Vector Calculus

VI: Integrals Over Paths and Surfaces

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1 The Path Integral

1.1 Definition of Path Integral

Suppose we have a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$, such that f sends points in $\mathbb{R}^3$ to real numbers. It would be useful to define the integral of such a function f along a path $\mathbf{c} : I = [a,b] \to \mathbb{R}^3$, where $\mathbf{c}(t) = (x(t), y(t), z(t))$.

The path integral, or the integral of $f(x,y,z)$ along the path $\mathbf{c}$, is defined when $\mathbf{c} : I = [a,b] \to \mathbb{R}^3$ is of class C^1 and when the composite function $t \mapsto f(x(t), y(t), z(t))$ is continuous on I. We define this path integral as

$$\int_{\mathbf{c}} f \; ds = \int_f^b (x(t), y(t), z(t))||\mathbf{c}'(t)||dt. \tag{1}$$

Sometimes $\int_{\mathbf{c}} f \; ds$ is denoted as

$$\int_{\mathbf{c}} f(x,y,z) \; ds \tag{2}$$

or

$$\int_a^b f(\mathbf{c}(t))||\mathbf{c}'(t)|| \; dt. \tag{3}$$

If $\mathbf{c}(t)$ is only piecewise C^1 or $f(\mathbf{c}(t))$ is piecewise continuous, we define $\int_{\mathbf{c}} f \; ds$ by breaking $[a,b]$ into pieces over which $f(\mathbf{c}(t))||\mathbf{c}'(t)||$ is continuous, and summing the integral over the pieces.

1.2 The Path Integral for Planar Curves

When the path $\mathbf{c}$ describes a plane curve. Suppose that all points $\mathbf{c}(t)$ lie in the xy plane and f is a real-valued function of two variables. The path integral of f along $\mathbf{c}$ is

$$\int_{\mathbf{c}} f(x,y) \; ds = \int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}dt. \tag{4}$$

When $f(x,y) \geq 0$, this integral can be geometrically interpreted as the "area of a fence".

2 Line Integrals

2.1 Definition of the Line Integral

Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ that is continuous on the C^1 path $\mathbf{c} : [a,b] \to \mathbb{R}^3$. We define $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, the line integral of $\mathbf{F}$ along $\mathbf{c}$, by the formula

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt. \tag{5}$$

That is, we integrate the dot product of $\mathbf{F}$ with $\mathbf{c}'$ over interval $[a,b]$. For paths $\mathbf{c}$ that satisfy $\mathbf{c}'(t) \neq \mathbf{0}$, there is another useful formula for the line integral. Namely, if $\mathbf{T}(t) = \mathbf{c}'(t)/||\mathbf{c}'(t)||$ denotes the unit tangent vector, we have

$$\begin{aligned} \int \mathbf{F} \cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt \\ &= \int_a^b \left[\mathbf{F}(\mathbf{c}(t)) \cdot \frac{\mathbf{c}'(t)}{||\mathbf{c}'(t)||}\right] ||\mathbf{c}'(t)||dt \\ &= \int_a^b [\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{T}(t)]||\mathbf{c}'(t)||dt. \end{aligned} \tag{6}$$

Either Equations (5) or (6) can be used, depending on whichever is easier.

Another common way of writing line integrals is

$$\int_{\mathbf{c}} F_1 dx + F_2 dy + F_3 dz = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}. \tag{7}$$

2.2 Reparametrizations

The line integral $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ depends not only on the field $\mathbf{F}$ but also on the path $\mathbf{c} : [a,b] \to \mathbb{R}^3$. In general, if $\mathbf{c}_1$ and $\mathbf{c}_2$ are two different paths in $\mathbb{R}^3$, $\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} \neq \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$. On the other hand, it is true that $\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} = \pm \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$ for every vector field $\mathbf{F}$ if $\mathbf{c}_1$ is what we call a reparametrization of $\mathbf{c}_2$. This means that $\mathbf{c}_1$ and $\mathbf{c}_2$ are different descriptions of the same geometric curve.

Let $h : I \to I_1$ be a C^1 real-valued function that is a one-to-one map of an interval $I = [a,b]$ onto another interview $I_1 = [a_1, b_1]$. Let $\mathbf{c} : I_1 \to \mathbb{R}^3$ be a piecewise C^1 path. Then we call the composition

$$\mathbf{p} = \mathbf{c} \circ h : I \to \mathbb{R}^3 \tag{8}$$

A reparametrization of $\mathbf{c}$. This means that $\mathbf{p}(t) = \mathbf{c}(h(t))$, and so h changes the variable.

It is implicit in the definition that h must carry the endpoints, where either $h(a) = a_1$ and $h(b) = b_1$, or $h(a) = b_1$ and $h(b) = a_1$. We thus distinguish between two types of parametrizations. If $\mathbf{c} \circ h$ is a reparametrization of $\mathbf{c}$, then either

$$(\mathbf{c} \circ h)(a) = \mathbf{c}(a_1) \text{ and } (\mathbf{c} \circ h)(b) = \mathbf{c}(b_1) \tag{9}$$

or

$$(\mathbf{c} \circ h)(a) = \mathbf{c}(b_1) \text{ and } (\mathbf{c} \circ h)(b) = \mathbf{c}(a_1) \tag{10}$$

In the first case, the reparametrization is orientation-preserving. In the second case, the reparametrization is orientation-reversing.

Let $\mathbf{F}$ be a vector field continuous on the C^1 path $\mathbf{c} : [a_1, b_1] \to \mathbb{R}^3$, and let $\mathbf{p} : [a, b] \to \mathbb{R}^3$ be a reparametrixation of $\mathbf{c}$. If $\mathbf{p}$ is orientation-preserving, then

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}, \tag{11}$$

and if $\mathbf{p}$ is orientation-reversing, then

$$\int_{\mathbf{p}} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}. \tag{12}$$

While the line integral is an oriented integral, the path integral does not have this property. Thus, if we let $\mathbf{c}$ be piecewise C^1, let f be a continuous (real-valued) function on the image $\mathbf{c}$, and let $\mathbf{p}$ be any reparametrization of $\mathbf{c}$, then

$$\int_{\mathbf{c}} f(x, y, z)\ ds = \int_{\mathbf{p}} f(x, y, z)\ ds. \tag{13}$$

2.3 Line Integrals of Gradient Fields

Recall that a vector field $\mathbf{F}$ is a gradient vector field if $\mathbf{F} = \nabla f$ for some real-valued function f. Thus,

$$\mathbf{F} = \frac{\delta f}{\delta x}\mathbf{i} + \frac{\delta f}{\delta y}\mathbf{j} + \frac{\delta f}{\delta z}\mathbf{k}. \tag{14}$$

Suppose g and G are real-valued continuous functions defined on a closed interval $[a, b]$, that is G is differentiable on (a, b) and that $G' = g$. Then by the fundamental theorem of calculus,

$$\int_a^b g(x)dx = G(b) - G(a). \tag{15}$$

Thus, the value of the integral of g depends on the value of G at the endpoints of the interval $[a, b]$. Because ∇f represents the derivative of f, we can ask whether $\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s}$ is completely determined by the value of f at the endpoints $\mathbf{c}(a)$ and $\mathbf{c}(b)$.

Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is of class C^1 and that $\mathbf{c} : [a, b] \to \mathbb{R}^3$ is a piecewise C^1 path. Then

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)). \tag{16}$$

2.4 Line Integrals Over Geometric Curves

We define a simple curve C to be the image of a piecewise C^1 map $\mathbf{c} : I \to \mathbb{R}^3$ that is one-to-one on an interval I; $\mathbf{c}$ is called a parametrization of C. Thus, a simple curve does not intersect itself. If $I = [a, b]$, we call $\mathbf{c}(a)$ and $\mathbf{c}(b)$ endpoints of the curve. Each simple curve C has two orientation or directions associated with it. If P and Q are the endpoints of the curve, then we can consider C as directed from P to Q or from Q to P. The simple curve C together with a sense of direction is called an oriented simple curve or directed simple curve.

By a simple closed curve we mean the image of a piecewise C^1 map $\mathbf{c} : [a, b] \to \mathbb{R}^3$ that is one-to-one $[a, b)$ and satisfies $\mathbf{c}(a) = \mathbf{c}(b)$. If $\mathbf{c}$ satisfies the condition $\mathbf{c}(a) = \mathbf{c}(b)$, but is not necessarily one-to-one on $[a, b)$, we call its image a closed curve. Simple closed curves have two orientations.

For line integrals and path integrals over oriented simple curves and simple closed curves C,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} \quad \text{and} \quad \int_C f \; ds = \int_{\mathbf{c}} f \; ds \tag{17}$$

These integrals do not depend on the choice of $\mathbf{c}$ as long as $\mathbf{c}$ is one-to-one. Although a curve must be parameterized to make integration along it tractable, it is not necessary to include the parametrization in our notation for the integral.

2.5 The $d\mathbf{r}$ Notation for Line Integrals

Sometimes we write a line integral using the notation

$$\int_C \mathbf{F} \cdot d\mathbf{r}. \tag{18}$$

This is because we think of a C^1 path $\mathbf{c}$ in terms of a moving position vector based at the origin and ending at the point $\mathbf{c}(t)$ at time t. Position vectors are often denoted by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, so the curve is described using the notation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ in place of $\mathbf{c}(t)$. By definition, the line integral is given by

$$\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt. \tag{19}$$

We can cancel out the dt's and replace the limits of integration with the geometric curve C, arriving at the notation $\int_C \mathbf{F} \cdot d\mathbf{r}$.

3 Parametrized Surfaces

3.1 Graphs Are Too Restrictive

One kind of surface is the graph of a function $f(x, y)$. However, many surfaces arise as level surfaces of functions. Suppose our surface S is a set of points (x, y, z), where $x - z + z^3 = 0$. Here S is not the graph of some functions $z = f(x, y)$, because this means that for each $(x_0, y_0) \epsilon \mathbb{R}^2$, there must be one z_0 with $(x_0, y_0, z_0) \epsilon S$. By extnding the definition of a graph to three variables, we can think of planes as being "pushed", "twisted", or "rolled" around.

3.2 Parametrized Surfaces as Mappings

Previously we dealt with mappings $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$. Taking $n = 2$ and $m = 3$ correspondings to the case of a 2D surface in 3D space.

A parametrization of a surface is a function $\mathbf{\Phi} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, where D is some domain in $\mathbb{R}^2$. The surface S corresponding to the function $\mathbf{\Phi}$ is its image: $S = \mathbf{\Phi}(D)$. We can write

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)). \tag{20}$$

If $\mathbf{\Phi}$ is differentiable or is of class C^1, we call S differentiable or a C^1 surface. Thus the $\mathbf{\Phi}$ function can be thought as a function taking a 2D plane and distorting it to generate a 3D surface.

3.3 Tangent Vectors to Parametrized Surfaces

Suppose that $\mathbf{\Phi}$ is a parametrized surface that is differentiable at $(u_0, v_0) \epsilon \mathbb{R}^2$. Fixing u at u_0, we get a map $\mathbb{R} \to \mathbb{R}^3$, given by $t \mapsto \mathbf{\Phi}(u_0, t) \epsilon \mathbb{R}^2$, whose image is a curve on the surface. The vector tangent to this curve at the point $\mathbf{\Phi}(u_0, v_0)$, which we denote by $\mathbf{T}_v$, is

$$\mathbf{T}_v = \frac{\delta \mathbf{\Phi}}{\delta v} = \frac{\delta x}{\delta v}(u_0, v_0)\mathbf{i} + \frac{\delta y}{\delta v}(u_0, v_0)\mathbf{j} + \frac{\delta z}{\delta v}(u_0, v_0)\mathbf{k}. \tag{21}$$

Similarly, if we fix v and consider the curve $t \mapsto \mathbf{\Phi}(t, v_0)$, we obtain the tangent vector to this curve at $\mathbf{\Phi}(u_0, v_0)$, given by

$$\mathbf{T}_u = \frac{\delta \mathbf{\Phi}}{\delta u} = \frac{\delta x}{\delta u}(u_0, v_0)\mathbf{i} + \frac{\delta y}{\delta u}(u_0, v_0)\mathbf{j} + \frac{\delta z}{\delta u}(u_0, v_0)\mathbf{k}. \tag{22}$$

3.4 Regular Surfaces

Because the vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ are tangent to two curves on the surface at a given point, the vector $\mathbf{T}_u \times \mathbf{T}_v$ ought to be normal to the surface at the same point.

We say that the surface S is regular or smooth at $\mathbf{\Phi}(u_0, v_0)$, provided that $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$ at (u_0, v_0). This surface is called regular if it is regular at all points $\mathbf{\Phi}(u_0, v_0) \epsilon S$.

3.5 Tangent Plane to a Parametrized Surface

We can use the fact that $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$ is normal to the surface to define the tangent plane and compute it.

If a parametrized surface $\mathbf{\Phi} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ is regular at $\mathbf{\Phi}(u_0, v_0)$ - that is, if $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$ - we define the tangent plane of the surface at $\mathbf{\Phi}(u_0, v_0)$ to be the plane determined by the vectors $\mathbf{T}_u$ and $\mathbf{T}_v$. Thus, $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$ is a normal vector, and an equation of the tangent plane at (x_0, y_0, z_0) is given by

$$(x - x_0, y - y_0, z - z_0) \cdot \mathbf{n} = 0, \tag{23}$$

Where n is evaluated at (u_0, v_0); that is, the tangent plane is the set of (x, y, z) satisfying the condition above. If $\mathbf{n} = (n_1, n_2, n_3) = n_a\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$, then the formula above becomes

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0. \tag{24}$$

4 Area of a Surface

4.1 Definition of Surface Area

We define a parametrized surface S to be the image of a function $\mathbf{\Phi} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, written as $\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v))$. The map $\mathbf{\Phi}$ was called the parametrization of S and S was said to be regular at $\mathbf{\Phi}(u, v)\ \epsilon\ S$ provided that $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$.

In the rest of this chapter and the next one, we will consider only piecewise regular surfaces that are unions of images of parametrized surfaces $\mathbf{\Phi}_i : D_i \to \mathbb{R}^3$ for which

1. D_i is an elementary region in the plane;
2. $\mathbf{\Phi}_i$ is of class C^1 and one-to-one, except possibly on the boundary of D_i; and
3. S_i, the image of $\mathbf{\Phi}_i$, is regular, except possibly at a finite number of points.

We define the surface area $A(S)$ of a parametrized surface by

$$A(S) = \iint_D ||\mathbf{T}_u \times \mathbf{T}_v|| du\ dv, \tag{25}$$

Where $||\mathbf{T}_u \times \mathbf{T}_v||$ is the norm of $\mathbf{T}_u \times \mathbf{T}_v$. If S is a union of surfaces S_i, its area is the sum of the areas of the S_i.

4.2 Surface Area of a Graph

A surface S given in the form $z = g(x, y)$, where $(x, y)\ \epsilon\ D$, admits the parametrization

$$x = u \quad y = v \quad z = g(u, v) \tag{26}$$

For $(u, v) \epsilon D$. When g is of class C^1, this parametrization is smooth and the formula for surface area reduces to

$$A(S) = \iint_D \left(\sqrt{\left(\frac{\delta g}{\delta x}\right)^2 + \left(\frac{\delta g}{\delta y}\right)^2 + 1} \right) dA, \tag{27}$$

After applying the formulas

$$\mathbf{T}_u = \mathbf{i} + \frac{\delta g}{\delta u}\mathbf{k}, \quad \mathbf{T}_v = \mathbf{i} + \frac{\delta g}{\delta v}\mathbf{k}, \tag{28}$$

and

$$\mathbf{T}_u \times \mathbf{T}_v = -\frac{\delta g}{\delta u}\mathbf{i} - \frac{\delta g}{\delta v}\mathbf{j} + \mathbf{k} = -\frac{\delta g}{\delta x}\mathbf{i} - \frac{\delta g}{\delta y}\mathbf{j} + \mathbf{k}. \tag{29}$$

4.3 Surfaces of Revolution

The lateral surface area generated by revolving the graph of a function $y = f(x)$ about the x axis is given by

$$A = 2\pi \int_a^b (|f(x)|\sqrt{1 + [f'(x)]^2})dx. \tag{30}$$

If the graph is revolved about the y axis, the surface area is

$$A = 2\pi \int_a^b (|x|\sqrt{1 + [f'(x)]^2}dx. \tag{31}$$

4.4 Integrals of Scalar Functions Over Surfaces

The integral of a scalar function f over a surface S is analogous to considering the path integral as a generation of arc length.

Let us start with a surface S parametrized by a mapping $\mathbf{\Phi} : D \to S \subset \mathbb{R}^3$, where D is an elementary region, which we write as $\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v))$.

If $f(x, y, z)$ is a real-valued continuous function defined on a parametrized surface S, we define the integral of f over S to be

$$\iint_S f(x,y,z)dS = \iint_S f\ dS = \iint_D f(\mathbf{\Phi}(u,v))||\mathbf{T}_u \times \mathbf{T}_v||du\ dv. \tag{32}$$

4.5 Surface Integrals Over Graphs

Suppose S is a graph of a C^1 function $z = g(x,y)$. Recall that we can parametrize S by setting

$$x = u, \quad y = v, \quad z = g(u,v), \tag{33}$$

And that in this case

$$||\mathbf{T}_u \times \mathbf{T}_v|| = \sqrt{1 + \left(\frac{\delta g}{\delta u}\right)^2 + \left(\frac{\delta g}{\delta v}\right)^2}, \tag{34}$$

So

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,g(x,y))\sqrt{1 + \left(\frac{\delta g}{\delta u}\right)^2 + \left(\frac{\delta g}{\delta v}\right)^2}\ dx\ dy. \tag{35}$$

4.6 Integrals Over Graphs

We can also develop another formula for surface integrals when the surface can be represented as a graph. We let S be the graph of $z = g(x,y)$ and consider the formula from the previous section. We claim that

$$\iint_S f(x,y,z)dS = \iint_D \frac{f(x,y,g(x,y))}{\cos\theta)}\ dx\ dy, \tag{36}$$

Where θ is the angle th enormal to the surface makes with the unit vector $\mathbf{k}$ at the point $(x,y,g(x,y))$. Describing the surface by the equation $\phi(x,y,z) = z - g(x,y) = 0$, a normal vector $\mathbf{N}$ is $\nabla\phi$; that is,

$$\mathbf{N} = -\frac{\delta g}{\delta x}\mathbf{i} - \frac{\delta g}{\delta y}\mathbf{j} + \mathbf{k}. \tag{37}$$

Thus,

$$\cos\theta = \frac{\mathbf{N}\cdot\mathbf{k}}{||\mathbf{N}||} = \frac{1}{\sqrt{(\delta g/\delta x)^2 + (\delta g/\delta y)^2 + 1}}. \tag{38}$$

Note that $\cos\theta = \mathbf{n}\cdot\mathbf{k}$, where $\mathbf{n} = \mathbf{N}/||\mathbf{N}||$ is the unit normal. Thus, we can write

$$d\mathbf{S} = \frac{dx\ dy}{\mathbf{n}\cdot\mathbf{k}}. \tag{39}$$

5 Surface Integrals of Vector Fields

5.1 Definition of the Surface Integral

Let $\mathbf{F}$ be a vector field defined on S, the image of a parametrized surface $\mathbf{\Phi}$. The surface integral of $\mathbf{F}$ over $\mathbf{\Phi}$ is denoted by

$$\iint_{\mathbf{\Phi}} \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{T}_u \times \mathbf{T}_v)\ du\ dv. \tag{40}$$

5.2 Orientation

An oriented surface is a two-sided surface with one side specified as the outside or positive side; the other side is the inside or negative side. At each point $(x, y, z)\ \epsilon\ S$ there are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, where $\mathbf{n}_1 = -\mathbf{n}_2$. To specify the side of a surface S, at each point we choose a unit normal vector $\mathbf{n}$ that points away from the positive side of S at that point.

Let $\mathbf{\Phi} : D \to \mathbb{R}^3$ be a parametrization of an oriented surface S and suppose S is regular at $\mathbf{\Phi}(u_0, v_0), (u_0, v_0)\ \epsilon\ D$; thus, the vector $(\mathbf{T}_{u_0} \times \mathbf{T}_{v_0})/||\mathbf{T}_{u_0} \times \mathbf{T}_{v_0}||$ is defined. If $\mathbf{n}(\mathbf{\Phi}(u_0, v_0))$ denotes the unit normal to S at $\mathbf{\Phi}(u_0, v_0)$, it follows that

$$(\mathbf{T}_{u_0} \times \mathbf{T}_{v_0})/||\mathbf{T}_{u_0} \times \mathbf{T}_{v_0}|| = \pm\mathbf{n}(\mathbf{\Phi}(u_0, v_0)). \tag{41}$$

The parametrization $\mathbf{\Phi}$ is orientation-preserving when using the $+$ sign. In other words, $\mathbf{\Phi}$ is orientation-preserving if the vector $\mathbf{T}_u \times \mathbf{T}_v$ points to the outside of the surface, and orientation-reversing if the vector $\mathbf{T}_u \times \mathbf{T}_v$ points to the inside of the surface.

5.3 Orientation and the Vector Surface Element of a Sphere

Consider the sphere of radius R, where $x^2 + y^2 + z^2 = R^2$. It is standard practice to orient the sphere with the outward unit normal. In terms of the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the outward unit normal is given by

$$\mathbf{n} = \frac{\mathbf{r}}{R}. \tag{42}$$

The surface-area element is then given by

$$d\mathbf{S} = \mathbf{n} \cdot (\mathbf{T}_\phi \times \mathbf{T}_\theta)\ d\phi\ d\theta = \mathbf{r}R\sin\phi\ d\phi\ d\theta = \mathbf{n}R^2\sin\phi\ d\phi\ d\theta. \tag{43}$$

5.4 Independence of Parametrization

The integral over an oriented surface is independent of the parametrization. Let S be an oriented surface and let $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ be two regular orientation-preserving parametrizations, with $\mathbf{F}$ a continuous vector field defined on S. Then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}. \tag{44}$$

If $\mathbf{\Phi}_1$ is orientation-preserving and $\mathbf{\Phi}_2$ is orientation-reversing, then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}. \tag{45}$$

If f is a real-valued continuous function defined on S, and if $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are parametrization of S, then

$$\iint_{\mathbf{\Phi}_1} f\ dS = \iint_{\mathbf{\Phi}_2} f\ dS. \tag{46}$$

5.5 Relation with Scalar Integrals

$\iint_S \mathbf{F} \cdot d\mathbf{S}$, the surface integral of $\mathbf{F}$ over S, is equal to the integral of the normal component of $\mathbf{F}$ over the surface. In short,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\ dS. \tag{47}$$

This observation can often save computational effort.

5.6 Surface Integrals Over Graphs

Let us derive the surface-integral formulas for vector fields $\mathbf{F}$ over surfaces S that are graphs of functions. Consider the surface S described by $z = g(x, y)$, where $(x, y) \epsilon D$, where S is oriented with the upward-pointing unit normal:

$$\mathbf{n} = \frac{-\frac{\delta g}{\delta x}\mathbf{i} - \frac{\delta g}{\delta y}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\frac{\delta g}{\delta x}\right)^2 + \left(\frac{\delta g}{\delta y}\right)^2 + 1}} \tag{48}$$

We have previously seen that we can parametrize S by $\mathbf{\Phi} : D \to \mathbb{R}^3$ given by $\mathbf{\Phi}(x, y) = (x, y, g(x, y))$. In this case, $\iint_S \mathbf{F} \cdot d\mathbf{S}$ can be written in a simple form. We have

$$\mathbf{T}_x = \mathbf{i} + \frac{\delta g}{\delta x}\mathbf{k}, \quad \mathbf{T}_y = \mathbf{j} + \frac{\delta g}{\delta y}\mathbf{k}. \tag{49}$$

Thus, $\mathbf{T}_x \times \mathbf{T}_y = -(\delta g/\delta x)\mathbf{i} - (\delta g)(\delta y)\mathbf{j} + \mathbf{k}$. If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a continuous vector field, we get that the surface integral of a vector field over a graph S is

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot (\mathbf{T}_x \times \mathbf{T}_y)\ dx\ dy \\ &= \iint_D \left[F_1\left(-\frac{\delta g}{\delta x}\right) + F_2\left(-\frac{\delta g}{\delta y}\right) + F_3\right]\ dx\ dy \end{aligned} \tag{50}$$

6 Summary: Formulas for Surface Integrals

Parametrized Surface: $\mathbf{\Phi}(u, v)$

1. Integral of a scalar function f:

$$\iint_S f\ dS = \iint_D f(\mathbf{\Phi}(u, v)||\mathbf{T}_u \times \mathbf{T}_v||\ du\ dv \tag{51}$$

2. Scalar surface element:

$$dS = ||\mathbf{T}_u \times \mathbf{T}_v||\ du\ dv \tag{52}$$

3. Integral of a vector field $\mathbf{F}$:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v)\ du\ dv \tag{53}$$

4. Vector surface element:

$$d\mathbf{S} = (\mathbf{T}_u \times \mathbf{T}_v)\ du\ dv = \mathbf{n}\ dS \tag{54}$$

Graph: $z = g(x, y)$

1. Integral of a scalar function f:

$$\iint_S f\ dS = \iint_D \frac{f(x, y, g(x, y))}{\cos\theta} dx\ dy \tag{55}$$

2. Scalar surface element:

$$dS = \frac{dx\ dy}{\cos\theta} = \sqrt{\left(\frac{\delta g}{\delta x}\right)^2 + \left(\frac{\delta g}{\delta y}\right)^2 + 1}\ dx\ dy \tag{56}$$

   Where $\cos\theta = \mathbf{n} \cdot \mathbf{k}$, and $\mathbf{n}$ is a unit normal vector to the surface.

3. Integral of a vector field $\mathbf{F}$:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-F_1 \frac{\delta g}{\delta x} - F_2 \frac{\delta g}{\delta y} + F_3\right) dx\ dy \tag{57}$$

4. Vector surface element:

$$d\mathbf{S} = \mathbf{n} \cdot dS = \left(-\frac{\delta g}{\delta x}\mathbf{i} - \frac{\delta g}{\delta y}\mathbf{j} + \mathbf{k}\right) dx\ dy \tag{58}$$

Sphere: $x^2 + y^2 + z^2 = R^2$

1. Scalar surface element:

$$dS = R^2 \sin\phi\ d\phi\ d\theta \tag{59}$$

2. Vector surface element:

$$d\mathbf{S} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) R \sin\phi\ d\phi\ d\theta = \mathbf{r} R \sin\phi\ d\phi\ d\theta = \mathbf{n}^2 \sin\phi\ d\phi\ d\theta \tag{60}$$