf{x_0}) = c\mathbf{D}f(\mathbf{x_0}). \tag{33}$$

**Sum Rule**
Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\mathbf{x_0}$. Then $h(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x})$ is differentiable at $\mathbf{x_0}$ and

$$\mathbf{D}h(\mathbf{x_0}) = \mathbf{D}f(\mathbf{x_0}) + \mathbf{D}g(\mathbf{x_0}). \tag{34}$$

**Product Rule** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\mathbf{x_0}$. Then $h(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ is differentiable at $\mathbf{x_0}$ and

$$\mathbf{D}h(\mathbf{x_0}) = g(\mathbf{x_0})\mathbf{D}f(\mathbf{x_0}) + f(\mathbf{x_0})\mathbf{D}g(\mathbf{x_0}). \tag{35}$$

**Quotient Rule**
Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\mathbf{x_0}$, where $g$ is never zero on $U$. Then $h(\mathbf{x}) = f(\mathbf{x})/g(\mathbf{x})$ is differentiable at $\mathbf{x_0}$ and

$$\mathbf{D}h(\mathbf{x_0}) = \frac{g(\mathbf{x_0})\mathbf{D}f(\mathbf{x_0}) - f(\mathbf{x_0})\mathbf{D}g(\mathbf{x_0})}{[g(\mathbf{x_0})]^2}. \tag{36}$$

## 5.2 Chain Rule

**General Chain Rule**
Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. Let $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $F : V \subset \mathbb{R}^m \to \mathbb{R}^p$ be given functions such that $g$ maps $U$ into $V$, so that $f \circ g$ is defined. Suppose $g$ is differentiable at $\mathbf{x_0}$ and $f$ is differentiable at $\mathbf{y_0} = g(\mathbf{x_0})$. Then $f \circ g$ is differentiable at $\mathbf{x_0}$ and

$$\mathbf{D}(f \circ g)(\mathbf{x_0}) = \mathbf{D}f(\mathbf{y_0})\mathbf{D}g(\mathbf{x_0}). \tag{37}$$

**Special Case I**
Suppose $\mathbf{c} : \mathbb{R} \to \mathbb{R}^3$ is a differentiable path and $f : \mathbb{R}^3 \to \mathbb{R}$. Let $h(t) = f(\mathbf{c}(t)) = f(x(t), y(t), z(t))$, where $\mathbf{c}(t) = (x(t), y(t), z(t))$. Then

$$\begin{aligned} \frac{dh}{dt} &= \frac{\delta f}{\delta x}\frac{\delta x}{\delta t} + \frac{\delta f}{\delta y}\frac{\delta y}{\delta t} + \frac{\delta f}{\delta z}\frac{\delta z}{\delta t} \\ &= \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t). \end{aligned} \tag{38}$$

Note that this is the special case of the chain rule when we take $\mathbf{c} = g$ and $f$ to be real-valued and $m = 3$. Notice that

$$\nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \mathbf{D}f(\mathbf{c}(t))\mathbf{D}\mathbf{c}(t). \tag{39}$$

**Special Case II**
Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}^3$. We write

$$g(x, y, z) = (u(x, y, z), v(x, y, z), w(x, y, z)), \tag{40}$$

And define $h : \mathbb{R}^3 \to \mathbb{R}$ by setting

$$h(x, y, z) = f(u(x, y, z), v(x, y, z), w(x, y, z)). \tag{41}$$

In this case, the chain rule states that

$$\begin{bmatrix} \frac{\delta h}{\delta x} & \frac{\delta h}{\delta y} & \frac{\delta h}{\delta z} \end{bmatrix} = \begin{bmatrix} \frac{\delta f}{\delta u} & \frac{\delta f}{\delta v} & \frac{\delta f}{\delta w} \end{bmatrix} \begin{bmatrix} \frac{\delta u}{\delta x} & \frac{\delta u}{\delta y} & \frac{\delta u}{\delta z} \\ \frac{\delta v}{\delta x} & \frac{\delta v}{\delta y} & \frac{\delta v}{\delta z} \\ \frac{\delta w}{\delta x} & \frac{\delta w}{\delta y} & \frac{\delta w}{\delta z}. \end{bmatrix} \tag{42}$$

In this special case, $n = m = 3$ and $p = 1$, and $U = \mathbb{R}^3$ and $V = \mathbb{R}^3$. Above, we have written out the matrix product of $[\mathbf{D}f(\mathbf{y_0})][\mathbf{D}g(\mathbf{y_0})]$ explicitly. To help illustrate the point, we can show $\frac{\delta h}{\delta x}$, $\frac{\delta h}{\delta y}$, and $\frac{\delta h}{\delta z}$, separately

$$\begin{aligned} \frac{\delta h}{\delta x} &= \frac{\delta f}{\delta u}\frac{\delta u}{\delta x} + \frac{\delta f}{\delta v}\frac{\delta v}{\delta x} + \frac{\delta f}{\delta w}\frac{\delta w}{\delta x}, \\ \frac{\delta h}{\delta x} &= \frac{\delta f}{\delta u}\frac{\delta u}{\delta y} + \frac{\delta f}{\delta v}\frac{\delta v}{\delta y} + \frac{\delta f}{\delta w}\frac{\delta w}{\delta y}, \\ \frac{\delta h}{\delta x} &= \frac{\delta f}{\delta u}\frac{\delta u}{\delta z} + \frac{\delta f}{\delta v}\frac{\delta v}{\delta z} + \frac{\delta f}{\delta w}\frac{\delta w}{\delta z}. \end{aligned} \tag{43}$$

# 6 Gradients and Directional Derivatives

## 6.1 Gradients

If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is differentiable, the **gradient** of $f$ at $(x, y, z)$ is the vector in space given by

$$\nabla f = \left( \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right). \tag{44}$$

Note that the gradient vector is always normal to the level surface. If $\nabla f(\mathbf{x}) \neq 0$, then $\nabla f(\mathbf{x})$ points in the direction along which $f$ is increase the fastest. This allows us to create a **gradient vector field**.

## 6.2 Directional Derivatives

If $f : \mathbb{R}^3 \to \mathbb{R}$, the **directional derivative** of $f$ at $\mathbf{x}$ along the vector $\mathbf{v}$ is given by

$$\frac{d}{dt} = f(\mathbf{x} + t\mathbf{v})|_{t=0}. \tag{45}$$

In the definition of a directional derivative, we usually choose $\mathbf{v}$ to be a unit vector, such that we are moving in the direction $\mathbf{v}$ with unit speed.

If $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable, then all directional derivatives exist. The directional derivative at $\mathbf{x}$ in the direction $\mathbf{v}$ is given by

$$\mathbf{D}f(\mathbf{x})\mathbf{v} = \nabla f(\mathbf{x}) \cdot \mathbf{v} = \left[ \frac{\delta f}{\delta x}(\mathbf{x}) \right] v_1 + \left[ \frac{\delta f}{\delta y}(\mathbf{x}) \right] v_2 + \left[ \frac{\delta f}{\delta z}(\mathbf{x}) \right] v_3, \tag{46}$$

Where $\mathbf{v} = (v_1, v_2, v_3)$.