Algorithms and Data Structures 2018/19
Coursework

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4 Asymptotics

4.1

Prove or disprove $2x^4$ is O(x^3+3x+2). Assuming $2x^4$ is O(x^3+3x+2) implies:

$$
\begin{aligned}
2x^4 &\leq C \cdot (x^3 + 3x + 2) \\
2x^4 &\leq Cx^3 + 3Cx + 2C \\
2x^4 - Cx^3 - 3Cx &\leq 2C \\
x(2x^3 - Cx^2 - 3C) &\leq 2C \\
x &\leq 2C
\end{aligned}
$$

and,

$$2x^3 - Cx^2 - 3C \leq 2C$$

Therefore,

$$k \leq x \leq 2C$$

This is a contradiction as x must not have an upper bound. Therefore, the statement is incorrect.

4.2

Prove or disprove $4x^3 + 2x^2 \cdot log(x) + 1$ is O(x^3). Assuming $4x^3 + 2x^2 \cdot log(x) + 1$ is O(x^3) implies:

$$4x^3 + 2x^2 \cdot \log x + 1 \leq Cx^3$$

Since $\log x < x$:

$$
\begin{aligned}
4x^3 + 2x^3 + x^3 &\leq Cx^3 \\
7x^3 &\leq Cx^3
\end{aligned}
$$

Suppose $C = 7$, the inequality holds for all $x \geq 1$ which means that $k = 1$.

4.3

Prove or disprove $3x^2+7x+1$ is $\omega(x \log x)$. Assuming $3x^2+7x+1$ is $\omega(x \log x)$ implies:

$$x \log x \text{ is } o(3x^2+7x+1)$$

$$\lim_{x\to\infty} \frac{x \log x}{3x^2+7x+1}$$

$$\lim_{x\to\infty} \frac{\frac{\log x}{x}}{3+\frac{7}{x}+\frac{1}{x^2}}$$

$$\frac{\lim_{x\to\infty} \frac{\log x}{x}}{\lim_{x\to\infty} 3+\frac{7}{x}+\frac{1}{x^2}}$$

As $\log x$ increases slower than x and by definition $\lim_{x\to\infty} \frac{1}{x} = 0$,

$$\frac{\lim_{x\to\infty} \frac{\log x}{x}}{\lim_{x\to\infty} 3+\frac{7}{x}+\frac{1}{x^2}} = \frac{0}{3+0+0} = 0$$

.
Therefore the statement is correct.

4.4

Prove or disprove x^2+4x is $\Omega(x \log x)$. Assuming x^2+4x is $\Omega(x \log x)$ implies:

$$x^2+4x \geq C \cdot x \log x$$

$$x+4 \geq C \log x$$

$$2^{x+4} \geq 2^{C \log x}$$

$$16 \cdot 2^x \geq x^C$$

Therefore the statement is correct as exponential functions always grow faster than polynomial functions for large x. A pair of witness that confirms this is $C=2$ and $k=0$.

4.5

Prove or disprove $f(x)+g(x)$ is $\Theta(f(x) \cdot g(x))$. Suppose:

$$f(x) = g(x) = x$$

$$f(x)+g(x) = 2x$$

$$f(x) \cdot g(x) = x^2$$

As $2x$ is not $\Theta(x^2)$, the statement is incorrect by counterexample.

5 Master Theorem

5.1

$$T(n) = 9T(\frac{n}{3}) + n^2$$

$$f(n) = n^2 \text{ and } n^{\log_b a} = n^2$$

As $f(n)$ is asymptotically the same as $n^{\log_b a}$, $T(n) = \Theta(n^2 \log n)$

5.2

$$T(n) = 4T(\frac{n}{2}) + 100n$$

$$f(n) = 100n \text{ and } n^{\log_b a} = n^2$$

As $f(n)$ is $O(n^2)$, $T(n) = \Theta(n^2)$

5.3

$$T(n) = 2^n T(\frac{n}{2}) + n^3$$

As 2^n is not a constant, the master theorem cannot be applied.

5.4

$$T(n) = 3T(\frac{n}{3}) + cn$$

$$f(n) = cn \text{ and } n^{\log_b a} = n$$

As $f(n)$ is asymptotically the same as $n^{\log_b a}$, $T(n) = \Theta(n \log n)$

5.5

$$T(n) = 0.99T(\frac{n}{7}) + \frac{1}{n^2}$$

Since $a < 1$, the master theorem cannot be applied.

6 Merge/Selection Sort

An example of a worst case input for this hybrid merge/selection sorting algorithm is $[8, 6, 4, 2, 7, 5, 3, 1]$. When considering the worst case, we can ignore the selection sort part of the algorithm as a selection sort on a list length n has a fixed number of comparisons of $\frac{1}{2}n(n-1)$. If you consider the merge sort, this input means that the most number of comparisons are carried out when the two lists of 4 sorted elements are merged. Hence, this is the worst case input.